1 The categories

- **Pos** is the category of posets and monotone maps.
- **RelPos** is the category of posets and monotone relations.
- **PosInv** is the category of posets equipped with an antitone involution and involution preserving maps.
- **RelPosInv** is the category of posets equipped with an antitone involution and monotone relations.

2 Pos

- **Pos** is cartesian closed.
- There is a functor $^{op}\colon \mathbf{Pos} \to \mathbf{Pos}$ that takes a poset to an opposite poset.
- There is a monad $D\colon \mathbf{Pos} \to \mathbf{Pos}$ that takes every poset to its poset of downsets ordered by inclusion.
- $\mathbf{Pos}^D \simeq \mathbf{Sup}$ -- the Eilenberg-Moore category.

3 RelPos

- **RelPos** is a compact category.
- The dual object is the dual poset, which is in this context denoted by *.
- **RelPos** is **Pos**-enriched.
- $\mathbf{RelPos} \simeq \mathbf{Pos}_D$ -- the Kleisli category.

4 PosInv

- The objects of **PosInv** can be represented as isomorphisms $'\colon P \to P^{op}$.
- The morphisms are then commutative squares.
- **PosInv** is cartesian closed.[1]
- The forgetful functor $U\colon \mathbf{Pos} \to \mathbf{PosInv}$ is both left and right adjoint.
- The left adjoint to U is $P \mapsto D(P) \oplus (D(P))^{op}$, where $\oplus$ is the coproduct in **Pos**.
- The right adjoint to U is $P \mapsto D(P) \times (D(P))^{op}$.

[1] What was the exponential object? I forgot.

5 RelPosInv

- The objects may be represented as **RelPos**-isomorphisms $P \to P^*$, morphisms remain the same.
- The contravariance of * then gives us a dagger: if $': P \to P^*$, $': Q \to Q^*$ are objects of **RelPosInv** and $f : P \to Q$ is a morphism (a monotone relation), then $f^\dagger : Q \to P$ is just $(')^{-1} \circ f^* \circ '$.
- This is a dagger-compact category.
- Effect algebras are certain dagger-Frobenius algebras in **RelPosInv**. [2].
- There is a chain of adjunctions **RelPos** $\leftrightarrow$ **Pos** $\leftrightarrow$ **PosInv**, inducing a monad T given by $T(P) = D(P) \times (D(P))^{op}$ on **PosInv**.
- $\mathbf{PosInv}_T \simeq \mathbf{RelPosInv}$
- $\mathbf{PosInv}^T \simeq \mathbf{Sup}$

[2] This was the original motivation to look at **RelPosInv**