Higher structures in higher Auslander-Reiten theory

I Preprojective algebras of hereditary algebras

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I Preprojective algebras of hereditary algebras

Fix $\Bbbk$: perfect field (e.g. $\Bbbk = \overline{\Bbbk}$)

A : finite-dimensional algebra (basic and connected, for simplicity)

$\operatorname{mod} A$: (abelian) category of finite-dimensional right A-modules

$\operatorname{proj} A$: (additive) category of finite-dimensional projective right A-modules

$\operatorname{inj} A$: (additive) category of finite-dimensional injective right A-modules

$$\begin{tikzcd}
\operatorname{proj} A \arrow[r, hook] \arrow[d, hook] & K^b(\operatorname{proj} A) \arrow[d, hook] \\
\operatorname{mod} A \arrow[r, hook] & \mathcal{D}^b(\operatorname{mod} A) \\
\operatorname{inj} A \arrow[r, hook] \arrow[u, hook] & K^b(\operatorname{inj} A) \arrow[u, hook]
\end{tikzcd}$$

$K^b(\operatorname{proj} A)$: (triangulated) bounded homotopy category of $\operatorname{proj} A$

$\mathcal{D}^b(\operatorname{mod} A)$: (triangulated) bounded derived category of $\operatorname{mod} A$

$K^b(\operatorname{inj} A)$: (triangulated) bounded homotopy category of $\operatorname{proj} A$

Rmk All of the above categories are additive, idempotent-complete and have finite-dimensional Hom-spaces. Consequently, the Krull-Remak-Schmidt Theorem holds in each of them (see e.g. Krause 2015)

Nakayama functor

$$D := \operatorname{Hom}_\Bbbk(-,\Bbbk) \rightsquigarrow \nu := -\otimes_A DA : \operatorname{proj} A \overset{\sim}{\longleftrightarrow} \operatorname{inj} A : \operatorname{Hom}_A(DA,-) =: \nu^-$$

$$\begin{tikzcd}
\operatorname{proj} A \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook] & \operatorname{inj} A \arrow[d, hook] \\
\nu : K^b(\operatorname{proj} A) \arrow[r, leftrightarrow, "\sim"] & K^b(\operatorname{inj} A) : \nu^-
\end{tikzcd}$$

In general, $\forall X \in \mathcal{D}^b(\operatorname{mod} A)$ $\forall P \in K^b(\operatorname{proj} A)$ $\operatorname{Hom}_A(X, \nu P) \cong D\operatorname{Hom}_A(P, X)$

Rmk A : selfinjective ($\operatorname{proj} A = \operatorname{inj} A$) $\rightsquigarrow \nu : \operatorname{proj} A \overset{\sim}{\longleftrightarrow} \operatorname{proj} A : \nu^-$

Rmk A : Iwanaga-Gorenstein ($K^b(\operatorname{proj} A) = K^b(\operatorname{inj} A)$) $\rightsquigarrow \nu : K^b(\operatorname{proj} A) \overset{\sim}{\longleftrightarrow} K^b(\operatorname{proj} A)$

$\Rightarrow$ A: Iwanaga-Gorenstein

Rmk gl.dim $A < \infty$ $\rightsquigarrow$ $K^b(\operatorname{proj} A) \xhookrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} A) \xhookleftarrow{\sim} K^b(\operatorname{inj} A)$

derived Nakayama functor

$\rightsquigarrow$ $\mathbb{S} := - \overset{\mathbb{L}}{\underset{A}{\otimes}} DA : \mathcal{D}^b(\operatorname{mod} A) \overset{\sim}{\longleftrightarrow} \mathcal{D}^b(\operatorname{mod} A) : \mathbb{R}\mathrm{Hom}_A(DA, -) =: \mathbb{S}^-$

Def (Bondal-Kapranov 1989) $\mathcal{T}$: triangulated cat. with fin.-dim. Hom-spaces

$\mathbb{S} : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ (exact) is a Serre functor if $\forall X, Y \in \mathcal{T}$ $\mathcal{T}(Y, \mathbb{S}X) \cong D\mathcal{T}(X, Y)$

Serre duality

Ex (Serre) $\mathbb{X}$: smooth projective variety & $\omega_{\mathbb{X}}$: canonical bundle

$\rightsquigarrow$ $- \overset{\mathbb{L}}{\underset{\mathcal{O}_{\mathbb{X}}}{\otimes}} \omega_{\mathbb{X}} [\dim \mathbb{X}] : \mathcal{D}^b(\operatorname{coh} \mathbb{X}) \xrightarrow{\sim} \mathcal{D}^b(\operatorname{coh} \mathbb{X})$ is a Serre functor

Ex A : Iwanaga-Gorenstein $\rightsquigarrow$ $\nu : K^b(\operatorname{proj} A) \xrightarrow{\sim} K^b(\operatorname{proj} A)$ is a Serre functor

Ex A: self-injective $\rightsquigarrow$ $\nu \circ \Omega : \underline{\operatorname{mod}} A \xrightarrow{\sim} \underline{\operatorname{mod}} A$ Serre functor

$\mathcal{T}$: triangulated cat. with split idempotents and fin.-dim. Hom-spaces
($\Rightarrow$ Krull-Remak-Schmidt Thm holds in $\mathcal{T}$)

$X \in \mathcal{T}$: indecomposable $\rightsquigarrow$ $\mathcal{T}(X, X)$: fin. dim. local algebra

$\delta \in \mathcal{T}(X, \mathbb{S}X) \cong D\mathcal{T}(X, X)$ non-zero element of simple socle

$:= \mathbb{S}X[-1]$: derived Auslander-Reiten translate

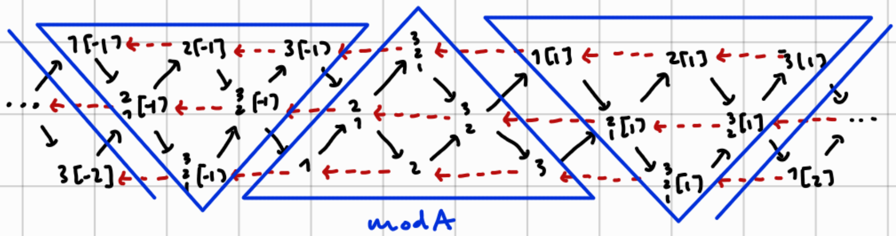
$\rightsquigarrow$ $\mathbb{S}_1 X \to E \to X \xrightarrow{\delta} \mathbb{S}X$ Auslander-Reiten triangle (Happel 1987)

Ex $A = \Bbbk(1 \to 2 \to 3)$

$$\begin{tikzcd} & Y \arrow[dr] & \\ \mathbb{S}_1 X \arrow[ur] \arrow[dr] & & X \arrow[ll, dashed] \\ & Z \arrow[ur] & \end{tikzcd}$$

$\rightsquigarrow$ $\mathbb{S}_1 X \to E \oplus F \to X \xrightarrow{\delta} \mathbb{S}X$: AR triangle

$\mathcal{D}^b(\operatorname{mod} A)$:



Auslander-Reiten quiver

Ex $A = \Bbbk(1 \rightrightarrows 2) \rightsquigarrow \mathcal{D}^b(\operatorname{mod} A) \overset{\text{(Beilinson 1978)}}{\simeq} \mathcal{D}^b(\operatorname{coh}\mathbb{P}^1)$ $\quad P_1 \mapsto \mathcal{O}$, $P_2 \mapsto \mathcal{O}(1)$ $\quad \operatorname{End}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1)) \cong A$

$\mathcal{P}$: preprojective component

$\mathcal{D}^b(\operatorname{mod}\Lambda): \cdots$  $\cdots$

Thm (Reiten - Van den Bergh 2002)

$\mathcal{T}$: triangulated cat. with split idempotents and fin.-dim. Hom-spaces

(1) $\exists\, \mathbb{S}: \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ Serre functor

$\Updownarrow$

(2) $\mathcal{T}$ admits Auslander-Reiten triangles

Suppose $\operatorname{gl.dim} A < \infty \rightsquigarrow \mathbb{S} = - \overset{\mathbb{L}}{\underset{A}{\otimes}} DA : \mathcal{D}^b(\operatorname{mod} A) \overset{\sim}{\longleftrightarrow} \mathcal{D}^b(\operatorname{mod} A) : \mathbb{R}\operatorname{Hom}_A(DA, -) =: \mathbb{S}^-$

By def, $\forall M \in \operatorname{mod} A$ $\quad H^*(\mathbb{S}X) \cong \operatorname{Tor}^A_*(M, DA)$ & $\boxed{H^*(\mathbb{S}^- M) \cong \operatorname{Ext}^*_A(DA, M)}$

Standing assumption until next section!

$\boxed{\text{Suppose } \operatorname{gl.dim} A \leq 1} \rightsquigarrow \forall\, X \in \mathcal{D}^b(\operatorname{mod} A)$: $X \simeq H^*(X)$ in $\mathcal{D}^b(\operatorname{mod} A)$

Exercise $\forall M \in \operatorname{mod} A$: indecomposable $\quad \mathbb{S}^- M \cong \begin{cases} \operatorname{Hom}_A(DA, M) & \text{if } M \text{ is injective} \\ \operatorname{Ext}^1_A(DA, M) & \text{if } M \text{ is \underline{not} injective} \end{cases}$

$M \in \operatorname{mod} A$: indecomposable $\rightsquigarrow 0 \to M \to I^0 \to I^1 \to 0$ min. injective coresolution

$$0 \to \operatorname{Hom}_A(DA, M) \to \operatorname{Hom}_A(DA, I^0) \to \operatorname{Hom}_A(DA, I^1) \to \operatorname{Ext}^1_A(DA, M) \to \cancel{\operatorname{Ext}^1_A(DA, I^0)}^{\,0}$$

$$\rightsquigarrow\ 0 \to \nu^-(M) \to \nu^-(I^0) \to \nu^-(I^1) \to \operatorname{Ext}^1_A(DA, M) \to 0$$

by def of τ^-

$\tau^- \cong \operatorname{Ext}^1_A(DA, -) : \operatorname{mod} A \longrightarrow \operatorname{mod} A$ $\quad$ inverse Auslander-Reiten translation

Exercise $\operatorname{Ext}^1_A(DA, -) \cong - \underset{A}{\otimes} \operatorname{Ext}^1_A(DA, A)$ on $\operatorname{Mod} A$

Def (Baer-Geigle-Lenzing 1987)

tensor algebra of Λ-bimodule

The preprojective algebra of A is $\Pi(A) := T_A \operatorname{Ext}^1_A(DA, A) \cong \bigoplus_{i \geq 0} \operatorname{Hom}_A(A, \tau^{-i}(A))$

Exercise

Rmk $\Pi(A)_A \cong \underset{\text{deg: } 0}{A} \oplus \underset{1}{\tau^{-}A} \oplus \underset{2}{\tau^{-2}A} \oplus \underset{3}{\tau^{-3}A} \oplus \underset{\cdots}{\cdots}$

$$\boxed{\dim_{\Bbbk} \Pi(A) < \infty \iff A \text{ is of finite representation type}}$$

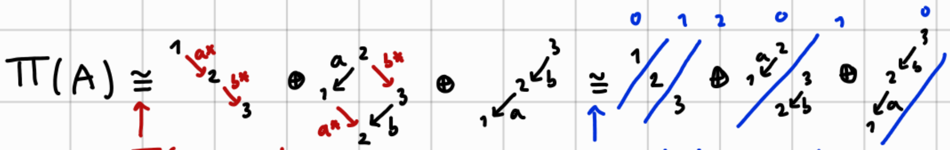
(Gelfand-Ponomarev 1979, Ringel 1998, Crawley-Boevey 1999) Q: finite acyclic quiver

$\Pi(\Bbbk Q) \cong \Bbbk \bar{Q}/I$ where $\bar{Q}$: double of Q & $I = \langle \sum_{a \in Q_1} [a, a^*] \rangle$

$\bar{Q}_0 = Q_0$, $\bar{Q}_1 = Q_1 \cup Q_1^*$

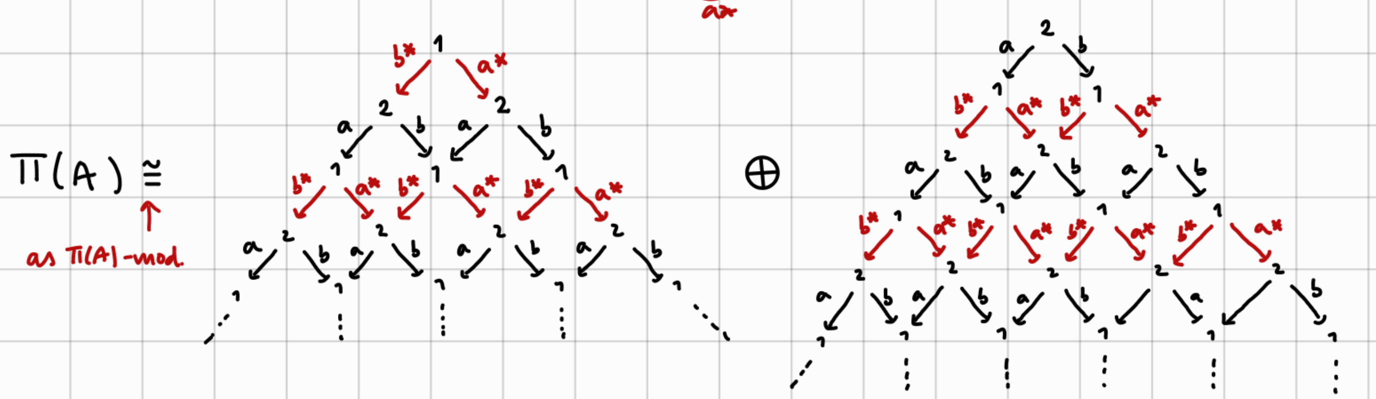
corresponds to "mesh relations" in AR quiver of mod $\Bbbk Q$

Ex $A = \Bbbk(1 \xrightarrow{a} 2 \xrightarrow{b} 3) \rightsquigarrow \Pi(A) \cong \Bbbk(1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2 \underset{b^*}{\overset{b}{\rightleftarrows}} 3) / \langle a^*a, aa^* - b^*b, bb^* \rangle$

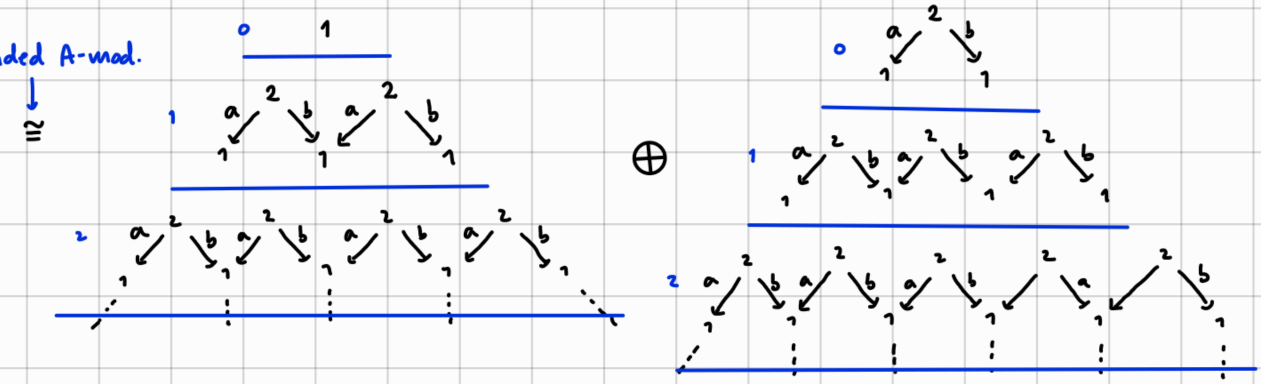


as $\Pi(A)$-mod. as graded A-mod.

