1 Basic Properties

Definition 1 *We assume U to be a fixed compact subset of $\mathbb{R}$ with $0 < \mu(U) < \infty$ (for μ the Lebesgue measure).*

Definition 2 *We define a permutation τ on a set $X \subseteq \mathbb{R}^n$ to be a function $\tau : X \to X$ which satisfies $\int_X f d\mu = \int_X f \circ \tau d\mu$ for all $f \in L_1(x, \mu)$.*

This is true if and only if τ preserves measure, so that $\mu(\tau(X)) = \mu(X)$ and is equivalent almost everywhere to an invertible function.

Definition 3 *For $\tau \in \sigma(U)$, we define τ^{-1} to be it's actual inverse if τ is invertible, and otherwise since $\tau \equiv g$ for some invertible function g, $\tau^{-1} = g^{-1}$. Another important caveat is that we require that $\tau \mapsto \tau^{-1}$ is an injective map.*

Definition 4 *We define $\sigma(S)$ for some $S \subseteq \mathbb{R}$ to be the set of all permutations on S.*

Proposition 1 *For all $\tau \in \sigma(X)$, we have that $\tau^{-1} \in \sigma(X)$.*

Proof. We see that $\int_X f(\tau^{-1}(x))d\mu = \int_X f(\tau^{-1}(\tau(x)))d\mu = \int_X f d\mu$. So, $\tau^{-1} \in \sigma(X)$.

Proposition 2 *For all $\tau, \gamma \in \sigma(X)$, we have that $\tau \circ \gamma \in \sigma(X)$*

Proof. We have that $\int_X f(\tau(\gamma(x)))d\mu = \int_X f(\gamma(x))d\mu = \int_X f d\mu$, so $\tau \circ \gamma \in \sigma(X)$.

Proposition 3 *Permutations need not be continuous anywhere, but if a permutation γ is continuous exactly on a set S, then for all $x \in S$ with $\gamma(x) \in S$, γ is differentiable with $\gamma'(x) = 1$.*

Proof. Consider $U = [0, 1]$ and $\tau(x) = x + \frac{1}{3} \mod 1$ if $x \in \mathbb{Q}$ and $\tau(x) = x$ otherwise. This is continuous nowhere, but is a permutation.

Now, for the second part assume that γ is continuous exactly on some set $S \subseteq U$. We have that S must then be open.

Define $f : U \to [0, \mu(U)]$ by $f(x) = \int_U \chi_{(-\infty,x]} d\mu$ where χ_S is the indicator function of a set S. For $x \in U$, we have that $f' = 1$.

Then, fix any $x \in S$ with $\gamma(x) \in S$. Now, by hypotheses, we have that $f(x) = \int_U \chi_{(-\infty,\gamma(x)]} d\mu = \mu(U \cap (-\infty, \gamma(x)])$. As γ is continuous on S, for some neighborhood of x we have that $\gamma(x) > a$ for some constant a, so that $f(x) = \mu(U \cap (-\infty, a]) + \mu((a, \gamma(x)] \cap U)$ for x sufficiently close. As S is open and we assume $\gamma(x) \in S$, we can choose b so that $(b, \gamma(x)] \subseteq S \subseteq U$ and with

$a < b$. Now, $f(x) = \mu(U \cap (-\infty, a]) + \mu((a, b] \cap U) + \mu((b, \gamma(x)]) = C + \gamma(x)$. Therefore, $\gamma'(x) = 1$.

Corollary. Since any $\tau \in \sigma(U)$ is measurable and bounded, there exists an open set S of measure $\mu(U)$ such that τ is continuous when restricted S, and is thus differentiable with derivative 1 a.e. when restricted to S.

Definition 5 *For any given $\tau \in \sigma(U)$, we denote the maximal such S on which τ is continuous (by inclusion) to be* $\mathrm{Reg}(\tau)$.

