1 Ordered vector spaces

Overall reference: [3]

Basic definitions

Let X be a real vector space. A subset $A \subseteq X$ is

- algebraically open (closed) if the intersection of any line with A is an open (closed) subset of the line
- linearly bounded if the intersection of A with any line is a bounded subset of the line

We say that $a \in A$ is an algebraic interior point of A if it is an interior point of the intersection of any line with A, that is, for any $x \in X$ there is some $\delta > 0$ such that $a + sx \in A$ for all $|s| \leq \delta$. The set of all such points is called the algebraic interior of A and is denoted by $aint(A)$. The algebraic closure of A is $acl(A) := X \setminus aint(X \setminus A)$. If A is convex, then

$$acl(A) = \{x \in X,\ \exists y \in X,\ x + \lambda y \in A,\ \forall \lambda \in (0, 1)\}.$$

A is algebraically open iff $A = aint(A)$ and algebraically closed iff $A = acl(A)$. If A is convex, then both $aint(A)$ and $acl(A)$ are convex as well.

Remark 1. (cf. [4, §16]) If A is convex, then $aint(A)$ is algebraically open, but in general $aint(aint(A)) \subsetneq aint(A)$. The algebraic closure is not necessarily algebraically closed even if A is convex. But if A is convex and $aint(A) \neq \emptyset$, then $acl(acl(A)) = acl(A)$.

Wedges, cones and orderings

A subset $P \subseteq X$ is called a wedge if $P + P \subseteq P$ and $\lambda P \subseteq P$ for any $\lambda \geq 0$. The preorder $x \leq y$ if $x - y \in P$ is compatible with the linear structure, such a preorder is called an ordering in X. Conversely, for any ordering, the set of positive elements is a wedge.

The pair (X, P) where P is a wedge is called an ordered vector space. The corresponding ordering is a partial order iff $P \cap -P = \{0\}$, in this case P is called a cone. X with this ordering is directed iff P is generating, that is, $X = P - P$.

Positive maps

Let (X, P) and (Y, Q) be ordered vector spaces. A linear map $F : X \to Y$ is called positive if $F(P) \subseteq Q$. Let (P, Q) denote the set of positive maps, then (P, Q) is a wedge in the vector space $L(X, Y)$ of all linear maps $X \to Y$. We have

Lemma 1. *(P, Q) is a cone if and only if P is generating and Q is a cone.*

Archimedean and almost Archimedean orderings

Let (X, P) be an ordered vector space. We say that the ordering (or P) is Archimedean if $x \leq \lambda y$ for some $y \in X$ and all $\lambda > 0$ implies that $x \leq 0$.

Proposition 1. *The following are equivalent.*

(i) *the ordering is Archimedean.*

(ii) $\exists y \in X, \epsilon > 0$ *such that* $x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x \leq 0$.

(iii) $\exists y \in P, \epsilon > 0$ *such that* $x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x \leq 0$.

(iv) $P = acl(P)$.

The ordering (or P) is almost Archimedean if $-\lambda y \leq x \leq \lambda y$ for some $y \in X$ and all $\lambda > 0$ implies $x = 0$.

Proposition 2. *The following are equivalent.*

(i) *the ordering is almost Archimedean.*

(ii) $\exists y \in X, \epsilon > 0$ *such that* $-\lambda y \leq x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x = 0$.

(iii) $\exists y \in P, \epsilon > 0$ *such that* $-\lambda y \leq x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x = 0$.

(iv) $acl(P)$ *is a cone.*

Remark 2. Note that an almost Archimedean wedge must be a cone. An Archimedean wedge is almost Archimedean iff it is a cone.

Order units and seminorms

An element $u \in X$ is an order unit in (X, P) if for any $x \in X$, there is some $\lambda \in \mathbb{R}^+$ such that $x \leq \lambda u$. This is equivalent to $u \in aint(P)$. If $aint(P) \neq \emptyset$, P is generating.