Ex $A = \Bbbk(1 \overset{a}{\underset{b}{\rightrightarrows}} 2) \rightsquigarrow \Pi(A) \cong \Bbbk(1 \rightrightarrows 2 \text{ with } a, b, a^*, b^*) / \langle aa^* - b^*b, a^*a - bb^* \rangle$



as $\Pi(A)$-mod



as graded A-mod.

Def (Kontsevich 1998)

$\mathcal{T}$: triangulated cat. with split idempotents and fin.-dim. Hom-spaces

For $n \in \mathbb{Z}$, $\mathcal{T}$ is n-Calabi-Yau if $[n] : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ is a Serre functor

Ex $\mathbb{X}$: smooth projective variety s.t. $\omega_{\mathbb{X}} \cong \mathcal{O}_{\mathbb{X}}$ $\rightsquigarrow$ $\mathbb{S} = - \overset{\mathbb{L}}{\underset{\mathbb{X}}{\otimes}} \omega_{\mathbb{X}} [\dim \mathbb{X}] \cong [\dim \mathbb{X}]$

$\rightsquigarrow$ $\mathcal{D}^b(\operatorname{coh} \mathbb{X})$ is a $(\dim \mathbb{X})$-Calabi-Yau triangulated category

$A \cong DA$ as A-bimods.

$\nu = - \otimes_A DA \cong \mathbb{1} = [0]$

Ex A: symmetric algebra $\rightsquigarrow$ $K^b(\operatorname{proj} A)$ is a 0-Calabi-Yau triangulated category

Ex (Erdmann-Skowroński 2006) A: self-injective algebra

$\rightsquigarrow$ $\underline{\operatorname{mod}} A$ is n-CY $\Leftrightarrow$ $\nu\Omega \cong \Omega^{-n}$ $\Leftrightarrow$ $\Omega^{n+1} \cong \nu^-$ (as functors on $\underline{\operatorname{mod}} A$)

Serre functor of $\underline{\operatorname{mod}} A$; n-fold suspension of $\underline{\operatorname{mod}} A$

Thm Suppose that A is of infinite representation type

(Baer-Geigle-Lenzing 1987) $\operatorname{gl.dim} \Pi(A) = 2$

$A = kQ$

(Crawley-Boevey 1998) $\mathcal{D}^{fd}(\Pi(A)) = \{ X \in \mathcal{D}(\operatorname{Mod} \Pi(A)) \mid \sum_{i \in \mathbb{Z}} \dim_k H^i(X) < \infty \}$

is a 2-Calabi-Yau triangulated category

$\Rightarrow A$: selfinjective

Def A is twisted m-periodic w.r.t. $\sigma : A \xrightarrow{\sim} A$ if $\Omega^m_{A^e}(A) \simeq {}_1A_\sigma$ in $\underline{\operatorname{mod}} A^e$

$A \otimes A^{op}$

$\Leftrightarrow$ $\Omega^m \cong \sigma^*$ on $\underline{\operatorname{mod}} A$

k: perfect

Thm Suppose that A is of finite representation type

$\mathcal{D}(Y, X[d]) \cong D\mathcal{D}(X, Y)$

$\cong D\mathcal{D}^{op}(Y, X)$

(Ringel-Schofield, Brenner-Butler-King 2002)

$\Pi(A)$ is twisted 3-periodic w.r.t. $\tilde{\nu}^- : \Pi(A) \xrightarrow{\sim} \Pi(A)$

the inverse Nakayama automorphism $(D\Pi(A) \cong \Pi(A)_{\tilde{\nu}})$

$\Omega^3 \cong (\tilde{\nu}^-)^* \cong \nu^-$

(Auslander-Reiten 1996) The stable category $\underline{\operatorname{mod}}\, \Pi(A)$ is 2-Calabi-Yau

$\boxed{\text{II Preprojective algebras of } d\text{-hereditary algebras}}$

Fix $d \geq 1$ an integer

Standing assumption until next section!

$\boxed{A : \text{finite-dimensional algebra s.t. gl.dim } \Lambda \leq d}$ (basic and connected)

(Iyama) $\mathbb{S}_d := \mathbb{S}[-d] = - \overset{\mathbb{L}}{\underset{A}{\otimes}} DA[-d] : \mathcal{D}^b(\mathrm{mod} A) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{mod} A)$

d-Auslander-Reiten translation ⇝ higher AR theory

$\tau_d := H^0(\mathbb{S}_d(-)) : \mathrm{mod} A \rightleftarrows \mathrm{mod} A : H^0(\mathbb{S}_d^-(-)) =: \tau_d^-$

$$\tau_d^- M = H^0(\underbrace{\mathbb{R}\mathrm{Hom}_A(DA, M)[d]}_{\mathbb{S}^-(M)}) = H^d(\mathbb{R}\mathrm{Hom}_A(DA, M)) \cong \mathrm{Ext}_A^d(DA, M)$$

$$\mathcal{D}^{d\mathbb{Z}}(\mathrm{mod} A) := \{ X \in \mathcal{D}^b(\mathrm{mod} A) \mid H^\ell(X) \neq 0 \Rightarrow \ell \in d\mathbb{Z} \} = \bigvee_{i \in \mathbb{Z}} (\mathrm{mod} A)[di]$$

$\underset{X}{\overset{\in}{}}\ \Longrightarrow\ X \simeq H^*(X)$ in $\mathcal{D}^b(\mathrm{mod} A)$

Rmk $d = 1$ ⇝ $\tau_1^{\pm} = \tau^{\pm}$ & $\mathcal{D}^{1\mathbb{Z}}(\mathrm{mod} A) = \mathcal{D}^b(\mathrm{mod} A)$

Def (Herschend - Iyama - Oppermann 2014)

(1) A is d-hereditary if $\forall i \geq 0$ $\mathbb{S}_d^{-i}(A) \in \mathcal{D}^{d\mathbb{Z}}(\mathrm{mod} A)$

(2) A is d-hereditary d-representation-finite (d-RF) if $\forall P \in \mathrm{proj} A$: indecomposable $\exists i \geq 0$ s.t. $\mathbb{S}_d^{-i}(A) \in \mathrm{inj} A$

(3) A is d-hereditary d-representation infinite (d-RI) if $\forall i \geq 0$ $\mathbb{S}_d^{-i}(A) \in \mathrm{mod} \Lambda$

Rmk $d = 1$ ⇝ d-hereditary d-RF = hereditary of finite rep. type

d-hereditary d-RI = hereditary of infinite rep. type

Exercise A : d-hereditary $\Rightarrow$ gl.dim $A = 0$ or gl.dim $A = d$

Exercise A : d-hereditary d-RF & d-RI $\Rightarrow$ $A = 0$

Thm (Herschend - Iyama - Oppermann 2014)

A : d-hereditary $\Rightarrow$ A is d-hereditary d-RF or d-hereditary d-RI

Def (Iyama 2007)

$M \in \operatorname{mod} A$ is d-cluster tilting if TFAE for all $X \in \operatorname{mod} A$

d-CT

(1) $X \in \operatorname{add} M = \{\text{direct summands of } M^{\oplus n},\ n \geq 0\}$

(2) $\forall\ 0<i<d \quad \operatorname{Ext}^i_A(X, M) = 0$

(3) $\forall\ 0<i<d \quad \operatorname{Ext}^i_A(M, X) = 0$

(Iyama - Yoshino 2008) d-cluster tilting in Hom-finite tri. cat's (same def)

$c \in \mathcal{T}$: d-cluster tilting $\Rightarrow$ $\operatorname{thick}(c) = \mathcal{T}$

Rmk M is 1-CT $\iff$ $\operatorname{add} M = \operatorname{mod} A$ (hence A is of finite rep. type)

Prop (Herschend - Iyama - Oppermann 2014)

(1) A is d-hereditary d-RF

$\Updownarrow$

finite direct sum!

(2) $\boxed{M(A) := \bigoplus_{i \geq 0} \tau_d^{-i} A} \in \operatorname{mod} A$ is d-cluster tilting (set $\boxed{\mathcal{M}(A) := \operatorname{add} M}$)

d-abelian cat (J 2006)

Thm (Iyama 2011) A : d-hereditary d-RF. TFSH

(1) $\mathcal{U}(A) := \operatorname{add}\{\mathbb{S}_d^{-i}(A) \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\operatorname{mod} A)$ is d-cluster tilting

(2) $\mathcal{U}(A) = \{X \in \mathcal{D}^{d\mathbb{Z}}(\operatorname{mod} A) \mid \forall i \in \mathbb{Z}\ \ H^{di}(X) \in \mathcal{M}(A)\}$ $\circlearrowright [\pm d]$

Rmk $d = 1 \rightsquigarrow \mathcal{U}(A) = \mathcal{D}^b(\operatorname{mod} A)$ is Happel's description of the derived cat.

more on these later

Thm (Geiss - Keller - Oppermann 2013) $\mathcal{U}(A)$ is a (d+2)-angulated category

Def (Auslander 1971 $d=1$, Iyama 2007) A : d-hereditary d-RF.

The **(d+1)-Auslander algebra** of A is $\mathrm{Aus}_{d+1}(A) := \mathrm{End}_A(M(A)) = \mathrm{End}_A\left(\bigoplus_{i\geq 0} \tau_d^{-i} A\right)$

$\mathrm{gl.dim}\, \mathrm{Aus}_{d+1}(A) = 0, d+1$

Def (Iyama-Oppermann 2011, Herschend-Iyama-Oppermann 2014)

The **(d+1)-preprojective algebra** of A is $\Pi_{d+1}(A) := T_A \mathrm{Ext}^d_A(DA, A)$

Rmk $\Pi_{d+1}(A) \cong \bigoplus_{i\geq 0} \mathrm{Hom}_A(A, \tau_d^{-i} A)$ & $\Pi_{d+1}(A)_A \cong \underset{\deg:\ 0}{A} \oplus \underset{1}{\tau_d^{-1} A} \oplus \underset{2}{\tau_d^{-2} A} \oplus \cdots$

Thm (Grant-Iyama 2020)

$$A : d\text{-hereditary } d\text{-RI} \implies \mathrm{gl.dim}\, \Pi_{d+1}(A) = d+1$$

Thm (Keller 2011, Herschend-Iyama-Oppermann 2014) A : d-hereditary d-RI

$\mathcal{D}^{fd}(\Pi_{d+1}(A))$ is a $(d+1)$-Calabi-Yau triangulated category.

Thm (Buchweitz-Hille) $\mathbb{X}$: smooth projective variety of $\dim \mathbb{X} = d$

$\mathrm{thick}(T) = \mathcal{D}^b(\mathrm{coh}\,\mathbb{X})$ & $\mathrm{Ext}^{>0}_{\mathbb{X}}(T,T) = 0$

$T \in \mathrm{coh}\,\mathbb{X}$: tilting & $A = \mathrm{End}_{\mathbb{X}}(T)$ s.t. $\mathrm{gl.dim}\, A = d \implies A$ is d-hereditary d-RI

Ex (Beilinson 1978, Herschend-Iyama-Oppermann 2014)

$$T = \bigoplus_{i=0}^{d} \mathcal{O}(i) \in \mathrm{coh}\,\mathbb{P}^d : \text{tilting} \quad \& \quad \underset{\substack{\uparrow \\ d\text{-hereditary } d\text{-RI}}}{\mathrm{End}(T)} \cong \Bbbk\left(0 \underset{x_d}{\overset{x_1}{\rightrightarrows}} 1 \cdots \underset{x_d}{\overset{x_0, x_1}{\rightrightarrows}} d\right)^{x_0} \Big/ \langle [x_i, x_j] \rangle$$

Thm (Iyama-Oppermann 2013)

A is d-hereditary d-RF $\iff$ $\Pi_{d+1}(A)$ is fin.-dim selfinj. & A is "vosnex" (LATER; empty cond. for $d=1,2$)

Thm A : d-hereditary d-RF. TFSH

(Dugas 2012) $\Pi_{d+1}(A)$ is twisted $(d+2)$-periodic w.r.t. $\tilde{\nu}^{-}: \Pi_{d+1}(A) \xrightarrow{\sim} \Pi_{d+1}(A)$

(Geiß-Leclerc-Schröer 2006, Iyama-Oppermann 2013)
The stable category $\underline{\mathrm{mod}}\, \Pi_{d+1}(A)$ is $(d+1)$-Calabi-Yau and admits a $(d+1)$-cluster tilting object

Recall (Q, W): $\underbrace{\text{quiver with potential}}_{\text{QP}}$ $\rightsquigarrow$ Q : finite quiver, W : possibly infinite lin comb. of cycles

$\rightsquigarrow$ $\mathrm{Jac}(Q,W) := \widehat{kQ} / \overline{\langle \partial_a(W) \mid a \in Q_1 \rangle}$ is its **Jacobian algebra**

(∂_a: cyclic derivative)

Thm (Herschend-Iyama 2011) $k = \bar{k}$

(Q,W) is selfinjective :=

(Q, W): QP s.t. $\mathrm{Jac}(Q,W)$ is fin. dim. & self-injective & $C \subseteq Q_1$: a "cut"

(1) $A = \mathrm{Jac}(Q,W)_C := \mathrm{Jac}(Q,W) / \overline{\langle a \mid a \in C \rangle}$ is 2-hereditary 2-RF and $\Pi_3(A) \cong \underbrace{\mathrm{Jac}(Q,W)}$ as graded algebras

$\deg a = 0$ if $a \notin C$, $\deg a = 1$ if $a \in C$

(2) Every 2-hereditary 2-RF algebra arises in this way

(Herschend-Iyama 2011) Selfinjective planar QP's

(Pasquali 2020) Selfinjective planar QP's from Postnikov diagrams in the disk

(Donovan-Wemyss 201?) Selfinjective QP's from crepant res'ns of compound Du Val sing.

