Invertible a.e. Measure-Preserving Maps as Permutations to construct Determinant on L^p Spaces

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1 Basic Properties

Definition 1 *Take $\Omega \subset \mathbb{R}^n$ to be an open and bounded Lipschitz Domain with $0 < \mu(\Omega) < \infty$.*

Definition 2 *The set of all invertible a.e. measure preserving maps (τ such that $\mu(X) = \mu(\tau(X))$) on a set Ω is denoted $\sigma(\Omega)$.*

Proposition 1 *For $\tau, \gamma \in \sigma(\Omega)$ we have that $\tau \circ \gamma \in \sigma(\Omega)$.*

Proposition 2 *For all $\tau \in \sigma(\Omega)$, there exists some function g such that $\tau \circ g = \mathrm{Id}$ almost everywhere. In addition, $g \in \sigma(\Omega)$ as well. g is usually denoted as τ^{-1} even though the actual inverse is not explicitly defined and need not exist.*

2 Rigidity and Form

Definition 3 *We let $I(\tau)$ be some any set $\subseteq \Omega$ with on which τ is equivalent a.e. to an invertible function and such that $\mu(I(\tau)) = \mu(\Omega)$.*

Proposition 3 *For all $\tau \in \sigma(\Omega)$ and any d-dimensional Hausdorff measurable set $K \subset I(\tau)$, we have that $H^d(K) = H^d(\tau(K))$ for all $1 \leq d \leq n$.*

Proof. We have the following.

$$H^d_\delta(K) = \inf \left\{ \sum_i (\operatorname{diam} U_i)^d : K \subseteq \cup_i U_i \wedge \operatorname{diam} U_i < \delta \right\}$$

Each U_i can be taken without loss of generality to be a countable union of balls so that their union is then again a countable union of balls. Therefore, we can assume without loss of generality that each U_i is just a ball.

$$= \inf \left\{ \sum_i (\operatorname{diam} B(x_i, r_i))^d : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\}$$

$$= \inf \left\{ \sum_i (C_n \mu(B(x_i, r_i)))^{d/n} : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\}$$

$$= \inf \left\{ \sum_i (C_n \mu(\tau(B(x_i, r_i))))^{d/n} : \tau(K) \subseteq \cup_i \tau(B(x_i, r_i)) \wedge 2r_i < \delta \right\}$$

For any set $X \subset \Omega$ we have that $(C_n \mu(X))^{1/n} \leq \operatorname{diam}(X)$ because if $\mu(B(x, \varepsilon)) = \mu(X)$, then $(C_n \mu(X))^{1/n} = (C_n \mu(B(x, \varepsilon)))^{1/n} = \operatorname{diam} B(x, \varepsilon) \leq \operatorname{diam} X$ as an n-sphere minimizes diameter for a given volume.

$$\leq \inf \left\{ \sum_i (C_n \mu(V_i))^{d/n} : \tau(K) \subseteq \cup_i V_i \wedge (C_n \mu(V_i))^{1/n} < \delta \right\}$$

$$\leq \inf \left\{ \sum_i (\operatorname{diam} V_i)^d : \tau(K) \subseteq \cup_i V_i \wedge \operatorname{diam}(V_i) < \delta \right\}$$

$$= H^d_\delta(\tau(K))$$

Now, taking limits as $\delta \to 0$ we have that $H^d(K) \le H^d(\tau(K))$.
Therefore, $H^d(\tau(K)) \le H^d(\tau^{-1}(\tau(K)) = H^d(K)$ so that $H^d(\tau(K)) = H^d(K)$.

Proposition 4 *For all $\tau \in \sigma(\Omega)$, there exists a function g_τ continuous on an open set S with $\mu(S) = \mu(\Omega)$ such that $g_\tau = \tau$ almost everywhere.*

Proof. We have that for all n there exists some $S_n \subset \Omega$ such that τ is continuous when restricted to S_n and $\mu(\Omega \setminus S_n) < 1/n$. Now, with $S = \cup_{i=1}^{\infty} S_i$ we have that τ is continuous on the S-inherited subspace topology and that $\mu(S) = \mu(\Omega)$ so that S is dense in Ω.

Now, let $G = \overline{\{(x, \tau(x)) \mid x \in S\}}$.
For any $x \in \Omega \setminus S$, we have that there exists some sequence $\{x_n\} \subset S$ convergent to x. Now, $\tau(x_n)$ is bounded and thus has some subsequence convergent to some y. Now, $(x, y) \in G$. Thus, with $p_1(x, y) = x$, we have that $p_1(G) = \Omega$.

