The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

Def (Reid 1983) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

is a *compound Du Val singularity* if

$$f = g(x,y,z) + t\,h(x,y,z,t)$$

arbitrary

$g=0$ Kleinian / Du Val singularity

The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

Def (Reid 1983) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

is a compound Du Val singularity if

$f = g(x,y,z) + t\,h(x,y,z,t)$

arbitrary

$g = 0$ Kleinian / Du Val singularity

e.g. $R = \mathbb{C}[[u,v,x,y]]/(uv - xy)$

Simplest example

The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

Def (Reid 1983) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

is a compound Du Val singularity if

$$f = g(x,y,z) + t\,h(x,y,z,t)$$

arbitrary

$g = 0$ Kleinian / Du Val singularity

e.g. $R = \mathbb{C}[\![u,v,x,y]\!]/(uv - xy)$

simplest example

Standing assumptions

- $\operatorname{Spec} R$ has an isolated singularity
- $\exists\, p : X \longrightarrow \operatorname{Spec} R$ crepant resolution

$p^* \omega_R \cong \omega_X$

The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

Def (Reid 1983) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

is a compound Du Val singularity if

$$f = g(x,y,z) + t\,h(x,y,z,t)$$

arbitrary

$g = 0$ Kleinian / Du Val singularity

e.g. $R = \mathbb{C}[[u,v,x,y]]/(uv - xy)$

simplest example

Standing assumptions

- Spec R has an isolated singularity
- $\exists\ p : X \longrightarrow \operatorname{Spec} R$ crepant resolution

  $p^*\omega_R \cong \omega_X$

Rmk In general, Spec R need not admit a crepant resolution and, if it does, it is not unique

The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

Def (Reid 1983) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

is a <u>compound Du Val singularity</u> if

$$f = g(x,y,z) + t\,h(x,y,z,t)$$

h: arbitrary

$g = 0$ Kleinian / Du Val singularity

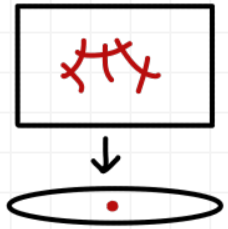
e.g. $R = \mathbb{C}[\![u,v,x,y]\!]/(uv - xy)$

simplest example

Standing assumptions

- $\operatorname{Spec} R$ has an isolated singularity
- $\exists\ p : X \longrightarrow \operatorname{Spec} R$ crepant resolution
  $p^*\omega_R \cong \omega_X$

Rmk In general, $\operatorname{Spec} R$ need not admit a crepant resolution and, if it does, it is not unique



$$X \xrightarrow{\ p\ } \operatorname{Spec} R$$

$$p^{-1}(0) = \bigcup_{i=1}^{n} C_i \ : \ \underline{\text{exc. fibre}}$$

$$C_i \cong \mathbb{P}^1$$

The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

Def (Reid 1983) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

is a compound Du Val singularity if

$$f = g(x,y,z) + t\,h(x,y,z,t)$$

arbitrary

$g=0$ Kleinian / Du Val singularity

e.g. $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

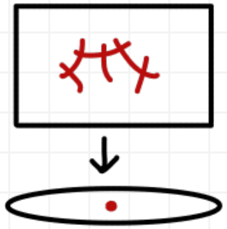
simplest example

Standing assumptions

- $\operatorname{Spec} R$ has an isolated singularity
- $\exists\, p : X \longrightarrow \operatorname{Spec} R$ crepant resolution

  $p^*\omega_R \cong \omega_X$

Rmk In general, $\operatorname{Spec} R$ need not admit a crepant resolution and, if it does, it is not unique



$$\begin{array}{c} X \\ \downarrow p \\ \operatorname{Spec} R \end{array} \qquad p^{-1}(0) = \bigcup_{i=1}^{n} C_i \ : \ \text{exc. fibre}$$

$C_i \cong \mathbb{P}^1$

Thm (Donovan-Wemyss 2016)

$\exists\, \Lambda(p)$: basic fin. dim. algebra that "controls" the NC deformations of $\mathcal{O}_{C_i}(-1) \in \operatorname{coh} X$

$\Lambda(p)$ is the contraction algebra of p.

The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

Def (Reid 1983) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

is a compound Du Val singularity if

$$f = g(x,y,z) + t\, h(x,y,z,t)$$

arbitrary

$g=0$ Kleinian / Du Val singularity

e.g. $R = \mathbb{C}[[u,v,x,y]]/(uv-xy)$

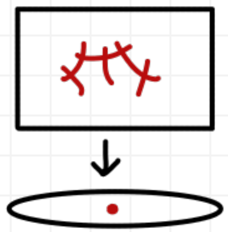
simplest example

Standing assumptions

- $\operatorname{Spec} R$ has an isolated singularity
- $\exists\, p : X \longrightarrow \operatorname{Spec} R$ crepant resolution

  $p^*\omega_R \cong \omega_X$

Rmk In general, $\operatorname{Spec} R$ need not admit a crepant resolution and, if it does, it is not unique



$$\begin{array}{c} X \\ \downarrow p \\ \operatorname{Spec} R \end{array}$$

$p^{-1}(0) = \bigcup_{i=1}^{n} C_i$: exc. fibre

$C_i \cong \mathbb{P}^1$

Thm (Donovan-Wemyss 2016)

$\exists\, \Lambda(p)$: basic fin. dim. algebra that "controls" the NC deformations of $\mathcal{O}_{C_i}(-1) \in \operatorname{coh} X$

$\Lambda(p)$ is the contraction algebra of p.

e.g $R = \mathbb{C}[[u,v,x,y]]/(uv-xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

$\Lambda(p) \cong \Lambda(q) \cong \mathbb{C}$

misleadingly simple...

The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

Def (Reid 1983) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

is a <u>compound Du Val singularity</u> if

$$f = g(x,y,z) + t\,h(x,y,z,t)$$

arbitrary

$g=0$ Kleinian / Du Val singularity

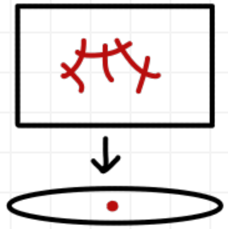
e.g. $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

simplest example

Standing assumptions

- Spec R has an isolated singularity
- $\exists\ p : X \longrightarrow \operatorname{Spec} R$ crepant resolution
  $p^*\omega_R \cong \omega_X$

Rmk In general, Spec R need not admit a crepant resolution and, if it does, it is not unique



$$\begin{array}{c} X \\ \downarrow p \\ \operatorname{Spec} R \end{array} \qquad p^{-1}(0) = \bigcup_{i=1}^{n} C_i : \text{exc. fibre}$$

$C_i \cong \mathbb{P}^1$

Thm (Donovan-Wemyss 2016)

$\exists\ \Lambda(p)$: basic fin. dim. algebra that "controls" the <u>NC deformations</u> of $\mathcal{O}_{C_i}(-1) \in \operatorname{coh} X$

$\Lambda(p)$ is the <u>contraction algebra</u> of p.

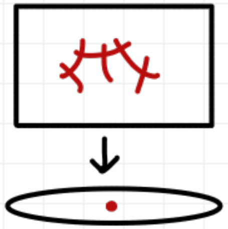
e.g. $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \operatorname{Spec} R & \end{array} \qquad \Lambda(p) \cong \Lambda(q) \cong \mathbb{C}$$

misleadingly simple...

Rmk $\Lambda(p)$ recovers all known numerical invariants associated with $p : X \to \operatorname{Spec} R$

$\boxed{\begin{array}{l}\mathrm{Spec}\, R : \text{cDV w/ isolated singularity} \\ \exists\, p : X \longrightarrow \mathrm{Spec}\, R \ \text{ crepant resolution}\end{array}}$



$$\begin{array}{c} X \\ \downarrow p \\ \mathrm{Spec}\, R \end{array} \qquad p^{-1}(0) = \bigcup_{i=1}^{n} C_i : \underline{\text{exc. fibre}}$$

$$C_i \cong \mathbb{P}^1$$

<u>Thm</u> (Donovan-Wemyss 2016)

$\exists\, \Lambda(p)$: basic fin. dim. algebra that "controls" the <u>NC deformations</u> of $\mathcal{O}_{C_i}(-1) \in \mathrm{coh}\, X$

$\Lambda(p)$ is the <u>contraction algebra</u> of p.

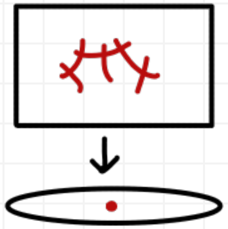
<u>e.g</u> $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

$$X \xrightarrow{p} \mathrm{Spec}\, R \xleftarrow{q} Y \qquad \Lambda(p) \cong \Lambda(q) \cong \mathbb{C}$$

misleadingly simple...