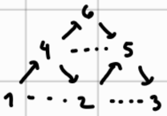
Rmk If $\underbrace{A = kQ/I}_{\mathrm{gl.dim}\, A \leq 2}$, then $\Pi_3(A)$ can be described by QP's (Keller 2011).

as a "relation completion"

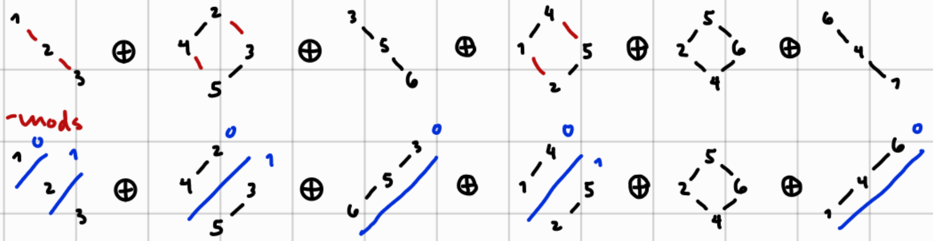
G. JASSO @ NORDFJORDEID / AUGUST 2023

Ex (Iyama 2011)

$\tilde{\nu}$ = rotation by $-\frac{2\pi}{3}$

A =  is 2-hereditary 2-RF & $\Pi_3(A) \cong$  $W = \sum$ ↗↘ − ↙↗

$= A_{4,2}$

$\Pi_3(A) \cong$ 

as $\Pi_3(A)$-mods

as graded A-mods

$M(A)$: 

 : fully commutative cube (complete with 0's if necessary)

$\cong \$X[-2]$ $\qquad\qquad\qquad\qquad$ $\cong \$X$

$\tau_2 X \to \cdot \to \cdot \to X \to \tau_2 X[-2]$

Auslander-Reiten 4-angle

$0 \leftarrow 1 \leftarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \leftarrow \begin{smallmatrix}&2&\\4&&3\\&5&\end{smallmatrix} \leftarrow \begin{smallmatrix}&4&\\1&&5\\&2&\end{smallmatrix} \leftarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \leftarrow 3 \leftarrow 0 \qquad \Omega^4_\Lambda(S_1) \cong S_3 = S_{\tilde{\nu}^{-}(1)}$

$0 \leftarrow 2 \leftarrow \begin{smallmatrix}&2&\\4&&3\\&5&\end{smallmatrix} \leftarrow \begin{smallmatrix}&4&\\1&&5\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \leftarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\2&&6\\&4&\end{smallmatrix} \leftarrow \begin{smallmatrix}&2&\\3&&4\\&5&\end{smallmatrix} \leftarrow 5 \leftarrow 0 \qquad \Omega^4_\Lambda(S_2) \cong S_5 = S_{\tilde{\nu}^{-}(2)}$

Green-Snashall-Solberg 2003 + Hanihara 2020

$\rightsquigarrow \forall i: \ \Omega^4_\Lambda(S_i) \cong S_{\tilde{\nu}^{-}(i)}$ $\quad$ equivalent condition for twisted 4-periodicity

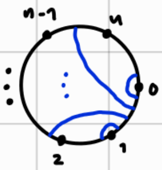
(uses k: perfect field)

Thm (Iyama 2011)

There exists a family $\{A_{n,d} \mid n \geq, d \geq 1\}$ of basic finite-dim. algebras s.t.

- $A_{n,1} = k(1 \to 2 \to \cdots \to n)$ & $A_{d,d} \cong k$
- $A_{n,d}$ is d-hereditary d-RF & $K_0(A_{n,d}) \cong \mathbb{Z}^{\binom{n}{d}}$
- $\mathrm{Aus}_{d+1}(A_{n,d}) \cong A_{n+1,d+1}$
- $A_{n,d}$ is Koszul and $A^!_{n,d} \cong A_{n,n-d}$

(Dyckerhoff - J - Lekili 2021)

- $\mathcal{D}^b(\mathrm{mod}\, \Lambda_{n,d}) \xrightarrow{\sim}$ Partially wrapped Fukaya category of :  2-dim disk with n+1 stops on its boundary

III Derived preprojective algebras and Calabi-Yau structures

A: differential graded algebra = cochain cpx. with compatible algebra structure
$= \mathrm{dg}$

$\mathcal{D}(A)$: (triangulated) unbounded derived cat. of dg A-modules

Key properties $\mathcal{D}(A)$ admits small coproducts

$\forall i \in \mathbb{Z}\quad \mathrm{Hom}_A(A, X[i]) = 0 \Rightarrow X = 0$

$A \in \mathcal{D}^c(A) := \{X \in \mathcal{D}(A) \mid \mathrm{Hom}_A(X,-)$ preserves small coproducts$\}$ — X is compact

perfect derived cat. $\quad \stackrel{\textcircled{=}}{}\ \mathrm{thick}(A)$ — Neemann (1992)

Ex A: ordinary algebra $\rightsquigarrow \mathcal{D}(A) = \mathcal{D}(\mathrm{Mod}\,A)$ & $\mathcal{D}^c(A) \simeq K^b(\mathrm{proj}\,A)$

Thm (Keller 1994) $\mathcal{T}$: triangulated cat. with small coproducts. TFAE

(1) $\exists A$: dg algebra s.t. $\mathcal{T} \simeq \mathcal{D}(A)$

(2) $\exists G \in \mathcal{T}$: compact object st. $\forall X \in \mathcal{T}\ (\forall i \in \mathbb{Z}\ \mathcal{T}(G, X[i]) = 0 \Rightarrow X = 0)$

Def (Kontsevich) A is homologically smooth if $A \in \mathcal{D}^c(A^e)$, $\quad A^e = A \otimes A^{op}$

Ex A: fin. dim. algebra (A is homologically smooth $\Leftrightarrow$ gl.dim $A < \infty$) — $\Leftarrow$ uses $\Bbbk$: perfect

$\Omega_A = A^\vee := \mathbb{R}\mathrm{Hom}_{A^e}(A, A^e) \in \mathcal{D}(A^e)$: inverse dualising dg A-bimodule

Fact A: homologically smooth $\rightsquigarrow$ Adjunctions

$$A \overset{\mathbb{L}}{\underset{\Bbbk}{\otimes}} - : \mathcal{D}(\Bbbk^{op}) \rightleftarrows \mathcal{D}((A^e)^{op}) : A^\vee \overset{\mathbb{L}}{\underset{A^e}{\otimes}} - \quad \& \quad - \overset{\mathbb{L}}{\underset{\Bbbk}{\otimes}} A^\vee : \mathcal{D}(\Bbbk) \rightleftarrows \mathcal{D}(A^e) : - \overset{\mathbb{L}}{\underset{A^e}{\otimes}} A$$

Lemma (Keller 2008) A: homologically smooth

$$\forall X \in \mathcal{D}^{fd}(A)\ \forall Y \in \mathcal{D}(A), \quad D\mathrm{Hom}_A(X, Y) \cong \mathrm{Hom}_A(Y \overset{\mathbb{L}}{\underset{A}{\otimes}} \Omega_A, X)$$

Def (Keller 2011) A: homologically smooth & $n\in\mathbb{Z}$

The <u>n-Calabi-Yau completion</u> of A is $\Pi_n(A) = \mathbb{L}T_A(\Omega_A[n-1])$ ← derived tensor dg alg.

quasi-iso & dg Morita invariant

Def (Ginzburg 2006) A: homologically smooth & $n\in\mathbb{Z}$

A is <u>n-Calabi-Yau as dg bimodule</u> if $\Omega_A[n]\simeq A$ in $\mathcal{D}(A^e)$

Prop (Keller 2008) A: n-CY as dg bimod $\Rightarrow$ $\mathcal{D}^{fd}(A)$ is an n-CY triangulated cat.

Def The <u>Hochschild homology</u> of A is $HH_*(A) := H^{-*}(A\overset{\mathbb{L}}{\otimes}_{A^e}A)$. Write $HH(A) := A\overset{\mathbb{L}}{\otimes}_{A^e}A$

derived cat. of <u>mixed complexes</u>

(Connes) $HH(A)$ admits an action of $\mathbb{k}[\varepsilon]$, $|\varepsilon| = -1$ $\rightsquigarrow$ $HH(A)\in\mathcal{D}(\mathbb{k}[\varepsilon])$

algebraic avatar of S^1-action

Def $HN_*(A) := H^{-*}(\mathbb{R}\mathrm{Hom}_{\mathbb{k}[\varepsilon]}(\mathbb{k}, HH(A)))$ is the <u>negative cyclic homology</u> of A

$HN(A) :=$

Augmentation $\mathbb{k}[\varepsilon]\to\mathbb{k}$ induces a canonical morphism $HN(A)\to HH(A)$ in $\mathcal{D}(\mathbb{k})$

Fact A: homologically smooth $\rightsquigarrow$ $A\overset{\mathbb{L}}{\otimes}_{A^e}A \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{A^e}(\Omega_A, A)$

$\rightsquigarrow$ can. map $HN_n(A)\to HH_n(A) = H^{-n}(A\overset{\mathbb{L}}{\otimes}_{A^e}A) \cong H^{-n}(\mathbb{R}\mathrm{Hom}_{A^e}(\Omega_A, A))$

$\tilde{\varphi}\longmapsto\varphi$

$\cong H^0(\mathbb{R}\mathrm{Hom}_{A^e}(\Omega_A, A)[-n]) \cong \mathrm{Hom}_{A^e}(\Omega_A[n], A)$

Def (Kontsevich-Vlassopoulos 2013) A: homologically smooth & $n\in\mathbb{Z}$

A <u>left n-CY structure</u> on A is a class $\tilde{\varphi}\in HN_n(A)$ s.t. $\varphi : \Omega_A[n]\xrightarrow{\sim} A$

Thm (Keller 2011) A: homologically smooth & $n\in\mathbb{Z}$

$\Pi_n(A)$ is homologically smooth & admits a canonical exact (see next page) left n-CY structure

Example $\Pi_n(\mathbb{k}) \underset{\text{qiso}}{\simeq} \mathbb{k}[x]$, $|x| = 1-n$, has a canonical left n-CY structure

(Brav-Dyckerhoff 2021) Left n-CY structures $\rightsquigarrow$ $(2-n)$-shifted symplectic structures

(Pantev-Toën-Vaquié-Vezzosi 2013)

The apparent triangle $k[1] \to k[\varepsilon] \to k \to k[2]$ in $\mathcal{D}(k[\varepsilon])$ induces, after applying the functor $A \overset{\mathbb{L}}{\underset{k[\varepsilon]}{\otimes}} - : \mathcal{D}(k[\varepsilon]) \longrightarrow \mathcal{D}(k)$, a triangle $HC(A)[1] \xrightarrow{B} HH(A) \longrightarrow HC(A) \longrightarrow HC(A)[2]$ in $\mathcal{D}(k)$ where the map B factors as $HC(A)[1] \xrightarrow{B} HN(A) \xrightarrow{\text{can}} HH(A)$, see also Hoyois (2015) for top. interpretation

Def (Van den Bergh 2015) A left n-CY structure $\tilde{\varphi} \in HN_n(A)$ on A is exact if it lies in the image of Conne's map $B : HC_{n-1}(A) \longrightarrow HN_n(A)$.

Rmk The Hochschild-Kostant-Rosenberg isomorphism $\Omega^*(A/k) \cong HH_*(A)$ for a smooth commutative algebra essentially of finite type over k suggests to interpret $HH_*(A)$ as a non-commutative analogue of differential forms. Similarly, $HN_*(A)$ gives analogues of closed forms. From this perspective, a CY structure is analogous to a non-degenerate closed form.

Write $\Pi = \Pi_n(A) = \mathbb{L}T_A(A^\vee[n-1])$ for Keller's n-CY completion of A.

A: smooth $\rightsquigarrow$ $\mathbb{R}\mathrm{Hom}_{A^e}(A, A)[n-1] \xleftrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{A^e}(A, A[n-1]) \xrightarrow{\sim} A^\vee[n-1] \overset{\mathbb{L}}{\underset{A^e}{\otimes}} A$

$\therefore$ $\mathrm{id}_A \in \mathbb{R}\mathrm{Hom}_{A^e}(A, A)$ induces a canonical class $c \in A^\vee[n-1] \overset{\mathbb{L}}{\underset{A^e}{\otimes}} A$

Fact (Keller 2018) The class c induces a canonical class $\bar{c} \in HC_{n-1}(\Pi)$

The canonical (exact) left n-CY structure on Π is $\tilde{\varphi} := B\bar{c} \in HN_n(\Pi)$ where $B : HC_{n-1}(\Pi) \longrightarrow HN_n(\Pi)$ is Connes' map.

