AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **38**, 3020 (1997); doi: 10.1063/1.532031
View online: http://dx.doi.org/10.1063/1.532031
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/38/6?ver=pdfcov
Published by the AIP Publishing

Effect algebras and statistical physical theories

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(Received 22 October 1996; accepted for publication 4 December 1996)

The dichotomic physical quantities of a physical system can be naturally hosted in a mathematical structure, called effect algebra, of which orthomodular posets and Boolean algebras are particular examples. We examine how effect algebras arise inside statistical physical theories and, conversely, we study to what extent an effect algebra can be taken as a primitive structure on which a satisfactory statistical physical model equipped with a convex set of states can be constructed. © *1997 American Institute of Physics.* [S0022-2488(97)03505-6]

I. INTRODUCTION

In the last few years the notion of effect algebra has received much attention within the studies on the mathematical foundations of quantum mechanics.[1–9] Effect algebras appear to be the natural outcome in the search of a mathematical structure that captures the fundamental aspects of the elementary two-valued physical quantities, or effects, pertaining to a physical system. The notion of effect algebra is sufficiently general to encompass the traditional order structures accompanying classical systems (Boolean algebras) and quantum systems (orthomodular posets), but it is sufficiently structured to carry a meaningful interplay with the physically relevant notions of states and of observables.

The purpose of this paper is to outline the relationship between the notion of effect algebra and the so-called operational, or convex, approach, in which the states of the physical system are taken as primitive elements and the convex set they form is the basic structure on which the descriptive model is built up. The approach we are referring to goes back to Ludwig (see Ref. 10); the name ''operational'' was proposed by Davies and Lewis[11] and is now widely used (see, e.g., Ref. 12). In this operational approach the observables are derived entities: we shall adopt the physically natural definition (introduced in Ref. 13) according to which an observable is an affine map from the convex set of states into the family of the probability measures on the space in which the observable takes values.

After reviewing in Section II the mathematical scheme of effect algebras, we examine in Section III how this scheme emerges from the convex approach. Section IV deals with the reversed problem: we examine the requirements that an effect algebra must meet in order to play the role of a basis for a satisfactory physical model. A crucial requirement is that the elements of the effect algebra should be separated by the probability measures on the effect algebra itself: this requirement singles out the class of the ''admissible'' effect algebras, which turn out to exhibit a number of significant properties. In Section V we discuss the conditions that make the convex approach equivalent to the framework based on an effect algebra.

From a formal point of view our results provide also a counterpart of the linearization procedure for orthomodular posets.[14] We find also a close connection with a theorem of Bennett and Foulis[15] (see also Ref. 1, p. 1373) stating that an effect algebra with an order determining set of probability measures is an interval effect algebra. Finally, our results contribute to answering a question advanced by Greechie and Foulis (the research project 4.6 of Ref. 1).

0022-2488/97/38(6)/3020/11/$10.00

II. EFFECT ALGEBRAS

An *effect algebra* is defined as a set $\mathcal{E}$ containing two special elements o,e and equipped with a partial binary operation $\oplus$ (to be called *sum*) satisfying the properties:

(i) $a\oplus b=b\oplus a$,
(ii) $a\oplus(b\oplus c)=(a\oplus b)\oplus c$,
(iii) for every $a \in \mathcal{E}$ there is in $\mathcal{E}$ a unique element, denoted $e\ominus a$, such that $a \oplus (e\ominus a)$ is defined and equals e,
(iv) if $a \oplus e$ is defined then $a=o$.

An effect algebra hosts in a natural way a partial order relation defined as follows: if $a,b \in \mathcal{E}$ we write $a\leqslant b$ if there is $c \in \mathcal{E}$ such that $a \oplus c$ is defined and $a \oplus c=b$.

As better seen in the next section, the elements of an effect algebra -in short the effects- will be naturally interpreted as dichotomic physical quantities, or events, pertaining to some physical system. To visualize how an event can be operationally generated one could associate it to a statement of the form: ‘‘the measurement of a given physical quantity gives an outcome that falls into a given numerical interval.’’ Of course, each state of the physical system will assign some probability of occurrence to every event. In this operational perspective the effect-sum $\oplus$ will be defined only for those pairs of effects such that, for each state of the physical system, the sum of the probabilities of their occurrence does not exceed 1 (we might say that these are ‘‘orthogonal’’ effects): in this case the sum of the two events will be the event whose probability of occurrence is just the sum of these two probabilities. Notice that the effect-sum $a \oplus b$ is obviously an upper bound of the effects a,b, with respect to the partial ordering said above; however it need not be the least upper bound.

We come now to the notion of infinite sums in an effect algebra: if $\{a_i:i=1,2,\ldots\}$ is an infinite sequence of elements of an effect algebra $\mathcal{E}$ we say that the sum $a_1 \oplus a_2 \oplus \ldots$ is defined in $\mathcal{E}$ if the finite sums $s_n:=a_1 \oplus a_2 \oplus \ldots \oplus a_n$ are defined for every n and the increasing sequence $\{s_n:n=1,2,\ldots\}$ has in $\mathcal{E}$ a least upper bound s under the order relation said above. Then we write $s=\Sigma_{i=1}^{\infty}a_i$.

An effect algebra $\mathcal{E}$ will be called σ-complete if every increasing sequence of its elements has a least upper bound.

Let us now consider the notion of morphism between effect algebras.

Definition 1: Given two effect algebras $\mathcal{E}_1$, $\mathcal{E}_2$, a map $\phi:\mathcal{E}_1\rightarrow\mathcal{E}_2$ is called a morphism if

(i) $\phi(e_1)=e_2$,
(ii) if $a \oplus b$ is defined in $\mathcal{E}_1$ then $\phi(a) \oplus \phi(b)$ is defined in $\mathcal{E}_2$ and $\phi(a) \oplus \phi(b)=\phi(a \oplus b)$.

