

Rigidity and Compactness for Almost Everywhere Invertible Measure Preserving Maps on Open Bounded Subsets of \mathbb{R}^n

Graham Bertele

1 Basic Properties

Definition 1 Take $\Omega \subset \mathbb{R}^n$ to be open and bounded with $0 < \mu(\Omega) < \infty$.

Definition 2 The set of all invertible a.e. measure preserving maps (τ such that $\mu(X) = \mu(\tau(X))$) on a set Ω is denoted $\sigma(\Omega)$.

Proposition 1 For $\tau, \gamma \in \sigma(\Omega)$ we have that $\tau \circ \gamma \in \sigma(\Omega)$.

Proposition 2 For all $\tau \in \sigma(\Omega)$, there exists some function g such that $\tau \circ g = \text{Id}$ almost everywhere. In addition, $g \in \sigma(\Omega)$ as well. g is usually denoted as τ^{-1} even though the actual inverse is not explicitly defined and need not exist.

2 Rigidity and Form

Proposition 3 For all $\tau \in \sigma(\Omega)$ and any set $K \subset \Omega$, we have that $H^d(K) = H^d(\tau(K))$ for all $1 \leq d \leq n$.

Proof. We have the following.

$$H_\delta^d(K) = \inf \left\{ \sum_i (\text{diam } U_i)^d : K \subseteq \cup_i U_i \wedge \text{diam } U_i < \delta \right\}$$

Each U_i can be taken without loss of generality to be a countable union of balls so that their union is then again a countable union of balls. Therefore, we can assume without loss of generality that each U_i is just a ball.

$$\begin{aligned} &= \inf \left\{ \sum_i (\text{diam } B(x_i, r_i))^d : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\} \\ &= \inf \left\{ \sum_i (C_n \mu(B(x_i, r_i)))^{d/n} : K \subseteq \cup_i B(x_i, r_i) \wedge 2r_i < \delta \right\} \\ &= \inf \left\{ \sum_i (C_n \mu(\tau(B(x_i, r_i))))^{d/n} : \tau(K) \subseteq \cup_i \tau(B(x_i, r_i)) \wedge 2r_i < \delta \right\} \end{aligned}$$

For any set $X \subset \Omega$ we have that $(C_n \mu(X))^{1/n} \leq \text{diam}(X)$ because if $\mu(B(x, \varepsilon)) = \mu(X)$, then $(C_n \mu(X))^{1/n} = (C_n \mu(B(x, \varepsilon)))^{1/n} = \text{diam } B(x, \varepsilon) \leq \text{diam } X$ as an n -sphere minimizes diameter for a given volume.

$$\begin{aligned} &\leq \inf \left\{ \sum_i (C_n \mu(V_i))^d : \tau(K) \subseteq \cup_i V_i \wedge (C_n \mu(V_i))^{1/n} < \delta \right\} \\ &\leq \inf \left\{ \sum_i (\text{diam } V_i)^d : \tau(K) \subseteq \cup_i V_i \wedge \text{diam}(V_i) < \delta \right\} \\ &= H_\delta^d(\tau(K)) \end{aligned}$$

Now, taking limits as $\delta \rightarrow 0$ we have that $H^d(K) \leq H^d(\tau(K))$.

Therefore, $H^d(\tau(K)) \leq H^d(\tau^{-1}(\tau(K))) = H^d(K)$ so that $H^d(\tau(K)) = H^d(K)$.

Proposition 4 *For all $\tau \in \sigma(\Omega)$, there exists a function g_τ continuous on an open set S with $\mu(S) = \mu(\Omega)$ such that $g_\tau = \tau$ almost everywhere.*

Proof. We have that for all n there exists some $S_n \subset \Omega$ such that τ is continuous when restricted to S_n and $\mu(\Omega \setminus S_n) < 1/n$. Now, with $S = \cup_{i=1}^{\infty} S_i$ we have that τ is continuous on the S -inherited subspace topology and that $\mu(S) = \mu(\Omega)$ so that S is dense in Ω .

Now, let $G = \overline{\{(x, \tau(x)) \mid x \in S\}}$.

For any $x \in \Omega \setminus S$, we have that there exists some sequence $\{x_n\} \subset S$ convergent to x . Now, $\tau(x_n)$ is bounded and thus has some subsequence convergent to some y . Now, $(x, y) \in G$. Thus, with $p_1(x, y) = x$, we have that $p_1(G) = \Omega$.

