

# Rigidity and Compactness for Almost Everywhere Invertible Measure Preserving Maps on Open Bounded Subsets of $\mathbb{R}^n$

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## 1 Basic Properties

**Definition 1** Take  $\Omega \subset \mathbb{R}^n$  to be open and bounded with  $0 < \mu(\Omega) < \infty$ .

**Definition 2** The set of all invertible a.e. measure preserving maps ( $\tau$  such that  $\mu(X) = \mu(\tau(X))$ ) on a set  $\Omega$  is denoted  $\sigma(\Omega)$ .

**Proposition 1** For  $\tau, \gamma \in \sigma(\Omega)$  we have that  $\tau \circ \gamma \in \sigma(\Omega)$ .

**Proposition 2** For all  $\tau \in \sigma(\Omega)$ , there exists some function  $g$  such that  $\tau \circ g = \text{Id}$  almost everywhere. In addition,  $g \in \sigma(\Omega)$  as well.  $g$  is usually denoted as  $\tau^{-1}$  even though the actual inverse is not explicitly defined and need not exist.

## 2 Rigidity and Form

**Proposition 3** For all  $\tau \in \sigma(\Omega)$  and any set  $K \subset \Omega$ , we have that  $H^d(K) = H^d(\tau(K))$  for all  $1 \leq d \leq n$ .

**Proof.** We have the following.

$$H_\delta^d(K) = \inf \left\{ \sum_i (\text{diam } U_i)^d : K \subseteq \cup_i U_i \wedge \text{diam } U_i < \delta \right\}$$

Each  $U_i$  can be taken without loss of generality to be a countable union of balls so that their union is then again a countable union of balls. Therefore, we can assume without loss of generality that each  $U_i$  is just a ball.

$$\begin{aligned} &= \inf \left\{ \sum_i (\text{diam } B(x_i, r_i))^d : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\} \\ &= \inf \left\{ \sum_i (C_n \mu(B(x_i, r_i)))^{d/n} : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\} \\ &= \inf \left\{ \sum_i (C_n \mu(\tau(B(x_i, r_i))))^{d/n} : \tau(K) \subseteq \cup_i \tau(B(x_i, r_i)) \wedge 2r_i < \delta \right\} \end{aligned}$$

For any set  $X \subset \Omega$  we have that  $(C_n \mu(X))^{1/n} \leq \text{diam}(X)$  because if  $\mu(B(x, \varepsilon)) = \mu(X)$ , then  $(C_n \mu(X))^{1/n} = (C_n \mu(B(x, \varepsilon)))^{1/n} = \text{diam } B(x, \varepsilon) \leq \text{diam } X$  as an  $n$ -sphere minimizes diameter for a given volume.

$$\begin{aligned} &\leq \inf \left\{ \sum_i (C_n \mu(V_i))^d : \tau(K) \subseteq \cup_i V_i \wedge (C_n \mu(V_i))^{1/n} < \delta \right\} \\ &\leq \inf \left\{ \sum_i (\text{diam } V_i)^d : \tau(K) \subseteq \cup_i V_i \wedge \text{diam}(V_i) < \delta \right\} \\ &= H_\delta^d(\tau(K)) \end{aligned}$$

Now, taking limits as  $\delta \rightarrow 0$  we have that  $H^d(K) \leq H^d(\tau(K))$ .  
Therefore,  $H^d(\tau(K)) \leq H^d(\tau^{-1}(\tau(K))) = H^d(K)$  so that  $H^d(\tau(K)) = H^d(K)$ .

**Proposition 4** *For all  $\tau \in \sigma(\Omega)$ , there exists a function  $g_\tau$  continuous on an open set  $S$  with  $\mu(S) = \mu(\Omega)$  such that  $g_\tau = \tau$  almost everywhere.*

**Proof.** We have that for all  $n$  there exists some  $S_n \subset \Omega$  such that  $\tau$  is continuous when restricted to  $S_n$  and  $\mu(\Omega \setminus S_n) < 1/n$ . Now, with  $S = \bigcup_{i=1}^\infty S_i$  we have that  $\tau$  is continuous on the  $S$ -inherited subspace topology and that  $\mu(S) = \mu(\Omega)$  so that  $S$  is dense in  $\Omega$ .

