Invertible a.e. Measure-Preserving Maps as Permutations

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1 Basic Properties

Definition 1 *Take $\Omega \subset \mathbb{R}^n$ to be an open and bounded Lipschitz Domain with $0 < \mu(\Omega) < \infty$.*

Definition 2 *By convention we take the range of functions in $L^p(\Omega)$ to $\mathbb{R}^n$. If the range is instead K, it will be denoted as $L^p(\Omega, K)$.*

Definition 3 *The set of all invertible a.e. measure preserving maps (τ such that $\mu(X) = \mu(\tau(X))$) on a set Ω is denoted $\sigma(\Omega)$.*

Proposition 1 *For $\tau, \gamma \in \sigma(\Omega)$ we have that $\tau \circ \gamma \in \sigma(\Omega)$.*

Proposition 2 *For all $\tau \in \sigma(\Omega)$, there exists some function g such that $\tau \circ g = \mathrm{Id}$ almost everywhere. In addition, $g \in \sigma(\Omega)$ as well. g is usually denoted as τ^{-1} even though the actual inverse is not explicitly defined and need not exist.*

2 Rigidity and Form

Definition 4 *We let $I(\tau)$ be some any set $\subseteq \Omega$ with on which τ is equivalent a.e. to an invertible function and such that $\mu(I(\tau)) = \mu(\Omega)$.*

Proposition 3 *For all $\tau \in \sigma(\Omega)$ and any d-dimensional Hausdorff measurable set $K \subset I(\tau)$, we have that $H^d(K) = H^d(\tau(K))$ for all $1 \leq d \leq n$.*

Proof. We have the following.

$$H^d_\delta(K) = \inf \left\{ \sum_i (\mathrm{diam}\, U_i)^d : K \subseteq \cup_i U_i \wedge \mathrm{diam}\, U_i < \delta \right\}$$

Each U_i can be taken without loss of generality to be a countable union of balls so that their union is then again a countable union of balls. Therefore, we can assume without loss of generality that each U_i is just a ball.

$$= \inf \left\{ \sum_i (\mathrm{diam}\, B(x_i, r_i))^d : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\}$$

$$= \inf \left\{ \sum_i (C_n \mu(B(x_i, r_i)))^{d/n} : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\}$$

$$= \inf \left\{ \sum_i (C_n \mu(\tau(B(x_i, r_i))))^{d/n} : \tau(K) \subseteq \cup_i \tau(B(x_i, r_i)) \wedge 2r_i < \delta \right\}$$

For any set $X \subset \Omega$ we have that $(C_n\mu(X))^{1/n} \leq \text{diam}(X)$ because if $\mu(B(x,\varepsilon)) = \mu(X)$, then $(C_n\mu(X))^{1/n} = (C_n\mu(B(x,\varepsilon)))^{1/n} = \text{diam} B(x,\varepsilon) \leq \text{diam} X$ as an n-sphere minimizes diameter for a given volume.

$$\leq \inf\left\{\sum_i (C_n\mu(V_i))^{d/n} : \tau(K) \subseteq \cup_i V_i \wedge (C_n\mu(V_i))^{1/n} < \delta\right\}$$

$$\leq \inf\left\{\sum_i (\text{diam } V_i)^d : \tau(K) \subseteq \cup_i V_i \wedge \text{diam}(V_i) < \delta\right\}$$

$$= H_\delta^d(\tau(K))$$

Now, taking limits as $\delta \to 0$ we have that $H^d(K) \leq H^d(\tau(K))$.

Therefore, $H^d(\tau(K)) \leq H^d(\tau^{-1}(\tau(K)) = H^d(K)$ so that $H^d(\tau(K)) = H^d(K)$.

Proposition 4 *For all $\tau \in \sigma(\Omega)$, there exists a function g_τ continuous on an open set S with $\mu(S) = \mu(\Omega)$ such that $g_\tau = \tau$ almost everywhere.*

Proof. We have that for all n there exists some $S_n \subset \Omega$ such that τ is continuous when restricted to S_n and $\mu(\Omega \setminus S_n) < 1/n$. Now, with $S = \cup_{i=1}^\infty S_i$ we have that τ is continuous on the S-inherited subspace topology and that $\mu(S) = \mu(\Omega)$ so that S is dense in Ω.

Now, let $G = \overline{\{(x,\tau(x)) \mid x \in S\}}$.

For any $x \in \Omega \setminus S$, we have that there exists some sequence $\{x_n\} \subset S$ convergent to x. Now, $\tau(x_n)$ is bounded and thus has some subsequence convergent to some y. Now, $(x,y) \in G$. Thus, with $p_1(x,y) = x$, we have that $p_1(G) = \Omega$.

Next, let $Q = \{x \in \Omega \mid ((x,y) \in G \wedge (x,z) \in G) \implies y = z\}$. If $x \in S$ then if $(x,s_1) \in G$ and $(x,s_2) \in G$ we have that there must be some sequences $(x_{in}, \tau(x_{in})) \to (x,s_i)$ for $x_n \in S$. As $|x_{in} - x| \to 0$ we have that because τ is continuous in S that $|\tau(x_{in}) - \tau(x)| \to 0$, and thus that $|\tau(x_{1n}) - \tau(x_{2n})| \to 0$ so that because $|\tau(x_{in}) - s_i| \to 0$ we have that $|s_1 - s_2|$ is arbitrarily small and thus $s_1 = s_2$. Therefore, $x \in Q$ so that $S \subseteq Q$.

