

1 Ordered vector spaces

Overall reference: [4]

1.1 Basic definitions

Let X be a real vector space. A subset $A \subseteq X$ is

- algebraically open (closed) if the intersection of any line with A is an open (closed) subset of the line
- linearly bounded if the intersection of A with any line is a bounded subset of the line

We say that $a \in A$ is an algebraic interior point of A if it is an interior point of the intersection of any line with A , that is, for any $x \in X$ there is some $\delta > 0$ such that $a + sx \in A$ for all $|s| \leq \delta$. The set of all such points is called the algebraic interior of A and is denoted by $\text{aint}(A)$. The algebraic closure of A is $\text{acl}(A) := X \setminus \text{aint}(X \setminus A)$. If A is convex, then

$$\text{acl}(A) = \{x \in X, \exists y \in X, x + \lambda y \in A, \forall \lambda \in (0, 1)\}.$$

A is algebraically open iff $A = \text{aint}(A)$ and algebraically closed iff $A = \text{acl}(A)$. If A is convex, then both $\text{aint}(A)$ and $\text{acl}(A)$ are convex as well.

Remark 1. (cf. [5, §16]) If A is convex, then $\text{aint}(A)$ is algebraically open, but in general $\text{aint}(\text{aint}(A)) \subsetneq \text{aint}(A)$. The algebraic closure is not necessarily algebraically closed even if A is convex. The counterexample is as follows. Let X be an infinite dimensional vector space with algebraic basis $\{x_\alpha\}$. Put

$$A = \{x = \sum_{\alpha} c_{\alpha} x_{\alpha}, c_{\alpha} \geq 0 \forall \alpha, \sum_{\alpha} c_{\alpha} \geq \frac{1}{n(x)}\}$$

where $n(x) = \#\{\alpha, x = \sum_{\alpha} c_{\alpha} x_{\alpha}, c_{\alpha} \neq 0\}$. Then A is convex, $\text{acl}(A) = \{x = \sum_{\alpha} c_{\alpha} x_{\alpha}, c_{\alpha} \geq 0 \forall \alpha, x \neq 0\}$ and $\text{acl}(\text{acl}(A))$ contains 0, so that $\text{acl}(A) \subsetneq \text{acl}(\text{acl}(A))$. On the other hand, if A is convex and $\text{aint}(A) \neq \emptyset$, then $\text{acl}(\text{acl}(A)) = \text{acl}(A)$.

Wedges, cones and orderings

A subset $P \subseteq X$ is called a wedge if $P + P \subseteq P$ and $\lambda P \subseteq P$ for any $\lambda \geq 0$. The preorder $x \leq y$ if $x - y \in P$ is compatible with the linear structure, such a preorder is called an ordering in X . Conversely, for any ordering, the set of positive elements is a wedge.

The pair (X, P) where P is a wedge is called an ordered vector space. The corresponding ordering is a partial order iff $P \cap -P = \{0\}$, in this case P is called a cone. X with this ordering is directed iff P is generating, that is, $X = P - P$.

Positive maps

Let (X, P) and (Y, Q) be ordered vector spaces. A linear map $F : X \rightarrow Y$ is called positive if $F(P) \subseteq Q$. If F is invertible with positive inverse, we say that F is an order isomorphism.

Let (P, Q) denote the set of positive maps, then (P, Q) is a wedge in the vector space $L(X, Y)$ of all linear maps $X \rightarrow Y$. We have

Lemma 1. *(P, Q) is a cone if and only if P is generating and Q is a cone.*

Archimedean and almost Archimedean orderings

Let (X, P) be an ordered vector space. We say that the ordering (or P) is Archimedean if $x \leq \lambda y$ for some $y \in X$ and all $\lambda > 0$ implies that $x \leq 0$.

Proposition 1. *The following are equivalent.*

- (i) *the ordering is Archimedean.*
- (ii) $\exists y \in X, \epsilon > 0$ such that $x \leq \lambda y$ for all $\epsilon \geq \lambda > 0 \implies x \leq 0$.
- (iii) $\exists y \in P, \epsilon > 0$ such that $x \leq \lambda y$ for all $\epsilon \geq \lambda > 0 \implies x \leq 0$.
- (iv) $P = \text{acl}(P)$.

The ordering (or P) is almost Archimedean if $-\lambda y \leq x \leq \lambda y$ for some $y \in X$ and all $\lambda > 0$ implies $x = 0$.

Proposition 2. *The following are equivalent.*

- (i) *the ordering is almost Archimedean.*

(ii) $\exists y \in X, \epsilon > 0$ such that $-\lambda y \leq x \leq \lambda y$ for all $\epsilon \geq \lambda > 0 \implies x = 0$.

