The equidistribution of nilsequences

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- What about polynomial progressions?
- How many primes in arithmetic progressions are there in $[N]$?
- Each of these problems involve the *nilpotent Hardy-Littlewood method*, a generalization of the *Hardy-Littlewood Circle method*.

Heuristic: a high dimensional circle method

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- Consider $F(\alpha n)$. We say that $F(\alpha n)$ is δ-*equidistributed* on scale N if

$$\left| \mathbb{E}_{n \in [N]} := \frac{1}{N} \sum_{n=0}^{N-1} F(n\alpha) - \int_{\mathbb{R}^d/\mathbb{Z}^d} F(x) dx \right| < \delta \|F\|_{Lip}.$$

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- We wish $F(\alpha n)$ to be *equidistributed* since $F(\alpha n)$ equidistributed behaves *randomly*, so is *easy* to study.

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- We wish to "approximate" $F(\alpha n)$ (possibly along progressions) by well-behaved objects.
- These well-behaved objects are of the form $\tilde{F}(\alpha' n)$ where α' is "very equidistributed" along a *rational subgroup* $\mathbb{R}^d/\mathbb{Z}^d$.

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- Otherwise, we may Fourier approximate

$$F(\alpha n) = \sum_{\xi \in \mathbb{Z}^d, |\xi| \leq \|F\|_{Lip} \delta^{-1-o(1)}} a_\xi e(\xi \cdot (\alpha n)) + O(\delta^{1+o(1)})$$

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with $|a_\xi| \leq 1$.
- Thus, there exists some nonzero ξ such that $\mathbb{E}_{n \in [N]} e(\xi \cdot \alpha n) \geq \delta^{O(d)}$. This rearranges to $\|\xi \cdot \alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{-O(d)}}{N}$.

Heuristic: a high dimensional circle method

- So we may write $\alpha = \epsilon + \alpha' + \gamma$ where $\|\epsilon\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O(d)}}{N}$, α' lies on a *subgroup* of $\mathbb{R}^d/\mathbb{Z}^d$ (that is $\delta^{-1-o(1)}$-*rational*), and γ is periodic modulo $\delta^{-1+o(1)}$.

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- Let q be the period of γ.
- Along arithmetic progressions of common difference q and length $\delta^{O(d)}$, $F(\alpha n)$ can be approximated by $F(\epsilon_0 + \alpha' n)$ for some constant ϵ_0.

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we would need to decrease the scale of equidistribution from δ to $\delta^{2+o(1)}$.
- Under an iteration, this would produce *at best* bounds of the shape δ^{2^d} since $\delta \mapsto \delta^2$ iterates to δ^{2^d}.
- Can we do better than this? Can we produce bounds *single exponential in dimensions*, i.e. $\delta^{O(d)^{O(1)}}$?

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- Something like $\delta \mapsto \delta^2$ is not allowed under iteration, since this iterates to δ^{2^d}.
- This process produces an equiditribution theory for the sequence (αn) rather than the sequence $F(\alpha n)$.

Observation

- If we define (αn) to be δ-equidistributed if for every Lipschitz function F such that

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} F(n\alpha) - \int_{\mathbb{R}^d/\mathbb{Z}^d} F(x)dx \right| < \delta \|F\|_{Lip}$$

a similar process to the work above would produce a factorization $\alpha = \epsilon + \alpha' + \gamma$ where α' is $\delta^{O(d)^{O(d)}}$-equidistrubted on a subgroup for *every* Lipschitz function on the subgroup.

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  a similar process to the work above would produce a factorization $\alpha = \epsilon + \alpha' + \gamma$ where α' is $\delta^{O(d)^{O(d)}}$-equidistrubted on a subgroup for *every* Lipschitz function on the subgroup.
- Such a factorization result is known as a *Ratner-type factorization theorem* in the literature.

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- If we do that, the number of complex exponentials we consider in fact *decreases*.
- Thus, one can prove an approximation result with bounds single exponential in dimension.

Main question

Question

What is the analogue of this heuristic in other contexts?

For instance, what can we say if instead of $\mathbb{R}^d/\mathbb{Z}^d$, we work with G/Γ where G is a Lie group, Γ a discrete cocompact subgroup (meaning that G/Γ is compact)?

