Definition 1 *For $a < b$, we take $U = [a, b]$. However, where possible, we prove things for U a general compact set with nonzero finite measure.*

Definition 2 *We define a permutation τ on a set $X \subseteq \mathbb{R}^n$ to be a function $\tau : X \to X$ which satisfies $\int_X f d\mu = \int_X f \circ \tau d\mu$ for all $f \in L_1(x, \mu)$.*

Note that we have that τ preserves the measure by substituting $f = \chi_S$ for χ_S the indicator function of a measurable set $S \subseteq U$.

Definition 3 *We define $\sigma(S)$ for some $S \subseteq \mathbb{R}$ to be the set of all permutations on S.*

Proposition 1 *For all $\tau, \gamma \in \sigma(X)$, we have that $\tau \circ \gamma \in \sigma(X)$*

Proof. We have that $\int_X f(\tau(\gamma(x)))d\mu = \int_X f(\gamma(x))d\mu = \int_X f d\mu$, so $\tau \circ \gamma \in \sigma(X)$.
The continuous differentiability almost everywhere follows immediately by restricting $\tau \circ \gamma$ to sufficiently small open neighborhoods of any point.

Definition 4 *For any given $\tau \in \sigma(U)$, by Lusin's theorem we have that for any $\varepsilon > 0$ there is some set S_ε such that τ restricted to S_ε is continuous (in the inherited subspace topology) and $\mu(U \setminus S_\varepsilon) < \varepsilon$. Now, let $\mathrm{Reg}(\tau) = \cup_{n=1}^{\infty} S_{1/n}$, and let the representative $\mathrm{Rep}(\tau)$ be τ restricted to $\mathrm{Reg}(\tau)$.*

Definition 5 *We define $D(f)$ to be the set of jump discontinuities of a function f.*

Proposition 2 *For any $\tau \in \sigma(U)$, we have that $\mathrm{Rep}(\tau)$ is locally continuously differentiable on $D(\mathrm{Rep}(\tau))$ with derivative ± 1.*

Proof. Let $g = \mathrm{Rep}(\tau)$ for simplicity. We know that $S = D(g)$ is open so for any $x \in S$, there is some $\delta > 0$ with $B(x, \delta) \subset S$.
Then, take any y with $|x - y| < \delta$. We have that $|g(x) - g(y)| = \mu([g(x), g(y)])$, assuming $g(x) \leq g(y)$ without loss of generality. Then, since g is invertible on $[x, y]$ it must be monotonic and since it's continuous we then have that $g([x, y]) = [g(x), g(y)]$ so that $|g(x) - g(y)| = \mu(g([x, y]) = \mu([x, y) = |x - y|$.
Therefore, by fixing x we see that $g(y) = \pm x + C$ for $y \in B(x, \varepsilon)$ so that g is locally differentiable with derivative ± 1.

Definition 6 *For a set $S \subseteq U$ with $\mu(S) = 0$, we define the cylindrical set of S to be $V_S = \{\tau \in \sigma(U) \mid D(\mathrm{Rep}(\tau)) \subseteq S \wedge \mathrm{Rep}(\tau)' \equiv 1\}$*

Definition 7 *For a set $S \subseteq U$ with $\mu(S) = 0$, a partition of components of $U \setminus S$ which are connected in U is denoted $\{A_\alpha\}_{\alpha \in I(S)}$ for some well-ordered set $I(S)$ whose cardinality corresponds to the cardinality of S.*

Definition 8 *We denote $\ell_p(A, B)$ to be the set of sequences in $B \subseteq \mathbb{R}$ indexed over a set A with a finite ℓ_p norm. We define $\ell_\infty(A, B)$ similarly.*

Proposition 3 *Let $A = \{x - y \mid x, y \in U\}$. Then, $(V_S, ||\cdot||_\infty)$ is isometric to a subset of $\ell_\infty(I(S), A)$.*

Proof. Let A_α be the crudest partition of $U \setminus S$ which is connected in U, and for any $\tau \in \sigma(U)$ let $\{C_{\tau\alpha}\}_{\alpha \in I(S)}$ be a sequence of constants so that $\tau(x) = x + C_{\tau\alpha}$ for $x \in A_\alpha$, which must exist since τ is continuous when restricted to each A_α and has continuous derivative 1 almost everywhere..
Now, consider the map $f(\tau) = C_\tau - M$ where M is a sequence with $M_\alpha = \inf A\alpha$.

$$\begin{aligned} ||\tau - \gamma||_\infty &= \sup_{\alpha \in I} \sup_{x \in A_\alpha} |\tau(x) - \gamma(x)| \\ &= \sup_{\alpha \in I} |C_{\tau\alpha} - C_{\gamma\alpha}| \\ &= \sup_{\alpha \in I} |C_{\tau\alpha} - M_\alpha - C_{\gamma\alpha} + M_\alpha| \\ &= ||(C_\tau - M) - (C_\gamma - M)||_{\ell_\infty(I)} \\ &= ||f(\tau) - f(\gamma)||_{\ell_\infty(I)} \end{aligned}$$

We denote $U - U = \{x - y \mid x, y \in U\}$. Now, since $f(\tau)_\alpha \in U$, we have that $f(\tau)_\alpha - f(\gamma)_\alpha \in U - U$, so that $\ell_\infty(I)$ can be replaced with $\ell_\infty(I, U - U)$ in this with no change.

