# 1 The objects

An involution on a poset $P$ is an antitone selfmap $x \mapsto x'$ such that $x'' = x$. A poset with involution is called an *involutive poset*.

Let $P$ be an involutive poset.

- We write $D(P)$ for the set of all order ideals/downsets/lower sets of $P$.
- The orthogonality relation $\perp$ on $P$ is given by the rule

$$x \perp y \Leftrightarrow x \leq y'$$

  Note that $\perp$ is symmetric.
- For a subset $X$ of $P$, we write $X' = \{x' : x \in X\}$.
- For a subset $X$ of $P$ and $y \in P$, we write $y \leq X$ to express the fact that $y$ is a lower bound of $X$; similarly for upped bounds.
- The set of all lower bounds of $X$ is denoted by $X^{\downarrow}$, $X^{\uparrow}$ are all upper bounds of $X$.
- The *orthoclosure* of a subset $X$ of $P$ is the set

$$X^{\perp} = \{y \in P : (\forall x \in X) x \perp y\}.$$

  Note that $X^{\perp} = X'^{\downarrow}$. In particular, $X^{\perp}$ is a lower set of $P$.
- $I \mapsto I^{\perp\perp}$ is a closure operator on $D(P)$.
- A lower set $I$ of $P$ is said to be *orthoclosed* iff $I = I^{\perp\perp}$.

**Lemma 1.1.** *Let $P$ be an involutive poset. For every $I \in D(P)$ $I^{\uparrow} = (I^{\perp})'$*

*Proof.* $x \in I^{\uparrow}$ iff $x \geq I$ iff $x' \leq I'$ iff $x' \in (I')^{\downarrow} = I^{\perp}$ iff $x \in (I^{\perp})'$. □

# 2 The adjunction

The categories are

- **PosInv**: objects are involutive posets and morphisms are monotone maps preserving the involution.
- **SupInv**: objects are involutive suplattices and morphisms preserve all joins and involution.

There is an obviously defined forgetful functor $G\colon \mathbf{SupInv} \to \mathbf{PosInv}$. For an involutive poset $P$, we write $F(P)$ for the set of all orthoclosed lower sets of $P$. It is easy to see that $I \mapsto I^{\perp}$ is an involution on $F(P)$. Moreover, for every element $x$ of $P$, $(x^{\downarrow}) \perp = (x')^{\downarrow}$. Thus, there is a **PosInv** morphism $\downarrow\colon P \to GF(P)$ that takes every element $x$ to its principal downset

$$\downarrow(x) = \{y \in P : y \leq x\}$$

**Theorem 2.1.** *Let $A$ be an object of* **PosInv** *and $B$ an object of* **SupInv**. *For every morphism $f\colon A \to G(B)$ there if a unique $\widehat{f}\colon F(A) \to B$ such that*

$$\begin{tikzcd}
GF(A) \arrow[r, "G(\widehat{f})"] & G(B) \\
A \arrow[u, "\downarrow"] \arrow[ur, "f"'] &
\end{tikzcd}$$

*commutes.*

*Let us prove that $\widehat{f}$ is unique.*