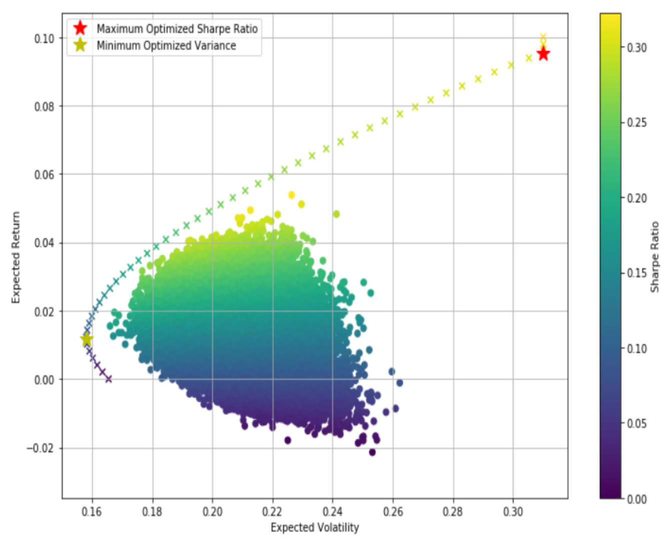
Project Report

CS357 Optimization lab

Portfolio Optimization



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Understanding Portfolio Optimization

An Algorithmic Investigation

Introduction

The financial market is the backbone of the modern economy. It helps control the flow of resources among various businesses. The finance sector is undergoing a revolution with the emergence of hedge funds and private investors. In this post-covid era, the market is very volatile.

Thus, it is essential for every investor to have a low-risk and advantageous portfolio.

The top domains of research in finance and computer science are

1. FinTech (Financial Technology) implementations and challenges.
2. Green finance/Social funding of green financing.
3. Blockchain implementations.
4. Portfolio Optimization
5. Cryptocurrencies (Bitcoin, Ethereum, etc.)

There are many rigorous algorithms developed for portfolio optimizations based on mathematical and statistical models that give high returns to the investors, as well as keep risk low.

In this project, we will look at some generic approaches to portfolio optimization and try to find out their shortcomings and benefits.

We will look at the Markowitz model who proposed the Modern Portfolio Theory(MPT). Also, we will develop the basis of efficient frontiers. And finally, we will look at the Monte Carlo approach to solving the above problem.

Problem Formulation

Our problem statement is to select the best portfolio (asset distribution), out of the set of all portfolios being considered, according to objectives set by the trader.

The objective typically maximizes factors such as expected return and minimizes costs like financial risk.

Terminologies

r_{it}	The anticipated return of i^{th} security, at a time t, per dollar invested in it
d_{it}	The rate at which return of i^{th} security is discounted back to the present
X_i	The relative amount of inventory invested in i^{th} security
R_i	Rate of return(per period) of i^{th} security
E(Y)	The expected value of random variable Y
V(Y)	The variance of random variable Y

Symbols

Historical Data: $s_j(t)$ = Stock price of the jth investment at time t

Return Data: $r_j(t) = s_j(t) / s_j(t-1)$

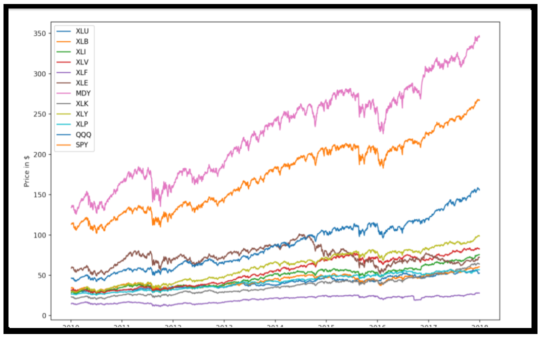


Fig1: Cost of security $s_j(t)$

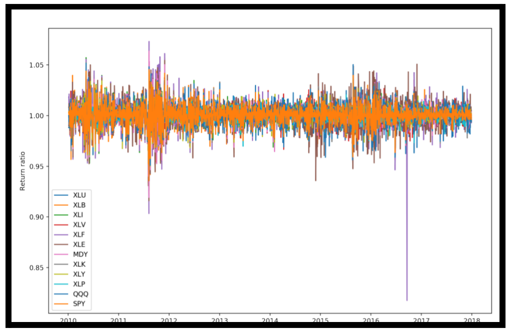


Fig2: Return on security $r_j(t)$

We exclude short sales

Anticipated return in a diversified portfolio - R

R is the weighted mean of μ_i (average return of i^{th} security) with respect to weights X_i ,