Next, let $Q = \{x \in \Omega \mid ((x, y) \in G \wedge (x, z) \in G) \implies y = z\}$. If $x \in S$ then if $(x, s_1) \in G$ and $(x, s_2) \in G$ we have that there must be some sequences $(x_{in}, \tau(x_{in})) \to (x, s_i)$ for $x_n \in S$. As $|x_{in} - x| \to 0$ we have that because τ is continuous in S that $|\tau(x_{in}) - \tau(x)| \to 0$, and thus that $|\tau(x_{1n}) - \tau(x_{2n})| \to 0$ so that because $|\tau(x_{in}) - s_i| \to 0$ we have that $|s_1 - s_2|$ is arbitrarily small and thus $s_1 = s_2$. Therefore, $x \in Q$ so that $S \subseteq Q$.

Finally, define $g_\tau : \Omega \to \Omega$ by $g_\tau(x) = y$ if $(x, y) \in G$, where y is chosen arbitrarily for $x \notin Q$.
We aim to show that g_τ is continuous on S. For $x \in S$ we have that for any $\varepsilon > 0$ there exists some δ such that $|x - y| < \delta$ for $y \in S$ implies that $|\tau(x) - \tau(y) < \varepsilon$. Then, take any $y \in \Omega \setminus S$ with $|x - y| < \delta$.

If $y \notin Q$, then assume (y, z_1) and (y, z_2) are in G. Now, there are sequences $\{s_n\}, \{t_n\} \subset B(x, \delta) \cap S$ such that $s_n, t_n \to y$, $\tau(s_n) \to z_1$, and $\tau(t_n) \to z_2$. Then, we have that $|\tau(s_n) - \tau(x)) < \varepsilon$, and for any $\varepsilon_2 > 0$ we have that there exists some N such that $n > N \implies |z_1 - \tau(s_n)| < \varepsilon_2$. Then, $|z_1 - \tau(x)| < \varepsilon + \varepsilon_2$ so that $|z_1 - \tau(x)| \le \varepsilon$. Similar logic shows that $|z_2 - \tau(x)| \le \varepsilon$.

If $y \in Q$, then $(y, z) \in G$. We have that there must be some sequence $\tau(s_n) \to z$ for $\{s_n\} \subset S \cap B(x, \delta)$ so that $|z - \tau(x)| \le |z - \tau(s_n)| + |\tau(s_n) - \tau(x)| \le \varepsilon_2 + \varepsilon \to \varepsilon$. Therefore, $|z - \tau(x)| \le \varepsilon$ as well.
Finally, regardless of the choice of value of $g_\tau(y)$ outside of Q, we have that g_τ is continuous at x.
Now, with S_2 the set of all x such that $g_\tau(x)$ is continuous at x, we have that S_2 is open and that $S \subseteq S_2$ so that $\mu(S_2) = \mu(\Omega)$. Therefore, S_2 is the desired set, and because $g_\tau(x) = \tau(x)$ for $x \in S$ we have that $g_\tau \equiv \tau$.

Definition 4 *For some $\tau \in \sigma(\Omega)$, we define $S(\tau)$ to be S_2 as above and g_τ to be g_τ as above.*

Corollary. We can choose the values of g_τ on $\Omega \setminus S(\tau)$ so that $g_\tau(\Omega) = \Omega$.
Corollary. $S(\tau) \subseteq I(\tau)$ by construction, so we can redefine $I(\tau) = S(\tau)$.
Corollary. $S(\tau^{-1}) = g_\tau(S(\tau))$

Proposition 5 *$S(\tau)$ can be partitioned into disjoint sets $\{A_\alpha\}_{\alpha \in I}$ such that $g_\tau|_{A_\alpha} = U_\alpha x + v_\alpha$ for U_α a unitary linear map and v_α a constant vector.*

Proof. We have that g_τ is continuous on an open set $S(\tau)$ with $\mu(S) = \mu(\Omega)$ so that $\partial S(\tau) = \Omega \setminus S(\tau)$.
Now, for any $x \in S(\tau)$, there is some $B_1 = B(x, \varepsilon_1) \subset S$.