Thm (Bozec - Calaque - Scherotzke 2023) A: dg algebra of "finite type" & $\underline{\mathrm{perf}}_A$: ∞-stack of perfect objects (Toën-Vaquié (2007))

(2-n)-shifted co-tangent stack

There is an equivalence $T^*[2-n](\underline{\mathrm{perf}}_A) \simeq \underline{\mathrm{perf}}_{\Pi_n(A)}$ of "(2-n)-shifted symplectic ∞-stacks"

Def The cyclic homology of A is $HC_*(A) := H^{-*}(HH(A) \overset{\mathbb{L}}{\underset{k[\varepsilon]}{\otimes}} k)$

Rmk Augmentation $k[\varepsilon] \to k$ induces map $HH(A) \longrightarrow HC(A)$ in $\mathcal{D}(k)$

$DA := \mathbb{R}\mathrm{Hom}_k(A,k) \rightsquigarrow \mathbb{R}\mathrm{Hom}_k(A \overset{\mathbb{L}}{\underset{A^e}{\otimes}} A, k) \simeq \mathbb{R}\mathrm{Hom}_{A^e}(A, \mathbb{R}\mathrm{Hom}_k(A,k)) = \mathbb{R}\mathrm{Hom}_{A^e}(A, DA)$

$\rightsquigarrow$ can. map $DHC_{-n}(A) \to DHH_{-n}(A) = H^{-n}\mathbb{R}\mathrm{Hom}_{A^e}(A, DA) \cong \mathrm{Hom}_{A^e}(A[n], DA)$

$\tilde{\psi} \mapsto \psi$

Def A is proper if $\sum_{i\in\mathbb{Z}} \dim_k H^i(A) < \infty$ $\left(\Rightarrow -\overset{\mathbb{L}}{\underset{A^e}{\otimes}} A : \mathcal{D}(A^e) \rightleftarrows \mathcal{D}(k) : -\overset{\mathbb{L}}{\underset{k}{\otimes}} DA \right)$ ($\perp$)

A is componentwise proper if $\forall i \in \mathbb{Z}$ $\dim_k H^i(A) < \infty$

Def (Kontsevich - Vlassopoulos 2013) A: componentwise proper & $n \in \mathbb{Z}$

A right n-CY structure on A is a class $\tilde{\psi} \in DHC_{-n}(A)$ st. $\psi : A[n] \xrightarrow{\sim} DA$

$\Rightarrow \mathcal{D}^c(A)$: n-CY tri. cat.

Thm (Brav - Dyckerhoff 2019) A: homologically smooth & $n \in \mathbb{Z}$

A left n-CY structure on A induces a right n-CY structure on $\mathcal{D}^{fd}(A)$

Example $\mathcal{D}^{fd}(\Pi_n(k)) = \mathcal{D}^{fd}(k[x]) \simeq \mathcal{D}^c(k[\varepsilon])$ has a right n-CY structure ($|x| = 1-n$, $|\varepsilon| = n$)

Thm (Amiot 2009 [$d=2$], Guo 2015 [$d\geq 2$], Keller 2005 [crucial], Iyama - Yang 2018 [$d=1$])

Γ : homologically smooth such that

- $\forall i > 0$ $H^i(\Gamma) = 0$ (Γ is connective) & $\dim_k H^0(\Gamma) < \infty$
- Γ is $(d+1)$-Calabi-Yau as dg bimodule.

Then TFSH

(1) $\mathcal{D}^{fd}(\Gamma) \subseteq \mathcal{D}^c(\Gamma) \rightsquigarrow \mathcal{C}(\Gamma) := \mathcal{D}^c(\Gamma)/\mathcal{D}^{fd}(\Gamma)$: AGK cluster cat. of Γ

(2) $\mathcal{C}(\Gamma)$ is a d-CY triangulated category

(3) $\Gamma \in \mathcal{C}(\Gamma)$ is a d-cluster tilting object with $\mathrm{End}_{\mathcal{C}(\Gamma)}(\Gamma) \cong H^0(\Gamma)$

Thm (Liu 2023, Keller - Liu 2023) Γ as in previous theorem

A left $(d+1)$-CY structure on Γ induces a right d-CY structure on $\mathcal{C}(\Gamma)$

Def $T \in \mathcal{T}$: basic $d\mathbb{Z}$-cluster tilting $:=$ d-cluster-tilting & $T[d] \cong T$

Thm (Iyama-Oppermann 2013 + Ladkani 2016) Γ as in previous theorem

$$(\operatorname{add}\Gamma)[d] = (\operatorname{add}\Gamma) \text{ in } \mathcal{C}(\Gamma) \iff \boxed{\forall\, 0 < i < d-1 \quad H^{-i}(\Gamma) = 0}$$

vosnex condition (empty if $d = 1,2$)

$\boxed{A : \text{finite-dimensional algebra with } \operatorname{gl.dim} A \leq d}$ $\Rightarrow$ A: homologically smooth & proper

ie: perfect

Fact ($\sim$ Keller's Lemma) $-\overset{\mathbb{L}}{\underset{A}{\otimes}} \Omega_A : \mathcal{D}(A) \xleftrightarrow{\sim} \mathcal{D}(A) : -\overset{\mathbb{L}}{\underset{A}{\otimes}} DA$ & $\Omega_A \cong \mathbb{R}\mathrm{Hom}_A(DA, A)$ in $\mathcal{D}(A^e)$

$\boldsymbol{\Pi}_{d+1}(A)$: derived $(d+1)$-preprojective algebra of A

$$\boldsymbol{\Pi}_{d+1}(A)_A \cong \bigoplus_{i \geq 0} \mathbb{S}_d^{-i}(A) \rightsquigarrow H^0(\boldsymbol{\Pi}_{d+1}(A)) \cong \Pi_{d+1}(A)$$

Exercise A: d-hereditary d-RI $\Rightarrow$ $\boldsymbol{\Pi}_{d+1}(A) \xrightarrow[\text{qiso}]{\sim} \Pi_{d+1}(A)$

Corollary A: d-hereditary d-RI $\Rightarrow$ $\mathcal{D}^{fd}(\Pi_{d+1}(A))$ is a $(d+1)$-CY tri. cat.

Thm (Iyama-Oppermann 2013) A: d-hereditary d-RF

$$\boldsymbol{\Pi}_{d+1}(A) \in \mathcal{C}(\boldsymbol{\Pi}_{d+1}(A)) \text{ is } d\mathbb{Z}\text{-cluster tilting} \;\&\; H^0(\boldsymbol{\Pi}_{d+1}(\Lambda)) \cong \Pi_{d+1}(\Lambda)$$

(Geiß-Keller-Oppermann 2013)

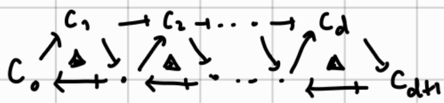
A $(d+2)$-angulated category is a triple $(\mathcal{F}, \Sigma, \pentagon)$ where

- $\mathcal{F}$ is an additive category
- $\Sigma : \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is an autoequivalence
- $\pentagon$ is a class of $(d+2)$-angles $X_0 \to X_1 \to \cdots \to X_d \to X_{d+1} \to \Sigma X_0$ in $\mathcal{F}$ that satisfies some axioms analogous to those of triang. categories

Rmk $(1+2)$-angulated categories $=$ triangulated categories

Thm (Geiß-Keller-Oppermann 2013) $\mathcal{T}$: triangulated cat. & $\mathcal{C} \subseteq \mathcal{T}$: $d\mathbb{Z}$-CT

$\Rightarrow (\mathcal{C}, [d], \pentagon)$ is a $(d+2)$-angulated category where

$\pentagon = \{$  with $C_i \in \mathcal{C} \}$

Ex A : d-hereditary d-RF

- $\mathcal{U}(A) = \operatorname{add}\{\mathbb{S}_d^i(A) \in \mathcal{D}^b(\operatorname{mod} A) \mid i \in \mathbb{Z}\} \circlearrowleft [\pm d]$ admits a $(d+2)$-angulation

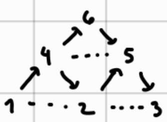
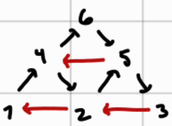
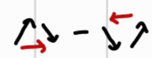
- $$\begin{array}{ccc} \mathcal{C}(\Pi_{d+1}(A)) \supseteq \operatorname{add} \Pi_{d+1}(A) & \xrightarrow{\sim} & \operatorname{proj} \Pi_{d+1}(A) \\ {\scriptstyle [d]} \downarrow & & \downarrow {\scriptstyle \nu} \\ \operatorname{add} \Pi_{d+1}(A) & \xrightarrow{\sim} & \operatorname{proj} \Pi_{d+1}(A) \end{array}$$ $\nu : \Pi_{d+1}(A) \xrightarrow{\sim} \Pi_{d+1}(A)$

  $\leftarrow$ admits a $(d+2)$-angulation

Prop (Freyd 1966 + Heller 1968 $d=1$, Geiß-Keller-Oppermann 2013)
Λ : finite-dim. alg. & $\sigma : \Lambda \xrightarrow{\sim} \Lambda \rightsquigarrow \Sigma := - \otimes_\Lambda \Lambda_\sigma : \operatorname{proj} \Lambda \xrightarrow{\sim} \operatorname{proj} \Lambda$
$(\operatorname{proj} \Lambda, \Sigma)$ admits a $(d+2)$-angulation $\Rightarrow$ Λ is twisted $(d+2)$-periodic w.r.t. σ

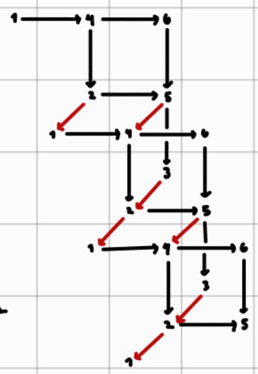
Cor A : d-hereditary d-RF $\Rightarrow$ $\Pi_{d+1}(A)$ is twisted $(d+2)$-periodic w.r.t. $\tilde{\nu}^{-} : \Pi_{d+1}(A) \xrightarrow{\sim} \Pi_{d+1}(A)$

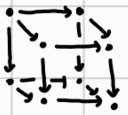
Cor A : d-hereditary d-RF $\Rightarrow$ The stable category $\underline{\operatorname{mod}}\, \Pi_{d+1}(A)$ is $(d+1)$-CY

Ex $A =$  $\rightsquigarrow \Lambda := \Pi_3(A) \cong$  $W = \Sigma$ 

$\circlearrowleft$
$\tilde{\nu}$ = rotation by $-2\pi/3$

$\operatorname{proj} \Lambda$:
$\circlearrowleft$
$\Sigma = - \otimes_\Lambda D\Lambda$



 fully commutative unit cube (complete with 0's if necessary)

$\rightsquigarrow$ Auslander-Reiten 4-angle

e.g. $P_1 \to P_4 \to P_2 \to P_1 \to P_6 = \Sigma P_1$

$P_4 \to P_2 \oplus P_6 \to P_1 \oplus P_5 \to P_4 \to P_5 = \Sigma P_4$

IV Relative derived preprojective algebras and relative Calabi-Yau structures

Recall dg category (= category enriched in cochain complexes of vsp's.).

$F : \mathcal{B} \to \mathcal{A}$ dg functor & $n \in \mathbb{Z}$ Write $(-)^\vee := \mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(-, \mathcal{A}^e) : \mathcal{D}(\mathcal{A}^e)^{op} \longrightarrow \mathcal{D}(\mathcal{A}^e)$

homologically smooth

$\overset{?}{\rightsquigarrow}$

$$\begin{tikzcd}
 & & \Pi_{n+1}(\mathcal{A},\mathcal{B}) & \\
 & \Pi_n(F) \arrow[ur] & & \mathcal{A} \arrow[ul] \\
\Pi_n(\mathcal{B}) \arrow[ur] & & \Pi_n(\mathcal{A}) \arrow[ul] \arrow[ur] & & \emptyset \arrow[ul]
\end{tikzcd}$$

Disclaimer The following discussion is expressed in the language of ∞-categories. The unfamiliar reader is advised to think in terms of Quillen model categories and homotopy (co)limits. In particular, $\mathcal{D}(\mathcal{A})$ is the ∞-cat. derived cat. of $\mathcal{A}$

not a dg category!

