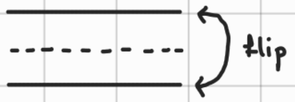
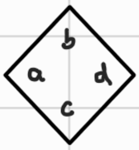
G. JASSO, Lund 2022-09-17

Frieze Patterns

	1		1		1		1		1		1		1		1		1		1		...
...		2		1		3		2		1		3		2		1		3		2	
	5		1		2		5		1		2		5		1		2		5		...
...		2		1		3		2		1		3		2		1		3		2	
	1		1		1		1		1		1		1		1		1		1		...

Q What do you notice in the above infinite pattern?

- Consists of positive integers
- Has horizontal symmetry: 

- It is periodic: 
- Has a glide symmetry:  (see next page)
- Every 2×2 diamond 

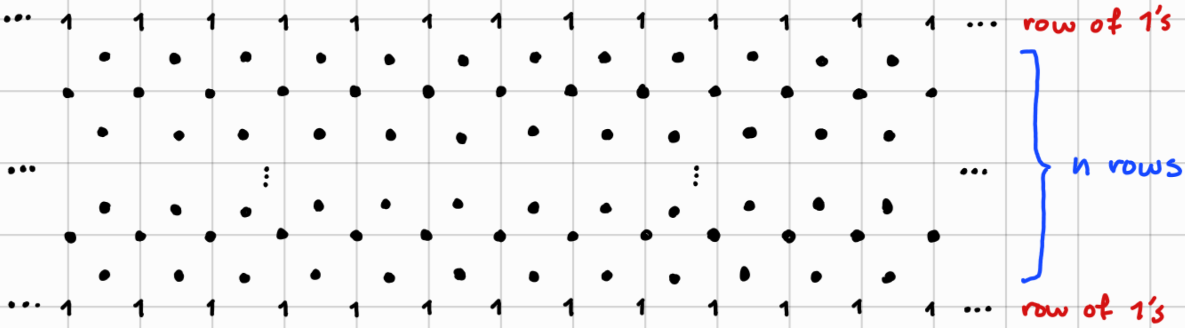
satisfies the unimodular rule $\boxed{ad - bc = 1}$

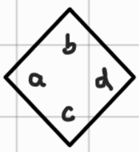
$$\left.\begin{aligned} d &= \frac{1+bc}{a} \\ c &= \frac{ad-1}{b} \end{aligned}\right\} \in \mathbb{Z}_{>0}$$

remarkable!

Def (Conway - Coxeter 1973)

An (integral) frieze pattern is an infinite array of positive integer numbers



where every 2×2 diamond 

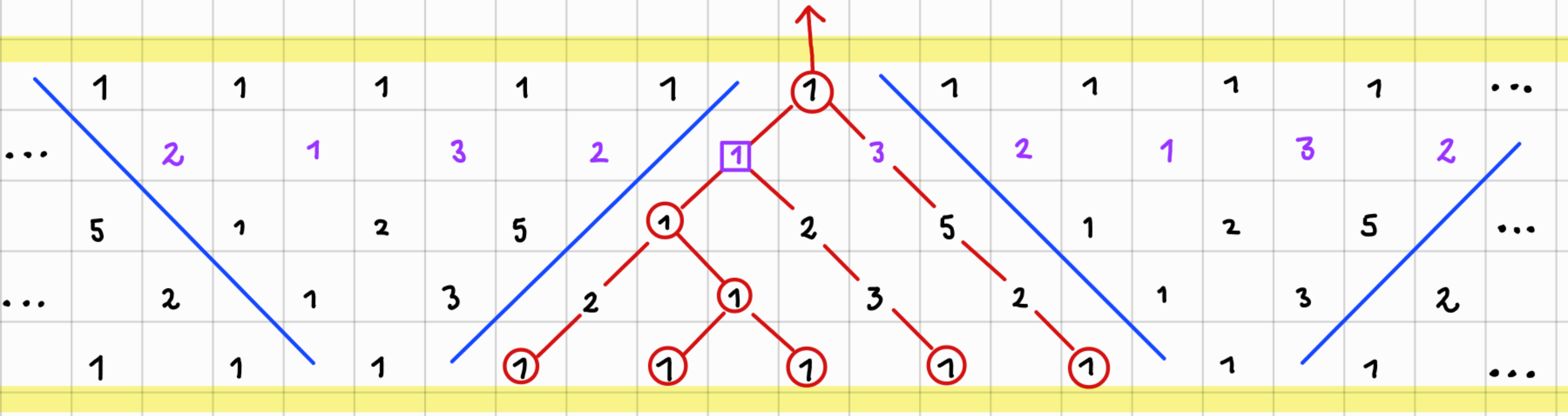
satisfies the unimodular rule $\boxed{ad - bc = 1}$

