2-d Collisions

Based on "2-Dimensional Elastic Collisions Without Trigonometry" by Chad Bercheck.
http://www.geocities.com/vobarian/2dcollisions/

If you are just looking for the equations to use in getting the new velocities for objects 1 and 2 after the collision, you need
- Equations A and B for object 1, see pages 7-8
- Equations C and D for object 2, see page 10

One-dimensional Collision:

Start with a collision in which the two objects collide "head-on". This can take place in two-dimensional (or three-dimensional) space.

Consider a collision where both particles are moving along the x-axis. In what follows, the two objects can be moving in opposite directions when they collide, or they can be moving in the same direction, with one overtaking the other.

In any collision, elastic or inelastic, momentum is always conserved:

$\mathbf{P}$ = momentum, a vector (has both magnitude and direction)
$\mathbf{P_1}$ = momentum of object 1 before the collision
$\mathbf{P_{1f}}$ = momentum of object 1 after the collision
$\mathbf{P} = m\mathbf{v}$
$\mathbf{P_1} + \mathbf{P_2} = \mathbf{P_{1f}} + \mathbf{P_{2f}}$ always true, elastic or inelastic collision
$m_1\mathbf{v_1} + m_2\mathbf{v_2} = m_1\mathbf{v_{1f}} + m_2\mathbf{v_{2f}}$ Equation 1

Kinetic energy is conserved for elastic collisions (it is not conserved for inelastic collisions). Kinetic energy for a mass is $\frac{1}{2}mv^2$. Since velocity is squared in this formula, it is no longer a vector quantity. Thus, kinetic energy is a scalar, not a vector, quantity. For two circular objects, the point of the collision is the place on each circle where the objects collide. Since we are only considering elastic collisions, there is no deformation of either circle.

KE = kinetic energy, a scalar (only magnitude, no direction)
$KE = \frac{1}{2}mv^2$
$KE_1 + KE_2 = KE_{1f} + KE_{2f}$
$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$ Equation 2

Using momentum (Equation 1):

$m_1v_1 + m_2v_2 = m_1v_{1f} + m_2v_{2f}$
$m_1(v_1 - v_{1f}) = m_2(v_{2f} - v_2)$ Equation 3

Using kinetic energy (Equation 2):

$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$
$m_1(v_1^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_2^2)$ Equation 4

Dividing Equation 4 by Equation 3:

$$\frac{m_1(v_1^2 - v_{1f}^2)}{m_1(v_1 - v_{1f})} = \frac{m_2(v_{2f}^2 - v_2^2)}{m_2(v_{2f} - v_2)}$$

$v_1 + v_{1f} = v_{2f} + v_2$
$v_{1f} = v_{2f} + v_2 - v_1$

Substitute for v_{1f} in Equation 3:

$m_1(v_1 - (v_{2f} + v_2 - v_1)) = m_2(v_{2f} - v_2)$
$2m_1v_1 - m_1v_{2f} - m_1v_2 = m_2v_{2f} - m_2v_2$
$2m_1v_1 - m_1v_2 + m_2v_2 = m_1v_{2f} + m_2v_{2f}$
$2m_1v_1 + v_2(m_2 - m_1) = v_{2f}(m_1 + m_2)$

$$v_{2f} = \frac{2m_1v_1 + v_2(m_2 - m_1)}{m_1 + m_2} \quad \text{Equation 5}$$

Repeat for v_{1f}:

$v_{2f} = v_1 + v_{1f} - v_2$
$m_1(v_1 - v_{1f}) = m_2(v_1 + v_{1f} - v_2 - v_2)$
$m_1v_1 - m_1v_{1f} = m_2v_1 + m_2v_{1f} - 2m_2v_2$
$m_1v_{1f} + m_2v_{1f} = m_1v_1 - m_2v_1 + 2m_2v_2$
$v_{1f}(m_1 + m_2) = v_1(m_1 - m_2) + 2m_2v_2$

$$v_{1f} = \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} \quad \text{Equation 6}$$

Note: For cases where $m_1 = m_2$, Equations 5 and 6 become:

$\mathbf{v_{2f} = v_1}$
$\mathbf{v_{1f} = v_2}$

Collisions in Two Dimensions:

Known:

Particle 1:

mass: m_1
velocity: $\mathbf{v_{1x}}, \mathbf{v_{1y}}$
position: x_1, y_1

Particle 2:

mass: m_2
velocity: $\mathbf{v_{2x}}, \mathbf{v_{2y}}$
position: x_2, y_2

Need, where f = final = after the collision:

Particle 1:

velocity: $\mathbf{v_{1xf}}, \mathbf{v_{1yf}}$

Particle 2:

velocity: $\mathbf{v_{2xf}}, \mathbf{v_{2yf}}$

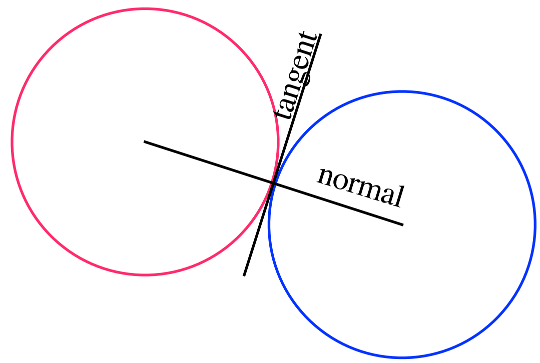
Note about mass: If $m_1 == m_2$, then equations do get simpler. I show below where this assumption comes into the equations. The final result given here does assume that $m_1 == m_2$.

Characteristics of an elastic collision between two circular masses:
Momentum is conserved for any collision, elastic or inelastic. Momentum for one object is mv, where m is the mass and v is the velocity vector. Mass is a scalar quantity, velocity is a vector quantity. Thus, momentum is a vector.

Now, draw two lines:

One line is drawn from the center of one object to the center of the other object. This is the "normal".

The second line is drawn perpendicular to the normal at the point of the collision. This is the "tangent".

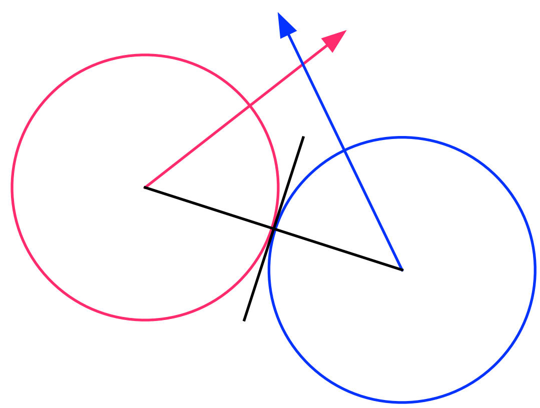


<u>The key point</u> in finding the velocities of the objects after an elastic collision:

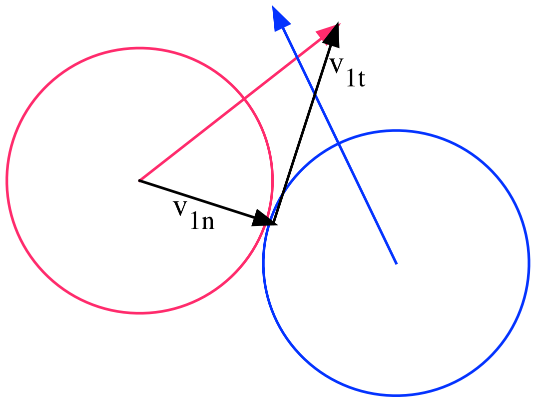
Treat the normal line as a one-dimensional elastic collision.

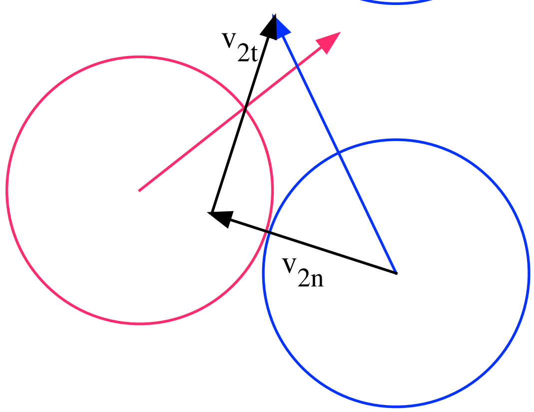
Find the new velocities along the normal for each of the two bodies.

