

# Rigidity and Compactness for Almost Everywhere Invertible Measure Preserving Maps on Open Bounded Subsets of $\mathbb{R}^n$

Graham Bertele

## 1 Basic Properties

**Definition 1** Take  $\Omega \subset \mathbb{R}^n$  to be open and bounded with  $0 < \mu(\Omega) < \infty$ .

**Definition 2** A permutation  $\tau : \Omega \rightarrow \Omega$  is some function such that  $\mu(\tau(X)) = \mu(X)$ .

**Definition 3** The set of all permutations on a set  $\Omega$  is denoted  $\sigma(\Omega)$ .

**Proposition 1** For  $\tau, \gamma \in \sigma(\Omega)$  we have that  $\tau \circ \gamma \in \sigma(\Omega)$ .

**Proposition 2** For all  $\tau \in \sigma(\Omega)$ , there exists some function  $g$  such that  $\tau \circ g = \text{Id}$  almost everywhere. In addition,  $g \in \sigma(\Omega)$  as well.  $g$  is usually denoted as  $\tau^{-1}$  even though the actual inverse is not explicitly defined and need not exist.

## 2 Rigidity and Form

**Proposition 3** For all  $\tau \in \sigma(\Omega)$  and any set  $K \subset \Omega$ , we have that  $H^d(K) = H^d(\tau(K))$  for all  $1 \leq d \leq n$ .

**Proof.** We have the following.

$$H_\delta^d(K) = \inf \left\{ \sum_i (\text{diam } U_i)^d : K \subseteq \cup_i U_i \wedge \text{diam } U_i < \delta \right\}$$

Each  $U_i$  can be taken without loss of generality to be a countable union of balls so that their union is then again a countable union of balls. Therefore, we can assume without loss of generality that each  $U_i$  is just a ball.

$$\begin{aligned} &= \inf \left\{ \sum_i (\text{diam } B(x_i, r_i))^d : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\} \\ &= \inf \left\{ \sum_i (C_n \mu(B(x_i, r_i)))^{d/n} : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\} \\ &= \inf \left\{ \sum_i (C_n \mu(\tau(B(x_i, r_i))))^{d/n} : \tau(K) \subseteq \cup_i \tau(B(x_i, r_i)) \wedge 2r_i < \delta \right\} \end{aligned}$$

For any set  $X \subset \Omega$  we have that  $(C_n \mu(X))^{1/n} \leq \text{diam}(X)$  because if  $\mu(B(x, \varepsilon)) = \mu(X)$ , then  $(C_n \mu(X))^{1/n} = (C_n \mu(B(x, \varepsilon)))^{1/n} = \text{diam } B(x, \varepsilon) \leq \text{diam } X$  as an  $n$ -sphere minimizes diameter for a given volume.

$$\begin{aligned} &\leq \inf \left\{ \sum_i (C_n \mu(V_i))^d : \tau(K) \subseteq \cup_i V_i \wedge (C_n \mu(V_i))^{1/n} < \delta \right\} \\ &\leq \inf \left\{ \sum_i (\text{diam } V_i)^d : \tau(K) \subseteq \cup_i V_i \wedge \text{diam}(V_i) < \delta \right\} \\ &= H_\delta^d(\tau(K)) \end{aligned}$$

Now, taking limits as  $\delta \rightarrow 0$  we have that  $H^d(K) \leq H^d(\tau(K))$ .  
Therefore,  $H^d(\tau(K)) \leq H^d(\tau^{-1}(\tau(K))) = H^d(K)$  so that  $H^d(\tau(K)) = H^d(K)$ .

**Proposition 4** *For all  $\tau \in \sigma(\Omega)$ , there exists a function  $g_\tau$  continuous on an open set  $S$  with  $\mu(S) = \mu(\Omega)$  such that  $g_\tau = \tau$  almost everywhere.*

**Proof.** We have that for all  $n$  there exists some  $S_n \subset \Omega$  such that  $\tau$  is continuous when restricted to  $S_n$  and  $\mu(\Omega \setminus S_n) < 1/n$ . Now, with  $S = \cup_{i=1}^\infty S_i$  we have that  $\tau$  is continuous on the  $S$ -inherited subspace topology and that  $\mu(S) = \mu(\Omega)$  so that  $S$  is dense in  $\Omega$ .

Now, let  $G = \overline{\{(x, \tau(x)) \mid x \in S\}}$ .

