Chapter 4
Church-Turing Thesis

We will now look into a much more powerful automata: the *Turing Machine,* which comes with a unlimited and unrestricted memory.

A Turing machine can do anything that a real computer can, with the difference being that ☺ it has all the time it needs to get something done. Efficiency is no longer an issue, a departure from *CS 3221 Algorithm Analysis*.

Anything it cannot solve is regarded beyond the theoretical limits of computation. In other words, it is not *solvable.* ☹

The Turing machine model uses an infinite tape as its memory and has a tape head that can read and write symbols and move both forward and *back* on the tape, similar to a large RAM in a real computer.

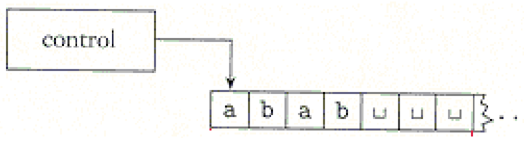
This bi-direction possibility changes everything.

How does it work?

Initially, the tape contains only the input string, and blanks,'␣' everywhere else. If the machine needs to store information, it just write this information on the tape.

To read the information that it once wrote on the tape, the machine can move its head *back and forth* over the tape, which makes all the difference as it may form a loop. ☺

The machine keeps on working until it has some output to send out. Then, it enters designated, *accepting* and/or *rejecting* states, to either accept, or reject, its input.



On the other hand, *if it doesn't either accept or reject a string, it may go on forever.* This is where the Hell breaks loose.... ☹

An example

Let $L_0 = \{w\#w : w \in \{0, 1\}\}$, which is not Context free. (Cf Example 2 on Page 80 of the CFL notes for a similar language. We now have a '#' to mark the middle position.)

Let's design a TM to accept A, by starting with a high level description of such a machine, M_1.

For a given string, s,
1. Scan the input to be sure that it contains a single # symbol. If not, reject.

2. Move across the tape to corresponding symbols on either side of the # symbol to check whether they match. If they don't, reject the string; otherwise, cross off those two symbols and continue.

3. When all symbols to the left of # have been crossed off, check for any remaining symbols to the right of the #. If any symbols remain, reject; otherwise, accept the input.

More details of M_1

The tape head begins with the leftmost symbol of the input. M_1 will record this symbol and cross it over by putting an x in its place. Then the head moves to the right until it sees the # and moves one symbol further. If it sees a blank after #, the input is not in the correct format, thus will be rejected.

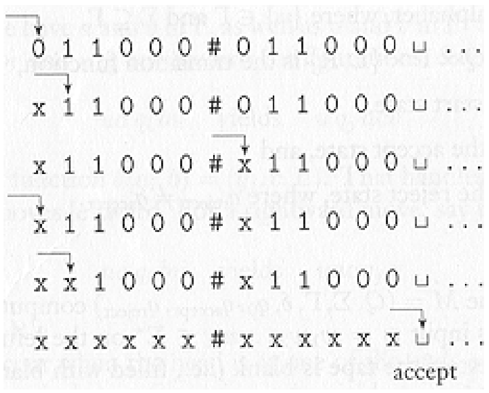
Otherwise, M_1 crosses *it* off by putting in a x, and moves back to the left until it comes to an x. Now, the head moves to the right *again*, trying to match the next pair of corresponding symbols, the same as it started.

It will go over all the way, beyond the # sign, to the last x. If there is nothing to the right of this last x, it rejects the input; otherwise, it puts down another x and comes back again.

When it goes left, if the symbol *it* sees after the rightmost x is a #, all the symbols to the left of # were successfully matched. Then, M_1 will move over any x's to the right of #.

If the first symbol after # and x's is either 0 or 1, the input string gets rejected; otherwise, if it is a blank, M_1 accepts the input by entering the accepted state.

Below shows how the input 011000#011000 gets processed.


0 1 1 0 0 0 # 0 1 1 0 0 0 ␣ ...
x 1 1 0 0 0 # 0 1 1 0 0 0 ␣ ...
x 1 1 0 0 0 # x 1 1 0 0 0 ␣ ...
x 1 1 0 0 0 # x 1 1 0 0 0 ␣ ...
x x 1 0 0 0 # x 1 1 0 0 0 ␣ ...
x x x x x x # x x x x x x ␣ ...
accept

Questions: Does this algorithm work for $D = \{ww : w \in \{0,1\}^*\}$? Where is the middle position?

