Chapter 0

1.

Let $T^2 = [0,1]^2$ with appropriate identifications. Suppose we remove a point from the interior. For any other point, draw a line from the removed point through it until it hits the boundary. This maps every point in $T^2 - \{p\}$ to the boundary, which is topologically $S^1 \vee S^1$. Denote this map f.

Now define

$$f_t : T^2 - \{p\} \to T^2 - \{p\} \qquad t \in [0,1]$$

by $$f_t(x) = f(x)t + x(1-t)$$

This map satisfies

- $f_0(x) = x$
- $f_1(x) \in S^1 \vee S^1$
- $f_t |_{S^1 \vee S^1} = \mathbb{1} \quad \forall t \in [0,1]$

Finally, $g : (T^2 - \{p\}) \times [0,1] \to T^2 - \{p\}$ given by $g(x,t) = f_t(x)$ is continuous on account that f is, so f_t is a deformation retraction.

2.

Similarly, define $f_t : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ by

$$f_t(x) = \frac{tx}{\|x\|} + (1-t)x$$

This satisfies

- $f_0(x) = x$
- $f_1(x) \in S^{n-1}$ $\qquad S^{n-1} = \{x, \|x\| = 1\}$
- $f_t|_{S^{n-1}} = \mathbb{1} \quad \forall t \in [0,1]$

Then on account that the norm is continuous, we have that $g : (\mathbb{R}^n - \{0\}) \times [0,1] \to \mathbb{R}^n - \{0\}$ is continuous and so f_t is a deformation retraction.

3. (a)

If X and Y are homotopy equivalent then there are maps

$$f : X \to Y \quad \text{and} \quad g : Y \to X$$

such that $fg \simeq id$ and $gf \simeq id$.

3(a) cont.

Explicitly, this means that we can 'interoplate' with functions F_t, G_t :

$$F_t : X \to X \qquad \text{with} \qquad F_0 = fg \qquad \text{and} \qquad F_1 = \mathrm{id}$$
$$G_t : Y \to Y \qquad \text{with} \qquad G_0 = gf \qquad \text{and} \qquad G_1 = \mathrm{id}$$

Similarly if Y is homotopy equivalent to Z then there are analogous functions h, k, H_t, K_t. Now consider the functions

$$\varphi : X \to Z \qquad \varphi = hf$$
$$\psi : Z \to X \qquad \psi = gk$$

As well as

$$\Phi_t = \begin{cases} g H_{2t} f & 0 \leq t \leq \frac{1}{2} \\ F_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$\Psi_t = \begin{cases} h G_{2t} k & 0 \leq t \leq \frac{1}{2} \\ K_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Note that

- $\Phi_t : X \to X$ and $\Psi_t : Z \to Z$
- $g H_1 f = gf = F_0$ and $h G_1 k = hk = K_0$
  So these maps are continuous (and well-defined) at $t = \frac{1}{2}$
- Consequently they are continuous for all t, x

Finally, they satisfy

$$\Phi_0 = gH_0 f = gkhf = \psi\varphi$$
$$\Phi_1 = F_1 = \mathrm{id}$$
$$\Psi_0 = hG_0 k = hfgk = \varphi\psi$$
$$\Psi_1 = K_1 = \mathrm{id}$$

So $\psi\varphi \simeq \mathrm{id}$ and $\varphi\psi \simeq \mathrm{id}$ and hence X and Z are homotopy equivalent. This verifies the transitivity axiom.

The symmetry axiom follows straight from the symmetry of the definition. To show that homotopy equivalence is reflexive, take

$$f : X \to X \qquad f = \mathrm{id}$$
$$g : X \to X \qquad g = \mathrm{id}$$

Then $fg = gf = \mathrm{id}$ and we know $\mathrm{id} \simeq \mathrm{id}$ since we can take

$$I_t : X \times [0,1] \to X \qquad I(x,t) = x \quad \forall t \in [0,1]$$

and this satisfies $I(x,0) = \mathrm{id}$ and $I(x,1) = \mathrm{id}$. Altogether we thus conclude that homotopy equivalence is an equivalence relation.

(b)

Suppose $f, g : X \to Y$ are homotopic and $g, h : X \to Y$ are homotopic. Then there are homotopies

$$F_t, G_t : X \to Y \qquad F_0 = f, \; F_1 = G_0 = g, \; G_1 = h$$

3 (b) cont.

Now define

$$\Phi_t : X \to Y \quad \text{by} \quad \Phi_t = \begin{cases} F_{2t} & 0 \leqslant t \leqslant 1/2 \\ G_{2t-1} & 1/2 \leqslant t \leqslant 1 \end{cases}$$

This satisfies

$$\Phi_0 = F_0 = f$$
$$\Phi_1 = G_1 = h$$

and it is continuous for all t, x. Hence it is a homotopy between f and h and so homotopy is transitive. It is symmetric, since if F_t is a homotopy between f and g then

$$\Phi_t = F_{1-t}$$

is a homotopy between g and f. Finally, it is reflexive, since we can take

$$\Phi_t = f \qquad \forall t \in [0,1]$$

as a homotopy between f and f. Thus homotopy is an equivalence relation.

(c)

Take spaces X, Y, maps $f, \varphi : X \to Y$ and $g : Y \to X$ and suppose

$$fg \simeq \mathrm{id} \qquad gf \simeq \mathrm{id} \qquad \varphi \simeq f$$

So f is a homotopy equivalence and X, Y are homotopy equivalent.