(iii) $\exists y \in P, \epsilon > 0$ such that $-\lambda y \leq x \leq \lambda y$ for all $\epsilon \geq \lambda > 0 \implies x = 0$.

(iv) $\text{acl}(P)$ is a cone.

Remark 2. Note that an almost Archimedean wedge must be a cone. An Archimedean wedge is almost Archimedean iff it is a cone.

1.2 Order units and bases

Order units and seminorms

An element $u \in X$ is an order unit in (X, P) if for any $x \in X$, there is some $\lambda \in \mathbb{R}^+$ such that $x \leq \lambda u$. This is equivalent to $u \in \text{aint}(P)$. If $\text{aint}(P) \neq \emptyset$, P is generating.

If u is an order unit, then P is (almost) Archimedean iff u is (almost) Archimedean: $x \leq \lambda u$ for all $\lambda > 0$ implies $x \leq 0$ (resp. $-\lambda u \leq x \leq \lambda u$ for all $\lambda > 0$ implies $x = 0$).

For an order unit u , put

$$\|x\|_u = \inf\{\lambda > 0, -\lambda u \leq x \leq \lambda u\}.$$

Then $\|\cdot\|_u$ is a seminorm in X . It is a norm iff u is almost Archimedean.

Remark 3. If $u_1, u_2 \in \text{aint}(P)$, the associated seminorms $\|\cdot\|_{u_1}$ and $\|\cdot\|_{u_2}$ are equivalent. The corresponding topology is thus a property of the ordering rather than the order unit. In fact, this topology is the finest locally convex topology making all order intervals bounded.

Lemma 2. Let u be Archimedean. Then $[-u, u] = \{x \in X, \|x\|_u \leq 1\}$ and the wedge P is closed in the topology given by $\|\cdot\|_u$.

Proof. Let $x \in [-u, u]$, then clearly $\|x\|_u \leq 1$. Conversely, assume that $\|x\|_u \leq 1$, then $-(1 + \epsilon)u \leq x \leq (1 + \epsilon)u$ for all $\epsilon > 0$. This implies that $\pm x - u \leq \epsilon u$ for all $\epsilon > 0$ and since u is Archimedean, this implies $\pm x \leq u$, that is, $x \in [-u, u]$.

For the second statement, let $x \in \bar{P}$ (the closure of P w.r. to $\|\cdot\|_u$). Then for all $n \in \mathbb{N}$, there is some $p_n \in P$ such that $\|x - p_n\|_u \leq \frac{1}{n}$. This implies that $-x \leq p_n - x \leq \frac{1}{n}u$ for all n and since u is Archimedean, $-x \leq 0$, so that $x \in P$.

□

Let (X_i, P_i) , $i = 1, 2$ be ordered vector spaces and let $u_i \in X_i$ be order units. A map $f : X_1 \rightarrow X_2$ is called unital if $f(u_1) = u_2$. The following is immediate.

Proposition 3. *Let (X_i, P_i, u_i) , $i = 1, 2$ be order unit spaces. Any positive unital map $f : X_1 \rightarrow X_2$ is a contraction with respect to the seminorms $\|\cdot\|_{u_1}$ and $\|\cdot\|_{u_2}$.*

A triple (X, P, u) where X is a vector space, $P \subseteq X$ an Archimedean cone and $u \in \text{int}(P)$ is called an order unit space. To summarize, in this case, $\|\cdot\|_u$ is a norm in X , $[-u, u]$ is the corresponding closed unit ball and P is norm closed. The category of order unit spaces with positive unital maps as morphisms will be denoted by **AOUS**.

Bases and seminorms

Let (X, P) be an ordered vector space. A convex subset $K \subset P$ is called a base of P if for any nonzero $p \in P$ there is a unique $\lambda > 0$ such that $\lambda p \in K$.