For any $y \in B_1$, let $\ell_1 = \{g_\tau(x) + a(g_\tau(y) - g_\tau(x)) \mid 0 \le a \le 1\}$. We have that $|g_\tau(x) - g_\tau(y)| = H^1(\ell_1) = H^1(g_\tau^{-1}(\ell_1))$. As $g_\tau^{-1}(\ell_1)$ is some continuous path between x and y, we have that $H^1(g_\tau^{-1}(\ell)) \ge |x - y|$. Therefore, $|g_\tau(x) - g_\tau(y)| \ge |x - y|$. Finally, we have that $|x - y| = |g_\tau^{-1}(g_\tau(x)) - g_\tau^{-1}(g_\tau(y))| \ge |g_\tau(x) - g_\tau(y)|$ so that $|x - y| = |g_\tau(x) - g_\tau(y)|$.

It is a known result that if $|g_\tau(x) - g_\tau(y)| = |x - y|$ for $g - \tau$ continuous that $g_\tau(x) = Ux + v$ for U a unitary map.

Choose some partitioning of $S(\tau)$ into balls such that $S = \cup_{\alpha\in I} B(x_\alpha, r_\alpha)$. Then, take $A_\alpha = B(x_\alpha, r_\alpha) \setminus (\cup_{i<\alpha} B(x_i, r_i))$ so that they are disjoint, $S = \cup_{\alpha\in I} A_\alpha$, and on each A_α we have that $g_\tau(x) = U_\alpha x + v_\alpha$.

Note. The partition I can be assumed to be countable because each of the A_α have been constructed to be pairwise disjoint.

Definition 5 *We take $P_{S(\tau)}$ to be the set of connected components of $S(\tau)$. Note that on each $C \in P_{S(\tau)}$ we must then have that τ is locally equivalent to a unitary map plus a shift.*

Proposition 6 *For any a.e. invertible L^p function $f : \Omega \to \Omega$, we have that $f \in \sigma(\Omega)$ if and only if for almost all $x \in \Omega$ there exists some $\varepsilon > 0$ such that there are $U \in U(n)$ and $v \in \mathbb{R}^n$ such that $f(y) \equiv Uy + vf$ for $y \in B(x, \varepsilon)$.*

Proof. We have proven the $\implies$ direction already. Now, conversely, assume that for all $x \in S$ there exists some $\varepsilon > 0$ such that $f(y) \equiv Uy + vf$ or $y \in B(x, \varepsilon)$, where $\mu(S) = \mu(\Omega)$.
Now, for all $y \in f(S)$ there exists some $\varepsilon > 0$ such that $f^{-1}(z) \equiv U^T z - U^T v$ for all $z \in B(y, \varepsilon)$. Then, $(f^{-1})'(y)$ is unitary.

$$
\begin{aligned}
\mu(f(A)) &= \int_\Omega 1_A \circ f^{-1} d\mu \\
&= \int_S 1_A \circ f^{-1} d\mu \\
&= \int_{f(S)} 1_A |\det(f^{-1})'| d\mu \\
&= \int_\Omega 1_A d\mu \\
&= \mu(A)
\end{aligned}
$$

And therefore, $f \in \sigma(\Omega)$.

Proposition 7 *For $\tau, \gamma \in \sigma(\Omega)$, we have that $S(\tau \circ \gamma) \subseteq \Omega \setminus (S(\gamma) \Delta g_\gamma^{-1}(S(\tau)))$.*

Proof. Assume $x \in S(\tau \circ \gamma)$. If g_γ is continuous at x we have that g_τ is continuous at $g_\gamma(x)$ so $x \in S(\gamma) \cap g_\gamma^{-1}(S(\tau))$.
Otherwise, if $x \notin S(\gamma)$ then if $g_\gamma(x) \in S(\tau)$ we have that $g_\tau(g_\gamma(x))$ cannot be continuous, a contradiction. Therefore, $x \in (\Omega \setminus S(\gamma)) \cap (\Omega \setminus g_\gamma^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \cup g_\gamma^{-1}(S(\tau)))$.
Finally, $S(\tau \circ \gamma) \subseteq (\Omega \setminus (S(\gamma) \cup g_\gamma^{-1}(S(\tau)))) \cup (S(\gamma) \cap g_\gamma^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \Delta g_\gamma^{-1}(S(\tau)))$.
Corollary. $S(\gamma) \cap g_\gamma^{-1}(S(\tau)) \subseteq S(\tau \circ \gamma)$

3 Estimates on Thickened Sets of Discontinuity

Definition 6 *We define $A(\tau, \varepsilon) = \{x \in \Omega \mid d(x, \partial(\Omega \setminus S(\tau)) < \varepsilon\}$.*

Note that $\partial(\Omega \setminus S(\tau)) = \Omega \setminus S(\tau)$, and are interchanged at times.

