Re-Derivation and Proof of The Bose-Einstein Integral

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1 Lemma 1

1. Given some $n \in \mathbb{N}$, $j = 0, 1, ..., n$, and $x \in \mathbb{C}$

$$C_j = \frac{1}{(0x - jx)(1x - jx)...((j-1)x - jx)((j+1)x - jx)...(nx - j)}$$

$$= \frac{(-1)^j}{n!x^n}\binom{n}{j}$$

1.1 Proof

Consider the terms $(0x - jx)(1x - jx)...((j-1)x - jx)$. By factoring out $(-1)^j x^j$ and reversing the order of the product, this is equal to $(-1)^j x^j (1)(2)...(j-1)(j) = (-1)^j x^j j!$. The same argument can be made for the remaining terms, so C_j can be represented as a much simpler fraction.

$$C_j = \frac{(-1)^j}{j!(n-j)!x^n} = \frac{(-1)^j}{n!x^n}\binom{n}{j}$$

2 Proof

Given a natural number n and unique $x_0, ..., x_n \in \mathbb{R}^+ \setminus \mathbb{Z}^-$, Let $P(x) = (x - x_0)(x - x_1)...(x - x_n)$ and now with partial fraction decomposition, the following is true.

$$\frac{1}{P(x)} = \sum_{j=0}^{n} \frac{C_j}{x + x_j} \tag{1}$$

where $C_j = \frac{1}{(x_0-x_j)(x_1-x_j)...(x_{j-1}-x_j)(x_{j+1}-x_j)...(x_n-x_j)}$ for $j = 0, 1, ..., n$. Now, using this:

$$\sum_{m=1}^{\infty} \frac{1}{P(m)} = \sum_{m=1}^{\infty} \sum_{j=0}^{n} \frac{C_j}{m+x_j}$$

$$= \sum_{j=0}^{n} \sum_{m=1}^{\infty} \left(\frac{C_j}{m+x_j} + \frac{C_j}{m} - \frac{C_j}{m} \right)$$

$$= \left(\sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \frac{1}{m} \right) - \sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+x_j} \right) *$$

$$= \left(\sum_{j=0}^{n} C_j \right) \left(\sum_{m=1}^{\infty} \frac{1}{m} \right) - \sum_{j=0}^{n} C_j H_{x_j}$$

where H_x is the xth harmonic number. The separation of sums in the starred step is the step we're unsure about. The convergence of the summation on the left seems to hold, since by replacing the ∞ with k, it can be seen that the term on the left will always equal 0. Then, taking the limit as k goes to infinity should mean that the left term equals 0 too. This is because, essentially by definition of C, the finite sum of C_js is zero. Given this reasoning, assume that the separation of sums was a valid step.

$$= -\sum_{j=0}^{n} C_j H_{x_j} = -\sum_{j=0}^{n} C_j \int_0^1 \frac{1-t^{x_j}}{1-t} dt$$

$$= -\int_0^1 \frac{\left(\sum_{j=0}^{n} C_j\right) - \sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$$

So, we have that $\sum_{m=0}^{\infty} \frac{1}{P(m)} = \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$. Assuming all other steps are valid, taking the limit as $(x_0, ..., x_n) \to (0...0)$ should converge with a P-series test of degree $n+1$. The summand is a sum of continuous rational functions defined when all $x_i = 0$. Because it converges, is continuous, and is defined as $\zeta(n+1)$, the limit must equal $\zeta(n+1)$. Because the limit exists, any path towards it will yield the same result. By parametrizating x_j into $x_j = jv$ for some v, the limit can be simplified significantly.

$$\zeta(n+1) = \lim_{(x_1...x_n)\to(0,...,0)} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$$

$$= \lim_{v\to 0} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{jv}}{1-t} dt$$

$$= \lim_{v\to 0} \frac{1}{n! v^n} \int_0^1 \frac{\sum_{j=0}^{n} \binom{n}{j} (-t^v)^j}{1-t} dt \text{ (with lemma 1)}$$

$$= \lim_{v\to 0} \frac{1}{n! v^n} \int_0^1 \frac{(1-t^v)^n}{1-t} dt$$

Now, using repeated application of L'Hopital's and induction:

$$= \lim_{v\to 0} \frac{1}{n!^2} \int_0^1 \frac{\ln(t)^n \sum_{j=0}^{n} \binom{n}{j} j^n (-t^v)^j}{1-t} dt$$

$$= \frac{1}{n!^2} \int_0^1 \frac{\ln(t)^n \sum_{j=0}^{n} \binom{n}{j} j^n (-1)^j}{1-t} dt$$

$$= \frac{(-1)^n}{n!} \int_0^1 \frac{\ln(t)^n}{1-t} dt$$

This is equivalent to the formula given using the Bose-Einstein integral shown below by applying the substitution $t = e^{-z}$ to the above integral.

$$\int_0^\infty \frac{t^x}{e^t - 1} dt = \Gamma(x+1)\zeta(x+1)$$

3 Generalizations

Using the parametrization $x_i = ix + a$, (3) can be derived as well through the same reasoning as above.

$$\Gamma(n)\zeta(n,a) = (-1)^{n-1} \int_0^1 \frac{t^{a-1} \ln(t)^{n-1}}{1-t} dt \tag{2}$$

$$\zeta(n,x) = \sum_{m=0}^{\infty} \frac{1}{(m+x)^n}$$

where $\zeta(n,a)$ is the Hurwitz Zeta function. The exponent of t in the numerator is $a-1$ instead of a because the sum in the Hurwitz Zeta function begins with 0. In addition, because the generalized harmonic number $H_{x,n} = \zeta(n,1) - \zeta(n,x+1)$, an identity with $H_{x,n}$ can be derived too. The generalized harmonic numbers are defined in (3) for $x \in \mathbb{N}$.

$$\Gamma(n)H_{x,n} = (-1)^{n-1}\int_0^1 \frac{(1-t^x)\ln(t)^{n-1}}{1-t}dt$$

$$H_{x,n} = \sum_{m=1}^{x} \frac{1}{m^n} \tag{3}$$

This can be verified for integer values of x by expanding the $(1-t^x)/(1-t)$ term into $1+t+...+t^{x-1}$, and then reversing the order of the sum and the integral. This gives $(-1)^{n-1}\sum_{i=0}^{x-1} G(i,n)$ where $G(i,n) = \int_0^1 t^i ln(t)^{n-1}$. Through integration by parts, $G(i,n) = \frac{1-n}{i+1}G(i,n-1)$. With induction, $G(n,i) = \frac{(n-1)!}{(i+1)^n}$. Now, $(-1)^{n-1}\sum_{i=0}^{x-1} G(i,n) = (-1)^{n-1}(n-1)!\sum_{i=1}^{x} \frac{1}{(1)^n} = \Gamma(n)H_{x,n}$.

To get the integrals to a form more similar to the Bose-Einstein Integral, the substitution $t = e^{-z}$ gives (4) and (5).

$$\Gamma(n)H_{x,n} = \int_0^\infty \frac{1-e^{-zx}}{e^z-1}z^{n-1}dz \tag{4}$$

$$\Gamma(n)\zeta(n,x) = \int_0^\infty \frac{e^{-z(x-1)}}{e^z-1}z^{n-1}dz \tag{5}$$