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Minimal A_∞-algebras of endomorphisms

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Lecture 1

Motivation: The reconstruction problem

$\mathcal{T}$: $\mathbf{k}$-linear Hom-finite Krull–Schmidt triangulated category

$G \in \mathcal{T}$: basic (classical) generator, $\mathrm{thick}(G) = \mathcal{T}$

$\mathrm{End}^{\bullet}_{\mathcal{T}}(G) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^i(G))$ $\quad g * f := \Sigma^j(g) \circ f, \quad |f| = j$

Problem: Reconstruct $\mathcal{T}$ from $\mathrm{End}^{\bullet}_{\mathcal{T}}(G)$ as a triangulated category.

In general, this is <u>NOT</u> possible!

$A = \mathbf{k}[x]/(x^{\ell}), \quad \ell \geq 3, \qquad \mathrm{thick}(S) = \mathrm{D^b}(\mathrm{mod}\, A) = \mathcal{T}$

$\mathrm{End}^{\bullet}_{\mathrm{D^b}(\mathrm{mod}\, A)}(S) \cong \mathrm{Ext}^{\bullet}_{A}(S, S) \cong \mathbf{k}[\varepsilon, t]/(\varepsilon^2), \quad |\varepsilon| = 1$ and $|t| = 2$

$\mathrm{End}^{\bullet}_{\mathrm{D^b}(\mathrm{mod}\, A)}(S)$ is <u>independent</u> of ℓ but $Z(A) = A$ is derived invariant.

Differential graded algebras

A differential graded algebra consists of a graded algebra

$$\mathbf{A} = \bigoplus_{i\in\mathbb{Z}} \mathbf{A}^i$$

$$\mathbf{A}^i \otimes \mathbf{A}^j \longrightarrow \mathbf{A}^{i+j}, \qquad x \otimes y \mapsto xy,$$

and a differential

$$\mathrm{d}\colon \mathbf{A} \longrightarrow \mathbf{A}(1), \qquad \mathrm{d}\circ\mathrm{d} = 0,$$

such that

$$\underbrace{\mathrm{d}(xy) = \mathrm{d}(x)y + (-1)^{|x|}x\mathrm{d}(y)}_{\text{graded Leibniz rule}}\,.$$

- Every differential graded algebra $\mathbf{A}$ has a triangulated derived category $\mathrm{D}(\mathbf{A})$.

$$\mathrm{Hom}_{\mathrm{D}(\mathbf{A})}(\mathbf{A}, \mathbf{A}[i]) \cong \mathrm{H}^i(\mathbf{A})$$

- $\mathrm{D}^{\mathrm{c}}(\mathbf{A}) := \mathrm{thick}(\mathbf{A}) \subseteq \mathrm{D}(\mathbf{A})$ is the perfect derived category.

$X^\bullet$: complex in an additive category

$$\mathrm{hom}(X^\bullet, X^\bullet) := \bigoplus_{i\in\mathbb{Z}} \mathrm{hom}(X^\bullet, X^\bullet)^i$$

$$\mathrm{hom}(X^\bullet, X^\bullet)^i := \prod_{j\in\mathbb{Z}} \mathrm{hom}(X^j, X^{i+j})$$

$$\partial(f) := d_{P^\bullet} \circ f - (-1)^{|f|} f \circ d_{P^\bullet}$$

Derived endomorphism algebras

Suppose that $\mathcal{T}$ is algebraic:

$\mathcal{T} \simeq \underline{\mathcal{E}}_{\mathcal{S}}$ for a $\mathbf{k}$-linear Frobenius exact category $(\mathcal{E}, \mathcal{S})$.

Choose a complete $\mathcal{S}$-projective resolution $P^\bullet$ of $G \in \mathcal{T} \simeq \underline{\mathcal{E}}_{\mathcal{S}}$:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & P^{-2} \arrow[rr] \arrow[dr, two heads] & & P^{-1} \arrow[rr] \arrow[dr, two heads] & & P^{0} \arrow[rr] \arrow[dr, two heads] & & P^{1} \arrow[r] \arrow[dr, two heads] & \cdots \\
\cdots \arrow[ur, tail] & & \Omega(G) \arrow[ur, tail] & & G \arrow[ur, tail] & & \Omega^{-1}(G) \arrow[ur, tail] & & \cdots
\end{tikzcd}$$

$\mathbf{R}\mathrm{End}_{(\mathcal{E},\mathcal{S})}(G) = \hom(P^\bullet, P^\bullet)$: differential graded algebra of endomorphisms

$\mathrm{H}^\bullet\big(\mathbf{R}\mathrm{End}_{(\mathcal{E},\mathcal{S})}(G)\big) \cong \mathrm{End}^\bullet_{\mathcal{T}}(G)$ as graded algebras

Keller's Reconstruction Theorem

Theorem (Keller 1994)

Set $\mathbf{A} := \mathbf{REnd}_{(\mathcal{E},\mathcal{S})}(G)$. There exists an exact equivalence

$$\mathcal{T} \xrightarrow{\sim} \mathrm{D}^{\mathrm{c}}(\mathbf{A}), \qquad G \longmapsto \mathbf{A}.$$

In general, the quasi-isomorphism type of $\mathbf{REnd}_{(\mathcal{E},\mathcal{S})}(G)$ is not determined by $\mathcal{T}$!

Problem: Classify the DG algebras $\mathbf{A}$ such that there exists an exact equivalence

$$\mathcal{T} \xrightarrow{\sim} \mathrm{D}^{\mathrm{c}}(\mathbf{A}), \qquad G \longmapsto \mathbf{A}.$$

Remark: This problem is intimately related to the question of uniqueness of differential graded enhancements for $\mathcal{T}$.

Formality of differential graded algebras

Definition

A differential graded algebra $\mathbf{A}$ is

- formal if it is quasi-isomorphic to its cohomology $\mathrm{H}^{\bullet}(A)$.
- intrinsically formal if every differential graded algebra $\mathbf{B}$ such that

$$\mathrm{H}^{\bullet}(\mathbf{A}) \cong \mathrm{H}^{\bullet}(\mathbf{B})$$

  is moreover quasi-isomorphic to $\mathbf{A}$.

Intrinsic formality $\implies$ Formality   The converse is false in general.