Proposition 8 *For any open set X with $X \subseteq S(\tau)$ and $\mu(X) = \mu(\Omega)$, we have that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\})$*

Proof. We have that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\}) = \mu(\{x \in g_\tau(X) \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\})$.
Now, assume for any $x \in g_\tau(X)$ that $y \in \Omega \setminus g_\tau(X)$ minimizes $|x - y|$. We see that $x \in C$ where C is taken to be the largest (by inclusion) connected component of $g_\tau(X)$ containing x. We can assume it is convex because of the construction given in Proposition 5. Then, $y \in \partial C$ by the minimality of $|x - y|$.
In addition, we have that $g_\tau(X) \subseteq g_\tau(S(\tau)) = S(\tau^{-1})$ so that $g_\tau^{-1}(x) = Ux + v$ on C.
Now, with $\ell = \{x + a(y - x) \mid a \in [0, 1)\}$, we have that $\ell \subset C$ by the convexity of ∂C (because of the convexity of C).
Thus, $|x - y| = H^1(\ell) = H^1(g_\tau^{-1}(\ell)) \geq |g_\tau^{-1}(x) - y_2|$ for some $y_2 \in \partial g_\tau^{-1}(C)$. Thus, $|x - y| \geq d(g_\tau^{-1}(x), g_\tau^{-1}(C)) \geq d(g_\tau^{-1}(x), \Omega \setminus g_\tau^{-1}(g_\tau(X))) = d(g_\tau^{-1}(x), \Omega \setminus X)$.

Finally, we have the following.

$$\begin{aligned} \implies \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\}) &\le \mu(\{x \in g_\tau(X) \mid d(g_\tau^{-1}(x), \Omega \setminus X) < \varepsilon\}) \\ &= \mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\}) \end{aligned}$$

Now, as $X \subseteq S(\tau)$ we see that $g_\tau(X) \subseteq S(\tau^{-1})$ so that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau^{-1}(g_\tau(X))) < \varepsilon\}) \le \mu(x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\})$.
Therefore, $\mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\})$

Proposition 9 *$\mu(A(\tau \circ \gamma^{-1}, \varepsilon)) \le \mu(A(\tau, \varepsilon)) + \mu(A(\gamma, \varepsilon))$ for all $\varepsilon > 0$ and $\tau, \gamma \in \sigma(\Omega)$.*

Proof.

$$\begin{aligned} \mu(A(\tau \circ \gamma^{-1}, \varepsilon)) &= \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\tau \circ \gamma^{-1})) < \varepsilon\}) \\ &\le \mu(\{x \in \Omega \mid d(x, \Omega \setminus (S(\gamma^{-1}) \cap g_\gamma(S(\tau)))) < \varepsilon\}) \\ &\le \mu(\{x \in \Omega \mid d(x, \Omega \setminus (S(\gamma^{-1}) \cap g_\gamma(S(\tau) \cap S(\gamma)))) < \varepsilon\}) \\ &= \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\gamma(S(\tau) \cap S(\gamma))) < \varepsilon\}) \end{aligned}$$

By **CITE PROP** we have that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\gamma(S(\tau) \cap S(\gamma))) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, (S(\tau) \cap S(\gamma))) < \varepsilon\})$ because $S(\tau) \cap S(\gamma) \subseteq S(\gamma)$ and has full measure.

$$\begin{aligned} \implies \mu(A(\tau \circ \gamma^{-1}, \varepsilon)) &\le \mu(\{x \in \Omega \mid d(x, \Omega \setminus (S(\tau) \cap S(\gamma))) < \varepsilon\}) \\ &\le \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\tau)) < \varepsilon\}) + \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\gamma)) < \varepsilon\}) \\ &= \mu(A(\tau, \varepsilon)) + \mu(A(\gamma, \varepsilon)) \end{aligned}$$

Proposition 10 *We have that $\mu(A(\gamma, \varepsilon)) = \mu(A(\gamma^{-1}, \varepsilon))$ for all $\gamma \in \sigma(\Omega)$ and $\varepsilon > 0$.*

Proof. We have the following estimate.

$$\begin{aligned} \mu(A(\gamma^{-1}, \varepsilon)) &= \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\gamma^{-1})) < \varepsilon\}) \\ &= \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\gamma(S(\gamma)) < \varepsilon\}) \end{aligned}$$

We see that $S(\gamma) \subseteq S(\gamma)$ so by **CITE PROP** we have that $\mu(A(\gamma^{-1}, \varepsilon)) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\gamma) < \varepsilon\}) = \mu(A(\gamma, \varepsilon))$.