Lemma 3. *Any wedge with a base is a cone.*

Proof. Let K be a base of a wedge P , and let $0 \neq x \in P \cap -P$. Then there are $\lambda, \mu > 0$ such that $\lambda x = x_1 \in K$ and $-\mu x = x_2 \in K$. It follows that $\lambda^{-1}x_1 = -\mu^{-1}x_2$ and then $\frac{\mu}{\lambda+\mu}x_1 + \frac{\lambda}{\lambda+\mu}x_2 = 0$. Since K is convex, we obtain $0 \in K$, but then for any $p \in K$, $\lambda p \in K$ for all $\lambda \in [0, 1]$. Hence P must be a cone. □

Proposition 4. *A wedge P has a base if and only if there exists a linear functional ξ on X which is strictly positive on P . In this case, we may put $K = \{p \in P, \xi(p) = 1\}$.*

Proof. Let K be a base of P . For $p \in P$, let $\xi(p)$ be the unique positive number such that $\xi(p)^{-1}p \in K$. Then clearly $\xi(sp) = s\xi(p)$. Further, let $p, q \in P$ and let $\alpha = \xi(p) + \xi(q)$, then

$$\alpha^{-1}(p + q) = \frac{\xi(p)}{\alpha}\xi(p)^{-1}p + \frac{\xi(q)}{\alpha}\xi(q)^{-1}q \in K,$$

so that $p \mapsto \xi(p)$ is an additive function $\xi : P \rightarrow \mathbb{R}^+$. The function ξ easily extends to $P - P$ and has an extension to all of X by Hahn-Banach theorem. This extension is obviously positive and $K = \{p \in P, \xi(p) = 1\}$.

Conversely, let $\xi : X \rightarrow \mathbb{R}$ be strictly positive, then $K = \{p \in P, \xi(p) = 1\}$ is a convex subset of P and $\xi(p)^{-1}p \in K$ for any $p \in P$. Uniqueness is obvious. □

Proposition 5. ([3]) *Let P be a generating cone in a vector space X and let K be a base of P . For $x \in X$, put*

$$\|x\|_K := \inf\{a + b, x = ap - bq, a, b \in \mathbb{R}^+, p, q \in K\}.$$

This defines a seminorm in X , which is a norm if and only if $S := \text{co}(K \cup -K)$ is linearly bounded.

Proof. It can be checked easily that $\|\cdot\|_K$ is a seminorm. Note also that $x \in S$ implies $\|x\|_K \leq 1$. Indeed, any $x \in S$ has the form $x = \lambda p - (1 - \lambda)q$ for some $\lambda \in [0, 1]$, $p, q \in K$ and then $\|x\|_K \leq \lambda + (1 - \lambda) = 1$. Assume that $\|\cdot\|_K$ is a norm and let $x_t := x + ty$ be a line in X . Then $\|y\|_K > 0$ and $x_t \in S$ implies that $1 \geq \|x_t\|_K \geq |\|x\|_K - |t|\|y\|_K|$, so that $|t| \leq \frac{1 + \|x\|_K}{\|y\|_K}$. Conversely, assume that S is linearly bounded and let $\|x\|_K = 0$. This implies $tx \in S$ for all $t \in \mathbb{R}$, hence we must have $x = 0$. □

The (semi)norm in the above proposition is called the base (semi)norm in X .

Remark 4. Note that $\|\cdot\|_K$ is the Minkowski functional of S , that is

$$\|x\|_K = \inf\{\lambda > 0, x \in \lambda S\}.$$

To see this, observe that $S = \{sp - (1 - s)q, s \in [0, 1], p, q \in K\}$. Denote the Minkowski functional by p_S . If $x = ap - bq$ for some $a, b \in \mathbb{R}^+$ and $p, q \in K$, then if $a + b = 0$, we must have $x = 0$ and the equality obviously holds. Otherwise,

$$x = (a + b)\left(\frac{a}{a + b}p - \frac{b}{a + b}q\right) \in (a + b)S,$$

so that $p_S(x) \leq \|x\|_K$. On the other hand, let $x \in \lambda S$ for some $\lambda > 0$. Then $x = \lambda(sp - (1 - s)q)$ for some $s \in [0, 1]$ and $p, q \in K$, so that

$$\|x\|_K \leq \lambda s + \lambda(1 - s) = \lambda,$$

hence $\|x\|_K \leq p_S(x)$.

Remark 5. Linear boundedness of K is in general not enough. There are some weird infinite dimensional examples such that K is linearly bounded but $\text{co}(K \cup -K)$ is not.

Let (X_i, P_i) be ordered vector spaces and let K_i be a base of P_i , $i = 1, 2$. A linear map $f : X_1 \rightarrow X_2$ is called base-preserving if $f(K_1) \subseteq K_2$.

Proposition 6. *Any base-preserving linear map is positive and contractive with respect to the base seminorms.*

A triple (X, P, K) , where X is a vector space, P a generating cone and K a base of P such that $\text{co}(K \cup -K)$ is linearly bounded is called a base-normed space. The category of base-normed spaces with base-preserving linear maps will be denoted by **BN**.

Some examples

The wedges X and $\{0\}$ are trivial.