Finally, define $g_\tau : \Omega \to \Omega$ by $g_\tau(x) = y$ if $(x,y) \in G$, where y is chosen arbitrarily for $x \notin Q$.

We aim to show that g_τ is continuous on S. For $x \in S$ we have that for any $\varepsilon > 0$ there exists some δ such that $|x - y| < \delta$ for $y \in S$ implies that $|\tau(x) - \tau(y) < \varepsilon$. Then, take any $y \in \Omega \setminus S$ with $|x - y| < \delta$.

If $y \notin Q$, then assume (y,z_1) and (y,z_2) are in G. Now, there are sequences $\{s_n\}, \{t_n\} \subset B(x,\delta) \cap S$ such that $s_n, t_n \to y$, $\tau(s_n) \to z_1$, and $\tau(t_n) \to z_2$. Then, we have that $|\tau(s_n) - \tau(x)) < \varepsilon$, and for any $\varepsilon_2 > 0$ we have that there exists some N such that $n > N \implies |z_1 - \tau(s_n)| < \varepsilon_2$. Then, $|z_1 - \tau(x)| < \varepsilon + \varepsilon_2$ so that $|z_1 - \tau(x)| \leq \varepsilon$. Similar logic shows that $|z_2 - \tau(x)| \leq \varepsilon$.

If $y \in Q$, then $(y,z) \in G$. We have that there must be some sequence $\tau(s_n) \to z$ for $\{s_n\} \subset S \cap B(x,\delta)$ so that $|z - \tau(x)| \leq |z - \tau(s_n)| + |\tau(s_n) - \tau(x)| \leq \varepsilon_2 + \varepsilon \to \varepsilon$. Therefore, $|z - \tau(x)| \leq \varepsilon$ as well.

Finally, regardless of the choice of value of $g_\tau(y)$ outside of Q, we have that g_τ is continuous at x.

Now, with S_2 the set of all x such that $g_\tau(x)$ is continuous at x, we have that S_2 is open and that $S \subseteq S_2$ so that $\mu(S_2) = \mu(\Omega)$. Therefore, S_2 is the desired set, and because $g_\tau(x) = \tau(x)$ for $x \in S$ we have that $g_\tau \equiv \tau$.

Definition 5 *For some $\tau \in \sigma(\Omega)$, we define $S(\tau)$ to be S_2 as above and g_τ to be g_τ as above.*

Corollary. We can choose the values of g_τ on $\Omega \setminus S(\tau)$ so that $g_\tau(\Omega) = \Omega$.
Corollary. $S(\tau) \subseteq I(\tau)$ by construction, so we can redefine $I(\tau) = S(\tau)$.
Corollary. $S(\tau^{-1}) = g_\tau(S(\tau))$

Proposition 5 *$S(\tau)$ can be partitioned into disjoint sets $\{A_\alpha\}_{\alpha \in I}$ such that $g_\tau|_{A_\alpha} = U_\alpha x + v_\alpha$ for U_α a unitary linear map and v_α a constant vector.*

Proof. We have that g_τ is continuous on an open set $S(\tau)$ with $\mu(S) = \mu(\Omega)$ so that $\partial S(\tau) = \Omega \setminus S(\tau)$.
Now, for any $x \in S(\tau)$, there is some $B_1 = B(x,\varepsilon_1) \subset S$.

For any $y \in B_1$, let $\ell_1 = \{g_\tau(x) + a(g_\tau(y) - g_\tau(x)) \mid 0 \le a \le 1\}$. We have that $|g_\tau(x) - g_\tau(y)| = H^1(\ell_1) = H^1(g_\tau^{-1}(\ell_1))$. As $g_\tau^{-1}(\ell_1)$ is some continuous path between x and y, we have that $H^1(g_\tau^{-1}(\ell)) \ge |x-y|$. Therefore, $|g_\tau(x) - g_\tau(y)| \ge |x-y|$. Finally, we have that $|x - y| = |g_\tau^{-1}(g_\tau(x)) - g_\tau^{-1}(g_\tau(y))| \ge |g_\tau(x) - g_\tau(y)|$ so that $|x - y| = |g_\tau(x) - g_\tau(y)|$.

It is a known result that if $|g_\tau(x) - g_\tau(y)| = |x - y|$ for $g - \tau$ continuous that $g_\tau(x) = Ux + v$ for U a unitary map.

Choose some partitioning of $S(\tau)$ into balls such that $S = \cup_{\alpha \in I} B(x_\alpha, r_\alpha)$. Then, take $A_\alpha = B(x_\alpha, r_\alpha) \setminus (\cup_{i < \alpha} B(x_i, r_i))$ so that they are disjoint, $S = \cup_{\alpha \in I} A_\alpha$, and on each A_α we have that $g_\tau(x) = U_\alpha x + v_\alpha$.

Note. The partition I can be assumed to be countable because each of the A_α have been constructed to be pairwise disjoint.

