COLLECTION

Honour School of Physics – Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy – Part B

B5: GENERAL RELATIVITY AND COSMOLOGY

3 hours

Attempt 3 questions. There are 5 questions in total. Each question carries equal weight.

The numbers in the margin indicate the weight that the Examiner expects to assign to each part of the question.

All questions use the $-+++$ metric convention and the convention $c = 1$.

Do NOT turn over until told that you may do so.

1. The energy-momentum tensor for a perfect fluid is given by

$$T^{\mu\nu} = Pg^{\mu\nu} + (\rho + P)U^\mu U^\nu \,,$$

where ρ and P are the rest-frame density and pressure of the fluid and U^μ is its four-velocity. For a fluid consisting of massive particles, explain why

$$U_\mu U^\mu = -1 \qquad \text{and} \qquad U_\mu \nabla_\nu U^\mu = 0\,.$$

Hence, using the conservation of energy-momentum, derive the fluid equations

$$\begin{aligned} U^\mu \nabla_\mu \rho + (\rho + P)\nabla_\mu U^\mu &= 0 \\ (\rho + P)U^\mu \nabla_\mu U^\nu + (g^{\mu\nu} + U^\mu U^\nu)\nabla_\mu P &= 0\,. \end{aligned} \tag{$*$}$$

(Hint: you may wish to contract the conservation equation with the four-velocity to obtain a scalar equation.)

[8]

Assume that this fluid lives in Minkowski spacetime (that is, assume that ρ and P are small enough that we can neglect their effect on the spacetime geometry). Write down the Minkowski metric in cylindrical polar coordinates (t, r, ϕ, z). Now write down the metric in "rotating coordinates" (t, χ, r, z) defined by $\chi = \phi - \omega t$, where ω is some constant.

[2]

Without explicit calculation, determine the components of the Riemann tensor in this coordinate system. Justify your answer.

[2]

Suppose that in these coordinates the fluid is at rest. Write down the components of the-four-velocity U^μ in these coordinates.

[2]

Suppose also that ρ and P depend only on r. By examining the $\nu = r$ component of equation $(*)$ in rotating coordinates, show that the pressure obeys the following equation

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{r\omega^2(\rho + P)}{1 - \omega^2 r^2}\,.$$

(You may need the expression for the connection coefficients in terms of the metric; see question 2 below.)

[6]

2. The geodesic equation for a particle moving freely in a curved spacetime is

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\sigma^2} + \Gamma^\mu{}_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\sigma} = 0\,,$$

where σ is an affine parameter along the geodesic and the connection coefficients are given by

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho} \right)\,.$$

Show that the geodesic equation can be obtained by applying the Euler-Lagrange equations to the Lagrangian

$$L = g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\,.$$

[8]

Consider the following metric, which describes a 1+1-dimensional spacetime,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}x^2\,.$$

Obtain the geodesic equations for this particular metric by applying the Euler-Lagrange equations to the Lagrangian as defined above. Hence, or otherwise, write down the connection coefficients for this metric.

[4]

From the connection coefficients we can compute the Ricci tensor. We find that $R_{xx} = a\ddot{a}$, $R_{tt} = -\ddot{a}/a$, and $R_{tx} = R_{xt} = 0$, where dot denotes differentiation with respect to t. Using these, solve the Einstein equations (neglecting the cosmological constant term) to determine the energy-momentum tensor that could give rise to this geometry.

[3]

This metric is a model of an expanding universe. Explain the concept of a particle horizon in this context. Compute the particle horizon in terms of the time today t_0 for the cases *i)* $a(t) = \sqrt{\alpha t}$, and *ii)* $a(t) = e^{\alpha t}$, for some positive constant α. Comment on your answers.

[5]

3. The Schwarzschild metric in Schwarzschild coordinates is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\,.$$

In this spacetime we can always assume our motion is in the plane $\theta = \pi/2$. In this case, the t and ϕ geodesic equations are

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}\sigma} = h$$
$$\left(1 - \frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\sigma} = k\,,$$

where σ is an affine parameter along the geodesic and h and k are constants. Instead of the geodesic equation for r, it is often convenient to use the normalisation condition on the four-velocity. Write this down for massive and massless particles.

[1]

Consider a massive particle dropped radially from rest at infinity. Calculate the proper time experienced by the particle between the points $r = R$ and $r = 0$.

[6]

Now consider a photon moving along a general geodesic. Show that the trajectory must satisfy

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\sigma}\right)^2 + V(r) = 0\,,$$

where $V(r)$ is a function which you should write down explicitly. By examining the form of $V(r)$, argue that there are no stable circular photon orbits. Show however that there is an unstable circular orbit at $r = 3GM$.