Then there are homotopies

$$F_t : X \to X, \quad F_0 = gf, \quad F_1 = \mathrm{id}$$

$$G_t : Y \to Y, \quad G_0 = fg, \quad G_1 = \mathrm{id}$$

$$\Phi_t : X \to Y, \quad \Phi_0 = f, \quad \Phi_1 = \varphi$$

Now define

$$F_t' : X \to X \quad \text{by} \quad F_t' = \begin{cases} g\,\Phi_{1-2t} & 0 \leq t \leq 1/2 \\ F_{2t-1} & 1/2 \leq t \leq 1 \end{cases}$$

$$G_t' : Y \to Y \quad \text{by} \quad G_t' = \begin{cases} \Phi_{1-2t}\,g & 0 \leq t \leq 1/2 \\ G_{2t-1} & 1/2 \leq t \leq 1 \end{cases}$$

These are continuous at $t = 1/2$ since $g\,\Phi_0 = gf = F_0$ and $\Phi_0 g = fg = G_0$ and so are continuous for all t, x. Furthermore,

$$F_0' = g\varphi \qquad F_1' = \mathrm{id}$$

$$G_0' = \varphi g \qquad G_1' = \mathrm{id}$$

So these are homotopies with $g\varphi \simeq \mathrm{id}$ and $\varphi g \simeq \mathrm{id}$. Hence the map φ is a homotopy equivalence, with g a homotopy inverse.

4.

A deformation retraction in the weak sense differs from a deformation retraction in that points in the subspace don't need to be left fixed, merely remain in the subspace.

4 Cont.

Define $g_t = f_t|A$ (note this is well-defined), and denote by f_1' the map f_1 considered as a map from X to A. Then we will show that

$$if_1' \simeq id \qquad \text{and} \qquad f_1'i \simeq id$$

Consider

$$F_t : X \to X, \; F_t = f_t \quad \text{and} \quad G_t : A \to A, \; G_t = g_t$$

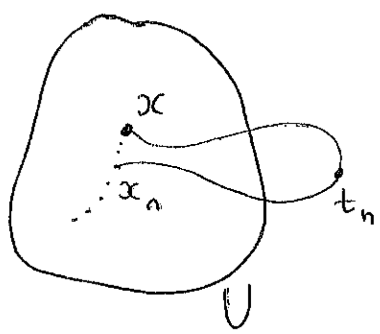
Then $f_0 = id$ and $f_1 = if_1'$, so $if_1' \simeq id$, and $g_0 = id$ and $g_1 = f_1'i$ so $f_1'i \simeq id$. Hence A and X are homotopy equivalent, with i a homotopy equivalence.

5.

If X deformation retracts to x then there is a map $F_t : X \to X$ with

$$F_0 = id, \qquad F_1(y) = x \;\; \forall y \in X \qquad F_t(x) = x \;\; \forall t \in [0,1]$$

We claim that for any neighbourhood U of x, there is a neighbourhood $V \subseteq U$ of x such that $F_t(V) \subseteq U$ $\forall t \in [0,1]$. We will first prove this metrically, and then in general.



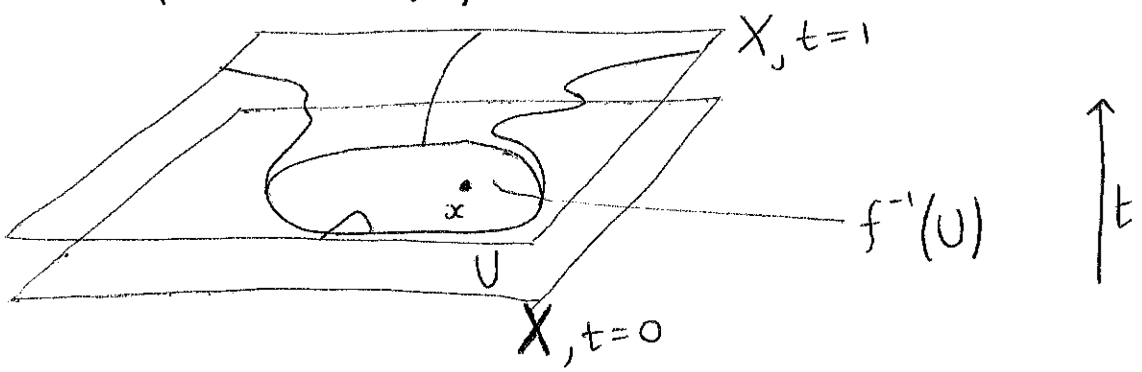
Suppose there is no such neighbourhood. ~~Suppose further~~ Then it must be that there is a sequence x_n converging to x ($x_n \neq x$ – if there is no such sequence then x is an isolated point and so $V = \{x\}$ suffices) with $F_{t_n}(x_n) \notin U$ for some t_n. Since $[0,1]$ is bounded, it must be that t_n has a convergent subsequence t_{n_j} by Bolzano-Weierstrass. Then by continuity,

$$F_{t_{n_j}}(x_{n_j}) \to \underset{\text{(for some } t)}{F_t(x)} = x$$

But this contradicts that $F_{t_n}(x_n)$ lies outside an open neighbourhood of x.
Now for a topological proof. Let

$$f : X \times [0,1] \to X \quad \text{be} \quad f(x,t) = F_t(x)$$

And consider the open set $f^{-1}(U)$:



For every $(\tilde{x},t) \in f^{-1}(U)$ there are open sets (in X and $[0,1]$) $U_{\tilde{x}}$ and U_t with $(\tilde{x},t) \in U_{\tilde{x}} \times U_t \subseteq f^{-1}(U)$, with

$$f^{-1}(U) = \bigcup_{(\tilde{x},t) \in f^{-1}(U)} U_{\tilde{x}} \times U_t$$

Now consider the U_t which cover $[0,1]$. By compactness these have a finite subcover, which we call U_{t_j}.

5 cont.

Then take the finite intersection

$$\bigcap_j U_{\tilde{x}_j} =: V$$

which is open. Note that

$$V \times [0,1] \subseteq \bigcup_j U_{\tilde{x}_j} \times U_{t_j} \subseteq f^{-1}(U)$$

Hence for all $v \in V$, $F_t(v) \in U$ $\forall t \in [0,1]$. We have almost proved the original claim, but it's not been established that $x \in V$. This requires a simple modification of the proof — we simply define

$$K = \bigcup_{(x,t) \in \{x\} \times [0,1]} U_x \times U_t \subseteq f^{-1}(U)$$

as an open neighbourhood of $\{x\} \times [0,1]$ and perform the same reasoning as before. Hence we have the desired neighbourhood V.