Next, let $Q = \{x \in \Omega \mid ((x, y) \in G \wedge (x, z) \in G) \implies y = z\}$. If $x \in S$ then if $(x, s_1) \in G$ and $(x, s_2) \in G$ we have that there must be some sequences $(x_{in}, \tau(x_{in})) \rightarrow (x, s_i)$ for $x_{in} \in S$. As $|x_{in} - x| \rightarrow 0$ we have that because τ is continuous in S that $|\tau(x_{in}) - \tau(x)| \rightarrow 0$, and thus that $|\tau(x_{1n}) - \tau(x_{2n})| \rightarrow 0$ so that because $|\tau(x_{in}) - s_i| \rightarrow 0$ we have that $|s_1 - s_2|$ is arbitrarily small and thus $s_1 = s_2$. Therefore, $x \in Q$ so that $S \subseteq Q$.

Finally, define $g_\tau : \Omega \rightarrow \Omega$ by $g_\tau(x) = y$ if $(x, y) \in G$, where y is chosen arbitrarily for $x \notin Q$.

We aim to show that g_τ is continuous on S . For $x \in S$ we have that for any $\varepsilon > 0$ there exists some δ such that $|x - y| < \delta$ for $y \in S$ implies that $|\tau(x) - \tau(y)| < \varepsilon$. Then, take any $y \in \Omega \setminus S$ with $|x - y| < \delta$.

If $y \notin Q$, then assume (y, z_1) and (y, z_2) are in G . Now, there are sequences $\{s_n\}, \{t_n\} \subset B(x, \delta) \cap S$ such that $s_n, t_n \rightarrow y$, $\tau(s_n) \rightarrow z_1$, and $\tau(t_n) \rightarrow z_2$. Then, we have that $|\tau(s_n) - \tau(x)| < \varepsilon$, and for any $\varepsilon_2 > 0$ we have that there exists some N such that $n > N \implies |z_1 - \tau(s_n)| < \varepsilon_2$. Then, $|z_1 - \tau(x)| < \varepsilon + \varepsilon_2$ so that $|z_1 - \tau(x)| \leq \varepsilon$. Similar logic shows that $|z_2 - \tau(x)| \leq \varepsilon$.

If $y \in Q$, then $(y, z) \in G$. We have that there must be some sequence $\tau(s_n) \rightarrow z$ for $\{s_n\} \subset S \cap B(x, \delta)$ so that $|z - \tau(x)| \leq |z - \tau(s_n)| + |\tau(s_n) - \tau(x)| \leq \varepsilon_2 + \varepsilon \rightarrow \varepsilon$. Therefore, $|z - \tau(x)| \leq \varepsilon$ as well.

Finally, regardless of the choice of value of $g_\tau(y)$ outside of Q , we have that g_τ is continuous at x .

Now, with S_2 the set of all x such that $g_\tau(x)$ is continuous at x , we have that S_2 is open and that $S \subseteq S_2$ so that $\mu(S_2) = \mu(\Omega)$. Therefore, S_2 is the desired set, and because $g_\tau(x) = \tau(x)$ for $x \in S$ we have that $g_\tau \equiv \tau$.

Definition 3 *For some $\tau \in \sigma(\Omega)$, we define $S(\tau)$ to be S_2 as above and g_τ to be g_τ as above.*

Proposition 5 *$S(\tau)$ can be partitioned into disjoint sets $\{A_\alpha\}_{\alpha \in I}$ such that $g_\tau|_{A_\alpha} = U_\alpha x + v_\alpha$ for U_α a unitary linear map and v_α a constant vector.*

Proof. We have that g_τ is continuous on an open set $S(\tau)$ with $\mu(S) = \mu(\Omega)$ so that $\partial S(\tau) = \Omega \setminus S(\tau)$. Now, for any $x \in S(\tau)$, there is some $B_1 = B(x, \varepsilon_1) \subset S$.

For any $y \in B_1$, let $\ell_1 = \{g_\tau(x) + a(g_\tau(y) - g_\tau(x)) \mid 0 \leq a \leq 1\}$. We have that $|g_\tau(x) - g_\tau(y)| = H^1(\ell_1) = H^1(g_\tau^{-1}(\ell_1))$. As $g_\tau^{-1}(\ell_1)$ is some continuous path between x and y , we have that $H^1(g_\tau^{-1}(\ell_1)) \geq |x - y|$. Therefore, $|g_\tau(x) - g_\tau(y)| \geq |x - y|$. Finally, we have that $|x - y| = |g_\tau^{-1}(g_\tau(x)) - g_\tau^{-1}(g_\tau(y))| \geq |g_\tau(x) - g_\tau(y)|$ so that $|x - y| = |g_\tau(x) - g_\tau(y)|$.