<u>Rmk</u> $\Lambda(p)$ recovers all known numerical invariants associated with $p : X \to \mathrm{Spec}\, R$

$\boxed{\begin{array}{l}\operatorname{Spec} R : \text{cDV w/ isolated singularity} \\ \exists\, p : X \longrightarrow \operatorname{Spec} R \quad \text{crepant resolution}\end{array}}$



$$\begin{array}{c} X \\ \downarrow p \\ \operatorname{Spec} R \end{array} \qquad p^{-1}(0) = \bigcup_{i=1}^{n} C_i \ : \ \underline{\text{exc. fibre}}$$

$$C_i \cong \mathbb{P}^1$$

<u>Thm</u> (Donovan-Wemyss 2016)

$\exists\, \Lambda(p)$: basic fin. dim. algebra that "controls" the <u>NC deformations</u> of $\mathcal{O}_{C_i}(-1) \in \operatorname{coh} X$

$\Lambda(p)$ is the <u>contraction algebra</u> of p.

<u>e.g</u> $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \operatorname{Spec} R & \end{array} \qquad \Lambda(p) \cong \Lambda(q) \cong \mathbb{C}$$

misleadingly simple ...

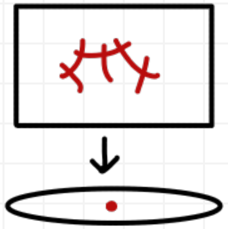
<u>Rmk</u> $\Lambda(p)$ recovers all known numerical invariants associated with $p : X \to \operatorname{Spec} R$

<u>The Donovan-Wemyss Conjecture</u> (2016, 2020)

$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing $(i=1,2)$

$$R_1 \cong R_2 \iff \mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

$\boxed{\begin{array}{l}\operatorname{Spec} R : \text{cDV w/ isolated singularity} \\ \exists\ p : X \to \operatorname{Spec} R \quad \text{crepant resolution}\end{array}}$



$$\begin{array}{c} X \\ \downarrow p \\ \operatorname{Spec} R \end{array} \qquad p^{-1}(0) = \bigcup_{i=1}^{n} C_i \ : \ \underline{\text{exc. fibre}}$$

$$C_i \cong \mathbb{P}^1$$

Thm (Donovan–Wemyss 2016)

$\exists\, \Lambda(p)$: basic fin. dim. algebra that "controls" the NC deformations of $\mathcal{O}_{C_i}(-1) \in \operatorname{coh} X$

$\Lambda(p)$ is the contraction algebra of p.

e.g $R = \mathbb{C}[[u, v, x, y]]/(uv - xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd} \qquad \Lambda(p) \cong \Lambda(q) \cong \mathbb{C}$$

misleadingly simple ...

Rmk $\Lambda(p)$ recovers all known numerical invariants associated with $p : X \to \operatorname{Spec} R$

The Donovan–Wemyss Conjecture (2016, 2020)

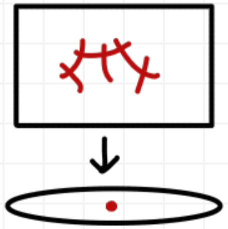
$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing ($i = 1, 2$)

$$R_1 \cong R_2 \iff \mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

($\Rightarrow$) Wemyss (2018) + Dugas (2015)

($\Leftarrow$) Reid (1983) in type A

$\boxed{\begin{array}{l}\operatorname{Spec} R : cDV \text{ w/ isolated singularity} \\ \exists\, p : X \to \operatorname{Spec} R \quad \text{crepant resolution}\end{array}}$



$$\begin{array}{c} X \\ \downarrow p \\ \operatorname{Spec} R \end{array} \qquad p^{-1}(0) = \bigcup_{i=1}^{n} C_i : \underline{\text{exc. fibre}}$$

$$C_i \cong \mathbb{P}^1$$

Thm (Donovan–Wemyss 2016)

$\exists\, \Lambda(p)$: basic fin.dim. algebra that "controls" the <u>NC deformations</u> of $\mathcal{O}_{C_i}(-1) \in \operatorname{coh} X$

$\Lambda(p)$ is the <u>contraction algebra</u> of p.

<u>e.g</u> $R = \mathbb{C}[\![u,v,x,y]\!]/(uv - xy)$

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \operatorname{Spec} R & \end{array} \qquad \Lambda(p) \cong \Lambda(q) \cong \mathbb{C}$$

misleadingly simple ...

<u>Rmk</u> $\Lambda(p)$ recovers all known numerical invariants associated with $p : X \to \operatorname{Spec} R$

<u>The Donovan–Wemyss Conjecture</u> (2016, 2020)

$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing $(i = 1, 2)$

$$R_1 \cong R_2 \iff \mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

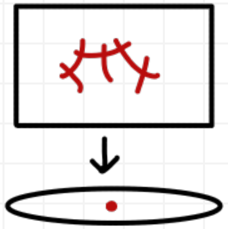
$(\Rightarrow)$ Wemyss (2018) + Dugas (2015)

$(\Leftarrow)$ Reid (1983) in type A

<u>Recall</u> $\mathcal{D}_{sg}(R) := \mathcal{D}^b(\operatorname{mod} R) / K^b(\operatorname{proj} R)$

is the <u>singularity cat</u> of R

$$\boxed{\begin{array}{l}\operatorname{Spec} R : cDV \text{ w/ isolated singularity} \\ \exists\ p : X \longrightarrow \operatorname{Spec} R \quad \text{crepant resolution}\end{array}}$$



$$\begin{array}{c} X \\ \downarrow p \\ \operatorname{Spec} R \end{array} \qquad p^{-1}(0) = \bigcup_{i=1}^{n} C_i : \underline{\text{exc. fibre}}$$

$$C_i \cong \mathbb{P}^1$$

Thm (Donovan–Wemyss 2016)

$\exists\, \Lambda(p)$: basic fin. dim. algebra that "controls" the NC deformations of $\mathcal{O}_{C_i}(-1) \in \operatorname{coh} X$

$\Lambda(p)$ is the contraction algebra of p.

e.g $R = \mathbb{C}[[u, v, x, y]]/(uv - xy)$

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \operatorname{Spec} R & \end{array} \qquad \Lambda(p) \cong \Lambda(q) \cong \mathbb{C}$$

misleadingly simple ...

Rmk $\Lambda(p)$ recovers all known numerical invariants associated with $p : X \to \operatorname{Spec} R$

The Donovan–Wemyss Conjecture (2016, 2020)

$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing $(i = 1, 2)$

$R_1 \cong R_2 \iff \mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$

$(\Rightarrow)$ Wemyss (2018) + Dugas (2015)

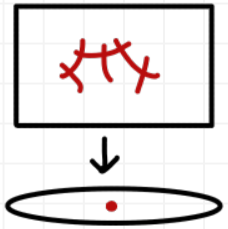
$(\Leftarrow)$ Reid (1983) in type A

Recall $\mathcal{D}_{sg}(R) := \mathcal{D}^b(\operatorname{mod} R) / K^b(\operatorname{proj} R)$

is the singularity cat of R

- Hom-finite & Krull-Schmidt
- 2-Calabi-Yau $\operatorname{Hom}(Y, X[2]) \cong D\operatorname{Hom}(X, Y)$
- $[2] \cong \mathbb{1}$
- $\forall X \in \mathcal{D}_{sg}(R)$ $\operatorname{End}(X)$ is symmetric since $\operatorname{Hom}(X, X) \cong \operatorname{Hom}(X, X[2]) \cong D\operatorname{Hom}(X, X)$

$\boxed{\begin{array}{l} \mathrm{Spec}\, R : \text{cDV w/ isolated singularity} \\ \exists\ p : X \longrightarrow \mathrm{Spec}\, R \ \text{ crepant resolution} \end{array}}$



$$\begin{array}{c} X \\ \downarrow p \\ \mathrm{Spec}\, R \end{array} \qquad p^{-1}(0) = \bigcup_{i=1}^{n} C_i : \underline{\text{exc. fibre}} \qquad C_i \cong \mathbb{P}^1$$

Thm (Donovan–Wemyss 2016)

$\exists\ \Lambda(p)$: basic fin. dim. algebra that "controls" the <u>NC deformations</u> of $\mathcal{O}_{C_i}(-1) \in \mathrm{coh}\, X$

$\Lambda(p)$ is the <u>contraction algebra</u> of p.

e.g $R = \mathbb{C}[[u, v, x, y]]/(uv - xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \mathrm{Spec}\, R & \end{tikzcd} \qquad \Lambda(p) \cong \Lambda(q) \cong \mathbb{C}$$

misleadingly simple ...