A morphism is called a σ-morphism if, for any infinite sequence $\{a_i:a_i \in \mathcal{E}_1,i=1,2,\ldots\}$ such that $\Sigma_{i=1}^{\infty}a_i$ is defined in $\mathcal{E}_1$, the sum $\Sigma_{i=1}^{\infty}\phi(a_i)$ is defined in $\mathcal{E}_2$ and $\phi(\Sigma_{i=1}^{\infty}a_i)=\Sigma_{i=1}^{\infty}\phi(a_i)$.

The unit interval $[0,1]$ of the real line $\mathbf{R}$ forms, under the addition operation of real numbers, a trivial example of effect algebra.

Definition 2: A probability measure on an effect algebra $\mathcal{E}$ is a σ-morphism of $\mathcal{E}$ into $[0,1]$. The set of all probability measures on $\mathcal{E}$ will be denoted $\mathcal{S}(\mathcal{E})$.

Notice that a weaker notion of probability measure on an effect algebra is sometimes used with only finite additivity assumed. In this paper the term *measure* will always imply σ-additivity.

An effect algebra need not admit probability measures. Indeed orthomodular posets are particular instances of effect algebras (they are effect algebras satisfying the ‘‘coherence’’ condition, see Ref. 1) and it is known[16] that there are orthomodular posets admitting no probability measures.

Remark 1: If an effect algebra $\mathcal{E}$ admits probability measures then $\mathcal{S}(\mathcal{E})$ is endowed with a natural convex structure: for any $\alpha_1,\alpha_2\in\mathcal{S}(\mathcal{E})$ and $\lambda_1,\lambda_2\in[0,1]$ we define $\lambda_1\alpha_1+\lambda_2\alpha_2$ to be the map $a\mapsto\lambda_1\alpha_1(a)+\lambda_2\alpha_2(a)$ of $\mathcal{E}$ into $[0,1]$.

In the sequel the following definition will be useful:

Definition 3: An effect algebra $\mathscr{E}$ is said to be admissible if it is separated by $\mathscr{S}(\mathscr{E})$, i.e., if for any $a,b \in \mathscr{E}, a \neq b$, there is $\alpha \in \mathscr{S}(\mathscr{E})$ such that $\alpha(a) \neq \alpha(b)$.

This admissibility notion will appear as a necessary requirement for any effect algebra of physical interest. In Section IV a number of relevant properties of admissible effect algebras will be given.

Let now Ξ be a measurable space and notice that the Boolean σ-algebra $\mathscr{B}(\Xi)$ of the measurable subsets of Ξ is a σ-complete effect algebra under the partial binary operation $\oplus$ defined only for pairs of disjoint elements and identified with the set-theoretic union (the zero element is the empty set while the unit element is the whole Ξ). In view of this fact we come now to a relevant definition.

Definition 4: Let $\mathscr{E}$ be an effect algebra and Ξ a measurable space. A σ-morphism of $\mathscr{B}(\Xi)$ into $\mathscr{E}$ is called an $\mathscr{E}$-valued measure on Ξ.

Now we have:

Lemma 1: Any $\mathscr{E}$-valued measure E on a measurable space Ξ defines an affine map $A^E:\mathscr{S}(\mathscr{E})\to M_1^+(\Xi)$, where $M_1^+(\Xi)$ is the convex set of all the probability measures on Ξ.

Proof: Let $\alpha \in \mathscr{S}(\mathscr{E})$. The set function $A^E\alpha:\mathscr{B}(\Xi)\to[0,1]$ defined by $A^E\alpha(X)=\alpha(E(X))$ is a probability measure on Ξ. Indeed, take a sequence $\{X_i:X_i \in \mathscr{B}(\Xi), i=1,2,\dots\}$ such that $X_i\cap X_j=\emptyset, i \neq j$, and notice that $\alpha(E(\cup_{i=1}^{\infty}X_i))=\alpha(\Sigma_{i=1}^{\infty}E(X_i))=\Sigma_{i=1}^{\infty}\alpha(E(X_i))$. It is clear that $A^E(\lambda\alpha_1+(1-\lambda)\alpha_2)=\lambda A^E\alpha_1+(1-\lambda)A^E\alpha_2$ for any $\alpha_1,\alpha_2 \in \mathscr{S}(\mathscr{E})$. □

The notion of effect algebra appeared independently (under various names) in several papers as a generalization of orthomodular posets and as an algebraic structure able to capture the basic properties of effects in operational statistical theories: we refer in particular to the papers of Cattaneo and Nistico'[3], Dalla Chiara and Giuntini,[4] Giuntini and Greuling,[5] Kopka and Chovanec,[6] Pulmannova,[7] besides the ones of Greechie and Foulis[1] and Gudder[2] which have also the character of review papers on effect algebras.

III. FROM STATES TO EFFECT ALGEBRAS

Typical of the convex, or operational, approach to a statistical theory is the fact that the states of the physical system under discussion are taken as primitive, undefined elements. Other quantities of interest, as the observables, are then defined on the basis of the set S of states. Convexity is a natural structure of S: it translates the physical possibility of mixing different preparations of the physical system.

The very notion of convexity presupposes that S is a convex subset of a real linear space, say L. The linear hull of S in L will be denoted $V(S)$ and we assume without loss of generality that the origin of $V(S)$ does not belong to S. About the properties of $V(S)$ we have the following.

Lemma 2:

(i) If the elements of the convex set S are separated by a subset of the convex set $A^b(S)$ of all real-valued bounded affine functions on S then $V(S)$ is a base-norm space;

(ii) if $V(S)$ is a base-norm space, then $A^b(S)$ is an order-unit Banach space.

Proof:

(i) We prove first that S is linearly bounded, that is the intersection of S with any affine line l lying in the hyperplane of S is bounded. In fact, should $S\cap l$ be unbounded, it would be impossible to separate its elements by a bounded function, contrary to the assumption that S is separated by a subset of $A^b(S)$. The proof that if S is linearly bounded then $V(S)$ is a base-norm space is given in Ref. 17.