4 Topological Properties

Definition 7 *We define for measurable $S \subseteq \Omega$ the cylindrical set $V_S = \{\tau \in \sigma(U) \mid S \subseteq S(\tau)\}$.*

Proposition 11 *If $X \subseteq \sigma(\Omega)$ is open under $||\cdot||_{L^p(\Omega)}$, then $X \circ \tau$ is as well for all $\tau \in \sigma(\Omega)$.*

Proof. We have that for any $\gamma \in X \circ \tau$ that $\gamma \circ \tau^{-1} \in X$ and thus there is some $B = B_{L^p(\Omega)}(\gamma \circ \tau^{-1}, \varepsilon) \subset X$. For $\lambda \in B$ we have that $||\lambda \circ \tau - \gamma||_{L^p(\Omega, \mathbb{R}^n)} = ||\lambda - \gamma \circ \tau^{-1}||_{L^p(\Omega)}$, so that $B_{L^p(\Omega)}(\gamma \circ \tau^{-1}, \varepsilon) \circ \tau = B_{L^p(\Omega)}(\gamma, \varepsilon) \subset X \circ \tau$. And thus there is an open neighborhood of $\gamma \in X \circ \tau$, so that $X \circ \tau$ is open as well.

Proposition 12 *If $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$, then $||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \to 0$ as well for all $\tau \in \sigma(\Omega)$.*

Proof. Note that $\mu(E_\delta) = \mu(\{x \in \Omega \mid |\gamma_n(x) - \gamma(x)| > \delta\}) < \delta^{-p} ||\gamma_n(x) - \gamma(x)||_{L^p(\Omega)}$ by Chebyshev's inequality.

$$
\begin{aligned}
||\tau \circ \gamma_n - \tau \circ \gamma||^p_{L^p(\Omega)} &= \int_\Omega |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu \\
&= \int_{E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu + \int_{\Omega \setminus E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu \\
&\leq \delta^{-p} ||\gamma_n - \gamma||_{L^p} \operatorname{diam}(\Omega)^p + I
\end{aligned}
$$

$$
I = \int_{(\Omega \setminus E_\delta) \setminus g_\gamma^{-1}(A(\tau, 2\delta))} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu + \int_{(\Omega \setminus E_\delta) \cap g_\gamma^{-1}(A(\tau, 2\delta))} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu
$$

Given $x \in (\Omega \setminus E_\delta) \setminus g_\gamma^{-1}(A(\tau, 2\delta))$, we see that $|g_{\gamma_n}(x) - g_\gamma(x)| < \delta$ a.e, and $g_\gamma(x) \in \Omega \setminus A(\tau, 2\delta)$ so that $d(g_\gamma(x), \Omega \setminus S(\tau)) \geq 2\delta$ and thus $d(g_{\gamma_n}(x), \Omega \setminus S(\tau)) \geq \delta$.
This means that $g_{\gamma_n}(x)$ and $g_\gamma(x)$ are in the same connected component of $S(\tau)$ so that $|g_\tau \circ g_{\gamma_n}(x) - g_\tau \circ g_\gamma(x)| = |g_{\gamma_n} - g_\gamma(x)| < \delta$.

$$
\begin{aligned}
\implies I &\leq \mu((\Omega \setminus E_\delta) \setminus g_\gamma^{-1}(A(\tau, 2\delta)))\delta^p + \mu((\Omega \setminus E_\delta) \cap g_\gamma^{-1}(A(\tau, 2\delta))) \operatorname{diam}(\Omega)^p \\
&\leq \mu(\Omega)\delta^p + \mu(A(\tau, 2\delta)) \operatorname{diam}(\Omega)^p \\
\implies ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} &\leq \delta^{-p} ||\gamma_n - \gamma||_{L^p(\Omega)} \operatorname{diam}(\Omega)^p + \mu(\Omega)\delta^p + \mu(A(\tau, 2\delta)) \operatorname{diam}(\Omega)^p
\end{aligned}
$$

