

Let M be an invertible matrix with a full set of distinct eigenvalues λ_i .

$$\begin{aligned}
\sum_i \lambda_i^n \dot{\lambda}_i &= \text{Tr}(M^n \dot{M}) \\
\vec{v}_i &= \text{Tr}(M^i \dot{M}) \wedge \vec{\lambda}_i = \lambda_i \wedge A_{ij} = \lambda_i^j \\
&\implies A \vec{\lambda} = \vec{v} \\
&\implies \dot{\lambda}_i = (A^{-1} \vec{v})_i = A_i^{-T} \cdot \vec{v} \\
A^{-1} &= \frac{1}{\det A} C^T \implies A_i^{-T} = \frac{1}{\det A} C_i \\
C_{ij} &= (-1)^j \left(\prod_{k < m \wedge k, m \neq i} (\lambda_m - \lambda_k) \right) \left(\sum_{(S \subseteq \{1, \dots, n\} \setminus \{i\} \wedge |S| = n-1-j)} \prod_{k \in S} \lambda_k \right) \\
p_B(x) &= \det(xI - B) \\
p_{\neq i}(x) &= \lim_{y \rightarrow x} \frac{p_M(y)}{\lambda_i - y} \\
&\implies \sum_{(S \subseteq \{1, \dots, n\} \setminus \{i\} \wedge |S| = n-1-j)} \prod_{k \in S} \lambda_k = \frac{p_{\neq i}^{(j)}(0)}{j!} \\
\prod_{k < m \wedge k, m \neq i} (\lambda_m - \lambda_k) &= \left(\prod_{m < k} (\lambda_m - \lambda_k) \right) / \left(\left(\prod_{m < i} (\lambda_i - \lambda_m) \right) \left(\prod_{i < m} (\lambda_m - \lambda_i) \right) \right) \\
&= (-1)^j \det(A) \left((-1)^j \prod_{m < i} (\lambda_m - \lambda_i) \right)^{-1} \left(\prod_{i < m} (\lambda_m - \lambda_i) \right)^{-1} \\
&= \det(A) \prod_{m \neq i} (\lambda_m - \lambda_i)^{-1} \\
&= \det(A) \frac{1}{p_{\neq i}(\lambda_i)} = \frac{-\det A}{p_M^{(1)}(\lambda_i)} \\
\implies C_{ij} &= -\frac{p_{\neq i}^{(j)}(0)}{j!} \frac{\det A}{p_M^{(1)}(\lambda_i)} \implies \dot{\lambda}_i = \frac{-1}{\det A} \sum_j \frac{p_{\neq i}^{(j)}(0)}{j!} \frac{\det A}{p_M^{(1)}(\lambda_i)} \text{Tr}(M^j \dot{M}) \\
&= \frac{-1}{p_M^{(1)}(\lambda_i)} \sum_j \frac{p_{\neq i}^{(j)}(0)}{j!} \text{Tr}(M^j \dot{M}) \tag{1} \\
&= \frac{-1}{p_M^{(1)}(\lambda_i)} \text{Tr}(\dot{M} P(M)) \\
P(x) &= \sum_j \frac{p_{\neq i}^{(j)}(0)}{j!} x^j = p_{\neq i}(x) \\
\implies P(M) &= p_M(M)(\lambda_i I - M)^{-1}
\end{aligned}$$

Since $\lambda_i I - M$ is not invertible and $p_{\neq i}$ is meant in the limiting sense, we take $P(M)$ to be a matrix which maps $x_j \rightarrow 0$ for $j \neq i$ and $x_i \rightarrow p_{\neq i}(\lambda_i)x = -\dot{p}_M(\lambda_i)x$ so that with Q the orthogonal projection onto x_i 's eigenspace, $P(M) = -\dot{p}_M(\lambda_i)Q$.

$$\begin{aligned} \implies \dot{\lambda}_i &= -\frac{-\dot{p}_M(\lambda_i)}{\dot{p}_M(\lambda_i)} \text{Tr}(\dot{M}Q) \\ &= \text{Tr}(\dot{M}Q) \end{aligned}$$

This last simplification requires knowledge of the eigenvectors of M in order to calculate Q , but the earlier representation in (1) only requires knowledge of the eigenvalues.