Definition 6 *We take $P_{S(\tau)}$ to be the set of connected components of $S(\tau)$. Note that on each $C \in P_{S(\tau)}$ we must then have that τ is locally equivalent to a unitary map plus a shift.*

Proposition 6 *For any a.e. invertible L^p function $f : \Omega \to \Omega$, we have that $f \in \sigma(\Omega)$ if and only if for almost all $x \in \Omega$ there exists some $\varepsilon > 0$ such that there are $U \in U(n)$ and $v \in \mathbb{R}^n$ such that $f(y) \equiv Uy + vf$ for $y \in B(x, \varepsilon)$.*

Proof. We have proven the $\implies$ direction already. Now, conversely, assume that for all $x \in S$ there exists some $\varepsilon > 0$ such that $f(y) \equiv Uy + vf$ or $y \in B(x, \varepsilon)$, where $\mu(S) = \mu(\Omega)$.
Now, for all $y \in f(S)$ there exists some $\varepsilon > 0$ such that $f^{-1}(z) \equiv U^T z - U^T v$ for all $z \in B(y, \varepsilon)$. Then, $(f^{-1})'(y)$ is unitary.

$$\mu(f(A)) = \int_\Omega 1_A \circ f^{-1} d\mu$$
$$= \int_S 1_A \circ f^{-1} d\mu$$
$$= \int_{f(S)} 1_A |\det(f^{-1})'| d\mu$$
$$= \int_\Omega 1_A d\mu$$
$$= \mu(A)$$

And therefore, $f \in \sigma(\Omega)$.

Proposition 7 *For $\tau, \gamma \in \sigma(\Omega)$, we have that $S(\tau \circ \gamma) \subseteq \Omega \setminus (S(\gamma) \Delta g_\gamma^{-1}(S(\tau)))$.*

Proof. Assume $x \in S(\tau \circ \gamma)$. If g_γ is continuous at x we have that g_τ is continuous at $g_\gamma(x)$ so $x \in S(\gamma) \cap g_\gamma^{-1}(S(\tau))$.
Otherwise, if $x \notin S(\gamma)$ then if $g_\gamma(x) \in S(\tau)$ we have that $g_\tau(g_\gamma(x))$ cannot be continuous, a contradiction.
Therefore, $x \in (\Omega \setminus S(\gamma)) \cap (\Omega \setminus g_\gamma^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \cup g_\gamma^{-1}(S(\tau)))$.
Finally, $S(\tau \circ \gamma) \subseteq (\Omega \setminus (S(\gamma) \cup g_\gamma^{-1}(S(\tau)))) \cup (S(\gamma) \cap g_\gamma^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \Delta g_\gamma^{-1}(S(\tau)))$.
Corollary. $S(\gamma) \cap g_\gamma^{-1}(S(\tau)) \subseteq S(\tau \circ \gamma)$

3 Estimates on Thickened Sets of Discontinuity

Definition 7 *We define $A(\tau, \varepsilon) = \{x \in \Omega \mid d(x, \partial(\Omega \setminus S(\tau))) < \varepsilon\}$.*

Note that $\partial(\Omega \setminus S(\tau)) = \Omega \setminus S(\tau)$, and are interchanged at times.

Proposition 8 *For any open set X with $X \subseteq S(\tau)$ and $\mu(X) = \mu(\Omega)$, we have that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\})$*

Proof. We have that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\}) = \mu(\{x \in g_\tau(X) \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\})$.
Now, assume for any $x \in g_\tau(X)$ that $y \in \Omega \setminus g_\tau(X)$ minimizes $|x - y|$. We see that $x \in C$ where C is taken to be the largest (by inclusion) connected component of $g_\tau(X)$ containing x. We can assume it is convex because of the construction given in Proposition 5. Then, $y \in \partial C$ by the minimality of $|x - y|$.
In addition, we have that $g_\tau(X) \subseteq g_\tau(S(\tau)) = S(\tau^{-1})$ so that $g_\tau^{-1}(x) = Ux + v$ on C.
Now, with $\ell = \{x + a(y - x) \mid a \in [0, 1)\}$, we have that $\ell \subset C$ by the convexity of ∂C (because of the convexity of C).
Thus, $|x - y| = H^1(\ell) = H^1(g_\tau^{-1}(\ell)) \geq |g_\tau^{-1}(x) - y_2|$ for some $y_2 \in \partial g_\tau^{-1}(C)$. Thus, $|x - y| \geq d(g_\tau^{-1}(x), g_\tau^{-1}(C)) \geq d(g_\tau^{-1}(x), \Omega \setminus g_\tau^{-1}(g_\tau(X))) = d(g_\tau^{-1}(x), \Omega \setminus X)$.
Finally, we have the following.

$$\implies \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\}) \leq \mu(\{x \in g_\tau(X) \mid d(g_\tau^{-1}(x), \Omega \setminus X) < \varepsilon\})$$
$$= \mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\})$$

Now, as $X \subseteq S(\tau)$ we see that $g_\tau(X) \subseteq S(\tau^{-1})$ so that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau^{-1}(g_\tau(X))) < \varepsilon\}) \leq \mu(x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\})$.
Therefore, $\mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\})$

Proposition 9 $\mu(A(\tau \circ \gamma^{-1}, \varepsilon)) \leq \mu(A(\tau, \varepsilon)) + \mu(A(\gamma, \varepsilon))$ for all $\varepsilon > 0$ and $\tau, \gamma \in \sigma(\Omega)$.