$\mathrm{H}^{\bullet}(\mathbf{A}) = \mathrm{H}^{0}(\mathbf{A}) \implies$ $\mathbf{A}$ is intrinsically formal   (corresponds to $G \in \mathcal{T}$ is tilting)

Derived endomorphism algebras of simple modules

Theorem (Keller 2001)

$A = \mathbf{k}Q/I$: finite-dimensional algebra

$S = S_1 \oplus \cdots \oplus S_n$ direct sum of the simple A-modules ($\mathrm{thick}(S) = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A)$)

$$\mathbf{R}\mathrm{Hom}_A(S,S) \text{ is formal} \iff A \text{ is Koszul}$$

A is Koszul $\iff$ $\mathrm{Ext}_A^{\bullet}(S,S)$ is generated in degrees 0 and 1

- Hereditary algebras
- Radical square-zero algebras
- Quadratic monomial algebras
- Exterior algebras
- Tensor products of Koszul algebras …

Kadeishvili's Intrinsic Formality Criterion

The Hochschild cohomology of a graded algebra $\Lambda^\star$ is the bigraded vector space

$$\mathrm{HH}^{\bullet,\star}(\Lambda^\star) := \mathrm{Ext}^{\bullet,\star}_{\Lambda^\star\text{-bimod}}(\Lambda^\star, \Lambda^\star).$$

Theorem (Kadeishvili 1988)

Suppose that

$$\mathrm{HH}^{p+2,-p}(\Lambda^\star) = 0, \qquad p > 0. \tag{$\dagger$}$$

Then, $\Lambda^\star$ is intrinsically formal as a differential graded algebra.

Theorem (Etgü–Lekili 2017, Lekili–Ueda 2022, J. Liu–Zh.Wang)

ADE zig-zag algebras in good characteristic satisfy condition ($\dagger$).

Intrinsic formality of Laurent polynomial algebras

Λ: arbitrary algebra

$$\Lambda[u^{\pm}] := \Lambda \otimes \mathbf{k}[u^{\pm}], \qquad |u| = d \geq 1$$

Remark: $\mathrm{D}(\Lambda[u^{\pm}])$ is the d-periodic derived category of Λ-modules.

Suppose that $\mathbf{1}_{\mathcal{T}} \cong \Sigma^{d}$ as additive functors and that $G \in \mathcal{T}$ satisfies

$$\mathrm{Hom}_{\mathcal{T}}(G, \Sigma^{i}(G)) = 0 \quad \text{for } i \notin d\mathbb{Z}.$$

Then $\mathrm{End}_{\mathcal{T}}^{\bullet}(G) \cong \mathrm{End}_{\mathcal{T}}(G)[u^{\pm}]$ with $|u| = d$.

Theorem (S. Saito 2023)

If Λ has projective dimension at most d as a Λ-bimodule, then $\Lambda[u^{\pm}]$ satisfies condition (†) and hence it is intrinsically formal as a differential graded algebra.

Twisted Laurent polynomial algebras

Λ an arbitrary algebra and $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$ an automorphism

$$\Lambda(\sigma, d) := \frac{\Lambda\langle u^{\pm}\rangle}{\langle xu - u\sigma(x) \mid x \in \Lambda\rangle}, \qquad |u| = d \geq 1$$

Suppose that $G \in \mathcal{T}$ satisfies

$\exists \varphi\colon G \xrightarrow{\sim} \Sigma^d(G)$ and $\mathrm{Hom}_{\mathcal{T}}(G, \Sigma^i(G)) = 0$ for $i \notin d\mathbb{Z}$.

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, dotted, "\sigma(f)"'] & \Sigma^d(G) \arrow[d, "\Sigma^d(f)"] \\
G & \Sigma^d(G) \arrow[l, "\varphi^{-1}"]
\end{tikzcd}$$

Define the automorphism

$$\sigma = \sigma_{\varphi}\colon \mathrm{End}_{\mathcal{T}}(G) \xrightarrow{\sim} \mathrm{End}_{\mathcal{T}}(G), \quad f \longmapsto \varphi^{-1} \circ \Sigma^d(f) \circ \varphi.$$

$$\mathrm{End}^{\bullet}_{\mathcal{T}}(G) \cong \mathrm{End}_{\mathcal{T}}(G)(\sigma, d), \qquad \varphi \longmapsto u$$

$d\mathbb{Z}$-cluster tilting objects

Definition (Iyama–Yoshino 2008)

A basic object $G \in \mathcal{T}$ is a <u>d-cluster tilting object</u> if

$$\begin{aligned} \operatorname{add}(G) &= \{X \in \mathcal{T} \mid \forall 0 < i < d,\ \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^i(G)) = 0\} \\ &= \{Y \in \mathcal{T} \mid \forall 0 < i < d,\ \operatorname{Hom}_{\mathcal{T}}(G, \Sigma^i(Y)) = 0\}. \end{aligned}$$

We call G a <u>$d\mathbb{Z}$-cluster tilting object</u> if, moreover,

- $\exists\, \varphi\colon G \xrightarrow{\sim} \Sigma^d(G)$ (Geiß–Keller–Oppermann 2013).

$G \in \mathcal{T}$ is $1\mathbb{Z}$-cluster tilting $\iff$ $\operatorname{add}(G) = \mathcal{T}$

Proposition (Iyama–Yoshino 2008)

$G \in \mathcal{T}$: $d\mathbb{Z}$-cluster tilting $\implies$ $\operatorname{thick}(G) = \mathcal{T}$

Triangulated categories with Serre functor

Suppose that $\exists\, \mathbb{S}\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ a Serre functor:

$$\mathrm{Hom}_{\mathcal{T}}(Y, \mathbb{S}X) \xrightarrow{\sim} D\mathrm{Hom}_{\mathcal{T}}(X, Y), \qquad \forall X, Y \in \mathcal{T}$$

Proposition (Iyama–Oppermann 2013)

The following are equivalent for a basic d-cluster tilting object $G \in \mathcal{T}$:

- G is a $d\mathbb{Z}$-cluster tilting object.
- There is an isomorphism $\mathbb{S}G \cong G$.
- $\mathrm{End}_{\mathcal{T}}(G)$ is self-injective and $\underbrace{\mathrm{Hom}_{\mathcal{T}}(\Sigma^i(G), G) \text{ for } 0 < i < d-1.}_{\text{vosnex property}}$

The vosnex property is vacuous for $d = 1, 2$

Examples of $1\mathbb{Z}$-cluster tilting objects

Triangulated categories of finite type: $\operatorname{add}(G) = \mathcal{T}$

- Stable module categories of self-injective algebras of finite representation type.
- Stable categories of maximal Cohen–Macaulay modules of complete local Gorenstein isolated singularities of finite Cohen–Macaulay type.
- Stable categories of Gorenstein-projective modules of finite-dimensional Iwanaga–Gorenstein algebras of finite Gorenstein-projective type.
- Cluster categories of hereditary algebras of finite representation type.

See **F. Muro's talk** next week for more on these.

Examples of $2\mathbb{Z}$-cluster tilting objects

Amiot cluster categories of self-injective quivers with potential

- (Barot–Kussin–Lenzing 2010, J .2015) Weighted projective lines of tubular tubular type $\neq (3,3,3)$.
- (Herschend–Iyama 2011) Certain planar quivers with potential.
- (Pasquali 2020) Rotationally-symmetric Postnikov diagrams on the disk.