We know that there exists a frieze pattern for $n=3$.

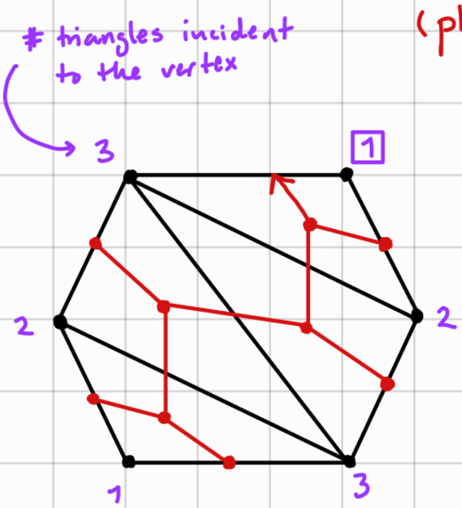
Q Do frieze patterns exist for every $n \geq 1$? Yes!

Q For fixed n, how many frieze patterns exist? Finitely many!

Q For fixed n, can we count them? Yes!



(planar) binary rooted tree!



$(1, 3, 2, 1, 3, 2)$

Quiddity sequence of the pattern

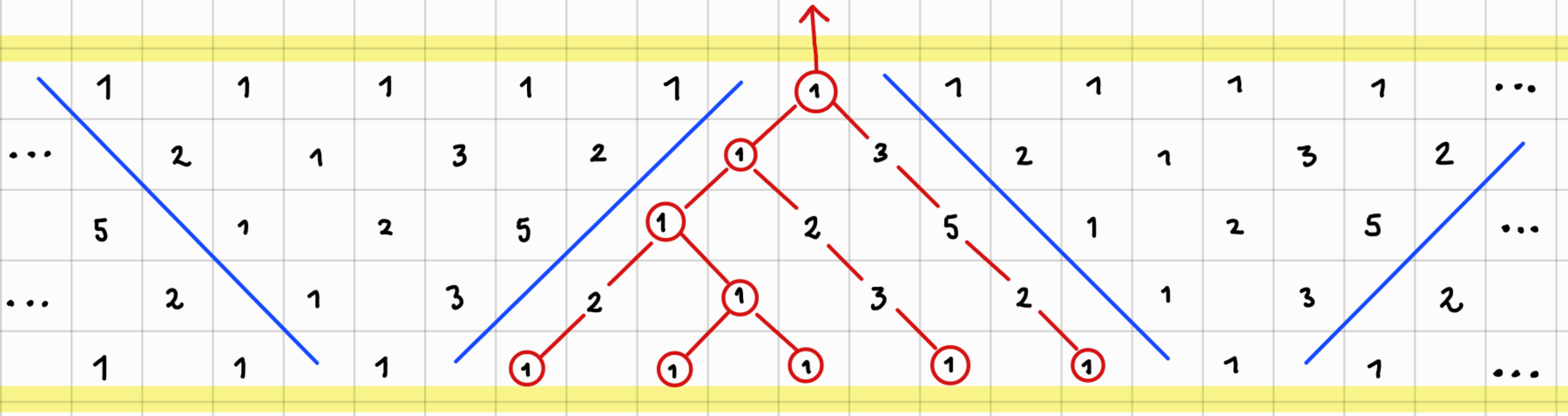
Thm (Conway - Coxeter 1973)

