1 Lemma 1

Given some $n \in \mathbb{N}$, $j = 0, 1, ..., n$, and $x \in \mathbb{C}$

$C_j = \frac{1}{(0x-jx)(1x-jx)...((j-1)x-jx)((j+1)x-jx)...(nx-j)}$

$= \frac{(-1)^j}{n!x^n}\binom{n}{j}$

1.1 Proof

Consider the terms $(0x - jx)(1x - jx)...((j - 1)x - jx)$
By factoring out $(-1)^j x^j$ and reversing the order
of the product, this is equal to $(-1)^j x^j (1)(2)...(j - 1)(j)$
$= (-1)^j x^j j!$. The same argument can be made for
the remaining terms. This shows that
$C_j = \frac{(-1)^j}{j!(n-j)!x^n} = \frac{(-1)^j}{n!x^n}\binom{n}{j}$

2 Proof

Given an integer n and unique $x_0, ..., x_n \in \mathbb{R}^+$
Let $P(x) = (x - x_0)(x - x_1)...(x - x_n)$ and now
with partial fraction decomposition,

$\frac{1}{P(x)} = \sum_{j=0}^{n} \frac{C_j}{x+x_j}$ (1)

where $C_j = \frac{1}{(x_0-x_j)(x_1-x_j)...(x_{j-1}-x_j)(x_{j+1}-x_j)...(x_n-x_j)}$

for $j = 0, 1, ..., n$. Now, using this:

$\sum_{m=1}^{\infty} \frac{1}{P(m)} = \sum_{m=1}^{\infty} \sum_{j=0}^{n} \frac{C_j}{m+x_j}$ (2)

$= \sum_{j=0}^{n} \sum_{m=1}^{\infty} \left(\frac{C_j}{m+x_j} + \frac{C_j}{m} - \frac{C_j}{m} \right)$

$= \left(\sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \frac{1}{m} \right) - \sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+x_j} \right) *$

$= \left(\sum_{j=0}^{n} C_j \right) \left(\sum_{m=1}^{\infty} \frac{1}{m} \right) - \sum_{j=0}^{n} C_j H_{x_j}$

where H_x is the xth harmonic number.
The separation of sums is the step we're unsure about.

By taking equation 1 and multiplying both sides by P(x) and then comparing coefficients of x^n, it can be seen that $\sum_{j=0}^{n} C_j = 0$. This step is analogous to algebraically cancelling it out, which can be seen by doing this process for a specific n. I'm not sure how to prove that this is valid generally though.

$$= -\sum_{j=0}^{n} C_j H_{x_j} = -\sum_{j=0}^{n} C_j \int_0^1 \frac{1-t^{x_j}}{1-t} dt$$

$$= -\int_0^1 \frac{\left(\sum_{j=0}^{n} C_j\right) - \sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$$

So, we have that $\sum_{m=0}^{\infty} \frac{1}{P(m)} = \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$

Assuming all other steps are valid, taking the limit as $(x_0, ..., x_n) \to (0...0)$ should yield $\zeta(n+1)$ from the LHS of equation 2, which converges with a P-series test. Now, by parametrizing x_j into $x_j = jv$ for some x. Using this:

$$\zeta(n+1) = \lim_{(x_1 \ldots x_n) \to (0,\ldots,0)} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$$

$$= \lim_{v \to 0} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{jv}}{1-t} dt \text{ with lemma 1}$$

$$= \lim_{v \to 0} \frac{1}{n! v^n} \int_0^1 \frac{\sum_{j=0}^{n} \binom{n}{j} (-t^v)^j}{1-t} dt$$

$$= \lim_{v \to 0} \frac{1}{n! v^n} \int_0^1 \frac{(1-t^v)^n}{1-t} dt$$

Now, using repeated application of L'Hopital's:

$$= \lim_{v \to 0} \frac{1}{n!^2} \int_0^1 \frac{ln(t)^n \sum_{j=0}^{n} \binom{n}{j} j^n (-t^v)^j}{1-t} dt$$

$$= \frac{1}{n!^2} \int_0^1 \frac{ln(t)^n \sum_{j=0}^{n} \binom{n}{j} j^n (-1)^j}{1-t} dt$$

$$= \frac{(-1)^n}{n!} \int_0^1 \frac{ln(t)^n}{1-t} dt$$

This can be verified for each n by using the infinite series for $(1-t)^{-1}$ and repeated integration by parts. It appears to hold for non-integer values of n with numerical evaluation, but I think that this proof requires n to be an integer.

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