Now we wish to show that the inclusion map $i : V \hookrightarrow U$ is nullhomotopic. Let $c : V \to U$ be defined by $c(v) = x$ $\forall v \in V$. We claim

$$i \simeq c$$

This is seen to follow straightforwardly from the existence of the map $F_t|V : V \to U$ which satisfies $F_0|V = i$ and $F_1|V = c$. So indeed the inclusion is nullhomotopic and V is contractible.

6.

(a)

For any point p in the segment $[0,1] \times \{0\}$ we can find a deformation retract onto it, namely

$$F_t(x) : X \to X$$

$$F_t(x,y) = \begin{cases} (x, y(1-2t)) & t \leq \frac{1}{2} \\ (p^{(2t-1)} + 2x(1-t), 0) & t \geq \frac{1}{2} \end{cases}$$

Which satisfies

$$F_0(x,y) = (x,y) \qquad F_1(x,y) = (p,0) \qquad F_t(p,0) = (p,0)$$

Now suppose X deformation retracted onto another point, x. From question 5 we know there must exist a neighbourhood U of x which is contractible inside any neighbourhood V of x. We can choose V so that it does not intersect the segment $[0,1] \times \{0\}$. Then U must be a subset of a series of lines at rational base points, that is

$$U \subseteq \bigcup_{r \in [0,1]} \{r\} \times (0, 1-r]$$

This set is 'totally disconnected' for the same reason the rationals are. Since U is a neighbourhood, so contains more than one point*, it is therefore disconnected. * more than one line at a rational base point.

6(a) cont.

Now a disconnected space is not contractible so we have a contradiction. To see this last point, choose a point y in a different connected component to x. If our space were contractible, there would be a continuous map F_t such that $F_0 = y$ and $F_1 = x$. This would make our space path-connected and hence connected, a contradiction.

(b)

Y does not deformation retract onto any point for the same reasons as above: for any point in Y, we can find a neighbourhood which contains a set of disconnected lines, where any neighbourhood inside it also contains some of these disconnected lines and so is not contractible.

The space itself is contractible, however. To show this, I will show that Y deformation retracts onto the zig-zag Z in the weak sense, which by problem 4 implies Y is a homotopy equivalent to Z. Then using problem 3(a) and the fact that Z is homotopy equivalent to a point we find that Y is also — that is, Y is contractible.

(c)

Let $x \in \mathbb{R}$ be a coordinate on Z such that the corners of the zig-zag correspond to $x \in \mathbb{Z}$ and x varies 'uniformly' between the corners. We can then label the 'spokes' by their basepoint $x \in \mathbb{Q}$ and a height $y = 1 - x \bmod 1$.

Now define the map $F_t : Y \to Y$ given by

$$F_t(x,y) = \begin{cases} (x, y-t) & t \leqslant y \\ (x+t-y, 0) & t \geqslant y \end{cases}$$

One can show that this is continuous. It also satisfies

$$F_0(x,y) = (x,y) \qquad F_1(x,y) \subseteq Z \qquad F_t(x,0) \subseteq Z$$

so is indeed a weak deformation retraction.

7.

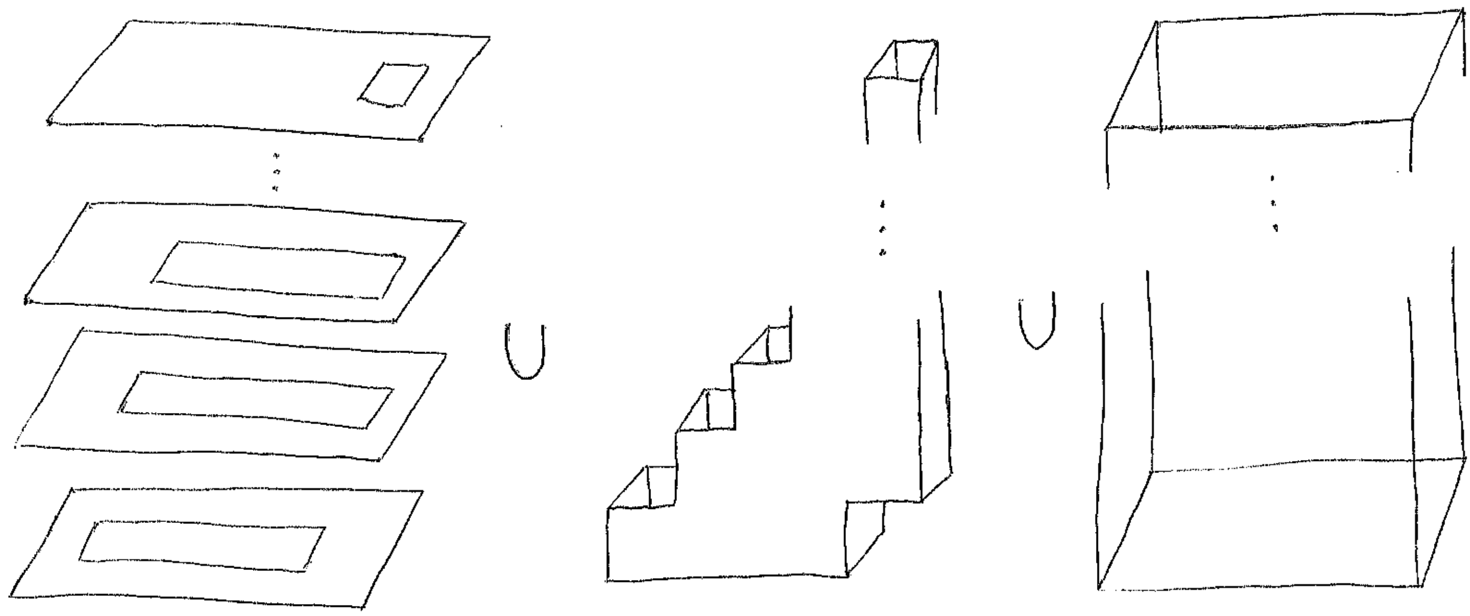
The Cantor set here acts just as the rationals did before. Considering the image of Y in the question to be 'top-down', a 'side-on' image looks thus

Y does not deformation retract onto any point for the same reason as before: for any point in the interior of the disc, we can find a neighbourhood such that any enclosed neighbourhood contains a disconnected set of 'fins'; for any point on the boundary of the disc, we can find a neighbourhood such that any enclosed neighbourhood contains a disconnected subset of the Cantor set which wraps the disc, and the same is true for points on this wrapping set.