$$R = \sum_{t=1}^{\infty} \sum_{i=1}^{N} d_{it} \, r_{it} \, X_i$$

Which can be written as,

$$R = \sum_{t=1}^{\infty} X_i r_i \qquad \text{where} \qquad \sum_{i=1}^{N} X_i = 1$$

Hence we get

$$E(R) = \sum_{t=1}^{\infty} X_i \mu_i \qquad \text{---(1)}$$

Variance in a diversified portfolio - V(R)

Introducing a new term covariance of R_i and R_j shown as $\boldsymbol{\sigma}_{ij}$

$\sigma_{ij} = E\{[R_i - E(R_i)] \; * \; [R_j - E(R_j)]\}$

Ie

$\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ where ρ_{ij} is correlation factor

Hence we get

$$V(R) = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \sigma_{ij} \quad \text{---(2)}$$

Classical formulation

$$min\, V(R) = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \sigma_{ij}$$

s.t the following conditions hold

$$R = \sum_{i=1}^{\infty} X_i r_i \geq \alpha \, , \quad \text{---(1)}$$

$$\sum_{i=1}^{N} X_i = 1, \quad \text{---(2)}$$

$$X_i \geq 0 \qquad \forall i = 1,2,3 \ldots \quad \text{---(3)}$$

Modern Portfolio Theory (MPT)

Also known as the Markowitz model

It is based on the classical formulation of the portfolio optimization problem as done by Henry Markowitz(1952). Note: It is a theoretical model.

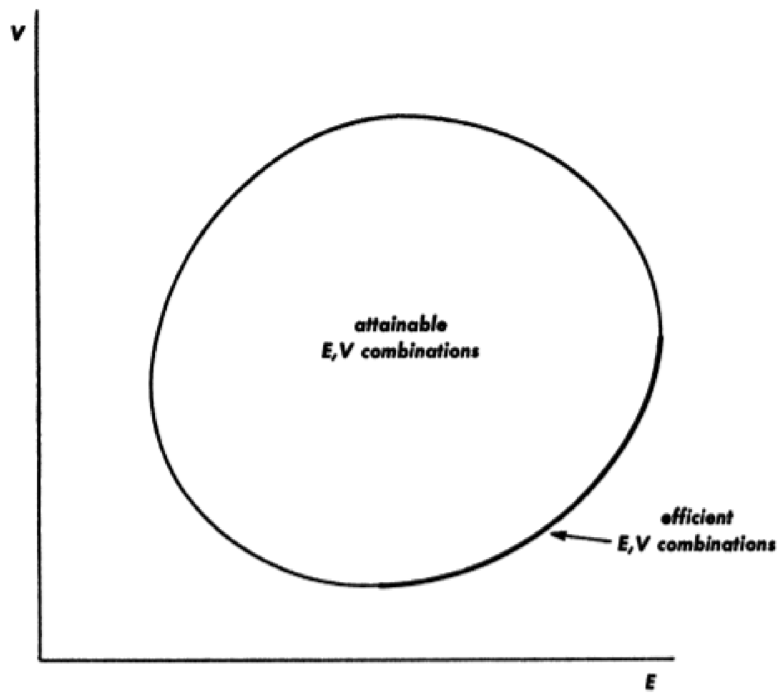


Fig. 1

Where E is Average returns and V is variance

The paper states that we are given (μ_i,δ_i) the parameters determined by the market. For the given investment scenario, the attainable E, V combinations can be depicted as in fig2. Maximizing return E, and minimizing V give us the point (set) called - **efficient combination**.

The paper provides a graphical method to solve this problem.

Dealing with a particular case of 3 securities.