The original velocities of both objects along the tangent *will not change*.



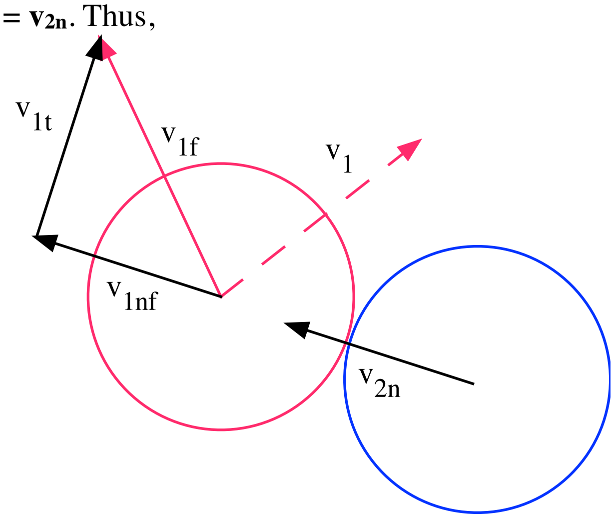
For each object, take its velocity vector and find the component of its velocity that is parallel to the normal, and the component that is parallel to the tangent. I.e., for the two objects above, consider their velocities:



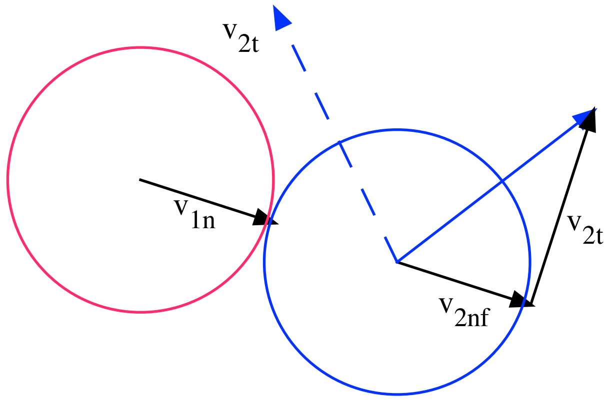


The two tangent components, $\mathbf{v_{1t}}$ and $\mathbf{v_{2t}}$, will be the same after the collision.
The two normal components, $\mathbf{v_{1n}}$ and $\mathbf{v_{2n}}$, will not be the same after the collision. These two velocities behave as a one-dimensional elastic collision.
Consider the case of $m_1 == m_2$:

> To find $\mathbf{v_{1f}}$, we need the vector sum of $\mathbf{v_{1t}}$ (which does not change) and $\mathbf{v_{1nf}}$. When the masses are the same, the final normal velocity for object 1 is the original normal velocity for object 2; i.e., $\mathbf{v_{1nf}} = \mathbf{v_{2n}}$. Thus,



Ditto for object 2, where $\mathbf{v_{2nf}} = \mathbf{v_{1n}}$:



We are not quite done. While the drawings above do show the correct final velocities, our calculations need $\mathbf{v_{1xf}}$, $\mathbf{v_{1yf}}$ and $\mathbf{v_{2xf}}$, $\mathbf{v_{2yf}}$. These are the x and y components of the final velocities for objects 1 and 2.

Find unit normal and unit tangent vectors:

n = normal vector

This is a vector running from the center of particle 1 to the center of particle 2. Note that it does not matter here which particle is 1 and which is 2.

n = $< x_2 - x_1, y_2 - y_1 >$ This is a vector add of a vector in the x-direction and a vector in the y-direction. Compute the two parts by subtracting the coordinates. When subtracting the x-coordinates, you are computing the x-direction; ditto for the y-direction.

un = unit normal vector. A unit vector is a vector with the same direction, but with magnitude 1.

