# Rigidity, Form, and Compactness of Invertible a.e. Measure-Preserving Maps of Bounded Lipschitz Domains in $\mathbb{R}^n$

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## 1  Basic Properties

**Definition 1** *Throughout the paper, $\Omega \subset \mathbb{R}^n$ always denotes a bounded Lipschitz Domain with $0 < \mu(\Omega) < \infty$ (for $\mu$ the usual Lebesgue measure).*

**Definition 2** *By convention we take the range of functions in $L^p(\Omega)$ to $\mathbb{R}^n$. If the range is instead $K$, it will be denoted as $L^p(\Omega, K)$.*

**Definition 3** *The set of all invertible a.e. measure preserving maps ($\tau$ such that $\mu(X) = \mu(\tau(X))$) on a set $\Omega$ is denoted $\sigma(\Omega)$.*

$\sigma(\Omega)$ also has a group structure as well.

**Proposition 1** *For $\tau, \gamma \in \sigma(\Omega)$ we have that $\tau \circ \gamma \in \sigma(\Omega)$.*

**Proposition 2** *For all $\tau \in \sigma(\Omega)$, there exists some function $g$ such that $\tau \circ g = \mathrm{Id}$ almost everywhere. In addition, $g \in \sigma(\Omega)$ as well. $g$ is usually denoted as $\tau^{-1}$ even though the actual inverse is not explicitly defined and need not exist.*

## 2  Rigidity and Form

We aim to characterize $\tau \in \sigma(\Omega)$ using a less measure-theoretic definition. Heuristically, if any such $\tau$ 'stretched or compressed space,' it would shrink or grow some sets and thus not preserve measure, so we expect each $\tau$ to be locally 'space-preserving.' We show that $\tau$ is locally a unitary map plus a shift, up to a null set.

**Definition 4** *Let $I(\tau)$ be any set $\subseteq \Omega$ on which $\tau$ is equivalent a.e. to an invertible function and such that $\mu(I(\tau)) = \mu(\Omega)$.*

If $\Omega$ is unbounded, there exist maps which preserve measure but lengthen certain distances, such as $\tau(x, y) = (x/2, 2y)$ from $\mathbb{R}^2 \to \mathbb{R}^2$. As $\Omega$ is bounded though, $\tau$'s expansion of space is heavily restricted. Consequently, $\tau$'s contraction of space, corresponding to $\tau^{-1}$'s expansion of space, is also heavily restricted. Therefore, we expect that $\tau$ does not stretch or expand distance. $\tau$ may still be discontinuous and 'cut' subspaces into parts though. Accordingly, we show that $\tau$ preserves Hausdorff measure.

**Proposition 3** *For all $\tau \in \sigma(\Omega)$ and any $d$-dimensional Hausdorff measurable set $K \subset I(\tau)$, we have that $H^d(K) = H^d(\tau(K))$ for all $1 \le d \le n$.*

**Proof.** We have the following.

$$H_\delta^d(K) = \inf \left\{ \sum_i (\operatorname{diam} U_i)^d : K \subseteq \cup_i U_i \wedge \operatorname{diam} U_i < \delta \right\}$$

Each $U_i$ can be taken without loss of generality to be a countable union of balls so that their union is then again a countable union of balls. Therefore, we can assume without loss of generality that each $U_i$ is just a

ball.

$$= \inf \left\{ \sum_i (\operatorname{diam} B(x_i, r_i))^d : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\}$$

$$= \inf \left\{ \sum_i (C_n \mu(B(x_i, r_i)))^{d/n} : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\}$$

$$= \inf \left\{ \sum_i (C_n \mu(\tau(B(x_i, r_i))))^{d/n} : \tau(K) \subseteq \cup_i \tau(B(x_i, r_i)) \wedge 2r_i < \delta \right\}$$

For any set $X \subset \Omega$ we have that $(C_n \mu(X))^{1/n} \leq \operatorname{diam}(X)$ because if $\mu(B(x, \varepsilon)) = \mu(X)$, then $(C_n \mu(X))^{1/n} = (C_n \mu(B(x, \varepsilon)))^{1/n} = \operatorname{diam} B(x, \varepsilon) \leq \operatorname{diam} X$ as an $n$-sphere minimizes diameter for a given volume.

$$\leq \inf \left\{ \sum_i (C_n \mu(V_i))^{d/n} : \tau(K) \subseteq \cup_i V_i \wedge (C_n \mu(V_i))^{1/n} < \delta \right\}$$

$$\leq \inf \left\{ \sum_i (\operatorname{diam} V_i)^d : \tau(K) \subseteq \cup_i V_i \wedge \operatorname{diam}(V_i) < \delta \right\}$$

$$= H_\delta^d(\tau(K))$$

Now, taking limits as $\delta \to 0$ we have that $H^d(K) \leq H^d(\tau(K))$.
Therefore, $H^d(\tau(K)) \leq H^d(\tau^{-1}(\tau(K)) = H^d(K)$ so that $H^d(\tau(K)) = H^d(K)$.

As $\tau \in \sigma(\Omega)$ cannot stretch or contract any distances too much, it would make sense for elements of $\sigma(\Omega)$ to be continuous in some sense as well.

**Proposition 4** *For all $\tau \in \sigma(\Omega)$, there exists a function $g_\tau$ continuous on an open set $S$ with $\mu(S) = \mu(\Omega)$ such that $g_\tau = \tau$ almost everywhere.*

**Proof.** We have that for all $n$ there exists some $S_n \subset \Omega$ such that $\tau$ is continuous when restricted to $S_n$ and $\mu(\Omega \setminus S_n) < 1/n$. Now, with $S = \cup_{i=1}^\infty S_i$ we have that $\tau$ is continuous on the $S$-inherited subspace topology and that $\mu(S) = \mu(\Omega)$ so that $S$ is dense in $\Omega$.

Now, let $G = \overline{\{(x, \tau(x)) \mid x \in S\}}$.
For any $x \in \Omega \setminus S$, we have that there exists some sequence $\{x_n\} \subset S$ convergent to $x$. Now, $\tau(x_n)$ is bounded and thus has some subsequence convergent to some $y$. Now, $(x, y) \in G$. Thus, with $p_1(x, y) = x$, we have that $p_1(G) = \Omega$.

Next, let $Q = \{x \in \Omega \mid ((x, y) \in G \wedge (x, z) \in G) \implies y = z\}$. If $x \in S$ then if $(x, s_1) \in G$ and $(x, s_2) \in G$ we have that there must be some sequences $(x_{in}, \tau(x_{in})) \to (x, s_i)$ for $x_n \in S$. As $|x_{in} - x| \to 0$ we have that because $\tau$ is continuous in $S$ that $|\tau(x_{in}) - \tau(x)| \to 0$, and thus that $|\tau(x_{1n}) - \tau(x_{2n})| \to 0$ so that because $|\tau(x_{in}) - s_i| \to 0$ we have that $|s_1 - s_2|$ is arbitrarily small and thus $s_1 = s_2$. Therefore, $x \in Q$ so that $S \subseteq Q$.