(ii) It is known that the Banach dual of a base-norm space is an order-unit Banach space (Ellis theorem, see, e.g., Ref. 17, p. 27). Hence the Banach dual $V(S)^*$ of $V(S)$ is an order-unit Banach space, the order unit being the linear functional on $V(S)$ which takes the value 1 on every element of S. The natural linear order and norm isomorphism τ between $V(S)^*$ and $A^b(S)$ is defined by taking for $\tau(\psi)$, $\psi \in V(S)^*$, the restriction of ψ to S while $\tau^{-1}(a)$, $a \in A^b(S)$, is the unique linear extension of a over $V(S)$. It is clear that the order unit of $V(S)^*$ is then mapped onto the unit function e of $A^b(S)$ (i.e., $e(\alpha)=1$ for all $\alpha \in S$). The order-unit norm $\|\psi\|$ of $\psi \in V(S)^*$ has to be equal to the Banach dual norm (the mentioned Ellis theorem), hence $\|\psi\| = \sup\{|\psi(\alpha)| : \alpha \in \mathrm{Conv}(S \cup -S)\} = \sup\{|\psi(\alpha)| : \alpha \in S\} = \sup\{|\tau(\psi))(\alpha)| : \alpha \in S\} = \|\tau(\psi\|$, the sup-norm on $A^b(S)$. We make use of the abbreviation Conv to denote ''the convex hull of.'' □

Come now to the notion of observable. From the physical point of view an observable has to determine, for every state of the physical system, a probability measure on the space (typically the real line) in which that observable takes values. Thus we are led to the following definition. Let Ξ be a measurable space, with the associated Boolean σ-algebra $\mathscr{B}(\Xi)$ of subsets of Ξ; an observable A on the convex set S of states is defined as an affine map $A:S \to M_1^+(\Xi)$ where $M_1^+(\Xi)$ is the convex set of all the probability measures on Ξ. Explicitly, the affinity conditions means that $A(\lambda\alpha_1 + (1-\lambda)\alpha_2) = \lambda A\alpha_1 + (1-\lambda)A\alpha_2$ for any $\alpha_1, \alpha_2 \in S$. The measurable space Ξ will be called the outcome space of the observable A.

This notion of observable, though more general than the ones adopted in standard classical statistical mechanics and in standard quantum mechanics, is not new in the literature[13,18–20] and covers the essential properties of physical and probabilistic concepts like those of coarsening,[21], fuzzy random variable,[22] Markov kernel,[23] etc.

Remark 2: The convex structure of $M_1^+(\Xi)$ induces the notion of convex combinations of observables having the same outcome space: if A_1, A_2 are two observables on S and Ξ is their common outcome space then $\lambda A_1 + (1-\lambda)A_2$, $\lambda \in [0,1]$, is the observable $S \to M_1^+(\Xi)$ whose action on $\alpha \in S$ is given by $\lambda A_1\alpha + (1-\lambda)A_2\alpha$.

We come now to the effect algebras that naturally arise inside the convex approach.

Definition 5: Given the convex set S, the elements of the order interval $[o_S, e_S] := \{o_S \leqslant a \leqslant e_S : a \in A^b(S)\}$, where o_S, e_S are the null and, respectively, the unit function on S, are called effects on S. In other words, the effects are the affine functions from S into $[0, 1]$; thus they form a class of fuzzy sets in S.

Notice that the simplest nontrivial observables on a convex set S are those which have a two-point outcome space $\Xi = \{\xi', \xi''\}$. Since a probability measure on $\{\xi', \xi''\}$ is uniquely determined by the value it takes at the singleton $\{\xi'\}$, any observable $A:S \to M_1^+(\{\xi', \xi''\})$ is uniquely determined by an affine function of S into $[0,1]$ so that the effects can be considered as elementary two-valued observables.

The set $[o_S, e_S]$, with the partial binary operation of addition of real functions (defined only on the pairs of elements whose sum is still in $[o_S, e_S]$), is obviously an effect algebra, specifically an *interval* effect algebra which is σ-complete (see Lemma 6 of the Appendix). Let us stress that the $\oplus$ operation abstractly introduced in the previous section becomes, for the effect algebra $[o_S, e_S]$, the familiar addition $+$.

Since $[o_S, e_S]$ is an effect algebra we can consider $[o_S, e_S]$-valued measures on some measurable space Ξ according to Definition 4, and we call them *effect-valued* measures.

Theorem 1: Let S be a convex set separated by $[o_S, e_S]$. Then

(i) every observable $A:S \to M_1^+(\Xi)$ defines an effect-valued measure $E^A : \mathscr{B}(\Xi) \to [o_S, e_S]$ on its outcome space by $(E^A(X))(\alpha) := (A\alpha)(X), \alpha \in S, X \in \mathscr{B}(\Xi)$;

(ii) every effect-valued measure $E : \mathscr{B}(\Xi) \to [o_S, e_S]$ defines an observable $A^E : S \to M_1^+(\Xi)$ by $(A^E\alpha)(X) := (E(X))(\alpha), \alpha \in S, X \in \mathscr{B}(\Xi)$;

(iii) $E^{(A^E)} = E$ and $A^{(E^A)} = A$ for any observable A and any effect-valued measure E.

Proof:

(i) If $\{X_i : i=1,2,\dots\}$ is any sequence of disjoint elements of $\mathscr{B}(\Xi)$, we have $(E^A(\cup_{i=1}^{\infty} X_i))(\alpha) = (A\alpha)(\cup_{i=1}^{\infty} X_i) = \Sigma_{i=1}^{\infty}(A\alpha)(X_i) = \Sigma_{i=1}^{\infty}(E^A(X_i))(\alpha) = (\Sigma_{i=1}^{\infty} E^A(X_i))(\alpha)$ for every $\alpha \in S$; this shows the σ-additivity of E^A.

(ii) Clearly the set function $A^E\alpha$ is a probability measure on Ξ and the map $A^E : S \to M_1^+(\Xi)$ is affine.

(iii) We have $(E^{(A^E)}(X))(\alpha) = (A^E\alpha)(X) = (E(X))(\alpha)$ and $(A^{(E^A)}\alpha)(X) = (E^A(X))(\alpha) = (A\alpha)(X)$. □

We can now speak of the probability measures on $[o_S, e_S]$ and it is natural to ask how they are related to the elements of S. To every $\alpha \in S$ we can associate a probability measure on $[o_S, e_S]$ as specified by the following lemma.