Now, let  $G = \overline{\{(x, \tau(x)) \mid x \in S\}}$ .

For any  $x \in \Omega \setminus S$ , we have that there exists some sequence  $\{x_n\} \subset S$  convergent to  $x$ . Now,  $\tau(x_n)$  is bounded and thus has some subsequence convergent to some  $y$ . Now,  $(x, y) \in G$ . Thus, with  $p_1(x, y) = x$ , we have that  $p_1(G) = \Omega$ .

Next, let  $Q = \{x \in \Omega \mid ((x, y) \in G \wedge (x, z) \in G) \implies y = z\}$ . If  $x \in S$  then if  $(x, s_1) \in G$  and  $(x, s_2) \in G$  we have that there must be some sequences  $(x_{in}, \tau(x_{in})) \rightarrow (x, s_i)$  for  $x_n \in S$ . As  $|x_{in} - x| \rightarrow 0$  we have that because  $\tau$  is continuous in  $S$  that  $|\tau(x_{in}) - \tau(x)| \rightarrow 0$ , and thus that  $|\tau(x_{1n}) - \tau(x_{2n})| \rightarrow 0$  so that because  $|\tau(x_{in}) - s_i| \rightarrow 0$  we have that  $|s_1 - s_2|$  is arbitrarily small and thus  $s_1 = s_2$ . Therefore,  $x \in Q$  so that  $S \subseteq Q$ .

Finally, define  $g_\tau : \Omega \rightarrow \Omega$  by  $g_\tau(x) = y$  if  $(x, y) \in G$ , where  $y$  is chosen arbitrarily for  $x \notin Q$ .

We aim to show that  $g_\tau$  is continuous on  $S$ . For  $x \in S$  we have that for any  $\varepsilon > 0$  there exists some  $\delta$  such that  $|x - y| < \delta$  for  $y \in S$  implies that  $|\tau(x) - \tau(y)| < \varepsilon$ . Then, take any  $y \in \Omega \setminus S$  with  $|x - y| < \delta$ .

If  $y \notin Q$ , then assume  $(y, z_1)$  and  $(y, z_2)$  are in  $G$ . Now, there are sequences  $\{s_n\}, \{t_n\} \subset B(x, \delta) \cap S$  such that  $s_n, t_n \rightarrow y$ ,  $\tau(s_n) \rightarrow z_1$ , and  $\tau(t_n) \rightarrow z_2$ . Then, we have that  $|\tau(s_n) - \tau(x)| < \varepsilon$ , and for any  $\varepsilon_2 > 0$  we have that there exists some  $N$  such that  $n > N \implies |z_1 - \tau(s_n)| < \varepsilon_2$ . Then,  $|z_1 - \tau(x)| < \varepsilon + \varepsilon_2$  so that  $|z_1 - \tau(x)| \leq \varepsilon$ . Similar logic shows that  $|z_2 - \tau(x)| \leq \varepsilon$ .

If  $y \in Q$ , then  $(y, z) \in G$ . We have that there must be some sequence  $\tau(s_n) \rightarrow z$  for  $\{s_n\} \subset S \cap B(x, \delta)$  so that  $|z - \tau(x)| \leq |z - \tau(s_n)| + |\tau(s_n) - \tau(x)| \leq \varepsilon_2 + \varepsilon \rightarrow \varepsilon$ . Therefore,  $|z - \tau(x)| \leq \varepsilon$  as well.

Finally, regardless of the choice of value of  $g_\tau(y)$  outside of  $Q$ , we have that  $g_\tau$  is continuous at  $x$ .

Now, with  $S_2$  the set of all  $x$  such that  $g_\tau(x)$  is continuous at  $x$ , we have that  $S_2$  is open and that  $S \subseteq S_2$  so that  $\mu(S_2) = \mu(\Omega)$ . Therefore,  $S_2$  is the desired set, and because  $g_\tau(x) = \tau(x)$  for  $x \in S$  we have that  $g_\tau \equiv \tau$ .