Proof.

$$\mu(A(\tau \circ \gamma^{-1}, \varepsilon)) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\tau \circ \gamma^{-1})) < \varepsilon\})$$
$$\leq \mu(\{x \in \Omega \mid d(x, \Omega \setminus (S(\gamma^{-1}) \cap g_\gamma(S(\tau)))) < \varepsilon\})$$
$$\leq \mu(\{x \in \Omega \mid d(x, \Omega \setminus (S(\gamma^{-1}) \cap g_\gamma(S(\tau) \cap S(\gamma)))) < \varepsilon\})$$
$$= \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\gamma(S(\tau) \cap S(\gamma))) < \varepsilon\})$$

By **CITE PROP** we have that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\gamma(S(\tau) \cap S(\gamma))) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, (S(\tau) \cap S(\gamma))) < \varepsilon\})$ because $S(\tau) \cap S(\gamma) \subseteq S(\gamma)$ and has full measure.

$$\implies \mu(A(\tau \circ \gamma^{-1}, \varepsilon)) \leq \mu(\{x \in \Omega \mid d(x, \Omega \setminus (S(\tau) \cap S(\gamma))) < \varepsilon\})$$
$$\leq \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\tau)) < \varepsilon\}) + \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\gamma)) < \varepsilon\})$$
$$= \mu(A(\tau, \varepsilon)) + \mu(A(\gamma, \varepsilon))$$

Proposition 10 We have that $\mu(A(\gamma, \varepsilon)) = \mu(A(\gamma^{-1}, \varepsilon))$ for all $\gamma \in \sigma(\Omega)$ and $\varepsilon > 0$.

Proof. We have the following estimate.

$$\mu(A(\gamma^{-1}, \varepsilon)) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\gamma^{-1})) < \varepsilon\})$$
$$= \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\gamma(S(\gamma))) < \varepsilon\})$$

We see that $S(\gamma) \subseteq S(\gamma)$ so by **CITE PROP** we have that $\mu(A(\gamma^{-1}, \varepsilon)) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\gamma) < \varepsilon\}) = \mu(A(\gamma, \varepsilon))$.

4   Topological Properties

Definition 8 We define for measurable $S \subseteq \Omega$ the cylindrical set $V_S = \{\tau \in \sigma(U) \mid S \subseteq S(\tau)\}$.

Proposition 11 If $X \subseteq \sigma(\Omega)$ is open under $||\cdot||_{L^p(\Omega)}$, then $X \circ \tau$ is as well for all $\tau \in \sigma(\Omega)$.

Proof. We have that for any $\gamma \in X \circ \tau$ that $\gamma \circ \tau^{-1} \in X$ and thus there is some $B = B_{L^p(\Omega)}(\gamma \circ \tau^{-1}, \varepsilon) \subset X$. For $\lambda \in B$ we have that $||\lambda \circ \tau - \gamma||_{L^p(\Omega, \mathbb{R}^n)} = ||\lambda - \gamma \circ \tau^{-1}||_{L^p(\Omega)}$, so that $B_{L^p(\Omega)}(\gamma \circ \tau^{-1}, \varepsilon) \circ \tau = B_{L^p(\Omega)}(\gamma, \varepsilon) \subset X \circ \tau$. And thus there is an open neighborhood of $\gamma \in X \circ \tau$, so that $X \circ \tau$ is open as well.

Proposition 12 *If $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$, then $||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \to 0$ as well for all $\tau \in \sigma(\Omega)$.*

Proof. Note that $\mu(E_\delta) = \mu(\{x \in \Omega \mid |\gamma_n(x) - \gamma(x)| > \delta\}) < \delta^{-p}||\gamma_n(x) - \gamma(x)||_{L^p(\Omega)}$ by Chebyshev's inequality.

$$||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)}^p = \int_\Omega |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu$$

$$= \int_{E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu + \int_{\Omega \setminus E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu$$

$$\leq \delta^{-p} ||\gamma_n - \gamma||_{L^p} \operatorname{diam}(\Omega)^p + I$$

$$I = \int_{(\Omega \setminus E_\delta) \setminus g_\gamma^{-1}(A(\tau, 2\delta))} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu + \int_{(\Omega \setminus E_\delta) \cap g_\gamma^{-1}(A(\tau, 2\delta))} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu$$

Given $x \in (\Omega \setminus E_\delta) \setminus g_\gamma^{-1}(A(\tau, 2\delta))$, we see that $|g_{\gamma_n}(x) - g_\gamma(x)| < \delta$ a.e, and $g_\gamma(x) \in \Omega \setminus A(\tau, 2\delta)$ so that $d(g_\gamma(x), \Omega \setminus S(\tau)) \geq 2\delta$ and thus $d(g_{\gamma_n}(x), \Omega \setminus S(\tau)) \geq \delta$.
This means that $g_{\gamma_n}(x)$ and $g_\gamma(x)$ are in the same connected component of $S(\tau)$ so that $|g_\tau \circ g_{\gamma_n}(x) - g_\tau \circ g_\gamma(x)| = |g_{\gamma_n} - g_\gamma(x)| < \delta$.

$$\implies I \leq \mu((\Omega \setminus E_\delta) \setminus g_\gamma^{-1}(A(\tau, 2\delta)))\delta^p + \mu((\Omega \setminus E_\delta) \cap g_\gamma^{-1}(A(\tau, 2\delta))) \operatorname{diam}(\Omega)^p$$

$$\leq \mu(\Omega)\delta^p + \mu(A(\tau, 2\delta)) \operatorname{diam}(\Omega)^p$$

$$\implies ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \leq \delta^{-p}||\gamma_n - \gamma||_{L^p(\Omega)} \operatorname{diam}(\Omega)^p + \mu(\Omega)\delta^p + \mu(A(\tau, 2\delta)) \operatorname{diam}(\Omega)^p$$

Now, choose $\delta = ||\gamma_n - \gamma||_{L^p(\Omega)}^{1/p+1}$.

$$\leq ||\gamma_n - \gamma||_{L^p(\Omega)}^{1/(p+1)} \operatorname{diam}(\Omega)^p + \mu(\Omega)||\gamma_n - \gamma||_{L^p(\Omega)}^{1/p+1} + \mu(A(\tau, 2||\gamma_n - \gamma||^{1/(p+1)})) \operatorname{diam}(\Omega)^p$$

This bound holds for all n. Now, we see that each term approaches 0 as $||\gamma_n - \gamma||$ does, so that $||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \to 0$ as $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$ for $1 \leq p < \infty$.