What does a TM look like?

Definition: A *Turing Machine* is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are all finite, where

1. Q is a set of states, including both q_{accept}, and q_{reject}.

2. Σ is the input alphabet not containing the special blank symbol, '␣'.

3. Γ is the tape alphabet containing '␣' and everything that occurs in the tape.

As always, the heart of the formal definition of a TM is the transition function. It takes the form: $Q \times \Gamma \mapsto Q \times \Gamma \times \{L, R\}$.

In particular, $\delta(q, a) = (r, b, L)$ means that if the machine is at state q, and the input symbol is a, then it replaces a with b, enters a new state r, and moves one position to the left.

An example

Let $L_1 = \{a^n b^n : n \geq 1\}$, we design a Turing machine that accepts it. We start with an algorithm.

For a given string, $s \in \{a, b\}^*$.

1. If the string starts with a b, reject.

2. For every a, move across the tape to look for a matching b. If we could not find such a b, reject.

3. When all the a's are matched, check for any remaining b. If any such b remains, reject; otherwise, accept the input.

Now, we will do some walking by implementing this algorithm into a detailed TM. ☺

An implementation

Let $TM_1 = (Q, \Sigma, \Gamma, \delta, q_0, \{q_4\}, \{q_5\})$, where $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$, $\Sigma = \{a, b\}$, $\Gamma = \{a, b, x, y, B\}$, and the δ function is given as follows:

$$\begin{array}{rcl}
1.\ \delta(q_0, b) & = & (q_5, b, R) \\
2.\ \delta(q_0, a) & = & (q_1, x, R) \\
3.\ \delta(q_0, y) & = & (q_3, y, R) \\
4.\ \delta(q_1, a) & = & (q_1, a, R) \\
5.\ \delta(q_1, b) & = & (q_2, y, L) \\
6.\ \delta(q_1, y) & = & (q_1, y, R) \\
7.\ \delta(q_1, B) & = & (q_5, B, L) \\
8.\ \delta(q_2, a) & = & (q_2, a, L) \\
9.\ \delta(q_2, x) & = & (q_0, x, R) \\
10.\ \delta(q_2, y) & = & (q_2, y, L) \\
11.\ \delta(q_3, y) & = & (q_3, y, R) \\
12.\ \delta(q_3, b) & = & (q_5, b, R) \\
13.\ \delta(q_3, B) & = & (q_4, B, L)
\end{array}$$

Question:I am dizzy... ☹ What is going on?

What does each state mean?

In q_0 we try to match off another a, and would fail if we cannot find an a by getting into q_5.

In q_1, holding an a, flipped to x, we trace through all the remaining a's and y's, flipped b's, trying to find a b to match with *this* a. If we find it, flip *it* to y; otherwise we reject the string in q_5.

In q_2, we have found the b we are looking for, so we go back to the beginning, bypassing all the y's and a's, trying to match off more a's, until we see an x, then we start again with q_0.

In q_3,all the a's have been matched, so we try to get to the end of the string to see if there are still b's there. If there is, the string is rejected in q_5; otherwise, it is accepted in q_4.

If we come over to q_4, we accept the string, and reject it if we end up with q_5.

Homework

1. Use *JFLAP* to set up TM_1 and run it with several input strings. Notice that *JFLAP* does not have a reject state, so rules 1, 7, and 12, where q_5 occurs, should be deleted. When the machine, e.g., is in state q_0, and the input is b, the machine will reject the input, since no matching rule can be applied.

2. Modify TM_1 so that 1) it will accept $L = \{a^n b^n | n \geq 0\}$, and 2) it will then restore the original string.

3. Design a TM that copies a string. For example, if it finds a string "`abaa`" on the input tape, it should leave "`abaa abaa`" in the tape when it stops.

Send in screen shots of your machines and your testing.

How does a TM work?

Initially, M receives its input $w = w_1 w_2 \cdots w_n \in \Sigma^*$ on the leftmost n squares of the tape, and the rest of the infinite tape is blank.