$\underline{\mathrm{dgcat}} := \mathrm{dgcat}[\mathrm{qeq}^{-1}]$: ∞-cat. of dg categories up to quasi-equivalence

$\underline{\mathrm{dgcat}}_{\mathcal{A}/} :=$ ∞-cat of dg categories under $\mathcal{A}$

$G : \mathcal{A} \to \mathcal{C} \rightsquigarrow U(G) : (a, a') \longmapsto \mathcal{C}(Ga, Ga') \rightsquigarrow \mathbb{L}T_{\mathcal{A}} : \mathcal{D}(\mathcal{A}^e) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \underline{\mathrm{dgcat}}_{\mathcal{A}/} : U$

$\mathbb{L}F_! : \underline{\mathrm{dgcat}}_{\mathcal{B}/} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \underline{\mathrm{dgcat}}_{\mathcal{A}/} : F^*$, $\quad \underset{\substack{F\downarrow \\ \mathcal{A}}}{\mathbb{L}F_!}(\mathcal{B} \to \mathcal{C}) = \mathcal{C} \coprod_{\mathcal{B}} \mathcal{A}$, $\quad F^*(\mathcal{A} \to \mathcal{C}) = \mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{C}$

$\mathbb{L}F_! : \mathcal{D}(\mathcal{B}^e) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{D}(\mathcal{A}^e) : F^*$, $\quad \mathbb{L}F_!(X) = X \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e$

$$\begin{tikzcd}
\mathcal{D}(\mathcal{A}^e) \arrow[d, "F^*"] \arrow[dr, phantom, "\circlearrowleft"] & \underline{\mathrm{dgcat}}_{\mathcal{A}/} \arrow[l, "U"'] \arrow[d, "F^*"] \\
\mathcal{D}(\mathcal{B}^e) & \underline{\mathrm{dgcat}}_{\mathcal{B}/} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
((a,a') \mapsto \mathcal{C}(Ga, Ga')) \arrow[d, mapsto] & (\mathcal{A} \xrightarrow{G} \mathcal{C}) \arrow[l, mapsto] \arrow[d, mapsto] \\
((b,b') \mapsto \mathcal{C}(GFb, GFb')) & (\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{C}) \arrow[l, mapsto]
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathcal{D}(\mathcal{A}^e) \arrow[r, "\mathbb{L}T_{\mathcal{A}}"] & \underline{\mathrm{dgcat}}_{\mathcal{A}/} \\
\mathcal{D}(\mathcal{B}^e) \arrow[u, "\mathbb{L}F_!"] \arrow[r, "\mathbb{L}T_{\mathcal{B}}"'] \arrow[ur, phantom, "\circlearrowleft"] & \underline{\mathrm{dgcat}}_{\mathcal{B}/} \arrow[u, "\mathbb{L}F_!"']
\end{tikzcd}$$

$$\boxed{\forall X \in \mathcal{D}(\mathcal{B}^e) \quad T_{\mathcal{A}}(X \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e) \simeq T_{\mathcal{B}}(X) \coprod_{\mathcal{B}} \mathcal{A}}$$

$\simeq \mathcal{A} \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \mathcal{B} \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \mathcal{A}$

$$\rightsquigarrow \quad \Omega_{\mathcal{A},\mathcal{B}} := \mathrm{cone}\left(\mathcal{A}^\vee \xrightarrow{\bar{F}^\vee} \mathcal{B}^\vee \underset{\mathcal{B}^e}{\otimes} \mathcal{A}^e\right)[-1] \in \mathcal{D}(\mathcal{A}^e)$$

$\bar{F}$: counit of adjunction $\; - \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \mathcal{A} : \mathcal{D}(\mathcal{B}) \rightleftarrows \mathcal{D}(\mathcal{A}) : F^* = F^*(-) \overset{\mathbb{L}}{\underset{\mathcal{B}}{\otimes}} \mathcal{B}$ (with $\perp$)

$$\begin{tikzcd}
\Pi_{n-1}(\mathcal{B}) \arrow[d, phantom, "\shortparallel"] \\
\mathbb{L}T_{\mathcal{B}}(\mathcal{B}^\vee[n-2]) \arrow[d] \\
\mathbb{L}T_{\mathcal{B}}(\mathcal{B}^\vee[n-2]) \amalg_{\mathcal{B}}^{\mathbb{L}} \mathcal{A}
\end{tikzcd}$$

Rmk $\mathcal{B} = \emptyset \Rightarrow \Omega_{\mathcal{A},\mathcal{B}} = \mathcal{A}^\vee = \Omega_{\mathcal{A}}$

$$\begin{tikzcd}
\mathcal{A}^\vee[n-2] \arrow[r, "F^\vee{[n-2]}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{B}^\vee \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e[n-2] \arrow[d] \\
0 \arrow[r] & \Omega_{\mathcal{A},\mathcal{B}}[n-1]
\end{tikzcd}
\quad \overset{\mathbb{L}T_{\mathcal{A}}}{\longmapsto} \quad
\begin{tikzcd}
\Pi_{n-1}(\mathcal{A}) = \mathbb{L}T_{\mathcal{A}}(\mathcal{A}^\vee[n-2]) \arrow[r, "\Pi_{n-1}(F)"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{L}T_{\mathcal{A}}\left(\mathcal{B}^\vee \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e[n-2]\right) \arrow[d] \\
\emptyset \arrow[r] & \mathcal{A} = \mathbb{L}T_{\mathcal{A}}(0) \arrow[r] & \mathbb{L}T_{\mathcal{A}}(\Omega_{\mathcal{A},\mathcal{B}}[n-1])
\end{tikzcd}$$

$\mathbb{L}T_{\mathcal{B}}(\mathcal{B}^\vee[n-2]) \amalg_{\mathcal{B}}^{\mathbb{L}} \mathcal{A} \cong \mathbb{L}T_{\mathcal{A}}\left(\mathcal{B}^\vee \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e[n-2]\right)$, $\quad \mathbb{L}T_{\mathcal{A}}(\Omega_{\mathcal{A},\mathcal{B}}[n-1]) =: \Pi_n(\mathcal{A},\mathcal{B})$

in $\mathcal{D}(\mathcal{A}^e)$ $\qquad\qquad$ in $\underline{\mathrm{dgcat}}_{\mathcal{A}/}$

Def (Yeung 2016)

The <u>relative n-CY completion</u> of F is $\Pi_{n-1}(\mathcal{B}) \to \Pi_n(\mathcal{A},\mathcal{B})$

Rmk $\mathcal{B} = \emptyset \Rightarrow \Pi_n(\mathcal{A},\mathcal{B}) = \Pi_n(\mathcal{A})$

$$F : \mathcal{B} \to \mathcal{A} \rightsquigarrow \underbrace{\mathbb{R}\mathrm{Hom}_{\Bbbk[\varepsilon]}\left(\Bbbk, \mathcal{B} \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{B}\right)}_{HN(\mathcal{B}) :=} \longrightarrow \underbrace{\mathbb{R}\mathrm{Hom}_{\Bbbk[\varepsilon]}\left(\Bbbk, \mathcal{A} \underset{\mathcal{A}^e}{\otimes} \mathcal{A}\right)}_{HN(\mathcal{A}) :=} \text{ in } \mathcal{D}(\Bbbk)$$

Def (Toën 2014, Brav-Dyckerhoff 2019) $\quad \mathcal{B} \xrightarrow{F} \mathcal{A} \xleftarrow{G} \mathcal{C}$ $\;(\ast)\;$ with $\mathcal{A},\mathcal{B},\mathcal{C}$: homologically smooth

A <u>left n-CY structure</u> on $(\ast)$ consists of a (coherent) commutative square

$$(\ast\ast) \quad \begin{tikzcd}
\Bbbk[n] \arrow[r, "\tilde{\varphi}_{\mathcal{B}}"] \arrow[d, "\tilde{\varphi}_{\mathcal{C}}"'] \arrow[dr, "\tilde{\varphi}_{\mathcal{A}}"] & HN(\mathcal{B}) \arrow[d, "HN(F)"] \\
HN(\mathcal{C}) \arrow[r, "HN(G)"'] & HN(\mathcal{A})
\end{tikzcd}
\quad \text{equivalently} \quad \Bbbk[n] \xrightarrow{\varphi} HN(\mathcal{B}) \times_{HN(\mathcal{A})} HN(\mathcal{C}) \text{ in } \mathcal{D}(\Bbbk)$$

(1) $\tilde{\varphi}_{\mathcal{B}} \in HN_n(\mathcal{B})$ and $\tilde{\varphi}_{\mathcal{C}} \in HN_n(\mathcal{C})$ are left n-CY structures

(2)

$$\begin{tikzcd}
\mathcal{A}^\vee[n] \arrow[r, "\bar{F}^\vee{[n]}"] \arrow[d, "\bar{G}^\vee{[n]}"'] \arrow[drr, "\varphi_{\mathcal{A}}"] & \mathcal{B}^\vee[n] \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e \arrow[r, "\varphi_{\mathcal{B}}", "\sim"'] & \mathcal{B} \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e \arrow[d, "\bar{F}"] \\
\mathcal{C}^\vee[n] \overset{\mathbb{L}}{\underset{\mathcal{C}^e}{\otimes}} \mathcal{A}^e \arrow[r, "\varphi_{\mathcal{C}}"', "\sim"] & \mathcal{C} \overset{\mathbb{L}}{\underset{\mathcal{C}^e}{\otimes}} \mathcal{A}^e \arrow[r, "\bar{G}"'] & \mathcal{A}
\end{tikzcd}
\quad \text{in } \mathcal{D}(\mathcal{A}^e) \text{ is (co)cartesian}$$

(∗∗∗) see appendix

We outline the construction of the (coherent) square (***):

Evaluating the counit of the adjunction $-\overset{\mathbb{L}}{\otimes}_{\mathcal{B}}\mathcal{A} = \mathbb{L}F_! : \mathcal{D}(\mathcal{B}) \rightleftarrows \mathcal{D}(\mathcal{A}) : F^* \simeq F^*(-)\overset{\mathbb{L}}{\otimes}_{\mathcal{B}}\mathcal{B}$ yields a morphism in $\mathcal{D}(\mathcal{A}^e)$

$$\overline{F} : \mathcal{B}\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e \simeq \mathcal{A}\overset{\mathbb{L}}{\otimes}_{\mathcal{B}}\mathcal{B}\overset{\mathbb{L}}{\otimes}_{\mathcal{B}}\mathcal{A} \xrightarrow{\sim} \mathcal{A}$$

Applying the functor $(-)^\vee = \mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(-,\mathcal{A}^e)$ to $\overline{F}$ yields a further morphism in $\mathcal{D}(\mathcal{A}^e)$

$$\overline{F}^\vee : \mathcal{A}^\vee \xrightarrow{\overline{F}^\vee} (\mathcal{B}\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e) \simeq \mathcal{B}^\vee\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e$$

On the other hand, since $\mathcal{A}$ and $\mathcal{B}$ are homologically smooth, there is a (coherently) commutative diagram in $\mathcal{D}(\Bbbk)$

$$\begin{tikzcd}
\mathrm{HN}(\mathcal{B}) \arrow[r] \arrow[dd, "\mathrm{HN}(F)"'] & \mathrm{HH}(\mathcal{B}) \arrow[r, "\sim"] \arrow[dd, "\mathrm{HH}(F)"] & \mathbb{R}\mathrm{Hom}_{\mathcal{B}^e}(\mathcal{B}^\vee,\mathcal{B}) \arrow[dd, dashed, "\Psi_F \ \text{(BD19)}"] \arrow[dr, "-\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e"] & \\
& & & \mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{B}^\vee\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e, \mathcal{B}\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e) \arrow[dl, "\overline{F}\circ ?\circ\overline{F}^\vee"] \\
\mathrm{HN}(\mathcal{A}) \arrow[r] & \mathrm{HH}(\mathcal{A}) \arrow[r, "\sim"] & \mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}^\vee,\mathcal{A}) &
\end{tikzcd}$$

We obtain a (coherently) commutative diagram in $\mathcal{D}(\Bbbk)$

$$\begin{tikzcd}
\Bbbk[n] \arrow[r, "\widetilde{\varphi}_{\mathcal{B}}"] \arrow[d, "\widetilde{\varphi}_{\mathcal{C}}"'] \arrow[dr, "\widetilde{\varphi}_{\mathcal{A}}"] & \mathrm{HN}(\mathcal{B}) \arrow[r] \arrow[d, "\mathrm{HN}(F)"] & \mathbb{R}\mathrm{Hom}_{\mathcal{B}^e}(\mathcal{B}^\vee,\mathcal{B}) \arrow[dd, "\Psi_F"] \arrow[dr, "-\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e"] & \\
\mathrm{HN}(\mathcal{C}) \arrow[r, "\mathrm{HN}(G)"'] \arrow[d] & \mathrm{HN}(\mathcal{A}) \arrow[dr] & & \mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{B}^\vee\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e, \mathcal{B}\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e) \arrow[dl, "\overline{F}\circ ?\circ\overline{F}^\vee"] \\
\mathbb{R}\mathrm{Hom}_{\mathcal{C}^e}(\mathcal{C}^\vee,\mathcal{C}) \arrow[rr, "\Psi_G"] \arrow[d, "-\overset{\mathbb{L}}{\otimes}_{\mathcal{C}^e}\mathcal{A}^e"'] & & \mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}^\vee,\mathcal{A}) & \\
\mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{C}^\vee\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e, \mathcal{C}\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e) \arrow[urr, "\overline{G}\circ ?\circ\overline{G}^\vee"'] & & &
\end{tikzcd}$$

The datum $\Psi_F(\varphi_{\mathcal{B}}) \simeq \Psi_G(\varphi_{\mathcal{C}})$ in $\mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}^\vee,\mathcal{A})$ provides the claimed (coherently) commutative diagram.

