In [2] we compute the excess for hereditary, radical square zero and monomial triangular algebras. For a bound quiver algebra Λ, a formula for the excess of Λ is obtained. We also give a criterion for Λ to be τ-rigid.

Let $\Lambda = kQ/I$ a bound quiver algebra, and let $Z\Lambda$ be its center. We have

$$\dim_k {}^{\tau}HH^1(\Lambda) = \mathsf{dim}_k Z\Lambda - \sum_{x\in Q_0} \mathsf{dim}_k x\Lambda x + \sum_{a\in Q_1} \mathsf{dim}_k t(a)\Lambda s(a).$$

Questions arise about Morita invariance, Morita derived invariance or derived invariance of ${}^{\tau}HH^1(\Lambda)$. Also about a possible Lie structure, and an eventual prolongation towards a cohomological theory.

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Universal Massey products in representation theory of algebras

Gustavo Jasso

(joint work with Fernando Muro)

We work over an arbitrary field. Recall Kadeishvili's Intrinsic Formality Criterion [Kad88]:

Theorem. Let A be a graded algebra whose Hochschild cohomology vanishes in the following bidegrees:

$$\mathrm{HH}^{p+2,-p}(A, A) = 0, \qquad p \geq 1.$$

Then, every minimal A_∞-algebra structure on A is gauge A_∞-isomorphic to the trivial A_∞-structure, whose higher operations $m_{p+2} = 0$, $p \geq 1$, vanish.

In our joint work we generalise Kadeishvili's Criterion as follows.

Definition. Fix an integer $d \geq 1$. A graded algebra is *d-sparse* if it is concentrated in degrees that are multiples of d (hence this condition is empty if $d = 1$). A *d-sparse Massey algebra* is a pair (A, m) consisting of a d-sparse graded algebra A and a Hochschild cohomology class

$$m \in \mathrm{HH}^{d+2,-d}(A, A), \qquad \mathrm{Sq}(m) = 0,$$

of bidegree $(d + 2, -d)$ whose Gerstenhaber square vanishes.

For example, if

$$(A, m_{d+2}, m_{2d+2}, m_{3d+2}, \dots)$$

is a minimal A_∞-algebra structure on a d-sparse graded algebra A (in which case $m_{i+2} = 0$, $i \notin d\mathbb{Z}$, for degree reasons), then $m_{d+2} \in \mathrm{C}^{d+2,-d}(A, A)$ is a Hochschild cocycle whose associated Hochschild cohomology class

$$\{m_{d+2}\} \in \mathrm{HH}^{d+2,-d},$$

its *universal Massey product (of length $d + 2$)*, has vanishing Gerstenhaber square

$$\mathrm{Sq}(\{m_{d+2}\}) = 0.$$

Consequently, the pair $(A, \{m_{d+2}\})$ is a d-sparse Massey algebra.

Remark. It is an easy consequence of the d-sparsity assumption that the universal Massey product of a minimal A_∞-algebra is invariant under A_∞-isomorphisms.

Remark. Universal Massey products of length 3 have been investigated previously in representation theory, see for example [BKS04].

Definition. The *Hochschild–Massey cohomology* of a d-sparse Massey algebra (A, m) is the cohomology

$$\mathrm{HH}^{\bullet,*}(A, m)$$

of the *Hochschild–Massey (cochain) complex*, that is the bigraded cochain complex with components

$$\mathrm{HH}^{p+2,*}(A, A), \qquad p \geq 0,$$

and differential

$$\mathrm{HH}^{\bullet,*}(A, A) \longrightarrow \mathrm{HH}^{\bullet+d+1,*-d}(A, A), \qquad x \longmapsto [m, x],$$

in source bidegrees different from $(d+1, -d)$, where the differential is instead given by the formula by

$$\mathrm{HH}^{d+1,-d}(A, A) \longrightarrow \mathrm{HH}^{2(d+1),-2d}(A, A), \qquad x \longmapsto [m, x] + x^2.$$

Remark. That the differential of the Hochschild–Massey complex squares to zero is a consequence of the Gerstenhaber relations and the assumption $\mathrm{Sq}(m) = 0$.

Theorem ([JKM22, Theorem B])**.** Let (A, m) be a d-sparse Massey algebra whose Hochschild–Massey cohomology vanishes in the following bidegrees:

$$\mathrm{HH}^{p+2,-p}(A,m)=0, \qquad p>d.$$

Then, any two minimal A_∞-algebras

$$(A, m_{d+2}, m_{2d+2}, m_{3d+2}, \dots) \qquad \text{and} \qquad (A, m'_{d+2}, m'_{2d+2}, m'_{3d+2}, \dots)$$

such that $\{m_{d+2}\} = m = \{m'_{d+2}\}$ are gauge A_∞-isomorphic.