**Definition 3** *For some  $\tau \in \sigma(\Omega)$ , we define  $S(\tau)$  to be  $S_2$  as above and  $g_\tau$  to be  $g_\tau$  as above.*

**Proposition 5**  *$S(\tau)$  can be partitioned into disjoint sets  $\{A_\alpha\}_{\alpha \in I}$  such that  $g_\tau|_{A_\alpha} = U_\alpha x + v_\alpha$  for  $U_\alpha$  a unitary linear map and  $v_\alpha$  a constant vector.*

**Proof.** We have that  $g_\tau$  is continuous on an open set  $S(\tau)$  with  $\mu(S) = \mu(\Omega)$  so that  $\partial S(\tau) = \Omega \setminus S(\tau)$ . Now, for any  $x \in S(\tau)$ , there is some  $B_1 = B(x, \varepsilon_1) \subset S$ .

For any  $y \in B_1$ , let  $\ell_1 = \{g_\tau(x) + a(g_\tau(y) - g_\tau(x)) \mid 0 \leq a \leq 1\}$ . We have that  $|g_\tau(x) - g_\tau(y)| = H^1(\ell_1) = H^1(g_\tau^{-1}(\ell_1))$ . As  $g_\tau^{-1}(\ell_1)$  is some continuous path between  $x$  and  $y$ , we have that  $H^1(g_\tau^{-1}(\ell_1)) \geq |x - y|$ . Therefore,  $|g_\tau(x) - g_\tau(y)| \geq |x - y|$ . Finally, we have that  $|x - y| = |g_\tau^{-1}(g_\tau(x)) - g_\tau^{-1}(g_\tau(y))| \geq |g_\tau(x) - g_\tau(y)|$  so that  $|x - y| = |g_\tau(x) - g_\tau(y)|$ .

It is a known result that if  $|g_\tau(x) - g_\tau(y)| = |x - y|$  for  $g - \tau$  continuous that  $g_\tau(x) = Ux + v$  for  $U$  a unitary map.

Choose some partitioning of  $S(\tau)$  into balls such that  $S = \bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)$ . Then, take  $A_\alpha = B(x_\alpha, r_\alpha) \setminus (\bigcup_{i < \alpha} B(x_i, r_i))$  so that they are disjoint,  $S = \bigcup_{\alpha \in I} A_\alpha$ , and on each  $A_\alpha$  we have that  $g_\tau(x) = U_\alpha x + v_\alpha$ .

**Note.** The partition  $I$  can be assumed to be countable because each of the  $A_\alpha$  have been constructed to be pairwise disjoint.

**Definition 4** We take  $P_{S(\tau)}$  to be the crudest such partition of  $S(\tau)$  such that the constructed partition above is a refinement of  $P_{S(\tau)}$ .

**Proposition 6** For any a.e. invertible  $L^p$  function  $f : \Omega \rightarrow \Omega$ , we have that  $f \in \sigma(\Omega)$  if and only if for almost all  $x \in \Omega$  there exists some  $\varepsilon > 0$  such that there are  $U \in U(n)$  and  $v \in \mathbb{R}^n$  such that  $f(y) \equiv Uy + vf$  for  $y \in B(x, \varepsilon)$ .

**Proof.** We have proven the  $\implies$  direction already. Now, conversely, assume that for all  $x \in S$  there exists some  $\varepsilon > 0$  such that  $f(y) \equiv Uy + vf$  or  $y \in B(x, \varepsilon)$ , where  $\mu(S) = \mu(\Omega)$ . Now, for all  $y \in f(S)$  there exists some  $\varepsilon > 0$  such that  $f^{-1}(z) \equiv U^T z - U^T v$  for all  $z \in B(y, \varepsilon)$ . Then,  $(f^{-1})'(y)$  is unitary.

$$\begin{aligned} \mu(f(A)) &= \int_{\Omega} 1_A \circ f^{-1} d\mu \\ &= \int_S 1_A \circ f^{-1} d\mu \\ &= \int_{f(S)} 1_A |\det(f^{-1})'| d\mu \\ &= \int_{\Omega} 1_A d\mu \\ &= \mu(A) \end{aligned}$$

And therefore,  $f \in \sigma(\Omega)$ .