For any  $x \in \Omega \setminus S$ , we have that there exists some sequence  $\{x_n\} \subset S$  convergent to  $x$ . Now,  $\tau(x_n)$  is bounded and thus has some subsequence convergent to some  $y$ . Now,  $(x, y) \in G$ . Thus, with  $p_1(x, y) = x$ , we have that  $p_1(G) = \Omega$ .

Next, let  $Q = \{x \in \Omega \mid ((x, y) \in G \wedge (x, z) \in G) \implies y = z\}$ . If  $x \in S$  then if  $(x, s_1) \in G$  and  $(x, s_2) \in G$  we have that there must be some sequences  $(x_{in}, \tau(x_{in})) \rightarrow (x, s_i)$  for  $x_n \in S$ . As  $|x_{in} - x| \rightarrow 0$  we have that because  $\tau$  is continuous in  $S$  that  $|\tau(x_{in}) - \tau(x)| \rightarrow 0$ , and thus that  $|\tau(x_{1n}) - \tau(x_{2n})| \rightarrow 0$  so that because  $|\tau(x_{in}) - s_i| \rightarrow 0$  we have that  $|s_1 - s_2|$  is arbitrarily small and thus  $s_1 = s_2$ . Therefore,  $x \in Q$  so that  $S \subseteq Q$ .

Finally, define  $g_\tau : \Omega \rightarrow \Omega$  by  $g_\tau(x) = y$  if  $(x, y) \in G$ , where  $y$  is chosen arbitrarily for  $x \notin Q$ .

We aim to show that  $g_\tau$  is continuous on  $S$ . For  $x \in S$  we have that for any  $\varepsilon > 0$  there exists some  $\delta$  such that  $|x - y| < \delta$  for  $y \in S$  implies that  $|\tau(x) - \tau(y)| < \varepsilon$ . Then, take any  $y \in \Omega \setminus S$  with  $|x - y| < \delta$ .

If  $y \notin Q$ , then assume  $(y, z_1)$  and  $(y, z_2)$  are in  $G$ . Now, there are sequences  $\{s_n\}, \{t_n\} \subset B(x, \delta) \cap S$  such that  $s_n, t_n \rightarrow y$ ,  $\tau(s_n) \rightarrow z_1$ , and  $\tau(t_n) \rightarrow z_2$ . Then, we have that  $|\tau(s_n) - \tau(x)| < \varepsilon$ , and for any  $\varepsilon_2 > 0$  we have that there exists some  $N$  such that  $n > N \implies |z_1 - \tau(s_n)| < \varepsilon_2$ . Then,  $|z_1 - \tau(x)| < \varepsilon + \varepsilon_2$  so that  $|z_1 - \tau(x)| \leq \varepsilon$ . Similar logic shows that  $|z_2 - \tau(x)| \leq \varepsilon$ .

If  $y \in Q$ , then  $(y, z) \in G$ . We have that there must be some sequence  $\tau(s_n) \rightarrow z$  for  $\{s_n\} \subset S \cap B(x, \delta)$  so that  $|z - \tau(x)| \leq |z - \tau(s_n)| + |\tau(s_n) - \tau(x)| \leq \varepsilon_2 + \varepsilon \rightarrow \varepsilon$ . Therefore,  $|z - \tau(x)| \leq \varepsilon$  as well.

Finally, regardless of the choice of value of  $g_\tau(y)$  outside of  $Q$ , we have that  $g_\tau$  is continuous at  $x$ .

Now, with  $S_2$  the set of all  $x$  such that  $g_\tau(x)$  is continuous at  $x$ , we have that  $S_2$  is open and that  $S \subseteq S_2$  so that  $\mu(S_2) = \mu(\Omega)$ . Therefore,  $S_2$  is the desired set, and because  $g_\tau(x) = \tau(x)$  for  $x \in S$  we have that  $g_\tau \equiv \tau$ .

**Definition 4** *For some  $\tau \in \sigma(\Omega)$ , we define  $S(\tau)$  to be  $S_2$  as above and  $g_\tau$  to be  $g_\tau$  as above.*

**Proposition 5**  *$S(\tau)$  can be partitioned into disjoint sets  $\{A_\alpha\}_{\alpha \in I}$  such that  $g_\tau|_{A_\alpha} = U_\alpha x + v_\alpha$  for  $U_\alpha$  a unitary linear map and  $v_\alpha$  a constant vector.*

**Proof.** We have that  $g_\tau$  is continuous on an open set  $S(\tau)$  with  $\mu(S) = \mu(\Omega)$  so that  $\partial S(\tau) = \Omega \setminus S(\tau)$ .