1. The only nontrivial wedges in \mathbb{R} are \mathbb{R}^+ and \mathbb{R}^- . \mathbb{R} with the usual ordering and norm coincides with both the order unit space $(\mathbb{R}, \mathbb{R}^+, 1)$ and the base-normed space $(\mathbb{R}, \mathbb{R}^+, \{1\})$.
2. **Function spaces:** Let S be a set, $X = \{f : S \rightarrow \mathbb{R}\}$, $P = \{f, f(S) \subseteq \mathbb{R}^+\}$. P is an Archimedean cone, (X, \leq) is a lattice. If S is not finite, $\text{aint}(P) = \emptyset$.
3. As 2, but bounded functions. In this case P is an Archimedean cone, $\text{aint}(P)$ is the set of strictly positive functions.
4. Let K be a convex set, we will denote the set of all affine functions $K \rightarrow \mathbb{R}$ by $A(K)$, the set of bounded affine functions by $A_b(K)$. If K is also a topological space, we denote the set of continuous affine functions by $A_c(K)$. We also denote by $A(K)^+$ ($A_b(K)^+$, $A_c(K)^+$) the set of positive affine (bounded, continuous) functions and 1_K the constant function $1_K(x) \equiv 1$. Then $(A_b(K), A_b^+(K), 1_K)$ is an order unit space, with $\|f\|_{1_K} = \sup_{x \in K} |f(x)|$. If K is also a compact Hausdorff topological space, then the same is true for $(A_c(K), A_c^+(K), 1_K)$.
5. $X = \{f : \mathbb{R} \rightarrow \mathbb{R}\}$, with the cone of nondecreasing functions.

6. **Sequence spaces:** X the set of all (or bounded, summable, convergent, converging to 0,...) sequences, with usual positive cone.
7. \mathbb{R}^2 with the usual or lexicographic ordering, with $P = \{(x, y), x > 0, y > 0\} \cup \{0\}$ or $P = \{(x, y), x > 0\} \cup \{0\}$.

Completeness

We give some sufficient conditions for completeness of order unit norms and base norms.

Proposition 7. [4] *Let (X, P) be an ordered vector space with an almost Archimedean order unit u . If every majorized increasing sequence in (X, P) has a supremum, then $(X, \|\cdot\|_u)$ is complete.*

Proof. We first show that any increasing Cauchy sequence has a limit. So let $\{x_n\}$ be such a sequence and let $\epsilon > 0$. Then $\|x_n - x_m\|_u < \epsilon$ for $m, n \geq N$. We then have for all $m \geq N$, $x_m - x_N \leq \epsilon u$, so that $x_m \leq x_N + \epsilon u$. It follows that $\{x_n\}$ is a majorized increasing sequence, so that there is some x_0 such that $x_0 = \sup_n x_n$. For all $m, n \geq N$, we have $x_n \leq x_m + \epsilon u$, hence $x_0 \leq x_m + \epsilon u$ and we have $0 \leq x_0 - x_m \leq \epsilon u$. This implies $\|x_0 - x_m\|_u \leq \epsilon$ for all $m \geq N$, so that $\lim_n x_n = x_0$.

Let now $\{x_n\}$ be any Cauchy sequence. Let $V_n = \{p - q, p, q \in [0, 2^{-n}]\}$, then V_n contains the ball with center 0 and radius 2^{-n+1} and is therefore a neighborhood of 0. Hence there is a subsequence such that $x_n - x_{n-1} \in V_n$. Let $a_n, b_n \in [0, 2^{-n}]$ be such that $x_n - x_{n-1} = a_n - b_n$. Then $\{\sum_{k=1}^n a_k\}$ and $\{\sum_{k=1}^n b_k\}$ are increasing Cauchy sequences and hence have a limit by the first part of the proof. Moreover, we have $x_n = \sum_{k=1}^n (a_k - b_k)$, so that x_n converges as well. □

Proposition 8. [1] *Let K be a base of P and assume that K is compact with respect to some Hausdorff topology τ , compatible with the linear structure of X . Then X is $\|\cdot\|_K$ -complete.*

Proof. Note that $S = co(K \cup -K)$ is also τ -compact, hence must be linearly bounded. It follows that $\|\cdot\|_K$ is a norm and it is easy to verify that S is the closed unit ball. Let $\{x_n\}$ be a Cauchy sequence, then it is norm-bounded, so we may assume that $\{x_n\} \subset S$. Let $y \in S$ be a τ -accumulation point of $\{x_n\}$.