The tape head starts on the leftmost square. As '$\sqcup$' $\not\in \Sigma$, so the first '$\sqcup$' marks the end of the input.(?)

Once M starts, the computation proceeds according to δ.

If M ever tries to move its head off the left end, the head stays in the same place for that move. *JFLAP* behaves differently, since it is infinite both ways. ☺ As we will see shortly, *this* does not lead to a more powerful machine.

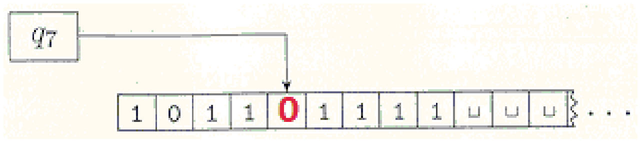
The computation continues until it either enters the accept or the reject state, at which it *halts.* If neither occurs, M goes on forever, which might drive us nuts. ☹

Homework: Exercise 3.1.

Configuration

At any point, the *configuration* of M is represented as uqv, which means that M is in the state q, and the current tape content is uv, and the tape head is located at the first symbol of v. The tape contains all blanks after v.

For example, $1011q_7 0 1111$ represents a configuration in which the tape contains a string 101101111, the current state is q_7, and the head is located at 0.



Question: How does a TM move from one configuration to next?

Answer: As usual, such a movement is specified by δ, the transition function.

The *yield* relation

Let $a, b \in \Sigma$, and $u, v \in \Sigma^*$. We have the following: 1. The configuration uaq_ibv yields to uq_jacv, if $\delta(q_i, b) = (q_j, c, L)$.

2. The configuration uaq_ibv yields to $uacq_jv$, if $\delta(q_i, b) = (q_j, c, R)$.

3. The case that when the head is in the left end might be special: we have that q_ibv yields to q_jcv even if the transition says to move to the left, $\delta(q_i, b) = (q_j, c, L)$.

4. The case that when the head in the right end is not special, as $uaq_i \equiv uaq_i\sqcup$. Thus, uaq_i yields to $uacq_j$, if $\delta(q_i, \sqcup) = (q_j, c, R)$.

The *start configuration* for any TM on input w is q_0w. The state in the *accepting*(*rejecting*) configuration is $q_{\text{accept}}(q_{\text{reject}})$, respectively, and are both *halting* configurations.

A *halting* configuration doesn't yield to any other configuration. We are done. ☹

The *accept* relation

Pretty much similar to the accept relation in the FA case (Cf. Page 17 in FA notes), a Turing Machine *accepts* an input s, if a sequence of configurations $C_1, C_2, \ldots, C_k$, exists, where

1. C_1 is the start configuration of M on s, q_0w,

2. each C_i yields to C_{i+1}, and

3. C_k is an accepting configuration, where the state is q_{accept}.

Finally, the collection of strings that M accepts is *the language of* M, denoted by $L(M)$.

For example, $L(TM_1) = L_1$ $(= \{a^n b^n : n \geq 1\})$.

Thus, a TM is also an algorithm, albeit a much more detailed one.... ☺

Homework: Exercise 3.4

Enumerable languages

Definition: A language is *(recursively) enumerable* if some Turing Machine accepts it.

In other words, a language L is enumerable if there exists a TM M that $L = L(M)$, the collection of all the words that M accepts.

Procedurally, given a word $w \in \Sigma^*$, if $w \in L$, M will accept it. If $w \notin L$, M will not accept it, i.e., M will not get into the accepting state.

Notice that, if it does not get into the accepting state, it does not mean it will necessarily get into the rejecting state.

Question: Is there a problem?

Answer: Yes, a big one. ☹

Decidable languages

There could be two cases when M does not accept w : we have no problem if it gets into the rejecting state.

But, if it gets into an infinite loop, we have no way to know if it has fallen into it, or it is just still in the process of working this out.

As a result, we prefer those machines that halt at, i.e., either accept or reject, all input strings. In other words, for all words $w \in \Sigma^*$, after working for a finite amount of time, M either gets into the accepting state, or rejecting state. Such a machine is called a *decider.* A decider that accepts some language also is said to *decide* that language.

Definition: A language is *decidable (recursive)* if some Turing Machine decides it. Procedurally, given a word $w \in \Sigma^*$, if $w \in L$, M will accept it. If $w \notin L$, M will reject it.