1) $$E = \sum_{i=1}^{3} X_i \mu_i$$

2) $$V = \sum_{i=1}^{3} \sum_{j=1}^{3} X_i X_j \sigma_{ij}$$

3) $$\sum_{i=1}^{3} X_i = 1$$

4) $X_i \geq 0$ for $i = 1, 2, 3$.

From (3) we get

3′) $$X_3 = 1 - X_1 - X_2$$

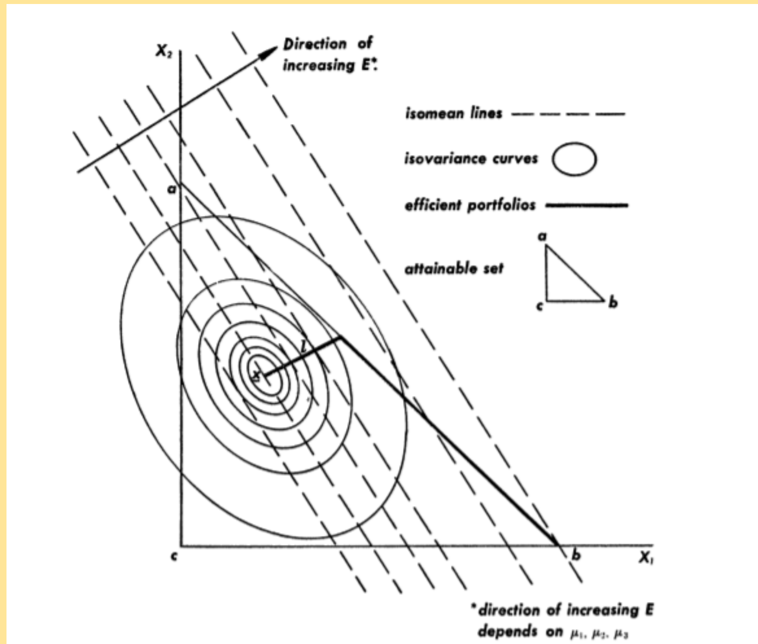
Substitution gives us the formulas for E and V in terms of X1, X2.

E = E(X1, X2)
V = V(X1, X2)
X1 ≥ 0, X2 ≥ 0, 1 - X1 - X2 ≥ 0

In the case of 3 securities, we have the following graph.

Figure 3

Ellipses are contours of V(X1,X2). At the center the variance is minimum. As we increase variance we keep maximizing Returns (E) along the dark line called the **efficient frontier**.



In the case of 4 securities, we have the following graph.

Figure 4

The dark line shows the efficient frontier directed from minimum variance to maximum returns.

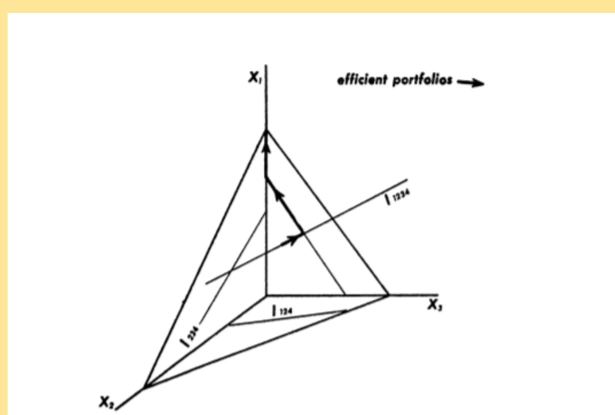


Fig. 4

Summary

The modern portfolio theory allows investors to construct more efficient portfolios. Every possible combination of assets can be plotted on a graph, with the portfolio's risk on the X-axis and the expected return on the Y-axis. This plot reveals the most desirable combinations for a portfolio.

It is possible to draw an upward sloping curve to connect all of the most efficient portfolios. This curve is called the **efficient frontier**.

Investing in a portfolio underneath the curve is not desirable because it does not maximize returns for a given level of risk.