Proposition 13 *If $X \subseteq \sigma(\Omega)$ is open under $||\cdot||_{L^p(\Omega)}$, then $\tau \circ X$ is as well for all $\tau \in \sigma(\Omega)$.*

Proof. Take any set X open in $||\cdot||_{L^p(\Omega)}$.
Now, take any sequence $\{\gamma_n\} \subseteq \sigma(\Omega) \setminus (\tau \circ X)$ convergent to some limit γ. We see that $\tau^{-1} \circ \gamma_n$ is then a sequence in $\sigma(\Omega) \setminus X$ convergent to $\tau^{-1} \circ \gamma$ by the previous proposition. Then, $\tau^{-1} \circ \gamma \in \Omega \setminus X$ as $\sigma(\Omega) \setminus X$ is closed. Finally, $\gamma \in \sigma(\Omega) \setminus (\tau \circ X)$, so that $\sigma(\Omega) \setminus (\tau \circ X)$ is closed and therefore $\tau \circ X$ is open.

Proposition 14 *$(\sigma(\Omega), ||\cdot||_{L^p(\Omega)})$ is a topological group.*

Proof. Assume that $||\tau_n - \tau||_{L^p} \to 0$. Now, we see that $||\tau^{-1} - \tau_n^{-1}||_{L^p(\Omega} = ||\tau^{-1} \circ \tau_n - \operatorname{Id}||_{L^p(\Omega)} = ||\tau^{-1} \circ \tau_n - \tau^{-1} \circ \tau||_{L^p(\Omega)} \to 0$ as well, so that $\tau \mapsto \tau^{-1}$ is continuous.
Now, assume that $||\tau_n - \tau||_{L^p} \to 0$ and $||\gamma_n - \gamma||_{L^p} \to 0$. We see that $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \leq ||\tau_n \circ \gamma_n - \tau \circ \gamma_n||_{L^p(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} = ||\tau_n - \tau||_{L^p(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)}$. As both sequences approach 0, we see that $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \to 0$ as well, so that $(\tau, \gamma) \mapsto \tau \circ \gamma$ is continuous too.

Therefore, $(\sigma(\Omega), ||\cdot||_{L^p(\Omega)})$ is a topological group.

Proposition 15 *For $X \subseteq \sigma(\Omega)$, if $\mu(A(\tau, \varepsilon)) \to 0$ uniformly on X as $\varepsilon \to 0$, then X is relatively compact under L^p for $1 \leq p < \infty$.*

Proof. Pick any arbitrary p and choose $q < p^* = np/(n-p)$.
Let n^ε denote the usual mollifier taken by dilating n, a positive smooth function such that $\int_{\mathbb{R}^n} n d\mu = 1$ and 0 outside of $B(0, 1)$. Take any sequence $\{\tau_n\} \subset X$ and let $\tau_n^\varepsilon = n^\varepsilon * \tau_n$.

First, we bound $||\tau_n^\varepsilon - \tau_n||_{L^q(\Omega)}$.

$$||\tau_n^\varepsilon - \tau_n||_{L^1(\Omega)} \leq \int_\Omega \int_{B(0,1)} n(y)|\tau_n(x - \varepsilon y) - \tau_n(x)|d\mu_y d\mu$$

$$\leq \int_{A(\tau,\varepsilon)} \int_{B(0,1)} n(y)|\tau_n(x - \varepsilon y) - \tau_n(x)|d\mu_y d\mu + \int_{B(0,1)} \int_{\Omega \setminus A(\tau,\varepsilon)} n(y)\varepsilon d\mu d\mu_y$$

$$\leq \int_{A(\tau,\varepsilon)} \int_{B(0,1)} n(y)|\tau_n(x - \varepsilon y) - \tau_n(x)|d\mu_y d\mu + \varepsilon\mu(\Omega)$$

$$\leq \mu(A(\tau,\varepsilon))\mu(B(0,1))\operatorname{diam}(\Omega) + \varepsilon\mu(\Omega)$$

As ε approaches 0^+, this upper bound approaches 0 by uniformly in n by hypothesis. Then, the rest of this proof follows exactly the same as Evan's proof of the Rellich-Kondrachov Compactness Theorem in 'Partial Differential Equations' **todo: cite**.

Therefore, there must be some subsequence τ_{m_n} convergent in the $L^q(\Omega)$ norm so that X is relatively compact in $L^q(\Omega)$.

Now, for any arbitrary p, we can choose p_1 such that $p < p_1^*$, and thus X is relatively compact in $L^p(\Omega)$.
Corollary. $V_S = \overline{V_S}$ is compact if $\mu(\partial(\Omega \setminus S)) = 0$.