Y is contractible, however, since there exists a weak deformation retraction onto the disc ~~a point~~. Schematically, move all points at the disc edge (or beyond) clockwise around the disc at fixed angular velocity. Move all points in the disc interior away from the compactification point and off the fins at fixed velocity, until at the equivalent point of the next outer 'crescent'. Then begin moving them clockwise around the disc centre (for the outer crescent, the points join the clockwise boundary motion).

8.

The house with n rooms should have n spaces that are simply connected and only accessible from the outside. It should be contractible.



9.

Suppose X is contractible and $f : X \to A \subseteq X$ is a retraction. There is a homotopy $F_t : X \to X$ with a point $p \in X$ such that

$$F_0 = \mathrm{id} \qquad F_1(x) = p \quad \forall x \in X \qquad \left(F_t(p) = p \quad \forall t \right)$$

↑ not in general

The homotopy $G_t = f \circ F_t : A \to A$ hence satisfies

$$\cancel{F}\ G_0 = \mathrm{id} \qquad G_1(a) = f(p) \quad \forall a \in A$$

But doesn't satisfy $G_t(f(p)) = f(p) \ \forall t$ necessarily. But this isn't necessary!

10.

Suppose X is contractible with homotopy F_t as before. Let $f: X \to Y$ and $g: Y \to X$ where Y is arbitrary. We want to show that f and g are nullhomotopic. Consider

$$f_t : X \to Y \qquad f_t = f \circ F_t$$

$$g_t : Y \to X \qquad g_t = F_t \circ g$$

Then

$$f_0 = f \qquad f_1(x) = f(p) \quad \forall x \in X$$
$$g_0 = g \qquad g_1(x) = p \quad \forall x \in Y$$

so these are the required homotopies. Conversely, assume for arbitrary Y we have homotopies f_t, g_t of f, g to constant maps. Then let $Y = X$ and $f = g = id$. We need to show

$$i_{\{p\}} \circ f_1 \simeq id \qquad \text{and} \qquad i_{\{q\}} \circ g_1 \simeq id$$

And f_t, g_t provide the required homotopies. So X is contractible. (Note that in the expressions directly above we're interpreting f_1 as a map from X to $\{p\}$ and g_1 as a map from X to $\{q\}$).

11.

We have maps $g, h : Y \to X$ with $fg \simeq 1$ and $hf \simeq 1$. Define

$$\tilde{f} : Y \to X \qquad \tilde{f} = h \circ f \circ g$$

Then

$$f\tilde{f} = fhfg \simeq fg \simeq 1$$

and

$$\tilde{f}f = hfgf \simeq hf \simeq 1$$

Where we have used that if $f_1 \simeq f_2$ then $hf_1g \simeq hf_2g$ for arbitrary h, g. Hence f is a homotopy equivalence with inverse $\tilde{f}$.

Now suppose fg and hf are homotopy equivalences. Then there exist

$$\tilde{g} : \cancel{X} \to \cancel{X} \qquad \tilde{h} : X \to X$$

with

$$\tilde{g}fg \simeq id \qquad hf\tilde{h} \simeq id$$
$$fg\tilde{g} \simeq id \qquad \tilde{h}hf \simeq id$$

Now consider the map

$$k : Y \to X \qquad k = \tilde{h} \circ h \circ f \circ g \circ \tilde{g}$$

Then

$$fk = f\underbrace{\tilde{h}hf}g\tilde{g} \simeq fg\tilde{g} \simeq \mathrm{id}$$

and

$$kf = \tilde{h}h\underbrace{fg\tilde{g}}f \simeq \tilde{h}hf \simeq \mathrm{id}$$

Where we used the result from before. Hence f is a homotopy equivalence with homotopy inverse k.

12.

We first prove f is a well-defined map. Namely, denoting path connectedness by $\sim$, we have that

$$x \sim y \Rightarrow f(x) \sim f(y)$$

Let $\Phi : I \to X$ be a path connecting x to y. Then $f \circ \Phi$ is a path connecting $f(x)$ to $f(y)$. Hence f defines a map from equivalence classes $[x] \in P_X$ to equivalence classes $[y] \in P_Y$.

The same applies to the homotopy inverse $g: Y \to X$. The key result is that since $fg \simeq \mathrm{id}$ and $gf \simeq \mathrm{id}$ we have $f(g(y)) \sim y$ and $g(f(x)) \sim x$: the homotopies between fg, gf and id defining the paths.

Thus f is surjective as $f : g([y]) \mapsto [y]$ for any $[y] \in P_Y$. And f is injective since if $f([x]) = f([\tilde{x}])$ then $g(f([x])) = g(f([\tilde{x}]))$ which means $[x] = [\tilde{x}]$, by the above. So f is bijective.

12 cont.

As before, we need to prove f is a well-defined map. This follows from the theorem that the continuous image of a connected set is connected: Suppose A, B are disjoint open sets covering Y. Then $f^{-1}(A), f^{-1}(B)$ are disjoint open sets covering X (assuming f surjective onto Y).

We know that $f(g(y))$ is path-connected to y, and likewise $g(f(x))$ is path-connected to x. But path-connected sets are connected, so the previous results apply with $\sim$ taken to mean 'in the same connected component'.

Finally, suppose the components and path-components of X coincide, and Y is homotopy equivalent to X. Then the above two bijections, along with this third one, induce a bijection between the components and path-components of Y. Actually we have something stronger than a bijection between components and path-components of X – they coincide.

Let's argue slightly differently. Suppose a component of Y contained (at least) two path-components. Then under the homotopy equivalence, they would be mapped to distinct path-components within the same component of X. This contradicts the coincidence of components and path-components of X, so the components and path-components of Y coincide.