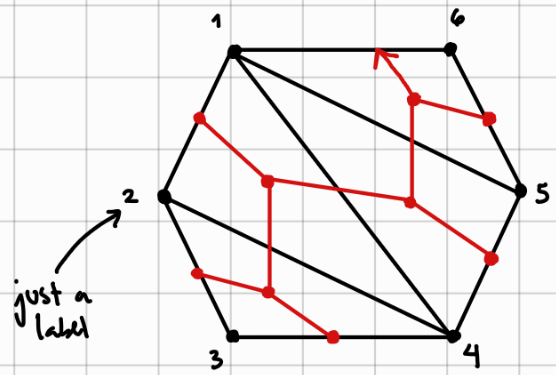
There is a bijection

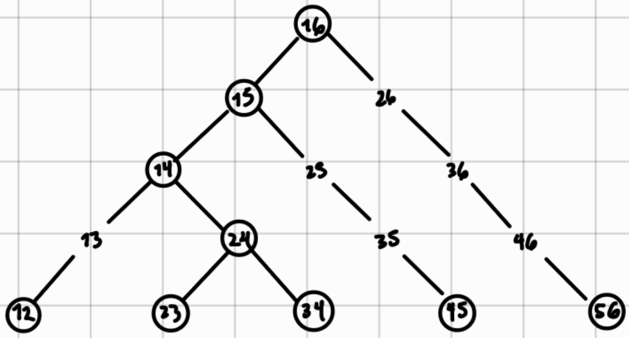
$$\{\text{frieze patterns with } n \text{ middle rows}\} \overset{1:1}{\longleftrightarrow} \{\text{triangulations of a convex } (n+3)\text{-gon}\}$$

- The unimodular rule implies that a frieze pattern is completely determined by its quiddity sequence.
- Frieze patterns with n middle rows have period $n+3$.

- $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ friezes with $n-1$ middle rows (n-th Catalan number)