An example

Let S be the collection of the names of all the people who committed some crime before.

Theoretically, such a language is decidable, FBI's system might be the decider: Given a name of a suspect, n, it will look at all the records. If the name turns up, $n \in S$; otherwise, $n \notin S$.

Realistically, the language is not even enumerable, in the sense that the above system will not pick up every ex-con, since, for some reasons, this one has never been caught. ☹

Thus, if a name shows up in some machine, $n \in S$; otherwise, we don't know.

We will show that some languages are *theoretically* undecidable, and even not enumerable. ☹

Question: Do you now see the difference between recursively enumerable and decidable (recursive) languages as shown on Page 89 of the *CFL* notes? We will fix it later... .

Variants of TMs

There are many variants of Turing Machines.

An astonishing fact is that the original model and those “reasonable” variants all have the same computing power, i.e, they accept the same class of languages.

In other words, they speak the same language.

For example, in the original model, we only allow the machine to move either left or right, what happens if we also allow it to stay in the same place?

For example, *JFLAP* provides this ‘`S`’ option of “same”, besides ‘`L`’ and ‘`R`’.

Question: Will this change let a Turing Machine accept more languages?

The answer is ‘No’...

For any transition rule to make the machine stay in the same place, we can replace it with a sequence of two rules, for a symbol $x \in \Sigma$, one moves the head to the right, and the next moves *it* back, where r is a new state.

$$\begin{aligned} & \delta(q,a) = (p,b,S) \\ \equiv \quad & (\delta(q,a) = (r,b,R), \delta(r,x) = (p,x,L)), \end{aligned}$$

We will further show the equivalence of the original model and several other models, by simulating each other, i.e., whatever we can do with one model, we can do it with the other one.

Thus, all such models are equivalent to each other. ☺

This tells us that computation is a natural process, which can be characterized by, e.g, TMs.

Multitape TMs

A *multitape TM* is like an ordinary TM, with several tapes, each of which has its own tape head to read and write. *JFlap* provides one.

Initially, the input is put on the first tape, while the other tapes are empty. The transition function is changed to allow for reading, writing, and moving the heads on all the tapes simultaneously. Formally, we have

$$\delta : Q \times \Gamma^k \;\mapsto\; Q \times \Gamma^k \times \{L, R\}^k,$$

where k is the number of tapes.

The expression

$$\delta(q_i, a_1, \ldots, a_k) = (q_j, b_1, \ldots, b_k, L, R, \ldots, L)$$

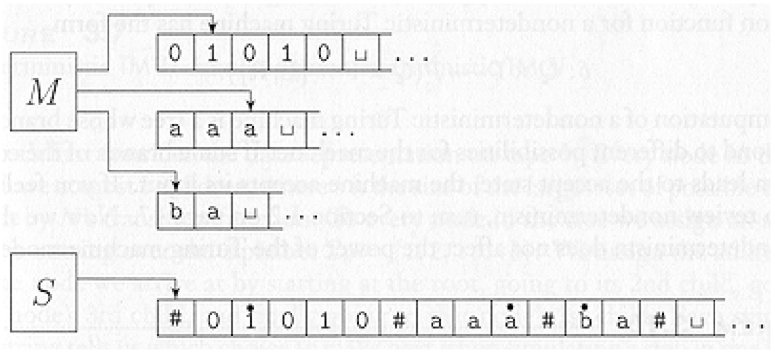
means that, if the TM is in state q_i, and heads 1 through k are reading symbols $a_1, \ldots, a_k$, then it goes to state q_j, writes $b_1, \ldots, b_k$, and moves each head to either left or right, as specified.

Theorem: A language is enumerable if and only if some multitape TM accepts it.

Proof: Since an ordinary TM is automatically a k-tape TM, when $k = 1$; the only thing we need to show is that, if a multitape TM, M, accepts a language, so does an ordinary one, S.

We show how to let S store all the information that M stores in its k tapes. We use $k + 1$ #'s as delimiters to separate the respective contents and use corresponding "dotted" tape symbols to keep track of the locations of those k heads.

The following shows how to use a one-tape machine to simulate one with multiple tapes.