13

Let r_t^0 and r_t^1 be deformation retractions of X onto A. I wanted to define a homotopy between them along the lines of 'do s seconds of r^1 followed by all of r^0', but as $s \to 1$ this meant doing r^0 infinitely quickly. The only way I could find of shrinking r^0 out of the picture as $s \to 1$ whilst still reaching A as $t \to 1$ was to define

$$g_t^s : X \to X \qquad g_t^s(x) = \begin{cases} r_t^0\left(r_{2ts}^1(x)\right) & s \leq \frac{1}{2} \\ r_{2t(1-s)}^0\left(r_t^1(x)\right) & s \geq \frac{1}{2} \end{cases}$$

We have

$$g_t^0(x) = r_t^0\left(r_0^1(x)\right) = r_t^0(x)$$

$$g_t^1(x) = r_0^0\left(r_t^1(x)\right) = r_t^1(x)$$

and for any given s,

$$g_0^s(x) = r_0^0\left(r_0^1(x)\right) = x$$

$$g_1^s(x) = r_1^0(y) \in A \ \left(s \leq \tfrac{1}{2}\right) \text{ or } r_{2(1-s)}^0(a) \in A \ \left(s \geq \tfrac{1}{2}\right)$$

$$g_t^s(a) = a$$

So g_t^s is indeed a deformation retraction. I will not prove continuity here, but it follows from the fact that we are composing continuous maps, with the piecewise nature causing no problems since the two pieces agree at $s = 1/2$.

14

I claim that any triple of integers (v, e, f) satisfying $v - e + f = 2$ can be written as a reference triple plus an integer linear combination of

$$(1, 1, 0) \quad \text{and} \quad (0, 1, 1)$$

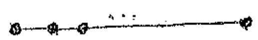
Restricting to positive integers, the two smallest reference triples are

$$(1, 1, 2) \quad \text{and} \quad (2, 1, 1)$$

Which can be realised as the following cell structures

Adding an arbitrary positive linear combination to the first gives the cell structure

Adding a single $(0,1,1)$ to the second yields a structure of this form, so we need only enumerate the cell structures for $(2,1,1) + k(1,1,0)$, which is



15

The cell structure in question has two n-cells for each n:

etc.

We see that the boundary of a given n-cell includes all lower-dimensional k-cells, and we must include the boundary since subcomplexes are closed.

Hence if n is the maximum dimension of a cell in our subcomplex, there are precisely three subcomplexes:

$$e_1^n \sqcup X^{n-1} \qquad e_2^n \sqcup X^{n-1} \qquad e_1^n \sqcup e_2^n \sqcup X^{n-1}$$

These have topologies

1) closed ball in $\mathbb{R}^n$
2) closed ball in $\mathbb{R}^n$
3) S^n.

Finally, if there is no such maximum dimension, then I claim the only subcomplex is the trivial one: S^∞.

To see this, note that if any cell e_α^n is missing from the subcomplex, then part of the boundary of e_β^m, for all $m > n$ and for both β, is also missing, which contradicts that the subcomplex is closed. The only way out is for all e_β^m $(m > n)$ to be missing as well, and so we're back to the case where a maximum dimension exists.

16.

This proof is taken directly from appendix 1.B of Hatcher.

Define $f_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$f_t(x_1, x_2, \dots) = (1-t)(x_1, x_2, \dots) + t(0, x_1, x_2, \dots)$$

This takes non-zero vectors to non-zero vectors, so $f_t / |f_t|$ is a well-defined map from the unit sphere to itself, and a homotopy from the identity to

$$(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$$

Next define $g_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$g_t(x_1, x_2, \dots) = (1-t)(0, x_1, x_2, \dots) + t(1, 0, 0, \dots)$$

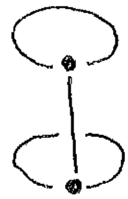
As before, $g_t / |g_t|$ is a well-defined map on the unit sphere. The piecewise map

$$h_t = \begin{cases} f_{2t} / |f_{2t}| & 0 \leq t \leq \frac{1}{2} \\ g_{2t-1} / |g_{2t-1}| & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a homotopy on S^∞ from the identity to a constant map, so S^∞ is contractible

17 (a)

We construct the mapping cylinder as follows. Take two 0-cells and three 1-cells and attach them thus:



Now take a 2-cell, thought of as a square, and attach it thus: attach the top edge to the upper circle, the left and right edges to the vertical line, and the lower edge to the lower circle via the map f. It is easy to check that these attaching maps are continuous at the corners of the square, so this gives a CW complex structure of the mapping cylinder of f.

(b)

Take the map f to have winding number 2 (concretely, take $\vartheta \mapsto 2\vartheta$). Then we claim the mapping cylinder of f is the Möbius band. To see this, view the upper circle as the single boundary of the band, and the lower circle as the central circle of the band.

One can take a third circle and the identity map $id: S^1 \to S^1$ and construct the mapping cylinder of this map, with the attachment to the lower circle of the previous space. We know from the book that a mapping cylinder deformation retracts onto the codomain of the map. Performing this deformation retract on the first mapping cylinder yields the cylinder, and on the second yields the Möbius band.

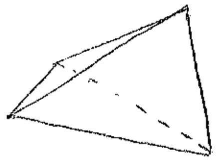
18.

We will assume in this question that the quotients involved in the smash product and those involved in defining spheres from disks can be performed in any order. I'm not sure how to justify this.

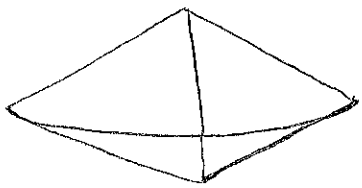
18 cont.

First consider the smash product of two intervals

$$I * I =$$



Identifying the ends of one of these intervals yields a 'top' shape:



That is, a 3-~~sphere~~ ball. Identifying the ends of the other interval involves identifying the top and bottom of the surface of the ball. By slicing the top in half we can view this as two 3-balls glued together on their 2-sphere boundaries, which gives the 3-sphere.

To answer the general question will require other arguments. First take

$$S^m \times S^n \times I$$

and at $S^m \times S^n \times \{1\}$, collapse S^n to a point. This yields

$$S^m \times D^{n+1} \qquad (+)$$

The final identification we need is at $S^m \times \partial D^{n+1}$, we collapse S^m to a point. Now if $n=0$, so $D^{n+1} = I$, this process is just the suspend of S^m, which we know is S^{m+1}. We will take advantage of this fact via induction.