Now, choose $\delta = ||\gamma_n - \gamma||^{1/p+1}_{L^p(\Omega)}$.

$$
\leq ||\gamma_n - \gamma||^{1/(p+1)}_{L^p(\Omega)} \operatorname{diam}(\Omega)^p + \mu(\Omega)||\gamma_n - \gamma||^{1/p+1}_{L^p(\Omega)} + \mu(A(\tau, 2||\gamma_n - \gamma||^{1/(p+1)})) \operatorname{diam}(\Omega)^p
$$

This bound holds for all n. Now, we see that each term approaches 0 as $||\gamma_n - \gamma||$ does, so that $||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \to 0$ as $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$ for $1 \leq p < \infty$.

Proposition 13 *If $X \subseteq \sigma(\Omega)$ is open under $||\cdot||_{L^p(\Omega)}$, then $\tau \circ X$ is as well for all $\tau \in \sigma(\Omega)$.*

Proof. Take any set X open in $||\cdot||_{L^p(\Omega)}$.
Now, take any sequence $\{\gamma_n\} \subseteq \sigma(\Omega) \setminus (\tau \circ X)$ convergent to some limit γ. We see that $\tau^{-1} \circ \gamma_n$ is then a sequence in $\sigma(\Omega) \setminus X$ convergent to $\tau^{-1} \circ \gamma$ by the previous proposition. Then, $\tau^{-1} \circ \gamma \in \Omega \setminus X$ as $\sigma(\Omega) \setminus X$ is closed. Finally, $\gamma \in \sigma(\Omega) \setminus (\tau \circ X)$, so that $\sigma(\Omega) \setminus (\tau \circ X)$ is closed and therefore $\tau \circ X$ is open.

Proposition 14 *$(\sigma(\Omega), ||\cdot||_{L^p(\Omega)})$ is a topological group.*

Proof. Assume that $||\tau_n - \tau||_{L^p} \to 0$. Now, we see that $||\tau^{-1} - \tau_n^{-1}||_{L^p(\Omega} = ||\tau^{-1} \circ \tau_n - \operatorname{Id}||_{L^p(\Omega)} = ||\tau^{-1} \circ \tau_n - \tau^{-1} \circ \tau||_{L^p(\Omega)} \to 0$ as well, so that $\tau \mapsto \tau^{-1}$ is continuous.
Now, assume that $||\tau_n - \tau||_{L^p} \to 0$ and $||\gamma_n - \gamma||_{L^p} \to 0$. We see that $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \leq ||\tau_n \circ \gamma_n - \tau \circ \gamma_n||_{L^p(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} = ||\tau_n - \tau||_{L^p(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)}$. As both sequences approach 0, we see that $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \to 0$ as well, so that $(\tau, \gamma) \mapsto \tau \circ \gamma$ is continuous too.

Therefore, $(\sigma(\Omega), ||\cdot||_{L^p(\Omega)})$ is a topological group.

Proposition 15 *For $X \subseteq \sigma(\Omega)$, if $\mu(A(\tau, \varepsilon)) \to 0$ uniformly as $\varepsilon \to 0$, then X is relatively compact under L^p for $1 \leq p < \infty$.*

Proof. Pick any arbitrary p and choose $q < p^* = np/(n-p)$.
Let n^ε denote the usual mollifier taken by dilating n, a positive smooth function such that $\int_{\mathbb{R}^n} n d\mu = 1$ and 0 outside of $B(0,1)$. Take any sequence $\{\tau_n\} \subset X$ and let $\tau_n^\varepsilon = n^\varepsilon * \tau_n$.
First, we bound $||\tau_n^\varepsilon - \tau_n||_{L^q(\Omega)}$.

$$||\tau_n^\varepsilon - \tau_n||_{L^1(\Omega)} \leq \int_\Omega \int_{B(0,1)} n(y)|\tau_n(x-\varepsilon y) - \tau_n(x)| d\mu_y d\mu$$
$$\leq \int_{A(\tau,\varepsilon)} \int_{B(0,1)} n(y)|\tau_n(x-\varepsilon y) - \tau_n(x)| d\mu_y d\mu + \int_{B(0,1)} \int_{\Omega \setminus A(\tau,\varepsilon)} n(y)\varepsilon d\mu d\mu_y$$
$$\leq \int_{A(\tau,\varepsilon)} \int_{B(0,1)} n(y)|\tau_n(x-\varepsilon y) - \tau_n(x)| d\mu_y d\mu + \varepsilon\mu(\Omega)$$
$$\leq \mu(A(\tau,\varepsilon))\mu(B(0,1)) \operatorname{diam}(\Omega) + \varepsilon\mu(\Omega)$$

As ε approaches 0^+, this upper bound approaches 0 by uniformly in n by hypothesis. Then, the rest of this proof follows exactly the same as Evan's proof of the Rellich-Kondrachov Compactness Theorem in 'Partial Differential Equations' **todo: cite**.