Main theorem (informal version)

Theorem (L. 2023+)

There is such an analogue in the case where G is nilpotent (connected and simply connected), and Γ a discrete cocompact subgroup.

We say G is s-step nilpotent if we take $s+1$ commutators $[G,[G,\cdots,[G,G]]] = id$.

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We say G is s-step nilpotent if we take $s+1$ commutators $[G,[G,\cdots,[G,G]]] = id$. We will see applications of this theorem in arithmetic combinatorics later.

Example of nilpotent Lie group: Heisenberg group

Simplest nontrivial example of a nilpotent Lie group is a Heisenberg group:

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

Here, G is two-step nilpotent and admits the *lower central series* $G_0 = G_1 = G, G_i = [G_{i-1}, G]$.

Terminology and example

A Lipschitz function F on G/Γ evaluated at an orbit $g^n\Gamma$ is referred to as a *nilsequence*. If G and Γ are as above, and we let

$$g = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}, g^n = \begin{pmatrix} 1 & \alpha n & \binom{n}{2}\alpha\beta \\ 0 & 1 & \beta n \\ 0 & 0 & 1 \end{pmatrix}$$

G/Γ admits a *parametrization* in $(-1/2, 1/2]^3$ as $(\{\alpha n\}, \{\beta n\}, \{\binom{n}{2}\alpha\beta - [\alpha n]\beta n\})$ where $\{x\} = x - [x]$, where $[x]$ is the nearest integer to x with $\{x\} \in (-1/2, 1/2]$.

Terminology and example

Thus, when we Fourier expand $F(g^n\Gamma)$ with respect to that parametrization, we obtain *bracket polynomials* as opposed to characters.

$$e(k[\alpha n]\{\beta n\} + k\binom{n}{2}\alpha\beta + \ell\alpha n + m\beta n).$$

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These bracket polynomials are *nilcharacters* (to be defined formally later).

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- In the one-step case (i.e. $\mathbb{R}^d/\mathbb{Z}^d$ case), it was an *equidistribution theory* for characters, that is, understanding sums of the form $\mathbb{E}_{n\in[N]}e(\alpha n)$ that led to an *equidistribution theory* for general Lipschitz functions.

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- In view of this, we shall aim to develop an equidistribution theory of *nilcharacters.*

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$$[X_i, X_j] \in \mathrm{Span}_{\mathbb{Q}}(X_{\max(i,j)+1}, \ldots, X_d).$$

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Furthermore, the elements $\prod_{i=1}^{d} \exp(t_i X_i)$ with $t_i \in \mathbb{R}$ generate G uniquely and when $t_i \in \mathbb{Z}$ generate Γ.

Definition of horizontal character

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Previous results on quantifying nilsequence equidistribution

Theorem (Green-Tao)

If $F : G/\Gamma$ is Lipschitz, and

$$\left|\mathbb{E}_{n\in[N]}F(g^n\Gamma) - \int_{G/\Gamma} F(x)dx\right| \geq \delta\|F\|_{Lip}$$

then there exists a nonzero horizontal character η of modulus at most $(\delta/M)^{-O(d)^{O(d)^{O(1)}}}$ such that

$$\|\eta(g)\|_{\mathbb{R}/\mathbb{Z}} \ll (\delta/M)^{-O(d)^{O(d)^{O(1)}}}/N.$$

Notes on Green-Tao's theorem

- Theorem works for more general *polynomial sequences* with respect to the filtration.

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- Theorem works for more general *polynomial sequences* with respect to the filtration.
- If G is degree two or step one, then bounds are single exponential in dimension.

Nilcharacter

Given a continuous homomorphism $\xi : G_s/\Gamma_s \to \mathbb{R}/\mathbb{Z}$, we define a *nilcharacter* of frequency ξ to be a Lipschitz function $F : G/\Gamma \to \mathbb{C}$ satisfying $F(g_s x) = e(\xi(g_s))F(x)$ (think, bracket polynomial with s iterated/nested brackets.)