Definition 9 *We denote* $||f||_{L_w^p(\Omega)} = \sup_{E \in \Sigma_\Omega} \mu(E)^{-1/p^*} \int_E |f|d\mu$ *to be the weak L^p norm.*

Proposition 16 *For a sequence $\{\gamma_n\} \subset \sigma(\Omega)$ and $\tau, \gamma \in \sigma(\Omega)$ such that $||\gamma_n - \gamma||_{L_w^p(\Omega)} \to 0$, we have that $||\tau \circ \gamma_n - \tau \circ \gamma||_{L_w^p(\Omega)} \to 0$ as well.*

Proof. Take any arbitrary measurable $A \subseteq \Omega$ with indicator function χ_A. We denote $E_\delta = \{x \in \Omega \mid |\gamma_n - \gamma| > \delta\}$ so that $\mu(E_\delta) < \delta^{-p}||\gamma_n - \gamma||_{L_w^p(\Omega)}^p$.

$$I_n = \int_A |\tau \circ \gamma_n - \tau \circ \gamma|d\mu = \int_\Omega |\tau \circ \gamma_n - \tau \circ \gamma|\chi_A d\mu$$

$$\leq \int_{\Omega \setminus E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma|\chi_A d\mu + \int_{E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma|\chi_A d\mu$$

$$\leq \int_{(\Omega \setminus E_\delta) \setminus A(\tau,2\delta)} |\tau \circ \gamma_n - \tau \circ \gamma|\chi_A d\mu + \int_{(\Omega \setminus E_\delta) \cap A(\tau,2\delta)} |\tau \circ \gamma_n - \tau \circ \gamma|\chi_A d\mu + \mu(E_\delta)\operatorname{diam}(\Omega)$$

$$\leq |\delta|^p\mu(\Omega) + \mu(A(\tau,2\delta))\operatorname{diam}(\Omega) + \delta^{-p}||\gamma_n - \gamma||_{L_w^p(\Omega)}^p \operatorname{diam}(\Omega)$$

We may choose $\delta_n = ||\gamma_n - \gamma||_{L_w^p(\Omega)}^{1/2}$.

$$\implies I_n \leq ||\gamma_n - \gamma||_{L_w^p(\Omega)}^{1/2}\mu(\Omega) + \mu(A(\tau, 2||\gamma_n - \gamma||_{L_w^p(\Omega)}^{1/2}))\operatorname{diam}(\Omega) + ||\gamma_n - \gamma||_{L_w^p(\Omega)}^{p/2}\operatorname{diam}(\Omega)$$

Therefore, since each term goes to 0 as $||\gamma_n - \gamma||_{L_w^p(\Omega)} \to 0$ we have that $\lim_{n \to \infty} I_n \to 0$. Now, we may substitute $A = E_r$ for any arbitrary $0 < r$. We see the following bounds.

$$M = \sup_{r > 0} r\mu(\{x \in \Omega \mid |\tau \circ \gamma_n - \tau \circ \gamma| > r\})^{1/p} = \sup_{0 < r \leq \operatorname{diam}(\Omega)} r\mu(\{x \in \Omega \mid |\tau \circ \gamma_n - \tau \circ \gamma| > r\})^{1/p}$$

$$\leq \sup_{0 < r \leq \operatorname{diam}(\Omega)} \left(\int_{E_r} |\tau \circ \gamma_n - \tau \circ \gamma|d\mu\right)^{1/p}$$

$$\leq I^{1/p}$$

Now, we have that $||\tau \circ \gamma_n - \tau \circ \gamma||_{L_w^p(\Omega)} \leq p^*M \leq p^*I_n^{1/p}$ by a well-known result **cite**. Therefore, $\lim_{n \to \infty} ||\tau \circ \gamma_n - \tau \circ \gamma||_{L_w^p(\Omega)} \leq p^* \lim_{n \to \infty} I_n^{1/p} = 0$.

Proposition 17 *For a sequence $\{\gamma_n\} \subset \sigma(\Omega)$ and $\tau, \gamma \in \sigma(\Omega)$ such that $||\gamma_n - \gamma||_{L_w^p(\Omega)} \to 0$, we have that $||\gamma_n \circ \tau - \gamma \circ \tau||_{L_w^p(\Omega)} \to 0$ as well.*

Proof. We see the following simplification for any function $f \in L_w^p(\Omega)$.

$$
\begin{aligned}
||f||_{L_w^p(\Omega)} &= \sup_{A \in \Sigma_\Omega} \mu(A)^{-1/p^*} \int_A |f| d\mu \\
&= \sup_{A \in \Sigma_\Omega} \mu(\tau^{-1}(A))^{-1/p^*} \int_{\tau^{-1}(A)} |f \circ \tau| d\mu \\
&= \sup_{A \in \Sigma_\Omega} \mu(A)^{-1/p^*} \int_A |f \circ \tau| d\mu \\
&= ||f \circ \tau||_{L_w^p(\Omega)}
\end{aligned}
$$

So, with $f = \gamma_n - \gamma$ we see that $||\gamma_n \circ \tau - \gamma \circ \tau||_{L_w^p(\Omega)} = ||\gamma_n - \gamma||_{L_w^p(\Omega)} \to 0$.

Proposition 18 $(\sigma(\Omega), || \cdot ||_{L_w^p(\Omega)})$ *is a topological group.*

Proof. We first prove that $\tau_n \to \tau$ implies that $\tau_n^{-1} \to \tau^{-1}$.
We see that $||\tau_n^{-1} - \tau^{-1}||_{L_w^p(\Omega)} = ||\operatorname{Id} - \tau^{-1} \circ \tau_n||_{L_w^p(\Omega)} = ||\tau^{-1} \circ \tau - \tau^{-1} \circ \tau_n||_{L_w^p(\Omega)}$ by the previous proposition.
Since we have that $||\tau_n - \tau||_{L_w^p(\Omega)} \to 0$, we see that $||\tau^{-1} \circ \tau - \tau^{-1} \circ \tau_n||_{L_w^p(\Omega)} \to 0$ as well by **cite prop**.
Next, assume that $||\tau_n - \tau||_{L_w^p(\Omega)} \to 0$ and $||\gamma_n - \gamma||_{L_w^p(\Omega)} \to 0$. Then, $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L_w^p(\Omega)} \leq ||\tau_n \circ \gamma_n - \tau \circ \gamma_n||_{L_w^p(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L_w^p(\Omega)}$. We see that this is then equal to $||\tau_n - \tau||_{L_w^p(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L_w^p(\Omega)}$. As both components go to 0 as $n \to \infty$, we see that $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L_w^p(\Omega)} \to 0$ as well so that $(\sigma(\Omega), || \cdot ||_{L_w^p(\Omega)})$ is a topological group.