(***)

$$\begin{tikzcd}
\mathcal{A}^\vee[n] \arrow[r, "\overline{F}^\vee[n]"] \arrow[d, "\overline{G}^\vee[n]"'] \arrow[ddrr, "\varphi_{\mathcal{A}}"] & \mathcal{B}^\vee[n]\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A} \arrow[dr, "\mathbb{L}F_!\varphi_{\mathcal{B}}", "\simeq"'] & \\
\mathcal{C}^\vee[n]\overset{\mathbb{L}}{\otimes}_{\mathcal{C}^e}\mathcal{A}^e \arrow[dr, "\simeq", "\mathbb{L}G_!\varphi_{\mathcal{C}}"'] & & \mathcal{B}\overset{\mathbb{L}}{\otimes}_{\mathcal{B}^e}\mathcal{A}^e \arrow[d, "\overline{F}"] \\
& \mathcal{C}\overset{\mathbb{L}}{\otimes}_{\mathcal{C}^e}\mathcal{A}^e \arrow[r, "\overline{G}"'] & \mathcal{A}
\end{tikzcd}$$

Def (Toën 2014, Brav-Dyckerhoff 2019) $\mathcal{B} \xrightarrow{F} \mathcal{A}$ with $\mathcal{A}, \mathcal{B}$: homologically smooth

A **relative left n-CY structure** on F is a left $(n-1)$-CY str. on $\mathcal{B} \xrightarrow{F} \mathcal{A} \xleftarrow{!} \emptyset$

Explicitly:

$$\begin{tikzcd} \Bbbk[n-1] \arrow[r, "\tilde{\varphi}_{\mathcal{B}}"] \arrow[d] \arrow[dr, phantom, "(**)"] & HN(\mathcal{B}) \arrow[d, "HN(F)"] \\ 0 \arrow[r] & HN(\mathcal{A}) \end{tikzcd} \quad \leftrightsquigarrow \quad \Bbbk[n-1] \xrightarrow{\tilde{\varphi}[-1]} \mathrm{fib}\,(HN(\mathcal{B}) \to HN(\mathcal{A})) =: HN(\mathcal{A}, \mathcal{B})[-1]$$

relative negative cyclic homology

(1) $\tilde{\varphi}_{\mathcal{A}} \in HN_{n-1}(\mathcal{A})$ is a left $(n-1)$-CY structure

(2)
$$\begin{tikzcd} \mathcal{A}^\vee[n-1] \arrow[r, "F^\vee{[n-1]}"] \arrow[d] \arrow[drr, phantom, "(***)"] & \mathcal{B}^\vee[n-1] \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e \arrow[r, "\varphi_{\mathcal{B}}", "\simeq"'] & \mathcal{B} \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e \arrow[d, "\bar{F}"] \\ 0 \arrow[rr] & & \mathcal{A} \end{tikzcd}$$
in $\mathcal{D}(\mathcal{A}^e)$ is (co)cartesian

Rmk (**) yields map $\tilde{\varphi} : \Bbbk[n] \longrightarrow HN(\mathcal{A}, \mathcal{B})$ in $\mathcal{D}(\Bbbk)$

$\mathcal{B} = \emptyset \rightsquigarrow HN(\mathcal{A}, \emptyset) = HN(\mathcal{A})$ & $\tilde{\varphi} : \Bbbk[n] \longrightarrow HN(\mathcal{A})$ i.e. $\tilde{\varphi} \in HN_n(\mathcal{A})$

(2) is equivalent to induced map

$\Omega_{\mathcal{A},\mathcal{B}}[n] = \mathrm{cone}\left(\mathcal{A}^\vee[n-1] \xrightarrow{F^\vee[n-1]} \mathcal{B}^\vee[n-1] \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e\right) \xrightarrow{\varphi} \mathcal{A}$ being an iso in $\mathcal{D}(\mathcal{A}^e)$

$\mathcal{B} = \emptyset \rightsquigarrow \Omega_{\mathcal{A}}[n] \xrightarrow{\varphi} \mathcal{A}$ is an isomorphism in $\mathcal{D}(\mathcal{A}^e)$

(2) also equivalent to induced morphism of triangles in $\mathcal{D}(\mathcal{A}^e)$ being and isomorphism

$$\begin{tikzcd} \mathcal{A}^\vee[n-1] \arrow[r, "\bar{F}^\vee{[n-1]}"] \arrow[d, "\psi"] & \mathcal{B}^\vee[n-1] \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e \arrow[r] \arrow[d, "\varphi_{\mathcal{B}} \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e"] & \Omega_{\mathcal{A},\mathcal{B}}[n] \arrow[r] \arrow[d, "\varphi"] & \mathcal{A}^\vee[n] \arrow[d, "\psi{[1]}"] \\ \mathrm{cone}(\bar{F})[-1] \arrow[r] & \mathcal{B} \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{A}^e \arrow[r, "\bar{F}"'] & \mathcal{A} \arrow[r] & \mathrm{cone}\,\bar{F} \end{tikzcd}$$

Rmk Left n-CY structure on $\mathcal{A}$ = Left $(n-1)$-CY structure on $\emptyset \to \mathcal{A} \leftarrow \emptyset$

Thm (Brav-Dyckerhoff 2019)

$$\begin{tikzcd} & & \mathcal{A} \amalg_{\mathcal{C}} \mathcal{D} & & \\ & \mathcal{A} \arrow[ur] \arrow[rr, phantom, "\lrcorner"] & & \mathcal{D} \arrow[ul] & \\ \mathcal{B} \arrow[ur] \arrow[rr, phantom, "(1)"] & & \mathcal{C} \arrow[ul] \arrow[ur] \arrow[rr, phantom, "(2)"] & & \mathcal{E} \arrow[ul] \end{tikzcd}$$

Left n-CY structures on (1) & (2) induce left n-CY structure on

$$\begin{tikzcd} & & \mathcal{A} \amalg_{\mathcal{C}} \mathcal{D} & & \\ & \mathcal{A} \arrow[ur] & & \mathcal{D} \arrow[ul] & \\ \mathcal{B} \arrow[ur] & & & & \mathcal{E} \arrow[ul] \end{tikzcd}$$

Thm (Yeung 2016, Bozec - Calaque - Scherotzke 2023) $F : \mathcal{B} \to \mathcal{A}$ with $\mathcal{A}, \mathcal{B}$: homologically smooth

$$
\begin{tikzcd}[column sep=small]
 & & \Pi_n(\mathcal{A},\mathcal{B}) & & \\
 & \Pi_{n-1}(F) \arrow[ur] \arrow[rr, phantom, "\vee"] & & \mathcal{A} \arrow[ul] & \\
\Pi_{n-1}(\mathcal{B}) \arrow[ur] \arrow[rr, phantom, "(1)"] & & \Pi_{n-1}(\mathcal{A}) \arrow[ul] \arrow[ur] \arrow[rr, phantom, "(2)"] & & \varnothing \arrow[ul]
\end{tikzcd}
$$

(1) & (2) admit left $(n-1)$-CY structures which yield a left $(n-1)$-CY structure on $\Pi_{n-1}(\mathcal{B}) \longrightarrow \Pi_n(\mathcal{A},\mathcal{B}) \longleftarrow \varnothing$

In other words, $\Pi_{n-1}(\mathcal{B}) \longrightarrow \Pi_n(\mathcal{A},\mathcal{B})$ admits a relative $(n-1)$-CY structure

Def $F : \mathcal{B} \longrightarrow \mathcal{A} \rightsquigarrow \operatorname{cofib} F := \mathcal{A}/\operatorname{im} F$: Drinfeld quotient
$$
\begin{tikzcd}
\mathcal{B} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{A} \arrow[d] \\
0 \arrow[r] & \operatorname{cofib} F
\end{tikzcd}
$$
in dgcat

Thm (Brav - Dyckerhoff 2019) $F : \mathcal{B} \longrightarrow \mathcal{A}$ with $\mathcal{A}, \mathcal{B}$ homologically smooth

A relative left n-CY structure on F induces a left n-CY structure on $\operatorname{cofib}(F)$ and a relative right n-CY structure on $F^* \mathcal{D}^{fd}(\mathcal{A}) \longrightarrow \mathcal{D}^{fd}(\mathcal{B})$

not define here due to lack of time

Prop (Wu 2023) $F : \mathcal{B} \longrightarrow \mathcal{A}$ with $\mathcal{A}, \mathcal{B}$ homologically smooth with rel. left n-CY structure

$$\forall\, Y \in \mathcal{D}(\mathcal{A}) \quad \forall\, X \in \mathcal{D}^{fd}(\mathcal{A}) \text{ s.t. } X|_{\operatorname{im}(F)} \simeq 0 \qquad D\operatorname{Hom}_{\mathcal{A}}(X, Y) \cong \operatorname{Hom}_{\mathcal{A}}(Y, X[n])$$

Prop (Wu 2023) $F : \mathcal{B} \longrightarrow \mathcal{A}$ with $\mathcal{A}, \mathcal{B}$ homologically smooth

$$\operatorname{cofib}\big(\Pi_{n-1}(\mathcal{B}) \longrightarrow \Pi_n(\mathcal{A},\mathcal{B})\big) \simeq \Pi_n(\operatorname{cofib} F)$$

$$\boxed{B : d\text{-hereditary } d\text{-rRF algebra} \quad \& \quad A := \operatorname{Aus}_{d+1}(B) = \operatorname{End}_B\Big(\bigoplus_{i \geq 0} \tau_d^{-i}(B)\Big)}$$

$f : B \hookrightarrow A$ (non-unital) algebra inclusion

Def The relative $(d+2)$-preprojective algebra of B is $\Pi_{d+2}(A,B)$ which comes equipped with the canonical morphism $\Pi_{d+1}(B) \longrightarrow \Pi_{d+2}(A,B)$

Prop (Wu 2023) $\forall N \in \mathcal{M}(B) = \operatorname{add}\left(\bigoplus_{i\geq 0} \tau_d^{-i}(B)\right) \quad N \overset{\mathbb{L}}{\underset{A}{\otimes}} \Omega_{A,B}[d+1] \underset{\text{qiso}}{\simeq} \tau_d^{-}(N)$

Thm (Wu 2023) Let $e = f(1_B)$. TFSH

(1) $\operatorname{cofib}(B \hookrightarrow A) \simeq A/\langle e\rangle = \underline{\operatorname{Aus}}_{d+1}(B)$

(2) $\Pi_{d+1}(B) = H^0(\boldsymbol{\Pi}_{d+1}(B)) \hookrightarrow H^0(\boldsymbol{\Pi}_{d+2}(A,B))$

(3) $H^*(\boldsymbol{\Pi}_{d+2}(A,B)) = H^0(\boldsymbol{\Pi}_{d+2}(A,B)) =: \Pi_{d+2}(A,B)$

(4) $\Pi_{d+2}(A,B)$ is finite dimensional

(5) $\Pi = \Pi_{d+2}(A,B)$ is internally bimodule $(d+2)$-CY w.r.t. e. (Pressland 2017)

- $\operatorname{gl.dim} \Pi \leq d+2$
- $\exists$ triangle $\Pi \to \Pi^\vee[n] \to C \to \Pi[1]$ in $\mathcal{D}(\Pi^e)$ s.t.

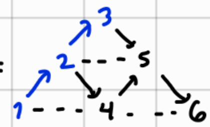
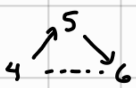
$$\forall X \in \mathcal{D}^{fd}(\Pi)_e \quad \forall Y \in \mathcal{D}^{fd}(\Pi^e)_e \quad \mathbb{R}\mathrm{Hom}_\Pi(C,X) \simeq 0 \simeq \mathbb{R}\mathrm{Hom}_{\Pi^{op}}(Y,C)$$

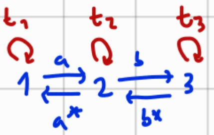
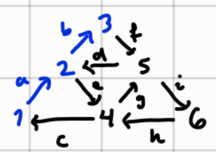
$$\mathcal{D}(\Pi)_e := \{X \in \mathcal{D}(\Pi) \mid \forall i \in \mathbb{Z} \ H^i(X) \in \operatorname{Mod}(\Pi/\langle e\rangle)\}$$
$$\mathcal{D}^{fd}(\Pi)_e = \mathcal{D}(\Pi)_e \cap \mathcal{D}^{fd}(\Pi)$$

New proof

(6) $\underline{\operatorname{mod}}\,\Pi_{d+1}(B) \simeq \mathcal{C}(\boldsymbol{\Pi}_{d+2}(\underline{\operatorname{Aus}}_{d+1}(B)))$ hence $\underline{\operatorname{mod}}\,\Pi_{d+1}(B)$ has a d-CT obj.

(Iyama–Oppermann 2013)

Example (Keller–Wang 2021) $B = 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \hookrightarrow A =$  & $\underline{A} =$ 

$\boldsymbol{\Pi}_2(B) =$  $\longrightarrow$  $= \boldsymbol{\Pi}_3(A,B)$

$|a| = |a^*| = |b| = |b^*| = 0$, $|t_i| = -1$

$\partial(t_1) = a^*a \quad \partial(t_2) = aa^* - b^*b \quad \partial(t_3) = bb^*$

$w = cea + dfb + hig - dge$

$\cancel{\partial_a} \ \partial_b$

 $= \boldsymbol{\Pi}_3(\underline{A})$

$w = hig \quad |g^*| = |h^*| = |i^*| = -1 \quad |t_i| = -2$

$\partial(g^*) = hi, \ \partial(h^*) = ig, \ \partial(i^*) = gh$

$\partial(t_4) = g^*g - hh^*, \ \partial(t_5) = \ldots$

$$\boxed{\begin{array}{c} \boldsymbol{\Pi}_{d+1}(A_{n,d}) \longrightarrow \boldsymbol{\Pi}_{d+2}(\operatorname{Aus}_{d+1}(A_{n,d}), A_{n,d}) \\ \downarrow \\ \boldsymbol{\Pi}_{d+2}(A_{n,d+1}) \end{array}}$$

V The Derived Auslander-Iyama Correspondence

Λ : fin.-dim. selfinjective algebra (basic and connected)

$\sigma : \Lambda \xrightarrow{\sim} \Lambda$ algebra automorphism $\rightsquigarrow$ $\Sigma := - \otimes_\Lambda {}_\sigma\Lambda : \operatorname{proj}\Lambda \xrightarrow{\sim} \operatorname{proj}\Lambda$

Recall $(\operatorname{proj}\Lambda, \Sigma)$ admits a $(d+2)$-angulation $\Rightarrow$ Λ is twisted $(d+2)$-periodic w.r.t. σ

Question What about the converse?

Thm (Amiot $d=1$, Lin 2015) Λ : twisted $(d+2)$-periodic w.r.t. σ

$$\delta : \quad 0 \to {}_1\Lambda_\sigma \to \underbrace{P_{d+1} \to P_d \to \cdots \to P_1 \to P_0}_{\text{projective } \Lambda\text{-bimods.}} \to \Lambda \to 0 \quad : \text{exact seq. in } \operatorname{mod}\Lambda^e$$

$\rightsquigarrow$ $\pentagon_\delta$: class of $(d+2)$-angles in $\operatorname{proj}\Lambda$ s.t. $(\operatorname{proj}\Lambda, \Sigma, \pentagon_\delta)$ is a $(d+2)$-ang. cat.

