A Technique of Derivation of The Zeta-Gamma/Bose Integral

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Abstract

We derive the Zeta-Gamma Integral $\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t-1}$ by considering the parametrized limit of an integral representation of the sum $\sum_{n=1}^\infty \frac{1}{(n+x_0)...(n+x_s)}$. This approach allows for the derivation of other analogous identities, including the analogous integrals for the Hurwitz zeta function, Harmonic Numbers

Lemma

Given some $n \in \mathbb{N}$, $j = 0, 1, ..., n$, and $x \in \mathbb{C}$

$$C_j = \frac{1}{(0x - jx)(1x - jx)...((j-1)x - jx)((j+1)x - jx)...(nx - j)}$$

$$= \frac{(-1)^j}{n!x^n}\binom{n}{j}$$

Let $j \in \{0, ..., n\}$ and consider the terms $(0x - jx)(1x - jx)...((j-1)x - jx)$. By factoring out $(-1)^j x^j$ and reversing the order of the product, this is equal to $(-1)^j x^j (1)(2)...(j-1)(j) = (-1)^j x^j j!$. C_j can be represented as a much simpler fraction.

$$C_j = \frac{(-1)^j}{j!(n-j)!x^n} = \frac{(-1)^j}{n!x^n}\binom{n}{j}$$

Derivation of Integral Identity

Given a natural number n and unique real $x_0, ..., x_n$ which aren't negative integers, Let $P(x) = (x - x_0)(x - x_1)...(x - x_n)$ and now with partial fraction decomposition, the following is true.

$$\frac{1}{P(x)} = \sum_{j=0}^{n} \frac{C_j}{x + x_j} \tag{1}$$

where $C_j = \frac{1}{(x_0 - x_j)(x_1 - x_j)...(x_{j-1} - x_j)(x_{j+1} - x_j)...(x_n - x_j)}$ for $j = 0, 1, ..., n$. So,

$$\sum_{m=1}^{\infty} \frac{1}{P(m)} = \sum_{m=1}^{\infty} \sum_{j=0}^{n} \frac{C_j}{m + x_j}$$

$$= \sum_{m=1}^{\infty} \sum_{j=0}^{n} \left(\frac{C_j}{m + x_j} + \frac{C_j}{m} - \frac{C_j}{m} \right)$$

$$= \sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{j=0}^{n} C_j \right) - \sum_{j=0}^{n} C_j \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m + x_j} \right)$$

$$= -\sum_{j=0}^{n} C_j H_{x_j} = -\sum_{j=0}^{n} C_j \int_0^1 \frac{1 - t^{x_j}}{1 - t} dt$$

$$= -\int_0^1 \frac{\left(\sum_{j=0}^{n} C_j \right) - \sum_{j=0}^{n} C_j t^{x_j}}{1 - t} dt$$

So, we have that $\sum_{m=1}^{\infty} \frac{1}{P(m)} = \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1-t} dt$. where H_x is the xth harmonic number. Taking the limit as $(x_0, ..., x_n) \to (0...0)$ converges to $\zeta(n+1)$, as shown by a P-series test of degree $n + 1$ and with the definition of $\zeta(n + 1)$. This means that the integral converges to $\zeta(n + 1)$, too. By parametrizing x_j into $x_j = jv$ for some v, the limit can be simplified significantly.

$$\begin{aligned}
\zeta(n+1) \quad &= \lim_{(x_1 ... x_n) \to (0,...,0)} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{x_j}}{1 - t} dt \\
&= \lim_{v \to 0} \int_0^1 \frac{\sum_{j=0}^{n} C_j t^{jv}}{1 - t} dt \\
&= \lim_{v \to 0} \frac{1}{n! v^n} \int_0^1 \frac{\sum_{j=0}^{n} \binom{n}{j} (-t^v)^j}{1 - t} dt \text{ (with lemma 1)} \\
&= \lim_{v \to 0} \frac{1}{n! v^n} \int_0^1 \frac{(1 - t^v)^n}{1 - t} dt
\end{aligned}$$

The v^n and $(1-t^v)^n$ are both of order n, so inductively using repeated application of L'Hôpital's rule

$$\begin{aligned}\zeta(n+1) &= \lim_{v\to 0}\frac{1}{n!^2}\int_0^1 \frac{\ln(t)^n \sum_{j=0}^{n}\binom{n}{j} j^n(-t^v)^j}{1-t}dt \\ &= \frac{1}{n!^2}\int_0^1 \frac{\ln(t)^n \sum_{j=0}^{n}\binom{n}{j} j^n(-1)^j}{1-t}dt \\ &= \frac{(-1)^n}{n!}\int_0^1 \frac{\ln(t)^n}{1-t}dt\end{aligned}$$

The simplification between the last two steps uses the result that $\sum_{j=0}^{n}\binom{n}{j} j^n(-1)^j = (-1)^n n!$. The integral is equivalent to the identity given by the Bose integral shown below by applying the substitution $t = e^{-z}$ to the above integral.

$$\int_0^\infty \frac{t^x}{e^t-1}dt = \Gamma(x+1)\zeta(x+1)$$

Generalizations

Using the parametrization $x_i = ix + a$, (2) can be derived as well through the same reasoning.

$$\Gamma(n)\zeta(n,a) = (-1)^{n-1}\int_0^1 \frac{t^{a-1}\ln(t)^{n-1}}{1-t}dt \tag{2}$$

$$\zeta(n,x) = \sum_{m=0}^{\infty}\frac{1}{(m+x)^n}$$

where $\zeta(n,a)$ is the Hurwitz Zeta function. Because the generalized harmonic number $H_{x,n} = \zeta(n,1) - \zeta(n,x+1)$, an identity with $H_{x,n}$ can be derived too. The generalized harmonic numbers are defined in (3) for $x \in \mathbb{N}$.

$$\Gamma(n)H_{x,n} = (-1)^{n-1}\int_0^1 \frac{(1-t^x)\ln(t)^{n-1}}{1-t}dt$$

$$H_{x,n} = \sum_{m=1}^{x}\frac{1}{m^n} \tag{3}$$

To get the integrals more similar to the Bose Integral in form and function class, the substitution $t = e^{-z}$ gives (4) and (5).

$$\Gamma(n)H_{x,n} = \int_0^\infty \frac{1-e^{-zx}}{e^z-1}z^{n-1}dz \tag{4}$$

$$\Gamma(n)\zeta(n,x) = \int_0^\infty \frac{e^{-z(x-1)}}{e^z-1}z^{n-1}dz \tag{5}$$