Voltage lifts of graphs from a category theory viewpoint

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VOLTAGE LIFTS OF GRAPHS FROM A CATEGORY THEORY VIEWPOINT

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ABSTRACT. We prove that the notion of a voltage graph lift comes from an adjunction between the category of voltage graphs and the category of group labeled graphs.

1. Introduction

In this paper, a *graph* means a structure sometimes called a *symmetric multidigraph* – that means that it may have multiple darts with the same source and target, and the set of all darts of the graph is equipped with an involutive mapping λ that maps every dart to a dart with source and target swapped.

A *voltage graph* is a graph in which every dart is labeled with an element of a group in a way that respects the involutive symmetry λ, so that the label of a dart d is inverse to the label of $\lambda(d)$. Similarly, a *group labeled graph* has all vertices labeled with elements of a group.

In [8], Gross introduced the construction of a *derived graph of a voltage graph*. Nowadays, derived voltage graphs are called *(ordinary) voltage graph lifts* – this is the terminology we will use in the present paper. Let us mention in passing that in [9], voltage graphs were generalized to a more general notion of *permutation voltage graphs*, in which the darts are labelled with permutations.

After their discovery, voltage graph lifts were extensively investigated in many papers. Voltage graph lifts were applied for example in the research concerning the degree-diameter problem [3, 4], lifting graph automorphisms [15] and several other areas of graph theory.

In the present paper, we prove that there is an adjunction

$$\begin{tikzcd} \mathbf{Lab} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{Volt} \arrow[l, bend left, "R"] \end{tikzcd}$$

between the category **Volt** of voltage graphs and a category **Lab** of group labeled graphs. We prove that for every object G of **Volt**, the underlying graph of the voltage graph $LR(G)$ is isomorphic to the voltage graph lift of G.

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Categories

Definition

A *category* $\mathcal{C}$ is a

- collection of objects $\mathcal{C}_0$, collection of morphisms $\mathcal{C}_1$
- each morphism f has a domain object and a codomain object

$$f\colon A \to B \qquad A \xrightarrow{f} B$$

- for every object A there is an identity morphism $\mathrm{id}_A\colon A \to A$
- We may compose morphisms

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Categories

Eilenberg, Mac Lane 1945

Definition

- the composition is associative
- the id_A is neutral with respect to composition

Examples:

- **Set**, **Grp**, **Top**, ...
- Every monoid is a category with a single object.
- Every poset is a category (at most one morphism between objects)

Homsets

- For a category $\mathcal{C}$, we write $\mathcal{C}(A, B)$ for the set of all morphisms with domain A and codomain B:

$$\mathcal{C}(A, B) = \{f : A \to B\}$$

- Composition is then a mapping

$$\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$$

Functors

morphisms between categories

A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is

- $F(A) \in \mathcal{D}_0$ for each $A \in \mathcal{C}_0$
- $F(f)\colon F(A) \to F(B)$ for each $(f : A \to B) \in \mathcal{C}_1$
- preserving the composition and the identity morphisms

$$F(f \circ g) = F(f) \circ F(g) \qquad F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$$

Natural transformations

morphisms between functors

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\alpha\colon F \to G$ is

- For every object $X \in \mathcal{C}_0$
- a morphism $\alpha_X\colon F(X) \to G(X)$, $\alpha_X \in \mathcal{D}_1$
- such that for every $f\colon A \to B$ in $\mathcal{C}_1$,

$$G(f) \circ \alpha_A = \alpha_B \circ F(f)$$

$$A \xrightarrow{f} B \qquad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

Graphs

A *graph* is a quadruple $G = (V, D, s, t, \lambda)$, where

- D is the *set of darts of G*
- V is the *set of vertices of G*
- $s, t\colon D \to V$ are the *source and target maps*, respectively.
- $\lambda\colon D \to D$ is a mapping such that $\lambda \circ \lambda = \mathrm{id}_D$.
- $s \circ \lambda = t$.

The mapping λ is called the *dart-reversing involution* of G.

Graphs

All the data in a graph (V, D, s, t, λ) can be expressed graphically by a commutative diagram:

$$
\begin{tikzcd}
D \arrow[rr, bend left, "\mathrm{id}_D"] \arrow[dr, "\lambda"] \arrow[ddr, bend right, "s"'] & & D \arrow[ddl, bend left, "s"] \\
& D \arrow[ur, "\lambda"'] \arrow[d, "t"] & \\
& V \arrow[loop below, "\mathrm{id}_V"] &
\end{tikzcd}
\tag{1}
$$

Morphisms of graphs

A *morphism of graphs* $f\colon G \to H$ is a pair of mappings (f^V, f^D), where

- $f^V\colon V(G) \to V(H)$
- $f^D\colon D(G) \to D(H)$
- for every dart $d \in D(G)$

$$
\begin{aligned}
s(f^D(d)) &= f^V(s(d)) \\
t(f^D(d)) &= f^V(t(d)) \\
\lambda(f^D(d)) &= f^D(\lambda(d))
\end{aligned}
$$

Clearly, graphs equipped with morphisms form a category, denoted by **Graph**.

