COLLECTION

Honour School of Mathematics – Prelims

GEOMETRY & CALCULUS

January 2021

3 hours

*Attempt **4 questions** from Section A and **3 questions** from Section B. There are 10 questions in total. Questions in Section A carry 10 marks each and those in Section B carry 20 marks each.*

The numbers in the margin indicate the weight that the Examiner expects to assign to each part of the question.

Do NOT turn over until told that you may do so.

There are no questions on this page.

Section A

1. Consider the locus of all points $\mathbf{x} \in \mathbb{R}^3$ such that

$$f(\mathbf{x}) \equiv x^2 + y^2 - z + 1 = 0\,.$$

Show that the cylindrical coordinates (z, ϕ) provide a smooth parametrisation of this surface, except when $y = 0$ and $x \geq 0$.

[3]

Now consider the region D bounded by the plane $z = 2$ and the paraboloid above:

$$D = \{\mathbf{x} \in \mathbb{R}^3 : z \leq 2, f(\mathbf{x}) \leq 0\}\,.$$

Find the maximum value of the function $g : \mathbb{R}^3 \to \mathbb{R}$ given by

$$g(\mathbf{x}) = \Big(2(x^2 + y^2)^2 - (x^2 + y^2)\Big)(2 - z)$$

on D.

[7]

2. Define the "star product" $*$ of two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ by

$$\mathbf{x} * \mathbf{y} = x_1 y_1 - x_2 y_2 - x_3 y_3\,.$$

We say that a point $\mathbf{x} \in \mathbb{R}^3$ is *lightlike* if $\mathbf{x} * \mathbf{x} = 0$. Describe the set of all lightlike points. Show that if $\mathbf{x}$ and $\mathbf{y}$ are lightlike points satisfying $\mathbf{x} * \mathbf{y} = 0$, then $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

[5]

For some $\theta \in \mathbb{R}$ let M be the matrix

$$M = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that M preserves the star product, in the sense that

$$(M\mathbf{x}) * (M\mathbf{y}) = \mathbf{x} * \mathbf{y} \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3\,.$$

Show that if $M\mathbf{x} = \lambda\mathbf{x}$ then either $\mathbf{x}$ is lightlike or $\lambda^2 = 1$. Hence, or otherwise, find the eigenvectors of M.

[5]

3. Let (r, ϕ, z) be cylindrical coordinates for $\mathbb{R}^3$. Let M be a real constant and define the surface S, known as an "Einstein-Rosen bridge", by the parametrisation

$$r(\rho, \phi) = \rho\left(1 + \frac{M}{2\rho}\right)^2$$

$$z(\rho, \phi) = \sqrt{2M\rho}\left(2 - \frac{M}{\rho}\right),$$

where $\rho > 0$ and $\phi \in [0, 2\pi)$. Let C_ρ be a curve of constant ρ on this surface. Show that C_ρ is a geodesic if and only if $\rho = M/2$.

[5]

Find the surface area of that part of S bounded by the curves $C_{M/2}$ and C_M.

[5]

4. Consider the integral

$$I = \int_0^{\pi^2} \int_{\sqrt{y}}^{\pi} \frac{\sin(x)}{x^2} \, \mathrm{d}x \, \mathrm{d}y \, .$$

Describe the region of integration geometrically. By reparametrising this region, exchange the order of integration and rewrite the integral as

$$I = \int_a^b \int_{c(x)}^{d(x)} \frac{\sin(x)}{x^2} \, \mathrm{d}y \, \mathrm{d}x \, ,$$

with limits $a, b, c(x), d(x)$ that you should determine. Hence evaluate I.

[5]

By considering the region of integration geometrically, evaluate the integral

$$J = \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} \arctan\left(\frac{y}{x}\right) \, \mathrm{d}y \, \mathrm{d}x \, .$$

[5]

Section B

5. Let $\mathbf{x}(t) \in \mathbb{R}^3$ denote the position of the Earth as a function of time t. Suppose $\mathbf{x}(t)$ obeys the differential equation

$$\ddot{\mathbf{x}} \equiv \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{x}(t) = -\frac{k}{r^3}\mathbf{x}(t)\,,$$

where $r = |\mathbf{x}|$ and k is a real constant. Show that

$$\mathbf{h} = \mathbf{x} \wedge \dot{\mathbf{x}}$$

is constant, and deduce that the Earth is confined to a plane through the origin.

[6]

Show that

$$\mathbf{e} = \frac{\dot{\mathbf{x}} \wedge \mathbf{h}}{k} - \frac{\mathbf{x}}{r}$$

is constant, and hence that

$$er\cos\theta = \frac{h^2}{k} - r\,,$$

where $e = |\mathbf{e}|$, $h = |\mathbf{h}|$, and θ denotes the angle between $\mathbf{x}$ and $\mathbf{e}$. Deduce that the Earth moves on a conic section. Characterise the conic section in terms of e, h, and k.