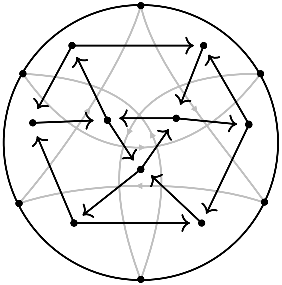


Figure by Colin Krawchuk

See **F. Muro's talk** for important examples from 3-dim birational geometry.

Examples of $d\mathbb{Z}$-cluster tilting objects

Definition (Iyama–Oppermann 2011)

A finite-dimensional algebra if d-representation-finite if it admits a d-cluster tilting module.

- (Geiß–Leclerc–Schroer 2007 for $d = 1$, Iyama–Oppermann 2013) Stable module categories of $(d+1)$-preprojective algebras of d-Auslander algebras of type $\mathbb{A}$.
- (Darpö–Iyama 2020) Stable module categories of certain self-injective d-representation-finite algebras.
- (J–Külshammer 2016) Stable module categories of self-injective d-Nakayama algebras.
- (Iyama–Oppermann 2013) d-Calabi–Yau Amiot–Guo–Keller cluster categories of Keller's derived $(d+1)$-preprojective algebras of d-representation-finite algebras of global dim d.

See the preprint `arXiv:2208.14413` (J-Muro) for more examples.

Twisted periodic algebras

Definition (Brenner–Butler, Green–Snashall–Solberg 2003)

A finite-dimensional algebra Λ is twisted $(d+2)$-periodic if there exists an automorphism $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$ such that

$$\Omega^{d+2}_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\sigma \qquad \text{in} \qquad \underline{\mathrm{mod}}\Lambda^e.$$

We say that A is $(d+2)$-periodic if $\sigma = \mathbf{1}$.

(Green–Snashall–Solberg 2003) Twisted periodic algebras are self-injective.

Proposition (Dugas 2012, Hanihara 2020 $d=1$, Chan–Darpö–Iyama–Marczinzik)

G: $d\mathbb{Z}$-cluster tilting object $\implies$ $\mathrm{End}_{\mathcal{T}}(G)$ is twisted $(d+2)$-periodic

Twisted fractionally CY algebras

A: finite-dimensional algebra of finite global dimension

The triangulated category $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$ admits the Serre functor

$$\mathbb{S} := - \otimes_A^{\mathrm{L}} DA \colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A).$$

Definition

Let $l \neq 0$ and m be integers. The algebra A is twisted fractionally $\frac{m}{\ell}$-Calabi–Yau if there exists an automorphism $\phi \colon A \xrightarrow{\sim} A$ such that

$$\mathbb{S}^{\ell} \cong [m] \circ \phi^*.$$

We say that A is fractionally $\frac{m}{\ell}$-Calabi–Yau if $\phi = \mathbf{1}$.

Periodic algebras from fractionally CY algebras

$T(A) := A \ltimes DA$ the trivial extension of A

Theorem (Chan–Darpö–Iyama–Marczinzik)

$$
\begin{array}{ccc}
A \text{ is fractionally CY} & \Longleftrightarrow & T(A) \text{ is periodic} \\
\text{trivial: } \sigma=1 \Downarrow & & \Downarrow \text{trivial: } \phi=1 \\
A \text{ is twisted fractionally CY} & \Longleftrightarrow & T(A) \text{ is twisted periodic}
\end{array}
\qquad \text{Open} \Uparrow
$$

Suppose that A is ring-indecomposable

Theorem (Herschend–Iyama 2011)

A is d-representation-finite of global dim d $\Longrightarrow$ A is twisted fractionally CY

$d\mathbb{Z}$-cluster tilting objects from twisted periodic algebras

Λ: basic twisted $(d+2)$-periodic algebra with respect to $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$

Problem 1: Does there exist a differential graded algebra $\mathbf{A}$ with $\mathrm{H}^\bullet(\mathbf{A}) \cong \Lambda(\sigma, d)$ and such that $\mathbf{A} \in \mathrm{D}^{\mathrm{c}}(\mathbf{A})$ is a $d\mathbb{Z}$-cluster tilting object?

Problem 2: Suppose that $\mathrm{H}^\bullet(\mathbf{A}) \cong \Lambda(\sigma, d)$. How to determine whether $\mathbf{A} \in \mathrm{D}^{\mathrm{c}}(\mathbf{A})$ is a $d\mathbb{Z}$-cluster tilting object?

Problem 3: Suppose that $\mathrm{H}^\bullet(\mathbf{A}) \cong \Lambda(\sigma, d)$ and that $\mathbf{A} \in \mathrm{D}^{\mathrm{c}}(\mathbf{A})$ is a $d\mathbb{Z}$-cluster tilting object.

What additional data is needed to reconstruct $\mathbf{A}$ from its cohomology $\mathrm{H}^\bullet(A)$, at least up to quasi-isomorphism?

Lecture 2

$d\mathbb{Z}$-cluster tilting objects from twisted periodic algebras

Λ: basic twisted $(d+2)$-periodic algebra with respect to $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$

Problem 1: Does there exist a differential graded algebra $\mathbf{A}$ with $\mathrm{H}^{\bullet}(\mathbf{A}) \cong \Lambda(\sigma, d)$ and such that $\mathbf{A} \in \mathrm{D}^{\mathrm{c}}(\mathbf{A})$ is a $d\mathbb{Z}$-cluster tilting object?

Problem 2: Suppose that $\mathrm{H}^{\bullet}(\mathbf{A}) \cong \Lambda(\sigma, d)$. How to determine whether $\mathbf{A} \in \mathrm{D}^{\mathrm{c}}(\mathbf{A})$ is a $d\mathbb{Z}$-cluster tilting object?

Problem 3: Suppose that $\mathrm{H}^{\bullet}(\mathbf{A}) \cong \Lambda(\sigma, d)$ and that $\mathbf{A} \in \mathrm{D}^{\mathrm{c}}(\mathbf{A})$ is a $d\mathbb{Z}$-cluster tilting object.

What additional data is needed to reconstruct $\mathbf{A}$ from its cohomology $\mathrm{H}^{\bullet}(A)$, at least up to quasi-isomorphism?