Finally, define $g_\tau : \Omega \to \Omega$ by $g_\tau(x) = y$ if $(x, y) \in G$, where $y$ is chosen arbitrarily for $x \notin Q$.
We aim to show that $g_\tau$ is continuous on $S$. For $x \in S$ we have that for any $\varepsilon > 0$ there exists some $\delta$ such that $|x - y| < \delta$ for $y \in S$ implies that $|\tau(x) - \tau(y) < \varepsilon$. Then, take any $y \in \Omega \setminus S$ with $|x - y| < \delta$.

If $y \notin Q$, then assume $(y, z_1)$ and $(y, z_2)$ are in $G$. Now, there are sequences $\{s_n\}, \{t_n\} \subset B(x, \delta) \cap S$ such that $s_n, t_n \to y$, $\tau(s_n) \to z_1$, and $\tau(t_n) \to z_2$. Then, we have that $|\tau(s_n) - \tau(x)) < \varepsilon$, and for any $\varepsilon_2 > 0$ we have that there exists some $N$ such that $n > N \implies |z_1 - \tau(s_n)| < \varepsilon_2$. Then, $|z_1 - \tau(x)| < \varepsilon + \varepsilon_2$ so that $|z_1 - \tau(x)| \leq \varepsilon$. Similar logic shows that $|z_2 - \tau(x)| \leq \varepsilon$.

If $y \in Q$, then $(y, z) \in G$. We have that there must be some sequence $\tau(s_n) \to z$ for $\{s_n\} \subset S \cap B(x, \delta)$ so that $|z - \tau(x)| \leq |z - \tau(s_n)| + |\tau(s_n) - \tau(x)| \leq \varepsilon_2 + \varepsilon \to \varepsilon$. Therefore, $|z - \tau(x)| \leq \varepsilon$ as well.
Finally, regardless of the choice of value of $g_\tau(y)$ outside of $Q$, we have that $g_\tau$ is continuous at $x$.

Now, with $S_2$ the set of all $x$ such that $g_\tau(x)$ is continuous at $x$, we have that $S_2$ is open and that $S \subseteq S_2$ so that $\mu(S_2) = \mu(\Omega)$. Therefore, $S_2$ is the desired set, and because $g_\tau(x) = \tau(x)$ for $x \in S$ we have that $g_\tau \equiv \tau$.

**Definition 5** *For some $\tau \in \sigma(\Omega)$, we define $S(\tau)$ to be $S_2$ as above and $g_\tau$ to be $g_\tau$ as above.*

**Corollary.** We can choose the values of $g_\tau$ on $\Omega \setminus S(\tau)$ so that $g_\tau(\Omega) = \Omega$.
**Corollary.** $S(\tau) \subseteq I(\tau)$ by construction, so we can redefine $I(\tau) = S(\tau)$.
**Corollary.** $S(\tau^{-1}) = g_\tau(S(\tau))$

Now, we are ready to prove the following characterization of $\tau \in \sigma(\Omega)$. Elements of $\sigma(\Omega)$ are locally continuous a.e and so we expect them to preserve metric distance, not just Hausdorff measure, in sufficiently small neighborhoods.

**Proposition 5** *$S(\tau)$ can be partitioned into disjoint sets $\{A_\alpha\}_{\alpha \in I}$ such that $g_\tau|_{A_\alpha} = U_\alpha x + v_\alpha$ for $U_\alpha$ a unitary linear map and $v_\alpha$ a constant vector.*

**Proof.** We have that $g_\tau$ is continuous on an open set $S(\tau)$ with $\mu(S) = \mu(\Omega)$ so that $\partial S(\tau) = \Omega \setminus S(\tau)$. Now, for any $x \in S(\tau)$, there is some $B_1 = B(x, \varepsilon_1) \subset S$.

For any $y \in B_1$, let $\ell_1 = \{g_\tau(x) + a(g_\tau(y) - g_\tau(x)) \mid 0 \le a \le 1\}$. We have that $|g_\tau(x) - g_\tau(y)| = H^1(\ell_1) = H^1(g_\tau^{-1}(\ell_1))$. As $g_\tau^{-1}(\ell_1)$ is some continuous path between $x$ and $y$, we have that $H^1(g_\tau^{-1}(\ell)) \ge |x-y|$. Therefore, $|g_\tau(x) - g_\tau(y)| \ge |x - y|$. Finally, we have that $|x - y| = |g_\tau^{-1}(g_\tau(x)) - g_\tau^{-1}(g_\tau(y))| \ge |g_\tau(x) - g_\tau(y)|$ so that $|x - y| = |g_\tau(x) - g_\tau(y)|$.

It is a known result that if $|g_\tau(x) - g_\tau(y)| = |x - y|$ for $g - \tau$ continuous that $g_\tau(x) = Ux + v$ for $U$ a unitary map.

Choose some partitioning of $S(\tau)$ into balls such that $S = \cup_{\alpha \in I} B(x_\alpha, r_\alpha)$. Then, take $A_\alpha = B(x_\alpha, r_\alpha) \setminus (\cup_{i < \alpha} B(x_i, r_i))$ so that they are disjoint, $S = \cup_{\alpha \in I} A_\alpha$, and on each $A_\alpha$ we have that $g_\tau(x) = U_\alpha x + v_\alpha$.

**Note.** The partition $I$ can be assumed to be countable because each of the $A_\alpha$ have been constructed to be pairwise disjoint.

**Definition 6** *We take $P_{S(\tau)}$ to be the set of connected components of $S(\tau)$. Note that on each $C \in P_{S(\tau)}$ we must then have that $\tau$ is locally equivalent to a unitary map plus a shift.*

The converse of our previous characterization holds true as well.

**Proposition 6** *For any a.e. invertible $L^p$ function $f : \Omega \to \Omega$, we have that $f \in \sigma(\Omega)$ if and only if for almost all $x \in \Omega$ there exists some $\varepsilon > 0$ such that there are $U \in U(n)$ and $v \in \mathbb{R}^n$ such that $f(y) \equiv Uy + v$ for $y \in B(x, \varepsilon)$.*

**Proof.** We have proven the $\implies$ direction already. Now, conversely, assume that for all $x \in S$ there exists some $\varepsilon > 0$ such that $f(y) \equiv Uy + vf$ or $y \in B(x, \varepsilon)$, where $\mu(S) = \mu(\Omega)$.
Now, for all $y \in f(S)$ there exists some $\varepsilon > 0$ such that $f^{-1}(z) \equiv U^T z - U^T v$ for all $z \in B(y, \varepsilon)$. Then, $(f^{-1})'(y)$ is unitary.

$$
\mu(f(A)) = \int_\Omega 1_A \circ f^{-1} d\mu
$$
$$
= \int_S 1_A \circ f^{-1} d\mu
$$
$$
= \int_{f(S)} 1_A |\det(f^{-1})'| d\mu
$$
$$
= \int_\Omega 1_A d\mu
$$
$$
= \mu(A)
$$

And therefore, $f \in \sigma(\Omega)$.