It is a known result that if $|g_\tau(x) - g_\tau(y)| = |x - y|$ for $g - \tau$ continuous that $g_\tau(x) = Ux + v$ for U a unitary map.

Choose some partitioning of $S(\tau)$ into balls such that $S = \cup_{\alpha \in I} B(x_\alpha, r_\alpha)$. Then, take $A_\alpha = B(x_\alpha, r_\alpha) \setminus (\cup_{i < \alpha} B(x_i, r_i))$ so that they are disjoint, $S = \cup_{\alpha \in I} A_\alpha$, and on each A_α we have that $g_\tau(x) = U_\alpha x + v_\alpha$.

Note. The partition I can be assumed to be countable because each of the A_α have been constructed to be pairwise disjoint.

Definition 4 We take $P_{S(\tau)}$ to be the crudest such partition of $S(\tau)$ such that the constructed partition above is a refinement of $P_{S(\tau)}$.

Proposition 6 For any a.e. invertible L^p function $f : \Omega \rightarrow \Omega$, we have that $f \in \sigma(\Omega)$ if and only if for almost all $x \in \Omega$ there exists some $\varepsilon > 0$ such that there are $U \in U(n)$ and $v \in \mathbb{R}^n$ such that $f(y) \equiv Uy + vf$ for $y \in B(x, \varepsilon)$.

Proof. We have proven the \implies direction already. Now, conversely, assume that for all $x \in S$ there exists some $\varepsilon > 0$ such that $f(y) \equiv Uy + vf$ or $y \in B(x, \varepsilon)$, where $\mu(S) = \mu(\Omega)$.

Now, for all $y \in f(S)$ there exists some $\varepsilon > 0$ such that $f^{-1}(z) \equiv U^T z - U^T v$ for all $z \in B(y, \varepsilon)$. Then, $(f^{-1})'(y)$ is unitary.

$$\begin{aligned}\mu(f(A)) &= \int_{\Omega} 1_A \circ f^{-1} d\mu \\ &= \int_S 1_A \circ f^{-1} d\mu \\ &= \int_{f(S)} 1_A |\det(f^{-1})'| d\mu \\ &= \int_{\Omega} 1_A d\mu \\ &= \mu(A)\end{aligned}$$

And therefore, $f \in \sigma(\Omega)$.

Proposition 7 For $\tau, \gamma \in \sigma(\Omega)$, we have that $S(\tau \circ \gamma) \subseteq \Omega \setminus (S(\gamma) \Delta g_{\gamma}^{-1}(S(\tau)))$.

Proof. Assume $x \in S(\tau \circ \gamma)$. If g_{γ} is continuous at x we have that g_{τ} is continuous at $g_{\gamma}(x)$ so $x \in S(\gamma) \cap g_{\gamma}^{-1}(S(\tau))$.

Otherwise, if $x \notin S(\gamma)$ then if $g_{\gamma}(x) \in S(\tau)$ we have that $g_{\tau}(g_{\gamma}(x))$ cannot be continuous, a contradiction. Therefore, $x \in (\Omega \setminus S(\gamma)) \cap (\Omega \setminus g_{\gamma}^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \cup g_{\gamma}^{-1}(S(\tau)))$.

Finally, $S(\tau \circ \gamma) \subseteq (\Omega \setminus (S(\gamma) \cup g_{\gamma}^{-1}(S(\tau)))) \cup (S(\gamma) \cap g_{\gamma}^{-1}(S(\tau))) = \Omega \setminus (S(\gamma) \Delta g_{\gamma}^{-1}(S(\tau)))$.

Corollary. $S(\gamma) \cap g_{\gamma}^{-1}(S(\tau)) \subseteq S(\tau \circ \gamma)$

3 Topological Properties

Definition 5 We define for measurable $S \subseteq \Omega$ the cylindrical set $V_S = \{\tau \in \sigma(U) \mid S \subseteq S(\tau)\}$.

Proposition 8 If $\mu(\partial\Omega) = 0$, then for sets $S \subseteq \Omega$ with $\mu(S) = \mu(\Omega)$, if $\mu(\partial(\Omega \setminus S)) = 0$, then V_S is relatively compact under $\|\cdot\|_{L^p(\Omega)}$ for all $1 \leq p < \infty$.

Proof. For this proof we extend $\tau \in \sigma(\Omega)$ to be defined $\mathbb{R}^n \rightarrow \Omega \cup \{0\}$ by taking $\tau(x) = 0$ for $x \notin \Omega$.