Lemma 3: For any $\alpha \in S$ the map $m_\alpha : [o_S, e_S] \to [0,1]$ given by $m_\alpha(a) = a(\alpha), a \in [o_S, e_S]$ defines a (σ-additive) probability measure on $[o_S, e_S]$.

Proof: Consider a sequence $\{a_i : a_i \in [o_S, e_S], i=1,2,\dots\}$ such that $\Sigma_{i=1}^{\infty} a_i := \sup\{\Sigma_{i=1}^{n} a_i : n=1,2,\dots\}$ exists in $[o_S, e_S]$. Known properties of order-unit spaces imply $\sup\{\Sigma_{i=1}^{n} a_i : n=1,2,\dots\} = w^* - \lim_{n\to\infty}\{\Sigma_{i=1}^{n} a_i : n=1,2,\dots\}$ and ensure the existence of the limit.[24] As any $\alpha \in S$ generates by evaluation a w^*-continuous functional on $A^b(S)$ we have that m_α is σ-additive. □

With reference to the notion of admissibility expressed by Definition 3 the following remark is worthwhile.

Remark 3: By definition, $[o_S, e_S]$ is separated by S: due to Lemma 3 it is also separated by the set $\mathscr{S}([o_S, e_S])$ of the probability measures on it. Thus $[o_S, e_S]$ is an admissible effect algebra. From the physical point of view this fact expresses the obvious requirement that two elementary observables can be recognized as distinct only if there is a preparation of the physical system (namely a state) that gives to them different probabilities.

Given that every $\alpha \in S$ determines a probability measure on $[o_S, e_S]$, the natural question arises whether every probability measure on $[o_S, e_S]$ comes from an element of S. A positive answer occurs with standard quantum mechanics where S is the set S_Q of all density operators on some separable complex Hilbert space $\mathscr{H}$: indeed it is known (see, e.g., Ref. 25) that $A^b(S_Q)$ can be identified with the space of bounded self-adjoint operators on $\mathscr{H}$ and all probability measures on $[o_{S_Q}, e_{S_Q}]$ are generated by density operators. But one can pick up situations in which the answer to the above question is negative: if S is an open segment on the reals, say $]0,1[$, it is easily realized that $\mathscr{S}([o_S, e_S])$ contains also the two Dirac measures concentrated at the boundary points 0,1 so that it is isomorphic to the closed segment $[0,1]$.

Though we can conceive convex sets for which not every probability measure on the effects comes from an element of the given convex set, we have the following theorem.

Theorem 2: Let S be a convex set separated by $[o_S, e_S]$. Then S is w^*-dense in the set $\mathscr{S}([o_S, e_S])$ of all the probability measures on the effect algebra $[o_S, e_S]$.

Proof: Lemma 3 provides an embedding of S, the base of $V(S)$, into $\mathscr{S}([o_S, e_S])$; we now prove that there is an embedding of $\mathscr{S}([o_S, e_S])$ into the base $\widetilde{S}$ of $V(S)^{**}$. Given $\mu \in \mathscr{S}([o_S, e_S])$ and $a \in [o_S, e_S]$, take an integer m and suppose that $ma \in [o_S, e_S]$: then the additivity of μ implies $\mu(ma) = m\mu(a)$. Similarly, if n is an integer, we have $\mu[(1/n)\,a] = (1/n)\,\mu(a)$. Hence also $\mu[(m/n)\,a] = (m/n)\,\mu(a)$. The σ-additivity of μ implies in turn the same property for real numbers: if λ is a real number such that $\lambda a \in [o_S, e_S]$ and if the sequence of rational numbers $\{\lambda_i : i=1,2,\dots\}$ converges to λ then we can write $\lambda a = \lambda_1 a + \Sigma_{i=1}^{\infty}(\lambda_{i+1} - \lambda_i) a$ and the σ-additivity together with the linearity of μ under multiplication by rationals imply now $\mu(\lambda a) = \mu(\lambda_1 a) + \mu(\Sigma_{i=1}^{\infty}(\lambda_{i+1} - \lambda_i) a) = (\lambda_1 + \Sigma_{i=1}^{\infty}(\lambda_{i+1} - \lambda_i))\mu(a) = \lambda\mu(a)$. Thus μ extends linearly over $A^b(S)$ and defines uniquely an element of the base $\widetilde{S}$ of the base norm space $A^b(S)^*$.

The composition of the map $S \to \mathscr{S}([o_S, e_S])$ of Lemma 3 with the map $\mathscr{S}([o_S, e_S]) \to \widetilde{S}$

discussed above is an embedding $S \to \widetilde{S}$ which is the restriction of the canonical embedding $\Phi: V(S) \to V(S)^{**}$, which in turn is known to be an isometric isomorphism (see, e.g., Ref. 26, Theorem III.4). Since S is order determining on $A^b(S)$ we can apply Theorem 1a of Ref. 27 to conclude that the image of S under Φ is w^*-dense in $\widetilde{S}$, hence also in $\mathscr{S}([o_S, e_S])$. □

Notice that the effect algebra $[o_S, e_S]$ is naturally endowed with a convex structure: in fact a convex combination of affine functions from S into $[0, 1]$ is still an affine function $S \to [0,1]$. This fact makes it possible to speak of extreme elements of the convex set $[o_S, e_S]$: they are often called *sharp* effects. Though not univocal, the notion of sharp observables and sharp effects, together with the complementary notion of fuzzy observables and fuzzy effects, proves to be relevant in foundational and logical aspects of physical theories (see, e.g., Refs. 2,4,12,13).

