

Definition 1 For $a < b$, we take $U = [a, b]$. However, where possible, we prove things for U a general compact set with nonzero finite measure.

Definition 2 We define a permutation τ on a set $X \subseteq \mathbb{R}^n$ to be a function $\tau : X \rightarrow X$ which satisfies $\int_X f d\mu = \int_X f \circ \tau d\mu$ for all $f \in L_1(x, \mu)$.

Note that we have that τ preserves the measure by substituting $f = \chi_S$ for χ_S the indicator function of a measurable set $S \subseteq U$.

Definition 3 We define $\sigma(S)$ for some $S \subseteq \mathbb{R}$ to be the set of all permutations on S .

Proposition 1 For all $\tau, \gamma \in \sigma(X)$, we have that $\tau \circ \gamma \in \sigma(X)$

Proof. We have that $\int_X f(\tau(\gamma(x))) d\mu = \int_X f(\gamma(x)) d\mu = \int_X f d\mu$, so $\tau \circ \gamma \in \sigma(X)$.

The continuous differentiability almost everywhere follows immediately by restricting $\tau \circ \gamma$ to sufficiently small open neighborhoods of any point.

Definition 4 For any given $\tau \in \sigma(U)$, by Lusin's theorem we have that for any $\varepsilon > 0$ there is some set S_ε such that τ restricted to S_ε is continuous (in the inherited subspace topology) and $\mu(U \setminus S_\varepsilon) < \varepsilon$. Now, let $\text{Reg}(\tau) = \cup_{n=1}^{\infty} S_{1/n}$, and let the representative $\text{Rep}(\tau)$ be τ restricted to $\text{Reg}(\tau)$.

Definition 5 We define $D(f)$ to be the set of jump discontinuities of a function f .

Proposition 2 For any $\tau \in \sigma(U)$, we have that $\text{Rep}(\tau)$ is locally continuously differentiable on $D(\text{Rep}(\tau))$ with derivative ± 1 .

Proof. Let $g = \text{Rep}(\tau)$ for simplicity. We know that $S = D(g)$ is open so for any $x \in S$, there is some $\delta > 0$ with $B(x, \delta) \subset S$.

Then, take any y with $|x - y| < \delta$. We have that $|g(x) - g(y)| = \mu([g(x), g(y)])$, assuming $g(x) \leq g(y)$ without loss of generality. Then, since g is invertible on $[x, y]$ it must be monotonic and since it's continuous we then have that $g([x, y]) = [g(x), g(y)]$ so that $|g(x) - g(y)| = \mu(g([x, y])) = \mu([x, y]) = |x - y|$.

Therefore, by fixing x we see that $g(y) = \pm x + C$ for $y \in B(x, \varepsilon)$ so that g is locally differentiable with derivative ± 1 .

Definition 6 For a set $S \subseteq U$ with $\mu(S) = 0$, we define the cylindrical set of S to be $V_S = \{\tau \in \sigma(U) \mid D(\text{Rep}(\tau)) \subseteq S\}$

Definition 7 For a set $S \subseteq U$ with $\mu(S) = 0$, a partition of components of $U \setminus S$ which are connected in U is denoted $\{A_\alpha\}_{\alpha \in I(S)}$ for some well-ordered set $I(S)$ whose cardinality corresponds to the cardinality of S .

Definition 8 We denote $\ell_p(A, B)$ to be the set of sequences in $B \subseteq \mathbb{R}$ indexed over a set A with a finite ℓ_p norm. We define $\ell_\infty(A, B)$ similarly.

Definition 9 We let $U_S = \{\tau \in V_S \mid \text{Rep}(\tau)' \equiv 1\}$.

Proposition 3 Let $A = \{x - y \mid x, y \in U\}$. Then, $(U_S, \|\cdot\|_\infty)$ is isometric to a subset of $\ell_\infty(I(S), A)$.

Proof. Let A_α be the crudest partition of $U \setminus S$ which is connected in U , and for any $\tau \in \sigma(U)$ let $\{C_{\tau\alpha}\}_{\alpha \in I(S)}$ be a sequence of constants so that $\tau(x) = x + C_{\tau\alpha}$ for $x \in A_\alpha$, which must exist since τ is continuous when restricted to each A_α and has continuous derivative 1 almost everywhere..

Now, consider the map $f(\tau) = C_\tau - M$ where M is a sequence with $M_\alpha = \inf A_\alpha$.

$$\begin{aligned} \|\tau - \gamma\|_\infty &= \sup_{\alpha \in I} \sup_{x \in A_\alpha} |\tau(x) - \gamma(x)| \\ &= \sup_{\alpha \in I} |C_{\tau\alpha} - C_{\gamma\alpha}| \\ &= \sup_{\alpha \in I} |C_{\tau\alpha} - M_\alpha - C_{\gamma\alpha} + M_\alpha| \\ &= \|(C_\tau - M) - (C_\gamma - M)\|_{\ell_\infty(I)} \\ &= \|f(\tau) - f(\gamma)\|_{\ell_\infty(I)} \end{aligned}$$

We denote $U - U = \{x - y \mid x, y \in U\}$. Now, since $f(\tau)_\alpha \in U$, we have that $f(\tau)_\alpha - f(\gamma)_\alpha \in U - U$, so that $\ell_\infty(I)$ can be replaced with $\ell_\infty(I, U - U)$ in this with no change.