Iterating Green-Tao's result

- We can again iterate to obtain a similar Ratner-type factorization theorem $g^n = \epsilon(n)g_1(n)\gamma(n)$, but now with bounds double exponential in dimension, even in the one-step case.

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- Since nilcharacters have integral zero, we may iterate this result to obtain a slightly stronger equidistribution theorem in this case.
- Unfortunately, inserting this result to the Fourier expanded nilcharacters in the two-step case doesn't do any better; the extra parameter, *complexity*, increases too fast.
- *induction on dimensions* is a huge issue everywhere.

Bracket polynomials and Bohr sets

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- Green and Tao show that degree two bracket polynomials are "morally equivalent" to quadratic functions on large generalized arithmetic progressions.
- In 2010, Gowers and Wolf apply an equidistribution theory for quadratic functions on generalized arithmetic progressions to the *true complexity problem*.

Bracket polynomials and Bohr sets

Let $[\vec{N}] = [N_1] \times [N_2] \times \cdots \times [N_d]$. Let $q(\vec{n}) = \sum_{ij} \alpha_{ij} n_i n_j$. We wish to study exponential sums

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The conclusion is that there exists some integer $q \ll \delta^{-O(d)^{O(1)}}$ such that

$$\|q\alpha_{ij}\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O(d)^{O(1)}}}{N_i N_j}.$$

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Bounds are good (single exponential in dimension).

Approaches

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- Can we understand this approach in terms of nilmanifolds?

Statement of Main Theorem

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- $F : G/\Gamma \to \mathbb{C}$ will be a *nilcharacter* of frequency ξ with $|\xi| \le (\delta/M)^{-1}$ (with δ some parameter). That is, $F(g_s x) = e(\xi(g_s))F(x)$ for $g_s \in G_{(s)}$.

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- If $\eta : G/\Gamma \to \mathbb{R}/\mathbb{Z}$ is a horizontal character, we identify it (via Mal'cev coordinates) with a vector $\vec{k} \in \mathbb{Z}^d$, so we may lift it to some $\tilde{\eta} : G \to \mathbb{R}$.

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- We say that $w \in G$ is *orthogonal* to η if $\tilde{\eta}(w) = 0$.

Statement of Main Theorem

- We can define notions of linear independent of horizontal characters by identifying them with vectors in $\mathbb{Z}^d$.
- By identifying $w \in \Gamma$ with a vector $k \in \mathbb{Z}^d$, we can also define modulus, and linear independence of w.

Statement of Main Theorem

Theorem

Let $\delta > 0$ and N an integer. Suppose

$$|\mathbb{E}_{n \in [N]} F(g^n \Gamma)| \geq \delta.$$

Then either $N \ll (\delta/M)^{-O_s(d)^{O_s(1)}}$ or there exists linearly independent horizontal characters $\eta_1, \dots, \eta_r$ of modulus at most $(\delta/M)^{-O_s(d)^{O_s(1)}}$ such that

$$\|\eta_j \circ g\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{(\delta/M)^{-O_s(d)^{O_s(1)}}}{N}$$

and if w_i are orthogonal to η_j, $\xi([w_1, \dots, w_s]) = 0$.

Statement of the Main Theorem, s = 2

Theorem

Let $\delta > 0$ and N an integer. Suppose G is two-step and

$$|\mathbb{E}_{n \in [N]} F(g^n \Gamma)| \geq \delta.$$

Then either $N \ll (\delta/M)^{-O(d)^{O(1)}}$ or there exists linearly independent horizontal characters $\eta_1, \dots, \eta_r$ of modulus at most $(\delta/M)^{-O(d)^{O(1)}}$, and $w_1, \dots, w_{d-r} \in \Gamma$ linearly independent and orthogonal to all of the η_i's and modulus at most $(\delta/M)^{-O(d)^{O(1)}}$ such that

$$\|\eta_j \circ g\|_{\mathbb{R}/\mathbb{Z}}, \|\xi([w_i, g])\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{(\delta/M)^{-O(d)^{O(1)}}}{N}.$$

Remark, s = 2

If we let $\tilde{G} = G/\ker(\xi)$, then

$$H := \{g \in \tilde{G} : \eta_i(g) = 0, \xi([w_i, g]) = 0 \forall i\}$$

is abelian. This is because if $g, h \in H$, then it suffices to check that $[g, h] = 0$. This follows since $\eta_i(g) = 0$ implies that g can be written $\pmod{[G, G]}$ as a combination of w_i's.

