

On length categories and their taxonomy

Def (Grothendieck 1957)

An additive cat \mathcal{A} is abelian if

$\forall f: X \rightarrow Y$ in $\mathcal{A} \exists \ker f \rightarrow X \exists Y \rightarrow \text{coker } f$ and

$$\begin{array}{ccccc} \ker f & \rightarrow & X & \xrightarrow{f} & Y \rightarrow \text{coker } f \\ & & \downarrow & = & \uparrow \\ & & \text{coim } f & \xrightarrow{\cong} & \text{im } f \end{array}$$

Ex R : ring $\rightsquigarrow \text{Mod}(R)$: cat of (right) R -modules

Ex X : top. space $\rightsquigarrow \text{Shv}(X)$: cat of sheaves of abelian groups on X

Gabriel 1973 : Indecomposable Representations II

Def $0 \neq S \in \mathcal{A}$ is simple if $\forall X \hookrightarrow S$ mono ($X = 0$ or $S/X = 0$)

Def (Gabriel 1962, 1973)

An abelian cat. \mathcal{A} is a length cat. if it is ess. small and

$\forall X \in \mathcal{A} \exists 0 = X_0 \hookrightarrow X_1 \hookrightarrow \dots \hookrightarrow X_l = X$ s.t.

$\forall 1 \leq i \leq l X_i/X_{i-1}$ is simple.

Ex R : ring $\rightsquigarrow \text{fl}(R)$: cat of finite-length R -modules

Ex Bernstein - Gel'fand - Gel'fand cat. (\mathcal{O}) (1976)

Def \mathcal{A} : abelian cat.

- $S \in \mathcal{A}$ is a brick if $\text{End}_\mathcal{A}(S)$ is a div. ring.
- S, T are orthogonal if $\text{Hom}_\mathcal{A}(S, T) = 0$

Thm (Ringel 1976) $\mathcal{S} \subseteq \mathcal{A}$: set of pairwise orthogonal bricks

$\text{Filt}(\mathcal{S}) = \{X \in \mathcal{S} \mid X \text{ admits an } \mathcal{S}\text{-filtration}\} \subseteq \mathcal{A}$
is an exact length subcat with simple objects $S \in \mathcal{S}$.

Def \mathcal{A} : length cat.

- The Loewy length of $X \in \mathcal{A}$ is the smallest $l \geq 0$ such that
$$\exists D = X_0 \hookrightarrow X_1 \hookrightarrow \dots \hookrightarrow X_l = X \text{ s.t. } \forall 1 \leq i \leq l \quad X_i/X_{i-1} \text{ is semi-simple}$$
- The height of \mathcal{A} is $\sup \{\text{Loewy length}(x) \mid x \in \mathcal{A}\}$

Rank The objects of Loewy length $N \geq 0$ form a length subcategory $\mathcal{A}_N \subseteq \mathcal{A}$ (need not be extension-closed)

Thm (Gabriel 1962, 1973) \mathcal{A} : length cat. TFAE

(1) $\exists R$: right artinian st. $\mathcal{A} \cong \text{fl}(R)$

(2) TFSH:

- The height of \mathcal{A} is finite
- There are only finitely many simple objects in \mathcal{A} up to iso
- $\forall S, T \in \mathcal{A}$: simple $\text{Ext}_\mathcal{A}^1(S, T)$ is finite length over $\text{End}(T)$

Def (Gabriel 1962, 1973)

- A complete Hausdorff top. ring R is pseudocompact if

$\forall 0 \in U \subseteq R : \text{open} \exists I \subseteq R^{\text{open}} \text{ s.t. } I \subseteq U \text{ & } R/I \text{ has finite length}$

R is basic if $R/\text{rad } R$ is a product of division rings.

- $\text{disc}(R)$: cat. of finite-length discrete R -modules ($\text{ann } M \subseteq R : \text{open}$)

Ex $R = \mathbb{C}[[X]]$ is a pseudocompact ring (\mathbb{Z} -adic topology)

Thm (Gabriel 1962) \mathcal{A} : abelian cat TFAE

- (1) $\exists ! R : \text{basic pseudocompact ring s.t. } \mathcal{A} \cong \text{disc}(R)$
- (2) \mathcal{A} is a length cat.