Rmk $\Lambda(p)$ recovers all known numerical invariants associated with $p : X \to \mathrm{Spec}\, R$

The Donovan–Wemyss Conjecture (2016, 2020)

$p_i : X_i \to \mathrm{Spec}\, R_i$ crepant res. of cDV's sing $(i = 1, 2)$

$$R_1 \cong R_2 \iff \mathcal{D}^b(\mathrm{mod}\, \Lambda(p_1)) \simeq \mathcal{D}^b(\mathrm{mod}\, \Lambda(p_2))$$

$(\Rightarrow)$ Wemyss (2018) + Dugas (2015)

$(\Leftarrow)$ Reid (1983) in type A

Recall $\mathcal{D}_{sg}(R) := \mathcal{D}^b(\mathrm{mod}\, R) / K^b(\mathrm{proj}\, R)$

is the <u>singularity</u> <u>cat</u> of R

- Hom-finite & Krull-Schmidt
- 2-Calabi-Yau $\mathrm{Hom}(Y, X[2]) \cong D\,\mathrm{Hom}(X, Y)$
- $[2] \cong \mathbb{1}$
- $\forall X \in \mathcal{D}_{sg}(R)$ $\mathrm{End}(X)$ is symmetric since $\mathrm{Hom}(X, X) \cong \mathrm{Hom}(X, X[2]) \cong D\,\mathrm{Hom}(X, X)$

Def (Iyama–Yoshino 2008) $T \in \mathcal{D}_{sg}(R)$: basic

$T \in \mathcal{D}_{sg}(R)$ is $2\mathbb{Z}$-cluster tilting object if

$\mathrm{add}\, T = \{ X \in \mathcal{D}_{sg}(R) \mid \mathrm{Hom}(T, X[1]) = 0 \}$

& $T \cong T[2]$

The Donovan-Wemyss Conjecture (2016, 2020)

$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing $(i=1,2)$

$R_1 \cong R_2 \iff \mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$

$(\Rightarrow)$ Wemyss (2018) + Dugas (2015)

$(\Leftarrow)$ Reid (1983) in type $\mathbb{A}$

Recall $\mathcal{D}_{sg}(R) := \mathcal{D}^b(\operatorname{mod} R) / K^b(\operatorname{proj} R)$

is the singularity cat of R

- Hom-finite & Krull-Schmidt
- 2-Calabi-Yau $\operatorname{Hom}(Y, X[2]) \cong D\operatorname{Hom}(X, Y)$
- $[2] \cong \mathbb{1}$
- $\forall X \in \mathcal{D}_{sg}(R)$ $\operatorname{End}(X)$ is symmetric since
  $\operatorname{Hom}(X,X) \cong \operatorname{Hom}(X, X[2]) \cong D\operatorname{Hom}(X,X)$

Def (Iyama-Yoshino 2008) $T \in \mathcal{D}_{sg}(R)$: basic

$T \in \mathcal{D}_{sg}(R)$ is $2\mathbb{Z}$-cluster tilting object if

$\operatorname{add} T = \{ X \in \mathcal{D}_{sg}(R) \mid \operatorname{Hom}(T, X[1]) = 0 \}$

& $T \cong T[2]$

The Donovan-Wemyss Conjecture (2016, 2020)

$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing $(i=1,2)$

$R_1 \cong R_2 \iff \mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$

$(\Rightarrow)$ Wemyss (2018) + Dugas (2015)

$(\Leftarrow)$ Reid (1983) in type $\mathbb{A}$

Recall $\mathcal{D}_{sg}(R) := \mathcal{D}^b(\operatorname{mod} R) / K^b(\operatorname{proj} R)$

is the singularity cat of R

- Hom-finite & Krull-Schmidt
- 2-Calabi-Yau $\operatorname{Hom}(Y, X[2]) \cong D\operatorname{Hom}(X, Y)$
- $[2] \cong \mathbb{1}$
- $\forall X \in \mathcal{D}_{sg}(R)$ $\operatorname{End}(X)$ is symmetric since $\operatorname{Hom}(X,X) \cong \operatorname{Hom}(X, X[2]) \cong D\operatorname{Hom}(X,X)$

Def (Iyama-Yoshino 2008) $T \in \mathcal{D}_{sg}(R)$: basic

$T \in \mathcal{D}_{sg}(R)$ is $2\mathbb{Z}$-cluster tilting object if

$\operatorname{add} T = \{ X \in \mathcal{D}_{sg}(R) \mid \operatorname{Hom}(T, X[1]) = 0 \}$

& $T \cong T[2]$

Prop (Iyama-Yoshino 2008)

$T \in \mathcal{D}_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$\Longrightarrow \forall X \in \mathcal{D}_{sg}(R)\ \exists\ \underbrace{T^{-1} \to T^0}_{\operatorname{add} T} \to X \to T[1]$

$(\Rightarrow \operatorname{thick}(T) = \mathcal{D}_{sg}(R))$

The Donovan-Wemyss Conjecture (2016, 2020)

$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing ($i=1,2$)

$R_1 \cong R_2 \iff \mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$

$(\Rightarrow)$ Wemyss (2018) + Dugas (2015)

$(\Leftarrow)$ Reid (1983) in type $\mathbb{A}$

Recall $\mathcal{D}_{sg}(R) := \mathcal{D}^b(\operatorname{mod} R) / K^b(\operatorname{proj} R)$

is the singularity cat of R

- Hom-finite & Krull-Schmidt
- 2-Calabi-Yau $\operatorname{Hom}(Y, X[2]) \cong D\operatorname{Hom}(X, Y)$
- $[2] \cong \mathbb{1}$
- $\forall X \in \mathcal{D}_{sg}(R)$ $\operatorname{End}(X)$ is symmetric since $\operatorname{Hom}(X,X) \cong \operatorname{Hom}(X, X[2]) \cong D\operatorname{Hom}(X,X)$

Def (Iyama-Yoshino 2008) $T \in \mathcal{D}_{sg}(R)$: basic

$T \in \mathcal{D}_{sg}(R)$ is $2\mathbb{Z}$-cluster tilting object if

$\operatorname{add} T = \{ X \in \mathcal{D}_{sg}(R) \mid \operatorname{Hom}(T, X[1]) = 0 \}$

& $T \cong T[2]$

Prop (Iyama-Yoshino 2008)

$T \in \mathcal{D}_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$\implies \forall X \in \mathcal{D}_{sg}(R)$ $\exists$ $\underbrace{T^{-1} \to T^0}_{\operatorname{add} T} \to X \to T[1]$

$(\Rightarrow \operatorname{thick}(T) = \mathcal{D}_{sg}(R))$

Thm (Wemyss 2018) There is a bijection

$$\begin{array}{ccc} \{ X \to \operatorname{Spec} R \mid \text{crepant resolution} \} / \cong & & p \\ \downarrow & & \downarrow \\ \{ T \in \mathcal{D}_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting} \} / \cong & & T(p) \end{array}$$

Moreover, $\operatorname{End}(T(p)) \cong \Lambda(p)$.

The Donovan-Wemyss Conjecture (2016, 2020)

$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing $(i=1,2)$

$$R_1 \cong R_2 \iff \mathcal{D}^b(\operatorname{mod}\Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod}\Lambda(p_2))$$

$(\Rightarrow)$ Wemyss (2018) + Dugas (2015)

$(\Leftarrow)$ Reid (1983) in type $\mathbb{A}$

Recall $\mathcal{D}_{sg}(R) := \mathcal{D}^b(\operatorname{mod} R) / K^b(\operatorname{proj} R)$

is the singularity cat of R

- Hom-finite & Krull-Schmidt
- 2-Calabi-Yau $\operatorname{Hom}(Y, X[2]) \cong D\operatorname{Hom}(X,Y)$
- $[2] \cong \mathbb{1}$
- $\forall X \in \mathcal{D}_{sg}(R)$ $\operatorname{End}(X)$ is symmetric since $\operatorname{Hom}(X,X) \cong \operatorname{Hom}(X,X[2]) \cong D\operatorname{Hom}(X,X)$

Def (Iyama-Yoshino 2008) $T \in \mathcal{D}_{sg}(R)$: basic

$T \in \mathcal{D}_{sg}(R)$ is $2\mathbb{Z}$-cluster tilting object if

$$\operatorname{add} T = \{ X \in \mathcal{D}_{sg}(R) \mid \operatorname{Hom}(T, X[1]) = 0 \}$$

& $T \cong T[2]$

Prop (Iyama-Yoshino 2008)

$T \in \mathcal{D}_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$$\Longrightarrow \forall X \in \mathcal{D}_{sg}(R) \ \exists \ \underbrace{T^{-1} \to T^0}_{\operatorname{add} T} \to X \to T[1]$$

$(\Rightarrow \operatorname{thick}(T) = \mathcal{D}_{sg}(R))$

Thm (Wemyss 2018) There is a bijection

$$\{ X \to \operatorname{Spec} R \mid \text{crepant resolution} \} / \cong \qquad p$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\{ T \in \mathcal{D}_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting} \} / \cong \qquad T(p)$$

Moreover, $\operatorname{End}(T(p)) \cong \Lambda(p)$.

e.g $R = \mathbb{C}[[u,v,x,y]]/(uv-xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

BMRRT Cluster cat.

$$\mathcal{D}_{sg}(R) \simeq \mathcal{D}^b(\operatorname{mod}\mathbb{C})/\nu[-2]$$

$T(p) \longmapsto \mathbb{C}$

$T(q) \longmapsto \mathbb{C}[1]$

The Donovan-Wemyss Conjecture (2016, 2020)

$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing $(i=1,2)$

$$R_1 \cong R_2 \iff D^b(\operatorname{mod} \Lambda(p_1)) \simeq D^b(\operatorname{mod} \Lambda(p_2))$$