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is abelian. This is because if $g, h \in H$, then it suffices to check that $[g, h] = 0$. This follows since $\eta_i(g) = 0$ implies that g can be written $\pmod{[G, G]}$ as a combination of w_i's. In fact, the map $(x, y) \mapsto \xi([x, y])$ is a *symplectic form* (after quotienting by degeneracies) and the theorem states that g morally lies in a Lagrangian (or rather *isotropic*) subspace with respect to the symplectic form.

Slogan

Theorem (Informal version)

If $F(g(n)\Gamma)$ is a nilcharacter of step s and

$$|\mathbb{E}_n F(g(n)\Gamma) - \int F| \geq \delta$$

then F is "morally" a nilsequence of step $s-1$ (with bounds single exponential in dimension).

Application: Polynomial Szemerédi

In 2022, L. showed:

Theorem

Let $P(x), Q(x) \in \mathbb{Z}[x]$ be two linearly independent polynomials with $P(0) = Q(0) = 0$. Suppose $A \subseteq \mathbb{Z}_N$ lacks a progression of the form $(x, x + P(y), x + Q(y), x + P(y) + Q(y))$. Then

$$|A| \ll_{P,Q} \frac{N}{\log_{m_{P,Q}}(N)}.$$

Here, $\log_{m_{P,Q}}(N)$ is an iterated logarithm with $m_{P,Q}$ times.

Application: Polynomial Szemerédi

Inserting this equidistribution theorem yields

Theorem (L, 2023+)

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In 2023, Peluse, Sah, and Sawhney showed:

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Suppose a subset $A \subseteq [N]$ lacks a progression of the form $(x, x + y^2 - 1, x + 2(y^2 - 1))$. Then

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They remark that a similar application of the equidistribution result would yield

$$|A| \ll_{P,Q} \frac{N}{\exp(\log\log(N)^c)}.$$

Application: Inverse theory of Gowers norm

In 2010, Green-Tao-Ziegler showed:

Theorem

Suppose $\|f\|_{U^{s+1}([N])} \geq \delta$. Then there exists a nilsequence $F(g^n\Gamma)$ of dimension $D(\delta)$ and complexity $M(\delta)$ such that

$$|\langle f, F(g^n\Gamma)\rangle| \geq c(\delta).$$

Application: Inverse theory of Gowers norm

- In 2010, Sanders shows that if $s = 2$, we may take $D(\delta) = \log(1/\delta)^{O(1)}$, $M(\delta) = O(1)$, and $c(\delta) = \exp(-\log(1/\delta)^{O(1)})$.

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- In 2018, Manners shows that we may generally take $D(\delta) = \delta^{-O_s(1)}$, $M(\delta) = \exp\exp(\delta^{-O_s(1)})$, and $c(\delta) = \exp(-\exp(\delta^{-O_s(1)}))$.

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- In the case of $s = 3$, Manners shows that we may take $M(\delta) = \exp(\delta^{-O(1)})$ and $c(\delta) = \exp(-\delta^{-O(1)})$.

Application: Inverse theory of Gowers norm

We can show:

Theorem (L., 2023+)

In the case of $s = 3$, we can take $M(\delta) = O(1)$, $D(\delta) = \exp(O(\log\log(1/\delta)^2))$, and $c(\delta) = \exp(-\exp(O(\log\log(1/\delta)^2)))$.

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Applying van der Corput gives that there exists $\delta^{O(1)} N$ many $h \in \mathbb{Z}_N$ such that

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Let us analyze $\phi(n+h)$.

Fourier Complexity and Bracket Polynomials

We can write

$$\alpha(n+h)[\beta(n+h)] = \alpha n[\beta(n+h)] + \alpha h[\beta(n+h)]$$

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The function $e(\{\alpha n\}\{\beta n\})$ can be written as $F(\{\alpha n\}, \{\beta n\})$ where $F(x,y) = e(xy)$. F is not defined on $(\mathbb{R}/\mathbb{Z})^2$, but if we approximate F with a *smoothed out* version of F near the boundary of $(-1/2, 1/2]^2$, it will be!