Explicit, but technical to define

(J-Muro 2022) Up to equivalence of $(d+2)$-ang. cat's, $\pentagon_\delta$ is independent of δ

Rmk Λ : selfinjective $\Rightarrow$ $\nu = - \otimes_\Lambda D\Lambda : \operatorname{proj}\Lambda \xrightarrow{\sim} \operatorname{proj}\Lambda$ is a Serre functor

$P^0 \in \operatorname{proj}\Lambda$: indec. & $S \in \operatorname{mod}\Lambda$: simple top of P^0 = simple socle of νP^0

Auslander-Reiten $(d+2)$-angle

twisted $\rightsquigarrow$ periodicity

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \nu_d P^{-1} \arrow[rr] \arrow[dr, two heads] & & \boxed{\nu_d P^0 \to P^{-d} \to \cdots \to P^{-1} \to P^0 \longrightarrow \nu P^0} \\
& & \Omega^{d+2}_\Lambda(S) \arrow[ur, hook] & S
\end{tikzcd}$$

: min. proj. resolution of S

Corollary (Hanihara 2020 $d=1$, J-Muro 2022) Λ : fin.-dim. algebra

Λ is twisted $(d+2)$-periodic $\Longleftrightarrow$ $\operatorname{proj}\Lambda$ admits $(d+2)$-angulated structure

$(\operatorname{proj}\Lambda, \Sigma, \pentagon)$: $(d+2)$-angulated category

$\rightsquigarrow \Lambda^* := \bigoplus_{i \in d\mathbb{Z}} \underbrace{\operatorname{Hom}_\Lambda(\Lambda, \Sigma^i(\Lambda))}_{\cong {}_{\sigma^i}\Lambda_1}$: graded algebra concentrated in multiples of d

Def A *dg enhancement* of $(\operatorname{proj}\Lambda, \Sigma, \pentagon)$ is a dg algebra A such that

(1) $A \in \mathcal{D}^c(A)$ is $d\mathbb{Z}$-cluster tilting

(2) $H^*(A) \cong \Lambda^*$ and the following diagram is part of an equivalence of $(d+2)$-angulated categories:

$$\begin{tikzcd} \mathcal{D}^c(A) \supseteq \operatorname{add} A \arrow[r, "\sim"] \arrow[d, "{[d]}"'] & \operatorname{proj}\Lambda \arrow[d, "\Sigma"] \\ \mathcal{D}^c(A) \supseteq \operatorname{add} A \arrow[r, "\sim"] & \operatorname{proj}\Lambda \end{tikzcd}$$

Rmk For $d=1$ we recover Bondal-Kapranov pre-triangulated dg enhancements

Thm (Muro 2022 $d=1$, J-Muro 2022) $(\operatorname{proj}\Lambda, \Sigma, \pentagon)$: Amiot-Lin $(d+2)$-ang. cat.

$(\operatorname{proj}\Lambda, \Sigma, \pentagon)$ admits a dg enhancement and it is unique up to quasi-iso.

Thm (Muro 2022 $d=1$, J-Muro 2022) There are bijective correspondences between:

(1) Dg algebras A s.t. $H^0(A)$ is basic fin. dim. and $A \in \mathcal{D}^c(A)$ is $d\mathbb{Z}$-CT / quasi-iso

(2) Pairs $(\mathcal{T} \ni c)$ where $\mathcal{T}$ is an algebraic tri. cat. with split idempotents and fin. dim. Hom-spaces & $c \in \mathcal{T}$ is $d\mathbb{Z}$-CT / equiv. compatible with $c \in \mathcal{T}$.

(3) Pairs $(\Lambda, \sigma\colon \Lambda \xrightarrow{\sim} \Lambda)$ s.t. Λ is twisted $(d+2)$-periodic w.r.t. σ / up to iso compatible with $[\sigma] \in \operatorname{Out}(\Lambda) := \operatorname{Aut}(\Lambda)/\operatorname{Inn}(\Lambda)$

induced by $[d] \circlearrowright \operatorname{add} c$

$$A \longmapsto (\mathcal{D}^c(A) \ni A), \quad (\mathcal{T} \ni c) \longmapsto (\mathcal{T}(c,c), \sigma)$$

Rmk $\overset{\text{as in (1)}}{A}$: d-CY as dg bimod. $\Rightarrow H^0(A)$: twisted $(d+2)$-periodic w.r.t. ν $\underset{\text{In progress}}{\overset{?}{\Longrightarrow}}$ A admits right d-CY str.

A : dg algebra s.t.

- $\Lambda^* = H^*(A) = \bigoplus_{d i \in d\mathbb{Z}} H^{di}(A)$ is concentrated in $d\mathbb{Z}$
- $\Lambda = H^0(A)$: basic fin. dim. & selfinjective
- $A \xrightarrow{\sim} A[d]$ in $\mathcal{D}^c(A)$

$\left(CC^{\bullet,*}(\Lambda^*,\Lambda^*), \underbrace{[-,-]}_{\text{Gerstenhaber bracket of bidegree } (-1,0)}, \partial_{Hoch}\right)$: Hochschild cochain complex $\quad \partial_{Hoch} = [\underbrace{m_2}_{\text{(sign-twisted) product in } \Lambda^*}, -]$

$CC^{p,q}(\Lambda^*,\Lambda^*) := \mathrm{Hom}_{\mathbb{k}}((\Lambda^*)^{\otimes p}, \Lambda^*[q])$: degree q multilinear operations with p inputs

(Kadeishvili 1982) $\Lambda^* = H^*(A)$ admits a minimal A_∞-algebra structure

$$m_{i+2} : (\Lambda^*)^{\otimes i+2} \xrightarrow{\sim} \Lambda^*[-i], \quad i \geq 1 \quad \text{s.t.} \quad A \underset{\text{qiso}}{\xrightarrow{\sim}} \Lambda^*.$$

$m = (m_i)$ must satisfy the Maurer-Cartan equation $\partial_{Hoch}(m) + \underbrace{\tfrac{1}{2}}_{\text{if char } \mathbb{k} \neq 2}[m,m] = 0.$

Exercise Λ^* : concentrated in $d\mathbb{Z}$ $\implies$ $\forall\, i \notin d\mathbb{Z}$ $m_{i+2} = 0$

Fact $\partial_{Hoch}(m_{d+2}) = 0 \rightsquigarrow \underbrace{\{m_{d+2}\}}_{\text{Universal Massey product (UMP)}} \in HH^{d+2,-d}(\Lambda^*,\Lambda^*) = H^{d+2,-d}(CC^{d+2,-d}(\Lambda^*,\Lambda^*))$

$$\begin{array}{ccc} j : \Lambda = \Lambda^0 \hookrightarrow \Lambda^* \rightsquigarrow j^* : HH^{d+2,-d}(\Lambda^*,\Lambda^*) & \longrightarrow & HH^{d+2,-d}(\Lambda,\Lambda^*) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \{m_{d+2}\} & \longmapsto & \underbrace{j^*\{m_{d+2}\}}_{\text{restricted Universal Massey product (rUMP)}} \end{array}$$

$j^*\{m_{d+2}\} \in HH^{d+2,-d}(\Lambda,\Lambda^*) \cong HH^{d+2}(\Lambda,\Lambda^{-d}) = \mathrm{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$

$\rightsquigarrow j^*\{m_{d+2}\} = [0 \to {}_1\Lambda_\sigma \to X \to \underbrace{P_d \to \cdots \to P_1 \to P_0}_{\text{projective } \Lambda\text{-bimods.}} \to \Lambda \to 0]$

Thm (J-Muro 2022) TFAE

(1) $A \in \mathcal{D}^c(A)$: $d\mathbb{Z}$-cluster tilting

(2) $j^*\{m_{d+2}\} = [0 \to {}_1\Lambda_\sigma \to P_{d+1} \to P_d \to \cdots \to P_1 \to P_0 \to \Lambda \to 0]$, $P_i \in \operatorname{proj}\Lambda^e$

(3) $j^*\{m_{d+2}\} \in HH^{d+2,-d}(\Lambda, \Lambda^*)$ is a unit

Hochschild-Tate cohomology

Thm (J-Muro 2022) Λ : twisted $(d+2)$-periodic w.r.t. $\sigma : \Lambda \xrightarrow{\sim} \Lambda$

$\delta : 0 \to {}_1\Lambda_\sigma \to P_{d+1} \to P_d \to \cdots \to P_1 \to P_0 \to \Lambda \to 0$, $P_i \in \operatorname{proj}\Lambda^e$

(1) There exists a minimal A_∞-algebra structure on Λ^* st. $j^*\{m_{d+2}\} = [\delta]$.
Moreover, such structure is unique up to A_∞-isomorphism with identity lin. part.

(2) Endowed with the above min. A_∞-algebra structure, any choice of dg algebra model for Λ^* is a dg enhancement of $(\operatorname{proj}\Lambda, \Sigma, \Delta_\delta)$.

Assume char $k \neq 2$ for simplicity.

Def A d-sparse Massey algebra is a pair (Λ^*, μ) where

- Λ^* is a graded algebra concentrated in $d\mathbb{Z}$
- $\mu \in HH^{d+2,-d}(\Lambda^*, \Lambda^*)$ s.t. $[\mu, \mu] = 0$

The Hochschild cochain complex of (A, μ) is $(HH^{\overset{\geq 2}{\bullet}, *}(\Lambda^*, \Lambda^*), \underbrace{[\mu, -]}_{\text{bidegree } (d+1,-d)})$ (if char $k \neq 2$)

Rmk Λ^* : d-sparse min. A_∞-algebra $\Rightarrow$ $(\Lambda^*, \{m_{d+2}\})$ is a d-sparse Massey algebra

Thm (J-Muro 2022) (Λ^*, μ) : d-sparse Massey algebra s.t. $HH^{p+2,-p}(\Lambda^*, \mu) = 0$ $\forall p > d$

A, B : min. A_∞-algebra structures on Λ^* s.t. $\{m^A_{d+2}\} = \mu = \{m^B_{d+2}\}$
$\Rightarrow$ $\exists\ A \xrightarrow{\sim} B$ an A_∞-isomorphism with identity linear part

Corollary (Kadeishvili 1988) Λ^*: graded algebra s.t. $HH^{p+2,-p}(\Lambda^*,\Lambda^*)=0 \ \forall p\geq 1$

A: min. A_∞-algebra str. on Λ^* $\Rightarrow \exists A \xrightarrow{\sim} \Lambda^*$ an A_∞-iso with identity linear part

Proof Apply previous thm to 1-sparse Massey algebra $(\Lambda^*, 0)$ and notice that $CC^{\circ,*}(A,0)=0$ since $HH^{p+2,-p}(\Lambda^*,\Lambda^*)=0 \ \forall p\geq 1$ by assumption ▨

$X_\infty :=$ Space of A_∞-algebra structures on graded v.sp. Λ^*

(Muro 2020) $\boxed{\pi_0(X_\infty) = \text{min. } A_\infty\text{-algebra str.'s on } \Lambda^* / A_\infty\text{-isos with id. lin. part}}$

$m \in X_\infty$:

$\pi_1(X_\infty, m) = A_\infty$-endomorphisms with id. lin. part / A_∞-htpies. with id. lin. part

$\pi_n(X_\infty, m) = HH^{2-n}_{\geq 2}(\Lambda^*, m) = H^*(CC^*_{\geq 2}(\Lambda^*, m))$

$\forall i \geq 0 \quad X_\infty \xrightarrow{\text{res}} X_i$: space of A_{i+2}-algebra structures on graded vsp Λ^*

$\rightsquigarrow X_\infty \xrightarrow{\sim} \lim(\cdots \to X_i \to \cdots \to X_2 \to X_1 \to X_0)$

(Muro 2020) Spectral sequence $E^{\circ,*}$ of Bousfield-Kan type for computing $\pi_*(X_\infty, m)$

Rmk Truncated spectral sequence can be defined for d-sparse Massey algebra

$E_r^{i+1,i}$: obstructions for extending A_{i+2}-algebra structure on Λ^* to an A_{i+3}-algebra structure with the same underlying A_{i-r+3}-algebra structure

$r=2$: $E_r^{i+1,i} = HH^{i+3,-i}(\Lambda^*)$ gives obstructions for extending A_{i+2}-algebra structure on Λ^* to an A_{i+3}-algebra structure with the same underlying A_{i+1}-algebra structure.

VI The Donovan-Wemyss Conjecture

$\Bbbk = \mathbb{C}$ & $d = 2$

Def (Reid 1983)

A compound Du Val (cDV) singularity is a complete local hypersurface

$$R \cong \mathbb{C}[\![x,y,z,t]\!]/(f), \quad f(x,y,z,t) = g(x,y,z) + t\,h(x,y,z,t)$$

where h is arbitrary and $R/(t) \cong \mathbb{C}[\![x,y,z]\!]/(g)$ is a Du Val sing.