$(\Rightarrow)$ Wemyss (2018) + Dugas (2015)

$(\Leftarrow)$ Reid (1983) in type $\mathbb{A}$

Recall $D_{sg}(R) := D^b(\operatorname{mod} R) / K^b(\operatorname{proj} R)$

is the singularity cat of R

- Hom-finite & Krull-Schmidt
- 2-Calabi-Yau $\operatorname{Hom}(Y, X[2]) \cong D\operatorname{Hom}(X, Y)$
- $[2] \cong \mathbb{1}$
- $\forall X \in D_{sg}(R)$ $\operatorname{End}(X)$ is symmetric since $\operatorname{Hom}(X,X) \cong \operatorname{Hom}(X, X[2]) \cong D\operatorname{Hom}(X,X)$

Def (Iyama-Yoshino 2008) $T \in D_{sg}(R)$: basic

$T \in D_{sg}(R)$ is $2\mathbb{Z}$-cluster tilting object if

$$\operatorname{add} T = \{ X \in D_{sg}(R) \mid \operatorname{Hom}(T, X[1]) = 0 \}$$

& $T \cong T[2]$

Prop (Iyama-Yoshino 2008)

$T \in D_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$\Longrightarrow$ $\forall X \in D_{sg}(R)$ $\exists$ $\underbrace{T^{-1} \to T^0}_{\operatorname{add} T} \to X \to T[1]$

$(\Rightarrow \operatorname{thick}(T) = D_{sg}(R))$

Thm (Wemyss 2018) There is a bijection

$$\begin{array}{ccc} \{ X \to \operatorname{Spec} R \mid \text{crepant resolution} \} / \cong & & p \\ \downarrow & & \downarrow \\ \{ T \in D_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting} \} / \cong & & T(p) \end{array}$$

Moreover, $\operatorname{End}(T(p)) \cong \Lambda(p)$.

e.g $R = \mathbb{C}[[u,v,x,y]] / (uv - xy)$

BMRRT Cluster cat.

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

$D_{sg}(R) \simeq D^b(\operatorname{mod} \mathbb{C}) / \nu[-2]$

$T(p) \longmapsto \mathbb{C}$

$T(q) \longmapsto \mathbb{C}[1]$

$\mathbb{C}[z^{\pm}]$, $|z| = -2$: dg algebra with trivial differential

$\rightsquigarrow$ $D_{sg}(R) \cong D^c(\mathbb{C}[z^{\pm}])$

compact / perfect der. cat.

Prop (Iyama-Yoshino 2008)

$T \in \mathcal{D}_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$\Longrightarrow \forall X \in \mathcal{D}_{sg}(R) \ \exists \ \underbrace{T^{-1} \to T^{0}}_{\mathrm{add}\, T} \to X \to T[1]$

$(\Rightarrow \mathrm{thick}(T) = \mathcal{D}_{sg}(R))$

Thm (Wemyss 2018) There is a bijection

$$\{X \to \operatorname{Spec} R \mid \text{crepant resolution}\}/\cong \qquad p$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\{T \in \mathcal{D}_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting}\}/\cong \qquad T(p)$$

Moreover, $\mathrm{End}(T(p)) \cong \Lambda(p)$.

e.g $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

BMRRT Cluster cat.

$\mathcal{D}_{sg}(R) \simeq \mathcal{D}^b(\mathrm{mod}\, \mathbb{C})/\nu[-2]$

$T(p) \longmapsto \mathbb{C}$

$T(q) \longmapsto \mathbb{C}[1]$

$\mathbb{C}[v^{\pm}]$, $|v| = -2$: dg algebra with trivial differential

$\rightsquigarrow \quad \mathcal{D}_{sg}(R) \cong \mathcal{D}^c(\mathbb{C}[v^{\pm}])$

compact / perfect der. cat.

Prop (Iyama-Yoshino 2008)

$T \in \mathcal{D}_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$$\Longrightarrow \forall X \in \mathcal{D}_{sg}(R) \;\; \exists \; \underbrace{T^{-1} \to T^{0}}_{\mathrm{add}\,T} \to X \to T[1]$$

$(\Rightarrow \mathrm{thick}(T) = \mathcal{D}_{sg}(R))$

Thm (Wemyss 2018) There is a bijection

$$\begin{array}{ccc} \{ X \to \operatorname{Spec} R \mid \text{crepant resolution} \} /_{\cong} & & p \\ \downarrow & & \mapsto \\ \{ T \in \mathcal{D}_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting} \} /_{\cong} & & T(p) \end{array}$$

Moreover, $\mathrm{End}(T(p)) \cong \Lambda(p)$.

e.g $R = \mathbb{C}[[u,v,x,y]]/(uv - xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

BMRRT Cluster cat.

$\mathcal{D}_{sg}(R) \simeq \mathcal{D}^b(\mathrm{mod}\,\mathbb{C})/\nu[-2]$

$T(p) \longmapsto \mathbb{C}$

$T(q) \longmapsto \mathbb{C}[1]$

$\mathbb{C}[z^{\pm}]$, $|z| = -2$: dg algebra with trivial differential

$$\rightsquigarrow \mathcal{D}_{sg}(R) \cong \mathcal{D}^c(\mathbb{C}[z^{\pm}])$$

compact / perfect der. cat.

Thm (August 2020)

$p : X \to \operatorname{Spec} R$ as above & Λ : basic fin. dim. alg.

$$\mathcal{D}^b(\mathrm{mod}\,\Lambda) \simeq \mathcal{D}^b(\mathrm{mod}\,\Lambda(p))$$

$$\Updownarrow$$

$$\exists \; q : Y \to \operatorname{Spec} R \text{ crepant resolution s.t. } \Lambda \cong \Lambda(q)$$

Prop (Iyama-Yoshino 2008)

$T \in \mathcal{D}_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$$\Longrightarrow \forall X \in \mathcal{D}_{sg}(R) \;\; \exists \; \underbrace{T^{-1} \to T^{0}}_{\mathrm{add}\,T} \to X \to T[1]$$

$(\Rightarrow \mathrm{thick}(T) = \mathcal{D}_{sg}(R))$

Thm (Wemyss 2018) There is a bijection

$$\{X \to \operatorname{Spec} R \mid \text{crepant resolution}\}/\cong \quad \downarrow \quad \{T \in \mathcal{D}_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting}\}/\cong$$

$$p \longmapsto T(p)$$

Moreover, $\operatorname{End}(T(p)) \cong \Lambda(p)$.

e.g $R = \mathbb{C}[[u,v,x,y]]/(uv-xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

BMRRT Cluster cat.

$\mathcal{D}_{sg}(R) \simeq \mathcal{D}^b(\mathrm{mod}\,\mathbb{C})/\nu[-2]$

$T(p) \longmapsto \mathbb{C}$

$T(q) \longmapsto \mathbb{C}[1]$

$\mathbb{C}[v^{\pm}]$, $|v| = -2$: dg algebra with trivial differential

$$\rightsquigarrow \quad \mathcal{D}_{sg}(R) \cong \mathcal{D}^c(\mathbb{C}[v^{\pm}])$$

compact / perfect der. cat.

Thm (August 2020)

$p : X \to \operatorname{Spec} R$ as above & Λ : basic fin. dim. alg.

$$\mathcal{D}^b(\mathrm{mod}\,\Lambda) \simeq \mathcal{D}^b(\mathrm{mod}\,\Lambda(p))$$

$$\Updownarrow$$

$$\exists\, q : Y \to \operatorname{Spec} R \text{ crepant resolution s.t. } \Lambda \cong \Lambda(q)$$

Corollary $p_i : X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$\mathcal{D}^b(\mathrm{mod}\,\Lambda(p_1)) \overset{\sim}{\longleftrightarrow} \mathcal{D}^b(\mathrm{mod}\,\Lambda(p_2))$$

$$\Downarrow$$

$$\exists\, q : Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

Prop (Iyama-Yoshino 2008)

$T \in \mathcal{D}_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$\Longrightarrow \forall X \in \mathcal{D}_{sg}(R)\ \exists\ \underbrace{T^{-1} \to T^{0}}_{\text{add}T} \to X \to T[1]$

$(\Rightarrow \text{thick}(T) = \mathcal{D}_{sg}(R))$

Thm (Wemyss 2018) There is a bijection

$$\begin{array}{ccc} \{X \to \operatorname{Spec} R \mid \text{crepant resolution}\}/\cong & & p \\ \downarrow & & \downarrow \\ \{T \in \mathcal{D}_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting}\}/\cong & & T(p) \end{array}$$

Moreover, $\operatorname{End}(T(p)) \cong \Lambda(p)$.

e.g $R = \mathbb{C}[[u,v,x,y]]/(uv-xy)$

BMRRT Cluster cat.