Definition 6 *The function continuous on* $\mathrm{Reg}(\tau)$ *which τ is equivalent to is denoted* $\mathrm{Rep}(\tau)$.

Definition 7 *Let $\mathrm{sgn} : \Sigma(U) \to C$ be some function on the set of measurable functions from U to U (denoted $\Sigma(U)$) which satisfies the following.*

(a) $\mathrm{sgn}(f) = 0$ if $f \notin \sigma(U)$.
(b) If $g \in \sigma(U)$, $\mathrm{sgn}(f \circ g) = \mathrm{sgn}(f)\,\mathrm{sgn}(g)$.
(c) For all sets of nonzero measure $S \subseteq U$, there exists some $\tau \in \sigma(U)$ for which $\tau(x) = x$ for $x \notin S$ and $\mathrm{sgn}(\tau) \neq 1$.
(d) If $\tau \equiv \gamma$ for $\tau, \gamma \in \Sigma(U)$, then $\mathrm{sgn}(\tau) = \mathrm{sgn}(\gamma)$.

Proposition 4 *With* Id *the identity permutation $x \mapsto x$ on U, we have that* $\mathrm{sgn}(\mathrm{Id}) = 1$.

Proof. This is because $\mathrm{sgn}(\mathrm{Id}) = \mathrm{sgn}(\mathrm{Id} \circ \mathrm{Id}) = \mathrm{sgn}(\mathrm{Id})^2$, so $\mathrm{sgn}(\mathrm{Id}) = 1$ or 0. Since there is guaranteed some γ with $0 \neq \mathrm{sgn}(\gamma) = \mathrm{sgn}(\gamma \circ \mathrm{Id}) = \mathrm{sgn}(\gamma)\,\mathrm{sgn}(\mathrm{Id})$, we have that $\mathrm{sgn}(\mathrm{Id}) = 1$.

Proposition 5 *If $\tau \in \sigma(U)$, then* $\mathrm{sgn}(\tau) \neq 0$.

We have that $1 = \mathrm{sgn}(\mathrm{Id}) = \mathrm{sgn}(\tau \circ \tau^{-1}) = \mathrm{sgn}(\tau)\,\mathrm{sgn}(\tau^{-1})$. Then, $\mathrm{sgn}(\tau) \neq 0$, so $\mathrm{sgn}(f) = 0$ iff $f \notin \sigma(U)$.

Definition 8 *Take $\mathbb{D}[\tau]$ (sometimes denoted $\mu_{\mathbb{D}}$ to be a measure on $\sigma(U)$ such that $\mu_{\mathbb{D}}(S) = \mu_{\mathbb{D}}(\tau \circ S)$ for all $\tau \in \sigma(U)$. Define the operator analogous to a determinant φ as follows for functions $f \in L_1(U^2, \mu^2, \mathbb{R})$.*

$$\varphi f = \int_{\sigma(U)} \mathrm{sgn}(\tau) \exp\left(\int_U \ln(f(x, \tau(x)))d\mu\right) \mathbb{D}[\tau]$$

We take some branch of ln such that $\exp(\ln(x)) = x$ for all x (including 0 and $\pm\infty$), so for example $\exp(\int_{[-1,1]} \ln(1_{[0,1]})d\mu) = 0$.
This can also be interpreted as $\exp(\int_{[-1,1]} \ln(1_{[0,1]})d\mu) = \exp(\int_{[-1,0]} \ln(0)d\mu) = \exp(-\infty\mu([-1,0])) = \exp(-\infty) = 0$.

