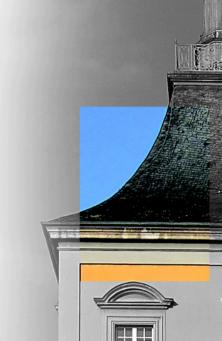


Deriving a theorem of Ladkani

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A theorem of Ladkani

R, S, E: DG rings (or, more generally, ring spectra)

$$_{S}M_{R}\in ext{D}\left(S^{ ext{op}}\otimes ^{L}R
ight)$$
 such that $M_{R}\in ext{perf}\left(R
ight)$

$$_{E}T_{R}\in ext{D}\left(E^{ ext{op}}\otimes^{L}R
ight)$$
 such that $-\otimes_{E}^{L}T\colon ext{D}\left(E
ight)\longrightarrow ext{D}\left(R
ight)$ is an equivalence

Theorem (Ladkani 2011 for rings)

There is a derived equivalence

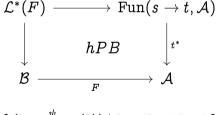
$$\operatorname{D}egin{pmatrix} S & M \ 0 & R \end{pmatrix} \simeq \operatorname{D}egin{pmatrix} E & \operatorname{RHom}_R(M,T) \ 0 & S \end{pmatrix}$$

Ladkani 2011: R, S, E rings, $M \in \operatorname{Mod}(S^{\operatorname{op}} \otimes^L R)$ and $\operatorname{RHom}_R(M, T) \in \operatorname{Mod}(E^{\operatorname{op}} \otimes S)$

Gluing along an exact functor

 $F \colon \mathcal{B} \longrightarrow \mathcal{A}$ exact functor between stable ∞ -categories

$$egin{aligned} \mathcal{L}_*(F) & \longrightarrow & \operatorname{Fun}(s
ightarrow t, \mathcal{A}) \ & & \downarrow & & \downarrow s^* \ \mathcal{B} & \longrightarrow & \mathcal{A} \end{aligned} \ \left\{ egin{aligned} (b,F(b) & \stackrel{arphi}{
ightarrow} a) \mid b \in \mathcal{B}, \ arphi \ ext{in } \mathcal{A} \end{array}
ight\} \end{aligned}$$



$$\{\,(b,a \stackrel{\psi}{
ightarrow} F(b))\,|\ b \in \mathcal{B},\ \psi\ \mathsf{in}\ \mathcal{A}\,\}$$

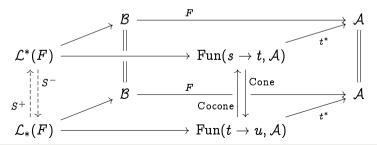
The universal BGP reflection functors

Lemma (Folklore, proof below from Dyckerhoff-J-Walde 2019)

There are canonical, mutually inverse equivalences

$$S^-\colon \mathcal{L}^*(F) \overset{\simeq}{\longleftrightarrow} \mathcal{L}_*(F)\colon S^+$$

$$S^-(b,a \xrightarrow{\psi} F(b)) = (b,F(b) o ext{cone}\,(\psi)) \quad S^+(b,F(b) \xrightarrow{arphi} a) = (b, ext{cocone}\,(arphi) o F(b))$$



Gluing along an adjunction

 $F:\mathcal{B}\rightleftarrows\mathcal{A}:G$ adjunction $F\dashv G$ between stable ∞ -categories

Lemma (Folklore)

There are canonical equivalences

$$\mathcal{L}_*(F) \overset{\simeq}{\longleftrightarrow} \mathcal{L}^*(G)$$

$$(b,F(b)\stackrel{arphi}{
ightarrow}a)\longmapsto (a,b\stackrel{\overline{arphi}}{
ightarrow}G(a)) \qquad (a,b\stackrel{\psi}{
ightarrow}G(a))\longmapsto (b,F(b)\stackrel{\overline{\psi}}{
ightarrow}a)$$

Remark. Identifying the corresp. hPB's with the ∞ -category of sections of a bicartesian fibration classifying $F \dashv G$, we even have $\mathcal{L}_*(F) = \mathcal{L}^*(G)$

Proof of the theorem (J. 2019)

$$egin{aligned} _SM_R \in & \mathbb{D}\left(S^{ ext{op}} \otimes^L R
ight) ext{s. t. } M_R \in \operatorname{perf}\left(R
ight) \ _ET_R \in & \mathbb{D}\left(E^{ ext{op}} \otimes^L R
ight) ext{ such that} \ & - \otimes^L_E T \colon & \mathbb{D}\left(E
ight) \stackrel{\simeq}{\longrightarrow} & \mathbb{D}\left(R
ight) ext{ is equiv.} \ & \mathbb{D}\left(E
ight) \stackrel{\operatorname{RHom}_R\left(M, - \otimes^L_E T
ight)}{\longrightarrow} & \mathbb{D}\left(S
ight) \ & - \otimes^L_E T \downarrow & & \parallel \ & \mathbb{D}\left(R
ight) \stackrel{\operatorname{RHom}_R\left(M, -
ight)}{\longrightarrow} & \mathbb{D}\left(S
ight) \end{aligned}$$

$$\operatorname{\mathsf{RHom}}_R(M,-\otimes^L_ET)\simeq -\otimes^L_E\operatorname{\mathsf{RHom}}_R(M,T)$$
 since $M_R\in\operatorname{\mathsf{perf}}(R)$ (Eilenberg-Watts)

$$\operatorname{D} \left(egin{matrix} S & M \ 0 & R \end{matrix}
ight) \overset{(1)}{\simeq} \mathcal{L}_* (-\otimes^L_S M)$$

adjunction $\simeq \mathcal{L}^*(\mathrm{RHom}_R(M,-))$

BGP reflection $\simeq \mathcal{L}_*(\mathrm{RHom}_R(M,-))$

functoriality $\simeq \mathcal{L}_*(-\otimes^L_E \operatorname{RHom}_R(M,T))$

$$\overset{(2)}{\simeq} \operatorname{D} \left(egin{matrix} E & \operatorname{RHom}_R(M,T) \ 0 & S \end{matrix}
ight)$$

Q.E.D.

(1), (2):

Recognition Theorem

- Keller 1994 (DG rings)
- Schwede-Shipley 2003 (ring spectra)

Simple computation using assoc. recollement

Extra: Recollement

 $egin{aligned} i_L \dashv i \dashv i_R & p_L \dashv p \dashv p_R \ & i, \, p_L, \, p_R ext{ are fully faithful} \end{aligned}$

$$egin{aligned} i_L(b,F(b)&\stackrel{arphi}{ o}a)&\simeq\operatorname{cone}\left(arphi
ight) &p_L(b)=(b,F(b)&\stackrel{1}{ o}F(b)) \ i(a)&=(0,F(0) o a) &p(b,F(b)&\stackrel{arphi}{ o}a)=b \ i_R(b,F(b)&\stackrel{arphi}{ o}a)&=a &p_R(b)=(b,F(b) o 0) \ (i_R\circ p_L)(b)&=F(b) & ext{and} &(i_L\circ p_R)(b)&\simeq F(b)[1] \end{aligned}$$

F has right adjoint $\iff i_R$ has right adjoint G. Jasso Deriving a theorem of Ladkani 6/6



Thank you for your attention!

Further reading:

Dyckerhoff, J., Walde – Generalised BGP reflection functors via the Grothendieck construction.

10.1093/imrn/rnz194

Ladkani – Derived equivalences of triangular matrix rings arising from extensions of tilting modules.

10.1007/s10468-009-9175-0