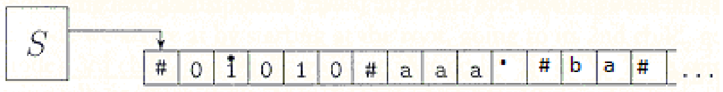


Here we go...

1. S uses its only tape to represent all the information stored in M in the following format:
$\# \overset{\bullet}{w}_1 \# \cdots \# \overset{\bullet}{w}_n \# \overset{\bullet}{B} \# \overset{\bullet}{B} \# \cdots \#$

2. To simulate a single move, S scans its tape from the first #, to the $(k+1)^{\text{st}}$ #, to determine the symbols which should be under the k dotted heads. Then, S scans the tape again to update the tapes according to the various transition rules.

3. If S tries to move across a #, S writes a '␣' on this tape cell, shifts all the tape contents, from this cell on, to its right by one cell, and continues as before. This is what happens when head 2 tries to move over...



It does take lots of time, but it works.

Non-deterministic TMs

We know that $NFA \equiv DFA$, but $NPDA \not\equiv PDA$(Cf. Page 27, CFL notes), we now show that $NTM \equiv TM$.

A non-deterministic TM, at any point, may proceed according to several possibilities. Thus, the transition function, δ, has the following format:

$$\delta : Q \times \Gamma \ \mapsto \ \mathcal{P}(Q \times \Gamma \times \{L, R\}).$$

Hence, the computation of such a TM is a tree, in which each branch is a configuration. If any branch of such a tree accepts an input, the TM will accept the input. (Cf. Pages 43-44 of the *NFA* notes)

Theorem: A language is enumerable if and only if some non-deterministic Turing Machine accepts it.

Again, we only need to prove the other half of the result, as TM is an NTM.

A rough idea..

Beginning with the starting configuration, we will let D, the simulating TM, try all branches of N, the NTM, *within a certain number of steps.*

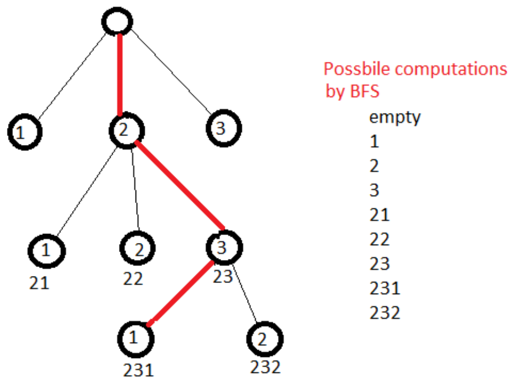
If D ever finds one branch with which N accepts the input, so does D. If all branches within that depth lead to rejection, D will try the next one. Thus D might go on forever.

The only way not to miss any possible halting outcome is to exhaustively traverse the tree in *breadth first search,* which we went over in *CS 3221 Algorithm Analysis*.

The key is to use an *address tape* which tells the location of next node to be traversed. Such a tape contains strings over $\Sigma^b = \{1, 2, \ldots, \}$, where b is the size of the largest set of possible choices decided by δ.

An example

For example, the string 231 means that starting at the root, we will try the second child, then that node's third child, finally, goes to the latter's first child.

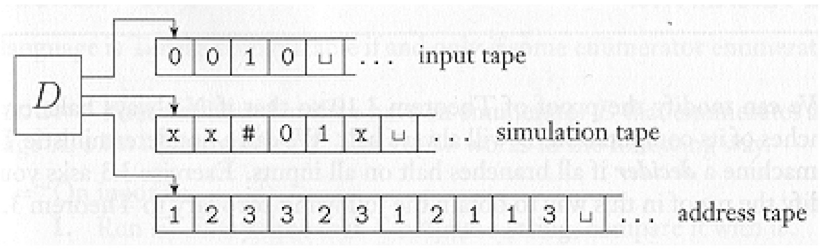


Let an empty string represent the root of such a tree, and let d be the maximum number of transition rules, we use 1, 2, ..., d, 11, 12, ..., 1d,, 21, 22, ..., 2d, etc., to exhaustively traverse all the possible paths (derivations) of a certain depth.

Notice this process is exhaustive, and maybe infinite, as this depth is infinite. ☹ But, if a string is accepted, it will be at some depth. ☺

How should we do it?