Thm (Kietpiński - Simson - Tyc 1973, Witkowski 1979) k : field

$$(\text{PCAAlg}_k)^{\text{op}} \xrightarrow{\sim} \text{Coalg}_k, \quad R \mapsto R^{\circ} := \underset{\text{continuous hom}}{\text{hom}}_k(R, k),$$

$\text{disc}(R) \cong R^{\circ} - \text{comod}$: cat of fin.-dim. R° -comodules

Thm (Takeuchi 1977) \mathcal{A} : abelian k -cat, k : field TFAE

- (1) $\exists C : \text{coalgebra st. } \mathcal{A} \cong C\text{-comod}$
- (2) \mathcal{A} is a length cat.

(Kleiner - Roiter 1977, Roiter 1979, Drozd 1979, ... ,

Crawley-Boevey 1988, Burt - Butler 1991, ...

König - Küskümmel - Ovsienko 2014, ...) BOCS \sim length cat's exact

Def (Gabriel 1973) \mathcal{A} : length cat.

- $s_i, i \in I$: complete set of representatives of the isoclasses of simple objects in \mathcal{A}
- $K_i := \text{End}_{\mathcal{A}}(s_i)$: division ring
- $jE_i := \text{Ext}_{\mathcal{A}}^1(s_i, s_j)$: $K_j - K_i$ -bimodule
- We say that \mathcal{A} is of species $(*) = ((K_i)_{i \in I}, (jE_i)_{i, j \in I})$
- A species is abstract data $(*)$ as above.
- A k -species, k a field, is a species s.t. $\forall i \in I$ K_i is a k -algebra
- A k -quiver is a k -species s.t. $\forall i \in I$ $K_i = k$.

\hookrightarrow quiver with $\begin{cases} \text{vertex set } I \\ \#(i \rightarrow j) = \dim_k(jE_i) \end{cases}$

Ex $\text{fl}(\mathbb{C}[[x]])$ and $\text{fl}(\mathbb{C}[x]/(x^n))$, $n \geq 2$,
are of the same species: $\bullet \circlearrowright$

Ex $\text{fl}\left(\begin{smallmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{smallmatrix}\right)$ is of species $\mathbb{R} \xleftarrow{\mathbb{C}} \mathbb{C}$ (Dynkin type B_2)

Def \mathcal{A} : length cat. The global dimension of \mathcal{A} is

$$\text{gldim } \mathcal{A} := \inf \{n > 0 \mid \forall i > d \quad \text{Ext}_{\mathcal{A}}^i = 0\}$$

Ex A is semisimple $\Leftrightarrow \text{gldim } A = 0$

In this case $A \simeq \prod_{i \in I} \text{fl}(K_i)$, K_i : division ring

Def A is hereditary if $\text{gldim } A \leq 1$

Construction (Gabriel 1973) $((K_i)_{i \in I}, (jE_i)_{i, j \in I})$: species

$R := \widehat{T}_k(\omega)$: completed tensor k -algebrz of ω where

$$K := \prod_{i \in I} K_i, \quad i\omega_j := \text{hom}_{K_i}(jE_i, K_i), \quad \omega := \prod_{i, j \in I} i\omega_j,$$

Then $\text{disc}(R)$ is a length category of species

$$((K_i)_{i \in I}, (jE_i)_{i, j \in I})$$

Rule For a k -giver, $\widehat{T}_k(\omega)$ is the completed path k -alg.

Thm (Gabriel 1973, 1980)

$$A : \text{fin.-dim alg } / k = \bar{k}$$

\downarrow k -giver of A

$$\Rightarrow A \text{ is Morita equivalent to } kQ_A/I, \quad I \subseteq J^2$$

J : arrow ideal

Rule If $k = \bar{k}$, the above thm reduces the rep. theory of finite-dim. algebras to the rep. theory of givers with admissible relations.

Prop (Folklore)

The Krull-Remak-Schmidt Theorem holds in a length cat:

$\forall X \in \mathcal{A} \quad \exists X \cong X_1 \oplus \cdots \oplus X_n, \quad \forall i : \text{End}(X_i) : \text{local ring}$
unique up to iso and permutation of the direct summands

Problem Given a length cat. \mathcal{A} , classify its indecomposable obj's.

Warning $\text{fl}(\mathcal{C}(X, Y))$ is intractable (wild)

Def \mathcal{A} is of finite type if there are only finitely many indecomposable objects in \mathcal{A} up to isomorphism.

Def An Artin algebra R is an algebra over a comm. artinian ring k s.t. R has finite length as k -module.