[14]

6. Prove that for any integer $n \geq 0$

$$\lim_{t\to\pm\infty} t^n \exp(-t^2) = 0\,.$$

[4]

Define the "state coach function" $E : \mathbb{R} \to \mathbb{R}$ by

$$E(t) = \begin{cases} \exp(-1/t^2) & t \neq 0 \\ 0 & t = 0\,. \end{cases}$$

Suppose that for all $t \neq 0$ and some integer $m \geq 0$ we have

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}E(t) = P_m(1/t)E(t)\,,$$

where P_m is a polynomial. Show that the result is also true for $m+1$ and give an expression for P_{m+1}. Use induction to prove that the result is true for all $m \geq 0$.

[6]

By considering

$$\frac{E^{(m)}(h) - E^{(m)}(0)}{h}\,,$$

use induction to prove that $E(t)$ is infinitely differentiable at $t = 0$. Hence conclude that the Taylor series of $E(t)$ about $t = 0$ has radius of convergence $R = 0$.

[10]

7. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \exp(z)\,.$$

Writing $z = x + iy$ and $\mathbf{x} = (x, y)$ we can consider this function as a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. Give this map explicitly.

[3]

Let $\mathbf{x}(t)$, $\mathbf{y}(t)$ be curves in $\mathbb{R}^2$ and let $\mathbf{X}(t) = f(\mathbf{x}(t))$, $\mathbf{Y}(t) = f(\mathbf{y}(t))$. Suppose that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ intersect when $t = T$. Show that the angle between the tangent vectors

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}$$

is the same as that between the tangent vectors to $\mathbf{X}(t)$ and $\mathbf{Y}(t)$, when $t = T$.

[8]

Consider the area element $\mathrm{d}x\,\mathrm{d}y$ at a point (x, y). Find the factor by which this area increases under the map f. Hence, or otherwise, find the area of the image under f of the triangular region bounded by the lines $y = 2x$, $x = 1$, and $y = 0$.

[9]

8. Consider the region Q of the (T, P) plane with $T > 0$, $P > 0$. Suppose

$$PV = T\,.$$

Show that any pair of coordinates (T, P), (P, V), or (V, T) gives a smooth coordinate system on Q.

[4]

Let $S : Q \to \mathbb{R}$ be a smooth function on this quarter-plane. Using the chain rule, show that

$$\left.\frac{\partial S}{\partial T}\right|_V = \left.\frac{\partial S}{\partial T}\right|_P + \frac{1}{V}\left.\frac{\partial S}{\partial P}\right|_T .$$

[6]

Let $G : Q \to \mathbb{R}$ be another smooth function on this quarter-plane, thought of as a function of T and P, and suppose its first partial derivatives are given by

$$\left.\frac{\partial G}{\partial T}\right|_P = -S \qquad \text{and} \qquad \left.\frac{\partial G}{\partial P}\right|_T = V\,.$$

Now define

$$C_V = T\left.\frac{\partial S}{\partial T}\right|_V$$
$$C_P = T\left.\frac{\partial S}{\partial T}\right|_P ,$$

and show that $C_P - C_V = 1$.

[10]

9. Find all solutions to the following differential equations:

i)

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + \frac{1}{x}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = x\,.$$

[6]

ii)

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{x^2} + \exp\left(-\frac{y}{x^2}\right)\,.$$

[8]

iii)

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 6x^2 = 0\,.$$

[6]

10. Define what is meant by an orthogonal matrix R in three dimensions. State the condition for R to be a rotation matrix and give an expression relating the trace of R to the angle of rotation.

[3]

Define the exponential of a matrix by the infinite series:

$$\exp(M) = I + M + \frac{1}{2}M^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}M^k\,.$$

You may assume $\exp(M)^{-1} = \exp(-M)$. Show that if M is antisymmetric then $\exp(M)$ is orthogonal.

[3]

Show that if λ is an eigenvalue of M then $\exp(\lambda)$ is an eigenvalue of $\exp(M)$ with the same eigenvector. Hence prove that if M is diagonalisable then

$$\det(\exp(M)) = \exp(\mathrm{tr}(M))\,,$$

and argue that no reflection matrix can be written as the exponential of an antisymmetric matrix.

[6]

Let $\alpha, \beta, \gamma \in \mathbb{R}$ and consider the orthogonal matrix

$$R = \exp(M) \qquad \text{where} \qquad M = \begin{pmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0 \end{pmatrix}\,.$$

Show that R has eigenvector $\mathbf{v} = (\alpha, \beta, \gamma)^T$ and compute the corresponding eigenvalue. Compute the eigenvalues of M and thereby characterise the rotation that R describes.

[8]