D uses three tapes, as shown below. Tape 1 always contains the input, Tape 2 maintains a copy of N's tape on some branch of its non-deterministic computation. Tape 3 keeps track of D's location in N's non-deterministic tree.



The deterministic TM D works as follows:

1. Initially Tape 1 contains the input w, and both Tape 2 and 3 are empty.

2. Copy Tape 1 to Tape 2.

Question: What does "12332312113" represent? ☺

3. Use Tape 2 to simulate N with input w on one branch of its non-deterministic computation, by using the address stored in Tape 3.

D accepts w if N *accepts it using the current address in Tape 3.*

If Tape 3 is empty, or the choice is invalid, or this branch rejects the input, abort this step and go to Step 4.

4. Replace the string in Tape 3, with the lexicographically next string. Simulate the next branch of N's computation by going back to Step 2.

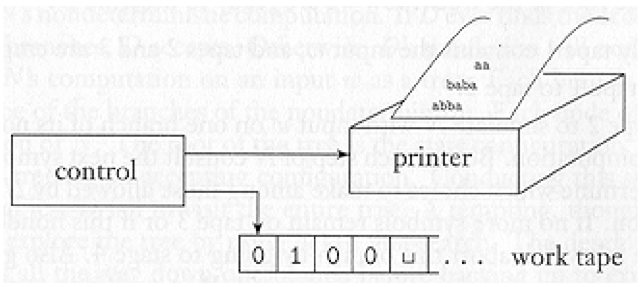
Thus, D accepts the same language as that of N.

By the previous result, there is also a single tape TM that accepts the original language. ☺

Enumerators

With the original model of TM, we focus on its input, thus a scanner? Roughly speaking, an *enumerator* is a Turing Machine that is attached to a printer.

Every time such an enumerator accepts a string, the machine prints it out. The language accepted by an enumerator is the collection of all the strings it ever prints out.



Thus, the concept of enumerable language in term of a TM is rather input oriented; while the concept of an enumerator is output oriented. However, this difference does not matter. ☺

Theorem: A language is enumerable if and only if some enumerator enumerates it.

One side is easy...

Proof: Assume a language is accepted by an enumerator, E, we construct the following TM, M, that works in the following way.

For any input w,
1. Run E, every time that E outputs a string, compare it with w,

2. If w *ever* appears in the output of E, accept.

If $w \in L(E)$, it will be printed out *eventually.*

Thus, M accepts a string when and if E prints it out, namely, it accepts the same set of strings that E enumerates; and will not accept anything that E doesn't print out.

In other words, for any enumerator E, we can construct a TM M such that $L(M) = L(E)$.

The other side is not so easy...

Now, assume that a TM M accepts a language, we construct the following enumerator, E.

Question: Given a word $w \in \Sigma^*$, should E print it out?

We give it to M, and let it run. If M accept it, we print it out; otherwise, if M rejects it, we won't.

Question: What happens if M runs it for a *long* time?

Do we know M has got into an infinite loop, so, we should not print it out; or is it still working on it, so we should wait a bit longer?

A Shakespearean moment: To do, or not to do. That is the question.

Scratching head.... ☹

What to do...

Let $s_1, \ldots, s_n, \ldots$ be a list of all the possible strings over Σ^*, we can construct the enumerator E as follows:

1. Repeat the following for $i = 1, 2, 3, \ldots,$

2. Run M for i steps for $s_1, s_2, \ldots, s_i$.

3. If any of the above computation accepts s_i, print it out.

Since, if a word w is accepted by M, it will be accepted *within a finite amount of steps,* it will be accepted with a certain fixed i. On the other hand, if M gets into an infinite loop, it will never be accepted by M, thus will not be printed.

Thus, E will print out a string if M accepts it; and will not print out anything that M does not accept.

Homework: Exercise 3.7.

More equivalence

Besides those presented so far, many other models for general computation have been presented. Some of them are very much like TMs, some others look quite different such as Church's λ-calculus (Cf. Course page).

All of them allow unrestricted and unlimited memory.