Definition 9 *For functions* $f, g \in L_1(U^2, \mu^2, \mathbb{R})$*, define* $f \times g \in L_1(U^2, \mu^2, \mathbb{R})$ *as follows.*

$$(f \times g)(x, y) = \int_U f(x, z) g(z, y) d\mu$$

Proposition 6 *For all functions* $g \in L_1(U, \mu, \mathbb{R})$ *and* $f \in L_1(U^2, \mu^2, \mathbb{R})$*,* $\varphi(f(x, y)g(x)) = \exp(\int_U \ln(g) d\mu)(\varphi f)$*.*

Proof.

$$\begin{aligned}
\varphi(fg) &= \int_{\sigma(U)} \operatorname{sgn}(\tau) \exp\left(\int_U \ln(f(x, \tau(x))g(x)) d\mu\right) \mathbb{D}[\tau] \\
&= \int_{\sigma(U)} \operatorname{sgn}(\tau) \exp\left(\int_U \ln(f(x, \tau(x))) d\mu + \int_U \ln(g) d\mu\right) \mathbb{D}[\tau] \\
&= \exp\left(\int_U \ln(g) d\mu\right) \int_{\sigma(U)} \operatorname{sgn}(\tau) \exp\left(\int_U \ln(f(x, \tau(x))) d\mu\right) \mathbb{D}[\tau] \\
&= \exp\left(\int_U \ln(g) d\mu\right) (\varphi f)
\end{aligned}$$

Proposition 7 *For any* $f \in L_1(U^2, \mu^2, \mathbb{R})$*, let* γ *be a permutation* U*. Then,* $\varphi(f(\gamma(x), y)) = \varphi(f) \operatorname{sgn}(\gamma)$*.*

Proof.

$$\varphi f(\gamma(x), y) = \int_{\sigma(U)} \operatorname{sgn}(\tau) \exp\left(\int_U \ln(f(\gamma(x), \tau(x)) d\mu\right) \mathbb{D}[\tau]$$

Now, let $x = \gamma^{-1}(y)$.

$$= \int_{\sigma(U)} \operatorname{sgn}(\tau) \exp\left(\int_U \ln(f(y, \tau(\gamma^{-1}(y))) d\mu\right) \mathbb{D}[\tau]$$

Then, with the substitution[1] $\tau = \Gamma \circ \gamma$, we have the following.

$$= \int_{\sigma(U)} \operatorname{sgn}(\Gamma) \operatorname{sgn}(\gamma) \exp\left(\int_U \ln(f(y, \Gamma(y))d\mu\right) \mathbb{D}[\Gamma]$$
$$= \operatorname{sgn}(\gamma)(\varphi f)$$

Proposition 8 *For any $f \in L_1(U^2, \mu^2, \mathbb{R})$, if there is some set $S \subseteq U$ with $\mu(S) > 0$ and $f(x, z) = f(y, z)$ $\forall z \in U, \forall x, y \in S$, then $\varphi f = 0$.*

Proof. Let γ be a permutation of S with $\operatorname{sgn}(\gamma) \neq 1$. Then, $f(\gamma(x), y) = f(x, y)$, so then $\varphi f = \varphi(f(\gamma(x), y)) = \varphi(f) \operatorname{sgn}(\gamma)$, so $\varphi f = 0$ since $\operatorname{sgn}(\gamma) \neq 1$.

Proposition 9 *For all functions $f, g \in L_1(U^2, \mu^2, \mathbb{R})$, we have that $\varphi(f \times g) = (\varphi f)(\varphi g)$.*

TODO

Proposition 10 $\operatorname{sgn}(\gamma) = \pm 1$ *for all $\gamma \in \sigma(U)$.*

We have that $\varphi(f(x, \gamma(x))) = \operatorname{sgn}(\gamma)\varphi f$ for all applicable f. We also have the following.

$$\varphi f = \int_{\sigma(U)} \operatorname{sgn}(\tau) \exp\left(\int_U \ln(f(x, \tau(x)))d\mu\right) \mathbb{D}[\tau]$$

Then, through the substitution $\tau = \gamma \circ \Gamma$, again assuming invariance under composition, we have the following.

$$= \operatorname{sgn}(\gamma) \int_{\sigma(U)} \operatorname{sgn}(\Gamma) \exp\left(\int_U \ln(f(x, \gamma(\Gamma(x))))d\mu\right) \mathbb{D}[\Gamma]$$
$$= \operatorname{sgn}(\gamma)\varphi(f(x, \gamma(x))$$
$$= \operatorname{sgn}(\gamma)^2 \varphi f$$

So, $\operatorname{sgn}(\gamma)^2 = 1$ and thus $\operatorname{sgn}(\gamma) = \pm 1$.

