Let $M(t)$ be a differentiable matrix-valued function such that $M(t)$ is always invertible with a full set of distinct eigenvalues $\lambda_i(t)$.

$$\sum_i \lambda_i^n \dot{\lambda}_i = \mathrm{Tr}(M^n \dot{M})$$

$$\vec{v}_i = \mathrm{Tr}(M^i \dot{M}) \wedge \vec{\dot{\lambda}}_i = \lambda_i \wedge A_{ij} = \lambda_i^j$$

$$\implies A\vec{\dot{\lambda}} = \vec{v}$$

$$\implies \dot{\lambda}_i = (A^{-1}\vec{v})_i = A_i^{-T} \cdot \vec{v}$$

$$A^{-1} = \frac{1}{\det A} C^T \implies A_i^{-T} = \frac{1}{\det A} C_i$$

$$C_{ij} = (-1)^j \left(\prod_{k<m \wedge k, m \neq i} (\lambda_m - \lambda_k) \right) \left(\sum_{(S \subseteq \{1,\ldots,n\} \setminus \{i\} \wedge |S| = n-1-j)} \prod_{k \in S} \lambda_k \right)$$

$$p_B(x) = \det(xI - B)$$

$$p_{\neq i}(x) = \lim_{y \to x} \frac{p_M(y)}{\lambda_i - y}$$

$$\implies \sum_{(S \subseteq \{1,\ldots,n\} \setminus \{i\} \wedge |S| = n-1-j)} \prod_{k \in S} \lambda_k = \frac{p_{\neq i}^{(j)}(0)}{j!}$$

$$\prod_{k<m \wedge k, m \neq i} (\lambda_m - \lambda_k) = \left(\prod_{m<k} (\lambda_m - \lambda_k) \right) \Big/ \left(\left(\prod_{m<i} (\lambda_i - \lambda_m) \right) \left(\prod_{i<m} (\lambda_m - \lambda_i) \right) \right)$$

$$= (-1)^j \det(A) \left((-1)^j \prod_{m<i} (\lambda_m - \lambda_i) \right)^{-1} \left(\prod_{i<m} (\lambda_m - \lambda_i) \right)^{-1}$$

$$= \det(A) \prod_{m \neq i} (\lambda_m - \lambda_i)^{-1}$$

$$= \det(A) \frac{1}{p_{\neq i}(\lambda_i)} = \frac{-\det A}{p_M^{(1)}(\lambda_i)}$$

$$\implies C_{ij} = -\frac{p_{\neq i}^{(j)}(0)}{j!} \frac{\det A}{p_M^{(1)}(\lambda_i)} \implies \dot{\lambda}_i = \frac{-1}{\det A} \sum_j \frac{p_{\neq i}^{(j)}(0)}{j!} \frac{\det A}{p_M^{(1)}(\lambda_i)} \mathrm{Tr}(M^j \dot{M})$$

$$= \frac{-1}{p_M^{(1)}(\lambda_i)} \sum_j \frac{p_{\neq i}^{(j)}(0)}{j!} \mathrm{Tr}(M^j \dot{M}) \tag{1}$$

$$= \frac{-1}{p_M^{(1)}(\lambda_i)} \mathrm{Tr}\left(\dot{M} P(M) \right)$$

$$P(x) = \sum_j \frac{p_{\neq i}^{(j)}(0)}{j!} x^j = p_{\neq i}(x)$$

$$\implies P(M) = p_M(M)(\lambda_i I - M)^{-1}$$

Since $\lambda_i I - M$ is not invertible and $p_{\neq i}$ is meant in the limiting sense, we take $P(M)$ to be a matrix which maps $x_j \to 0$ for $j \neq i$ and $x_i \to p_{\neq i}(\lambda_i)x = -\dot{p}_M(\lambda_i)x$ so that with Q the orthogonal projection onto x_i's eigenspace, $P(M) = -\dot{p}_M(\lambda_i)Q$.

$$
\begin{aligned}
\implies \dot{\lambda}_i &= -\frac{-\dot{p}_M(\lambda_i)}{\dot{p}_M(\lambda_i)} \operatorname{Tr}(\dot{M}Q) \\
&= \operatorname{Tr}(\dot{M}Q)
\end{aligned}
$$

This last simplification requires knowledge of the eigenvectors of M in order to calculate Q, but the earlier representation in (1) only requires knowledge of the eigenvalues.