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd} \qquad \begin{array}{l} \mathcal{D}_{sg}(R) \simeq \mathcal{D}^b(\text{mod}\,\mathbb{C})/\nu[-2] \\ T(p) \longmapsto \mathbb{C} \\ T(q) \longmapsto \mathbb{C}[1] \end{array}$$

$\mathbb{C}[z^{\pm}]$, $|z| = -2$: dg algebra with trivial differential

$\rightsquigarrow \mathcal{D}_{sg}(R) \cong \mathcal{D}^c(\mathbb{C}[v^{\pm}])$

compact / perfect der. cat.

Thm (August 2020)

$p: X \to \operatorname{Spec} R$ as above & Λ: basic fin. dim. alg.

$$\mathcal{D}^b(\text{mod}\,\Lambda) \simeq \mathcal{D}^b(\text{mod}\,\Lambda(p))$$

$$\Updownarrow$$

$\exists\ q: Y \to \operatorname{Spec} R$ crepant resolution s.t. $\Lambda \cong \Lambda(q)$

Corollary $p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$\mathcal{D}^b(\text{mod}\,\Lambda(p_1)) \xleftrightarrow{\sim} \mathcal{D}^b(\text{mod}\,\Lambda(p_2))$$

$$\Downarrow$$

$\exists\ q: Y \to \operatorname{Spec} R_1$ s.t. $\Lambda(p_2) \cong \Lambda(q)$

$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$

Prop (Iyama-Yoshino 2008)

$T \in \mathcal{D}_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$\Longrightarrow \forall X \in \mathcal{D}_{sg}(R) \ \exists \ \underbrace{T^{-1} \to T^{0}}_{\mathrm{add}\,T} \to X \to T[1]$

$(\Rightarrow \mathrm{thick}(T) = \mathcal{D}_{sg}(R))$

Thm (Wemyss 2018) There is a bijection

$$\{X \to \operatorname{Spec} R \mid \text{crepant resolution}\}/\cong \quad p$$
$$\downarrow \quad \quad \downarrow$$
$$\{T \in \mathcal{D}_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting}\}/\cong \quad T(p)$$

Moreover, $\operatorname{End}(T(p)) \cong \Lambda(p)$.

e.g $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

BMRRT Cluster cat.

$\mathcal{D}_{sg}(R) \simeq \mathcal{D}^b(\mathrm{mod}\,\mathbb{C})/\nu[-2]$

$T(p) \longmapsto \mathbb{C}$

$T(q) \longmapsto \mathbb{C}[1]$

$\mathbb{C}[u^{\pm}]$, $|u| = -2$: dg algebra with trivial differential

$\rightsquigarrow \quad \mathcal{D}_{sg}(R) \cong \mathcal{D}^c(\mathbb{C}[u^{\pm}])$

compact / perfect der. cat.

Thm (August 2020)

$p: X \to \operatorname{Spec} R$ as above & Λ : basic fin. dim. alg.

$$\mathcal{D}^b(\mathrm{mod}\,\Lambda) \simeq \mathcal{D}^b(\mathrm{mod}\,\Lambda(p))$$
$$\Updownarrow$$
$$\exists\, q: Y \to \operatorname{Spec} R \text{ crepant resolution s.t. } \Lambda \cong \Lambda(q)$$

Corollary $p_i : X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$\mathcal{D}^b(\mathrm{mod}\,\Lambda(p_1)) \xleftrightarrow{\sim} \mathcal{D}^b(\mathrm{mod}\,\Lambda(p_2))$$
$$\Downarrow$$
$$\exists\, q: Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\mathrm{mod}\,R)_{dg} / K^b(\mathrm{proj}\,R)_{dg}$

is the dg singularity cat of R

Prop (Iyama - Yoshino 2008)

$T \in \mathcal{D}_{sg}(R)$: $2\mathbb{Z}$ - cluster tilting

$\Longrightarrow \forall X \in \mathcal{D}_{sg}(R) \ \exists \ \underbrace{T^{-1} \to T^0}_{\mathrm{add}\,T} \to X \to T[1]$

$(\Rightarrow \mathrm{thick}(T) = \mathcal{D}_{sg}(R))$

Thm (Wemyss 2018) There is a bijection

$$\begin{array}{ccc} \{X \to \operatorname{Spec} R \mid \text{crepant resolution}\}/\cong & & p \\ \downarrow & & \downarrow \\ \{T \in \mathcal{D}_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting}\}/\cong & & T(p) \end{array}$$

Moreover, $\operatorname{End}(T(p)) \cong \Lambda(p)$.

e.g $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

BMRRT Cluster cat.

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

$\mathcal{D}_{sg}(R) \simeq \mathcal{D}^b(\mathrm{mod}\,\mathbb{C})/\nu[-2]$

$T(p) \longmapsto \mathbb{C}$

$T(q) \longmapsto \mathbb{C}[1]$

$\mathbb{C}[u^{\pm}]$, $|u| = -2$: dg algebra with trivial differential

$\rightsquigarrow \ \mathcal{D}_{sg}(R) \cong \mathcal{D}^c(\mathbb{C}[u^{\pm}])$

compact / perfect der. cat.

Thm (August 2020)

$p : X \to \operatorname{Spec} R$ as above & Λ : basic fin. dim. alg.

$\mathcal{D}^b(\mathrm{mod}\,\Lambda) \simeq \mathcal{D}^b(\mathrm{mod}\,\Lambda(p))$

$\Updownarrow$

$\exists\, q : Y \to \operatorname{Spec} R$ crepant resolution s.t. $\Lambda \cong \Lambda(q)$

Corollary $p_i : X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$\mathcal{D}^b(\mathrm{mod}\,\Lambda(p_1)) \overset{\sim}{\longleftrightarrow} \mathcal{D}^b(\mathrm{mod}\,\Lambda(p_2))$

$\Downarrow$

$\exists\, q : Y \to \operatorname{Spec} R_1$ s.t. $\Lambda(p_2) \cong \Lambda(q)$

$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\mathrm{mod}\,R)_{dg} / K^b(\mathrm{proj}\,R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg category

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}))$ is the

Hochschild cohomology of $\mathcal{A}$

Thm (August 2020)

$p: X \to \operatorname{Spec} R$ as above & Λ: basic fin. dim. alg.

$$\mathcal{D}^b(\operatorname{mod} \Lambda) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p))$$

$$\Updownarrow$$

$\exists\, q: Y \to \operatorname{Spec} R$ crepant resolution s.t. $\Lambda \cong \Lambda(q)$

Corollary $p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$\mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \overset{\sim}{\longleftrightarrow} \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

$$\Downarrow$$

$$\exists\, q: Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

$$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\operatorname{mod} R)_{dg} / K^b(\operatorname{proj} R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg category

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}))$ is the

Hochschild cohomology of $\mathcal{A}$

Thm (August 2020)

$p : X \to \operatorname{Spec} R$ as above & Λ : basic fin. dim. alg.

$$\mathcal{D}^b(\operatorname{mod} \Lambda) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p))$$

$$\Updownarrow$$

$\exists\, q : Y \to \operatorname{Spec} R$ crepant resolution s.t. $\Lambda \cong \Lambda(q)$

Corollary $p_i : X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$\mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \xleftrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

$$\Downarrow$$

$$\exists\, q : Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

$$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\operatorname{mod} R)_{dg} / K^b(\operatorname{proj} R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg category

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}))$ is the

Hochschild cohomology of $\mathcal{A}$

Thm (Hua - Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (August 2020)

$p : X \to \operatorname{Spec} R$ as above & Λ : basic fin. dim. alg.

$$\mathcal{D}^b(\operatorname{mod} \Lambda) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p))$$

$$\Updownarrow$$

$$\exists\, q : Y \to \operatorname{Spec} R \text{ crepant resolution s.t. } \Lambda \cong \Lambda(q)$$

Corollary $p_i : X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$\mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \xleftrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

$$\Downarrow$$

$$\exists\, q : Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

$$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\operatorname{mod} R)_{dg} / K^b(\operatorname{proj} R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg category

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\operatorname{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}))$ is the

Hochschild cohomology of $\mathcal{A}$

Thm (Hua - Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather - Yau 1982)

T_f + dim R determine R up to isomorphism

Thm (August 2020)

$p: X \to \operatorname{Spec} R$ as above & Λ: basic fin. dim. alg.

$$\mathcal{D}^b(\operatorname{mod}\Lambda) \simeq \mathcal{D}^b(\operatorname{mod}\Lambda(p))$$

$$\Updownarrow$$

$$\exists\, q: Y \to \operatorname{Spec} R \text{ crepant resolution s.t. } \Lambda \cong \Lambda(q)$$

Corollary $p_i: X_i \to \operatorname{Spec} R_i$, $i=1,2$ as in DW Conj.

$$\mathcal{D}^b(\operatorname{mod}\Lambda(p_1)) \overset{\sim}{\longleftrightarrow} \mathcal{D}^b(\operatorname{mod}\Lambda(p_2))$$

$$\Downarrow$$

$$\exists\, q: Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

$$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\operatorname{mod} R)_{dg} / K^b(\operatorname{proj} R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg category

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A},\mathcal{A}))$ is the

Hochschild cohomology of $\mathcal{A}$

Thm (Hua - Keller 2018) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

$$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f \text{ : Tyurina algebra of } f$$

Thm (Mather-Yau 1982)

T_f + dim R determine R up to isomorphism

$\rightsquigarrow$ Putting everything together ...