**Proposition 7** For  $\tau, \gamma \in \sigma(\Omega)$ , we have that  $S(\tau \circ \gamma) \subseteq \Omega \setminus (S(\gamma) \Delta g_{\gamma}^{-1}(S(\tau)))$ .

**Proof.** Assume  $x \in S(\tau \circ \gamma)$ . If  $g_{\gamma}$  is continuous at  $x$  we have that  $g_{\tau}$  is continuous at  $g_{\gamma}(x)$  so  $x \in S(\gamma) \cap g_{\gamma}^{-1}(S(\tau))$ .

Otherwise, if  $x \notin S(\gamma)$  then if  $g_{\gamma}(x) \in S(\tau)$  we have that  $g_{\tau}(g_{\gamma}(x))$  cannot be continuous, a contradiction. Therefore,  $x \in (\Omega \setminus S(\gamma)) \cap (\Omega \setminus g_{\gamma}^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \cup g_{\gamma}^{-1}(S(\tau)))$ .

Finally,  $S(\tau \circ \gamma) \subseteq (\Omega \setminus (S(\gamma) \cup g_{\gamma}^{-1}(S(\tau)))) \cup (S(\gamma) \cap g_{\gamma}^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \Delta g_{\gamma}^{-1}(S(\tau)))$ .

**Corollary.**  $S(\gamma) \cap g_{\gamma}^{-1}(S(\tau)) \subseteq S(\tau \circ \gamma)$

### 3 Topological Properties

**Definition 5** We define for measurable  $S \subseteq \Omega$  the cylindrical set  $V_S = \{\tau \in \sigma(U) \mid S \subseteq S(\tau)\}$ .

**Proposition 8** If  $\mu(\partial\Omega) = 0$ , then for sets  $S \subseteq \Omega$  with  $\mu(S) = \mu(\Omega)$ , if  $\mu(\partial(\Omega \setminus S)) = 0$ , then  $V_S$  is relatively compact under  $\|\cdot\|_{L^p(\Omega)}$  for all  $1 \leq p < \infty$ .

**Proof.** For this proof we extend  $\tau \in \sigma(\Omega)$  to be defined  $\mathbb{R}^n \rightarrow \Omega \cup \{0\}$  by taking  $\tau(x) = 0$  for  $x \notin \Omega$ .

We use the Kolmogorov-Reisz compactness theorem for the proof. Take  $T_h f(x) = f(x + h)$ . As  $\Omega$  is bounded we only need to show that  $\|T_h \tau - \tau\|_{L^p(\mathbb{R}^n)} \rightarrow 0$  uniformly on  $V_S$  as  $|h| \rightarrow 0$  in  $\mathbb{R}^n$ . For  $h \in \mathbb{R}^n$  define  $A_h = \{x \in S \mid B(x, |h|) \not\subseteq S\}$  and define  $B_h = \{x \notin \Omega \mid B(x, |h|) \cap \Omega \neq \emptyset\}$ .

$$\begin{aligned} \|T_h \tau - \tau\|_{L^p(\mathbb{R}^n)}^p &= \int_{\mathbb{R}^n} |T_h \tau - \tau|^p d\mu \\ &= \int_{\mathbb{R}^n \setminus \Omega} |T_h \tau|^p d\mu + \int_{\Omega \setminus A_{|h|}} |T_h \tau - \tau|^p d\mu + \int_{A_{|h|}} |T_h \tau - \tau|^p d\mu \end{aligned}$$

Now, we can bound each of those 3 integrals.