The Derived Auslander–Iyama Correspondence

Theorem (Muro 2022 for $d = 1$, J–Muro for $d \geq 1$)

Suppose that the field $\mathbf{k}$ is perfect. The map

$$\mathbf{A} \longmapsto (\mathrm{H}^0(\mathbf{A}),\ \mathrm{H}^{-d}(\mathbf{A})) = (\mathrm{Hom}_{\mathrm{D}(\mathbf{A})}(\mathbf{A},\mathbf{A}),\ \mathrm{Hom}_{\mathrm{D}(\mathbf{A})}(\mathbf{A},\mathbf{A}[-d]))$$

induces a bijection between the following:

1. Quasi-isomorphism classes of DG algebras $\mathbf{A}$ such that:
   - $\mathrm{H}^0(\mathbf{A})$ is a basic finite-dimensional algebra.
   - $\mathbf{A} \in \mathrm{D}^{\mathrm{c}}(\mathbf{A})$ is a $d\mathbb{Z}$-cluster tilting object.
2. Pairs (Λ, σ) such that
   - Λ is a basic self-injective algebra and
   - $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$ such that $\Omega^{d+2}_{\Lambda^e}(\Lambda) \simeq {}_1\Lambda_\sigma$ in $\underline{\mathrm{mod}}\Lambda^e$,

   up to algebra isomorphisms compatible with

$$\overline{\sigma} \in \mathrm{Out}(\Lambda) := \mathrm{Aut}(\Lambda)/\mathrm{Inn}(\Lambda). \qquad (\ \mathrm{H}^{-d}(\mathbf{A}) \cong {}_1\mathrm{H}^0(\mathbf{A})_\sigma\ )$$

Constructing the inverse of the correspondence

Λ: twisted $(d+2)$-periodic with respect to $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$

$$\Lambda(\sigma, d) \cong \bigoplus_{di \in d\mathbb{Z}} {}_{\sigma^i}\Lambda_1, \qquad x * y := \sigma^j(x)y, \quad |y| = dj$$

We aim to construct a differential graded algebra $\mathbf{A}$ such that

$$\mathrm{H}^\bullet(\mathbf{A}) \cong \Lambda(\sigma, d)$$

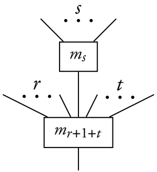
and $\mathbf{A} \in \mathrm{D}^c(\mathbf{A})$ is a $d\mathbb{Z}$-cluster tilting object.

These properties should determine $\mathbf{A}$ up to quasi-isomorphism.

Stasheff's A_∞-algebras

An A_∞-algebra structure on a graded vector space $\Lambda^\star$ consists of homogeneous morphisms of degree $2-n$

$$m_n \colon \underbrace{\Lambda^\star \otimes \cdots \otimes \Lambda^\star}_{n \text{ times}} \longrightarrow \Lambda^\star, \qquad n \geq 1,$$

$\sum \pm$  $= 0$

such that the A_∞-equations are satisfied:

$$\sum_{n=r+s+t} (-1)^{r+st} m_{r+1+t} \circ (\mathbf{1}^r \otimes m_s \otimes \mathbf{1}^t) = 0 \qquad (n \geq 1)$$

$$\begin{aligned} m_1 \circ m_1 &= 0 \\ m_1 \circ m_2 &= m_2 \circ (m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1) \\ \underbrace{m_2 \circ (\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1})}_{\text{Associator for } m_2} &= \underbrace{m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1)}_{\partial(m_3) \text{ in } \hom(\Lambda^\star \otimes \Lambda^\star \otimes \Lambda^\star,\, \Lambda^\star) \qquad (\Lambda^\star, m_1)} \end{aligned}$$

Remarks on the definition of A_∞-algebras

$\Lambda^\star = \Lambda^0 \quad \Longrightarrow \quad m_n = 0$ for $n \neq 2$ for degree reasons.

$m_1 = 0 \quad \Longrightarrow \quad (\Lambda^\star, 0, m_2)$ is an associative graded algebra.

$(\Lambda^\star, m_1, m_2)$: differential graded algebra $\Longleftrightarrow$ $(\Lambda^\star, m_1, m_2, 0, \dots)$: A_∞-algebra.

There are several sign conventions in use: Stasheff, Keller–Lefèvre-Hasegawa*, Kontsevich–Merkulov, Fukaya–Seidel.

See `Polishchuk's Field Guide` for details.

… one may equivalently consider shifted A_∞-structures to dispense with most signs.

Morphisms between A_∞-algebras

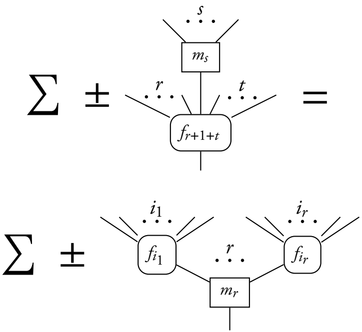
An A_∞-morphism between A_∞-algebras

$$f\colon (\Lambda_1^\star, m^{(1)}) \rightsquigarrow (\Lambda_2^\star, m^{(2)})$$

consists of degree $1-n$ morphisms

$$f_n\colon \underbrace{\Lambda_1^\star \otimes \cdots \otimes \Lambda_1^\star}_{n \text{ times}} \longrightarrow \Lambda_2^\star, \qquad n \geq 1,$$

that satisfy the following equations:



$$\sum (-1)^{r+st} f_{r+1+t} \circ (\mathbf{1}^r \otimes m_s \otimes \mathbf{1}^t) = \sum (-1)^s m_r \circ (f_{i_1} \otimes \cdots \otimes f_{i_r}) \qquad (n \geq 1)$$

We say that f is an A_∞-quasi-isomorphism if f_1 is a quasi-isomorphism.

Minimal models of differential graded algebras

An A_∞-algebra is minimal if $\quad m_1 = 0.$

A minimal model of a differential graded algebra $\mathbf{A}$ is an A_∞-quasi-isomorphism

$$f\colon (\mathrm{H}^\bullet(\mathbf{A}), m_2, m_3, m_4, m_5, \dots) \rightsquigarrow \mathbf{A}$$

such that f_1 induces the identity in cohomology: $\mathrm{H}^\bullet(f_1) = \mathbf{1}.$

Homotopy Transfer Theorem (Kadeishvili 1982)

Every differential graded algebra admits a minimal model.

$$\begin{tikzcd} \mathrm{H}^\bullet(\mathbf{A}) \arrow[r, shift left, "i"] & \mathbf{A} \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd}$$

$$\begin{aligned} |i| &= |p| = 0, & |h| &= -1 \\ \partial(i) &= 0 & \partial(p) &= 0 \\ p \circ i &= \mathbf{1} & \partial(h) &= \mathbf{1} - i \circ p \end{aligned}$$

Minimal models are unique up to A_∞-isomorphism.

A_∞-algebras vs differential graded algebras

A_∞-category $\equiv$ A_∞-algebra with many objects

Theorem (Lefèvre-Hasegawa 2003, …, Canonaco–Ornaghi–Stellari 2019 Pascaleff 2024)

The canonical functor $\mathrm{dgcat} \to A_\infty\text{-}\mathrm{cat}$ induces an equivalence of $(\infty, 1)$-categories after ∞-localising at the corresponding classes of quasi-equivalences.

This means that the notions of "differential graded category" and of "A_∞-category" are equivalent in a very strong sense.