**Proposition 7** *For $\tau, \gamma \in \sigma(\Omega)$, we have that $S(\tau \circ \gamma) \subseteq \Omega \setminus (S(\gamma) \Delta g_\gamma^{-1}(S(\tau)))$.*

**Proof.** Assume $x \in S(\tau \circ \gamma)$. If $g_\gamma$ is continuous at $x$ we have that $g_\tau$ is continuous at $g_\gamma(x)$ so $x \in S(\gamma) \cap g_\gamma^{-1}(S(\tau))$.

Otherwise, if $x \notin S(\gamma)$ then if $g_\gamma(x) \in S(\tau)$ we have that $g_\tau(g_\gamma(x))$ cannot be continuous, a contradiction.

Therefore, $x \in (\Omega \setminus S(\gamma)) \cap (\Omega \setminus g_\gamma^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \cup g_\gamma^{-1}(S(\tau)))$.

Finally, $S(\tau \circ \gamma) \subseteq (\Omega \setminus (S(\gamma) \cup g_\gamma^{-1}(S(\tau)))) \cup (S(\gamma) \cap g_\gamma^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \Delta g_\gamma^{-1}(S(\tau)))$.

**Corollary.** $S(\gamma) \cap g_\gamma^{-1}(S(\tau)) \subseteq S(\tau \circ \gamma)$

# 3 Estimates on Measures of Thickened Sets of Discontinuity

We now wish to study the topological and compactness properties of $\sigma(\Omega)$, treating it as a subset of $L^p$. With the Rellich Kondrachov theorem, we see that compactness of sets in $L^p$ is in some sense correlated to the uniform convergence of $\|\tau(x+h) - \tau(x)\|_{L^p(\Omega)} \to 0$ as $|h| \to 0$. For $h$ sufficiently small so that it $B(x, h)$ contains no discontinuities of $g_\tau$, $|g_\tau(x+h) - g_\tau(x)| = |h|$, so the uniformity is almost entirely determined by the size of the set of discontinuities of $g_\tau$. Thus, in order to study compactness, we first study the size of the sets of discontinuity of each $\tau$.

**Definition 7** *We define $A(\tau, \varepsilon) = \{x \in \Omega \mid d(x, \partial(\Omega \setminus S(\tau))) < \varepsilon\}$.*

Note that $\partial(\Omega \setminus S(\tau)) = \Omega \setminus S(\tau)$, and are interchanged at times.

**Lemma 1** *For any open set $X$ with $X \subseteq S(\tau)$ and $\mu(X) = \mu(\Omega)$, we have that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\})$*

**Proof.** We have that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\}) = \mu(\{x \in g_\tau(X) \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\})$.

Now, assume for any $x \in g_\tau(X)$ that $y \in \Omega \setminus g_\tau(X)$ minimizes $|x - y|$. We see that $x \in C$ where $C$ is taken to be the largest (by inclusion) connected component of $g_\tau(X)$ containing $x$. We can assume it is convex because of the construction given in Proposition 5. Then, $y \in \partial C$ by the minimality of $|x - y|$.

In addition, we have that $g_\tau(X) \subseteq g_\tau(S(\tau)) = S(\tau^{-1})$ so that $g_\tau^{-1}(x) = Ux + v$ on $C$.

Now, with $\ell = \{x + a(y - x) \mid a \in [0, 1)\}$, we have that $\ell \subset C$ by the convexity of $\partial C$ (because of the convexity of $C$).

Thus, $|x - y| = H^1(\ell) = H^1(g_\tau^{-1}(\ell)) \geq |g_\tau^{-1}(x) - y_2|$ for some $y_2 \in \partial g_\tau^{-1}(C)$. Thus, $|x - y| \geq d(g_\tau^{-1}(x), g_\tau^{-1}(C)) \geq d(g_\tau^{-1}(x), \Omega \setminus g_\tau^{-1}(g_\tau(X))) = d(g_\tau^{-1}(x), \Omega \setminus X)$.

Finally, we have the following.

$$\implies \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\}) \leq \mu(\{x \in g_\tau(X) \mid d(g_\tau^{-1}(x), \Omega \setminus X) < \varepsilon\})$$
$$= \mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\})$$

Now, as $X \subseteq S(\tau)$ we see that $g_\tau(X) \subseteq S(\tau^{-1})$ so that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau^{-1}(g_\tau(X))) < \varepsilon\}) \leq \mu(x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\})$.

Therefore, $\mu(\{x \in \Omega \mid d(x, \Omega \setminus X) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\tau(X)) < \varepsilon\})$.

**Proposition 8** $\mu(A(\tau \circ \gamma^{-1}, \varepsilon)) \leq \mu(A(\tau, \varepsilon)) + \mu(A(\gamma, \varepsilon))$ *for all $\varepsilon > 0$ and $\tau, \gamma \in \sigma(\Omega)$.*

**Proof.**

$$\mu(A(\tau \circ \gamma^{-1}, \varepsilon)) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\tau \circ \gamma^{-1})) < \varepsilon\})$$
$$\leq \mu(\{x \in \Omega \mid d(x, \Omega \setminus (S(\gamma^{-1}) \cap g_\gamma(S(\tau)))) < \varepsilon\})$$
$$\leq \mu(\{x \in \Omega \mid d(x, \Omega \setminus (S(\gamma^{-1}) \cap g_\gamma(S(\tau) \cap S(\gamma)))) < \varepsilon\})$$
$$= \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\gamma(S(\tau) \cap S(\gamma))) < \varepsilon\})$$

By Lemma 1 we have that $\mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\gamma(S(\tau) \cap S(\gamma))) < \varepsilon\}) = \mu(\{x \in \Omega \mid d(x, (S(\tau) \cap S(\gamma))) < \varepsilon\})$ because $S(\tau) \cap S(\gamma) \subseteq S(\gamma)$ and has full measure.

$$\implies \mu(A(\tau \circ \gamma^{-1}, \varepsilon)) \leq \mu(\{x \in \Omega \mid d(x, \Omega \setminus (S(\tau) \cap S(\gamma))) < \varepsilon\})$$
$$\leq \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\tau)) < \varepsilon\}) + \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\gamma)) < \varepsilon\})$$
$$= \mu(A(\tau, \varepsilon)) + \mu(A(\gamma, \varepsilon))$$

Intuitively, we expect that $A(\gamma, \varepsilon)$ and $A(\gamma^{-1}, \varepsilon)$ are very similar because $S(\gamma^{-1}) = g_\gamma(S(\gamma))$ and because $g_\gamma \in \sigma(\Omega)$ are locally unitary and invertible, they tend to preserve the geometry of sets in some sense.