We use the Kolmogorov-Reisz compactness theorem for the proof. Take $T_h f(x) = f(x + h)$. As Ω is bounded we only need to show that $\|T_h \tau - \tau\|_{L^p(\mathbb{R}^n)} \rightarrow 0$ uniformly on V_S as $|h| \rightarrow 0$ in \mathbb{R}^n .

For $h \in \mathbb{R}^n$ define $A_h = \{x \in S \mid B(x, |h|) \not\subseteq S\}$ and define $B_h = \{x \notin \Omega \mid B(x, |h|) \cap \Omega \neq \emptyset\}$.

$$\begin{aligned}\|T_h \tau - \tau\|_{L^p(\mathbb{R}^n)}^p &= \int_{\mathbb{R}^n} |T_h \tau - \tau|^p d\mu \\ &= \int_{\mathbb{R}^n \setminus \Omega} |T_h \tau|^p d\mu + \int_{\Omega \setminus A_{|h|}} |T_h \tau - \tau|^p d\mu + \int_{A_{|h|}} |T_h \tau - \tau|^p d\mu\end{aligned}$$

Now, we can bound each of those 3 integrals.

$$\begin{aligned}
I_1 &= \int_{\Omega \setminus A_{|h|}} |T_h \tau - \tau|^p d\mu = \sum_m \int_{(P_S)_m \setminus A_{|h|}} |T_h \tau - \tau|^p d\mu \\
&= \sum_m \int_{(P_S)_m \setminus A_{|h|}} |(U_m(x+h) + v_m) - (U_m x + v_m)|^p d\mu \\
&\leq \sum_m \int_{(P_S)_m} |h|^p d\mu \\
&= |h|^p \mu(S) \rightarrow 0
\end{aligned}$$

We do similarly for the third integral.

$$\begin{aligned}
I_2 &= \int_{A_{|h|}} |T_h \tau - \tau|^p d\mu \leq \left(2 \sup_{u \in \Omega} |u| \right)^p \mu(A_{|h|}) \\
&\quad \lim_{|h| \rightarrow 0} \mu(A_{|h|}) \leq \mu(\cap_{n=1}^{\infty} A_{1/n})
\end{aligned}$$

If $x \in \cap_{n=1}^{\infty} A_{1/n}$, then for all $\varepsilon > 0$ we have that there is some $y \in B(x, \varepsilon)$ which also satisfies $y \notin S$. Then, $x \in \partial(\Omega \setminus S)$.

$$\implies I_2 \leq \left(2 \sup_{u \in \Omega} |u| \right)^p \mu(\partial(\Omega \setminus S)) = 0$$

$$\begin{aligned}
I_3 &= \int_{\mathbb{R}^n \setminus \Omega} |T_h \tau|^p d\mu = \int_{B_{|h|}} |T_h \tau|^p d\mu \leq \mu(B_{|h|}) \left(\sup_{u \in \Omega} |u| \right)^p \\
&\quad \lim_{|h| \rightarrow 0} \mu(B_{|h|}) \leq \mu(\cap_{n=1}^{\infty} B_{1/n}) \\
&= \mu(\partial\Omega \setminus \Omega) = \mu(\partial\Omega) = 0
\end{aligned}$$

Therefore, if $\mu(\partial S) = 0$ then $\|T_h \tau - \tau\|_{L^p(\mathbb{R}^n)} \rightarrow 0$ uniformly on V_S as $|h| \rightarrow 0$ so that V_S is relatively compact under $\|\cdot\|_{L^p(\Omega)}$.

Corollary. For $X \subseteq \sigma(\Omega)$, let $A = \cup_{\tau \in X} (\Omega \setminus S(\tau))$. If $\mu(\partial A) = 0$ then $X \subseteq V_A$ is relatively compact under $\|\cdot\|_{L^p(\Omega)}$.

Proposition 9 For $1 \leq k, p \leq \infty$ and $\tau \in \sigma(\Omega)$, we have that $\|\tau\|_{W^{k,p}(\Omega, \mathbb{R}^n)} = \|\text{Id}\|_{W^{k,p}(\Omega, \mathbb{R}^n)}$.