$p_i: X_i \to \operatorname{Spec} R_i$, $i=1,2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

Thm (August 2020)

$p: X \to \operatorname{Spec} R$ as above & Λ: basic fin. dim. alg.

$$\mathcal{D}^b(\operatorname{mod} \Lambda) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p))$$

$$\Updownarrow$$

$\exists\, q: Y \to \operatorname{Spec} R$ crepant resolution s.t. $\Lambda \cong \Lambda(q)$

Corollary $p_i: X_i \to \operatorname{Spec} R_i$, $i=1,2$ as in DW Conj.

$$\mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \xleftrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

$$\Downarrow$$

$$\exists\, q: Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

$$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\operatorname{mod} R)_{dg} / K^b(\operatorname{proj} R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg category

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\operatorname{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}))$ is the Hochschild cohomology of $\mathcal{A}$

Thm (Hua - Keller 2018) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather - Yau 1982)

T_f + dim R determine R up to isomorphism

⇝ Putting everything together ...

$p_i: X_i \to \operatorname{Spec} R_i$, $i=1,2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$\underset{\mathbb{Z}\text{-CT}}{T(p_1)} \in \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \Lambda(p_1) := \mathbb{R}\operatorname{End}(T(p_1))$$

$$\underset{\mathbb{Z}\text{-CT}}{T(p_2)} \in \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \Lambda(p_2) := \mathbb{R}\operatorname{End}(T(p_2))$$

Thm (August 2020)

$p: X \to \operatorname{Spec} R$ as above & Λ: basic fin. dim. alg.

$$\mathcal{D}^b(\operatorname{mod} \Lambda) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p))$$

$$\Updownarrow$$

$\exists\, q: Y \to \operatorname{Spec} R$ crepant resolution s.t. $\Lambda \cong \Lambda(q)$

Corollary $p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$\mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \xleftrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

$$\Downarrow$$

$$\exists\, q: Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

$$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\operatorname{mod} R)_{dg} / K^b(\operatorname{proj} R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg category

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\operatorname{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}))$ is the

Hochschild cohomology of $\mathcal{A}$

Thm (Hua - Keller 2018) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather - Yau 1982)

T_f + dim R determine R up to isomorphism

⇝ Putting everything together ...

$p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_1)} \in \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \mathcal{L}(p_1) := \mathbb{R}\operatorname{End}(T(p_1))$$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_2)} \in \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \mathcal{L}(p_2) := \mathbb{R}\operatorname{End}(T(p_2))$$

$$H^*(\mathcal{L}(p_1)) \cong H^*(\mathcal{L}(p_2)) \cong \Lambda[z^{\pm}], \quad |z| = -2$$

Thm (August 2020)

$p: X \to \operatorname{Spec} R$ as above & Λ: basic fin. dim. alg.

$$\mathcal{D}^b(\operatorname{mod} \Lambda) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p))$$

$$\Updownarrow$$

$\exists\ q: Y \to \operatorname{Spec} R$ crepant resolution s.t. $\Lambda \cong \Lambda(q)$

Corollary $p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$\mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \xleftrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

$$\Downarrow$$

$$\exists\ q: Y \to \operatorname{Spec} R_1 \ \text{ s.t. } \ \Lambda(p_2) \cong \Lambda(q)$$

$$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\operatorname{mod} R)_{dg} / K^b(\operatorname{proj} R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg category

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}))$ is the

Hochschild cohomology of $\mathcal{A}$

Thm (Hua - Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather - Yau 1982)

T_f + $\dim R$ determine R up to isomorphism

$\rightsquigarrow$ Putting everything together ...

$p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_1)} \in \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \mathcal{A}(p_1) := \mathbb{R}\mathrm{End}(T(p_1))$$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_2)} \in \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \mathcal{A}(p_2) := \mathbb{R}\mathrm{End}(T(p_2))$$

$$H^*(\mathcal{A}(p_1)) \cong H^*(\mathcal{A}(p_2)) \cong \Lambda[z^{\pm}], \ |z| = -2$$

(Keller 1994) $\mathcal{D}^c(\mathcal{A}(p_1))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_1)_{dg}$

$$\mathcal{D}^c(\mathcal{A}(p_2))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_2)_{dg}$$

Thm (August 2020)

$p: X \to \operatorname{Spec} R$ as above & Λ : basic fin. dim. alg.

$$\mathcal{D}^b(\operatorname{mod} \Lambda) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p))$$

$$\Updownarrow$$

$$\exists\, q: Y \to \operatorname{Spec} R \text{ crepant resolution s.t. } \Lambda \cong \Lambda(q)$$

Corollary $p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$\mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \overset{\sim}{\longleftrightarrow} \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$$

$$\Downarrow$$

$$\exists\, q: Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

$$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$$

Recall $\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\operatorname{mod} R)_{dg} / K^b(\operatorname{proj} R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg category

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\operatorname{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}))$ is the

Hochschild cohomology of $\mathcal{A}$

Thm (Hua – Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather – Yau 1982)

T_f + $\dim R$ determine R up to isomorphism

$\rightsquigarrow$ Putting everything together ...

$p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$T(p_1) \overset{2\mathbb{Z}\text{-CT}}{\in} \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \boldsymbol{\Lambda}(p_1) := \mathbb{R}\operatorname{End}(T(p_1))$$

$$T(p_2) \underset{2\mathbb{Z}\text{-CT}}{\in} \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \boldsymbol{\Lambda}(p_2) := \mathbb{R}\operatorname{End}(T(p_2))$$

$$H^*(\boldsymbol{\Lambda}(p_1)) \cong H^*(\boldsymbol{\Lambda}(p_2)) \cong \Lambda[u^{\pm}], \quad |u| = -2$$

(Keller 1994) $\mathcal{D}^c(\boldsymbol{\Lambda}(p_1))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_1)_{dg}$

$\mathcal{D}^c(\boldsymbol{\Lambda}(p_2))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_2)_{dg}$

$$\boxed{\mathcal{D}^c(\boldsymbol{\Lambda}(p_1))_{dg} \simeq \mathcal{D}^c(\boldsymbol{\Lambda}(p_2))_{dg} : \text{quasi-eq} \implies \text{DW Conj.}}$$

<u>Thm</u> (Hua - Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: <u>Tyurina algebra</u> of f

<u>Thm</u> (Mather - Yau 1982)

T_f + dim R determine R up to isomorphism

$\rightsquigarrow$ Putting everything together ...

$p_i : X_i \to \operatorname{Spec} R_i$, $i = 1,2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$\underset{\mathbb{Z}\text{-CT}}{T(p_1)} \in \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \varLambda(p_1) := \mathbb{R}\mathrm{End}(T(p_1))$$

$$\underset{\mathbb{Z}\text{-CT}}{T(p_2)} \in \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \varLambda(p_2) := \mathbb{R}\mathrm{End}(T(p_2))$$

$$H^*(\varLambda(p_1)) \cong H^*(\varLambda(p_2)) \cong \Lambda[z^{\pm}], \quad |z| = -2$$

(Keller 1994) $\mathcal{D}^c(\varLambda(p_1))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_1)_{dg}$

$\mathcal{D}^c(\varLambda(p_2))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_2)_{dg}$

$$\boxed{\mathcal{D}^c(\varLambda(p_1))_{dg} \simeq \mathcal{D}^c(\varLambda(p_2))_{dg} : \text{quasi-eq} \Longrightarrow \text{DW Conj.}}$$

Thm (Hua-Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather-Yau 1982)

T_f + dim R determine R up to isomorphism

$\rightsquigarrow$ Putting everything together ...

$p_i : X_i \to \operatorname{Spec} R_i$, $i = 1, 2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$T(p_1) \underset{}{\overset{2\mathbb{Z}\text{-CT}}{\in}} \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \mathbb{\Lambda}(p_1) := \mathbb{R}\mathrm{End}(T(p_1))$$

$$T(p_2) \underset{2\mathbb{Z}\text{-CT}}{\in} \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \mathbb{\Lambda}(p_2) := \mathbb{R}\mathrm{End}(T(p_2))$$

$$H^*(\mathbb{\Lambda}(p_1)) \cong H^*(\mathbb{\Lambda}(p_2)) \cong \Lambda[z^{\pm}], \quad |z| = -2$$

(Keller 1994) $\mathcal{D}^c(\mathbb{\Lambda}(p_1))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_1)_{dg}$

$$\mathcal{D}^c(\mathbb{\Lambda}(p_2))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_2)_{dg}$$

$$\boxed{\mathcal{D}^c(\mathbb{\Lambda}(p_1))_{dg} \simeq \mathcal{D}^c(\mathbb{\Lambda}(p_2))_{dg} : \text{quasi-eq} \implies \text{DW Conj.}}$$

Thm (J-Muro 2022)

R: isolated cDV singularity & $p : X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[z^{\pm}]$, $|z| = -2$

Thm (Hua-Keller 2018) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather-Yau 1982)

T_f + dim R determine R up to isomorphism

$\rightsquigarrow$ Putting everything together ...