**Proposition 9** *We have that $\mu(A(\gamma, \varepsilon)) = \mu(A(\gamma^{-1}, \varepsilon))$ for all $\gamma \in \sigma(\Omega)$ and $\varepsilon > 0$.*

**Proof.** We have the following estimate.

$$\mu(A(\gamma^{-1}, \varepsilon)) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\gamma^{-1})) < \varepsilon\})$$
$$= \mu(\{x \in \Omega \mid d(x, \Omega \setminus g_\gamma(S(\gamma))) < \varepsilon\})$$

We see that $S(\gamma) \subseteq S(\gamma)$ so by Lemma 1 we have that $\mu(A(\gamma^{-1}, \varepsilon)) = \mu(\{x \in \Omega \mid d(x, \Omega \setminus S(\gamma)) < \varepsilon\}) = \mu(A(\gamma, \varepsilon))$.

# 4 Topological Properties

We are now ready to study the topological properties of $\sigma(\Omega)$.

**Proposition 10** *If $X \subseteq \sigma(\Omega)$ is open under $|| \cdot ||_{L^p(\Omega)}$, then $X \circ \tau$ is as well for all $\tau \in \sigma(\Omega)$.*

**Proof.** We have that for any $\gamma \in X \circ \tau$ that $\gamma \circ \tau^{-1} \in X$ and thus there is some $B = B_{L^p(\Omega)}(\gamma \circ \tau^{-1}, \varepsilon) \subset X$. For $\lambda \in B$ we have that $||\lambda \circ \tau - \gamma||_{L^p(\Omega, \mathbb{R}^n)} = ||\lambda - \gamma \circ \tau^{-1}||_{L^p(\Omega)}$, so that $B_{L^p(\Omega)}(\gamma \circ \tau^{-1}, \varepsilon) \circ \tau = B_{L^p(\Omega)}(\gamma, \varepsilon) \subset X \circ \tau$. And thus there is an open neighborhood of $\gamma \in X \circ \tau$, so that $X \circ \tau$ is open as well.

Next, for $\tau, \gamma, \lambda \in \sigma(\Omega)$ we see that $||\tau \circ \lambda - \gamma \circ \lambda||_{L^p(\Omega)} = ||\tau - \gamma||_{L^p(\Omega)}$ so that convergence in $|| \cdot ||_{L^p(\Omega)}$ is right-invariant. This raises the question of whether or not convergence is left-invariant as well. We expect that because distances $|\lambda(x) - \lambda(y)|$ are preserved (a.e) if $|x - y|$ is small enough and $||\tau - \gamma||_{L^p(\Omega)}$ being small forces $|\tau - \gamma|$ to be small as well that $|\lambda \circ \tau - \lambda \circ \gamma|$ must be small as well, constrained by $||\tau - \gamma||_{L^p(\Omega)}$.

**Proposition 11** *If $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$, then $||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \to 0$ as well for all $\tau \in \sigma(\Omega)$.*

**Proof.** Note that $\mu(E_\delta) = \mu(\{x \in \Omega \mid |\gamma_n(x) - \gamma(x)| > \delta\}) < \delta^{-p}||\gamma_n(x) - \gamma(x)||_{L^p(\Omega)}$ by Chebyshev's inequality.

$$||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)}^p = \int_\Omega |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu$$
$$= \int_{E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu + \int_{\Omega \setminus E_\delta} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu$$
$$\leq \delta^{-p}||\gamma_n - \gamma||_{L^p} \operatorname{diam}(\Omega)^p + I$$
$$I = \int_{(\Omega \setminus E_\delta) \setminus g_\gamma^{-1}(A(\tau, 2\delta))} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu + \int_{(\Omega \setminus E_\delta) \cap g_\gamma^{-1}(A(\tau, 2\delta))} |\tau \circ \gamma_n - \tau \circ \gamma|^p d\mu$$

Given $x \in (\Omega \setminus E_\delta) \setminus g_\gamma^{-1}(A(\tau, 2\delta))$, we see that $|g_{\gamma_n}(x) - g_\gamma(x)| < \delta$ a.e, and $g_\gamma(x) \in \Omega \setminus A(\tau, 2\delta)$ so that $d(g_\gamma(x), \Omega \setminus S(\tau)) \geq 2\delta$ and thus $d(g_{\gamma_n}(x), \Omega \setminus S(\tau)) \geq \delta$.
This means that $g_{\gamma_n}(x)$ and $g_\gamma(x)$ are in the same connected component of $S(\tau)$ so that $|g_\tau \circ g_{\gamma_n}(x) - g_\tau \circ$

$g_\gamma(x)| = |g_{\gamma_n} - g_\gamma(x)| < \delta.$

$$\implies I \le \mu((\Omega \setminus E_\delta) \setminus g_\gamma^{-1}(A(\tau, 2\delta)))\delta^p + \mu((\Omega \setminus E_\delta) \cap g_\gamma^{-1}(A(\tau, 2\delta))) \operatorname{diam}(\Omega)^p$$
$$\le \mu(\Omega)\delta^p + \mu(A(\tau, 2\delta)) \operatorname{diam}(\Omega)^p$$
$$\implies ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \le \delta^{-p}||\gamma_n - \gamma||_{L^p(\Omega)} \operatorname{diam}(\Omega)^p + \mu(\Omega)\delta^p + \mu(A(\tau, 2\delta)) \operatorname{diam}(\Omega)^p$$

Now, choose $\delta = ||\gamma_n - \gamma||_{L^p(\Omega)}^{1/p+1}$.

$$\le ||\gamma_n - \gamma||_{L^p(\Omega)}^{1/(p+1)} \operatorname{diam}(\Omega)^p + \mu(\Omega)||\gamma_n - \gamma||_{L^p(\Omega)}^{1/p+1} + \mu(A(\tau, 2||\gamma_n - \gamma||^{1/(p+1)})) \operatorname{diam}(\Omega)^p$$

This bound holds for all $n$. Now, we see that each term approaches 0 as $||\gamma_n - \gamma||$ does, so that $||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \to 0$ as $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$ for $1 \le p < \infty$.

**Proposition 12** *If $X \subseteq \sigma(\Omega)$ is open under $|| \cdot ||_{L^p(\Omega)}$, then $\tau \circ X$ is as well for all $\tau \in \sigma(\Omega)$.*

**Proof.** Take any set $X$ open in $|| \cdot ||_{L^p(\Omega)}$.
Now, take any sequence $\{\gamma_n\} \subseteq \sigma(\Omega) \setminus (\tau \circ X)$ convergent to some limit $\gamma$. We see that $\tau^{-1} \circ \gamma_n$ is then a sequence in $\sigma(\Omega) \setminus X$ convergent to $\tau^{-1} \circ \gamma$ by the previous proposition. Then, $\tau^{-1} \circ \gamma \in \Omega \setminus X$ as $\sigma(\Omega) \setminus X$ is closed. Finally, $\gamma \in \sigma(\Omega) \setminus (\tau \circ X)$, so that $\sigma(\Omega) \setminus (\tau \circ X)$ is closed and therefore $\tau \circ X$ is open.