- Each A_∞-algebra A has a triangulated derived category $\mathrm{D}(A)$.
- A_∞-quasi-isomorphic A_∞-algebras have equivalent derived categories:

$$A \simeq B \quad\Longrightarrow\quad \mathrm{D}(A) \simeq \mathrm{D}(B)$$

Constructing the inverse of the Correspondence

Λ: twisted $(d+2)$-periodic with respect to $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$

$$\Lambda(\sigma,d) \cong \bigoplus_{di\in d\mathbb{Z}} {}_{\sigma^i}\Lambda_1, \qquad x*y := \sigma^j(x)y, \quad |y| = dj$$

We aim to construct a minimal A_∞-algebra $A = (\Lambda(\sigma,d), m)$ such that $A \in \mathrm{D}^c(A)$ is a $d\mathbb{Z}$-cluster tilting object.

This property should determine $A = (\Lambda(\sigma,d), m)$ up to A_∞-isomorphism.

See **F. Muro's talk** for details on the existence of such an A.

Minimal A_∞-structures on Yoneda algebras of simples

Theorem (Keller 2001)

A: basic finite-dimensional algebra

$S = S_1 \oplus \cdots \oplus S_n$ direct sum of the simple A-modules

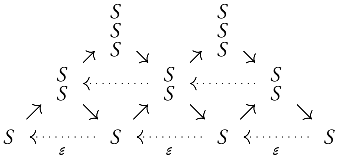
Every minimal model of $\mathbf{R}\mathrm{Hom}_A(S, S)$ is generated in deg 0 and 1 as A_∞-algebra.

See arXiv:2402.14004 (J) for a proof using AR theory of Nakayama algebras.

$$A = \mathbf{k}[x]/(x^\ell), \quad \ell \geq 3$$

$$\mathrm{Ext}_A^\bullet(S, S) \cong \mathbf{k}[\varepsilon, t]/(\varepsilon^2), \quad |\varepsilon| = 1 \text{ and } |t| = 2$$

$$m_\ell(\varepsilon, \varepsilon, \dots, \varepsilon) = \pm t \quad \text{and} \quad m_k = 0 \quad \text{for} \quad k \neq 2, \ell$$



Minimal A_∞-structures on Yoneda algebras of simples

Theorem (Keller 2001)

$A = \mathbf{k}Q/I$: finite-dimensional algebra

$S = S_1 \oplus \cdots \oplus S_n$ direct sum of the simple A-modules

$(\mathrm{Ext}^\bullet_A(S,S), 0)$ is a minimal model of $\mathbf{R}\mathrm{Hom}_A(S,S)$ $\iff$ A is Koszul

Sketch of proof of the theorem:

$(\Longrightarrow)$ Immediate from the previous theorem.

$(\Longleftarrow)$ Bigraded Homotopy Transfer Theorem.

$$\forall n \geq 0 \quad \forall i \neq n$$
$$\mathrm{Ext}^n_{\mathrm{Gr}\,A}(S, S\langle i\rangle) = 0$$

See **Jan Thomm's talk** for A_∞-structures on Yoneda algebras of rep. generators.

Question: What is the significance of the first non-vanishing higher operation?

An old example, revisited

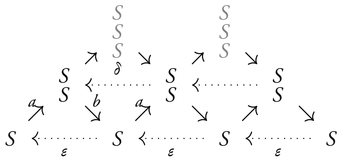
$$A = \mathbf{k}[x]/(x^3), \qquad G = S \oplus \begin{smallmatrix} S \\ S \end{smallmatrix} \in \underline{\mathrm{mod}} A, \qquad \mathrm{add}(G) = \underline{\mathrm{mod}} A$$

$$\Lambda = \underline{\mathrm{End}}_A(G) \cong \mathbf{k}\Big(S \underset{b}{\overset{a}{\rightleftarrows}} \begin{smallmatrix} S \\ S \end{smallmatrix} \Big)/(ba,\ ab) = \Pi(\mathbb{A}_2)$$

(Schofield, Erdmann–Snashall 1998, Brenner–Butler–King 2002)

The preprojective algebra $\Pi(\mathbb{A}_2)$ is twisted 3-periodic w.r.t.

$$\sigma(S) = \begin{smallmatrix} S \\ S \end{smallmatrix}, \quad \sigma\big(\begin{smallmatrix} S \\ S \end{smallmatrix}\big) = S, \qquad \sigma(a) = -b, \qquad \sigma(b) = -a.$$



$(\underline{\mathrm{End}}^{\bullet}_A(G), m)$: minimal A_∞-algebra

$$m_3(\varepsilon, \varepsilon, \varepsilon) = t_S \qquad m_3(\partial, \partial, \partial) = t_{\begin{smallmatrix} S \\ S \end{smallmatrix}}$$

$$m_3(\varepsilon, b, a) = \mathbf{1}_S \qquad m_3(\partial, a, b) = \mathbf{1}_{\begin{smallmatrix} S \\ S \end{smallmatrix}}$$

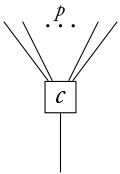
The Hochschild cochain complex

The bigraded Hochschild (cochain) complex of a graded algebra $\Lambda^\star$ has components

$$C^{p,q}(\Lambda^\star) = C^{p,q}(\Lambda^\star, \Lambda^\star) := \mathrm{Hom}_{\Bbbk}((\Lambda^\star)^{\otimes p}, \Lambda^\star[q]) \qquad p \geq 0, \quad q \in \mathbb{Z}.$$

Thus, a (p,q)-Hochschild cochain is a degree q morphism of graded vector spaces

$$c\colon \underbrace{\Lambda^\star \otimes \cdots \otimes \Lambda^\star}_{p \text{ times}} \longrightarrow \Lambda^\star.$$



The bidegree $(1,0)$ Hochschild differential is, for $c \in C^{p,\star}(\Lambda^\star)$,

$$\mathrm{d}_{\mathrm{Hoch}}c(x_1,\dots,x_p,x_{p+1}) := \pm x_1 c(x_2,\dots,x_{p+1}) + \textstyle\sum_{i=1}^{p} \pm c(\dots,x_i x_{i+1},\dots,) + \pm c(x_1,\dots,x_p)x_{p+1}$$

The Hochschild cochain complex (cont.)

For $c_1 \in C^{p,q}(\Lambda^\star)$ and $c_2 \in C^{s,t}(\Lambda^\star)$ define $c_1\{c_2\} \in C^{p+s-1,q+t}(\Lambda^\star)$ by

$$c_1\{c_2\}(x_1,\ldots,x_{p+s-1}) := \sum_{i=1}^{p} \pm c_1(\ldots,x_{i-1},c_2(x_i,\ldots,x_{i-1+s}),x_{i+s},\ldots)$$

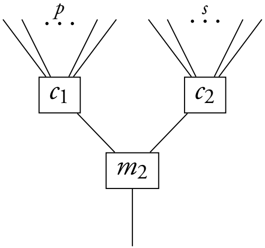
- The bidegree $(-1,0)$ Gerstenhaber bracket is

$$[c_1,c_2] := c_1\{c_2\} \pm c_2\{c_1\}.$$

- The bidegree $(0,0)$ cup product is

$$c_1 \cdot c_2 = c_1 \smile c_2 := \pm m_2\{c_1,c_2\},$$

where $m_2 \colon \Lambda^\star \otimes \Lambda^\star \to \Lambda^\star$ is the multiplication.