Therefore, there must be some subsequence τ_{m_n} convergent in the $L^q(\Omega)$ norm so that X is relatively compact in $L^q(\Omega)$.

Now, for any arbitrary p, we can choose p_1 such that $p < p_1^*$, and thus X is relatively compact in $L^p(\Omega)$.
Corollary. $V_S = \overline{V_S}$ is compact if $\mu(\partial(\Omega \setminus S)) = 0$.

Definition 8 *We denote* $||f||_{L^p_w(\Omega)} = \sup_{E \in \Sigma_\Omega} \mu(E)^{-1/p^*} \int_E |f| d\mu$ *to be the weak* L^p *norm.*

Proposition 16 *For a sequence* $\{\gamma_n\} \subset \sigma(\Omega)$ *and* $\tau, \gamma \in \sigma(\Omega)$ *such that* $||\gamma_n - \gamma||_{L^p_w(\Omega)} \to 0$*, we have that* $||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p_w(\Omega)} \to 0$ *as well.*

Proof. Take any arbitrary measurable $A \subseteq \Omega$ with indicator function χ_A. We denote $E_\delta = \{x \in \Omega \mid |\gamma_n - \gamma| > \delta\}$ so that $\mu(E_\delta) < \delta^{-p}||\gamma_n - \gamma||^p_{L^p_w(\Omega)}$.

$$I_n = \int_A |\tau \circ \gamma_n - \tau \circ \gamma| d\mu = \int_\Omega |\tau \circ \gamma_n - \tau \circ \gamma| \chi_A d\mu$$
$$\leq \int_{\Omega \setminus E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma| \chi_A d\mu + \int_{E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma| \chi_A d\mu$$
$$\leq \int_{(\Omega \setminus E_\delta) \setminus A(\tau, 2\delta)} |\tau \circ \gamma_n - \tau \circ \gamma| \chi_A d\mu + \int_{(\Omega \setminus E_\delta) \cap A(\tau, 2\delta)} |\tau \circ \gamma_n - \tau \circ \gamma| \chi_A d\mu + \mu(E_\delta) \operatorname{diam}(\Omega)$$
$$\leq |\delta|^p \mu(\Omega) + \mu(A(\tau, 2\delta)) \operatorname{diam}(\Omega) + \delta^{-p} ||\gamma_n - \gamma||^p_{L^p_w(\Omega)} \operatorname{diam}(\Omega)$$

We may choose $\delta_n = ||\gamma_n - \gamma||^{1/2}_{L^p_w(\Omega)}$.

$$\implies I_n \leq ||\gamma_n - \gamma||^{1/2}_{L^p_w(\Omega)} \mu(\Omega) + \mu(A(\tau, 2||\gamma_n - \gamma||^{1/2}_{L^p_w(\Omega)})) \operatorname{diam}(\Omega) + ||\gamma_n - \gamma||^{p/2}_{L^p_w(\Omega)} \operatorname{diam}(\Omega)$$

Therefore, since each term goes to 0 as $||\gamma_n - \gamma||_{L^p_w(\Omega)} \to 0$ we have that $\lim_{n\to\infty} I_n \to 0$. Now, we may substitute $A = E_r$ for any arbitrary $0 < r$. We see the following bounds.

$$M = \sup_{r>0} r\mu(\{x \in \Omega \mid |\tau \circ \gamma_n - \tau \circ \gamma| > r\})^{1/p} = \sup_{0<r\leq \operatorname{diam}(\Omega)} r\mu(\{x \in \Omega \mid |\tau \circ \gamma_n - \tau \circ \gamma| > r\})^{1/p}$$
$$\leq \sup_{0<r\leq \operatorname{diam}(\Omega)} \left(\int_{E_r} |\tau \circ \gamma_n - \tau \circ \gamma| d\mu \right)^{1/p}$$
$$\leq I^{1/p}$$

Now, we have that $||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p_w(\Omega)} \leq p^* M \leq p^* I_n^{1/p}$ by a well-known result **cite**. Therefore, $\lim_{n\to\infty} ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p_w(\Omega)} \leq p^* \lim_{n\to\infty} I_n^{1/p} = 0$.