Definition 10 *For any $f \in L_1(U^2, \mu^2, \mathbb{R})$, we define $f^T(x, y) = f(y, x)$.*

Proposition 11 $\varphi f = \varphi f^T$

[1] Assuming no chain-rule type effect with the measure $\mathbb{D}[\tau]$.

Proof.

$$\varphi f = \int_{\sigma(U)} \operatorname{sgn}(\tau) \exp\left(\int_U \ln(f(x,\tau(x))d\mu\right) \mathbb{D}[\tau]$$

In the inner integral, let $x = \tau^{-1}(u)$. Note that $\operatorname{sgn}(\tau) = \operatorname{sgn}(\tau^{-1})^{-1} = \operatorname{sgn}(\tau^{-1})$ since $\operatorname{sgn}(\tau) = \pm 1$.

$$= \int_{\sigma(U)} \operatorname{sgn}(\tau^{-1}) \exp\left(\int_U \ln(f(\tau^{-1}(x),x)d\mu\right) \mathbb{D}[\tau]$$

Then, let $\tau = \gamma^{-1}$.

$$\begin{aligned} &= \int_{\sigma(U)} \operatorname{sgn}(\gamma) \exp\left(\int_U \ln(f(\gamma(x),x)d\mu\right) \mathbb{D}[\gamma] \\ &= \int_{\sigma(U)} \operatorname{sgn}(\gamma) \exp\left(\int_U \ln(f^T(x,\gamma(x))d\mu\right) \mathbb{D}[\gamma] \\ &= \varphi f^T \end{aligned}$$

Proposition 12 *The multiplicative identity*[2] *is $I(x,y) = \delta_{x-y}$.*

Proof. For any $f \in L_1(U^2, \mu^2, \mathbb{R})$, we have that $(f \times I)(x,y) = \int_U f(x,z)\delta_{z-y} d\mu = f(x,y)$.

Proposition 13 *Let $M \in L_1(U^2, \mu^2, \mathbb{R})$ be diagonal*[3]*, so that $M(x,y) = \delta_{x-y} g(x)$. Then, $\varphi M = \exp(\int_U \ln(g) d\mu)$*

Definition 11 *For some $M \in L_1(U^2, \mu^2, \mathbb{R})$ and $v \in L_1(U, \mu, \mathbb{R})$, we define $(M \times v)(x) = \int_U M(x,y)v(y)d\mu = \langle M_x, v\rangle$ where $\langle \cdot, \cdot \rangle$ is the L_1 inner product.*

Proposition 14 *We have that $M(x+y) = Mx + My$ and for $c \in \mathbb{R}$, $Mcv = cMv$.*

Definition 12 *For any $\tau \in \sigma(U)$, we define*[4] *the corresponding permutation matrix to be $P_\tau(x,y) = \delta_{y-\sigma(x)}$.*

Proposition 15 *For a permutation matrix P_γ, we have that $\varphi P_\gamma = \operatorname{sgn}(\gamma)\varphi I$.*

[2] This assumes we have defined these operations for distributions, which we have not.

[3] Using distributions in this way is not at all rigorous, and to do so we would need to define exp and ln for them.

[4] Again, there are issues with using distributions here.

Proof. We have that $P_\gamma = I(\gamma(x), y)$. Then, $\varphi P_\gamma = \text{sgn}(\gamma)\varphi I$.