$p_i : X_i \to \operatorname{Spec} R_i$, $i = 1,2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_1)} \in \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \mathbb{\Lambda}(p_1) := \mathbb{R}\mathrm{End}(T(p_1))$$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_2)} \in \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \mathbb{\Lambda}(p_2) := \mathbb{R}\mathrm{End}(T(p_2))$$

$$H^*(\mathbb{\Lambda}(p_1)) \cong H^*(\mathbb{\Lambda}(p_2)) \cong \Lambda[z^\pm], \quad |z| = -2$$

(Keller 1994) $\mathcal{D}^c(\mathbb{\Lambda}(p_1))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_1)_{dg}$

$\mathcal{D}^c(\mathbb{\Lambda}(p_2))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_2)_{dg}$

$$\boxed{\mathcal{D}^c(\mathbb{\Lambda}(p_1))_{dg} \simeq \mathcal{D}^c(\mathbb{\Lambda}(p_2))_{dg} \text{ : quasi-eq} \Longrightarrow \text{DW Conj.}}$$

Thm (J-Muro 2022)

R: isolated cDV singularity & $p : X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[z^\pm]$, $|z| = -2$

Strategy Prove $\mathbb{R}\mathrm{End}(T) \simeq \mathbb{\Lambda}(p)$ for $T \in \mathcal{A}$ satisfying (1) & (2)

Thm (Hua-Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather-Yau 1982)

T_f + dim R determine R up to isomorphism

$\rightsquigarrow$ Putting everything together ...

$p_i : X_i \to \operatorname{Spec} R_i$, $i = 1, 2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_1)} \in \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \mathbb{\Lambda}(p_1) := \mathbb{R}\mathrm{End}(T(p_1))$$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_2)} \in \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \mathbb{\Lambda}(p_2) := \mathbb{R}\mathrm{End}(T(p_2))$$

$$H^*(\mathbb{\Lambda}(p_1)) \cong H^*(\mathbb{\Lambda}(p_2)) \cong \Lambda[u^{\pm}], \ |u| = -2$$

(Keller 1994) $\mathcal{D}^c(\mathbb{\Lambda}(p_1))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_1)_{dg}$

$\mathcal{D}^c(\mathbb{\Lambda}(p_2))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_2)_{dg}$

$$\boxed{\mathcal{D}^c(\mathbb{\Lambda}(p_1))_{dg} \simeq \mathcal{D}^c(\mathbb{\Lambda}(p_2))_{dg} : \text{quasi-eq} \implies \text{DW Conj.}}$$

Thm (J-Muro 2022)

R: isolated cDV singularity & $p : X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[u^{\pm}]$, $|u| = -2$

Strategy Prove $\mathbb{R}\mathrm{End}(T) \simeq \mathbb{\Lambda}(p)$ for $T \in \mathcal{A}$ satisfying (1) & (2)

Sketch of proof $\mathbb{\Lambda} = \mathbb{\Lambda}(p)$ & $\Lambda = \Lambda(p)$

$$\rightsquigarrow H^*(\mathbb{\Lambda}) \cong \Lambda[u^{\pm}], \ |u| = -2.$$

Thm (Hua-Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather-Yau 1982)

T_f + dim R determine R up to isomorphism

$\rightsquigarrow$ Putting everything together ...

$p_i : X_i \to \operatorname{Spec} R_i$, $i = 1, 2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_1)} \in \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \mathbb{\Lambda}(p_1) := \mathbb{R}\mathrm{End}(T(p_1))$$

$$\underset{2\mathbb{Z}\text{-CT}}{T(p_2)} \in \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \mathbb{\Lambda}(p_2) := \mathbb{R}\mathrm{End}(T(p_2))$$

$$H^*(\mathbb{\Lambda}(p_1)) \cong H^*(\mathbb{\Lambda}(p_2)) \cong \Lambda[u^{\pm}], \ |u| = -2$$

(Keller 1994) $\mathcal{D}^c(\mathbb{\Lambda}(p_1))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_1)_{dg}$

$\mathcal{D}^c(\mathbb{\Lambda}(p_2))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_2)_{dg}$

$\boxed{\mathcal{D}^c(\mathbb{\Lambda}(p_1))_{dg} \simeq \mathcal{D}^c(\mathbb{\Lambda}(p_2))_{dg} \text{ : quasi-eq} \implies \text{DW Conj.}}$

Thm (J-Muro 2022)

R: isolated cDV singularity & $p : X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[u^{\pm}]$, $|u| = -2$

Strategy: Prove $\mathbb{R}\mathrm{End}(T) \simeq \mathbb{\Lambda}(p)$ for $T \in \mathcal{A}$ satisfying (1) & (2)

Sketch of proof: $\mathbb{\Lambda} = \mathbb{\Lambda}(p)$ & $\Lambda = \Lambda(p)$

$$\rightsquigarrow H^*(\mathbb{\Lambda}) \cong \Lambda[u^{\pm}], \ |u| = -2.$$

(Kadeishvili 1982)

$\mathbb{\Lambda}$ induces a minimal A_∞-algebra structure on $\Lambda[u^{\pm}]$ s.t. $\mathbb{\Lambda} \simeq \Lambda[u^{\pm}]$ as A_∞-algebras

Thm (Hua-Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$$HH^0(\mathcal{D}_{sg}(R)_{dg}) \cong T_f \; : \text{Tyurina algebra of } f$$

Thm (Mather-Yau 1982)

T_f + dim R determine R up to isomorphism

$\rightsquigarrow$ Putting everything together ...

$p_i : X_i \to \operatorname{Spec} R_i$, $i = 1, 2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

$$T(p_1) \overset{2\mathbb{Z}\text{-CT}}{\in} \mathcal{D}_{sg}(R_1)_{dg} \rightsquigarrow \boldsymbol{\Lambda}(p_1) := \mathbb{R}\mathrm{End}(T(p_1))$$

$$T(p_2) \underset{2\mathbb{Z}\text{-CT}}{\in} \mathcal{D}_{sg}(R_2)_{dg} \rightsquigarrow \boldsymbol{\Lambda}(p_2) := \mathbb{R}\mathrm{End}(T(p_2))$$

$$H^*(\boldsymbol{\Lambda}(p_1)) \cong H^*(\boldsymbol{\Lambda}(p_2)) \cong \Lambda[u^{\pm}], \; |u| = -2$$

(Keller 1994) $\mathcal{D}^c(\boldsymbol{\Lambda}(p_1))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_1)_{dg}$

$$\mathcal{D}^c(\boldsymbol{\Lambda}(p_2))_{dg} \xrightarrow{\sim} \mathcal{D}_{sg}(R_2)_{dg}$$

$$\boxed{\mathcal{D}^c(\boldsymbol{\Lambda}(p_1))_{dg} \simeq \mathcal{D}^c(\boldsymbol{\Lambda}(p_2))_{dg} \; : \text{quasi-eq} \implies \text{DW Conj.}}$$

Thm (J-Muro 2022)

R: isolated cDV singularity & $p : X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[u^{\pm}]$, $|u| = -2$

Strategy Prove $\mathbb{R}\mathrm{End}(T) \simeq \boldsymbol{\Lambda}(p)$ for $T \in \mathcal{A}$ satisfying (1) & (2)

Sketch of proof $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(p)$ & $\Lambda = \Lambda(p)$

$$\rightsquigarrow H^*(\boldsymbol{\Lambda}) \cong \Lambda[u^{\pm}], \; |u| = -2.$$

(Kadeishvili 1982)

$\boldsymbol{\Lambda}$ induces a minimal A_∞-algebra structure on $\Lambda[u^{\pm}]$ s.t. $\boldsymbol{\Lambda} \simeq \Lambda[u^{\pm}]$ as A_∞-algebras

$m_n : \Lambda[u^{\pm}]^{\otimes n} \longrightarrow \Lambda[u^{\pm}]$ of degree $2-n$, $n \geq 2$

$\rightsquigarrow \forall\, n \notin 2\mathbb{Z}$, $m_n = 0$

$$\boxed{\boldsymbol{\Lambda} = (\Lambda[u^{\pm}], m_4, m_6, m_8, \dots)}$$

Thm (J-Muro 2022)

R: isolated cDV singularity & $p : X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[z^{\pm}]$, $|z| = -2$

Strategy Prove $\mathbb{R}\mathrm{End}(T) \simeq \mathbb{\Lambda}(p)$ for $T \in \mathcal{A}$ satisfying (1) & (2)

Sketch of proof $\mathbb{\Lambda} = \mathbb{\Lambda}(p)$ & $\Lambda = \Lambda(p)$

$$\rightsquigarrow H^*(\mathbb{\Lambda}) \cong \Lambda[z^{\pm}], \ |z| = -2.$$

(Kadeishvili 1982)

$\mathbb{\Lambda}$ induces a minimal A_∞-algebra structure on $\Lambda[z^{\pm}]$ s.t. $\mathbb{\Lambda} \simeq \Lambda[z^{\pm}]$ as A_∞-algebras

$m_n : \Lambda[z^{\pm}]^{\otimes n} \longrightarrow \Lambda[z^{\pm}]$ of degree $2-n$, $n \geq 2$

$\rightsquigarrow \forall n \notin 2\mathbb{Z}$, $m_n = 0$

$$\boxed{\mathbb{\Lambda} = (\Lambda[z^{\pm}], m_4, m_6, m_8, \dots)}$$

Thm (J-Muro 2022)

R: isolated cDV singularity & $p : X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[z^{\pm}]$, $|z| = -2$

Strategy Prove $\mathbb{R}\mathrm{End}(T) \simeq \mathbb{\Lambda}(p)$
for $T \in \mathcal{A}$ satisfying (1) & (2)

Sketch of proof $\mathbb{\Lambda} = \mathbb{\Lambda}(p)$ & $\Lambda = \Lambda(p)$

$\rightsquigarrow H^*(\mathbb{\Lambda}) \cong \Lambda[z^{\pm}]$, $|z| = -2$.