Algorithmic Design

Type of problem defined

1. Constrained
2. Linear
3. Deterministic
4. Continuous
5. Global

Modeling as LPP

We introduce the L_1 risk(absolute deviation) function -

$$w(X) = E\left[\left|\sum_{j=1}^{n} R_j X_j - E\left[\sum_{j=1}^{n} R_j X_j\right]\right|\right]$$

instead of the L_2 risk(standard deviation) function of the earning out of portfolio. These two measures are essentially the same if $(R_1,R_2,\ldots,R_n)$ are multivariate normally distributed.
If $(R_1,R_2,\ldots,R_n)$ are multivariate normally distributed then following relation is true-

$$w(X) = \sqrt{\frac{2}{\Pi}}\sigma(X)$$

The above relation implies that minimizing w(X) is equivalent to minimizing σ(X) if (R_1,R_2,...,R_n) are multivariate normally distributed. Thus we are led to an alternative L_1 risk minimization problem.

$$
\begin{array}{ll}
\text{minimize} & w(X) = E\left[\left|\sum_{j=1}^{n} R_j X_j - E\left[\sum_{j=1}^{n} R_j X_j\right]\right|\right] \\
\text{subject to} & \sum_{j=1}^{n} E(R_j) X_j \geq \alpha \\
& \sum_{j=1}^{n} X_j = 1 \\
& X_j \geq 0, \quad j = 1, 2, ..., N
\end{array}
\quad \text{-----(LPP1)}
$$

Let r_{jt} be the realization of random variable R_j during period t(t= 1,...,T) which we assume to be available through the historical data or from some future projection. We also assume that the expected value of the random variable can be approximated by the average derived from these data.

Let,

$$r_j = E[R_j] = \sum_{t=1}^{T} \frac{r_{jt}}{T}$$

Then w(X) can be approximated as:

$$E\left[\left|\sum_{j=1}^{n} R_j X_j - E\left[\sum_{j=1}^{n} R_j X_j\right]\right|\right] = \frac{1}{T} \sum_{t=1}^{T} \left|\sum_{j=1}^{n} (r_{jt} - r_j) X_j\right|$$

Let us denote

$$a_{jt} = r_{jt} - r_j, \;\; j = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, T$$

Then (LPP1) leads to the following minimization problem

$$\begin{aligned}
&\text{minimize} \quad w(X) = \frac{\sum_{t=1}^{T} \left| \sum_{j=1}^{n} (a_{jt}) X_j \right|}{T} \\
&\text{subject to} \quad \sum_{j=1}^{n} r_j X_j \geq \alpha \\
&\qquad\qquad\quad \sum_{j=1}^{n} X_j = 1 \\
&\qquad\qquad\quad X_j \geq 0, \quad j = 1, 2, \ldots, N
\end{aligned}$$

which is equivalent to the linear program

$$\begin{aligned}
&\text{minimize} \quad w(X) = \sum_{t=1}^{T} \frac{y_t}{T} \\
&\text{subject to} \quad y_t + \sum_{j=1}^{n} a_{jt} X_j \geq 0, \;\; t = 1, 2, \ldots, T \\
&\qquad\qquad\quad y_t - \sum_{j=1}^{n} a_{jt} X_j \geq 0, \;\; t = 1, 2, \ldots, T \\
&\qquad\qquad\quad \sum_{j=1}^{n} r_j X_j \geq \alpha \\
&\qquad\qquad\quad \sum_{j=1}^{n} X_j = 1 \\
&\qquad\qquad\quad X_j \geq 0, \quad j = 1, 2, \ldots, N
\end{aligned} \quad \text{-----(LPP2)}$$

Simplex

We can state that the optimization model (LPP2) is one of linear type. Obviously, comparing the obtained model (LPP2) with the classical Markowitz ‘s model, we can say that the linear one certainly is more simple to solve both in terms of methodology, and of the solving time. In order to solve the model (LPP2), we can quite effectively use the simplex algorithm for solving linear programming problems.
Simplex algorithm is as follows-