Proposition 16 *For permutation matrices P_τ and P_γ, we have that $P_\tau \times P_\gamma = P_{\gamma\circ\tau}$.*

Proof.

$$
\begin{aligned}
(P_\tau \times P_\gamma)(x, y) &= \int_U P_\tau(x, z) P_\gamma(z, y) d\mu_z \\
&= \int_U \delta_{z-\tau(x)} \delta_{y-\gamma(z)} d\mu_z
\end{aligned}
$$

The leftmost δ is nonzero if and only if $z = \tau(x)$.

$$
\begin{aligned}
&= \delta_{y-\gamma(\tau(x))} \\
&= P_{\gamma\circ\tau}
\end{aligned}
$$

Proposition 17 *Fix some matrix $M \in L_1(U^2, \mu^2, \mathbb{R})$ and assume that for all $y \in U$, we have that $M \times f_y(x) = \lambda(y) f_y(x)$. Assume also that there is some $g \in L_1(U^2, \mu^2, \mathbb{R})$ such that $g \times f_y(x) = I$. Then, $\langle M_x, v\rangle = (f \times (\Lambda \times (g \times v)))(x)$ for all vectors $v(y)$.*

Proof. Let[5] $\Lambda(z_1, z_2) = \lambda(z_1) I(z_1, z_2)$.
Now, for all $v(y) \in L_1(U, \mu, \mathbb{R})$, we compute $M(x, y) \times v(y)$.

[5] Here, we are also implicitly using distributions, which is unjustified, but we only assume some previously half-proven properties of their use. The inner product definition of matrix-vector multiplication allows for some leeway, but not much.

$$
\begin{aligned}
\ell(x) &= \langle g_x, v\rangle \\
\implies \langle f_x, \ell\rangle &= v \\
\implies \int_U f(x,y)\ell(y)d\mu_y &= v(x) \\
\langle M_x, v\rangle &= \int_U M(x,z)\ell(z)d\mu_z \\
&= \int_U M(x,z)\int_U f(z,y)\ell(y)d\mu_y d\mu_z \\
&= \int_U \ell(y)\int_U M(x,z)f(z,y)d\mu_z d\mu_y \\
&= \int_U \ell(y)\langle M_x, f_y\rangle d\mu_y \\
&= \int_U \ell(y)\lambda(y)f(x,y)d\mu_y \\
&= \langle f_x, \lambda\ell\rangle \\
&= f \times (\lambda\ell) \\
&= f \times (\Lambda \times \ell) \\
&= f \times (\Lambda \times (g \times v))
\end{aligned}
$$

Definition 13 *We define the values of $\lambda(y)$ to be the eigenvalues of M, and the values of $f(x,y)$ for each y to be the corresponding eigenvectors of M.*

2 Properties of $\sigma(U)$ and $\mu_{\mathbb{D}}$

Denote $\mu_{\mathbb{D}}$ to be the measure usually denoted above as $\mathbb{D}[\tau]$, a measure on subsets of $\sigma(U)$. We have used $\mu_{\mathbb{D}}$ without constructing it, only assuming invariance of measure under composition so that $\mu_{\mathbb{D}}(X) = \mu_{\mathbb{D}}(\tau \circ X)$. Given the assumed invariance of $\mu_{\mathbb{D}}$, we want it to be a measure on $\sigma(U)$ as a topological group equipped with some topology and the group operation of composition. Then, we can extend it arbitrarily from a σ-algebra on $\sigma(U)$ to a larger one on $L_1(U, \mu, U)$.