**Proposition 13** *$(\sigma(\Omega), || \cdot ||_{L^p(\Omega)})$ is a topological group.*

**Proof.** Assume that $||\tau_n - \tau||_{L^p} \to 0$. Now, we see that $||\tau^{-1} - \tau_n^{-1}||_{L^p(\Omega} = ||\tau^{-1} \circ \tau_n - \operatorname{Id}||_{L^p(\Omega)} = ||\tau^{-1} \circ \tau_n - \tau^{-1} \circ \tau||_{L^p(\Omega)} \to 0$ as well, so that $\tau \mapsto \tau^{-1}$ is continuous.
Now, assume that $||\tau_n - \tau||_{L^p} \to 0$ and $||\gamma_n - \gamma||_{L^p} \to 0$. We see that $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \le ||\tau_n \circ \gamma_n - \tau \circ \gamma_n||_{L^p(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} = ||\tau_n - \tau||_{L^p(\Omega)} + ||\tau \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)}$. As both sequences approach 0, we see that $||\tau_n \circ \gamma_n - \tau \circ \gamma||_{L^p(\Omega)} \to 0$ as well, so that $(\tau, \gamma) \mapsto \tau \circ \gamma$ is continuous too.

Therefore, $(\sigma(\Omega), || \cdot ||_{L^p(\Omega)})$ is a topological group.

Now, the continuity of the group operations have been observed. As mentioned earlier, the compactness of subsets of $\sigma(\Omega)$ under the $L^p$ norm is expected to be governed mostly by the sets of discontinuity of each of the elements. Accordingly, we prove the following theorem for relative compactness.

**Proposition 14** *For $X \subseteq \sigma(\Omega)$, if $\mu(A(\tau, \varepsilon)) \to 0$ uniformly on $X$ as $\varepsilon \to 0$, then $X$ is relatively compact under $L^p$ for $1 \le p < \infty$.*

**Proof.** Pick any arbitrary $p$ and choose $q < p^* = np/(n - p)$.
Let $n^\varepsilon$ denote the usual mollifier taken by dilating $n$, a positive smooth function such that $\int_{\mathbb{R}^n} n d\mu = 1$ and 0 outside of $B(0, 1)$. Take any sequence $\{\tau_n\} \subset X$ and let $\tau_n^\varepsilon = n^\varepsilon * \tau_n$.
First, we bound $||\tau_n^\varepsilon - \tau_n||_{L^q(\Omega)}$.

$$||\tau_n^\varepsilon - \tau_n||_{L^1(\Omega)} \le \int_\Omega \int_{B(0,1)} n(y)|\tau_n(x - \varepsilon y) - \tau_n(x)| d\mu_y d\mu$$
$$\le \int_{A(\tau, \varepsilon)} \int_{B(0,1)} n(y)|\tau_n(x - \varepsilon y) - \tau_n(x)| d\mu_y d\mu + \int_{B(0,1)} \int_{\Omega \setminus A(\tau, \varepsilon)} n(y)\varepsilon d\mu d\mu_y$$
$$\le \int_{A(\tau, \varepsilon)} \int_{B(0,1)} n(y)|\tau_n(x - \varepsilon y) - \tau_n(x)| d\mu_y d\mu + \varepsilon\mu(\Omega)$$
$$\le \mu(A(\tau, \varepsilon))\mu(B(0, 1)) \operatorname{diam}(\Omega) + \varepsilon\mu(\Omega)$$

As $\varepsilon$ approaches $0^+$, this upper bound approaches 0 by uniformly in $n$ by hypothesis. Then, the rest of this proof follows exactly the same as Evan's proof of the Rellich-Kondrachov Compactness Theorem in 'Partial

Differential Equations'.[2, p. 286]

Therefore, there must be some subsequence $\tau_{m_n}$ convergent in the $L^q(\Omega)$ norm so that $X$ is relatively compact in $L^q(\Omega)$.

Now, for any arbitrary $p$, we can choose $p_1$ such that $p < p_1^*$, and thus $X$ is relatively compact in $L^p(\Omega)$.

Next, we have seen that left composition by $\tau \in \sigma(\Omega)$ does not affect the $L^p$ convergence of a sequence $\gamma_n \to \gamma$ in $\sigma(\Omega)$. We are able to prove an even stronger result as well.

**Proposition 15** *Take any $1 \leq p, q < \infty$. For all $f \in L^q(\Omega)$, if $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$ then $||f \circ \gamma_n - f \circ \gamma||_{L^q(\Omega)} \to 0$.*

**Proof.** We first bound $||f \circ \gamma_n - f \circ \gamma||_{L^q(\Omega)}$. Choose any continuous and bounded function $r : \Omega \to \mathbb{R}$ such that $||f - r||_{L^q(\Omega)} < \delta_1$. Next, choose any closed set $A \subseteq \Omega$ with $\mu(\Omega \setminus A) < \delta_2$. In addition, since $r$ is then uniformly continuous on $A$ because $A$ is compact, choose $\delta_3$ so that $|x - y| < \delta_3 \implies |r(x) - r(y)| < \delta_4$ on $E$. In addition, let $E = \{x \in \Omega \mid |\gamma_n - \gamma| \geq \delta_3\}$ so that $\mu(E) \leq \delta_3^{-p}||\gamma_n - \gamma||_{L^p(\Omega)}^p$.
Then, we have the following.

$$||f \circ \gamma_n - f \circ \gamma||_{L^q(\Omega)} \leq ||f \circ \gamma_n - r \circ \gamma_n||_{L^q(\Omega)} + ||r \circ \gamma_n - r \circ \gamma||_{L^q(\Omega)} + ||r \circ \gamma - f \circ \gamma||_{L^q(\Omega)}$$
$$< ||r \circ \gamma_n - r \circ \gamma||_{L^q(\Omega)} + 2\delta_1$$
$$||r \circ \gamma_n - r \circ \gamma||_{L^q(\Omega)}^q = \int_{\Omega \setminus g_\gamma^{-1}(A)} |r \circ \gamma_n - r \circ \gamma|^q d\mu + \int_{g_\gamma^{-1}(A)} |r \circ \gamma_n - r \circ \gamma|^q d\mu$$

$$\leq (4 \sup_{x \in \Omega} |r(x)|^q) \mu(g_\gamma^{-1}(\Omega \setminus A)) + \int_{g_\gamma^{-1}(A) \setminus E} |r \circ \gamma_n - r \circ \gamma|^q d\mu + \int_{g_\gamma^{-1}(A) \cap E} |r \circ \gamma_n - r \circ \gamma|^q d\mu$$
$$\leq (4 \sup_{x \in \Omega} |r(x)|^q)(\mu(\Omega \setminus A) + \mu(E)) + \int_{g_\gamma^{-1}(A) \setminus E} |r \circ \gamma_n - r \circ \gamma|^q d\mu$$
$$\leq (4 \sup_{x \in \Omega} |r(x)|^q)(\delta_2 + \delta_3^{-p}||\gamma_n - \gamma||_{L^p(\Omega)}^p) + \int_{(g_\gamma^{-1}(A) \setminus E) \cap g_{\gamma_n}^{-1}(A)} |r \circ \gamma_n - r \circ \gamma|^q d\mu + \int_{(g_\gamma^{-1}(A) \setminus E) \setminus g_{\gamma_n}^{-1}(A)} |r \circ \gamma_n - r \circ \gamma|^q d\mu$$