Definition 9 *Under the above isometry, we denote* $\mathrm{Seq}(S) = f(V_S) \subseteq \ell_\infty(I(S), U - U)$

Proposition 4 $(\sigma(U), |\cdot|_{L_1(U,\mu)})$ *is isometric to* $(\sigma([0,\mu(U)]), |\cdot|_{L_1([0,\mu(U)],\mu)})$ *where* $U \subseteq \mathbb{R}$ *is any compact set of nonzero measure.*

Proof. Consider the function $D : \mathbb{R} \to [0, \mu(U)]$ with $D(x) = \mu((-\infty, x) \cap U)$.
We have that D must be invertible when restricted to some set $K \subseteq U$ with $\mu(K) = \mu(U)$. Now, there is a clear identity isometry between $\sigma(K)$ and $\sigma(U)$, so it suffices to prove that $\sigma(K)$ is isometric to $\sigma([0, \mu(U)])$.
In addition, D is Lipchitz since for $y > x$ we have $D(y) - D(x) = \mu((x, y) \cap U) \leq y - x$.
Then, D must be differentiable almost everywhere. Since K has nonzero measure, it must be differentiable almost everywhere on K.
It's clear that its derivative can only be 0 or 1, and on K it must be 1 almost everywhere.
Consider the function $J : \sigma(K) \mapsto \sigma([0, \mu(U)])$ with $J(\tau) = \tau \circ D$.
Now, we have the following.

$$
\begin{aligned}
||J(\tau) - J(\gamma)||_{L_1([0,\mu(U),\mu)]} &= \int_{[0,\mu(U)]} |J(\tau) - J(\gamma)| d\mu \\
&= \int_U |\tau - \gamma| D' d\mu \\
&= ||\tau - \gamma||_{L_1(U,\mu)}
\end{aligned}
$$

In addition, the mapping J is bijective with $J^{-1}(\tau) = \tau \circ D^{-1}$.
Therefore, $\sigma(K)$ is isometric to $\sigma([0, \mu(U)])$.

Corollary. $\sigma(U)$ is isometric to $\sigma([0, 1])$ up to a dilation of the metric.
We have through the mapping $J_2 : \sigma([0, \mu(U)]) \to \sigma([0, 1])$ with $J_2(\tau) = \tau(\mu(U)x)/\mu(U)$ that J_2 is a dilation.
Therefore, convergence of the corresponding sequence in $\sigma([0, 1])$ holds if and only if convergence in $\sigma([0, \mu(U)])$ holds.

Proposition 5 $\sigma(U)$ *is not compact under the* L_1 *norm.*

Proof. We have that $\sigma(U)$ is closed under the L_1 norm, so it suffices to show some sequence has no convergent subsequence.
By the previous proposition, it suffices to show that that $\sigma([0, 1])$ is not compact under $L_1([0, 1], \mu)$.
Let U_{ij} be a sequence of points of $[0, 1]$ for $0 \leq j \leq 2^i$. Let $U_{i0} = 0$ and $U_{i2^i} = 1$. Then, we define $U_{i(2j)} = U_{(i-1)j}$ and $U_{i(2j+1)} = (U_{(i-1)j} + U_{(i-1)(j+1)})/2$.
Now, we have a sequence of partitions $V_{ij} = (U_{i(j-1)}, U_{ij})$ for $1 \leq j \leq 2^i$.
Let τ_n be the permutation which transposes $V_{n(2k+1)}$ with $V_{n(2^n-2k-1)}$, or equivalently $\tau_n(x) = x$ on $V_{n(2k)}$ and $\tau_n(x) = x + 0.5 \mod 1$ on $V_{n(2k+1)}$. Now, for any distinct $n, m \in \mathbb{N}$, we have that $||\tau_n - \tau_m||_1 = 1$, so that there is no convergent subsequence.
Therefore, $\sigma([0, 1])$ is not compact, so that $\sigma(U)$ is not compact.

Proposition 6 *For any* $S \subseteq U$ *with* $\mu(S) = 0$*,* V_S *is relatively compact under the* $|\cdot|_1$ *norm if and only if* $\mu(\partial SU) = 0$.