Write $D^{n+1} = D^n \times I$ and at $\partial D^n \times I$, collapse I to a point. This doesn't change the topology, and amounts to viewing D^{n+1} in a 'round' sense.

In this decomposition,

$$\partial D^{n+1} = D^n \times \partial I \qquad (\text{with } \partial D^n \times \partial I \text{ collapsed to } \partial D^n)$$

Returning to our original space, we have

$$S^m \times D^{n+1} = S^m \times I \times D^n$$

where at $D^n \times \partial I$ we collapse S^m to a point, and at $\partial D^n \times I$ we collapse $S^m \times I$ to a point. The former collapse amounts to a suspend, so this process results in the space

$$S^{m+1} \times D^n \qquad \text{where we collapse } S^{m+1} \text{ to a point at } S^{m+1} \times \partial D^n$$

This is precisely the place we were at with $(+)$, except we've shifted a dimension from a disk to a sphere. Iterating this process, we hence find

$$S^m * S^n = S^{m+n+1}$$

19.

We use proposition 0.18 from the book that if attaching maps f, g are homotopic, then the attached spaces are homotopy equivalent.

We also claim that any map $S^1 \to S^2$ is homotopic to a constant map map (and we take that constant to be the north pole). This is the statement that S^2 is simply connected, and is clear geometrically.

19 cont.

Hence any space obtained by attaching a 2-cell along a circle in S^2 is homotopy equivalent to the space with a 2-cell attached along its boundary to the north pole in S^2, which gives $S^2 \vee S^2$.

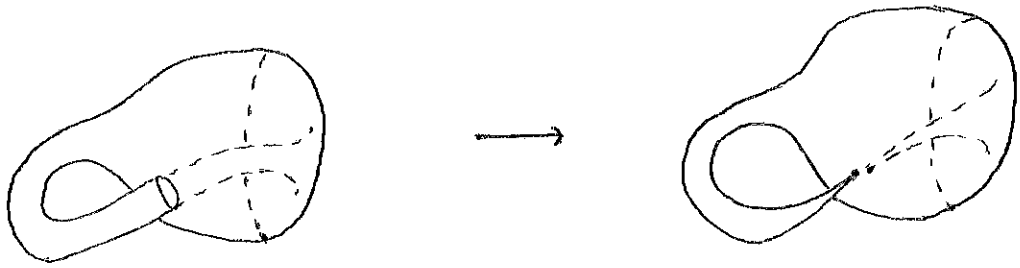
With n 2-cells, this therefore gives the wedge sum of $n+1$ 2-spheres.

20.

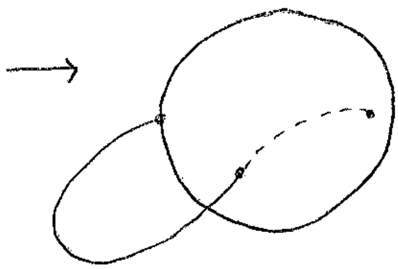
Crucially, the disk where the Klein bottle intersects itself is not removed in this picture, as I originally assumed!

We make repeated use of proposition 0.18 in the book that we can always collapse a contractible subcomplex without changing homotopy type.

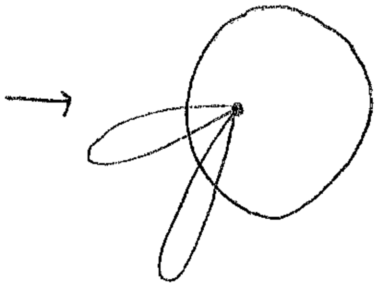
First collapse the intersection disk:



We can then introduce some contractible intervals:



We can finally collapse the two lines between the intersection points, lying within the sphere, to obtain the final space:



$= \quad S^1 \vee S^1 \vee S^2$

21.

By representing each S^2 as a point and the connectivity by lines joining the points, we create a graph of connectivity.

Attaching the 2-spheres to these points yields a space homotopy equivalent to the original one. To see this, we can collapse all the lines of connectivity to yield a space of spheres connected at points. Even though the spheres in the original space may have connected to different spheres at different points, we can collapse lines within the spheres so that each one connects to the others at a single point.

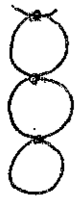
As in example 0.7 in the book, we can collapse lines in the graph of connectivity to generate a bouquet of circles (a single one, since the original space was connected). The result is a wedge sum of S^1's and S^2's, homotopy equivalent to the original space.

22.

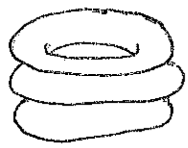
We consider each connected component of the graph separately, and consider the two cases that the ~~grap~~ component intersects the edge of P at no vertices or at some vertices, in turn.

22 cont.

If a component X has no vertices on the edge of P then the surface of revolution is simply $S^1 \times X$. Since X is homotopy equivalent to a wedge sum of circles, or equivalently

$X \simeq$ 

The surface of revolution is equivalent to a stack of tori:



Now suppose X intersects the edge of P. A line connecting an edge vertex to another vertex becomes a disk upon rotation, which is contractible. We can hence collapse lines in the graph except those connecting two edge vertices, without changing homotopy type:

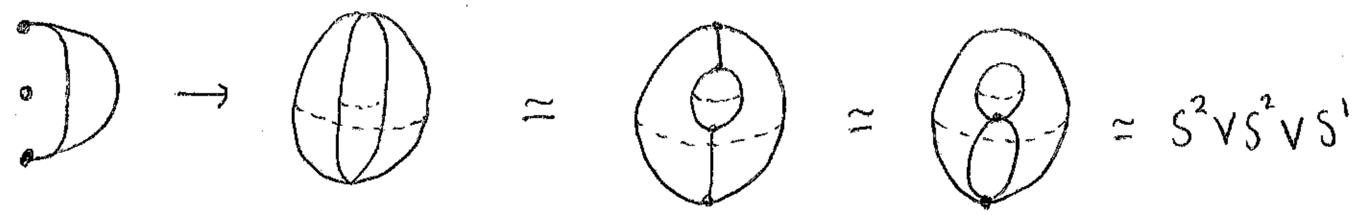
$X \simeq$ 

There are two types of line here: those connecting a point to itself and those connecting to another point. Upon rotation