If u is an order unit, then P is (almost) Archimedean iff u is (almost) Archimedean: $x \leq \lambda u$ for all $\lambda > 0$ implies $x \leq 0$ (resp. $-\lambda u \leq x \leq \lambda u$ for all $\lambda > 0$ implies $x = 0$).

For an order unit u, put

$$\|x\|_u = \inf\{\lambda > 0,\ -\lambda u \leq x \leq \lambda u\}.$$

Then $\|\cdot\|_u$ is a seminorm in X. It is a norm iff u is almost Archimedean.

Remark 3. If $u_1, u_2 \in aint(P)$, the associated seminorms $\|\cdot\|_{u_1}$ and $\|\cdot\|_{u_2}$ are equivalent. The corresponding topology is thus a property of the ordering rather than the order unit. In fact, this topology is the finest locally convex topology making all order intervals bounded.

Lemma 2. *Let u be Archimedean. Then $[-u, u] = \{x, \in X,\ \|x\|_u \leq 1\}$ and the wedge P is closed in the topology given by $\|\cdot\|_u$.*

Proof. Let $x \in [-u, u]$, then clearly $\|x\|_u \leq 1$. Conversely, assume that $\|x\|_u \leq 1$, then $-(1+\epsilon)u \leq x \leq (1+\epsilon)u$ for all $\epsilon > 0$. This implies that $\pm x - u \leq \epsilon u$ for all $\epsilon > 0$ and since u is Archimedean, this implies $\pm x \leq u$, that is, $x \in [-u, u]$.

For the second statement, let $x \in \bar{P}$ (the closure of P w.r. to $\|\cdot\|_u$). Then for all $n \in \mathbb{N}$, there is some $p_n \in P$ such that $\|x - p_n\|_u \leq \frac{1}{n}$. This implies that $-x \leq p_n - x \leq \frac{1}{n}u$ for all n and since u is Archimedean, $-x \leq 0$, so that $x \in P$.

□

Order unit spaces

A triple (X, P, u) where X is a vector space, $P \subseteq X$ an Archimedean cone and $u \in aint(P)$ is called an order unit space. To summarize, in this case, $\|\cdot\|_u$ is a norm in X, $[-u, u]$ is the corresponding closed unit ball and P is norm closed. If (X_i, P_i, u_i), $i = 1, 2$ are order unit spaces, a linear map $f : X_1 \to X_2$ is called unital if $f(u_1) = u_2$.

Proposition 3. *Let (X_i, P_i, u_i), $i = 1, 2$ be order unit spaces. Any positive unital map $f : X_1 \to X_2$ is a contraction.*

Bases and seminorms

Let (X, P) be an ordered vector space. A convex subset $K \subset P$ is called a base of P if for any nonzero $p \in P$ there is a unique $\lambda > 0$ such that $\lambda p \in K$.

Lemma 3. *The wedge P has a base if and only if P is a cone and there exists a linear functional ξ on X which is strictly positive on P. Then $K = \{p \in P, \xi(p) = 1\}$.*

Proof. Let K be a base of P, and let $0 \neq x \in P \cap -P$. Then there are $\lambda, \mu > 0$ such that $\lambda x = x_1 \in K$ and $-\mu x = x_2 \in K$. It follows that $\lambda^{-1} x_1 = -\mu^{-1} x_2$ and then $\frac{\mu}{\lambda+\mu} x_1 + \frac{\lambda}{\lambda+\mu} x_2 = 0$. Since K is convex, we obtain $0 \in K$, but then for any $p \in K$, $\lambda p \in K$ for all $\lambda \in [0, 1]$. Hence P must be a cone. For $p \in P$, let $\xi(p)$ be the unique positive number such that $\xi(p)^{-1} p \in K$. Then clearly $\xi(sp) = s\xi(p)$. Further, let $p, q \in P$ and let $\alpha = \xi(p) + \xi(q)$, then

$$\alpha^{-1}(p + q) = \frac{\xi(p)}{\alpha} \xi(p)^{-1} p + \frac{\xi(q)}{\alpha} \xi(q)^{-1} q \in K,$$

so that $p \mapsto \xi(p)$ is an additive function $\xi : P \to \mathbb{R}^+$. The function ξ easily extends to $P - P$ and has an extension to all of X by Hahn-Banach theorem. This extension is obviously positive and $K = \{p \in P,\ \xi(p) = 1\}$.