$$\begin{aligned}
I_1 &= \int_{\Omega \setminus A_{|h|}} |T_h \tau - \tau|^p d\mu = \sum_m \int_{(P_S)_m \setminus A_{|h|}} |T_h \tau - \tau|^p d\mu \\
&= \sum_m \int_{(P_S)_m \setminus A_{|h|}} |(U_m(x+h) + v_m) - (U_m x + v_m)|^p d\mu \\
&\leq \sum_m \int_{(P_S)_m} |h|^p d\mu \\
&= |h|^p \mu(S) \rightarrow 0
\end{aligned}$$

We do similarly for the third integral.

$$\begin{aligned}
I_2 &= \int_{A_{|h|}} |T_h \tau - \tau|^p d\mu \leq \left( 2 \sup_{u \in \Omega} |u| \right)^p \mu(A_{|h|}) \\
&\lim_{|h| \rightarrow 0} \mu(A_{|h|}) \leq \mu(\cap_{n=1}^{\infty} A_{1/n})
\end{aligned}$$

If  $x \in \cap_{n=1}^{\infty} A_{1/n}$ , then for all  $\varepsilon > 0$  we have that there is some  $y \in B(x, \varepsilon)$  which also satisfies  $y \notin S$ . Then,  $x \in \partial(\Omega \setminus S)$ .

$$\implies I_2 \leq \left( 2 \sup_{u \in \Omega} |u| \right)^p \mu(\partial(\Omega \setminus S)) = 0$$

$$\begin{aligned}
I_3 &= \int_{\mathbb{R}^n \setminus \Omega} |T_h \tau|^p d\mu = \int_{B_{|h|}} |T_h \tau|^p d\mu \leq \mu(B_{|h|}) \left( \sup_{u \in \Omega} |u| \right)^p \\
&\lim_{|h| \rightarrow 0} \mu(B_{|h|}) \leq \mu(\cap_{n=1}^{\infty} B_{1/n}) \\
&= \mu(\partial\Omega \setminus \Omega) = \mu(\partial\Omega) = 0
\end{aligned}$$

Therefore, if  $\mu(\partial S) = 0$  then  $\|T_h \tau - \tau\|_{L^p(\mathbb{R}^n)} \rightarrow 0$  uniformly on  $V_S$  as  $|h| \rightarrow 0$  so that  $V_S$  is relatively compact under  $\|\cdot\|_{L^p(\Omega)}$ .

**Corollary.** For  $X \subseteq \sigma(\Omega)$ , let  $A = \cup_{\tau \in X} (\Omega \setminus S(\tau))$ . If  $\mu(\partial A) = 0$  then  $X \subseteq V_A$  is relatively compact under  $\|\cdot\|_{L^p(\Omega)}$ .

**Proposition 9** For  $1 \leq k, p \leq \infty$  and  $\tau \in \sigma(\Omega)$ , we have that  $\|\tau\|_{W^{k,p}(\Omega, \mathbb{R}^n)} = \|\text{Id}\|_{W^{k,p}(\Omega, \mathbb{R}^n)}$ .

**Proof.** We use  $\text{Tr}(A^T A)$  as the norm for matrices. We assume  $p$  is finite because the  $p = \infty$  case follows by taking from the limit.

$$\begin{aligned}
\|\tau\|_{W^{k,p}(S(\tau))} &= \|g_\tau\|_{W^{k,p}(S(\tau))} \\
&= \|g_\tau\|_{W^{1,p}(S(\tau))} \\
&= \|\nabla g_\tau\|_{L^p(S(\tau))} + \|g_\tau\|_{L^p(S(\tau))} \\
\int_{S(\tau)} |g_\tau|^p d\mu &= \int_{S(\tau)} |g_\tau \circ g_\tau^{-1}|^p d\mu \\
&= \int_{S(\tau)} |\text{Id}|^p d\mu \\
\implies \|g_\tau\|_{L^p(S(\tau))} &= \|\text{Id}\|_{L^p(S(\tau))} = \|\text{Id}\|_{L^p(\Omega)} \\
\int_{S(\tau)} \text{Tr}((\nabla g_\tau)^T (\nabla g_\tau))^p d\mu &= \sum_m \int_{(P_{S(\tau)})_m} \text{Tr}(U_m^T U_m)^p d\mu \\
&= n^p \sum_m \mu((P_{S(\tau)})_m) \\
&= n^p \mu(\Omega) \\
&= \int_\Omega \text{Tr}((\nabla \text{Id})^T (\nabla \text{Id}))^p d\mu \\
\implies \|\nabla g_\tau\|_{L^p(S(\tau))} &= \|\nabla \text{Id}\|_{L^p(S(\tau))} \\
\implies \|g_\tau\|_{W^{k,p}(S(\tau), \mathbb{R}^n)} &= \|\nabla \text{Id}\|_{L^p(\Omega)} + \|\text{Id}\|_{L^p(\Omega)} \\
&= \|\text{Id}\|_{W^{k,p}(\Omega)}
\end{aligned}$$