$\mathbf{un_x}$ = x-component of unit normal vector
$\mathbf{un_y}$ = y-component of unit normal vector

The unit normal vector has the same direction as the normal vector, but with a magnitude of 1. It is computed by taking the x- and y-components and dividing each by the magnitude of the normal vector. The magnitude is found by computing the distance between the two points. The distance between the two points is sqrt($(x_2 - x_1)^2 + (y_2 - y_1)^2$). To avoid typing this formula repeatedly, it is represented as sqrt[] in the formulas below.

un = $< (x_2 - x_1)$ / sqrt[], $(y_2 - y_1)$ / sqrt[] $>$
$\mathbf{un_x}$ = $(x_2 - x_1)$ / sqrt[] = the unit normal vector component in the x-direction.
$\mathbf{un_y}$ = $(y_2 - y_1)$ / sqrt[] = the unit normal vector component in the y-direction.

t = tangent vector

This is a vector perpendicular to the normal vector described above. It takes its name from the tangent line at the point of collision of two circular masses.

ut = unit tangent vector

$\mathbf{ut_x}$ = x-component of unit tangent vector
$\mathbf{ut_y}$ = y-component of unit tangent vector

t = $<-(y_2 - y_1), (x_2 - x_1)>$

ut = $< -(y_2 - y_1)$ / sqrt[], $(x_2 - x_1)$ / sqrt[] $>$
$\mathbf{ut_x}$ = $-(y_2 - y_1)$ / sqrt[]
$\mathbf{ut_y}$ = $(x_2 - x_1)$ / sqrt[]

Find normal and tangent for the original $\mathbf{v_1}$ and $\mathbf{v_2}$:

For each of the two particles, find the magnitude of their original velocities relative to the unit normal and unit tangent directions. This is done by taking a dot product of $\mathbf{v_1} \bullet \mathbf{n}$ and $\mathbf{v_1} \bullet \mathbf{t}$ (and analogously for $\mathbf{v_2}$). Since we know the x- and y-components of $\mathbf{v_1}$ and $\mathbf{v_2}$, we compute these as follows:

$$v_{1n} = \mathbf{un} \bullet \mathbf{v_1} = un_x v_{1x} + un_y v_{1y} = [v_{1x}(x_2 - x_1) + v_{1y}(y_2 - y_1)] / \text{sqrt}[]$$
$$v_{2n} = \mathbf{un} \bullet \mathbf{v_2} = un_x v_{2x} + un_y v_{2y} = [v_{2x}(x_2 - x_1) + v_{2y}(y_2 - y_1)] / \text{sqrt}[]$$
$$v_{1t} = \mathbf{ut} \bullet \mathbf{v_1} = ut_x v_{1x} + ut_y v_{1y} = [-v_{1x}(y_2 - y_1) + v_{1y}(x_2 - x_1)] / \text{sqrt}[]$$
$$v_{2t} = \mathbf{ut} \bullet \mathbf{v_2} = ut_x v_{2x} + ut_y v_{2y} = [-v_{2x}(y_2 - y_1) + v_{2y}(x_2 - x_1)] / \text{sqrt}[]$$

Note: v_{1n}, v_{1t}, v_{2n}, and v_{2t} are scalar values, not vectors. They can become vectors if we include their directions. For v_{1n} and v_{2n}, the direction is either in the same direction as, or the opposite direction of, the unit normal vector, **un**. For v_{1t} and v_{2t}, the direction is either in the same direction as, or the opposite direction of, the unit tangent vector, **ut**.

Find normal and tangent <u>after</u> collision for $\mathbf{v_1}$ and $\mathbf{v_2}$:

Find the normal and tangent values for both particles after the collision. This will give us $\mathbf{v_{1nf}}$, $\mathbf{v_{1tf}}$, $\mathbf{v_{2nf}}$ and $\mathbf{v_{2tf}}$, where f indicates the final velocity after the collision.

- The tangential velocities after the collision are the same for $\mathbf{v_{1f}}$ and $\mathbf{v_{2f}}$ as before the collision; this result derives from the conservation of momentum.
- The normal velocities of $\mathbf{v_{1f}}$ and $\mathbf{v_{2f}}$ are different after the collision; this result derives from the one-dimension collision and uses conservation of momentum and kinetic energy.