Conversely, let $\xi : X \to \mathbb{R}$ be strictly positive, then $K = \{p \in P, \xi(p) = 1\}$ is a convex subset of P and $\xi(p)^{-1} p \in K$ for any $p \in P$. Uniqueness is obvious.

□

Proposition 4. *([2]) Let P be a generating cone in a vector space X and let K be a base of P. For $x \in X$, put*

$$\|x\|_K := \inf\{a + b,\ x = ap - bq,\ a, b \in \mathbb{R}^+, p, q \in K\}.$$

This defines a seminorm in X, which is a norm if and only if $S := co(K \cup -K)$ is linearly bounded.

Proof. It can be checked easily that $\| \cdot \|_K$ is a seminorm. Note also that $x \in S$ implies $\|x\|_K \leq 1$. Indeed, any $x \in S$ has the form $x = \lambda p - (1 - \lambda) q$ for some $\lambda \in [0, 1]$, $p, q \in K$ and then $\|x\|_K \leq \lambda + (1 - \lambda) = 1$. Assume that $\| \cdot \|_K$ is a norm and let $x_t := x + ty$ be a line in X. Then $\|y\|_K > 0$ and $x_t \in S$ implies that $1 \geq \|x_t\|_K \geq |\|x\|_K - |t|\|y\|_K|$, so that $|t| \leq \frac{1+\|x\|_K}{\|y\|_K}$. Conversely, assume that S is linearly bounded and let $\|x\|_K = 0$. This implies $tx \in S$ for all $t \in \mathbb{R}$, hence we must have $x = 0$.

□

The (semi)norm in the above proposition is called the base (semi)norm in X.

Remark 4. Linear boundedness of K is in general not enough. There are some weird infinite dimensional examples such that K is linearly bounded but $co(K \cup -K)$ is not.

Base-normed spaces

A triple (X, P, K), where X is a vector space, P a generating cone and K a base of P such that $co(K \cup -K)$ is linearly bounded is called a base-normed space. Let (X_i, P_i, K_i) be base-normed spaces. A linear map $f : X_1 \to X_2$ is called base-preserving if $f(K_1) \subset K_2$.

Proposition 5. *Let (X_i, P_i, K_i), $i = 1, 2$ be base-normed spaces. Any base-preserving linear map $f : X_1 \to X_2$ is a positive contraction.*

The order dual

Let (X, P) be an ordered vector space and let X' denote the algebraic dual of X. Then the dual wedge of P is defined as

$$P' := \{\varphi \in X', \varphi(p) \geq 0, \forall p \in P\}$$

Then (X', P') is an ordered vector space: the order dual of X. Note that $P' = (P, \mathbb{R}^+)$ and it follows by Lemma 1 that P' is a cone iff P is generating. Further, note that $p \in P \cap -P$ implies that $\varphi(p) = 0$ for all $\varphi \in P'$, hence if P' is generating, P must be a cone. The converse is not true in general.

The dual of a vector space with an order unit norm

Let (X, P) be an ordered vector space with an order unit u such that $\|\cdot\|_u$ is a norm. This implies that P is generating and almost Archimedean, hence a cone. We do not assume that (X, P, u) is an order unit space, so u does not have to be Archimedean.

Let X^* be the normed space dual of $(X, \|\cdot\|_u)$ and let $\|\cdot\|_u^*$ be the norm in X^*.

Lemma 4. *(i) Any $\varphi \in P'$ is bounded, with $\|\varphi\|_u^* = \varphi(u)$.*

(ii) If $\varphi \in X^*$ is such that $\|\varphi\|_u^* = \varphi(u)$, then $\varphi \in P'$.