Graphs are functors to **Set**

This

$$
\begin{tikzcd}
D \arrow[rr, bend left, "\mathrm{id}_D"] \arrow[dr, "\lambda"] \arrow[ddr, bend right, "s"'] & & D \arrow[ddl, bend left, "s"] \\
& D \arrow[ur, "\lambda"] \arrow[d, "t"] & \\
& V \arrow[loop below, "\mathrm{id}_V"] &
\end{tikzcd}
$$

is a category **gph** with $\mathbf{gph}_0 = \{D, V\}$ and $\mathbf{gph}_1 = \{s, t, \lambda, \mathrm{id}_D, \mathrm{id}_V\}$.

A graph G is functor from **gph** to **Set**:

- $G(D)$ is the set of darts.
- $G(V)$ is the set of vertices.
- $G(s), G(t), G(\lambda), \ldots$ are then maps between those sets.
- The equations are valid, they are valid in **gph** and composition is preserved by the functor G.

Morphisms of graphs are natural transformations of functors

That means, for $G, H\colon \mathbf{gph} \to \mathbf{Set}$

- $f^D\colon G(D) \to H(D)$
- $f^V\colon G(V) \to H(V)$
- Naturality square for s (there are similar for t, λ)

$$D \xrightarrow{s} V \qquad \begin{array}{ccc} G(D) & \xrightarrow{G(s)} & G(V) \\ {\scriptstyle f^D}\downarrow & & \downarrow{\scriptstyle f^V} \\ H(D) & \xrightarrow[H(s)]{} & H(V) \end{array}$$

Graph is a topos

As a consequence of this, $\mathbf{Graph} = \mathbf{Set}^{\mathbf{gph}}$ is a very nice category:

- **Graph** all small limits and colimits.
- **Graph** is exponentially closed.
- **Graph** has a subobject classifier.

Voltage graphs

A *voltage graph* over a group Γ is a triple (G, Γ, α), where

- G is a graph
- Γ is a group
- $\alpha \colon D(G) \to \Gamma$ is a mapping such that

$$\alpha(\lambda(d)) = (\alpha(d))^{-1}$$

The mapping α is called a Γ-*voltage on* G.

Morphisms of voltage graphs

Γ is fixed

A morphism of voltage graphs over Γ

$$f\colon (G,\alpha) \to (G',\alpha')$$

is a morphism of graphs $f\colon G \to G'$ that preserves the voltage

- for all $d \in D(G)$, $\alpha(d) = \alpha'(f^D(d))$.

$$\begin{tikzcd}
D(G) \arrow[rr, "f^D"] \arrow[dr, "\alpha"'] & & D(G') \arrow[dl, "\alpha'"] \\
& \Gamma &
\end{tikzcd}$$

- The category of voltage graphs over Γ is denoted by $\mathbf{Volt}_\Gamma$.

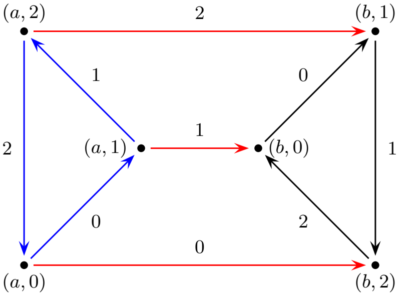
Voltage lift

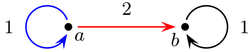
Definition

[Gross, 1974] Let (G, Γ, α) be a voltage graph. There is a *lift of G*, denoted by $(G^\alpha, \Gamma, \alpha')$

- $V(G^\alpha) = V(G) \times \Gamma$
- $D(G^\alpha) = D(G) \times \Gamma$
- $s(d, x) = (s(d), x)$
- $t(d, x) = (t(d), x.\alpha(d))$
- $\lambda(d, x) = (\lambda(d), x.\alpha(d))$
- $\alpha(d, x) = \alpha(d)$

An example; the group is $\mathbb{Z}_3$





- There is always a projection map: $((a, x) \mapsto a)\colon G^\alpha \to G$
- The projection map is a very nice surjection (a *covering*).

Cayley graphs are lifts

- Let $S \subseteq \Gamma$ be closed with respect to inversion.
- C_S is a one-vertex graph with $D(C_S) = S$, λ is the inversion.
- The lift of C_S is the Cayley graph of S.

Group labeled graphs

We should call them *charge graphs*

A *group labeled graph* is a triple (G, Γ, β), where G is a graph, Γ is a group and $\beta \colon V(G) \to \Gamma$ is a mapping, called a Γ-*labeling on* G.