Proof. We use $\text{Tr}(A^T A)$ as the norm for matrices. We assume p is finite because the $p = \infty$ case follows by taking from the limit.

$$\begin{aligned}
\|\tau\|_{W^{k,p}(S(\tau))} &= \|g_\tau\|_{W^{k,p}(S(\tau))} \\
&= \|g_\tau\|_{W^{1,p}(S(\tau))} \\
&= \|\nabla g_\tau\|_{L^p(S(\tau))} + \|g_\tau\|_{L^p(S(\tau))} \\
\int_{S(\tau)} |g_\tau|^p d\mu &= \int_{S(\tau)} |g_\tau \circ g_\tau^{-1}|^p d\mu \\
&= \int_{S(\tau)} |\text{Id}|^p d\mu \\
\implies \|g_\tau\|_{L^p(S(\tau))} &= \|\text{Id}\|_{L^p(S(\tau))} = \|\text{Id}\|_{L^p(\Omega)} \\
\int_{S(\tau)} \text{Tr}((\nabla g_\tau)^T (\nabla g_\tau))^p d\mu &= \sum_m \int_{(P_{S(\tau)})_m} \text{Tr}(U_m^T U_m)^p d\mu \\
&= n^p \sum_m \mu((P_{S(\tau)})_m) \\
&= n^p \mu(\Omega) \\
&= \int_{\Omega} \text{Tr}((\nabla \text{Id})^T (\nabla \text{Id}))^p d\mu \\
\implies \|\nabla g_\tau\|_{L^p(S(\tau))} &= \|\nabla \text{Id}\|_{L^p(S(\tau))} \\
\implies \|g_\tau\|_{W^{k,p}(S(\tau), \mathbb{R}^n)} &= \|\nabla \text{Id}\|_{L^p(\Omega)} + \|\text{Id}\|_{L^p(\Omega)} \\
&= \|\text{Id}\|_{W^{k,p}(\Omega)}
\end{aligned}$$

Corollary. $V_{S(\tau)} \subset \partial B_{W^{k,p}(\Omega, \mathbb{R}^n)}(0, \|\text{Id}\|_{W^{k,p}(\Omega, \mathbb{R}^n)})$.

Definition 6 For $1 \leq p < \infty$, define the topology \mathcal{T}_p on $\sigma(\Omega)$ by $X \in \mathcal{T}_p$ if and only if $X \cap V_S$ is open under the $L^p(\Omega)$ subspace topology on V_S for all open $S \subseteq \Omega$ with $\mu(S) = \mu(\Omega)$.

Proposition 10 If $X \in \mathcal{T}_p$ then $X \circ \tau \in \mathcal{T}_p$ as well for all $\tau \in \sigma(\Omega)$.

Proof. Take any open $S \subseteq \Omega$ with $\mu(S) = \mu(\Omega)$, and consider $V_S \circ \tau$. We have that if $\lambda \in V_S$ then $S(\lambda \circ \tau^{-1}) \supseteq S(\tau^{-1}) \cap g_\tau(S(\lambda)) \supseteq S(\tau^{-1}) \cap g_\tau(S)$ so that $\lambda \circ \tau \in S(\tau) \cap g_\tau^{-1}(S)$. Then, there must be some $\varepsilon > 0$ such that $B_p(\lambda \circ \tau^{-1}, \varepsilon) \cap V_{S(\tau) \cap g_\tau^{-1}(S)} \subseteq X \cap V_{S(\tau) \cap g_\tau^{-1}(S)}$. Now, composing by τ on both sides, we get the following.

$$\begin{aligned}
(B_p(\lambda \circ \tau^{-1}, \varepsilon) \circ \tau) \cap (V_{S(\tau) \cap g_\tau^{-1}(S)} \circ \tau) &\subseteq (X \circ \tau) \cap (V_{S(\tau) \cap g_\tau^{-1}(S)} \circ \tau) \\
\implies B_p(\lambda, \varepsilon) \cap (V_{S(\tau) \cap g_\tau^{-1}(S)} \circ \tau) &\subseteq (X \circ \tau) \cap (V_{S(\tau) \cap g_\tau^{-1}(S)} \circ \tau) \\
V_S \circ \tau^{-1} &\subseteq V_{S(\tau) \cap g_\tau^{-1}(S)} \implies V_S \subseteq V_{S(\tau) \cap g_\tau^{-1}(S)} \\
\implies B_p(\lambda, \varepsilon) \cap V_S &\subseteq (X \circ \tau) \cap V_S
\end{aligned}$$

And, therefore, there is a neighborhood of λ in $(V_S, \|\cdot\|_{L^p(\Omega)})$ contained in $X \circ \tau$ so that $X \circ \tau \in \mathcal{T}_p$ as well.

Next Steps: Show $(\sigma(\Omega), \mathcal{T}_p)$ is a topological group, then show it's Locally Compact, then quotient by equivalence classes and define a haar measure on it