Proof. Let $T_h f = f(x + h)$, and take $\tau(x) = 0$ for all $\tau \in \sigma(U)$ and $x \notin U$. By the Kolmogorov-Riesz Compactness Theorem, it suffices to show that $||T_h f - f||_1 \to 0$ uniformly (equicontinuously) on V_S.
For any $h \in \mathbb{R}$, let $A_h = \{x \in U \setminus S \mid d(x, \mathbb{R} \setminus (U \setminus S) < |h|\}$, so that if $x \notin A_h$ then $|T_h \operatorname{Rep}(\tau) - \operatorname{Rep}(\tau)| = |h|$ since we would then have that $\operatorname{Rep}(\tau)$ is continuous with derivative ± 1 a.e. on the $|h|$-neighborhood around x.

$$
\begin{aligned}
\int_U |T_h \tau - \tau| d\mu &= \int_{U \setminus S} |T_h \operatorname{Rep}(\tau) - \operatorname{Rep}(\tau)| d\mu \\
= \int_{(U \setminus S) \setminus A_h} &|T_h \operatorname{Rep}(\tau) - \operatorname{Rep}(\tau)| d\mu + \int_{A_h} |T_h \operatorname{Rep}(\tau) - \operatorname{Rep}(\tau)| d\mu \\
&\leq |h| \mu((U \setminus S) \setminus A_h) + 2\mu(A_h)(\sup_{x \in U} |x|)
\end{aligned}
$$

So, to prove that $||T_h\tau - \tau||_1 \to 0$ uniformly on V_S, it suffices to show that $\lim_{h\to 0}\mu(A_h) = 0$.

$$\begin{aligned}\lim_{h\to 0}\mu(A_h) &= \mu\left(\cap_{n=1}^{\infty} A_{1/n}\right)\\ &= \mu(U\cap\partial S)\\ &= \mu(\partial S)\end{aligned}$$

So, if $\mu(\partial S) = 0$, it converges uniformly on V_S.
Otherwise, assume $\mu(\partial S) > 0$. Let A_α be the crudest partition of $U \setminus S$ which is connected in U. Now, ∂S is a dense subset of some closed set $K_1 \subseteq U$ which is also compact with $0 < \mu(K_1) < \infty$. Now, let $K = \{x \in K_1 \mid x \in A_\alpha \implies A_\alpha \subseteq K$. This set K is a closed set with some number of open sets subtracted from it, and thus is itself closed and thus compact.
Let $J = \{\alpha \in I(S) \mid A_\alpha \subseteq K\}$. By the previous proposition, we have that V_S is isometric to $\mathrm{Seq}(S) \subseteq \ell_\infty(I(S))$.
Now, consider the following subset of $\mathrm{Seq}(S)$ and mapping.

$$\begin{gathered}M = \{C \in \ell_\infty(I(S)) \mid C_\alpha = \inf A_\alpha \forall\alpha \notin J\}\\ Z : \sigma(K) \to M\\ Z(\tau) = \mathrm{arginf}_{C_\gamma\in M}\,||\gamma - \tau||_{L_1(K,\mu)}\end{gathered}$$

Now, $||Z(\tau) - Z(\gamma)||_{\ell_\infty(J)} = ||\tau - \gamma||_{L_1(K,\mu)}$, so that $\sigma(K)$ is isometric to at least a subset of M. Then, since $\sigma(K)$ is not relatively compact, M cannot be relatively compact, and therefore $\mathrm{Seq}(S)$ cannot be relatively compact. Finally, it follows that V_S cannot be relatively compact.

Proposition 7 *Assume that for some $X \subseteq \sigma(U)$, we have that $\mathrm{Rep}(\tau)' \equiv 1$ for all $\tau \in X$. Then, X is relatively compact under $|\cdot|_1$ if and only if $\mu\left(\partial \cup_{\tau\in X} D(\mathrm{Rep}(\tau))\right) = 0$.*

Proof. If $\mu\left(\partial \cup_{\tau\in X} D(\mathrm{Rep}(\tau))\right) = 0$, then $X \subseteq V_{\cup_{\tau\in X} D(\mathrm{Rep}(\tau))}$ so that X is relatively compact by the previous proposition.
Otherwise, if X is not relatively compact, then there exists some sequence $\{\tau_n\} \subseteq X$ such that $\{\tau_n\} \nsubseteq V_S$ for all sets $S \subseteq U$ with $\mu(\partial S) = 0$.
Now, $\mu\left(\partial \cup_{n=1}^{\infty} D(\mathrm{Rep}(\tau_n))\right) > 0$, otherwise $\{\tau_n\}$ would be relatively compact and thus contain a convergent subsequence.
Finally, we have the following.

$$\mu\left(\partial \cup_{\tau\in X} D(\mathrm{Rep}(\tau))\right) \geq \mu\left(\partial \cup_{n=1}^{\infty} D(\mathrm{Rep}(\tau_n))\right) > 0$$