Morphisms of group labeled graphs

Γ is fixed

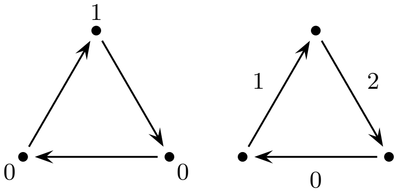
A morphism of Γ-labeled graphs $(G, \beta) \to (G', \beta')$ is a morphism of graphs $f\colon G \to G'$ such that for all $v \in V(G)$, $\beta(v) = \beta'(f^V(v))$.

$$\begin{tikzcd}
V(G) \arrow[rr, "f^V"] \arrow[dr, "\beta"'] & & V(G') \arrow[dl, "\beta'"] \\
& \Gamma &
\end{tikzcd}$$

The category of group labeled graphs is denoted by $\mathbf{Lab}_\Gamma$.

From group labeled graphs to voltage graphs

For every Γ-labeled graph (G, β), there is a voltage graph $L(G, \beta) = (G, \alpha)$, with the voltage α given by the rule $\alpha(d) = \beta(s(d))^{-1}\beta(t(d))$.



L is a functor $\mathbf{Lab}_\Gamma \to \mathbf{Volt}_\Gamma$.

Can we recognize the objects in the range of L?

- Which voltage graphs can be represented as $L(G, \beta)$?
- Not all of them.
- Obviously, the product of voltages along every closed walk must be equal to $1 \in \Gamma$.
- Every voltage lift arises as $L(R(G, \alpha))$ for a certain labelled graph $R(G, \alpha)$.

Adjunction

Kan 1956

Definition

[Lane, 1971, (ii) of Theorem IV.1] Let $\mathcal{C}, \mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. We say that F *is left adjoint to* G, or that G *is right adjoint to* F, in symbols $F \dashv G$, if there is a family

$$\{\epsilon_Y\colon FG(Y) \to Y\}_{Y \in \mathrm{obj}(\mathcal{D})}$$

of $\mathcal{D}$-morphisms, such that for every $\mathcal{C}$-object X and a $\mathcal{D}$-morphism $f\colon F(X) \to Y$ there is a unique $\mathcal{C}$-morphism $u\colon X \to G(Y)$ such that

$$\begin{tikzcd} F(X) \arrow[d, dotted, "F(u)"'] \arrow[dr, "f"] & \\ FG(Y) \arrow[r, "\epsilon_Y"'] & Y \end{tikzcd}$$

commutes.

Adjunction

For each voltage graph (G, α) over Γ, we need to find

- a Γ-labeled graph $R(G, \alpha)$ and
- a morphism of voltage graphs $\epsilon_{G,\alpha}\colon L(R(G, \alpha)) \to (G, \alpha)$

such that

- For every Γ-labeled graph (H, β) and
- every morphism of voltage graphs over Γ $f : L(H, \beta) \to (G, \alpha)$
- there exists a unique morphism of Γ-labeled graphs $u\colon (H, \beta) \to R(G, \alpha)$

such that

$$\epsilon_{G,\alpha} \circ L(u) = f$$

Adjunction

From this, we obtain (for free!)

- R is a functor
- $\epsilon\colon (L \circ R) \to \mathrm{id}_{\mathbf{Volt}}$ is a natural transformation:

$$\begin{tikzcd} LR(G,\alpha) \arrow[r, "LR(f)"] \arrow[d, "\epsilon_{G,\alpha}"'] & LR(G',\alpha') \arrow[d, "\epsilon_{G',\alpha'}"] \\ (G,\alpha) \arrow[r, "f"'] & (G',\alpha') \end{tikzcd}$$

- $\mathbf{Volt}_\Gamma(L(H), G) \simeq \mathbf{Lab}_\Gamma(H, R(G))$.

Adjunctions are everywhere

$$\begin{tikzcd} \mathbf{Lab}_\Gamma \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{Volt}_\Gamma \arrow[l, bend left, "R"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "J"] \arrow[r, phantom, "\perp"] & \mathbf{Rel} \arrow[l, bend left, "\mathcal{P}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Grp} \arrow[l, bend left, "U"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "Ult"] \arrow[r, phantom, "\perp"] & \mathbf{CHaus} \arrow[l, bend left, "U"] \end{tikzcd}$$

A voltage graph is a morphism in **Graph**

- Consider the digraph $\ell(\Gamma)$ with a single vertex v and $D(\ell(\Gamma)) = \Gamma$.
- $\lambda\colon D(\ell(\Gamma)) \to D(\ell(\Gamma))$ is given by $\lambda(a) = a^{-1}$.
- ℓ is then a functor from **Grp** to **Graph**.
- A voltage graph (G, α) can be represented as a morphism of graphs

$$\alpha\colon G \to \ell(\Gamma)$$

- Under this representation, morphisms in $\mathbf{Volt}_\Gamma$ are exactly commuting triangles.