 $\longrightarrow$  $= S^2$

And

where the first equivalence follows by collapsing the vertical line. In the total graph X, we get a wedge sum of such spaces, with one complication being lines that connect points already connected:



So the surface of revolution is homotopy equivalent to a wedge sum of circles and 2-spheres. We can count these. Suppose our graph has k edges connected to the same vertex, and m edges connecting different vertices. Suppose X intersects the edge of P in n vertices.

each loop gives $S^1 \vee S^2$

each other edge gives S^2

each redundant edge gives an extra S^1

With n edge vertices, there are $m+1-n$ redundant edges. So our surface is

$$\bigvee^{k+m} S^2 \vee \bigvee^{k+m+1-n} S^1$$

Finally, thinking in terms of the original graph, our collapsing process doesn't change the quantities $F = k+m+1-n$ (number of faces) or $E-V = k+m-n$ (edges minus vertices). So we can write our space as

$$\bigvee^{F+n-1} S^2 \vee \bigvee^{F} S^1$$

23.

Let A, B be contractible subcomplexes and X their intersection. By proposition 0·17,

$$A/X \simeq A \qquad\qquad B/X \simeq B$$

So these quotients are both contractible. Now collapsing X in the total complex yields

$$(A/X) \vee (B/X)$$

which is the wedge sum of contractible spaces so is contractible. By proposition 0·17 again,

$$(A \cup B)/X \simeq A \cup B$$

So $A \cup B$ is contractible.

24.

We write both spaces as $X \times Y \times I$ with appropriate identifications.

$X * Y / (X * \{y_0\} \cup \{x_0\} * Y)$:

$$\left.\begin{array}{l} X \times Y \times \{0\} \to \{x_0\} \times Y \times \{0\} \\ X \times Y \times \{1\} \to X \times \{y_0\} \times \{1\} \end{array}\right\} \text{join}$$

$$\left.\begin{array}{l} X \times \{y_0\} \times I \to \{pt\} \\ \{x_0\} \times Y \times I \to \{pt\} \end{array}\right\} \text{quotient}$$

The latter two identifications can be used to further simplify the former two:

$$X \times Y \times \{0\} \longrightarrow \{pt\}$$
$$X \times Y \times \{1\} \longrightarrow \{pt\}$$

$S(X \wedge Y) / S(\{x_0\} \wedge \{y_0\})$:

$$\left.\begin{array}{l} \{x_0\} \times Y \times I \to \{pt\} \\ X \times \{y_0\} \times I \to \{pt\} \end{array}\right\} \text{smash product}$$
$$\left.\begin{array}{l} X \times Y \times \{0\} \to \{pt\} \\ X \times Y \times \{1\} \to \{pt\} \end{array}\right\} \text{suspension}$$
$$\left.\{x_0\} \times \{y_0\} \times I \to \{pt\}\right\} \text{quotient}$$

The final identification is clearly accounted for by the first two. Ignoring this, we have the same four identifications as before so the two spaces are homeomorphic.

We next argue that the two quotients are contractible:

1) $S(\{x_0\} \wedge \{y_0\}) \cong S(\{pt\}) \cong I \simeq \{pt\}$

2a) $X * \{y_0\} \cong C(X) \simeq \{pt\}$ and likewise for $\{x_0\} * Y$

2b) $(X * \{y_0\} \cap \{x_0\} * Y) = \{x_0\} \times \{y_0\} \times I \simeq \{pt\}$

Now 2a and 2b, combined with problem 23, imply that $X * \{y_0\} \cup \{x_0\} * Y$ is contractible. With these results, we can use proposition 0.17 to conclude that

$$X*Y/(X*\{y_0\} \cup \{x_0\} * Y) \simeq X * Y \qquad \longrightarrow \quad X*Y \simeq S(X \wedge Y)$$
$$S(X \wedge Y)/S(\{x_0\} \wedge \{y_0\}) \simeq S(X \wedge Y)$$

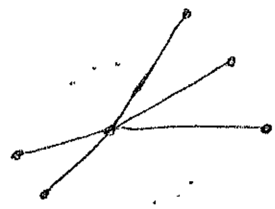
25.

It's not clear what is meant by

$$\bigvee_\alpha SX_\alpha$$

If X is a circle with one 0-cell and one 1-cell, then suspending the individual components yields a line and a disc. No wedge sum of a graph, line, and disc yields the suspension of X, a 2-sphere.

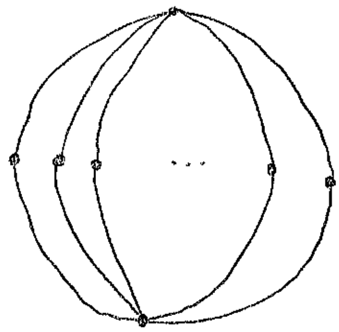
Hence this notation must mean the suspension of each disconnected component of X. Clearly the suspend of X produces the suspension of each component, but also identifies the 'north poles' of each component as well as the 'south poles'. Let Z_n be the maximally connected graph with n vertices. No don't do this. Let Z_n be a 'spoke graph' with $n+1$ vertices:



Then attaching spaces to the outer vertices gives a space homotopy equivalent to the wedge sum of those spaces, since the 'spokes' are contractible. If X has n connected components, then SX is homotopy equivalent to the space obtained by attaching one copy of Z_n to the north poles and one to the south poles. By moving the two attaching points together through the respective components, we see that SX is homotopy equivalent to

$$Y \vee_\alpha SX_\alpha$$

Where

$$Y = \bigvee^{n-1} S^1 =$$ 

Now suppose X is a finite graph. If X is connected, then it ~~it~~ is homotopy equivalent to a wedge sum of circles, as argued in the chapter. Then SX is homotopy equivalent to a wedge sum of 2-spheres ~~and circles~~. To see this, first note that

$$S(S^1) = S^2$$

With a wedge sum of circles, the suspend acts to join these ~~two~~ 2-spheres along 'longitudinal lines'. These lines are contractible so in fact

$$S\left(\bigvee^n S^1\right) \simeq \bigvee^n S^2$$

If X is disconnected, then the result of the first part of the question tells us that SX is homotopy equivalent to a wedge sum of circles and 2-spheres as stated, according to

$$SX = \bigvee^{n-1} S^1 \vee \bigvee^m S^2$$

where n is the number of disconnected components, $m = \sum_\alpha m_\alpha$ with m_α the number of loops in each component graph.

