Re-Derivation and Proof of The Bose-Einstein Integral

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Lemma

Given some $n \in \mathbb{N}$, $j = 0, 1, ..., n$, and $x \in \mathbb{C}$

$$C_j = \frac{1}{(0x - jx)(1x - jx)...((j-1)x - jx)((j+1)x - jx)...(nx - j)}$$

$$= \frac{(-1)^j}{n!x^n}\binom{n}{j}$$

Let $j \in \{0, ..., n\}$ and consider the terms $(0x - jx)(1x - jx)...((j-1)x - jx)$. By factoring out $(-1)^j x^j$ and reversing the order of the product, this is equal to $(-1)^j x^j (1)(2)...(j-1)(j) = (-1)^j x^j j!$. C_j can be represented as a much simpler fraction.

$$C_j = \frac{(-1)^j}{j!(n-j)!x^n} = \frac{(-1)^j}{n!x^n}\binom{n}{j}$$

Derivation of Bose-Einstein Integral Identity

Given a natural number n and unique real $x_0, ..., x_n$ which aren't negative integers, Let $P(x) = (x - x_0)(x - x_1)...(x - x_n)$ and now with partial fraction decomposition, the following is true.

$$\frac{1}{P(x)} = \sum_{j=0}^{n} \frac{C_j}{x + x_j} \tag{1}$$

where $C_j = \frac{1}{(x_0 - x_j)(x_1 - x_j)...(x_{j-1} - x_j)(x_{j+1} - x_j)...(x_n - x_j)}$ for $j = 0, 1, ..., n$. So,

$$\sum_{m=1}^{\infty} \frac{1}{P(m)} = \sum_{m=1}^{\infty} \sum_{j=0}^{n} \frac{C_j}{m + x_j}$$

$$= \sum_{j=0}^{n} \sum_{m=1}^{\infty} \left(\frac{C_j}{m + x_j} + \frac{C_j}{m} - \frac{C_j}{m} \right)$$

$$= \left(\sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \frac{1}{m} \right) - \sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m + x_j} \right) *$$

$$= \left(\sum_{j=0}^{n} C_j \right) \left(\sum_{m=1}^{\infty} \frac{1}{m} \right) - \sum_{j=0}^{n} C_j H_{x_j}$$

$$= - \sum_{j=0}^{n} C_j H_{x_j} = - \sum_{j=0}^{n} C_j \int_0^1 \frac{1 - t^{x_j}}{1 - t} dt$$

$$= - \int_0^1 \frac{\left(\sum_{j=0}^{n} C_j \right) - \sum_{j=0}^{n} C_j t^{x_j}}{1 - t} dt$$

So, we have that $\sum_{m=1}^{\infty} \frac{1}{P(m)} = \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$. where H_x is the xth harmonic number. Taking the limit as $(x_0, ..., x_n) \to (0...0)$ converges to $\zeta(n+1)$, as shown by a P-series test of degree $n+1$ and with the definition of $\zeta(n+1)$. This means that the integral converges to $\zeta(n+1)$, too. By parametrizing x_j into $x_j = jv$ for some v, the limit can be simplified significantly.

$$\begin{aligned} \zeta(n+1) \quad &= \lim_{(x_1 \ldots x_n) \to (0, \ldots, 0)} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1 - t} dt \\ &= \lim_{v \to 0} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{jv}}{1 - t} dt \\ &= \lim_{v \to 0} \frac{1}{n! v^n} \int_0^1 \frac{\sum_{j=0}^{n} \binom{n}{j} (-t^v)^j}{1 - t} dt \text{ (with lemma 1)} \\ &= \lim_{v \to 0} \frac{1}{n! v^n} \int_0^1 \frac{(1 - t^v)^n}{1 - t} dt \end{aligned}$$

Now, using repeated application of L'Hopital's and induction,

$$\begin{aligned} \zeta(n+1) \quad &= \lim_{v \to 0} \frac{1}{n!^2} \int_0^1 \frac{\ln(t)^n \sum_{j=0}^{n} \binom{n}{j} j^n (-t^v)^j}{1 - t} dt \\ &= \frac{1}{n!^2} \int_0^1 \frac{\ln(t)^n \sum_{j=0}^{n} \binom{n}{j} j^n (-1)^j}{1 - t} dt \\ &= \frac{(-1)^n}{n!} \int_0^1 \frac{\ln(t)^n}{1 - t} dt \end{aligned}$$

This is equivalent to the formula given by the Bose-Einstein integral shown below by applying the substitution $t = e^{-z}$ to the above integral.

$$\int_0^\infty \frac{t^x}{e^t - 1} dt = \Gamma(x+1)\zeta(x+1)$$

Generalizations

Using the parametrization $x_i = ix + a$, (2) can be derived as well through the same reasoning.

$$\Gamma(n)\zeta(n,a) = (-1)^{n-1} \int_0^1 \frac{t^{a-1} \ln(t)^{n-1}}{1-t} dt \tag{2}$$

$$\zeta(n,x) = \sum_{m=0}^{\infty} \frac{1}{(m+x)^n}$$

where $\zeta(n,a)$ is the Hurwitz Zeta function. Because the generalized harmonic number $H_{x,n} = \zeta(n,1) - \zeta(n,x+1)$, an identity with $H_{x,n}$ can be derived too. The generalized harmonic numbers are defined in (3) for $x \in \mathbb{N}$.

$$\Gamma(n)H_{x,n} = (-1)^{n-1} \int_0^1 \frac{(1-t^x)\ln(t)^{n-1}}{1-t} dt$$

$$H_{x,n} = \sum_{m=1}^{x} \frac{1}{m^n} \tag{3}$$

This can be verified for integer values of x by expanding the $(1-t^x)/(1-t)$ term into $1 + t + ... + t^{x-1}$, and then reversing the order of the sum and the integral. This gives $(-1)^{n-1}\sum_{i=0}^{x-1} G(i,n)$ where $G(i,n) = \int_0^1 t^i ln(t)^{n-1}$. Through integration by parts, $G(i,n) = \frac{1-n}{i+1} G(i, n-1)$. With induction, $G(n,i) = \frac{(n-1)!}{(i+1)^n}$. Now, $(-1)^{n-1}\sum_{i=0}^{x-1} G(i,n) = (-1)^{n-1}(n-1)!\sum_{i=1}^{x} \frac{1}{(1)^n} = \Gamma(n)H_{x,n}$.

To get the integrals to a form more similar to the Bose-Einstein Integral, the substitution $t = e^{-z}$ gives (4) and (5).

$$\Gamma(n)H_{x,n} = \int_0^\infty \frac{1-e^{-zx}}{e^z - 1} z^{n-1} dz \tag{4}$$

$$\Gamma(n)\zeta(n,x) = \int_0^\infty \frac{e^{-z(x-1)}}{e^z - 1} z^{n-1} dz \tag{5}$$