$m_2\{c_1,c_2\}$

Hochschild cohomology of graded algebras

The Hochschild cohomology of $\Lambda^\star$ is the cohomology of the Hochschild complex:

$$\mathrm{HH}^{\bullet,\star}(\Lambda^\star) := \mathrm{H}^{\bullet,\star}\big(\mathrm{C}^{\bullet,\star}(\Lambda^\star)\big) \cong \mathrm{Ext}^{\bullet,\star}_{\Lambda^\star\text{-bimod}}(\Lambda^\star, \Lambda^\star)$$

The Hochschild cohomology is a Gerstenhaber algebra w.r.t the total degree $\bullet + \star$:

- $\mathrm{HH}^{\bullet,\star}(\Lambda^\star)[1]$ is a graded Lie algebra with the Gerstenhaber bracket.
- $\mathrm{HH}^{\bullet,\star}(\Lambda^\star)$ is a graded commutative algebra with the cup product.
- The Gerstenhaber square $\mathrm{Sq}(c)$ induced by $c \longmapsto c\{c\}$.

$$\mathrm{Sq}(x+y) = \mathrm{Sq}(x) + \mathrm{Sq}(y) + [x,y]$$
$$\mathrm{Sq}(x\cdot y) = \mathrm{Sq}(x)\cdot y^2 + x\cdot[x,y]\cdot y + x^2\cdot \mathrm{Sq}(y)$$
$$[\mathrm{Sq}(x), y] = [x,[x,y]]$$

In $\mathrm{char}(\mathbf{k}) \neq 2$, $\quad \mathrm{Sq}(x) = \frac{1}{2}[x,x]$.

Minimal A_∞-algebras, revisited

A minimal A_∞-algebra structure on $\Lambda^\star$ consists of Hochschild cochains

$$m_n \in C^{n,2-n}(\Lambda^\star), \qquad n \geq 3,$$

such that the (formal) Hochschild cochain

$$m = (m_3, m_4, m_5, \dots) \in \prod_{n\geq 3} C^{n,\star}(\Lambda^\star)$$

satisfies the Maurer–Cartan equation

$$d_{\mathrm{Hoch}}(m) = \pm m\{m\}.$$

$d_{\mathrm{Hoch}}(m_n) = 0$ if $m_k = 0$ for $2 < k < n$

Shifted A_∞-structures are implicit here.

Lecture 3

Minimal A_∞-algebras, revisited

A minimal A_∞-algebra structure on $\Lambda^\star$ consists of Hochschild cochains

$$m_n \in \mathrm{C}^{n,2-n}\left(\Lambda^\star\right), \qquad n \geq 3,$$

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satisfies the Maurer–Cartan equation

$$\mathrm{d}_{\mathrm{Hoch}}(m) = \pm m\{m\} \overset{\mathrm{char}\,\Bbbk\neq 2}{=} \tfrac{1}{2}[m, m].$$

$\mathrm{d}_{\mathrm{Hoch}}(m_n) = 0$ if $m_k = 0$ for $2 < k < n$

Shifted A_∞-structures are implicit here.

The universal Massey product

A graded algebra is d-sparse if it is concentrated in degrees $d\mathbb{Z}$.

Definition

The universal Massey product (UMP) of a d-sparse minimal A_∞-algebra $(\Lambda^\star, m)$ is the Hochschild class

$$\overline{m_{d+2}} \in \mathrm{HH}^{d+2,-d}(\Lambda^\star)$$

of the first possibly non-trivial higher operation.

The UMP satisfies $\mathrm{Sq}(\overline{m_{d+2}}) = 0$ and is invariant under A_∞-isomorphisms.

Remark: For $d = 1$, Benson–Krause–Schwede (2004), Keller (2005, 2006), …

The restricted universal Massey product

$$j\colon \Lambda := \Lambda^0 \hookrightarrow \Lambda^\star \text{ inclusion of the degree } 0 \text{ component}$$

$$j^*\colon \mathrm{HH}^{\bullet,\star}(\Lambda^\star,\Lambda^\star) \longrightarrow \mathrm{HH}^{\bullet,\star}(\Lambda,\Lambda^\star)$$

Definition

The restricted universal Massey product (rUMP) of a d-sparse minimal A_∞-algebra $(\Lambda^\star, m)$ is the Hochschild class

$$j^*(\overline{m_{d+2}}) \in \mathrm{HH}^{d+2,-d}(\Lambda,\Lambda^\star).$$

$$\mathrm{HH}^{d+2,-d}(\Lambda,\Lambda^\star) \cong \mathrm{HH}^{d+2}(\Lambda,\Lambda^{-d}) \cong \mathrm{Ext}^{d+2}_{\Lambda\text{-bimod}}(\Lambda,\Lambda^{-d})$$

The Unit Theorem

Λ: twisted $(d+2)$-periodic w.r.t. $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$

$A = (\Lambda(\sigma, d), m)$: minimal A_∞-algebra

Theorem (J–Muro)

Suppose that $\mathbf{k}$ is perfect. The following are equivalent:

1. $A \in \mathrm{D}^{\mathrm{c}}(A)$ is a $d\mathbb{Z}$-cluster tilting object.
2. The rUMP

   $$j^*(\overline{m_{d+2}}) \in \mathrm{HH}^{d+2}(\Lambda, {}_1\Lambda_\sigma) \cong \underline{\mathrm{Hom}}_{\Lambda^e}(\Omega^{d+2}_{\Lambda^e}(\Lambda),\ {}_1\Lambda_\sigma)$$

   is invertible in $\underline{\mathrm{mod}}\Lambda^e$.
3. $j^*(\overline{m_{d+2}})$ is invertible in Hochschild–Tate cohomology $\underline{\mathrm{HH}}^{\bullet,\star}(\Lambda, \Lambda^\star)$.

$j^*(\overline{m_{d+2}}) = 0$ is an isomorphism $\implies$ Λ is semi-simple

The bijectivity of the correspondence

Λ: twisted $(d+2)$-periodic w.r.t. $\sigma\colon\Lambda\xrightarrow{\sim}\Lambda$

Theorem (J–Muro)

1. There exists a minimal A_∞-algebra structure $(\Lambda(\sigma,d),m)$ s.t. the rUMP
$$j^*(\overline{m_{d+2}})\in \mathrm{HH}^{d+2}(\Lambda,{}_1\Lambda_\sigma)\cong \underline{\mathrm{Hom}}_{\Lambda^e}(\Omega^{d+2}_{\Lambda^e}(\Lambda),\ {}_1\Lambda_\sigma)$$
is invertible in $\underline{\mathrm{mod}}\Lambda^e$.
2. Any two minimal A_∞-algebras as above are A_∞-isomorphic.