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & G' \arrow[dl, "\alpha'"] \\
& \ell(\Gamma) &
\end{tikzcd}$$

A group labeled graph is a morphism in **Graph**

Let Γ be a group. Let $\mathring{k}(\Gamma)$ be the graph on the vertex set Γ with

- $V(\mathring{k}(\Gamma)) = \Gamma$
- $D(\mathring{k}(\Gamma)) = \Gamma \times \Gamma$
- $s(x_1, x_2) = x_1$
- $t(x_1, x_2) = x_2$
- $\lambda(x_1, x_2) = (x_2, x_1)$

Clearly, $\mathring{k}$ is a functor from **Grp** to **Graph**.
A Γ-labeling β on a graph G is the same thing as a morphism of graphs $\beta\colon G \to \mathring{k}(\Gamma)$.

L as a composition

For every group, there is a morphism of graphs $q_\Gamma^V \colon \mathring{k}(\Gamma) \to \ell(\Gamma)$:

- Every vertex is mapped to the single vertex of $\ell(\Gamma)$.
- The dart (a, b) in $\mathring{k}(\Gamma)$ is mapped to the dart $a^{-1}b$ in $\ell(\Gamma)$.
- Then, L can be represented as a composition with q_Γ:

$$L(G,\beta) = q_\Gamma \circ \beta \qquad \begin{tikzcd} G \arrow[d, "\beta"'] \arrow[dr, "L(G{,}\beta)"] & \\ \mathring{k}(\Gamma) \arrow[r, "q_\Gamma"'] & \ell(\Gamma) \end{tikzcd}$$

R as a pullback along q_Γ

$$
\begin{tikzcd}
H \arrow[drr, bend left, "f"] \arrow[dr, dotted, "u"] \arrow[ddr, bend right, "\beta"'] & & \\
& R(G) \arrow[r, "\epsilon_{G,\alpha}"] \arrow[d, "R(\alpha)"'] & G \arrow[d, "\alpha"] \\
& \mathring{k}(\Gamma) \arrow[r, "q_\Gamma"'] & \ell(\Gamma)
\end{tikzcd}
$$

The projection is a covering

An *in-neighbourhood* $N(v)$ of a vertex v of a graph is the set of darts with target v. A morphism of graphs $f\colon G' \to G$ is a *fibration* if for every vertex $v \in V(G')$, f^D restricted to $N(v)$ is a bijection from $N(v)$ to $N(f(v))$. A fibration is a *covering* if and only if it is surjective on vertices.

Theorem

[Boldi and Vigna, 2002] A morphism of graphs is a fibration iff the square

$$\begin{tikzcd}
D(G) \arrow[r, "t"] \arrow[d, "f^D"'] & V(G) \arrow[d, "f^V"] \\
D(G') \arrow[r, "t"'] & V(G')
\end{tikzcd}$$

is a pullback in **Set***. A covering is a vertex-surjective fibration.*

The projection is a covering

Theorem

[Boldi and Vigna, 2002, Theorem 45] A pullback of a fibration in **Graph** *along an arbitrary morphism is a fibration.*

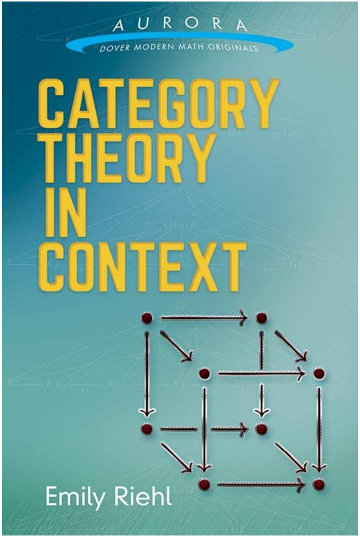
Corollary

The underlying graph morphism of the projection $\epsilon_{G,\alpha} \colon R(G,\alpha) \to (G,\alpha)$ *is a covering.*

Proof.

- q_Γ is a fibration.
- $\epsilon_{G,\alpha}$ is a pullback of q_Γ, and it is surjective.

$\square$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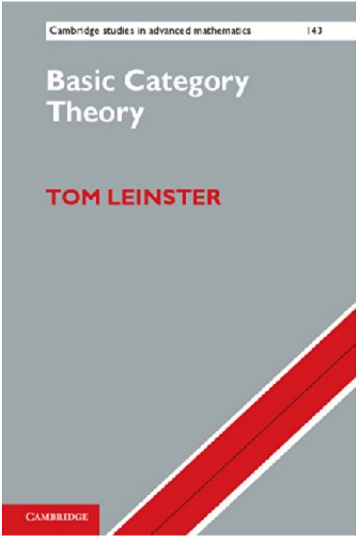
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