Now, for any  $x \in S(\tau)$ , there is some  $B_1 = B(x, \varepsilon_1) \subset S$ .

For any  $y \in B_1$ , let  $\ell_1 = \{g_\tau(x) + a(g_\tau(y) - g_\tau(x)) \mid 0 \leq a \leq 1\}$ . We have that  $|g_\tau(x) - g_\tau(y)| = H^1(\ell_1) = H^1(g_\tau^{-1}(\ell_1))$ . As  $g_\tau^{-1}(\ell_1)$  is some continuous path between  $x$  and  $y$ , we have that  $H^1(g_\tau^{-1}(\ell_1)) \geq |x - y|$ . Therefore,  $|g_\tau(x) - g_\tau(y)| \geq |x - y|$ . Finally, we have that  $|x - y| = |g_\tau^{-1}(g_\tau(x)) - g_\tau^{-1}(g_\tau(y))| \geq |g_\tau(x) - g_\tau(y)|$  so that  $|x - y| = |g_\tau(x) - g_\tau(y)|$ .

It is a known result that if  $|g_\tau(x) - g_\tau(y)| = |x - y|$  for  $g - \tau$  continuous that  $g_\tau(x) = Ux + v$  for  $U$  a unitary map.

Choose some partitioning of  $S(\tau)$  into balls such that  $S = \cup_{\alpha \in I} B(x_\alpha, r_\alpha)$ . Then, take  $A_\alpha = B(x_\alpha, r_\alpha) \setminus (\cup_{i < \alpha} B(x_i, r_i))$  so that they are disjoint,  $S = \cup_{\alpha \in I} A_\alpha$ , and on each  $A_\alpha$  we have that  $g_\tau(x) = U_\alpha x + v_\alpha$ .

**Definition 5** *We take  $P_{S(\tau)}$  to be the crudest such partition of  $S(\tau)$  such that the constructed partition above is a refinement of  $P_{S(\tau)}$ .*

**Proposition 6** For any a.e. invertible  $L^p$  function  $f : \Omega \rightarrow \Omega$ , we have that  $f \in \sigma(\Omega)$  if and only if for almost all  $x \in \Omega$  there exists some  $\varepsilon > 0$  such that there are  $U \in U(n)$  and  $v \in \mathbb{R}^n$  such that  $f(y) \equiv Uy + vf$  for  $y \in B(x, \varepsilon)$ .

**Proof.** We have proven the  $\implies$  direction already. Now, conversely, assume that for all  $x \in S$  there exists some  $\varepsilon > 0$  such that  $f(y) \equiv Uy + vf$  or  $y \in B(x, \varepsilon)$ , where  $\mu(S) = \mu(\Omega)$ . Now, for all  $y \in f(S)$  there exists some  $\varepsilon > 0$  such that  $f^{-1}(z) \equiv U^T z - U^T v$  for all  $z \in B(y, \varepsilon)$ . Then,  $(f^{-1})'(y)$  is unitary.

$$\begin{aligned}\mu(f(A)) &= \int_{\Omega} \chi_A \circ f^{-1} d\mu \\ &= \int_S \chi_A \circ f^{-1} d\mu \\ &= \int_{f(S)} \chi_A |\det(f^{-1})'| d\mu \\ &= \int_{\Omega} \chi_A d\mu \\ &= \mu(A)\end{aligned}$$

And therefore,  $f \in \sigma(\Omega)$ .

### 3 Topological Properties

**Definition 6** We define for measurable  $S \subseteq \Omega$  the cylindrical set  $V_S = \{\tau \in \sigma(U) \mid \Omega \setminus S(\tau) \subseteq S\}$ .

**Proposition 7** If  $\mu(\partial\Omega) = 0$ , then for sets  $S \subseteq \Omega$  with  $\mu(S) = 0$ , if  $\mu(\partial S) = 0$  then  $V_S$  is relatively compact under  $\|\cdot\|_{L^p(\Omega)}$  for all  $1 \leq p < \infty$ .

**Proof.** For this proof we extend  $\tau \in \sigma(\Omega)$  to be defined  $\mathbb{R}^n \rightarrow \Omega \cup \{0\}$  by taking  $\tau(x) = 0$  for  $x \notin \Omega$ .