In analogy to Remark 2 we can also notice that the natural convex structure of $[o_S, e_S]$ induces the notion of convex combination of effect-valued measures on a measurable space Ξ: if $E_1, E_2: \mathscr{B}(\Xi) \to [o_S, e_S]$ are two effect-valued measures on Ξ, then their convex combination $\lambda E_1 + (1-\lambda) E_2$ is the effect-valued measure on Ξ that takes the value $\lambda E_1(X) + (1-\lambda) E_2(X)$ at $X \in \mathscr{B}(\Xi)$. In view of the correspondence between observables and effect-valued measures specified by Theorem 1 we can expect that the notion of convex combinations of observables having the same outcome space is strictly related to the notion of convex combinations of effect-valued measures on a same measurable space. In fact, it is immediate to check that, if $A_1, A_2: S \to M_1^+(\Xi)$ and $A = \lambda A_1 + (1-\lambda) A_2$, $\lambda \in [0,1]$, then $E^A = \lambda E^{A_1} + (1-\lambda) E^{A_2}$; conversely, if $E_1, E_2: \mathscr{B}(\Xi) \to [0_S, e_S]$ and $E = \lambda E_1 + (1-\lambda) E_2$, then $A^E = \lambda A^{E_1} + (1-\lambda) A^{E_2}$.

IV. FROM EFFECT ALGEBRAS TO CONVEXITY FRAMEWORKS

In this section we shall discuss to what extent an effect algebra can generate a satisfactory description of a physical system; more specifically, to what extent an effect algebra can be taken as the primitive structure carrying a convexity framework of a statistical theory.

As in Section II, let $\mathscr{E}$ be an effect algebra, that we take as the fundamental structure of a physical model, and we interpret its elements as elementary two-valued physical quantities.

It is natural to assume that every probability measure on $\mathscr{E}$ represents a state of the physical system: the set $\mathscr{S}(\mathscr{E})$ introduced in Section II is thus taken as the set of states and its convex structure (outlined in Remark 1) corresponds to the statistical mixing of samples of the physical system produced by different preparation procedures.

On physical grounds we have to assume, as a minimal requirement, that $\mathscr{E}$ is admissible, in the sense of Definition 3. Indeed, the admissibility condition now says that distinct elementary observables must be separated by some state of the physical system. Thus the restriction to admissible effect algebras appears crucial to allow an effect algebra to be the basis of a physical model.

We shall now list a number of facts that make the class of admissible effect algebras an interesting one, both from the mathematical and the physical point of view.

A first fact is that for an admissible effect algebra $\mathscr{E}$ it is meaningful to speak of the set $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$ of the affine functions from $\mathscr{S}(\mathscr{E})$ (which is now ensured to be nonempty) into $[0,1]$, that is the effects (in the sense of Definition 5) on the convex set $\mathscr{S}(\mathscr{E})$. It is then natural to ask whether there is a correspondence between the elements of $\mathscr{E}$ and the elements of $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$. The answer is partially contained in the next theorem.

Theorem 3: If $\mathscr{E}$ is an admissible effect algebra then the evaluation of its elements on $\mathscr{S}(\mathscr{E})$ defines the natural injective morphism (the evaluation map) $v: \mathscr{E} \to [o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$ given by $va(\alpha) := \alpha(a)$ for every $a \in \mathscr{E}, \alpha \in \mathscr{S}(\mathscr{E})$.

Proof: Clearly the evaluation map v is an embedding due to the fact that $\mathscr{S}(\mathscr{E})$ separates $\mathscr{E}$. If for $a,b \in \mathscr{E}$ the effect algebraic sum $a \oplus b$ is defined then $\alpha(a \oplus b) = \alpha(a) + \alpha(b)$ for every $\alpha \in \mathscr{S}(\mathscr{E})$. This implies that v preserves the effect algebra structure of $\mathscr{E}$. □

The fact expressed by the above theorem, that an admissible effect algebra can be embedded into an effect algebra of functions from a set into $[0,1]$, can be seen as a particular case of a more general fact. We have indeed the following lemma.

Lemma 4: An effect algebra is admissible if and only if there is an injective morphism of it into an effect algebra of functions from a set into $[0,1]$.

Proof: Having in mind Theorem 3, we have only to prove that every effect algebra $\mathscr{F}$ of functions from a set T into $[0,1]$ is admissible. Indeed, if $f_i : T \to [0,1], i=1,2,\ldots,$ belongs to $\mathscr{F}$ and if the pointwise sum $f=\Sigma_{i=1}^{\infty} f_i$ does exist in $\mathscr{F}$, then $f(\alpha)=\Sigma_{i=1}^{\infty} f_i(\alpha)$ for any $\alpha \in T$. This means that any $\alpha \in T$ defines a probability measure on $\mathscr{F}$, and moreover $\mathscr{F}$ is separated by these probability measures, so that it is admissible. □

The above lemma can be paraphrased by saying that every admissible effect algebra admits a representation in terms of fuzzy sets in some set: representations of this kind have been recently worked out by Dvurecenskij.[28] Notice that the representation of elementary observables by means of functions on the set of states appears also in the framework of quantum logic.[29,30]

According to Theorem 3 every element of an admissible effect algebra $\mathscr{E}$ generates an element of $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$, but the converse need not be true: it is not guaranteed that every element of $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$ comes from an element of $\mathscr{E}$. A counterexample can be found in the mathematical edifice of standard quantum mechanics: the projectors in a Hilbert space form an admissible effect algebra and Gleason's theorem says that (if the dimension of the Hilbert space is not less than 3) the probability measures on this effect algebra are just the density operators, but the affine functions on the density operators with values in $[0,1]$ are known to be in a one-to-one correspondence with the positive operators having mean value at every state not bigger than 1. The family of these positive operators is definitely bigger than the family of projectors.

Having in mind that for an admissible effect algebra $\mathscr{E}$ we can only assert that it can be embedded into $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$, we can ask which elements should be added to $\mathscr{E}$ in order to approach the whole $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$. As shown by the next theorem the answer is that what should be added are the convex combinations of the elements of $\mathscr{E}$ with respect to the convexity inherited from $\mathscr{S}(\mathscr{E})$.

Theorem 4: If $\mathscr{E}$ is an admissible effect algebra then the convex hull of $v(\mathscr{E})$ is w^*-dense in $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$.