Fourier Complexity and Bracket Polynomials

We may thus Fourier approximate the smoothed out $\tilde{F}$ to obtain

$$\tilde{F}(x, y) = \sum_{|\eta| \leq \delta^{-1}} a_\eta e(\eta \cdot (x, y)) + O_{L^\infty[\mathbb{T}^2]}(\delta)$$

with $|a_\eta| \leq 1$ assuming that α, β are denominator N, we have

$$F(\{\alpha n\}, \{\beta n\}) = \sum_{|\eta| \leq \delta^{-1}} a_\eta e(\eta \cdot (\alpha n, \beta n)) + O_{L^1[N]}(\delta).$$

Fourier Complexity and Bracket Polynomials

Thus, $e(\{\alpha n\}(\{\beta n\} + \{\beta h\} - \{\beta(n+h)\})$ is *lower order* and may be Fourier expanded into linear phases. One can show that

$$e(\phi(n+h) - \phi(n)) = e(\sum_{i=1}^{d} \alpha_i n\{\beta_i h\} - \beta_i n\{\alpha_i h\} + \beta nh).$$

Thus, letting $a = (\alpha_i, -\beta_i)$ and $\alpha = (\{\beta_i h\}, \{\alpha_i n\})$, we have

$$|\mathbb{E}_{n \in [N]} e(an \cdot \{\alpha h\} + \beta nh)| \geq \delta^{O(d)^{O(1)}}.$$

This implies that

$$\|\beta h + a \cdot \{\alpha h\}\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{-O(d)^{O(1)}}}{N}.$$

Refined Bracket Polynomial Lemma

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- Green and Tao show that either $|a| \ll \delta^{-O(d)^{O(1)}}/N$, or that there exists some character $\eta \ll \delta^{-O(d)^{O(1)}}$ such that $\|\eta \cdot \alpha\| \ll \frac{\delta^{-O(d)^{O(1)}}}{N}$.

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- Can we do better?
- Gowers-Wolf suggests that we may be able to.

Refined Bracket Polynomial Lemma

Lemma

Let $\frac{1}{10} > \delta > 0$ and N be a prime. Suppose $\alpha, a \in \mathbb{R}^d$ are of denominator N, $|a| \leq \delta^{-1}$,

$$\|\beta + a \cdot \{\alpha h\}\|_{\mathbb{R}/\mathbb{Z}} = 0$$

for δN many $h \in [N]$. The either $N \ll \delta^{-O(d)^{O(1)}}$ or else there exists linearly independent $w_1, \ldots, w_r$ and $\eta_1, \ldots, \eta_{d-r}$ in $\mathbb{Z}^d$ with size at most $\delta^{-O(d)^{O(1)}}$ such that $\langle w_i, \eta_j \rangle = 0$ and

$$\|\eta_j \cdot \alpha\|_{\mathbb{R}/\mathbb{Z}} = 0, \quad |w_i \cdot a| = 0.$$

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- Tao has a simple proof (in the denominator N case) using Minkowski's second theorem. This does not generalize so simply.
- L.'s proof is quite intricate, at one point involving an iteration

$$(\delta_j, M_j, K_j, N_j, L_j, q_j)$$

$$= (\delta_{j-1}/4, M_{j-1}, (2q_{j-1}K_1/2^d)^{O(jd^2)}, N_{j-1}/(L_{j-1}q_{j-1}), \\ jL_{j-1}(\delta_{j-1}/2^d M)^{-O(d)}, (\delta_{j-1}/2^d M)^{-O(d)}q_{j-1}).$$

Remarks and questions

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- It is possible (though extremely painful) to rewrite this proof using purely bracket polynomial formalism.
- Is it possible to improve the upper bounds for $r_5(N)$, the size of the largest subset of $[N]$ which avoids 5-term arithmetic progressions?
- Is it possible to improve $U^{s+1}(\mathbb{Z}/N\mathbb{Z})$ inverse theorem for all s?

Thank you!