$\mathbf{v_{1tf} = v_{1t}}$

$\mathbf{v_{2tf} = v_{2t}}$

$\mathbf{v_{1nf}} = [v_{1n}(m_1 - m_2) + 2m_2 v_{2n}] / (m_1 + m_2)$

If $m_1 == m_2$, this equation becomes:

$\mathbf{v_{1nf}} = [v_{1n}(m - m) + 2mv_{2n}] / (m + m) = 2mv_{2n} / (2m)$

$\mathbf{v_{1nf} = v_{2n}}$

$\mathbf{v_{2nf}} = [v_{2n}(m_1 - m_2) + 2m_1 v_{1n}] / (m_1 + m_2)$

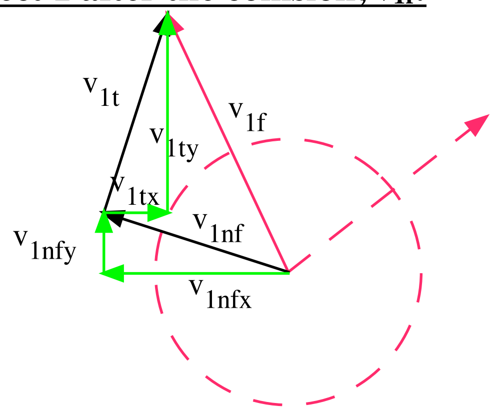
If $m_1 == m_2$, this equation becomes:

$v_{2nf} = [v_{2n}(m - m) + 2mv_{1n}] / (m + m) = 2mv_{1n} / (2m)$

$\mathbf{v_{2nf} = v_{1n}}$

From this point forward, assume $m_1 == m_2$.

Find the velocity of Object 1 after the collision, $\mathbf{v_{1f}}$:



For particle 1, we know: $\mathbf{v_{1nf}}$ and $\mathbf{v_{1t}}$. We can use these two to find:

$$\mathbf{v_{1f}} = \mathbf{v_{1nf}} + \mathbf{v_{1t}}$$

as shown in the drawing above.

But, we need $\mathbf{v_{1xf}}$ and $\mathbf{v_{1yf}}$; that is, we need the x- and y-components of $\mathbf{v_{1f}}$. We compute:

$$\mathbf{v_{1nf}} = \langle v_{1nfx}, v_{1nfy} \rangle$$

and

$$\mathbf{v_{1t}} = \langle v_{1tx}, v_{1ty} \rangle$$

Then, we can add vectors to get:

$$\mathbf{v_{1f}} = \langle v_{1nfx} + v_{1tx}, v_{1nfy} + v_{1ty} \rangle.$$

Compute $\mathbf{v_{1nf}}$:

$\mathbf{v_{1nf}}$ is composed of 4 vectors which are then added together: $\mathbf{v_{1nfx}}$, $\mathbf{v_{1nfy}}$, $\mathbf{v_{1tfx}}$, and $\mathbf{v_{1tfy}}$. These are the four arrows shown in green in the previous drawing.

Find $\mathbf{v_{1nfx}}$. This is done by multiplying $\mathbf{v_{1nf}}$ by the x-component of the unit normal vector:

$$\mathbf{v_{1nfx}} = \frac{\mathbf{v_{1nf}}(x_2 - x_1)}{\text{sqrt}[]} = \frac{\mathbf{v_{2n}}(x_2 - x_1)}{\text{sqrt}[]} =$$

$$\frac{v_{2x}(x_2 - x_1) + v_{2y}(y_2 - y_1)}{\text{sqrt}[]} \; \frac{(x_2 - x_1)}{\text{sqrt}[]} =$$

$$\frac{v_{2x}(x_2 - x_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Find $\mathbf{v_{1nfy}}$. This is done by multiplying $\mathbf{v_{1nf}}$ by the y-component of the unit normal vector:

$$\mathbf{v_{1nfy}} = \frac{\mathbf{v_{1nf}}(y_2 - y_1)}{\text{sqrt[]}} = \frac{\mathbf{v_{2n}}(y_2 - y_1)}{\text{sqrt[]}} =$$

$$\frac{v_{2x}(x_2 - x_1) + v_{2y}(y_2 - y_1)}{\text{sqrt[]}} \; \frac{(y_2 - y_1)}{\text{sqrt[]}} =$$

$$\frac{v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(y_2 - y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Compute $\mathbf{v_{1t}}$:

The tangent velocity after the collision is the same as the tangent velocity before the collision; i.e., $\mathbf{v_{1tf}} = \mathbf{v_{1t}}$. We need the x- and y-components of the tangent velocity to add to the x- and y-components of the normal velocity. To find these two components of the tangent velocity, we use:

$$\mathbf{ut} = \left< \frac{-(y_2 - y_1)}{\text{sqrt[]}}, \frac{-(x_2 - x_1)}{\text{sqrt[]}} \right> = \text{unit tangent vector}$$

$\mathbf{ut_x} = -(y_2 - y_1) / \text{sqrt[]}$
$\mathbf{ut_y} = (x_2 - x_1) / \text{sqrt[]}$

$\mathbf{v_{1t}} = \mathbf{ut} \bullet \mathbf{v_1} = ut_x v_{1x} + ut_y v_{1y}$

$$\mathbf{v_{1t}} = \frac{v_{1x}(-(y_2 - y_1)) + v_{1y}(x_2 - x_1)}{\text{sqrt[]}}$$

separating $\mathbf{v_{1t}}$ into its x- and y-components gives:

$$\mathbf{v_{1tx}} = \frac{v_{1x}(-(y_2 - y_1))}{\text{sqrt[]}}$$

$$\mathbf{v_{1ty}} = \frac{v_{1y}(x_2 - x_1)}{\text{sqrt[]}}$$

Find $\mathbf{v_{1tx}}$, then $\mathbf{v_{1ty}}$.

$$\mathbf{v_{1tx}} = \frac{\mathbf{v_{1t}}(-(y_2 - y_1))}{\text{sqrt[]}} = \frac{-v_{1x}(y_2 - y_1) + v_{1y}(x_2 - x_1)}{\text{sqrt[]}} \; \frac{-(y_2 - y_1)}{\text{sqrt[]}} =$$

$$\frac{v_{1x}(y_2 - y_1)^2 - v_{1y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{1ty}} = \frac{v_{1t}(x_2 - x_1)}{\text{sqrt[]}} = \frac{-v_{1x}(y_2 - y_1) + v_{1y}(x_2 - x_1)}{\text{sqrt[]}} \; \frac{(x_2 - x_1)}{\text{sqrt[]}} =$$

$$\frac{-v_{1x}(y_2 - y_1)(x_2 - x_1) + v_{1y}(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Compute x- and y-velocities for Object 1 after the collision:

We now have the four pieces needed to compute $\mathbf{v_{1f}}$:

$\mathbf{v_{1f}} = \mathbf{v_{1xf}} + \mathbf{v_{1yf}} = \langle v_{1nfx} + v_{1tx}, v_{1nfy} + v_{1ty} \rangle$

$$\mathbf{v_{1nfx}} = \frac{v_{2x}(x_2 - x_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \mathbf{v_{1tx}} = \frac{v_{1x}(y_2 - y_1)^2 - v_{1y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{1fx}} = \frac{v_{2x}(x_2 - x_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \frac{v_{1x}(y_2 - y_1)^2 - v_{1y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{1fx}} = \frac{v_{2x}(x_2 - x_1)^2 + v_{2y}(x_2 - x_1)(y_2 - y_1) + v_{1x}(y_2 - y_1)^2 - v_{1y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Equation A}$$

$\mathbf{v_{1fy}} = \mathbf{v_{1nfy}} + \mathbf{v_{1ty}}$ Note below that both terms have the same denominator:

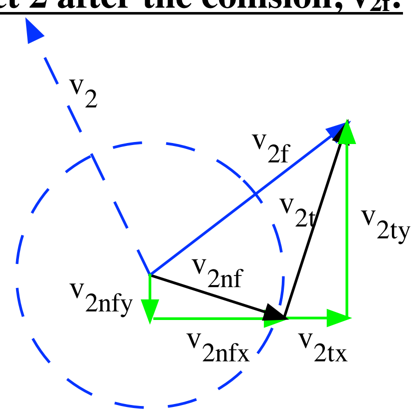
$$\mathbf{v_{1nfy}} = \frac{v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(y_2 - y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \mathbf{v_{1ty}} = \frac{-v_{1x}(y_2 - y_1)(x_2 - x_1) + v_{1y}(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{1fy}} = \frac{v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(y_2 - y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \frac{-v_{1x}(y_2 - y_1)(x_2 - x_1) + v_{1y}(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{1fy}} = \frac{v_{2x}(x_2 - x_1)(y_2 - y_1) + v_{2y}(y_2 - y_1)^2 - v_{1x}(y_2 - y_1)(x_2 - x_1) + v_{1y}(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Equation B}$$

$\mathbf{v_{1fx}}$ and $\mathbf{v_{1fy}}$ from above are the formulas needed to compute the x and y components of object 1 after the elastic collision.

<u>Find the velocity of Object 2 after the collision, $\mathbf{v_{2f}}$:</u>



For particle 2, we know: $\mathbf{v_{2nf}}$ and $\mathbf{v_{2t}}$. We can use these two to find:

$\mathbf{v_{2f}} = \mathbf{v_{2nf}} + \mathbf{v_{2t}}$

as shown in the drawing above.

But, we need $\mathbf{v_{2xf}}$ and $\mathbf{v_{2yf}}$; that is, we need the x- and y-components of $\mathbf{v_{2f}}$. We compute:

$\mathbf{v_{2nf}} = \langle v_{2nfx}, v_{2nfy} \rangle$

and

$\mathbf{v_{2t}} = \langle v_{2tx}, v_{2ty} \rangle$

Then, we can add vectors to get:

$\mathbf{v_{2f}} = \langle v_{2nfx} + v_{2tx}, v_{2nfy} + v_{2ty} \rangle$.

Compute $\mathbf{v_{2nf}}$:

$\mathbf{v_{2nf}}$ is composed of 4 vectors which are then added together: $\mathbf{v_{2nfx}}$, $\mathbf{v_{2nfy}}$, $\mathbf{v_{2tfx}}$, and $\mathbf{v_{2tfy}}$. These are the four arrows shown in green in the previous drawing.

Find $\mathbf{v_{2nfx}}$. This is done by multiplying $\mathbf{v_{2nf}}$ by the x-component of the unit normal vector:

$$\mathbf{v_{2nfx}} = \frac{\mathbf{v_{2nf}}(x_2 - x_1)}{\text{sqrt}[]} = \frac{\mathbf{v_{1n}}(x_2 - x_1)}{\text{sqrt}[]} =$$

$$\frac{v_{1x}(x_2 - x_1) + v_{1y}(y_2 - y_1)}{\text{sqrt}[]} \; \frac{(x_2 - x_1)}{\text{sqrt}[]} =$$

$$\frac{v_{1x}(x_2 - x_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Find $\mathbf{v_{2nfy}}$. This is done by multiplying $\mathbf{v_{2nf}}$ by the y-component of the unit normal vector:

$$\mathbf{v_{2nfy}} = \frac{\mathbf{v_{2nf}}(y_2 - y_1)}{\text{sqrt}[]} = \frac{\mathbf{v_{1n}}(y_2 - y_1)}{\text{sqrt}[]} =$$

$$\frac{v_{1x}(x_2 - x_1) + v_{1y}(y_2 - y_1)}{\text{sqrt}[]} \; \frac{(y_2 - y_1)}{\text{sqrt}[]} =$$

$$\frac{v_{1x}(x_2 - x_1)(y_2 - y_1) + v_{1y}(y_2 - y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Compute $\mathbf{v_{2t}}$:

The tangent velocity after the collision is the same as the tangent velocity before the collision; i.e., $\mathbf{v_{1tf}} = \mathbf{v_{1t}}$. We need the x- and y-components of the tangent velocity to add to the x- and y-components of the normal velocity. To find these two components of the tangent velocity, we use:

$$\mathbf{ut} = \left\langle \frac{-(y_2 - y_1)}{\text{sqrt}[]}, \frac{-(x_2 - x_1)}{\text{sqrt}[]} \right\rangle = \text{unit tangent vector}$$