For $x \in (g_\gamma^{-1}(A) \setminus E) \cap g_{\gamma_n}^{-1}(A)$ we have that $g_\gamma(x), g_{\gamma_n}(x) \in A$ and $|g_\gamma(x) - g_{\gamma_n}(x)| < \delta_3$ a.e.so that $|r \circ \gamma_n - r \circ \gamma| < \delta_4$ a.e.

$$\leq (4 \sup_{x \in \Omega} |r(x)|^q)(\delta_2 + \delta_3^{-p}||\gamma_n - \gamma||_{L^p(\Omega)}^p) + \int_{(g_\gamma^{-1}(A) \setminus E) \cap g_{\gamma_n}^{-1}(A)} \delta_4^q d\mu + \int_{\Omega \setminus g_{\gamma_n}^{-1}(A)} |r \circ \gamma_n - r \circ \gamma|^q d\mu$$
$$\leq (4 \sup_{x \in \Omega} |r(x)|^q)(2\delta_2 + \delta_3^{-p}||\gamma_n - \gamma||_{L^p(\Omega)}^p) + \mu(\Omega)\delta_4^q$$

Note that these hold regardless of the $\delta_1, \delta_2, \delta_4 > 0$ chosen and so long as $\delta_3$ is sufficiently small (with upper bound dependent upon $\delta_1$ and $\delta_4$).
Fix any $\varepsilon > 0$, and for any $\varepsilon > 0$ choose $\delta_1 > 0$ (and thus $r$) with $\delta_1 < \varepsilon/2$. Next, choose $\delta_4$ (and thus $\delta_3$) with $\delta_4 < \varepsilon^{1/q}\mu(\Omega)^{-1/q}$ and $\delta_2 < \varepsilon/(8 \sup_{x \in \Omega} |r(x)|^q)$ so that we have the following.

$$||f \circ \gamma_n - f \circ \gamma||_{L^q(\Omega)} < \left(2\varepsilon + \delta_3^{-p}(4 \sup_{x \in \Omega} |r(x)|^q)||\gamma_n - \gamma||_{L^p(\Omega)}\right)^{1/q} + \varepsilon$$

Finally, we may choose $N$ such that $n > N \implies \delta_3^{-p}(4 \sup_{x \in \Omega} |r(x)|^q)||\gamma_n - \gamma||_{L^p(\Omega)} < \varepsilon$, so that we have that $n > N$ implies the following.

$$||f \circ \gamma_n - f \circ \gamma||_{L^q(\Omega)} < (3\varepsilon)^{1/q} + \varepsilon$$

Then, as $\varepsilon > 0$ was arbitrary we have that $\lim_{n\to\infty} ||f \circ \gamma_n - f \circ \gamma||_{L^q(\Omega)} \to 0$.

We have seen that $L^p$ compactness is much simpler on $\sigma(\Omega)$, as it is governed mostly by the size of the sets of discontinuity. We see that in $L^2$ there is an even nicer characterization of compactness, due to the fact that $L^2$ is induced by an inner product.

**Definition 8** *We define $\mathcal{T}_w$ to be the topology on $\sigma(\Omega)$ such that $\tau_n \to \tau$ if and only if $\langle \tau_n, f \rangle \to \langle \tau, f \rangle$ for all $f \in L^2(\Omega)$.*

**Proposition 16** *$\mathcal{T}_w$ is equivalent to the $L^2$ topology on $\sigma(\Omega)$.*

**Proof.** It's clear that $L^2$ convergence implies convergence in $\mathcal{T}_w$. For the other direction, assume $\tau_n \to \tau$ in $\mathcal{T}_w$.

$$\implies \langle \tau_n, \tau \rangle = \frac{1}{2}(||\tau_n||^2_{L^2(\Omega)} + ||\tau||^2_{L^2(\Omega)} - ||\tau_n - \tau||^2_{L^2(\Omega)})$$

$$= ||\operatorname{Id}||^2_{L^2(\Omega)} - \frac{1}{2}||\tau_n - \tau||^2_{L^2(\Omega)}$$

$$\to \langle \tau, \tau \rangle = ||\tau||^2_{L^2(\Omega)}$$

$$= ||\operatorname{Id}||^2_{L^2(\Omega)}$$

$$\implies ||\tau_n - \tau||_{L^2(\Omega)} \to 0$$

Therefore $\tau_n \to \tau$ in $\mathcal{T}_w$ if and only if $||\tau_n - \tau||_{L^2(\Omega)} \to 0$.

**Proposition 17** *$(\sigma(\Omega), || \cdot ||_{L^2(\Omega)})$ is a compact topological group.*

**Proof.** It has already been proven that it is a topological group.
Because the weak and weak-* topologies coincide on $L^2$, we have by the Banach-Alaoglu Theorem [3, p. 235] that $B = \{f \in L^2 \mid \sup_{||g||_{L^2(\Omega)} < 1/||\operatorname{Id}||_{L^2(\Omega)}} \langle f, g \rangle \leq 1\})$ is compact in the weak topology.
We see that $\sigma(\Omega) \subseteq B$ so that $\sigma(\Omega)$ has compact closure in the weak-topology. Finally, since $\mathcal{T}_w$ is the subspace topology inherited from the weak topology on $L^2$, $(\sigma(\Omega), \mathcal{T}_w) = (\sigma(\Omega), || \cdot ||_{L^2(\Omega)})$ is compact.

**Definition 9** *As $(\sigma(\Omega), || \cdot ||_{L^2(\Omega)})$ is a compact topological group, there must exist some right-invariant measure on subsets of $\sigma(\Omega)$, denoted $\mu_{\mathbb{D}}$, and normalized so that $\mu_{\mathbb{D}}(\sigma(\Omega)) = 1$.*

# 5 Algebraic Properties

Next, each $\tau$ induces an isometry in $L^p$ by the mapping $f \mapsto f \circ \tau$. This begs the question of which isometries are induced by $\tau \in \sigma(\Omega)$ in some form.