Proof: The normed linear spaces $V(\mathscr{S}(\mathscr{E}))$ and $A^b(\mathscr{S}(\mathscr{E}))$ form a dual pair, hence the polar $[\mathrm{Conv}(\mathscr{S}(\mathscr{E}) \cup -\mathscr{S}(\mathscr{E}))]^o$ equals the unit ball $[-e_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}] = (2[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}] - e_{\mathscr{S}(\mathscr{E})})$ of $A^b(\mathscr{S}(\mathscr{E})$ (Banach-Alaoglu theorem, see e.g., Ref. 31). On the other hand $(2v(\mathscr{E})-1)^o = \mathrm{Conv}(\mathscr{S}(\mathscr{E}) \cup -\mathscr{S}(\mathscr{E}))$ and the bipolar theorem (see, e.g., Ref. 32) implies the assertion. □

Let us remark that not only an admissible effect algebra $\mathscr{E}$ is separated by $\mathscr{S}(\mathscr{E})$ but also, conversely, $\mathscr{S}(\mathscr{E})$ is separated by $\mathscr{E}$, since the former is a set of distinct probability measures on $\mathscr{E}$. Taking now into account the previous Theorem 3, we see that $\mathscr{S}(\mathscr{E})$ is also separated by $v(\mathscr{E}) \subset A^b(\mathscr{S}(\mathscr{E}))$ so that, by Lemma 2, $V(\mathscr{S}(\mathscr{E}))$ is a base-norm space and $A^b(\mathscr{S}(\mathscr{E}))$ is an order-unit space. Thus we see that an admissible effect algebra generates the dual pair $A^b(\mathscr{S}(\mathscr{E}))$, $V(\mathscr{S}(\mathscr{E}))$ typical of the convex, or operational, approach.

The connection between effect algebras and the structure of base-norm spaces, with their dual order-unit spaces, is further specified by the next theorem.

Theorem 5: Let $\mathscr{E}$ be an admissible effect algebra and let $W(\mathscr{E})$ denote the linear subspace of $A^b(\mathscr{S}(\mathscr{E}))$ spanned by $v(\mathscr{E})$. Then

(i) $\|\alpha\| = \sup\{|\alpha(a)| : a \in \mathscr{E}\}$ for every α belonging to the base-norm space $V(\mathscr{S}(\mathscr{E}))$;

(ii) $W(\mathscr{E})$ is an order-unit space under the norm inherited from $A^b(\mathscr{S}(\mathscr{E}))$.

Proof:

(i) The base norm is defined by $\|\alpha\| := \inf\{\lambda : \lambda \geq 0,\ \alpha \in \lambda \mathrm{Conv}(\mathscr{S}(\mathscr{E}) \cup -\mathscr{S}(\mathscr{E}))\}$ and we have $\mathrm{Conv}(\mathscr{S}(\mathscr{E}) \cup -\mathscr{S}(\mathscr{E})) = \{\alpha : |\alpha(a)| \leq 1,\ a \in \mathscr{E}\}$ so that $\alpha \in \lambda \mathrm{Conv}(\mathscr{S}(\mathscr{E}) \cup -\mathscr{S}(\mathscr{E}))$ if and only if $|\alpha(a)| \leq \lambda$ for every $a \in \mathscr{E}$. This implies that $\|\alpha\| = \sup\{|\alpha(a)| : a \in \mathscr{E}\}$.

(ii)The order-unit norm on $A^b(\mathscr{S}(\mathscr{E}))$ is clearly the sup-norm (see Lemma 2) and the same holds for the inherited norm on $W(\mathscr{E})$. Thus for any $a \in W(\mathscr{E})$ we have $\|a\| = \sup\{|a(\alpha)| : \alpha \in \mathscr{S}(\mathscr{E})\} = \inf\{\lambda : \lambda \geqslant 0,\ -\lambda \leqslant a(\alpha) \leqslant \lambda,\ \alpha \in \mathscr{S}(\mathscr{E})\} = \inf\{\lambda : \lambda \geqslant 0,\ -\lambda e(\alpha) \leqslant a(\alpha) \leqslant \lambda e(\alpha),\ \alpha \in \mathscr{S}(\mathscr{E})\} = \inf\{\lambda : \lambda \geqslant 0,\ a \in \lambda[-e,e]\}$. □

The above remarks, together with Lemma 2 provide a contribution to a research project formulated by Greechie and Foulis (see Ref. 1, item 4.6).

Come now to the notion of observable. It is natural to define the observables on the basis of the convex set $\mathscr{S}(\mathscr{E})$ which is interpreted as the set of states: explicitly, if Ξ is a measurable space then an observable taking values in Ξ is an affine map $\mathscr{S}(\mathscr{E}) \to M_1^+(\Xi)$.

In view of Lemma 1 every $\mathscr{E}$-valued measure on a measurable space Ξ defines an observable, but does every observable come from a $\mathscr{E}$-valued measure on some measurable space? In general the answer is no. A paradigmatic counterexample lies again in the framework of standard quantum mechanics. Take the effect algebra formed by all projection operators in a Hilbert space so that the $\mathscr{E}$-valued measures on $\mathbf{R}$ are simply the projection-valued measures on $\mathbf{R}$ (PV measures): a general observable on the set of states, namely an affine map from the convex set of density operators into $M_1^+(\mathbf{R})$, is now associated to a positive-operator-valued measure (POV measure) on $\mathbf{R}$, and the class of POV measures is definitely bigger than the class of PV measures.

Thus the property expressed by item (i) of Theorem 1 for a particular class of effect algebras (the ones occurring within the convex approach) does not hold for a generic admissible effect algebra.

Summing up, we have seen that for an admissible effect algebra $\mathscr{E}$ it is not guaranteed that every element of $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$ comes from an element of $\mathscr{E}$, nor it is guaranteed that every observable on $\mathscr{S}(\mathscr{E})$ comes from an $\mathscr{E}$-valued measure. These two facts are however strictly correlated, as specified by the following theorem.

Theorem 6: Let $\mathscr{E}$ be an admissible effect algebra. Every affine map $\mathscr{S}(\mathscr{E}) \to M_1^+(\Xi)$ defines an $\mathscr{E}$-valued measure on the measurable space Ξ if and only if every element of $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$ defines an element of $\mathscr{E}$.

