1 Proof

Given an integer n and unique $x_0, ..., x_n \in \mathbf{R}$
Let $P(x) = (x - x_0)(x - x_1)...(x - x_n)$ and now with partial fraction decomposition,

$\frac{1}{P(x)} = \sum_{j=0}^{n} \frac{C_j}{x+x_j}$ (1)

where $C_j = \frac{1}{(x_0-x_j)(x_1-x_j)...(x_{j-1}-x_j)(x_{j+1}-x_j)...(x_n-x_j)}$

Now, using this, you can get that:

$\sum_{m=1}^{\infty} \frac{1}{P(m)} = \sum_{m=1}^{\infty} \sum_{j=0}^{n} \frac{C_j}{m+x_j}$ (2)

$= \sum_{j=0}^{n} \sum_{m=1}^{\infty} \frac{C_j}{m+x_j} + \frac{C_j}{m} - \frac{C_j}{m}$

$= \sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \frac{1}{m} - \sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \frac{1}{m} - \frac{1}{m+x_j}$

$= \left(\sum_{j=0}^{n} C_j\right) \left(\sum_{m=1}^{\infty} \frac{1}{m}\right) - \sum_{j=0}^{n} C_j H_{x_j}$

where H_x is the xth harmonic number. The separation of sums is the step we're unsure about. By taking equation 1 and multiplying both sides by P(x) and then comparing coefficients of x^n, it can be seen that the first sum in the product goes to 0. This step is analogous to algebraically cancelling it out, which can be seen by doing this process for a specific n. I'm not sure how to prove this is valid though.

$= -\sum_{j=0}^{n} C_j H_{x_j} = -\sum_{j=0}^{n} C_j \int_0^1 \frac{1-t^{x_j}}{1-t} dt$

$= -\int_0^1 \frac{\left(\sum_{j=0}^{n} C_j\right) - \sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$

$= \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$

Assuming all other steps are valid, taking the limit as $(x_0, ..., x_n) \to (0...0)$ should yield $\zeta(n+1)$ from the LHS of equation 2, which converges with a P-series test. Now, by parametrizing x_j into $x_j = jx$ for some x. Using this:

$C_j = \frac{1}{(0x-jx)(1x-jx)...((j-1)x-jx)((j+1)x-jx)...(nx-j)}$

$= \frac{(-1)^j}{j!(n-j)!x^n} = \frac{(-1)^j}{n!x^n}\binom{n}{j}$

$\zeta(n+1) = \lim_{(x_1 \ldots x_n)\to(0,\ldots,0)} \int_0^1 \frac{\sum_{j=0}^n C_j t^{x_j}}{1-t} dt$

$= \lim_{x\to 0} \int_0^1 \frac{\sum_{j=0}^n C_j t^{jx}}{1-t} dt$

$= \lim_{x\to 0} \frac{1}{n!x^n} \int_0^1 \frac{\sum_{j=0}^n \binom{n}{j}(-t^x)^j}{1-t} dt$

$= \lim_{x\to 0} \frac{1}{n!x^n} \int_0^1 \frac{(1-t^x)^n}{1-t} dt$

Now, using repeated application of L'Hopital's:

$= \lim_{x\to 0} \frac{1}{n!^2} \int_0^1 \frac{ln(t)^n \sum_{j=0}^n \binom{n}{j} j^n (-t^x)^j}{1-t} dt$

$= \frac{1}{n!^2} \int_0^1 \frac{ln(t)^n \sum_{j=0}^n \binom{n}{j} j^n (-1)^j}{1-t} dt$

$= \frac{(-1)^n}{n!} \int_0^1 \frac{ln(t)^n}{1-t} dt = \zeta(n+1)$

I believe this can be verified for each n by using the infinite series for $(1-x)^{-1}$ and a lot of integration by parts. It seems to work for non-integer values of n by numerically evaluating, but I don't think this proves that too.

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