Definition 14 *For $f \in L_1(U, \mu, U)$, we define $[f] = \{g \in L_1(U, \mu, U) \mid g \equiv f\}$.*

Proposition 18 *$\mu_{\mathbb{D}}([\tau]) = 0$ for $\tau \in \sigma(U)$.*

Proof. Given that $\mu_{\mathbb{D}}$ is invariant under composition, $\mu_{\mathbb{D}}([\tau]) = \mu_{\mathbb{D}}([\mathrm{Id}])$.
Now, with δ_{xy} the Kroenecker Delta, we have the following.

$$
\begin{aligned}
\varphi 0 &= \varphi \delta_{xy} \\
&= \int_{\sigma(U)} \operatorname{sgn}(\tau) \exp\left(\int_U \ln(\delta_{x\tau(x)}) d\mu \right) \mathbb{D}[\tau]
\end{aligned}
$$

Now, if $x \not\equiv \tau(x)$, the integrand will be 0, and if $x \equiv \tau(x)$, we have that $\exp(\ln(\cdots)) = \mu(U)$.
Given that $\tau \equiv \mathrm{Id}$, we have that $\operatorname{sgn}(\tau) = 1$ as well, so we can then integrate only over [Id] and simplify.

$$
\begin{aligned}
&= \int_{[\mathrm{Id}]} \operatorname{sgn}(\tau) \exp\left(\int_U \ln(\delta_{x\tau(x)}) d\mu \right) \mathbb{D}[\tau] \\
&= \mu(U) \int_{[\mathrm{Id}]} \mathbb{D}[\tau] \\
&= \mu(U)\mu_{\mathbb{D}}([\mathrm{Id}])
\end{aligned}
$$

Since $\varphi 0 = 0$, we have that $\mu_{\mathbb{D}}([\tau]) = \mu_{\mathbb{D}}(\mathrm{Id}) = 0$.

Definition 15 *We define $D(f)$ to be the set of jump discontinuities of a function f.*

Definition 16 *For a set $S \subseteq U$ with $\mu(S) = 0$, we define $V_S = \{\tau \in \sigma(U) \mid D(\mathrm{Rep}(\tau)) \subseteq S\}$*

Definition 17 *For a set $S \subseteq U$ with $\mu(S) = 0$ partition of components of $U \setminus S$ which are connected in U is denoted $\{A_\alpha\}_{\alpha \in I(S)}$ for some well-ordered set $I(S)$ whose cardinality corresponds to the cardinality of S.*

Definition 18 *We denote $\ell_p(A, B)$ to be the set of sequences in $B \subseteq \mathbb{R}$ indexed over a set A with a finite ℓ_p norm. We define $\ell_\infty(A, B)$ similarly.*

Proposition 19 *$(V_S, ||\cdot||_\infty)$ is isometric to a subset of $\ell_\infty(I(S), U - U)$.*

Proof. Let A_α be the corresponding partition of $U \setminus S$, and for any $\tau \in \sigma(U)$ let $\{C_{\tau\alpha}\}_{\alpha \in I(S)}$ be a sequence of constants so that $\tau(x) = x + C_{\tau\alpha}$ for $x \in A_\alpha$, which must exist since τ is continuous on each A_α and has derivative 1 a.e.
Now, consider the map $f(\tau) = C_\tau - M$ where M is a sequence with $M_\alpha = \inf A\alpha$.

$$
\begin{aligned}
||\tau - \gamma||_\infty &= \sup_{\alpha \in I} \sup_{x \in A_\alpha} |\tau(x) - \gamma(x)| \\
&= \sup_{\alpha \in I} |C_{\tau\alpha} - C_{\gamma\alpha}| \\
&= \sup_{\alpha \in I} |C_{\tau\alpha} - M_\alpha - C_{\gamma\alpha} + M_\alpha| \\
&= ||(C_\tau - M) - (C_\gamma - M)||_{\ell_\infty(I)} \\
&= ||f(\tau) - f(\gamma)||_{\ell_\infty(I)}
\end{aligned}
$$

Now, since $f(\tau)_\alpha \in U$, we have that $f(\tau)_\alpha - f(\gamma)_\alpha \in U - U$, so that $\ell_\infty(I)$ can be replaced with $\ell_\infty(I, U-U)$ in this with no change.