Proof: Suppose that every affine map $\mathscr{S}(\mathscr{E}) \to M_1^+(\Xi)$ defines an $\mathscr{E}$-valued measure on Ξ. For arbitrary $a \in [o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$ and $\xi_1, \xi_2 \in \Xi$ take the $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$-valued measure E_a on Ξ defined by: $E_a(X) = a$ if $\xi_1 \in X$ and $\xi_2 \notin X$, $E_a(X) = e_{\mathscr{S}(\mathscr{E})} - a$ if $\xi_1 \notin X$ and $\xi_2 \in X$, $E_a(X) = \varnothing$ if $\xi_1, \xi_2 \notin X$, $E_a(X) = [o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$ if $\xi_1, \xi_2 \in X$. This $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$-valued measure E_a defines an affine map $A^{E_a} : \mathscr{S}(\mathscr{E}) \to M_1^+(\Xi)$ by $(A^{E_a}\alpha)(X) := (E_a(X))(\alpha)$, $\alpha \in \mathscr{S}(\mathscr{E})$, $X \in \mathscr{B}(\Xi)$. But the hypothesis ensures that E_a has to be $\upsilon(\mathscr{E})$-valued, hence both a and $e_{\mathscr{S}(\mathscr{E})} - a$ have to belong to $\upsilon(\mathscr{E})$. As a was arbitrary we conclude that υ is surjective.

To prove now the reverse property suppose that every element of $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$ defines an element of $\mathscr{E}$. By Theorem 1 (i) we know that any affine map $A : \mathscr{S}(\mathscr{E}) \to M_1^+(\Xi)$ defines a $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$-valued measure on Ξ. But our hypothesis now ensures that this is also a $\mathscr{E}$-valued measure. □

V. CONSISTENCY PROPERTIES

The following definitions will be useful.

Definition 6: A convex set S will be called consistent if there is an affine bijection between S and $\mathscr{S}([o_S, e_S])$, in short if $S = \mathscr{S}([o_S, e_S])$.

Definition 7: An admissible effect algebra $\mathscr{E}$ will be called consistent if there is a bijective morphism between $\mathscr{E}$ and $[o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$, in short if $\mathscr{E} = [o_{\mathscr{S}(\mathscr{E})}, e_{\mathscr{S}(\mathscr{E})}]$.

A familiar example of a consistent set of states occurs in the standard quantum description, while an example of consistent effect algebra is provided by the operational quantum mechanics as described in Ref. 12.

There is an intertwining between the above notions of consistency, as specified by the following lemma.

Lemma 5:

(i) If a convex set S is consistent then the admissible effect algebra $[o_S, e_S]$ is consistent;

(ii) if an admissible effect algebra $\mathscr{E}$ is consistent then the convex set $\mathscr{S}(\mathscr{E})$ is consistent.

Proof:

(i) Denote $[o_S,e_S]$ by $\mathscr{E}$. Then the consistency of S reads $S=\mathscr{S}(\mathscr{E})$, hence $\mathscr{E}=[o_{\mathscr{S}(\mathscr{E})},e_{\mathscr{S}(\mathscr{E})}]$, which is the consistency condition for $\mathscr{E}$;

(ii) denote $\mathscr{S}(\mathscr{E})$ by S. Then the consistency condition for $\mathscr{E}$ reads $\mathscr{E}=[o_S,e_S]$, hence $S=\mathscr{S}([o_S;e_S])$, which is the consistency condition for S. $\square$

Notice that Theorem 6 can now be rephrased by saying that an admissible effect algebra $\mathscr{E}$ is consistent if and only if every observable on $\mathscr{S}(\mathscr{E})$ comes from an $\mathscr{E}$-valued measure.

In Section III we have seen that the convex approach, based on a convex set S of states of the physical system, naturally generates the admissible effect algebra $[o_S,e_S]$. The density property expressed by Theorem 2 says that the states (the elements of S) are physically indistinguishable from the probability measures on $[o_S,e_S]$ since every element of $\mathscr{S}([o_S,e_S])$ can be approached with arbitrary accuracy by elements of S. This makes natural, and avoiding mathematical complications, to assume from the outset that the convex set S is consistent. Notice that the family of consistent convex sets includes the convex sets which are the convex hull of their extreme elements: in our context these are the sets of states which have pure states and such that every mixed state is a convex combination of pure states. Thus the restriction to consistent convex sets of states encompasses a familiar pattern of statistical physical models.

Notice also that a consistent convex set S is obviously separated by $[o_S,e_S]$, so that the hypothesis often made in Section III, notably in Theorem 1, is automatically met.

Thus the convex approach based on a consistent set of states S is perfectly closed with respect to the effect algebra naturally arising in it: the probability measures on $[o_S,e_S]$ give back the elements of S and the effect-valued measures are just the observables on S (see Theorem 1).

When one adopts an effect algebra $\mathscr{E}$, in particular an admissible one, as the basis of a physical model, the relevant facts are the ones reviewed in Section IV. Now the model is, in general, not closed: an admissible effect algebra $\mathscr{E}$ is, in general, a structure not sufficiently rich to ensure the equivalence between $\mathscr{E}$ itself and the effect algebra $[o_{\mathscr{S}(\mathscr{E})},e_{\mathscr{S}(\mathscr{E})}]$ built on the set $\mathscr{S}(\mathscr{E})$ of states, nor the equivalence between $\mathscr{E}$-valued measures and observables on $\mathscr{S}(\mathscr{E})$. The root of this situation lies in the fact that an effect algebra, even if admissible, does not carry an intrinsic convex structure. As a consequence, also the set of $\mathscr{E}$-valued measures (on some measurable space) is not intrinsically endowed with a notion of convexity.

To build a satisfactory physical model we have to start from a more structured object, that is from a consistent effect algebra. In this case the model becomes closed: the effects on the convex set of states $\mathscr{S}(\mathscr{E})$ correspond to the elements of $\mathscr{E}$, and the $\mathscr{E}$-valued measures are just the observables on $\mathscr{S}(\mathscr{E})$.