**Corollary.**  $V_{S(\tau)} \subset \partial B_{W^{k,p}(\Omega, \mathbb{R}^n)}(0, \|\text{Id}\|_{W^{k,p}(\Omega, \mathbb{R}^n)})$ .

**Definition 6** For  $1 \leq p < \infty$ , define the topology  $\mathcal{T}_p$  on  $\sigma(\Omega)$  by  $X \in \mathcal{T}_p$  if and only if  $X \cap V_S$  is open under the  $L^p(\Omega)$  subspace topology on  $V_S$  for all open  $S \subseteq \Omega$  with  $\mu(S) = \mu(\Omega)$ .

**Proposition 10** If  $X \in \mathcal{T}_p$  then  $X \circ \tau \in \mathcal{T}_p$  as well for all  $\tau \in \sigma(\Omega)$ .

**Proof.** Take any open  $S \subseteq \Omega$  with  $\mu(S) = \mu(\Omega)$ , and consider  $V_S \circ \tau$ . We have that if  $\lambda \in V_S$  then  $S(\lambda \circ \tau^{-1}) \supseteq S(\tau^{-1}) \cap g_\tau(S(\lambda)) \supseteq S(\tau^{-1}) \cap g_\tau(S)$  so that  $\lambda \circ \tau \in S(\tau) \cap g_\tau^{-1}(S)$ . Then, there must be some  $\varepsilon > 0$  such that  $B_p(\lambda \circ \tau^{-1}, \varepsilon) \cap V_{S(\tau) \cap g_\tau^{-1}(S)} \subseteq X \cap V_{S(\tau) \cap g_\tau^{-1}(S)}$ . Now, composing by  $\tau$  on both sides, we get the following.

$$\begin{aligned}
(B_p(\lambda \circ \tau^{-1}, \varepsilon) \circ \tau) \cap (V_{S(\tau) \cap g_\tau^{-1}(S)} \circ \tau) &\subseteq (X \circ \tau) \cap (V_{S(\tau) \cap g_\tau^{-1}(S)} \circ \tau) \\
\implies B_p(\lambda, \varepsilon) \cap (V_{S(\tau) \cap g_\tau^{-1}(S)} \circ \tau) &\subseteq (X \circ \tau) \cap (V_{S(\tau) \cap g_\tau^{-1}(S)} \circ \tau) \\
V_S \circ \tau^{-1} \subseteq V_{S(\tau) \cap g_\tau^{-1}(S)} &\implies V_S \subseteq V_{S(\tau) \cap g_\tau^{-1}(S)} \\
\implies B_p(\lambda, \varepsilon) \cap V_S &\subseteq (X \circ \tau) \cap V_S
\end{aligned}$$

And, therefore, there is a neighborhood of  $\lambda$  in  $(V_S, \|\cdot\|_{L^p(\Omega)})$  contained in  $X \circ \tau$  so that  $X \circ \tau \in \mathcal{T}_p$  as well.  
**Next Steps:** Show  $(\sigma(\Omega), \mathcal{T}_p)$  is a topological group, then show it's Locally Compact, then quotient by equivalence classes and define a haar measure on it