Remark. Kadeishvili's Intrinsic Formality Criterion is indeed a corollary of the above theorem: Take $d = 1$ and notice that the hypothesis in the criterion implies that every minimal A_∞-algebra structure on A has vanishing universal Massey product $\{m_3\} = 0$.

The proof of the theorem relies in an essential way on an enhanced A_∞-obstruction theory developed by F. Muro in [Mur20a]. We also mention that the theorem is one of the key ingredients in the proof of the main theorem in [JKM22] which, as explained by B. Keller in the Appendix to *loc. cit.*, in a special case yields the final step in the proof of the Donovan–Wemyss Conjecture in the context of the Homological Minimal Model Program for threefolds [DW16, Wem23].

The aforementioned applications of the theorem rely on the following observation: The Hochschild–Massey cochain is equipped with a canonical bidegree $(d + 2, -d)$ endomorphism given by

$$\mathrm{HH}^{\bullet,*}(A,A) \longrightarrow \mathrm{HH}^{\bullet+d+2,*-d}(A,A), \qquad x \longmapsto m \smile x,$$

in source bidegrees different from $(d + 1, -d)$, where it is given by

$$\mathrm{HH}^{d+1,-d}(A,A) \longrightarrow \mathrm{HH}^{2(d+1)+1,-2d}(A,A), \qquad m \smile x + \{\delta_{/d}\} \smile x^2.$$

Here,

$$\delta_{/d} \in \mathrm{C}^{1,0}(A,A), \qquad x \longmapsto \tfrac{|x|}{d}x,$$

is the fractional Euler derivation (notice that $\frac{|x|}{d}$ is an integer due to the assumption that the graded algebra A is d-sparse). The above endomorphisms is in fact null-homotopic. An explicit bidegree $(1, 0)$ null-homotopy is given by

$$\mathrm{HH}^{\bullet,*}(A,A) \longrightarrow \mathrm{HH}^{\bullet+1,*}(A,A), \qquad x \longmapsto \{\delta_{/d}\} \smile x.$$

Thus, a sufficient condition for the assumptions in the theorem to be satisfied is that the components of above endomorphism of the Hochschild–Massey complex of (A, m) are bijective in all non-trivial source bidegrees. The latter condition is satisfied by the d-sparse Massey algebras investigated in [JKM22].

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The homotopy theory of operated algebras

Guodong Zhou

A general philosophy of deformation theory of mathematical structures, as evolved from ideas of Gerstenhaber, Nijenhuis, Richardson, Deligne, Schlessinger, Stasheff, Goldman, Millson etc, is that the deformation theory of any given mathematical object can be described by a certain differential graded (=dg) Lie algebra or more generally an L_∞-algebra associated to the mathematical object (whose underlying complex is called the deformation complex). This philosophy has been made into a theorem in characteristic zero by Lurie [17] and Pridham [18], expressed in terms of infinity categories. It is an important problem to construct explicitly the dg Lie algebra or L_∞-algebra governing deformation theory of the mathematical object under consideration.

Another important problem about algebraic structures is to study their homotopy versions, just like A_∞-algebras for usual associative algebras. From the perspective of operad theory, specifically, the task is to formulate a cofibrant resolution for the operad of an algebraic structure. The most desirable outcome would be providing a minimal model of the operad governing the algebraic structure. When this operad is Koszul, there exists a general theory, the so-called Koszul duality for operads [9, 8], which defines a homotopy version of this algebraic structure via the cobar construction of the Koszul dual cooperad, which, in this case, is a minimal model. However, when a operad is NOT Koszul, essential difficulties arise and few examples of minimal models have been worked out.

These two problems, say, describing controlling L_∞-algebras and constructing homotopy versions, are closed related. In fact, given a cofibrant resolution, in particular a minimal model, of the operad in question, one can form the deformation complex of the algebraic structure and construct its L_∞-structure as explained by Kontsevich and Soibelman [14] and van der Laan [24, 25]. However, in practice, a minimal model or a small cofibrant resolution is not known a priori.

Recently, we succeeded in resolving completely the above two problems for a large class of non-Koszul operads, say operads of operated algebras (that is, associative or Lie algebras endowed with certain kinds of linear operators), such as Rota-Baxter algebras with arbitrary weight and differential algebras with nonzero weight. Surprisingly, our method returns to the original method of Gerstenhaber [6, 7]. The method consists of several steps.