The assumption of consistency for the basic effect algebra is physically equivalent to complete $\mathscr{E}$ (more exactly its image $v(\mathscr{E})$ in $[o_{\mathscr{S}(\mathscr{E})},e_{\mathscr{S}(\mathscr{E})}]$) by all the convex combinations of its elements: in fact Theorem 4 ensures that every element of $[o_{\mathscr{S}(\mathscr{E})},e_{\mathscr{S}(\mathscr{E})}]$ can be approached with arbitrary accuracy by elements of the convex hull of $v(\mathscr{E})$. The completion of the effect algebra by these convex combinations is physically meaningful: the physical relevance of convex combinations of effects and of effect-valued measures is discussed, e.g., in Ref. 12 (in particular the Example 3, p. 10).

Summing up, the convex framework based on a consistent set of states is fully equivalent to the framework based on a consistent effect algebra: each one generates the other in a circular way.

ACKNOWLEDGMENTS

We are grateful to S. Gudder for critical remarks on a first draft of this paper; we thank also G. Cattaneo, M. L. Dalla Chiara, R. Giuntini, and M. Maczynski for very useful comments and suggestions.

APPENDIX: σ-MORPHISMS OF ADMISSIBLE EFFECT ALGEBRAS

We are going to comment on how a σ-morphism of admissible effect algebras can be implemented by means of an affine map between the convex sets of probability measures on the algebras.

Lemma 6: Let S be a linearly bounded convex set: the interval effect algebra $[o_S,e_S]$ is σ-complete.

Proof: According to Lemma 2, the linear boundedness of S implies that $A^b(S)$ is an order-unit Banach space, dual to the base norm space $V(S)$. In this context it is known that any increasing sequence in $[o_S,e_S]$ converges pointwise to its least upper bound.[24] □

Lemma 7: Let S_1,S_2 be linearly bounded convex sets. Any affine map $A:S_1\to S_2$ defines a σ-morphism $[o_{S_2},e_{S_2}]\to[o_{S_1},e_{S_1}]$.

Proof: Any affine map $A:S_1\to S_2$ extends linearly to the map $\widetilde{A}$: $V(S_1)\to V(S_2)$ and then defines the Banach dual $\widetilde{A}^*:A^b(S_2)\to A^b(S_1)$. Let $A^*:[o_{S_2},e_{S_2}]\to[o_{S_1},e_{S_1}]$ denote the restriction of $\widetilde{A}^*$ to $[o_{S_2},e_{S_2}]$. As $\widetilde{A}$ is norm continuous, the dual $\widetilde{A}^*$ is w*-continuous and consequently its restriction A^* is pointwise continuous. This implies that any pointwise convergent sequence of elements of $[o_{S_2},e_{S_2}]$ is transformed into a similar sequence in $[o_{S_1},e_{S_1}]$, what in turn implies that A^* is a σ-morphism. □

Theorem 7: Let $\mathscr{E}_1,\mathscr{E}_2$ be two admissible effect algebras, and $v_1:\mathscr{E}_1\to A^b(\mathscr{S}(\mathscr{E}_1))$, $v_2:\mathscr{E}_2\to A^b(\mathscr{S}(\mathscr{E}_2))$ the corresponding evaluation maps.

(i) Any σ-morphism $\phi:\mathscr{E}_1\to\mathscr{E}_2$ of the two admissible effect algebras defines an affine map $A^\phi:\mathscr{S}(\mathscr{E}_2)\to\mathscr{S}(\mathscr{E}_1)$ such that $\phi(a)=v_2^{-1}((A^\phi)^*(v_1(a)))$ for every $a\in\mathscr{E}_1$.

(ii) Any affine map $A:\mathscr{S}(\mathscr{E}_2)\to\mathscr{S}(\mathscr{E}_1)$ defines a σ-morphism $\phi^A:\mathscr{E}_1\to A^b(\mathscr{S}(\mathscr{E}_2))$ such that $A^{\phi^A}=A$.

Proof: (i) The composition of two σ-morphisms is a σ-morphism, hence $\alpha\circ\phi:\mathscr{E}_1\to\mathscr{E}_2\to[0,1]$ is a probability measure on $\mathscr{E}_1$ for any $\alpha\in\mathscr{S}(\mathscr{E}_2)$. Define $A^\phi:\mathscr{S}(\mathscr{E}_2)\to\mathscr{S}(\mathscr{E}_1)$ by $A^\phi(\alpha):=\alpha\circ\phi$; obviously A^ϕ is affine. For every $\alpha\in\mathscr{S}(\mathscr{E}_2)$ and every $a\in\mathscr{E}_1$ we have: $\alpha(v_2^{-1}((A^\phi)^*(v_1(a))))=((A^\phi)^*(v_1(a)))(\alpha)=(v_1(a))(A^\phi(\alpha))=(A^\phi(\alpha))(a)=\alpha(\phi(a))$, hence $\phi(a)=v_2^{-1}((A^\phi)^*(v_1(a)))$ because $\mathscr{S}(\mathscr{E}_2)$ separates $\mathscr{E}_2$ (the admissibility condition).

(ii) For any $a\in\mathscr{E}_1$, the composition $v_1(a)\circ A:\mathscr{S}(\mathscr{E}_2)\to\mathscr{S}(\mathscr{E}_1)\to[0,1]$ is an affine function on $\mathscr{S}(\mathscr{E}_2)$ hence it belongs to $A^b(\mathscr{S}(\mathscr{E}_2))$. The map $\mathscr{E}_1\to A^b(\mathscr{S}(\mathscr{E}_2))$ obtained in this way will be denoted ϕ^A: clearly $\phi^A=A^*\circ v_1$ and ϕ^A is a σ-morphism because both A^* and v_1 are σ-morphisms. For every $\alpha\in\mathscr{S}(\mathscr{E}_2)$ and $a\in\mathscr{E}_1$ we have: $(A^{\phi^A}\alpha)(a)=(\phi^A(a))(\alpha)=(A^*(v_1(a)))(\alpha)=(v_1(a))(A\alpha)=(A\alpha)(a)$, hence $A^{\phi^A}=A$. □

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