For the first direction, we assume  $\mu(\partial S) = 0$ . We use the Kolmogorov-Reisz compactness theorem. Take  $T_h f(x) = f(x + h)$ . As  $\Omega$  is bounded we only need to show that  $\|T_h \tau - \tau\|_{L^p(\mathbb{R}^n)} \rightarrow 0$  uniformly on  $V_S$  as  $|h| \rightarrow 0$  in  $\mathbb{R}^n$ .

For  $h \in \mathbb{R}$  define  $A_h = \{x \in \Omega \setminus S \mid B(x, |h|) \not\subseteq \Omega \setminus S\}$  and define  $B_h = \{x \notin \Omega \mid B(x, |h|) \cap \Omega \neq \emptyset\}$ .

$$\begin{aligned}\|T_h \tau - \tau\|_{L^p(\mathbb{R}^n)}^p &= \int_{\mathbb{R}^n} |T_h \tau - \tau|^p d\mu \\ &= \int_{\mathbb{R}^n \setminus \Omega} |T_h \tau|^p d\mu + \int_{\Omega \setminus A_{|h|}} |T_h \tau - \tau|^p d\mu + \int_{A_{|h|}} |T_h \tau - \tau|^p d\mu\end{aligned}$$

Now, we can bound each of those 3 integrals.

$$\begin{aligned}I_1 &= \int_{\Omega \setminus A_{|h|}} |T_h \tau - \tau|^p d\mu = \sum_{\alpha \in I} \int_{V_{\alpha} \setminus A_{|h|}} |T_h \tau - \tau|^p d\mu \\ &= \sum_{\alpha \in I} \int_{V_{\alpha} \setminus A_{|h|}} |(U_{\alpha}(x + h) + v_{\alpha}) - (U_{\alpha}x + v_{\alpha})|^p d\mu \\ &\leq \sum_{\alpha \in I} \int_{V_{\alpha}} |h|^p d\mu \\ &= |h|^p \mu(\Omega \setminus S) \rightarrow 0\end{aligned}$$

We do similarly for the third integral.

$$I_2 = \int_{A_{|h|}} |T_h \tau - \tau|^p d\mu \leq \left( 2 \sup_{u \in \Omega} |u| \right)^p \mu(A_{|h|})$$

$$\lim_{|h| \rightarrow 0} \mu(A_{|h|}) \leq \mu(\cap_{n=1}^{\infty} A_{1/n})$$

If  $x \in \cap_{n=1}^{\infty} A_{1/n}$ , then for all  $\varepsilon > 0$  we have that there is some  $y \in B(x, \varepsilon)$  which also satisfies  $y \notin \Omega \setminus S$ . Then,  $x \in \partial(\Omega \setminus S) \cap (\Omega \setminus S) \subseteq \partial S$ .

$$\implies I_2 \leq \left( 2 \sup_{u \in \Omega} |u| \right)^p \mu(\partial S) = 0$$

$$I_3 = \int_{\mathbb{R}^n \setminus \Omega} |T_h \tau|^p d\mu = \int_{B_{|h|}} |T_h \tau|^p d\mu \leq \mu(B_{|h|}) \left( \sup_{u \in \Omega} |u| \right)^p$$

$$\lim_{|h| \rightarrow 0} \mu(B_{|h|}) \leq \mu(\cap_{n=1}^{\infty} B_{1/n})$$

$$= \mu(\partial \Omega \setminus \Omega) = \mu(\partial \Omega) = 0$$

Therefore, if  $\mu(\partial S) = 0$  then  $\|T_h \tau - \tau\|_{L^p(\mathbb{R}^n)} \rightarrow 0$  uniformly on  $V_S$  as  $|h| \rightarrow 0$  so that  $V_S$  is relatively compact under  $\|\cdot\|_{L^p(\Omega)}$ .

**Corollary.** For  $X \subseteq \sigma(\Omega)$ , let  $A = \cup_{\tau \in X} (\Omega \setminus S(\tau))$ . If  $\mu(\partial A) = 0$  then  $X \subseteq V_A$  is relatively compact under  $\|\cdot\|_{L^p(\Omega)}$ .

**TO ADD:  $\sigma(\Omega)$  UNDER SOBOLEV NORM IS HOMEOMORPHIC TO  $(U(n) \otimes \mathbb{R}^n)^{P_S}$  UNDER BOX-TOPOLOGY INDUCED BY NORMS -¿ LIMIT TO PRODUCT TOPOLOGY SOMEHOW?**