Rigidity and Compactness for Almost Everywhere Invertible Measure Preserving Maps on Open Bounded Subsets of $\mathbb{R}^n$

Graham Bertele

Definition 1 *Take $\Omega \subset \mathbb{R}^n$ to be open and bounded with $0 < \mu(\Omega) < \infty$.*

Definition 2 *A permutation $\tau : \Omega \to \Omega$ is some function such that $\mu(\tau(X)) = \mu(X)$.*

Definition 3 *The set of all permutations on a set Ω is denoted $\sigma(\Omega)$.*

Proposition 1 *For $\tau, \gamma \in \sigma(\Omega)$ we have that $\tau \circ \gamma \in \sigma(\Omega)$.*

Proposition 2 *For all $\tau \in \sigma(\Omega)$, there exists some function g such that $\tau \circ g = \mathrm{Id}$ almost everywhere. In addition, $g \in \sigma(\Omega)$ as well. g is usually denoted as τ^{-1} even though the actual inverse is not explicitly defined and need not exist.*

Proposition 3 *For all $\tau \in \sigma(\Omega)$ and any set $K \subset \Omega$, we have that $H^d(K) = H^d(\tau(K))$ for all $1 \leq d \leq n$.*

Proof. We have the following.

$$H^d_\delta(K) = \inf \left\{ \sum_i (\operatorname{diam} U_i)^d : K \subseteq \cup_i U_i \wedge \operatorname{diam} U_i < \delta \right\}$$

Each U_i can be taken without loss of generality to be a finite union of balls as Ω is bounded.

$$= \inf \left\{ \sum_i (\operatorname{diam} B(x_i, r_i))^d : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\}$$
$$= \inf \left\{ \sum_i (C_n \mu(B(x_i, r_i)))^{d/n} : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\}$$
$$= \inf \left\{ \sum_i (C_n \mu(\tau(B(x_i, r_i))))^{d/n} : \tau(K) \subseteq \cup_i \tau(B(x_i, r_i)) \wedge 2r_i < \delta \right\}$$

For any set $X \subset \Omega$ we have that $(C_n\mu(X))^{1/n} \leq \operatorname{diam}(X)$ because if $\mu(B(x, \varepsilon)) = \mu(X)$, then $(C_n\mu(X))^{1/n} = (C_n\mu(B(x, \varepsilon)))^{1/n} = \operatorname{diam} B(x, \varepsilon) \leq \operatorname{diam} X$ as an n-sphere minimizes diameter for a given volume.

$$\leq \inf \left\{ \sum_i (C_n \mu(V_i))^d : \tau(K) \subseteq \cup_i V_i \wedge (C_n\mu(V_i))^{1/n} < \delta \right\}$$
$$\leq \inf \left\{ \sum_i (\operatorname{diam} V_i)^d : \tau(K) \subseteq \cup_i V_i \wedge \operatorname{diam}(V_i) < \delta \right\}$$
$$= H^d_\delta(\tau(K))$$

Now, taking limits as $\delta \to 0$ we have that $H^d(K) \leq H^d(\tau(K))$.
Therefore, $H^d(\tau(K)) \leq H^d(\tau^{-1}(\tau(K)) = H^d(K)$ so that $H^d(\tau(K)) = H^d(K)$.

Proposition 4 *For all $\tau \in \sigma(\Omega)$, there exists a function g_τ continuous on an open set S with $\mu(S) = \mu(\Omega)$ such that $g_\tau = \tau$ almost everywhere.*

Proof. We have that for all n there exists some $S_n \subset \Omega$ such that τ is continuous when restricted to S_n and $\mu(\Omega \setminus S_n) < 1/n$. Now, with $S = \cup_{i=1}^{\infty} S_i$ we have that τ is continuous on the S-inherited subspace topology and that $\mu(S) = \mu(\Omega)$ so that S is dense in Ω.

Now, let $G = \overline{\{(x, \tau(x)) \mid x \in S\}}$.
For any $x \in \Omega \setminus S$, we have that there exists some sequence $\{x_n\} \subset S$ convergent to x. Now, $\tau(x_n)$ is bounded and thus has some subsequence convergent to some y. Now, $(x, y) \in G$. Thus, with $p_1(x, y) = x$, we have that $p_1(G) = \Omega$.

Next, let $Q = \{x \in \Omega \mid ((x, y) \in G \wedge (x, z) \in G) \implies y = z\}$. If $x \in S$ then if $(x, s_1) \in G$ and $(x, s_2) \in G$ we have that there must be some sequences $(x_{in}, \tau(x_{in})) \to (x, s_i)$ for $x_n \in S$. As $|x_{in} - x| \to 0$ we have that because τ is continuous in S that $|\tau(x_{in}) - \tau(x)| \to 0$, and thus that $|\tau(x_{1n}) - \tau(x_{2n})| \to 0$ so that because $|\tau(x_{in}) - s_i| \to 0$ we have that $|s_1 - s_2|$ is arbitrarily small and thus $s_1 = s_2$. Therefore, $x \in Q$ so that $S \subseteq Q$.