26.

We know that (X, A) having the homotopy extension property is equivalent to the existence of a retraction from

$$X \times I \quad \text{to} \quad X \times \{0\} \cup A \times I$$

Given the inclusion

$$\imath : A \to X$$

this latter space is merely the mapping cylinder $M_\imath$. Consider also the natural inclusion

$$\jmath : M_\imath \to X \times I$$

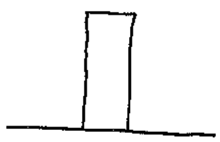
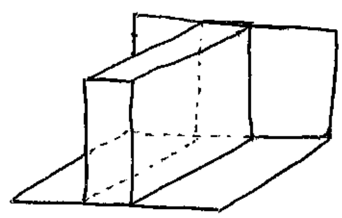
This has mapping cylinder

$$M_\jmath = X \times I \times \{0\} \cup X \times \{0\} \times I \cup A \times I \times I$$

For example, take

$X =$ $\qquad A =$

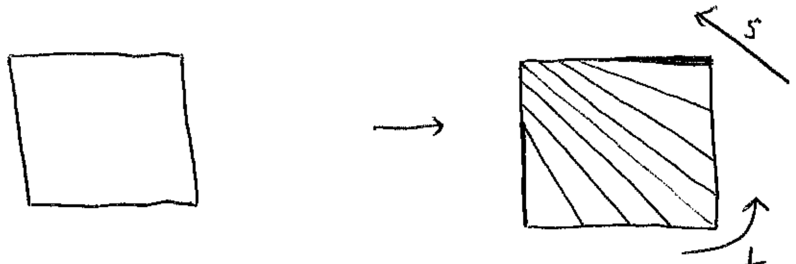
Then

$M_\imath =$  $\qquad M_\jmath =$ 

The crucial argument is that

$$M_\jmath \text{ is homeomorphic to } M_\imath \times I$$

This is clear from the diagram. It follows since we can reparametrise $I \times I$ thus



That is, with coordinates s measuring how far along one of these lines the point is and t measuring how far around the square the line attaches.

This has two important consequences

A) $X \times I \times I$ retracts onto M_J (since $X \times I$ retracts onto M_2)

B) M_J deformation retracts onto M_2 (by collapsing the I)

From here we have

$A \Rightarrow (X \times I, M_2)$ has the homotopy extension property

$B \Rightarrow J$ is a homotopy equivalence (by corollary 0.21)

Now from corollary 0.20 it follows that

$$M_2 \text{ is a deformation retract of } X \times I$$

as desired. From here it follows trivially that proposition 0.18 follows for any (X_1, A) with the homotopy extension property since the only relevance in the original proof of (X_1, A) being a CW pair was that it had this property, as guaranteed by proposition 0.16 (namely the property of there existing a deformation retraction from $X_1 \times I$ to M_2).

27.

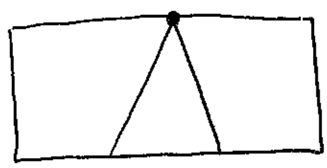
Let n be the natural map $X \to B \sqcup_f X$. Consider the mapping cylinder M_n. Note that this is simply $X \times I$ attached to M_f according to the map

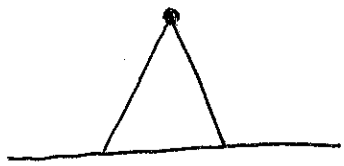
$$g(a,t) = \begin{cases} (a,t) & t < 1 \\ (f(a),t) & t = 1 \end{cases}$$

where $a \in A$. Since (X, A) has the homotopy extension property, there is a deformation retraction from $X \times I$ to $X \times \{0\} \cup A \times I$, and this induces a deformation retraction from

$$M_n \quad \text{to} \quad X \times \{0\} \cup M_f$$

In terms of the example spaces from the previous question, taking B to be a point,

M_n = 

$X \times \{0\} \cup M_f$ = 

Now since the map f is a homotopy equivalence, M_f deformation retracts to A (by corollary 0.21). The composition of these two deformation retractions gives a deformation retraction from M_n to X, which guarantees (again by corollary 0.21) that the map n is a homotopy equivalence $\#$.

28.

This question seems to me to be completely trivial — I must be missing something. We can attach $X_0 \times I$ along $A \times I$, and the retract from $X \times I$ to $X \times \{0\} \cup A \times I$ (guaranteed by the homotopy extension property) induces a retract from $(X_0 \sqcup_f X_1) \times I$ to $(X_0 \sqcup_f X_1) \times \{0\} \cup (X_0 \sqcup_f A) \times I$. This implies that the pair $(X_0 \sqcup_f X_1, X_0 \sqcup_f A)$ has the homotopy extension property.

29.

We get an extension of a homotopy by composing the the map from $X \times \{0\} \cup A \times I \to Y$ with a (deformation) retraction from $X \times I \to X \times \{0\} \cup A \times I$. Taking this deformation retraction to be the radial projection in the proof of proposition 0·16, we see that, for given $x \in e^n$, the point (x, t) first moves radially out from the centre of e^n until it meets the boundary, and then moves up the interval I, staying at fixed $a \in A$.

Note the point (x, t) only reaches the boundary if x is closer to the boundary than ~~to~~ the centre to begin with, and only reaches the top of the interval if x was on the boundary to begin with. The centre of the disc doesn't move at all.

Thus, composing this retraction with our map $X \times \{0\} \cup A \times I \to Y$, we see that under the extended homotopy, the path $f_t(x)$ involves the point x moving radially outward to the edge of the n-cell for some initial period, and then upon meeting the subcomplex A, follows the initial portion of the homotopy $f_t(a)$, where a is the point on the boundary that x meets.