1 Lemma 1

1. Given some $n \in \mathbb{N}$, $j = 0, 1, ..., n$, and $x \in \mathbb{C}$

$$C_j = \frac{1}{(0x - jx)(1x - jx)...((j-1)x - jx)((j+1)x - jx)...(nx - j)}$$

$$= \frac{(-1)^j}{n!x^n}\binom{n}{j}$$

1.1 Proof

Consider the terms $(0x - jx)(1x - jx)...((j-1)x - jx)$. By factoring out $(-1)^j x^j$ and reversing the order of the product, this is equal to $(-1)^j x^j (1)(2)...(j-1)(j) = (-1)^j x^j j!$. The same argument can be made for the remaining terms. This then shows the following.

$$C_j = \frac{(-1)^j}{j!(n-j)!x^n} = \frac{(-1)^j}{n!x^n}\binom{n}{j}$$

2 Proof

Given an integer n and unique $x_0, ..., x_n \in \mathbb{R}^+$, Let $P(x) = (x - x_0)(x - x_1)...(x - x_n)$ and now with partial fraction decomposition, the following is true.

$$\frac{1}{P(x)} = \sum_{j=0}^{n} \frac{C_j}{x + x_j} \tag{1}$$

where $C_j = \frac{1}{(x_0 - x_j)(x_1 - x_j)...(x_{j-1} - x_j)(x_{j+1} - x_j)...(x_n - x_j)}$ for $j = 0, 1, ..., n$. Now, using this:

$$\sum_{m=1}^{\infty} \frac{1}{P(m)} = \sum_{m=1}^{\infty}$$

$$\sum_{j=0}^{n} \frac{C_j}{m + x_j} (2)$$

$$= \sum_{j=0}^{n} \sum_{m=1}^{\infty} \left(\frac{C_j}{m + x_j} + \frac{C_j}{m} - \frac{C_j}{m} \right)$$

$$= \left(\sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \frac{1}{m}\right) - \sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+x_j}\right) *$$

$$= \left(\sum_{j=0}^{n} C_j\right)\left(\sum_{m=1}^{\infty} \frac{1}{m}\right) - \sum_{j=0}^{n} C_j H_{x_j}$$

where H_x is the xth harmonic number. The separation of sums in the starred step is the step we're unsure about. By taking equation 1 and multiplying both sides by P(x), then comparing coefficients of x^n, it can be seen that $\sum_{j=0}^{n} C_j = 0$. This step is analogous to algebraically cancelling it out, which can be seen by doing this process for a specific n. I'm not sure how to prove that this is valid for any n though. Assuming it's valid though, the following steps give a result.

$$= -\sum_{j=0}^{n} C_j H_{x_j} = -\sum_{j=0}^{n} C_j \int_0^1 \frac{1-t^{x_j}}{1-t} dt$$

$$= -\int_0^1 \frac{\left(\sum_{j=0}^{n} C_j\right) - \sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$$

So, we have that $\sum_{m=0}^{\infty} \frac{1}{P(m)} = \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$. Assuming all other steps are valid, taking the limit as $(x_0, ..., x_n) \to (0...0)$ should yield $\zeta(n+1)$ from the left of equation 2, which converges with a P-series test. Now, by parametrizating x_j into $x_j = jv$ for some x. Using this:

$$\zeta(n+1) = \lim_{(x_1 \ldots x_n) \to (0,\ldots,0)} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$$

$$= \lim_{v \to 0} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{jv}}{1-t} dt \text{ with lemma 1}$$

$$= \lim_{v \to 0} \frac{1}{n! v^n} \int_0^1 \frac{\sum_{j=0}^{n} \binom{n}{j} (-t^v)^j}{1-t} dt$$

$$= \lim_{v \to 0} \frac{1}{n! v^n} \int_0^1 \frac{(1-t^v)^n}{1-t} dt$$

Now, using repeated application of L'Hopital's:

$$= \lim_{v \to 0} \frac{1}{n!^2} \int_0^1 \frac{\ln(t)^n \sum_{j=0}^{n} \binom{n}{j} j^n (-t^v)^j}{1-t} dt$$

$$= \frac{1}{n!^2} \int_0^1 \frac{\ln(t)^n \sum_{j=0}^{n} \binom{n}{j} j^n (-1)^j}{1-t} dt$$

$$= \frac{(-1)^n}{n!} \int_0^1 \frac{\ln(t)^n}{1-t} dt$$

This is equivalent to the formula given using the Bose-Einstein integral shown below by applying the substitution $t = e^{-z}$ to the above integral.

$$\int_0^\infty \frac{t^x}{e^t - 1} dt = \Gamma(x+1)\zeta(x+1)$$

Using the parametrization $x_i = ix + a$, the following can be seen as well through the same reasoning as above.

$$\Gamma(n)\zeta(n,a) = (-1)^{n-1} \int_0^1 \frac{t^{a-1} \ln(t)^{n-1}}{1-t} dt$$

where $\zeta(n,a)$ is the Hurwitz Zeta function. The exponent of t in the numerator is $a-1$ instead of a because the sum in the Hurwitz Zeta function begins with 0. In addition, because the generalized harmonic number $H_{x,n} = \zeta(n,1) - \zeta(n,x+1)$, an identity with $H_{x,n}$ can be derived too.

$$\Gamma(n)H_{x,n} = (-1)^{n-1} \int_0^1 \frac{(1-t^x) \ln(t)^{n-1}}{1-t} dt$$

Graham Bertele - gbertele26@sidwell.edu