Definition 10 Under the above isometry, we denote $\text{Seq}(S) = f(U_S) \subseteq \ell_\infty(I(S), U - U)$

Proposition 4 $\sigma(U)$ is not compact under the L_1 norm.

Proof. We have that $\sigma(U)$ is closed under the L_1 norm, so it suffices to show some sequence has no convergent subsequence.

Because we take $U = [a, b]$, it suffices to show that $\sigma([0, 1])$ is not compact under $L_1([0, 1], \mu)$ since we can scale all elements of $\sigma([0, 1])$ to/from corresponding elements of $\sigma(U)$ bijectively.

Let U_{ij} be a sequence of points of $[0, 1]$ for $0 \leq j \leq 2^i$. Let $U_{i0} = 0$ and $U_{i2^i} = 1$. Then, we define $U_{i(2j)} = U_{(i-1)j}$ and $U_{i(2j+1)} = (U_{(i-1)j} + U_{(i-1)(j+1)})/2$.

Now, we have a sequence of partitions $V_{ij} = (U_{i(j-1)}, U_{ij})$ for $1 \leq j \leq 2^i$.

Let τ_n be the permutation which transposes $V_{n(2k+1)}$ with $V_{n(2n-2k-1)}$, or equivalently $\tau_n(x) = x$ on $V_{n(2k)}$ and $\tau_n(x) = x + 0.5 \pmod{1}$ on $V_{n(2k+1)}$. Now, for any distinct $n, m \in \mathbb{N}$, we have that $\|\tau_n - \tau_m\|_1 = 1$, so that there is no convergent subsequence.

Therefore, $\sigma([0, 1])$ is not compact, so that $\sigma(U)$ is not compact.

Proposition 5 For any $S \subseteq U$ with $\mu(S) = 0$, V_S is relatively compact under the $|\cdot|_1$ norm if $\mu(\partial SU) = 0$.

Proof. Let $T_h f = f(x + h)$, and take $\tau(x) = 0$ for all $\tau \in \sigma(U)$ and $x \notin U$. By the Kolmogorov-Riesz Compactness Theorem, it suffices to show that $\|T_h f - f\|_1 \rightarrow 0$ uniformly (equicontinuously) on V_S .

For any $h \in \mathbb{R}$, let $A_h = \{x \in U \setminus S \mid d(x, \mathbb{R} \setminus (U \setminus S)) < |h|\}$, so that if $x \notin A_h$ then $|T_h \text{Rep}(\tau) - \text{Rep}(\tau)| = |h|$ since we would then have that $\text{Rep}(\tau)$ is continuous with derivative ± 1 a.e. on the $|h|$ -neighborhood around x .

$$\begin{aligned} \int_U |T_h \tau - \tau| d\mu &= \int_{U \setminus S} |T_h \text{Rep}(\tau) - \text{Rep}(\tau)| d\mu \\ &= \int_{(U \setminus S) \setminus A_h} |T_h \text{Rep}(\tau) - \text{Rep}(\tau)| d\mu + \int_{A_h} |T_h \text{Rep}(\tau) - \text{Rep}(\tau)| d\mu \\ &\leq |h| \mu((U \setminus S) \setminus A_h) + 2\mu(A_h) (\sup_{x \in U} |x|) \end{aligned}$$

So, to prove that $\|T_h \tau - \tau\|_1 \rightarrow 0$ uniformly on V_S , it suffices to show that $\lim_{h \rightarrow 0} \mu(A_h) = 0$.

$$\begin{aligned} \lim_{h \rightarrow 0} \mu(A_h) &= \mu(\cap_{n=1}^\infty A_{1/n}) \\ &= \mu(U \cap \partial S) \\ &= \mu(\partial S) \end{aligned}$$

So, if $\mu(\partial S) = 0$, it converges uniformly on V_S .

Proposition 6 $X \subseteq \sigma(U)$ is relatively compact under $|\cdot|_1$ if and only if $\mu(\partial \cup_{\tau \in X} D(\text{Rep}(\tau))) = 0$.

Proof. If $\mu(\partial \cup_{\tau \in X} D(\text{Rep}(\tau))) = 0$, then $X \subseteq V_{\cup_{\tau \in X} D(\text{Rep}(\tau))}$ so that X is relatively compact by the previous proposition.

Otherwise, if X is not relatively compact, then there exists some sequence $\{\tau_n\} \subseteq X$ such that $\{\tau_n\} \not\subseteq V_S$ for all sets $S \subseteq U$ with $\mu(\partial S) = 0$.

Now, $\mu(\partial \cup_{n=1}^\infty D(\text{Rep}(\tau_n))) > 0$, otherwise $\{\tau_n\}$ would be relatively compact and thus contain a convergent subsequence.

Finally, we have the following.

$$\mu(\partial \cup_{\tau \in X} D(\text{Rep}(\tau))) \geq \mu(\partial \cup_{n=1}^\infty D(\text{Rep}(\tau_n))) > 0$$