Thm (Auslander 1971)

There is a 1-1 correspondence between:

(1) Morita eq. classes of Artin alg's. R s.t.

$\text{fl}(R)$ is of finite type

(2) Γ s.t.

$$\text{gl. dim } \Gamma \leq 2 \leq \text{dom.dim } \Gamma$$

(1) \rightarrow (2) is given by $R \mapsto \text{End}_R(M)$, $\text{add}(M) = \text{fl}(R)$.

(Auslander - Reiten 1975)

Almost-split sequences \rightsquigarrow AR theory

"Projective resolutions of simple Γ -modules of proj. dim 2"

(Iyama 2007)

Replacing "2" by " $d+1$ " leads to the discovery
of d -cluster tilting modules and higher AR theory

"Projective resolutions of simple Γ -modules of proj. dim $d+1$ "

Thm (Yoshii 1956, Bäckström, Gabriel, Kleiner 1972)

k : field , Q : connected acyclic quiver . TFAE

(1) The path algebra kQ is of finite representation type.

(2) The underlying graph of Q is a Dynkin diagram
of type $A_n, D_n, E_n, n=6,7,8$

$$A_n : \bullet - \circ - \bullet - \cdots - \bullet$$

$$D_n : \bullet - \overset{!}{\circ} - \bullet - \cdots - \bullet$$

$$E_n : \bullet - \circ - \overset{!}{\circ} - \bullet - \cdots - \bullet$$

$$n=6,7,8$$

(Tits?)

In this case, the number of indec. rep's of kQ equals the
number of positive roots of the corresponding root system.

(Bernstein - Gelfand - Ponomarev 1973)

Proof using the concept of reflection functors.

(Dlab - Ringel 1976)

Extension to k -species. All Dynkin types can appear, and do appear for suitable fields.

(Dowbor - Ringel - Simson 1980)

Extension to hereditary artinian rings. All finite Coxeter types can appear. $I_2(5)$ can be realised (Scholfield)

$I_2(p)$, $p \geq 7$ seems to be open

- Kac's Thesis (1980)

- Ringel - Hall algebras (1990)

- Fomin - Zelevinsky cluster algebras (2002)

- Additive categorifications :

- Buan - Marsh - Reiten - Reineke - Todorov (2006)

- Derksen - Weyman - Zelevinsky (2008, 2010)

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- Geiß - Leclerc - Schröer (2006 - 2018)

- Demonet (2010)

- Labardini - Fragoso - Zelevinsky (2016)

(*) Variants of the concept of species :

(GLS) species over truncated polynomial algebras

(Dem) species over group algebras

(LF-Z) species with potential

- Pro-species of algebras (Külshammer 2017)

Prop (Beilinson - Bernstein - Deligne - Gabber 1982)

$\mathcal{D}^b(\mathbb{A})$ admits a bounded t-structure with heart \mathbb{A}

Rmk $\mathcal{S} = \{S_i \mid i \in I\} \subseteq \mathbb{A}$: simples of \mathbb{A} .

$$\rightsquigarrow \text{thick}(\mathcal{S}) = \mathcal{D}^b(\mathbb{A})$$

$$(\text{Keller 1994}) \quad \text{perf}(\mathcal{S}_{dg}) \xrightarrow{\sim} \mathcal{D}^b(\mathbb{A})$$

Thm (Al-Nofayel 2009, Schnürer 2011, Keller - Nicolás 2013, Keller - Yang 2014, Rickard - Rouquier 2017, Su - Yang 2019)

\mathcal{T} : ess. small tri. cat.

\cup

$\mathcal{S} = \{S_i \mid i \in I\}$ $\xleftarrow{\text{set}}$ pairwise non-isomorphic s.t.

$$\bullet \quad \forall i, j \in I \quad \mathcal{T}(S_i, S_j) = \begin{cases} \text{division ring} & i=j \\ 0 & i \neq j \end{cases}$$

$$\bullet \quad \forall i, j \in I \quad \mathcal{T}(S_i, \Sigma^{<0}(S_j)) = 0$$

\Rightarrow

$\text{tri}(\mathcal{S})$ admits a bounded t-structure with length heart

Rmk Bounded t-structures with length heart play an important role in the study of Bridgeland's stability manifold.

Ex $D^{\text{fd}}(A)$, A : proper connective dga, \mathcal{S} : simple $H^0(A)$ -mod's