Definition 19 *Under the above isometry, we denote* $\mathrm{Seq}(S) = f(V_S) \subseteq \ell_\infty(I(S), U-U)$

Proposition 20 $(\sigma(U), |\cdot|_{L_1(U,\mu)})$ *is isometric to* $(\sigma([0,\mu(U)]), |\cdot|_{L_1([0,\mu(U)],\mu)})$.

Proof. Consider the function $D : \mathbb{R} \to [0, \mu(U)]$ with $D(x) = \mu((-\infty, x) \cap U)$. We have that D must be invertible when restricted to some set $K \subseteq U$ with $\mu(K) = \mu(U)$. Now, there is a clear identity isometry between $\sigma(K)$ and $\sigma(U)$, so it suffices to prove that $\sigma(K)$ is isometric to $\sigma([0,\mu(U)])$.
In addition, D is Lipchitz since for $y > x$ we have $D(y) - D(x) = \mu((x,y) \cap U) \leq y - x$.
Then, D must be differentiable almost everywhere. Since K has nonzero measure, it must be differentiable almost everywhere on K.
It's clear that its derivative can only be 0 or 1, and on K it must be 1 almost everywhere.
Consider the function $J : \sigma(K) \mapsto \sigma([0, \mu(U)])$ with $J(\tau) = \tau \circ D$.
Now, we have the following.

$$
\begin{aligned}
||J(\tau) - J(\gamma)||_{L_1([0,\mu(U),\mu)]} &= \int_{[0,\mu(U)]} |J(\tau) - J(\gamma)| d\mu \\
&= \int_U |\tau - \gamma| D' d\mu \\
&= ||\tau - \gamma||_{L_1(U,\mu)}
\end{aligned}
$$

In addition, the mapping J is bijective with $J^{-1}(\tau) = \tau \circ D^{-1}$.

Corollary. $\sigma(U)$ is isometric to $\sigma([0,1])$ up to a dilation.
Now, we have through the mapping $J_2 : \sigma([0,\mu(U)]) \to \sigma([0,1])$ with $J_2(\tau) = \tau(\mu(U)x)/\mu(U)$ that J_2 is a dilation with scaling factor $\frac{1}{\mu(U)^2}$.
Therefore, convergence in $\sigma([0,1])$ holds if and only if convergence in $\sigma([0,\mu(U)])$ holds.

Proposition 21 $\sigma(U)$ *is not relatively compact under the* L_1 *norm.*

Proof. We have that $\sigma(U)$ is closed under the L_1 norm, so it suffices to show some sequence has no convergent subsequence.
By the previous proposition, it suffices to show that that $\sigma([0,1])$ is not compact under $L_1([0,1],\mu)$.
Let U_{ij} be a sequence of points of $[0,1]$ for $0 \leq j \leq 2^i$. Let $U_{i0} = 0$ and $U_{i2^i} = 1$.

Then, we define $U_{i(2j)} = U_{(i-1)j}$ and $U_{i(2j+1)} = (U_{(i-1)j} + U_{(i-1)(j+1)})/2$.
Now, we have a sequence of partitions $V_{ij} = (U_{i(j-1)}, U_{ij})$ for $1 \leq j \leq 2^i$.
Let τ_n be the permutation which transposes $V_{n(2k+1)}$ with $V_{n(2^n-2k-1)}$, or equivalently $\tau_n(x) = x$ on $V_{n(2k)}$ and $\tau_n(x) = x + 0.5 \mod 1$ on $V_{n(2k+1)}$. Now, for any distinct $n, m \in \mathbb{N}$, we have that $||\tau_n - \tau_m||_1 = 1$, so that there is no convergent subsequence.
Therefore, $\sigma([0,1])$ is not relatively compact, so that $\sigma(U)$ is not relatively compact.