**Proposition 18** *For any linear map $M : L^2(\Omega, \mathbb{R}) \to L^2(\Omega, \mathbb{R})$ which preserve $|| \cdot ||_{L^1(\Omega, \mathbb{R})}$, $|| \cdot ||_{L^\infty(\Omega, \mathbb{R})}$, $\langle \cdot, \cdot \rangle$, and such that $f \geq 0 \implies Mf \geq 0$, we have that $Mf = f \circ \gamma^{-1}$ for $\gamma \in \sigma(\Omega)$.*

**Proof.** Take such an $M$. It preserves $|| \cdot ||_{L^2(\Omega, \mathbb{R})}$ as well since it preserves inner products.
Now, take any measurable set $A \in \Sigma_\Omega$ with nonzero measure and let $\chi_A$ be its indicator function. We see that $0 \leq M\chi_A \leq 1$ almost everywhere since $||M\chi_A||_{L^{\infty, \mathbb{R}}} = 1$.
Assume for a contradiction that $K = \{x \in \Omega \mid 0 < M\chi_A < 1\}$ has nonzero measure. Then, we have the following.

$$\mu(A) = \langle M\chi_A, M\chi_A \rangle$$

$$= \int_{\Omega \setminus K} (M\chi_A)^2 d\mu + \int_K (M\chi_A)^2 d\mu$$

$$= \int_{\Omega \setminus K} M\chi_A d\mu + \int_K (M\chi_A)^2 d\mu$$

$$< \int_{\Omega \setminus K} M\chi_A d\mu + \int_K M\chi_A d\mu$$

$$= \int_\Omega |M\chi_A| d\mu$$

$$= ||M\chi_A||_{L^1(\Omega,\mathbb{R})} = \mu(A)$$

Then, we have a contradiction, so that $\mu(K) = 0$ and thus $M\chi_A \in \{0,1\}$. Now, we may define $\tau : \Sigma_\Omega \to \Sigma_\Omega$ by $\tau(A) = \{x \in \Omega \mid M\chi_A = 1\}$. Note that we now have that $\mu(A) = ||M\chi_A||_{L^1(\Omega,\mathbb{R})} = ||\chi_{\tau(A)}||_{L^1(\Omega,\mathbb{R})} = \mu(\tau(A))$.

For disjoint $A, B$ we have by the linearity of $M$ that $M\chi_{A\cup B} \equiv M\chi_A + M\chi_B \equiv \chi_{\tau(A)} + \chi_{\tau(B)}$ so that $\tau(A), \tau(B)$ are disjoint as well (modulo a null set) and thus $\tau(A \cup B) = \tau(A) \cup \tau(B)$ (again modulo a null set).

The same logic shows that $\tau(A \setminus B) = \tau(A) \setminus \tau(B)$.

Thus, by Theorem 9.5.1 in [1, p. 295] we have that there exists some $\gamma : \Omega \to \Omega$ such that $\mu(\tau(A)\Delta\gamma(A)) = 0$ for all $A \in \Sigma_\Omega$.

Then, $|M\chi_A| \equiv \chi_{\gamma(A)} \equiv \chi_A \circ \gamma^{-1}$, so that $|Mf| \equiv f \circ \gamma^{-1}$ for simple functions $f \in L^\infty(\Omega,\mathbb{R})$ as well, and therefore $|Mf| \equiv f \circ \gamma^{-1}$ for all $f \in L^\infty(\Omega,\mathbb{R})$ by taking limits of simple functions.

**Definition 10** *We denote the set of linear maps $M$ as in the hypotheses of the previous proposition to be $L(\Omega)$, the linear representation of $\Omega$. We have that $L(\Omega)$ is a subgroup of the group $G(\Omega) = U(L^2(\Omega,\mathbb{R})) \cap U(L^1(\Omega,\mathbb{R})) \cap U(L^\infty(\Omega,\mathbb{R}))$ (with the group action being multiplication/composition).*

Even if $M \in G(\Omega) \setminus L(\Omega)$, we still see that $M$ must be induced (in a way) by some $\tau \in \sigma(\Omega)$.

**Proposition 19** *For all $M \in G(\Omega)$, we have that $M = \lambda N$ for $\lambda \in L^\infty(\Omega,\mathbb{R})$ with $\lambda = \pm 1$ a.e., and $N \in L(\Omega)$.*

**Proof.** Take any $M \in G(\Omega)$. Take any measurable $A \in \Sigma_\Omega$, so we see that $\mu(A) = ||M\chi_A||_{L^1(\Omega,\mathbb{R})}$.

Then, consider $M\chi_\Omega$. If $|M\chi_\Omega| < 1$ on a set $T$ of nonzero measure then since $||M\chi_\Omega||_{L^\infty(\Omega,\mathbb{R})} = 1$ we have that $\mu(\Omega) = \int_\Omega |M\chi_\Omega| d\mu = \int_T |M\chi_\Omega| d\mu + \int_{\Omega \setminus T} |M\chi_\Omega| d\mu < \mu(T) + \mu(\Omega \setminus T) = \mu(\Omega)$, a contradiction. Then, $|M\chi_\Omega| = 1$ almost everywhere so that $(M\chi_\Omega)M \in G(\Omega)$ as well.

Define $K = \{x \in \Omega \mid (M\chi_\Omega)(M\chi_A) < 0\}$. If $x \in K$ then since $|(M\chi_\Omega)(M\chi_A)| = 0$ or 1 we have that $(M\chi_\Omega)(M\chi_A) = -1$. Then, $\langle (M\chi_\Omega)(M\chi_A), \chi_K \rangle = -\mu(K)$. Next, we have the following.

$$\langle M\chi_\Omega M\chi_A, 1 \rangle = \langle M\chi_\Omega M\chi_A, \chi_K \rangle + \langle M\chi_\Omega M\chi_A, \chi_{\Omega \setminus K} \rangle$$

$$\leq -\mu(K) + ||M\chi_\Omega M\chi_A||_{L^1(\Omega,\mathbb{R})} ||\chi_{\Omega \setminus K}||_{L^\infty(\Omega,\mathbb{R})}$$

$$= \mu(A) - \mu(K)$$

However, we also have that $\langle M\chi_\Omega M\chi_A, 1 \rangle = \langle M\chi_\Omega, M\chi_A \rangle = \langle \chi_\Omega, \chi_A \rangle = \mu(A)$. Therefore, since $\mu(A) \leq \mu(A) - \mu(K)$, we have that $\mu(K) = 0$, and that $M\chi_\Omega M\chi_A \geq 0$ a.e.

Then, $M\chi_\Omega Mf \geq 0$ for all simple functions $f \geq 0$, and thus by taking limits we see that $M\chi_\Omega Mf \geq 0$ for all $f \in L^1(\Omega,\mathbb{R})$ which are $\geq 0$.

Finally, we have shown that $(M\chi_\Omega)M \in L(\Omega)$ and $M\chi_\Omega = \pm 1$ almost everywhere.

**Corollary.** Any $N \in G(\Omega)$ is thus also in $U(L^p(\Omega,\mathbb{R}))$ for all other $1 \leq p < \infty$, since $Nf = (N\chi_\Omega)f \circ \tau^{-1}$ for some $\tau \in \sigma(\Omega)$ so that $||Nf||_{L^p(\Omega,\mathbb{R})}^p = \int_\Omega |(N\chi_\Omega)f \circ \tau^{-1}|^p d\mu = \int_\Omega |f \circ \tau^{-1}|^p d\mu = ||f||_{L^p(\Omega)}^p$.

Therefore, with $U(L^p(\Omega,\mathbb{R}))$ the group of $|| \cdot ||_{L^p(\Omega)}$ preserving linear transformations (with the operation

being composition), we have that $\cap_{p \in [1,\infty]} U(L^p(\Omega, \mathbb{R})) = U(L^1(\Omega, \mathbb{R})) \cap U(L^2(\Omega, \mathbb{R})) \cap U(L^\infty(\Omega, \mathbb{R}))$.