Proof. (i) is quite easy. For (ii), we may assume $\varphi(u) = 1$. Let $x \in P$ and let $\lambda > 0$ be such that $0 \leq x \leq \lambda u$. Then $\|x - \lambda u\|_u \leq \lambda$ and we have

$$|\varphi(x) - \lambda| = |\varphi(x - \lambda u)| \leq \|\varphi\|_u^* \|x - \lambda u\|_u \leq \lambda.$$

This implies $\varphi(x) \geq 0$.

□

Theorem 1. *P' has a w^*-compact base K such that (X^*, P', K) is a base-normed space and $\|\cdot\|_K = \|\cdot\|_u^*$.*

Proof. [?] The set $K = \{\varphi \in P', \varphi(u) = 1\}$ is a w^*-compact base of P'. We will show that the base seminorm $\|\cdot\|_K$ equals to the dual norm in X^* and hence is itself a norm.

Let $Y = X \times X$ be ordered by the wedge $Q = P \times P$, then (u, u) is an order unit in (Y, Q). Let

$$Z = \{t(u, u) - (x, -x),\ t \in \mathbb{R}, x \in X\},$$

then Z is a linear subspace in Y containing the order unit. For $\varphi \in X^*$, put

$$F_\varphi(z) = t\|\varphi\|_u^* - \varphi(x), \qquad z = t(u, u) - (x, -x) \in Z$$

This defines a linear functional on Z. Moreover, note that $z = t(u, u) - (x, -x) \in Q$ iff $\|x\|_u \leq t$ and then $F_\varphi(z) \geq (t - \|x\|_u)\|\varphi\|_u^* \geq 0$. Since Z contains the order unit, F_φ extends to a positive linear functional on Y (e.g. Krein's theorem). Put

$$\psi_1(x) = F_\varphi(x, 0), \quad \psi_2(x) = F_\varphi(0, x), \qquad x \in X.$$

Then $\psi_1, \psi_2 \in P'$ and $\varphi = \psi_2 - \psi_1$, this shows that P' is generating in X^*. Moreover, $F_\varphi(u, u) = \|\varphi\|_u^*$

$$\|\varphi\|_u^* = F_\varphi(u, u) = \psi_1(u) + \psi_2(u) \geq \|\varphi\|_K$$

On the other hand, let $\varphi = a\varphi_1 - b\varphi_2$ with $a, b \geq 0$, $\varphi_1, \varphi_2 \in K$, then $\|\varphi\|_u \leq a + b$, this shows the opposite inequality.

□

The dual of a base-normed space

Let (X, P, K) be a base-normed space and let X^* be the normed space dual of $(X, \|\cdot\|_K)$. Let $P^* = P' \cap X^*$.

Theorem 2. *There is an order unit $u \in X^*$ such that (X^*, P^*, u) is an order unit space.*

Proof. Let (X, P, K) be a base-normed space. Note first that for any $\varphi \in X'$, we have

$$\|\varphi\|_K^* = \sup_{x\in S} |\varphi(x)| = \sup_{x\in K} |\varphi(x)|,$$

where $S = co(K \cup -K)$. There is a strictly positive functional $u \in X'$ such that $K = \{p \in P, u(p) = 1\}$. Note that u is a base-preserving linear map into the base-normed space $(\mathbb{R}, \mathbb{R}^+, 1)$, hence is a positive contraction. Moreover, for $\varphi \in X^*$ and $x \in K$, we have $-\|\varphi\|_K \leq \varphi(x) \leq \|\varphi\|_K$, so that $-\|\varphi\|_K u \leq \varphi \leq \|\varphi\|_K u$, it follows that u is an order unit in $(X^*, P' \cap X^*)$ and $\|\varphi\|_u \leq \|\varphi\|_K^*$. Conversely, $-\lambda u \leq \varphi \leq \lambda u$ implies that $\sup_{x\in K} |\varphi(x)| \leq \lambda$, so that $\|\varphi\|_u = \|\varphi\|_K^*$. To show that u is Archimedean, let $\varphi \leq \lambda u$ for all $\lambda > 0$. Then for $x \in K$, $\varphi(x) \leq \lambda$ for any $\lambda > 0$, hence $\varphi(x) \leq 0$.