Proposition 22 *For any $S \subseteq U$ with $\mu(S) = 0$, V_S is relatively compact under the $|\cdot|_1$ norm if and only if $\mu(\partial SU) = 0$*

Proof. Let $T_h f = f(x+h)$, and take $\tau(x) = 0$ for all $\tau \in \sigma(U)$ and $x \notin U$. By the Kolmogorov-Riesz Compactness Theorem, it suffices to show that $||T_h f - f||_1 \to 0$ uniformly (equicontinuously) on V_S.
For any $h \in \mathbb{R}$, let $A_h = \{x \in U \setminus S \mid d(x, \mathbb{R} \setminus (U \setminus S) < |h|\}$, so that if $x \notin A_h$ then $T_h \operatorname{Rep}(\tau) - \operatorname{Rep}(\tau) = h$ since we would then have that $\operatorname{Rep}(\tau)$ is continuous with derivative 1 a.e. on the h-neighborhood around x.

$$\int_U |T_h\tau - \tau| d\mu = \int_{U \setminus S} |T_h \operatorname{Rep}(\tau) - \operatorname{Rep}(\tau)| d\mu$$
$$= \int_{(U\setminus S)\setminus A_h} |T_h \operatorname{Rep}(\tau) - \operatorname{Rep}(\tau)| d\mu + \int_{A_h} |T_h \operatorname{Rep}(\tau) - \operatorname{Rep}(\tau)| d\mu$$
$$\leq |h|\mu((U \setminus S) \setminus A_h) + 2\mu(A_h)(\sup_{x \in U} |x|)$$

So, to prove that $||T_h\tau - \tau||_1 \to 0$ uniformly on V_S, it suffices to show that $\lim_{h \to 0} \mu(A_h) = 0$.

$$\lim_{h \to 0} \mu(A_h) = \mu\left(\cap_{n=1}^{\infty} A_{1/n}\right)$$
$$= \mu(U \cap \partial S)$$
$$= \mu(\partial S)$$

So, if $\mu(\partial S) = 0$, it converges uniformly on V_S.
Otherwise, assume $\mu(\partial S) > 0$. Let A_α be the crudest partition of $U \setminus S$ which is connected in U. Now, ∂S is a dense subset of some closed set $K_1 \subseteq U$ which is also compact with $0 < \mu(K_1) < \infty$. Now, let $K = \{x \in K_1 \mid x \in A_\alpha \implies A_\alpha \subseteq K$. This set K is a closed set with some number of open sets subtracted from it, and thus is itself closed and thus compact.
Let $J = \{\alpha \in I(S) \mid A_\alpha \subseteq K\}$. By the previous proposition, we have that V_S is isometric to $\operatorname{Seq}(S) \subseteq \ell_\infty(I(S))$.

Now, consider the following subset of Seq(S) and mapping.

$$M = \{C \in \ell_\infty(I(S)) \mid C_\alpha = \inf A_\alpha \forall \alpha \notin J\}$$
$$Z : \sigma(K) \to M$$
$$Z(\tau) = \operatorname{arginf}_{C_\gamma \in M} ||\gamma - \tau||_{L_1(K,\mu)}$$

Now, $||Z(\tau) - Z(\gamma)||_{\ell_\infty(J)} = ||\tau - \gamma||_{L_1(K,\mu)}$, so that $\sigma(K)$ is isometric to at least a subset of M. Then, since $\sigma(K)$ is not relatively compact, M cannot be relatively compact, and therefore Seq(S) cannot be relatively compact. Finally, it follows that V_S cannot be relatively compact.

Proposition 23 *Given some $f \in L_1(U^2, \mu^2, \mathbb{R})$, if there exists some $g \in L_1(U, \mu, \mathbb{R})$ which is nonzero on some subset S of U with nonzero measure such that $\int_S fgd\mu = 0$ then $\varphi f = 0$.*

Proof. Given g, let h be 1 if $g = 0$ and $g = h$ otherwise. Now, $\int_U \exp(\ln(h)) \neq 0$. We have the following.

$$\varphi f(x, y) = C\varphi(f(h(x), y)$$