Remarkably, all models with this feature turns out to be equivalent to each other in the sense that whatever you can do with one of the models, you can do the same with any of the others. ☺

This observation shows that, although there are different ways to characterize the general computation process, *the class of algorithms that they are describing is unique and natural.*

Just an example

For example, we can use either *C* (*CS 2470*) or *Java* (*CS 2381*) to write programs and these two languages look quite different from each (Cf. Check out their grammars in the resource section of NPDA notes.)

However, as we can write a compiler of *C* in *Java* and write an interpreter of *Java* in *C*, the programs in either language can be turned into its equivalent in the other language, or eventually in a machine language in the host machine.

Hence, those two languages are really equivalent: If you can write a program to implement one algorithm in one language, you can write another to implement the same algorithm in the other language.

Homework: Exercise 3.2

The definition of algorithms

Informally, an *algorithm* is a collection of simple instructions for carrying out some task.

This notion is informally used in place of a procedure or a recipe.

For a long time, there has been no precise definition of algorithm. People only use it in a vague and intuitive sense. ☹ For example, if we know, intuitively, there exists a way to do something, we say something is "computable."

It is quite fine in this positive sense. *The problem is how to use it in the negative sense.* For example, when we try to prove that something is not "computable", how could we show there does not exist an algorithm?

Could that be the case that you have not worked hard and/or long enough? ☺

In these places, we need a precise mathematic definition of algorithm. Either "yes" or "no".

Hilbert's tenth problem

In 1900, a mathematician David Hilbert presented a talk at the International Congress of Mathematicians in Paris, in which he identified 23 problems and posed them as a challenge for the coming 20th century, i.e., the one that was over almost 24 years ago. ☺

His tenth problem is to devise an algorithm that tests if a polynomial has an integral root.

For example, the following polynomial

$$6x^3yz^2 + 3xy^2 - x^3 - 10$$

has a root at $x = 5, y = 3$, and $z = 0$.

In his talk, he implicitly assumed that such an algorithm, or "process", must exist, the only issue is how, and when, to dig *it* out. ☺

It was shown, back in 1970, that no such algorithm exists. In other words, *it is algorithmically unsolvable.* ☹

It is acceptable,...

Let D be $\{p : p$ is a polynomial with an integral root $\}$.

Then, Hilbert's tenth problem is simply asking if D is decidable.

This problem is enumerable, e.g., there is a Turing machine, M_1, that will accept the language D_1, which is defined as $\{p : p$ is a polynomial over x with an integral root $\}$.

Below is the Turing machine that accepts D_1.

$M_1 =$ "The input is a polynomial p over the variable x.

1. Evaluate p with x set successively to the values 0, 1, -1, ...

2. If at any point the polynomial evaluates to 0, accept."

..., but not decidable.

M could be similarly constructed. Both M and M_1 are enumerators, or recognizers, but not deciders for the respective language. ☹

But, M_1, on just one variable, can be converted to a decider, by calculating the bounds within which the roots of a *single variable* polynomial must lie, and evaluating the polynomial against only that *finite bound.* ☺

As a matter of fact, the involved range is

$$(-k\frac{c_{\max}}{c_1}, k\frac{c_{\max}}{c_1}),$$

where k is the number of terms, $c_{\max}$ is the coefficient with largest absolute value, and c_1 is the coefficient of the highest order term.

For $6x^3 - 31x^2 - x - 10$, $k = 4, c_1 = 6$, and $c_{\max} = 31$.

However, it has been shown in 1970 that it is *impossible* to calculate such bounds for *multi-variate* polynomial. ☹

The Thesis

The definition of algorithm was suggested in the 1936 paper by Alan Turing in terms of his machine, while Alonzo Church used, also in 1936, a notational system, called the λ-calculus, to define algorithms.

These two definitions were shown to be equivalent to each other, as well as to many other alternative definition of "computation", as we saw some of them earlier in the notes.

This connection between an informal notion of algorithms and the precise definition has come to be called the *Church-Turing Thesis:*

The informal notation of algorithms equals that of Turing Machines algorithms.

What does it mean?

This Thesis cannot be proved. It can only either be accepted or rejected. ☺

It proposes that we give the necessarily informal notion of algorithms some precise meaning.

A Turing machine is of a mechanical nature, thus is certainly an algorithm. On the other hand, if we accept this thesis, then whenever we see some intuitive procedures, we agree that it actually characterizes a formal TM. *However, we must fill in all the necessary details, when needed.*

This Thesis has not failed us so far....