□

Completeness

We give some sufficient conditions for completeness of order unit norms and base norms.

Preduals

We next discuss the Banach space preduals of order unit and base-normed spaces. Here $(X, \|\cdot\|)$ is a Banach space and $(X^*, \|\cdot\|^*)$ the dual space. If $P \in X$ is a wedge, we will denote $P^* := \{\varphi \in X^*,\ \varphi(p) \geq 0,\ \forall p \in P\} = P' \cap X^*$. Similarly, if Q is a wedge in X^*, we will denote $Q_* := \{x \in X,\ q(x) \geq 0,\ \forall q \in Q\}$. It is clear that P^* and Q_* are wedges. Moreover, $(P^*)_* = \bar{P}$ and $(Q_*)^*$ is the weak*-closure of Q.

Theorem 3. *[2, 1] Let X^* be an order unit space with weak*-closed positive cone. Then X is base-normed. More precisely, if there is an Archimedean weak*-closed cone $Q \subset X^*$ with an order unit u such that $\|\cdot\|^* = \|\cdot\|_u$, then $Q_* \subset X$ has a base $K = \{p \in Q_*, u(p) = 1\}$ and (X, Q_*, K) is a base-normed space with $\|\cdot\| = \|\cdot\|_K$.*

Proof. Let $p \in Q_*$ be such that $u(p) = 0$, then for any $\varphi \in Q$,

$$0 \le \varphi(p) \le \|\varphi\|_u \varphi(u) = 0.$$

Since $X^* = Q - Q$ separates points in X, we obtain $p = 0$. Hence u defines a strictly positive linear functional on (X, Q_*) and K is a base of Q_*. For $p \in Q_*$, we have

$$\|p\| = \sup_{\varphi \in [-u,u]} |\varphi(p)| = u(p),$$

it follows that $S = co(K \cup -K)$ is a subset of the unit ball of X. Hence $\|\cdot\| \le \|\cdot\|_K$ (since $\|\cdot\|_K$ is the Minkowski functional of S). Since $Q = (Q_*)^*$, we have for $\varphi \in X^*$:

$$\begin{aligned}\|\varphi\|_u &= \inf\{\lambda > 0,\ \lambda u \pm \varphi \in Q\} = \inf\{\lambda > 0,\ (\lambda u \pm \varphi)(p) \ge 0,\ \forall p \in Q_*\} \\ &= \inf\{\lambda > 0,\ |\varphi(p)| \le \lambda,\ \forall p \in K\} = \sup_{p \in K} |\varphi(p)|.\end{aligned}$$

Assume that $x_0 \in X$ is such that $\|x_0\| \le 1$ and $x_0 \ne \bar{S}$, then by Hahn-Banach separation theorem, there is some $\varphi \in X^*$ such that

$$\|\varphi\|_u = \sup_{p \in K} |\varphi(p)| = \sup_{x \in S} \varphi(x) < \varphi(x_0) \le \|\varphi\|^* = \|\varphi\|_u.$$

It follows that S is dense in the unit ball X_1 of X. Choose any $\alpha > 1$ and let $\alpha_n > 0$ be a sequence such that $1 + \sum_n \alpha_n < \alpha$. There is some element $x_1 \in S$ such that $\|x_0 - x_1\| < \alpha_1$. Similarly, there is some $x_2 \in \alpha_1 S$ such that $\|x_0 - x_1 - x_2\| < \alpha_2$. Continuing by induction, we obtain a sequence $\{x_n\}$ in X such that $\|x_n\|_K \le \alpha_{n-1}$ and $\|x_0 - \sum_n x_n\| < \alpha_n \to 0$. Hence

$$\|x_0\|_K = \|\sum_n x_n\|_K \le \sum_n \|x_n\|_K \le 1 + \sum_n \alpha_n < \alpha,$$

so that $X_1 \subset \alpha S$ and consequently $X = Q_* - Q_*$. Since the above inequality holds for all $\alpha > 1$, we have $\|\cdot\| = \|\cdot\|_K$.

□

Categories of ordered vector spaces

References

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