Finally, define $g_\tau : \Omega \to \Omega$ by $g_\tau(x) = y$ if $(x, y) \in G$, where y is chosen arbitrarily for $x \notin Q$.
We aim to show that g_τ is continuous on S. For $x \in S$ we have that for any $\varepsilon > 0$ there exists some δ such that $|x - y| < \delta$ for $y \in S$ implies that $|\tau(x) - \tau(y) < \varepsilon$. Then, take any $y \in \Omega \setminus S$ with $|x - y| < \delta$.

If $y \notin Q$, then assume (y, z_1) and (y, z_2) are in G. Now, there are sequences $\{s_n\}, \{t_n\} \subset B(x, \delta) \cap S$ such that $s_n, t_n \to y$, $\tau(s_n) \to z_1$, and $\tau(t_n) \to z_2$. Then, we have that $|\tau(s_n) - \tau(x)) < \varepsilon$, and for any $\varepsilon_2 > 0$ we have that there exists some N such that $n > N \implies |z_1 - \tau(s_n)| < \varepsilon_2$. Then, $|z_1 - \tau(x)| < \varepsilon + \varepsilon_2$ so that $|z_1 - \tau(x)| \leq \varepsilon$. Similar logic shows that $|z_2 - \tau(x)| \leq \varepsilon$.

If $y \in Q$, then $(y, z) \in G$. We have that there must be some sequence $\tau(s_n) \to z$ for $\{s_n\} \subset S \cap B(x, \delta)$ so that $|z - \tau(x)| \leq |z - \tau(s_n)| + |\tau(s_n) - \tau(x)| \leq \varepsilon_2 + \varepsilon \to \varepsilon$. Therefore, $|z - \tau(x)| \leq \varepsilon$ as well.
Finally, regardless of the choice of value of $g_\tau(y)$ outside of Q, we have that g_τ is continuous at x.
Now, with S_2 the set of all x such that $g_\tau(x)$ is continuous at x, we have that S_2 is open and that $S \subseteq S_2$ so that $\mu(S_2) = \mu(\Omega)$. Therefore, S_2 is the desired set, and because $g_\tau(x) = \tau(x)$ for $x \in S$ we have that $g_\tau \equiv \tau$.

Definition 4 *For some $\tau \in \sigma(\Omega)$, we define $S(\tau)$ to be S_2 as above and g_τ to be g_τ as above.*

Proposition 5 *$S(\tau)$ can be partitioned into disjoint sets $\{A_\alpha\}_{\alpha \in I}$ such that $g_\tau|_{A_\alpha} = U_\alpha x + v_\alpha$ for U_α a unitary linear map and v_α a constant vector.*

Proof. We have that g_τ is continuous on an open set $S(\tau)$ with $\mu(S) = \mu(\Omega)$ so that $\partial S(\tau) = \Omega \setminus S(\tau)$.
Now, for any $x \in S(\tau)$, there is some $B_1 = B(x, \varepsilon_1) \subset S$.

For any $y \in B_1$, let $\ell_1 = \{g_\tau(x) + a(g_\tau(y) - g_\tau(x)) \mid 0 \leq a \leq 1\}$. We have that $|g_\tau(x) - g_\tau(y)| = H^1(\ell_1) = H^1(g_\tau^{-1}(\ell_1))$. As $g_\tau^{-1}(\ell_1)$ is some continuous path between x and y, we have that $H^1(g_\tau^{-1}(\ell)) \geq |x - y|$. Therefore, $|g_\tau(x) - g_\tau(y)| \geq |x - y|$. Finally, we have that $|x - y| = |g_\tau^{-1}(g_\tau(x)) - g_\tau^{-1}(g_\tau(y))| \geq |g_\tau(x) - g_\tau(y)|$ so that $|x - y| = |g_\tau(x) - g_\tau(y)|$.

It is a known result that if $|g_\tau(x) - g_\tau(y)| = |x - y|$ for $g - \tau$ continuous that $g_\tau(x) = Ux + v$ for U a unitary map.

Choose some partitioning of $S(\tau)$ into balls such that $S = \cup_{\alpha \in I} B(x_\alpha, r_\alpha)$. Then, take $A_\alpha = B(x_\alpha, r_\alpha) \setminus (\cup_{i < \alpha} B(x_i, r_i))$ so that they are disjoint, $S = \cup_{\alpha \in I} A_\alpha$, and on each A_α we have that $g_\tau(x) = U_\alpha x + v_\alpha$.