Given $\mathcal{B}, \mathcal{N}, x_{\mathrm{B}} = B^{-1}b \geq 0, x_{\mathrm{N}} = 0$;
Solve $B^T \lambda = c_{\mathrm{B}}$ for λ,
Compute $s_{\mathrm{N}} = c_{\mathrm{N}} - N^T \lambda$; (* pricing *)
if $s_{\mathrm{N}} \geq 0$
    stop; (* optimal point found *)
Select $q \in \mathcal{N}$ with $s_q < 0$ as the entering index;
Solve $Bd = A_q$ for d;
if $d \leq 0$
    stop; (* problem is unbounded *)
Calculate $x_q^+ = \min_{i \mid d_i > 0} (x_{\mathrm{B}})_i / d_i$, and use p to denote the minimizing i;
Update $x_{\mathrm{B}}^+ = x_{\mathrm{B}} - d x_q^+$, $x_{\mathrm{N}}^+ = (0, \ldots, 0, x_q^+, 0, \ldots, 0)^T$;
Change $\mathcal{B}$ by adding q and removing the basic variable corresponding to column p of B.

This method(converting to lpp and using simplex) that we have used is accurate and efficient enough to solve realistic problems in a reasonable amount of time.

Monte Carlo

Forward-looking Monte Carlo Simulation: This is a **statistical technique** that uses pseudo-random uniform variables for a given statistical distribution based on risk and returns to predict the future value of the portfolio. The underlying concept is to use randomness to solve problems that might be deterministic in principle.

With Monte Carlo simulation we will generate portfolio weighting allocations and record the expected return and variance.

SHARPE RATIO

The expected excess return of the portfolio over the risk-free short rate, divided by the expected standard deviation of the portfolio over a given discount rate.

It is the ratio of returns to standard deviations

Formula and Calculation of Sharpe Ratio

$$Sharpe\ Ratio = \frac{R_p - R_f}{\sigma_p}$$

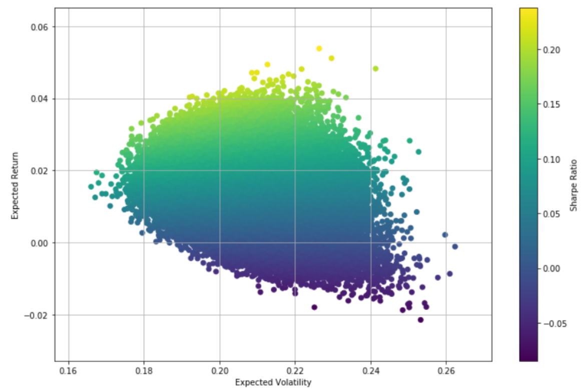
where:

R_p = return of portfolio

R_f = risk-free rate

σ_p = standard deviation of the portfolio's excess return

The Monte Carlo method performs repeated random sampling in this open_source dataset.



We will use scipy.optimize library to generate weights to maximize Sharpe ratio

Note - weights (X_i) are the actual amount we will invest in the i^{th} security.

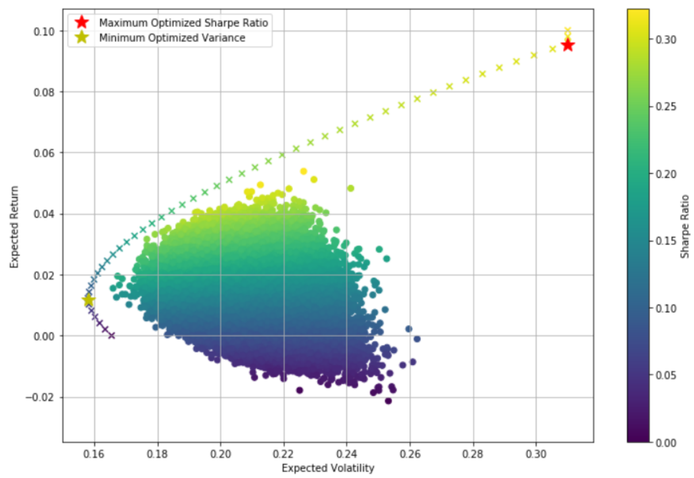
```
We first give random values to weights
weights = np.random.random(noa)

Define required function to optimize
import scipy.optimize as sco
def negative_func_sharpe(weights):
   return -statistics(weights)[2]

Run optimizer function
Opt = sco.minimize(negative_func_sharpe, x0, method='SLSQP')
```

Results obtained

The image shows efficient frontiers that will maximize the Sharpe ratio.



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