Thus, from now on, we will focus on algorithms, rather on the detailed description of Turing machines.

Coding of inputs

The input to a TM is always a string. If we want to provide an object other than a string as an input, we must *encode* this object into a string.

Strings can easily represent numbers, polynomials, graphs, grammars, automata, and any combination of those objects.

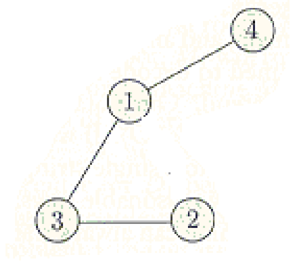
A TM can be constructed to both *encode* an object into a string, and *decode* it back so that it can be interpreted in the intended way by a user.

Our notation for the encoding of an object, O, into a string is $\langle O \rangle$. If we have several objects, $O_1, \ldots, O_k$, we denote their encoding into a single object as $\langle O_1, \ldots, O_k \rangle$.

How to encode a graph?

A graph $G(V, E)$ is typically represented as an ordered pair, whose first element, V, is the collection of all the nodes, while the second element, E, is the collection of all the edges, each of which is also a pair of nodes.

Hence, one way to encode a graph is to represent it as a list of its nodes, followed by another list, which represents the edges. For example, below is a graph.



The code for such a graph could be $\langle G \rangle = (1, 2, 3, 4)\ ((2, 3), (1, 3), (1, 4))$.

Is *this* a graph?

When M receives its input, it has to check the input to make sure that it represents a graph, i.e., the input consists of two lists. The first one should be a list of distinct numbers, and the second should be a list of pairs of numbers, each has occurred in the node list.

It also checks other things. For example, no repetition in the first list, and every component that occurs in the second list must occur in the first list: Every edge is a pair of vertices of that graph.

Once all such checking went through, we know this is a correct representation of a graph, and we can use it to solve some graph related problems.

We are ready... . ☺

An example

G is connected if there is a path between any pair of its vertices. Let

$$A = \{\langle G\rangle : G \text{ is a connected graph.}\}.$$

The following is an algorithm that decides A.

Given an input $\langle G\rangle$, the encoding of a graph G,

1. Select the first node of G and mark it.

2. Repeat the following step until no new nodes can be marked.

2.1. For each node in G, mark it if this node is associated via an edge to a node that is already marked.

3. Scan all the nodes of G to determine whether they are all marked, if they are, accept; otherwise reject.

Question: Can a computer understand this algorithm? ☺ Probably not. ☹

Let’s give *it* more details...

1. Mark the first node, with a dot on the leftmost digit, e.g., $\overset{\circ}{1}$.

2. Scan the list of nodes to look for an undotted one.

2.1. If all the nodes are dotted, accept the input.

3. Let one of the undotted nodes be n_1, and underline this node, e.g., $n_1 = \underline{2}$.

3.1. Scan the list of nodes to find a dotted one, n_2, by underlining it, too, e.g., $n_2 = \underline{\overset{\circ}{1}}$.

3.2. For each edge, check if (n_1, n_2) is an edge.

3.3. If they are, dot n_1, remove its underlining, and go back to Step 2.

3.4. Otherwise, if the edge list is not over, yet, check for the next edge.

3.5. If there is no more edge left, (n_1, n_2) is not an edge in G, remove the underlining of n_2.

Since $(1, 2)$ is not an edge, we will keep on looking in the edge list until the cow comes home, when $\underline{2}$ is decrowned. ☹

3.6. If there are still undotted nodes that have not been underlined, repeat this step. Otherwise reject.

We will go back to Step 3, set $n_1 = \underline{3}$. Since $(1, 3)$ is an edge, we get $\overset{\circ}{3}$. We then go back to Step 2 to set $n_1 = \underline{2}$ and will get $\overset{\circ}{2}$, since $(2, 3)$ is an edge; and go back to Step 2, and set $n_1 = \underline{4}$ and will get $\overset{\circ}{4}$, since $(1, 4)$ is an edge.

Now all the nodes are dotted, Step 2.1 will accept this graph.

By the Thesis, there is no need to dig any further. This problem is solvable.