Proposition 17 *For a sequence $\{\gamma_n\} \subset \sigma(\Omega)$ and $\tau, \gamma \in \sigma(\Omega)$ such that $||\gamma_n - \gamma||_{L^p_w(\Omega)} \to 0$, we have that $||\gamma_n \circ \tau - \gamma \circ \tau||_{L^p_w(\Omega)} \to 0$ as well.*

Proof. We see the following simplification for any function $f \in L^p_w(\Omega)$.

$$\begin{aligned}
||f||_{L^p_w(\Omega)} &= \sup_{A \in \Sigma_\Omega} \mu(A)^{-1/p^*} \int_A |f| d\mu \\
&= \sup_{A \in \Sigma_\Omega} \mu(\tau^{-1}(A))^{-1/p^*} \int_{\tau^{-1}(A)} |f \circ \tau| d\mu \\
&= \sup_{A \in \Sigma_\Omega} \mu(A)^{-1/p^*} \int_A |f \circ \tau| d\mu \\
&= ||f \circ \tau||_{L^p_w(\Omega)}
\end{aligned}$$

So, with $f = \gamma_n - \gamma$ we see that $||\gamma_n \circ \tau - \gamma \circ \tau||_{L^p_w(\Omega)} = ||\gamma_n - \gamma||_{L^p_w(\Omega)} \to 0$.

Proposition 18 $(\sigma(\Omega), ||\cdot||_{L^p_w(\Omega)})$ *is a topological group.*

Proof. We first prove that $\tau_n \to \tau$ implies that $\tau_n^{-1} \to \tau^{-1}$.
We see that $||\tau_n^{-1} - \tau^{-1}||_{L^p_w(\Omega)} = ||\operatorname{Id} - \tau^{-1} \circ \tau_n||_{L^p_w(\Omega)} = ||\tau^{-1} \circ \tau - \tau^{-1} \circ \tau_n||_{L^p_w(\Omega)}$ by the previous proposition. Since we have that $||\tau_n - \tau||_{L^p_w(\Omega)} \to 0$, we see that $||\tau^{-1} \circ \tau - \tau^{-1} \circ \tau_n||_{L^p_w(\Omega)} \to 0$ as well by **cite prop**.
Next, assume that $||\tau_n - \tau||_{L^p_w(\Omega)} \to 0$ and $||\gamma_n - \gamma||_{L^p_w(\Omega)} \to 0$. Then, $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L^p_w(\Omega)} \leq ||\tau_n \circ \gamma_n - \tau \circ \gamma_n||_{L^p_w(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p_w(\Omega)}$. We see that this is then equal to $||\tau_n - \tau||_{L^p_w(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p_w(\Omega)}$. As both components go to 0 as $n \to \infty$, we see that $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L^p_w(\Omega)} \to 0$ as well so that $(\sigma(\Omega), ||\cdot||_{L^p_w(\Omega)})$ is a topological group.

Now, since for all $\tau \in \sigma(\Omega)$ we see that $||\tau||_{L^p_w(\Omega)} = ||\tau \circ \tau^{-1}||_{L^p_w(\Omega)} = ||\operatorname{Id}||_{L^p_w(\Omega)} < \infty$, we have that $\sigma(\Omega)$ has a compact closure in the weak L^p_w topology by the Banach-Alaoglu theorem.
Therefore, $\sigma(\Omega)$ is a compact topological group under the subspace topology inherited from the weak L^p_w topology. Taking quotients by equivalence classes, we can then construct an invariant probability measure on $\sigma(\Omega)$.

Definition 9 *We define the left-invariant probability measure on $\sigma(\Omega)$ to be $\mu_{\mathbb{D}} : \Sigma_\sigma \to \mathbb{R}^+$ by $\mu_{\mathbb{D}}(X) = \lim_{n\to\infty} \frac{[X:G_n]}{[\sigma(\Omega):G_n]}$ where $G_n = B_{L^p_w(\Omega)}(\operatorname{Id}, 2^{-n}) \cap \sigma(\Omega)$.*

TODO: explicitly construct measure
define sgn : $\sigma(\Omega) \to \mathbb{C}$ -¿ axiomatically?