Definition 5 *We take $P_{S(\tau)}$ to be the crudest such partition of $S(\tau)$ such that the constructed partition above is a refinement of $P_{S(\tau)}$.*

Definition 6 *We define for measurable $S \subseteq \Omega$ the cylindrical set $V_S = \{\tau \in \sigma(U) \mid \Omega \setminus S(\tau) \subseteq S\}$.*

Proposition 6 *If $\mu(\partial\Omega) = 0$, then for sets $S \subseteq \Omega$ with $\mu(S) = 0$, if $\mu(\partial S) = 0$ then V_S is relatively compact under $|| \cdot ||_{L^p(\Omega)}$ for all $1 \leq p < \infty$.*

Proof. For this proof we extend $\tau \in \sigma(\Omega)$ to be defined $\mathbb{R}^n \to \Omega \cup \{0\}$ by taking $\tau(x) = 0$ for $x \notin \Omega$.

For the first direction, we assume $\mu(\partial S) = 0$. We use the Kolmogorov-Reisz compactness theorem. Take $T_h f(x) = f(x+h)$. As Ω is bounded we only need to show that $||T_h\tau - \tau||_{L^p(\mathbb{R}^n)} \to 0$ uniformly on V_S as $|h| \to 0$ in $\mathbb{R}^n$.
For $h \in \mathbb{R}$ define $A_h = \{x \in \Omega \setminus S \mid B(x,|h|) \not\subset \Omega \setminus S\}$ and define $B_h = \{x \notin \Omega \mid B(x,|h|) \cap \Omega \neq \emptyset\}$.

$$||T_h\tau - \tau||^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |T_h\tau - \tau|^p d\mu$$
$$= \int_{\mathbb{R}^n \setminus \Omega} |T_h\tau|^p d\mu + \int_{\Omega \setminus A_{|h|}} |T_h\tau - \tau|^p d\mu + \int_{A_{|h|}} |T_h\tau - \tau|^p d\mu$$

Now, we can bound each of those 3 integrals.

$$I_1 = \int_{\Omega \setminus A_{|h|}} |T_h\tau - \tau|^p d\mu = \sum_{\alpha \in I} \int_{V_\alpha \setminus A_{|h|}} |T_h\tau - \tau|^p d\mu$$
$$= \sum_{\alpha \in I} \int_{V_a \setminus A_{|h|}} |(U_\alpha(x+h) + v_\alpha) - (U_\alpha x + v_\alpha)|^p d\mu$$
$$\leq \sum_{\alpha \in I} \int_{V_a} |h|^p d\mu$$
$$= |h|^p \mu(\Omega \setminus S) \to 0$$

We do similarly for the third integral.

$$I_2 = \int_{A_{|h|}} |T_h\tau - \tau|^p d\mu \leq \left(2 \sup_{u \in \Omega} |u|\right)^p \mu(A_{|h|})$$
$$\lim_{|h| \to 0} \mu(A_{|h|}) \leq \mu\left(\cap_{n=1}^\infty A_{1/n}\right)$$

If $x \in \cap_{n=1}^\infty A_{1/n}$, then for all $\varepsilon > 0$ we have that there is some $y \in B(x,\varepsilon)$ which also satisfies $y \notin \Omega \setminus S$. Then, $x \in \partial(\Omega \setminus S) \cap (\Omega \setminus S) \subseteq \partial S$.

$$\implies I_2 \leq \left(2 \sup_{u \in \Omega} |u|\right)^p \mu(\partial S) = 0$$

$$I_3 = \int_{\mathbb{R}^n \setminus \Omega} |T_h\tau|^p d\mu = \int_{B_{|h|}} |T_h\tau|^p d\mu \leq \mu(B_{|h|}) \left(\sup_{u \in \Omega} |u|\right)^p$$
$$\lim_{|h| \to 0} \mu(B_{|h|}) \leq \mu\left(\cap_{n=1}^\infty B_{1/n}\right)$$
$$= \mu(\partial\Omega) = 0$$

Therefore, if $\mu(\partial S) = 0$ then $||T_h\tau - \tau||_{L^p(\mathbb{R}^n)} \to 0$ uniformly on V_S as $|h| \to 0$ so that V_S is relatively compact under $||\cdot||_{L^P(\Omega)}$.

Corollary. For $X \subseteq \sigma(\Omega)$, let $A = \cup_{\tau \in X}(\Omega \setminus S(\tau))$. If $\mu(\partial A) = 0$ then $X \subseteq V_A$ is relatively compact under $||\cdot||_{L^P(\Omega)}$.