$\mathbf{ut_x} = -(y_2 - y_1) / \text{sqrt}[]$

$\mathbf{ut_y} = (x_2 - x_1) / \text{sqrt}[]$

$\mathbf{v_{1t}} = \mathbf{ut} \bullet \mathbf{v_1} = ut_x v_{1x} + ut_y v_{1y}$

$$\mathbf{v_{2t}} = \frac{v_{2x}(-(y_2 - y_1)) + v_{2y}(x_2 - x_1)}{\text{sqrt}[]}$$

separating $\mathbf{v_{1t}}$ into its x- and y-components gives:

$$\mathbf{v_{2tx}} = \frac{v_{2x}(-(y_2 - y_1))}{\text{sqrt}[]}$$

$$\mathbf{v_{2ty}} = \frac{v_{2y}(x_2 - x_1)}{\text{sqrt}[]}$$

Find $\mathbf{v_{2tx}}$, then $\mathbf{v_{2ty}}$.

$\mathbf{t} = \langle -(y_2 - y_1), (x_2 - x_1) \rangle$

$$\mathbf{v_{2tx}} = \frac{\mathbf{v_{2t}}(-(y_2 - y_1))}{\text{sqrt}[]} = \frac{-v_{2x}(y_2 - y_1) + v_{2y}(x_2 - x_1)}{\text{sqrt}[]} \quad \frac{-(y_2 - y_1)}{\text{sqrt}[]} =$$

$$\frac{v_{2x}(y_2 - y_1)^2 - v_{2y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{2ty}} = \frac{v_{2t}(x_2 - x_1)}{\text{sqrt}[]} = \frac{-v_{2x}(y_2 - y_1) + v_{2y}(x_2 - x_1)}{\text{sqrt}[]} \quad \frac{(x_2 - x_1)}{\text{sqrt}[]} =$$

$$\frac{-v_{2x}(y_2 - y_1)(x_2 - x_1) + v_{2y}(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Compute x- and y-velocities for Object 2 after the collision:

We now have the four pieces needed to compute $\mathbf{v_{2f}}$:

$\mathbf{v_{2f}} = \mathbf{v_{2xf}} + \mathbf{v_{2yf}} = \langle v_{2nfx} + v_{2tx}, v_{2nfy} + v_{2ty} \rangle$

$$\mathbf{v_{2nfx}} = \frac{v_{1x}(x_2 - x_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \mathbf{v_{2tx}} = \frac{v_{2x}(y_2 - y_1)^2 - v_{2y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{2fx}} = \frac{v_{1x}(x_2 - x_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \frac{v_{2x}(y_2 - y_1)^2 - v_{2y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{2fx}} = \frac{v_{1x}(x_2 - x_1)^2 + v_{1y}(x_2 - x_1)(y_2 - y_1) + v_{2x}(y_2 - y_1)^2 - v_{2y}(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Equation C}$$

$\mathbf{v_{2fy}} = \mathbf{v_{2nfy}} + \mathbf{v_{2ty}}$ Note below that both terms have the same denominator:

$$\mathbf{v_{2nfy}} = \frac{v_{1x}(x_2 - x_1)(y_2 - y_1) + v_{1y}(y_2 - y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \mathbf{v_{2ty}} = \frac{-v_{2x}(y_2 - y_1)(x_2 - x_1) + v_{2y}(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{2fy}} = \frac{v_{1x}(x_2 - x_1)(y_2 - y_1) + v_{1y}(y_2 - y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \frac{-v_{2x}(y_2 - y_1)(x_2 - x_1) + v_{2y}(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\mathbf{v_{2fy}} = \frac{v_{1x}(x_2 - x_1)(y_2 - y_1) + v_{1y}(y_2 - y_1)^2 - v_{2x}(y_2 - y_1)(x_2 - x_1) + v_{2y}(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Equation D}$$

$\mathbf{v_{2fx}}$ and $\mathbf{v_{2fy}}$ from above are the formulas needed to compute the x and y components of object 2 after the elastic collision.