Now, since for all $\tau \in \sigma(\Omega)$ we see that $||\tau||_{L_w^p(\Omega)} = ||\tau \circ \tau^{-1}||_{L_w^p(\Omega)} = ||\operatorname{Id}||_{L_w^p(\Omega)} < \infty$, we have that $\sigma(\Omega)$ has a compact closure in the weak L_w^p topology by the Banach-Alaoglu theorem.
Therefore, $\sigma(\Omega)$ is a compact topological group under the subspace topology inherited from the weak L_w^p topology. Taking quotients by equivalence classes, it is then compact and hausdorff. Therefore, there exists a left-invariant probability measure on $\sigma(\Omega)$.

5   Algebraic Properties

Proposition 19 *For any linear map $M : L^2(\Omega, \mathbb{R}) \to L^2(\Omega, \mathbb{R})$ which preserve $|| \cdot ||_{L^1(\Omega)}, || \cdot ||_{L^\infty(\Omega)}, \langle \cdot, \cdot \rangle$, and such that $f \geq 0 \implies Mf \geq 0$, we have that $Mf = f \circ \gamma^{-1}$ for $\gamma \in \sigma(\Omega)$.*

Proof. Take such an M. It preserves $|| \cdot ||_{L^2(\Omega)}$ as well since it preserves inner products.
Now, take any measurable set $A \in \Sigma_\Omega$ with nonzero measure and let χ_A be its indicator function. We see that $0 \leq M\chi_A \leq 1$ almost everywhere since $||M\chi_A||_{L^\infty} = 1$.
Assume for a contradiction that $K = \{x \in \Omega \mid 0 < M\chi_A < 1\}$ has nonzero measure. Then, we have the following.

$$
\begin{aligned}
\mu(A) &= \langle M\chi_A, M\chi_A \rangle \\
&= \int_{\Omega \setminus K} (M\chi_A)^2 d\mu + \int_K (M\chi_A)^2 d\mu \\
&= \int_{\Omega \setminus K} M\chi_A d\mu + \int_K (M\chi_A)^2 d\mu \\
&< \int_{\Omega \setminus K} M\chi_A d\mu + \int_K M\chi_A d\mu \\
&= \int_\Omega |M\chi_A| d\mu \\
&= ||M\chi_A||_{L^1(\Omega)} = \mu(A)
\end{aligned}
$$

7

Then, we have a contradiction, so that $\mu(K) = 0$ and thus $M\chi_A \in \{0, 1\}$. Now, we may define $\tau : \Sigma_\Omega \to \Sigma_\Omega$ by $\tau(A) = \{x \in \Omega \mid M\chi_A = 1\}$. Note that we now have that $\mu(A) = ||M\chi_A||_{L^1(\Omega)} = ||\chi_{\tau(A)}||_{L^1(\Omega)} = \mu(\tau(A))$.

For disjoint A, B we have by the linearity of M that $M\chi_{A \cup B} \equiv M\chi_A + M\chi_B \equiv \chi_{\tau(A)} + \chi_{\tau(B)}$ so that $\tau(A), \tau(B)$ are disjoint as well (modulo a null set) and thus $\tau(A \cup B) = \tau(A) \cup \tau(B)$ (again modulo a null set).

The same logic shows that $\tau(A \setminus B) = \tau(A) \setminus \tau(B)$.

Thus, by Theorem 9.5.1 in **CITE SOURCE 3** we have that there exists some $\gamma : \Omega \to \Omega$ such that $\mu(\tau(A) \Delta \gamma(A)) = 0$ for all $A \in \Sigma_\Omega$.

Then, $|M\chi_A| \equiv \chi_{\gamma(A)} \equiv \chi_A \circ \gamma^{-1}$, so that $|Mf| \equiv f \circ \gamma^{-1}$ for simple functions $f \in L^\infty(\Omega)$ as well, and therefore $|Mf| \equiv f \circ \gamma^{-1}$ for all $f \in L^\infty(\Omega)$ by taking limits of simple functions.

Definition 10 *We denote the set of linear maps M as in the hypotheses of the previous proposition to be $L(\Omega)$, the linear representation of Ω. We have that $L(\Omega)$ is a subgroup of the group $G(\Omega) = U(L^2(\Omega, \mathbb{R})) \cap U(L^1(\Omega, \mathbb{R})) \cap U(L^\infty(\Omega, \mathbb{R}))$ (with the group action being multiplication/composition).*

Proposition 20 *For all $M \in G(\Omega)$, we have that $M = \lambda N$ for $\lambda \in L^\infty(\Omega)$ with $\lambda = \pm 1$ a.e., and $N \in L(\Omega)$.*

Proof. Take any $M \in G(\Omega)$. Take any measurable $A \in \Sigma_\Omega$, so we see that $\mu(A) = ||M\chi_A||_{L^1(\Omega)}$.