(Kadeishvili 1982)

$\mathbb{\Lambda}$ induces a minimal A_∞-algebra structure on $\Lambda[z^{\pm}]$ s.t. $\mathbb{\Lambda} \simeq \Lambda[z^{\pm}]$ as A_∞-algebras

$m_n : \Lambda[z^{\pm}]^{\otimes n} \longrightarrow \Lambda[z^{\pm}]$ of degree $2-n$, $n \geq 2$

$\rightsquigarrow \forall\, n \notin 2\mathbb{Z}$, $m_n = 0$

$$\boxed{\mathbb{\Lambda} = (\Lambda[z^{\pm}], m_4, m_6, m_8, \dots)}$$

$m_4 \in CC^{4,-2}(\Lambda[z^{\pm}], \Lambda[z^{\pm}])$: Hochschild cochain

$\partial_{\mathrm{Hoch}}(m_4) = 0 \rightsquigarrow \{m_4\} \in HH^{4,-2}(\Lambda[z^{\pm}], \Lambda[z^{\pm}])$

$\{m_4\}$: Universal Massey Product (UMP)

Thm (J-Muro 2022)

R: isolated cDV singularity & $p : X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[u^{\pm}]$, $|u| = -2$

Strategy Prove $\mathbb{R}\mathrm{End}(T) \simeq \mathbb{A}(p)$ for $T \in \mathcal{A}$ satisfying (1) & (2)

Sketch of proof $\mathbb{A} = \mathbb{A}(p)$ & $\Lambda = \Lambda(p)$

$$\rightsquigarrow H^*(\mathbb{A}) \cong \Lambda[u^{\pm}],\ |u| = -2.$$

(Kadeishvili 1982)

$\mathbb{A}$ induces a minimal A_∞-algebra structure on $\Lambda[u^{\pm}]$ s.t. $\mathbb{A} \simeq \Lambda[u^{\pm}]$ as A_∞-algebras

$m_n : \Lambda[u^{\pm}]^{\otimes n} \longrightarrow \Lambda[u^{\pm}]$ of degree $2-n$, $n \geq 2$

$\rightsquigarrow \forall n \notin 2\mathbb{Z}$, $m_n = 0$

$$\boxed{\mathbb{A} = (\Lambda[u^{\pm}], m_4, m_6, m_8, \ldots)}$$

$m_4 \in CC^{4,-2}(\Lambda[u^{\pm}], \Lambda[u^{\pm}])$: Hochschild cochain

$\partial_{\mathrm{Hoch}}(m_4) = 0 \rightsquigarrow \{m_4\} \in HH^{4,-2}(\Lambda[u^{\pm}], \Lambda[u^{\pm}])$

$\{m_4\}$: <u>Universal Massey Product</u> (UMP)

$j : \Lambda \hookrightarrow \Lambda[u^{\pm}]$ inclusion of degree 0 component

$j^* : HH^{\bullet,\bullet}(\Lambda[u^{\pm}], \Lambda[u^{\pm}]) \longrightarrow HH^{\bullet,\bullet}(\Lambda, \Lambda[u^{\pm}])$

$j^*\{m_4\}$: <u>restricted Universal Massey Product</u> (rUMP)

Thm (J-Muro 2022)

R: isolated cDV singularity & $p: X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[z^{\pm}]$, $|z| = -2$

Strategy Prove $\mathbb{R}\mathrm{End}(T) \simeq \mathcal{A}(p)$ for $T \in \mathcal{A}$ satisfying (1) & (2)

Sketch of proof $\mathcal{A} = \mathcal{A}(p)$ & $\Lambda = \Lambda(p)$

$\rightsquigarrow H^*(\mathcal{A}) \cong \Lambda[z^{\pm}]$, $|z| = -2$.

(Kadeishvili 1982)

$\mathcal{A}$ induces a minimal A_∞-algebra structure on $\Lambda[z^{\pm}]$ s.t. $\mathcal{A} \simeq \Lambda[z^{\pm}]$ as A_∞-algebras

$m_n : \Lambda[z^{\pm}]^{\otimes n} \to \Lambda[z^{\pm}]$ of degree $2-n$, $n \geq 2$

$\rightsquigarrow \forall n \notin 2\mathbb{Z}$, $m_n = 0$

$$\boxed{\mathcal{A} = (\Lambda[z^{\pm}], m_4, m_6, m_8, \dots)}$$

$m_4 \in CC^{4,-2}(\Lambda[z^{\pm}], \Lambda[z^{\pm}])$: Hochschild cochain

$\partial_{Hoch}(m_4) = 0 \rightsquigarrow \{m_4\} \in HH^{4,-2}(\Lambda[z^{\pm}], \Lambda[z^{\pm}])$

$\{m_4\}$: Universal Massey Product (UMP)

$j : \Lambda \hookrightarrow \Lambda[z^{\pm}]$ inclusion of degree 0 component

$j^* : HH^{\bullet,\bullet}(\Lambda[z^{\pm}], \Lambda[z^{\pm}]) \longrightarrow HH^{\bullet,\bullet}(\Lambda, \Lambda[z^{\pm}])$

$j^*\{m_4\}$: restricted Universal Massey Product (rUMP)

Thm (J-Muro 2022)

$j^*\{m_4\} \in \underline{HH}^{\bullet,\bullet}(\Lambda, \Lambda[z^{\pm}])$ is a unit

Hochschild-Tate cohomology

Thm (J-Muro 2022)

R: isolated cDV singularity & $p : X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[u^{\pm}]$, $|u| = -2$

Strategy Prove $\mathbb{R}\mathrm{End}(T) \simeq \mathbb{A}(p)$ for $T \in \mathcal{A}$ satisfying (1) & (2)

Sketch of proof $\mathbb{A} = \mathbb{A}(p)$ & $\Lambda = \Lambda(p)$

$$\rightsquigarrow H^*(\mathbb{A}) \cong \Lambda[u^{\pm}],\ |u| = -2.$$

(Kadeishvili 1982)

$\mathbb{A}$ induces a minimal A_∞-algebra structure on $\Lambda[u^{\pm}]$ s.t. $\mathbb{A} \simeq \Lambda[u^{\pm}]$ as A_∞-algebras

$m_n : \Lambda[u^{\pm}]^{\otimes n} \to \Lambda[u^{\pm}]$ of degree $2-n$, $n \geq 2$

$\rightsquigarrow \forall n \notin 2\mathbb{Z},\ m_n = 0$

$$\boxed{\mathbb{A} = (\Lambda[u^{\pm}], m_4, m_6, m_8, \dots)}$$

$m_4 \in CC^{4,-2}(\Lambda[u^{\pm}], \Lambda[u^{\pm}])$: Hochschild cochain

$\partial_{Hoch}(m_4) = 0 \rightsquigarrow \{m_4\} \in HH^{4,-2}(\Lambda[u^{\pm}], \Lambda[u^{\pm}])$

$\{m_4\}$: Universal Massey Product (UMP)

$j : \Lambda \hookrightarrow \Lambda[u^{\pm}]$ inclusion of degree 0 component

$j^* : HH^{\bullet,\bullet}(\Lambda[u^{\pm}], \Lambda[u^{\pm}]) \to HH^{\bullet,\bullet}(\Lambda, \Lambda[u^{\pm}])$

$j^*\{m_4\}$: restricted Universal Massey Product (rUMP)

Thm (J-Muro 2022)

$$j^*\{m_4\} \in \underline{HH}^{\bullet,\bullet}(\Lambda, \Lambda[u^{\pm}]) \text{ is a unit}$$

Hochschild-Tate cohomology

Moreover, a minimal A_∞-algebra

$$A = (\Lambda[u^{\pm}], m_4^A, m_6^A, m_8^A, \dots)$$

is A_∞-isomorphic to $\mathbb{A}$ if and only if

$$j^*\{m_4^A\} \in \underline{HH}^{\bullet}(\Lambda, \Lambda[u^{\pm}]) \text{ is a unit}$$