See **F. Muro's talk** next week for more details on this and the previous theorem, where the crucial role of Geiß–Keller–Oppermann $\underline{(d+2)\text{-angulated categories}}$ will be explained.

Kadeishvili's Intrinsic Formality Criterion, revisited

Theorem (Kadeishvili 1988)

Suppose that

$$\mathrm{HH}^{p+2,-p}(\Lambda^\star) = 0, \qquad p > 0.$$

Then, every minimal A_∞-structure on $\Lambda^\star$ is A_∞-isomorphic to $(\Lambda^\star, 0)$.

$$\overline{m_3} \in \mathrm{HH}^{3,-1}(\Lambda^\star) = 0 \quad \Longrightarrow \quad \exists\, f_2 \in \mathrm{C}^{2,-1}(\Lambda^\star) \quad \text{such that} \quad \pm \mathrm{d}_{\mathrm{Hoch}}(f_2) = m_3.$$

$$(\mathbf{1}, f_2, 0, \dots)\colon (\Lambda^\star, m_3, m_4, m_5, \dots) \rightsquigarrow (\Lambda^\star, 0, m_4', m_5', \dots)$$

Aim: Generalise Kadeishvili's Theorem to deal with the case

$$0 \neq \overline{m_{d+2}} \in \mathrm{HH}^{d+2,-d}(\Lambda^\star).$$

d-sparse Massey algebras

A graded algebra is d-sparse if it is concentrated in degrees $d\mathbb{Z}$.

Definition (J–Muro)

A d-sparse Massey algebra is a pair $(\Lambda^\star, \overline{c})$ consisting of:

- A d-sparse graded algebra $\Lambda^\star$.
- A Hochschild class

$$\overline{c} \in \mathrm{HH}^{d+2,-d}(\Lambda^\star)$$

such that $\mathrm{Sq}(\overline{c}) = 0$.

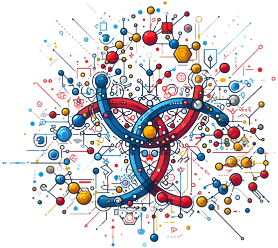

Figure by DALL·E

$(\Lambda^\star, m)$: d-sparse min. A_∞-algebra $\Longrightarrow$ $(\Lambda^\star, \overline{m_{d+2}})$: d-sparse Massey algebra

The Hochschild–Massey complex of a Massey algebra

Aim: Generalise Kadeishvili's Theorem to d-sparse Massey algebras.

The Hochschild–Massey complex of a d-sparse Massey algebra $(\Lambda^\star, \bar{c})$ is

$$\mathrm{C}^{p,q}(\Lambda^\star, \bar{c}) := \mathrm{HH}^{p,q}(\Lambda^\star) \qquad p \geq 0, \quad q \in \mathbb{Z}.$$

The bidegree $(d+1, -d)$ Hochschild–Massey differential is (almost everywhere)

$$\bar{x} \longmapsto [\bar{c}, \bar{x}].$$

The Hochschild–Massey cohomology of $(\Lambda^\star, \bar{c})$ is

$$\mathrm{HH}^{\bullet,\star}(\Lambda^\star, \bar{c}) := \mathrm{H}^{\bullet,\star}\left(\mathrm{C}^{\bullet,\star}(\Lambda^\star, \bar{c})\right).$$

A Kadeishvili-type theorem for sparse Massey algebras

$(\Lambda^\star, \overline{c})$: d-sparse Massey algebra

Theorem (J–Muro)

Suppose that

$$\mathrm{HH}^{p+2,-p}(\Lambda^\star, \overline{c}) = 0, \qquad p > d. \tag{††}$$

Then, any two minimal A_∞-algebras

$$(\Lambda^\star, m_{d+2}^{(1)}, m_{2d+2}^{(1)}, \dots) \quad \text{and} \quad (\Lambda^\star, m_{d+2}^{(2)}, m_{2d+2}^{(2)}, \dots)$$

such that $\overline{m_{d+2}}^{(1)} = \overline{c} = \overline{m_{d+2}}^{(2)}$ are (gauge) A_∞-isomorphic.

Recovering Kadeishvili's Theorem

$(\Lambda^\star, \overline{c})$: d-sparse Massey algebra

$$\mathrm{HH}^{p+2,-p}(\Lambda^\star, \overline{0}) = 0, \qquad p > d \quad \Longleftrightarrow \quad \mathrm{HH}^{p+2,-p}(\Lambda^\star) = 0, \qquad p > d$$

If this condition is satisfied, the theorem shows that a minimal A_∞-algebra $(\Lambda^\star, m)$ such that $\overline{m_{d+2}} = 0$ is formal.

Proof of Kadeishvili's Thm: Let $\Lambda^\star$ be a (1-sparse) graded algebra such that

$$\mathrm{HH}^{p+2,-p}(\Lambda^\star) = 0, \qquad p > 0.$$

- The vanishing for $p = 1$ implies $(\Lambda^\star, \overline{0})$ is the unique Massey algebra structure.
- The vanishing for $p > 1$ implies the Kadeishvili-type theorem applies.

On the proof of the Kadeishvili-type Theorem

$(\Lambda^\star, m_3, m_4, m_5, \ldots)$: minimal A_∞-algebra

The equations of an A_∞-morphism imply that an arbitrary collection

$$f_1 = \mathbf{1}, \quad f_2 \in \mathrm{C}^{2,-1}\left(\Lambda^\star\right), \quad f_3 \in \mathrm{C}^{3,-2}\left(\Lambda^\star\right), \quad \ldots$$

determines a unique minimal A_∞-algebra structure

$$(\Lambda^\star, m_3', m_4', m_5', \ldots)$$

such that

$$f = (\mathbf{1}, f_2, f_3, \ldots)\colon (\Lambda^\star, m) \rightsquigarrow (\Lambda^\star, m')$$

is an A_∞-isomorphism.

For example, $\quad m_3' = m_3 \pm \mathrm{d}_{\mathrm{Hoch}}(f_2)$

On the proof of the Kadeishvili-type Theorem (cont.)

The gauge A_∞-isomorphisms group

$$\mathfrak{G}(\Lambda^\star) := \{f \in \textstyle\prod_{n=1}^{\infty} C^{n,1-n}(\Lambda^\star) \mid f_1 = \mathbf{1}\}$$

acts on the set of minimal A_∞-structures on $\Lambda^\star$.

Tautologically, two minimal A_∞-structures are gauge A_∞-isomorphic if and only if they have the same $\mathfrak{G}(\Lambda^\star)$-orbit.

Question: How can we leverage this observation?