Then, consider $M\chi_\Omega$. If $|M\chi_\Omega| < 1$ on a set T of nonzero measure then since $||M\chi_\Omega||_{L^\infty(\Omega)} = 1$ we have that $\mu(\Omega) = \int_\Omega |M\chi_\Omega| d\mu = \int_T |M\chi_\Omega| d\mu + \int_{\Omega \setminus T} |M\chi_\Omega| d\mu < \mu(T) + \mu(\Omega \setminus T) = \mu(\Omega)$, a contradiction. Then, $|M\chi_\Omega| = 1$ almost everywhere so that $(M\chi_\Omega) M \in G(\Omega)$ as well.

Define $K = \{x \in \Omega \mid (M\chi_\Omega)(M\chi_A) < 0\}$. If $x \in K$ then since $|(M\chi_\Omega)(M\chi_A)| = 0$ or 1 we have that $(M\chi_\Omega)(M\chi_A) = -1$. Then, $\langle (M\chi_\Omega)(M\chi_A), \chi_K \rangle = -\mu(K)$. Next, we have the following.

$$\langle M\chi_\Omega M\chi_A, 1 \rangle = \langle M\chi_\Omega M\chi_A, \chi_K \rangle + \langle M\chi_\Omega M\chi_A, \chi_{\Omega \setminus K} \rangle$$
$$\leq -\mu(K) + ||M\chi_\Omega M\chi_A||_{L^1(\Omega)} ||\chi_{\Omega \setminus K}||_{L^\infty(\Omega)}$$
$$= \mu(A) - \mu(K)$$

However, we also have that $\langle M\chi_\Omega M\chi_A, 1 \rangle = \langle M\chi_\Omega, M\chi_A \rangle = \langle \chi_\Omega, \chi_A \rangle = \mu(A)$. Therefore, since $\mu(A) \leq \mu(A) - \mu(K)$, we have that $\mu(K) = 0$, and that $M\chi_\Omega M\chi_A \geq 0$ a.e.

Then, $M\chi_\Omega Mf \geq 0$ for all simple functions $f \geq 0$, and thus by taking limits we see that $M\chi_\Omega Mf \geq 0$ for all $f \in L^1(\Omega)$ which are ≥ 0.

Finally, we have shown that $(M\chi_\Omega) M \in L(\Omega)$ and $M\chi_\Omega = \pm 1$ almost everywhere.

Corollary. Any $N \in G(\Omega)$ is thus also in $U(L^p(\Omega, \mathbb{R}))$ for all other $1 \leq p < \infty$, since $Nf = (N\chi_\Omega) f \circ \tau^{-1}$ for some $\tau \in \sigma(\Omega)$ so that $||Nf||_{L^p(\Omega)}^p = \int_\Omega |(N\chi_\Omega) f \circ \tau^{-1}|^p d\mu = \int_\Omega |f \circ \tau^{-1}|^p d\mu = ||f||_{L^p(\Omega)}^p$.

Therefore, we have that $U(L^\infty(\Omega, \mathbb{R})) \cap \cap_p U(L^p(\Omega, \mathbb{R})) = U(L^1(\Omega, \mathbb{R})) \cap U(L^2(\Omega, \mathbb{R})) \cap U(L^\infty(\Omega, \mathbb{R}))$.

Proposition 21 *We construct a nontrivial homomorphism $\sigma(\Omega)$ from to S^1*

Proof. Consider $\sigma(\Omega)$ as a subgroup of $U(L^2(\Omega))$. Take the projection onto the abelianization $\pi_1 : U(L^2(\Omega)) \to U(L^2(\Omega))/[U(L^2(\Omega)), U(L^2(\Omega))] = U(L^2(\Omega))^{ab}$. Because $PU(L^2(\Omega)) \cong U(L^2(\Omega))/Z$ by **Cite result, nlab** with Z the centralizer of $U(L^2(\Omega))$, we have that $\pi_1(PU(L^2(\Omega))) \cong U(L^2(\Omega))^{ab}/\pi_1(Z)$.

We see that $PU(L^2(\Omega))$ is simple and non-abelian by **cite** but $U(L^2(\Omega))^{ab}/\pi_1(Z)$ is abelian, so that it must be trivial.

Then, since the quotient is trivial we see that $U(L^2(\Omega))^{ab} \cong \pi_1(Z) \cong Z$. It is known that $Z \cong S^1$ so that therefore, we have a nontrivial homomorphism $U(L^2(\Omega)) \to S^1$

Definition 11 *Let $\zeta_1 : U(L^2(\Omega)) \to S^1$ be the previously constructed homomorphism, $\zeta_2 : \sigma(\Omega) \to L(\Omega)$ be the canonical homomorphism, and ζ_0 be the unique isomorphism from S^1 to the complex unit circle. Then, we define $\mathrm{sgn} : \sigma(\Omega) \to S^1$ by $\mathrm{sgn} = \zeta_0 \circ \zeta_1 \circ \zeta_2$.*

Note that, if $L^2(\Omega)$ is replaced by the Hilbert space $\mathbb{R}^n$, the definitions of sign coincide with that of finite permutations in S_n.

For some $p \in S_n$, we have that the analogous ζ_2 would map p to $\zeta_2(p) = P$, the permutation matrix corresponding to p, so that $\zeta_1(P) = \det P = \operatorname{sgn}(p)$.

The sgn function may be constructed similarly on any unitary group $U(H)$ for a Hilbert space H, which suggests that many of these results may be generalizable to such a setting.

TODO: explicitly construct $\mu_{\mathbb{D}}$