[7]

Consider two identical computers in Schwarzschild spacetime, one dropped radially from rest at infinity and one held at rest at $r = 3GM$. When the falling computer passes the computer at rest, both begin computing the digits of pi using an identical algorithm, and additionally the computer at rest fires a laser pulse in the tangential direction (such as not to affect the motion of the computer). Assuming that the falling computer is destroyed when it hits the singularity, and the computer at rest is destroyed when the laser pulse returns and hits it from behind, which computer manages to compute more digits of pi before being destroyed?

[4]

Briefly explain how we know that $r = 0$ is a real, physical singularity, whilst $r = 2GM$ is not.

[2]

4. In harmonic gauge the linearised Einstein equations are

$$\partial_\rho \partial^\rho \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \qquad \text{where} \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\rho{}_\rho \,.$$

Find the time-independent spherically-symmetric solution to these equations for a stationary point mass, with energy-momentum tensor

$$T_{\mu\nu} = M\delta(x)\delta(y)\delta(z)\mathrm{diag}(1,0,0,0)\,,$$

assuming that the metric tends to Minkowski spacetime at large distances. Here $\delta(x)$ is the one-dimensional Dirac delta function.

[6]

Now suppose the point mass undergoes non-relativistic simple harmonic motion, so that $T_{\mu\nu} = M\delta(x)\delta(y)\delta(z - z_0\sin(\omega t))\mathrm{diag}(1,0,0,0)$. What is the average rate at which energy is lost to gravitational waves by the mass?

[4]

Next, find all solutions which depend only on the cylindrical coordinate $r = \sqrt{x^2+y^2}$ to the linearised Einstein equations for a "cosmic string", with energy-momentum tensor

$$T_{\mu\nu} = \mu\delta(x)\delta(y)\mathrm{diag}(1,0,0,-1)\,,$$

where μ is a small positive constant. In particular, show that the metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g(r)(\mathrm{d}r^2 + r^2\mathrm{d}\phi^2) + \mathrm{d}z^2\,,$$

where $g(r) = 1 - 8\mu G\ln(r/r_0)$.

[5]

Perform a coordinate transformation, giving explicit forms of the functions $u(r)$ and $\psi(\phi)$, to bring the metric for the cosmic string above to the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}u^2 + u^2\mathrm{d}\psi^2 + \mathrm{d}z^2\,.$$

Work only to first order in μ throughout. Is this Minkowski spacetime? Argue that distant objects may give rise to double images.

[5]

(*Hint: it may help to recall the form of the electric potential for point and line charges in electromagnetism.*)

5. The Friedmann-Lemaître-Robertson-Walker metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin^2 \theta \mathrm{d}\phi^2 \right) \qquad k = -1, 0, 1 \,.$$

Explain qualitatively how this metric is obtained from some basic initial assumptions.

[2]

The Friedmann equation for a universe with cosmological constant Λ, spatial curvature k, and (pressureless) matter density ρ is

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3} \,.$$

Using this and the continuity equation for the matter, show that a obeys the equation

$$\frac{1}{2} \dot{a}^2 + V(a) = 0 \qquad \text{where} \qquad V(a) = -\frac{4\pi G}{3} \frac{\rho_0}{a} + \frac{k}{2} - \frac{\Lambda}{6} a^2 \,,$$

where dot denotes differentiation with respect to t, ρ_0 denotes the matter density today, and where we assume the scale factor today satisfies $a_0 = 1$. Sketch $V(a)$ for the three cases *i)* $k = 0$, $\Lambda = 0$, *ii)* $k = 1$, $\Lambda = 0$, and *iii)* $k = 0$, $\Lambda > 0$. Assuming the universe "starts" such that a is increasing at early times, describe the evolution of the universe in each case. Where applicable determine the maximal value of the scale factor.

[10]

We now consider the thermodynamics of an expanding universe. For this part of the question we take $\hbar = k_B = 1$. Suppose that the universe contains only photons, free protons, free electrons, and hydrogen atoms. Assume thermal equilibrium, so that the number density of particle species i at temperature T is given by

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T} \right) \,,$$

where m_i, μ_i, and g_i are the mass, chemical potential, and number of internal degrees of freedom of particle species i. Show that the number density of hydrogen is given by

$$n_H \simeq \left(\frac{2\pi}{m_e T} \right)^{3/2} \exp\left(\frac{B}{T} \right) n_p^2 \,,$$

where B is the binding energy of hydrogen.

[4]

Describe what happens to the proton and hydrogen number densities as the universe cools. What are the implications for the behaviour of the photons? Assuming that this "recombination" was mostly complete once the temperature dropped to around 2% the binding energy of hydrogen, determine the redshift of recombination (note that 1 eV $\simeq$ 12,000 K).

[4]