ADE classification

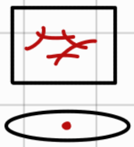
Standing assumption R has an isolated singularity

$\omega_X = p^* \omega_R$

$\exists\, p : X \to \operatorname{Spec} R$: (smooth) crepant resolution

(Donovan-Wemyss 2016) $\Lambda = \Lambda(p)$: contraction algebra

$$\begin{array}{c} X \\ \downarrow p \\ \operatorname{Spec} R \end{array}$$



$p^{-1}(0) = \bigcup_{i=1}^{n} C_i$

Λ is defined in terms of the "non-commutative deformations of $\bigoplus_{i=1}^{n} \mathcal{O}_{C_i} \in \operatorname{coh} X$"

Rmk The contraction algebra refines known numerical invariants of p,

e.g. Ried's width, Gopakumar-Vafa invariants (Toda 2015, Hua-Toda 2018)

Donovan-Wemyss Conjecture (2016)

R_1, R_2 : isolated cDV singularities with crepant resolutions $p_i : X_i \to \operatorname{Spec} R_i$

$$\mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p_2)) \iff R_1 \cong R_2$$

($\Leftarrow$) Follows from results of Wemyss (2018) & August (2020)

($\Rightarrow$) Known in type A (Reid 1983)

Rmk "DG enhanced" variants of the conjecture are known to hold

(Hua 2018, Hua-Keller 2018, Booth 2019)

Examples (Donovan - Wemyss 2016)

Name	Eq'n for Spec R	Δ	Λ	Λ^{ab}	$\dim_{\mathbb{C}} \Lambda$	$\dim_{\mathbb{C}} \Lambda^{ab}$
Atiyah flop		$\mathbb{A}_1$	$\mathbb{C}$	$\mathbb{C}$	1	1
Pagoda flop	$uv = (x-y^n)(x+y^n)$	$\mathbb{A}_1$	$\mathbb{C}[x]/(x^n)$	$\mathbb{C}[x]/(x^n)$	n	n
Laufer flop	$u^2+v^2y = x(x^2+y^{2n+1})$	$\mathbb{D}_4$	$\mathbb{C}\langle x,y\rangle / \begin{smallmatrix} xy=-yx \\ x^2=y^{2n+1}\end{smallmatrix}$	$\mathbb{C}[x,y] / \begin{smallmatrix} xy=0 \\ x^2=y^{2n+1}\end{smallmatrix}$	$3(2n+1)$	$2n+3$

R: isolated cDV singularity $\rightsquigarrow$ $\mathcal{D}_{sg}(R) := \mathcal{D}^b(\mathrm{mod}\, R)/K^b(\mathrm{proj}\, R)$ <u>singularity category</u>

<u>Facts</u> R: complete local $\Rightarrow$ $\mathcal{D}_{sg}(R)$ has split idempotents & fin.dim. Hom-spaces

(Auslander 1978) $\dim R = 3 \Rightarrow \mathcal{D}_{sg}(R)$ is a 2-CY triangulated category

(Eisenbud 1980) R: hypersurface $\Rightarrow$ $\mathcal{D}_{sg}(R) \circlearrowright [2] \cong \mathbb{1}$

<u>Exercise</u> $\forall X \in \mathcal{D}_{sg}(R)$: End(X) is a symmetric algebra

<u>Thm</u> (Wemyss 2018) R: isolated cDV singularity with crepant resolution

$$\{ X \to \mathrm{Spec}\, R : \text{crepant res} \}/_{\cong} \xleftrightarrow{\sim} \{ T \in \mathcal{D}_{sg}(R) : 2\mathbb{Z}\text{-cluster tilting} \}/_{\cong}$$
$$T \longmapsto T(p)$$
$$\mathrm{End}(T(p)) \cong \Lambda(p)$$

$\mathcal{D}_{sg}(R)$ admits a <u>canonical</u> dg enhancement $\boxed{\mathcal{D}_{sg}(R)_{dg} := \mathcal{D}^b(\mathrm{mod}\, R)_{dg} / K^b(\mathrm{proj}\, R)_{dg}}$

<u>Corollary</u> (J-Muro 2022) $\mathcal{D}_{sg}(R)$ admits a <u>unique</u> dg enhancement.

<u>Thm</u> (August 2020) R: isolated cDV singularity

The contraction algebras of R form a single and complete derived equivalence class of basic algebras

Thm (Hua–Keller 2023) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$: isolated cDV singularity

$$HH^0(\mathcal{D}_{sg}(R)) \cong \underbrace{\boxed{\mathbb{C}[\![x,y,z,t]\!]/(f, \partial_x f, \partial_y f, \partial_z f, \partial_t f)}}_{\text{Tyurina algebra of } f}$$

(Mather–Yau 1982) Since $\dim R = 3$ is known, the Tyurina alg. determines R up to iso.

Proof of the DW Conjecture (after Keller)

R_1, R_2 : isolated cDV's with crepant res. $p_i : X_i \to \operatorname{Spec} R_i$, $i=1,2$

Suppose that $\mathcal{D}^b(\operatorname{mod} \Lambda(p_1)) \simeq \mathcal{D}^b(\operatorname{mod} \Lambda(p_2))$

(Wemyss 2018) $\left.\begin{array}{l} \exists\ T_1 \in \mathcal{D}_{sg}(R_1) \\ T_2' \in \mathcal{D}_{sg}(R_2) \end{array}\right\}$ $2\mathbb{Z}$-cluster tilting

$$\Lambda(p_1) \cong \operatorname{End}(T_1) \quad \& \quad \Lambda(p_2) \cong \operatorname{End}(T_2')$$

(August 2020) $\exists\ T_2 \in \mathcal{D}_{sg}(R_2)$: $2\mathbb{Z}$-cluster tilting such that $\operatorname{End}(T_2) \cong \Lambda(p_1)$

Set $\Lambda := \Lambda(p_1) \cong \operatorname{End}_{\mathcal{D}_{sing}(R_1)}(T_1) \cong \operatorname{End}_{\mathcal{D}_{sing}(R_2)}(T_2)$

$$\mathbb{A}_1 := \mathbf{R}\!\operatorname{End}(T_1) \quad \& \quad \mathbb{A}_2 := \mathbf{R}\!\operatorname{End}(T_2)$$

Keller $\rightsquigarrow$ 1994

$$\begin{tikzcd}
T_1 \arrow[r, mapsto] \arrow[d, phantom, "\cap" rotate=-90] & \mathbb{A}_1 \arrow[d, phantom, "\cap" rotate=-90] \\
\mathcal{D}_{sg}(R_1)_{dg} \arrow[r, phantom, "\simeq"] & \mathcal{D}^c(\mathbb{A}_1)_{dg} \arrow[d, phantom, "\wr\!\!\wr"] \\
\mathcal{D}_{sg}(R_2)_{dg} \arrow[r, phantom, "\simeq"] & \mathcal{D}^c(\mathbb{A}_2)_{dg} \\
T_2 \arrow[r, mapsto] \arrow[u, phantom, "\cup" rotate=90] & \mathbb{A}_2 \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}
\implies
\begin{array}{c}
HH^0(\mathbb{A}_1) \cong \text{Tyurina of } R_1 \\
\wr\!\wr \\
HH^0(\mathbb{A}_2) \cong \text{Tyurina of } R_2 \\
\Downarrow \\
R_1 \cong R_2 \quad \blacksquare
\end{array}$$

Derived Auslander–Iyama Correspondence

Def R: isolated cDV singularity with crepant resolution $p : X \to \operatorname{Spec} R$

The 2ℤ-derived contraction algebra of p is $\Lambda(p) := \operatorname{REnd}(T(p))$

Rmk $\tau^{\leq 0}\Lambda(p)$ is the derived contraction algebra of p

Exercise $H^*(\Lambda(p)) \cong \Lambda(p)[z^{\pm}]$, $|z| = -2$

Prop (J-Keller-Muro) $\Lambda(p)$ is formal $\overset{(1)}{\Leftrightarrow} \Lambda(p) \cong \mathbb{C} \overset{(2)}{\Leftrightarrow}$ R is the base of Atiyah flop

Proof (2) Instance of the DW conjecture

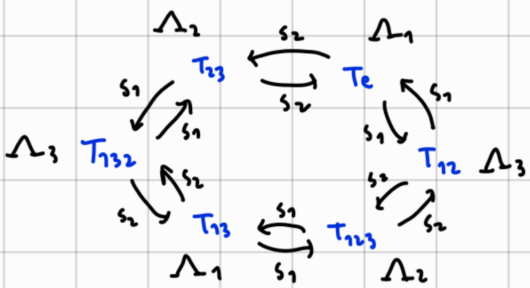
(1) $(\Leftarrow)$ $\Lambda(p) \cong \mathbb{C} \Rightarrow H^*(\Lambda(p)) \cong \mathbb{C}[z^{\pm}]$, $|z| = -2$,
which is intrinsically formal

$(\Rightarrow)$ $\Lambda(p)$: formal $\Rightarrow$ rUMP of $\Lambda(p) = j^*\{m_{d+2}\} = 0$

$\Rightarrow \underbrace{HH^{\bullet,*}}_{\text{rUMP is a unit}}(\Lambda(p)^*, \Lambda(p)^*) = 0$ where $\Lambda(p)^* = H^*(\Lambda(p))$

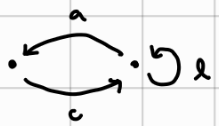
$\therefore H^0(\Lambda(p)) \cong \Lambda(p)$ is semisimple $\overset{\Lambda(p):\text{ basic and connected}}{\Longrightarrow} \Lambda(p) \cong \mathbb{C}$ $\blacksquare$

Example (Iyama-Wemyss 2018, August 2020) R: $\mathbb{C}[\![u,v,x,y]\!]/(uv - xy(x^2+y^3))$



$\mathcal{D}_{sg}(R)$ admits 6 2ℤ-CT objects

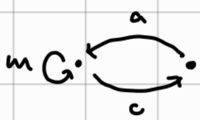
$T \xrightarrow{s_i} T'$ if they are related by "mutation at i"

$\Lambda_1 \cong$ 

$l^2 + cacaca = 0$, $lc = 0$, $al = 0$

$\Lambda_2 \cong$ 

$l^2 = ca$, $m^3 = ac$, $al = 0$, $ma = 0$, $lc = 0$, $cm = 0$

$\Lambda_3 \cong$ 

$m^3 + caca = 0$, $mc = 0$, $al = 0$

In particular, $\Lambda(p)$ determines R but not $p : X \to \operatorname{Spec} R$

Appendix: Dualisability in the Morita category

Recall Dg category = category enriched in cochain complexes of $\Bbbk$-vsp's.

$F : \mathcal{A} \to \mathcal{B}$ dg functor $\rightsquigarrow$ $\mathbb{L}F_! : \mathcal{D}(\mathcal{A}) \rightleftarrows \mathcal{D}(\mathcal{B}) : \mathbb{R}F^* = F^*$ ($\perp$)

Def / Prop $F : \mathcal{A} \to \mathcal{B}$ dg functor TFAE

(1) $\mathbb{L}F_! : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B})$ is an equivalence

(2) $\mathbb{L}F_! : \mathcal{D}^c(\mathcal{A}) \longrightarrow \mathcal{D}^c(\mathcal{B})$ is an equivalence

If these equivalent conditions are satisfied, F is a <u>Morita equivalence</u>

Thm (Tabuada 2005) dgcat : cat. of small dg cat's & dg functors between them
W : Morita equivalences of dg categories

$\rightsquigarrow$ $\mathrm{Hmo} := \mathrm{dgcat}[W^{-1}]$ can be presented by a cofibrantly gen. model structure.
Morita cat. of dg category $\rightsquigarrow$ can be promoted to ∞-cat $\mathbb{H}\mathrm{mo}$

Rmk In Hmo, the dg Yoneda embedding $\mathcal{A} \hookrightarrow \mathcal{D}^c(\mathcal{A})_{dg}$ is an isomorphism.

Thm (Toën 2007)
Hmo is a closed symmetric monoidal category with tensor product induced by the tensor product of dg categories.

$$\forall\, \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Hmo} \rightsquigarrow \underline{\mathrm{Hom}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\mathcal{A}, \underline{\mathrm{Hom}}(\mathcal{B}, \mathcal{C}))$$

Toën's internal Hom

Thm (Kontsevich, Faonte 2017)
The internal Hom-functor $(\mathcal{A}, \mathcal{B}) \mapsto \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}) \cong \mathrm{Fun}_{A_\infty}(\mathcal{A}, \mathcal{D}^c(\mathcal{B}))$ of Hmo is given by the dg category of A_∞-functors $\mathcal{A} \to \mathcal{D}^c(\mathcal{A})_{dg}$

Example A : dg alg $\rightsquigarrow$ $\underline{\mathrm{Hom}}(A^{op}, \Bbbk) \cong \mathrm{Fun}_{A_\infty}(A^{op}, \mathcal{D}^c(\Bbbk)) \cong \mathcal{D}^{fd}(A)_{dg}$ in Hmo

Def $\mathcal{A} \in \mathrm{Hmo}$ is **dualisable** if $\exists\, \mathcal{A}^* \in \mathrm{Hmo}$ & $ev : \mathcal{A} \otimes \mathcal{A}^* \to k$, $coev : k \to \mathcal{A}^* \otimes \mathcal{A}$ st.

$$
\begin{tikzcd}
 & \mathcal{A} \otimes \mathcal{A}^* \otimes \mathcal{A} \arrow[dr] & \\
\mathcal{A} \otimes k \cong \mathcal{A} \arrow[ur, "1 \otimes coev"] \arrow[rr, "1"'] & = & \mathcal{A} \cong 1 \otimes \mathcal{A}
\end{tikzcd}
\qquad \& \qquad
\begin{tikzcd}
 & \mathcal{A}^* \otimes \mathcal{A} \otimes \mathcal{A}^* \arrow[dr] & \\
1 \otimes \mathcal{A}^* \cong \mathcal{A}^* \arrow[ur] \arrow[rr, "1"'] & & \mathcal{A}^* \simeq \mathcal{A}^* \otimes 1
\end{tikzcd}
$$

Fact $\mathcal{A} \in \mathrm{Hmo}$: dualisable $\rightsquigarrow$ $- \otimes \mathcal{A} : \mathrm{Hmo} \rightleftarrows \mathrm{Hmo} : \mathcal{A}^* \otimes -$ (with $\perp$)

$$\forall\, \mathcal{B} \in \mathrm{Hmo} \quad \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}) \cong \mathcal{A}^* \otimes \mathcal{B} \quad \rightsquigarrow \quad \mathcal{D}^{fd}(\mathcal{A}^{op}) \cong \underline{\mathrm{Hom}}(\mathcal{A}, k) \cong \mathcal{A}^* \otimes k \cong \mathcal{A}^*$$

Thm (Cisinski - Tabuada 2012) $\mathcal{A}$: small dg category

$\mathcal{A}$ is dualisable in Hmo $\Longleftrightarrow$ $\mathcal{A}$ is homologically smooth & proper

In this case the dual of $\mathcal{A}$ is the opposite dg category $\mathcal{A}^{op}$.

Corollary (Toën - Vaquié 2007)

$\mathcal{A}$: homologically smooth & proper dg category $\Longrightarrow$ $\mathcal{A} \cong \mathcal{D}^{fd}(\mathcal{A})_{dg}$ in Hmo

Proof $\mathcal{A} \cong (\mathcal{A}^*)^* \cong (\mathcal{A}^{op})^* \cong \underline{\mathrm{Hom}}(\mathcal{A}^{op}, k) \cong \mathcal{D}^{fd}(\mathcal{A})_{dg}$ in Hmo $\blacksquare$