The set of minimal A_∞-algebra structures on $\Lambda^\star$ are the vertices of a CW complex $\mathfrak{A}_\infty(\Lambda^\star)$ whose 1-cells are the gauge A_∞-isomorphisms!

The $\mathfrak{G}(\Lambda^\star)$-orbits are the path-connected components $\pi_0(\mathfrak{A}_\infty(\Lambda^\star))$.

With a little help from my friends

The CW complex $\mathfrak{A}_\infty(\Lambda^\star)$ is the homotopy limit of a tower of fibrations

$$\mathfrak{A}_\infty(\Lambda^\star) \simeq \operatorname{holim} \mathfrak{A}_n(\Lambda^\star) \longrightarrow \cdots \longrightarrow \mathfrak{A}_n(\Lambda^\star) \longrightarrow \cdots \longrightarrow \mathfrak{A}_4(\Lambda^\star) \longrightarrow \mathfrak{A}_3(\Lambda^\star)$$

where $\mathfrak{A}_n(\Lambda^\star)$ is the CW complex of minimal A_n-algebra structures on $\Lambda^\star$:

- A minimal A_3-algebra structure consists of a Hochschild cochain $m_3 \in C^{3,-1}(\Lambda^\star)$.
- A minimal A_4-algebra structure consists of a Hochschild cocycle $m_3 \in C^{3,-1}(\Lambda^\star)$ and a Hochschild cochain $m_4 \in C^{4,-2}(\Lambda^\star)$.
- …

We can leverage techniques from **Algebraic Topology / Homotopy Theory** such as the Milnor exact sequence

$$* \longrightarrow \varprojlim{}^1 \pi_1(\mathfrak{A}_n(\Lambda^\star)) \longrightarrow \pi_0(\mathfrak{A}_\infty(\Lambda^\star)) \longrightarrow \varprojlim \pi_0(\mathfrak{A}_n(\Lambda^\star)) \longrightarrow *$$

There is a spectral sequence …

The existence of Milnor exact sequences

$$* \longrightarrow \varprojlim{}^1 \pi_{k+1}(\mathfrak{A}_n(\Lambda^\star)) \longrightarrow \pi_k(\mathfrak{A}_\infty(\Lambda^\star)) \longrightarrow \varprojlim \pi_k(\mathfrak{A}_n(\Lambda^\star)) \longrightarrow *$$

can be leveraged thanks to the (fringed) Bousfield–Kan spectral sequence (1972) of the tower

$$\mathfrak{A}_\infty(\Lambda^\star) \simeq \operatorname{holim} \mathfrak{A}_n(\Lambda^\star) \longrightarrow \cdots \longrightarrow \mathfrak{A}_n(\Lambda^\star) \longrightarrow \cdots \longrightarrow \mathfrak{A}_4(\Lambda^\star) \longrightarrow \mathfrak{A}_3(\Lambda^\star)$$

Idea of proof of the Kadeishvili-type theorem:

- Two d-sparse minimal A_∞-algebra structures $(\Lambda^\star, m^{(1)})$ and $(\Lambda^\star, m^{(2)})$ such that

  $$\overline{m_{d+2}}^{(1)} = \overline{m_{d+2}}^{(2)}$$

  lie in the pointed kernel of the map $\pi_0(\mathfrak{A}_\infty(\Lambda^\star)) \longrightarrow \varprojlim \pi_0(\mathfrak{A}_\infty(\Lambda^\star))$.
- Condition (††) yields the vanishing of $\varprojlim{}^1 \pi_1(\mathfrak{A}_n(\Lambda^\star))$ — this uses Muro's extended Bousfield–Kan spectral sequece (2020).

Muro's extended Bousfield–Kan spectral sequence

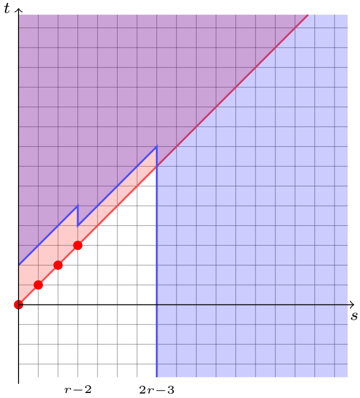

Figure by Fernando Muro

$$\mathfrak{A}_\infty(\Lambda^\star) \simeq \operatorname{holim} \mathfrak{A}_n(\Lambda^\star)$$

- Pointed sets along the line $t - s = 0$
- Groups along the line $t - s = 1$
- Abelian groups elsewhere in the red region
- **Vector spaces** in the extended blue region

$$E_{d+2}^{p,p} = \mathrm{HH}^{p+2,-p}(\Lambda^\star, \bar{c}) \qquad p > d$$

$$\pi_0(\mathfrak{A}_\infty(\Lambda^\star)) \cong \varprojlim \pi_0(\mathfrak{A}_n(\Lambda^\star))$$

Concluding remarks and an invitation

Working with minimal A_∞-algebras instead of differential graded algebras provides access to new invariants and thus we may formulate new properties:

"The rUMP of the d-sparse minimal A_∞-algebra $(\Lambda(\sigma, d), m)$ is invertible."

I invite the audience to consider the following questions:

Let $\mathbf{A}$ be a differential graded algebra such that $\mathbf{A} \in \mathrm{D}^{\mathrm{c}}(\mathbf{A})$ is a generator of a preferred type (P), for example a d-cluster tilting object.

Question 1: Can we detect property (P) in terms of the minimal models of $\mathbf{A}$?

Question 2: Is there a derived correspondence for generators of type (P)?

Question 3: Are there properties of a minimal A_∞-algebra A that imply an interesting novel property of $A \in \mathrm{D}^{\mathrm{c}}(A)$?

The Kontsevich–Soibelman perspective

A minimal A_∞-algebra structure on a graded algebra $\Lambda^\star$

$$m \in \prod_{n\geq 3} \mathrm{C}^{n,2-n}(\Lambda^\star)$$

has total degree 1 in the differential graded Lie algebra $\mathrm{C}^{\bullet,\star}(\Lambda^\star)[1]$ and is a solution to the Maurer–Cartan equation

$$\mathrm{d}_{\mathrm{Hoch}}(m) = \pm m\{m\} \overset{\mathrm{char}\,\mathbf{k}\neq 2}{=} \pm\tfrac{1}{2}[m,m].$$

"An A_∞-algebra is the same as a non-commutative formal graded manifold X over, say, field $\mathbf{k}$, having a marked $\mathbf{k}$-point pt equipped with [a degree 1 homological vector field]. … It is an interesting problem to make a dictionary from the pure algebraic language of A_∞-algebras and A_∞-categories to the language of non-commutative geometry."

Kontsevich–Soibelman (2009)

Perhaps certain qualitative properties of such vector fields allow to extend the dictionary to include some aspects of the representation theory of FD algebras!

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