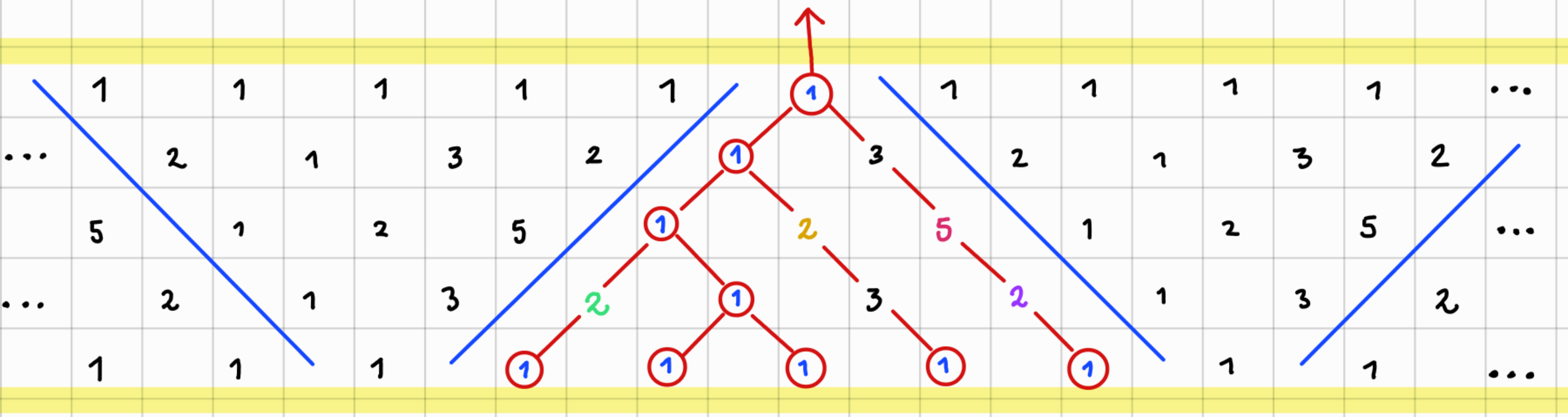




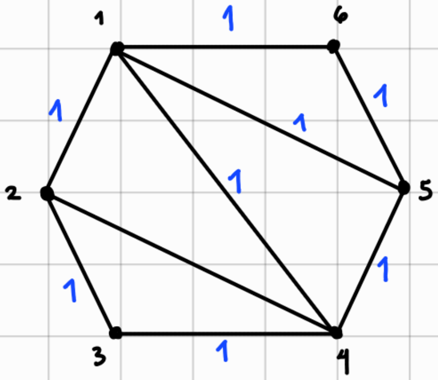


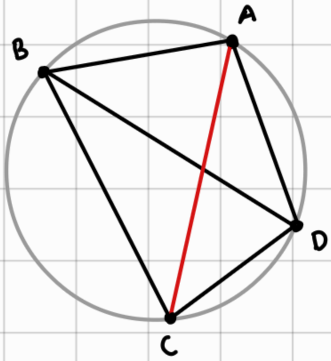
Q What can we say about larger diamonds?

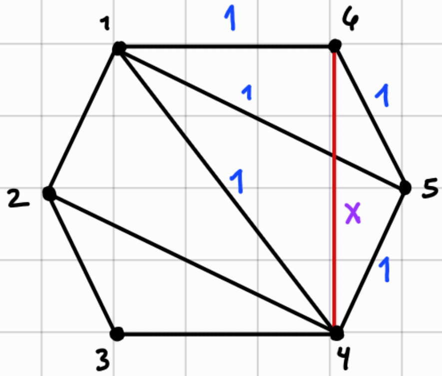
(ij) if i — j is in the triangulation



Recall Ptolemy's Theorem:

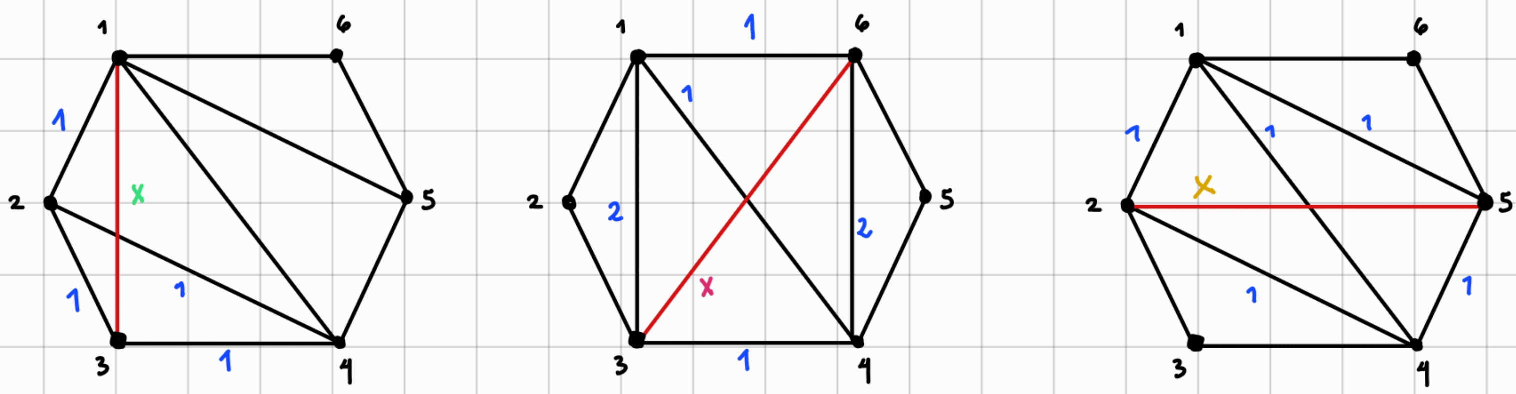






$AC \cdot BD = AB \cdot CD + AD \cdot BC$

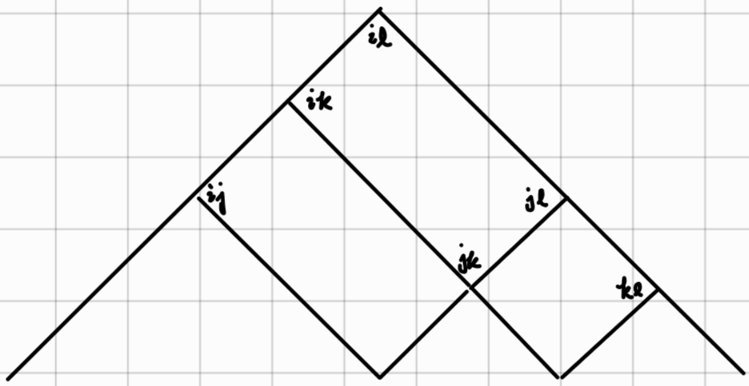
$x \cdot 1 = 1 \cdot 1 + 1 \cdot 1 \Rightarrow x = 2$



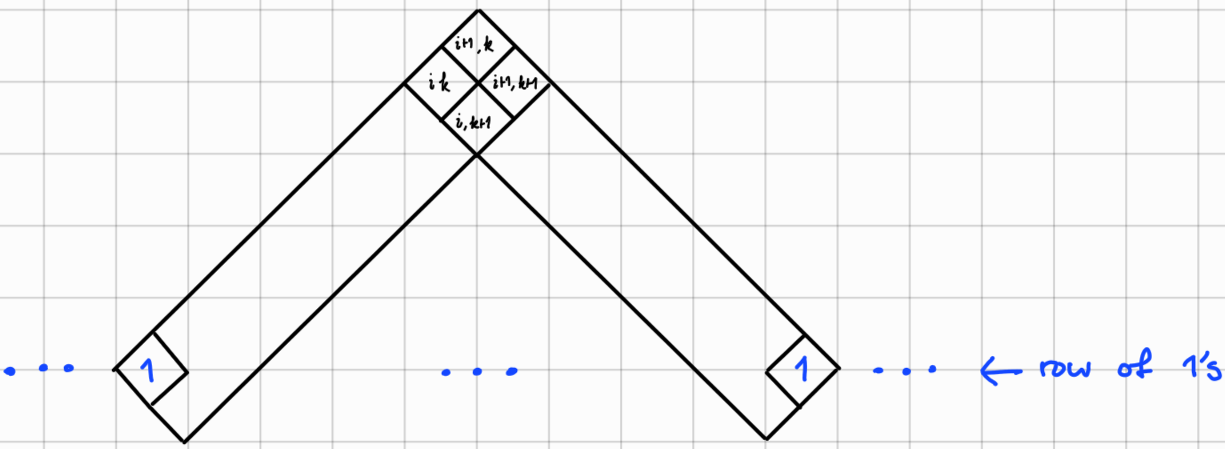
$x \cdot 1 = 1 \cdot 1 + 1 \cdot 1 \Rightarrow x = 2$ $\quad$ $x \cdot 1 = 1 \cdot 1 + 2 \cdot 2 \Rightarrow x = 5$ $\quad$ $x \cdot 1 = 1 \cdot 1 + 1 \cdot 1 \Rightarrow x = 2$

In general, for $1 \leq i < j < k < l \leq n+3$ we have

$$\boxed{ik \cdot jl = il \cdot jk + ij \cdot kl}$$



and the unimodular rule is the special case $1 \leq i < i+1 < k < k+1 \leq n+3$



Have fun proving it all!