**Proposition 20** *The homomorphism $\zeta : \sigma(\Omega) \to L(\Omega)$ given by $\zeta(\tau)f = f \circ \tau^{-1}$ is a homeomorphism when $L(\Omega)$ is given the strong/pointwise topology and $\sigma(\Omega)$ is given the $||\cdot||_{L^p(\Omega)}$ topology.*

**Proof.** Take any sequence $\{\gamma_n\} \subset \sigma(\Omega)$ convergent in $||\cdot||_{L^p(\Omega)}$ to some $\gamma \in \sigma(\Omega)$. Now, take any $f \in L^2(\Omega)$. Because $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$ and $(\sigma(\Omega), ||\cdot||_{L^p(\Omega)})$ is a topological group we see that $||\gamma_n^{-1} - \gamma^{-1}||_{L^p(\Omega)} \to 0$ as well so that by the previous proposition we have that $||f \circ \gamma_n^{-1} - f \circ \gamma||_{L^2(\Omega)} \to 0$ so that $\zeta(\gamma_n)f \to \zeta(\gamma)f$ and thus $\zeta(\gamma_n) \to \zeta(\gamma)$ in the strong/pointwise topology on $U(L^2(\Omega))$.

Conversely, assume that $\zeta(\gamma_n) \to \zeta(\gamma)$ in the strong/pointwise topology on $L(\Omega)$. We see that $||\operatorname{Id} \circ \gamma_n^{-1} - \operatorname{Id} \circ \gamma^{-1}||_{L^2(\Omega)} \to 0$ so that $||\gamma_n - \gamma||_{L^2(\Omega)} \to 0$ as well since $\sigma(\Omega)$ is a topological group. Then, $||\gamma_n - \gamma||_{L^1(\Omega)} = \int_\Omega |\gamma_n - \gamma| d\mu \leq ||1||_{L^2(\Omega)}||\gamma_n - \gamma||_{L^2(\Omega)} \to 0$, so that $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$ for all $p \geq 1$. Finally, this implies that $||\gamma_n - \gamma||_{L^p(\Omega)} \to 0$ as well for all $p \geq 1$, so that $\zeta(\gamma_n)$ converges if and only if $\gamma_n$ converges in $||\cdot||_{L^p(\Omega)}$ so that $\sigma(\Omega)$ is homeomorphic to $\zeta(\sigma(\Omega)) = L(\Omega)$.

**Proposition 21** *The center $Z$ of $\sigma(\Omega)$ is precisely* Id.

**Proof.** Assume $\tau \in Z$, and assume for a contradiction that for some $A \subseteq S(\tau)$ we have that $g_\tau(A) \neq A$ and $\mu(A) > 0$. Then, there must exist some open $C \subseteq A$ such that $g_\tau(C) \cap C = \emptyset$ and $\mu(C) > 0$.
Now, there must be some $B(y, r) \subseteq C$ so that we may take $C = B(y, r)$. Because $g_\tau$ is continuous on $C$ we have that $g_\tau(B(y, r)) = B(g_\tau(y), r)$. We can take $r > 0$ small enough so that $B(g_\tau(y), r) \cap B(y, r) = \emptyset$.
Let $B(z, r) \subseteq S(\tau)$ be another set disjoint from both $B(y, r)$ and $B(g_\tau(y), r)$, and let $\gamma$ be the permutation with $\gamma(x) = x + z - g_\tau(y)$ for $x \in B(g_\tau(y), r)$, $\gamma(x) = x + g_\tau(y) - z$ for $x \in B(z, r)$, and $\gamma(x) = x$ otherwise. We see that $g_\tau(\gamma(B(y, r))) = g_\tau(B(y, r)) = B(g_\tau(y), r)$ and $\gamma(g_\tau(B(y, r))) = \gamma(B(g_\tau(y), r)) = B(z, r)$. Therefore, we see that $g_\tau \circ \gamma \neq \gamma \circ g_\tau$, a contradiction to the fact that $g_\tau \in Z$ (which must be true because $\tau \in Z$).
Therefore, $g_\tau(A) = A$ for all $A \subseteq \Omega$ with $\mu(A) \geq 0$ so that $\tau$ is equivalent a.e. to Id.

# 6    Future Directions

In some sense, functions in $\sigma(\Omega)$ are closely analogous to permutations $p \in S_n$. For example, $\sum_{m=1}^n f(m) = \sum_{m=1}^n f(p(m))$ is analogous to $\int_\Omega f d\mu = \int_\Omega f \circ \tau d\mu$. In addition, $\{1, \dots, n\}$ can be partitioned into intervals $[a, b]$ such that $p$ has slope $\pm 1$, similar to how $\Omega$ can be partitioned into sets so that $\tau$ has a unitary slope. Moreover, the problem originally arose when trying to generalize the finite-dimensional determinant to an infinite-dimensional one, with matrices being functions $A : \Omega^2 \to \mathbb{C}$ instead of $A : \{1, \dots, n\}^2 \to \mathbb{C}$. Symbolically, the generalization using the Leibniz formula for the determinant is straightforward.

$$\sum_{p \in S_n} \operatorname{sgn}(p) \prod_i A_{ip(i)}$$

$$\int_{\sigma(\Omega)} \operatorname{sgn}(\tau) \exp\left(\int_\Omega \ln(A(x, \tau(x))) d\mu\right) d\mu_{\mathbb{D}}$$

However, proving properties about the determinant (even $\det(AB) = \det(A)\det(B)$ or some similar version) has proven difficult. Heuristically, the proof that $\det(AB) = \det(A)\det(B)$ using the Leibniz determinant involves a sum over $x \in \{1, \dots, n\}^n$. The counting measure on $\{1, \dots, n\}^n$ reduces to the counting measure on $S_n$ when each element of $S_n$ is identified with the corresponding element of $\{1, \dots, n\}^n$, and so for the infinite case we would want a measure on $\Omega^\Omega$ (or some similarly large set) which reduces to $\mu_{\mathbb{D}}$ on $\sigma(\Omega)$.

The natural log in the product integral also introduces issues stemming from the complex logarithm being multivalued. Using alternate product integrals, like the Volterra integral, has not yielded much more.

Care must be taken to define sgn appropriately as well. Attempts to define sgn have involved defining

sgn as a homomorphism $\sigma(\Omega) \to GL(n, \mathbb{R})$. Fathi (1978) showed that $\sigma(\Omega)$ is a simple (and thus perfect) group, so that Im(sgn) cannot be abelian.

Further developments may prove the same results for a larger class of $\Omega$ including the finite sets $\{1, \ldots, n\}$ so that the analogy between $\sigma(\Omega)$ and the $S_n$ can be made more rigorous. These may also allow a generalization of Leibniz' determinant formula to infinite-dimensional 'matrices' (functions $\Omega^2 \to \mathbb{C}$).

# References

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