For $\epsilon > 0$, $\|x_n - x_m\|_K < \epsilon$ for $n, m \geq N$. This implies that $x_n \in x_N + \epsilon S$ for $n \geq N$. Since S is τ -closed, $y \in x_N + \epsilon S$. It follows that

$$\|y - x_n\|_K \leq \|y - x_N\|_K + \|x_N - x_n\|_K \leq 2\epsilon,$$

this finishes the proof. □

1.3 Duality

Positive functionals

Let (X, P) be an ordered vector space and let X' denote the algebraic dual of X . The dual wedge of P is defined as

$$P' := \{\varphi \in X', \varphi(p) \geq 0, \forall p \in P\}$$

Note that $P' = (P, \mathbb{R}^+)$ and it follows by Lemma 1 that P' is a cone iff P is generating.

Remark 6. To see the above duality in this specific case, note that P is a generating wedge in the subspace $P - P$, whose algebraic dual can be identified with the quotient space $X'|_{(P-P)^\perp}$. Here

$$(P - P)^\perp = \{\varphi \in X', \varphi(x) = 0, \forall x \in P - P\} = P' \cap -P'.$$

If $P - P = X$, then $P' \cap -P' = (P - P)^\perp = \{0\}$, so P' is a cone. Conversely, if P' is a cone, then

$$P - P = (P - P)^{\perp\perp} = \{0\}^\perp = X,$$

this holds since any subspace $E \subseteq X$ satisfies $E^{\perp\perp} = E$. However, this is no longer true for subspaces in X' ([5]), so a dual statement does not hold. More precisely, it is easily checked that $P \cap -P \subseteq (P' - P')^\perp$, so that if P' is generating, P must be a cone. The converse is not true in general: we only have $P' - P' \subseteq (P' - P')^{\perp\perp} = (P \cap -P)^\perp$, so P' may be not generating even if P is a cone (there are indeed counterexamples).

It is also clear from the definition that P' is Archimedean.

The dual of a vector space with an order unit

Let (X, P) be an ordered vector space with an order unit u . Positive unital linear functionals are called states, the set of all states will be denoted by $\mathcal{S}(X, P, u)$. We will study the subspace of $\|\cdot\|_u$ -bounded functionals

$$X^* := \{\varphi \in X', \|\varphi\|_u^* := \sup_{\|x\|_u \leq 1} |\varphi(x)| < \infty\}.$$

Note that $\|\cdot\|_u^*$ is a norm in X^* and $(X, \|\cdot\|_u^*)$ is a Banach space.

Lemma 4. (i) Any $\varphi \in P'$ is $\|\cdot\|_u$ -bounded, with

$$\|\varphi\|_u^* := \sup_{\|x\|_u \leq 1} |\varphi(x)| = \varphi(u)$$

(ii) $\mathcal{S}(X, P, u)$ is a base of P' .

(iii) If $\varphi \in X'$ is such that $\|\varphi\|_u^* = \varphi(u)$, then $\varphi \in P'$.

(iv) For $x \in X$, $\|x\|_u = \sup_{\varphi \in \mathcal{S}(X, P, u)} |\varphi(x)|$.

Proof. (i) is quite easy. This also implies that u is strictly positive over P' , hence the set of states forms a base of P' by Proposition 4. For (iii), we may assume $\varphi(u) = 1$. Let $x \in P$ and let $\lambda > 0$ be such that $0 \leq x \leq \lambda u$. Then $\|x - \lambda u\|_u \leq \lambda$ and we have

$$|\varphi(x) - \lambda| = |\varphi(x - \lambda u)| \leq \|\varphi\|_u^* \|x - \lambda u\|_u \leq \lambda.$$

This implies $\varphi(x) \geq 0$. For (iv), let $x \in X$ be such that $-\lambda u \leq x \leq \lambda u$, then $|\varphi(x)| \leq \lambda$ for any $\varphi \in \mathcal{S}(X, P, u)$, so that $\sup_{\varphi \in \mathcal{S}(X, P, u)} |\varphi(x)| \leq \|x\|_u$. Assume that this inequality is strict for some x_0 , we may put $\|x_0\|_u = 1$. Then there is some $0 < a < 1$ such that $|\varphi(x_0)| < a$ for any state φ . Let $H = \{x \in X, x \leq au\}$, then we have either $x_0 \notin H$ or $-x_0 \notin H$. Note that $H = au - P$ is a convex set such that $\text{aint}(H) \neq \emptyset$. If $y \notin H$, then by a separation theorem by Edelheit [4, 0.2.4], there is some nonzero $\psi \in X'$ such that $\sup_{x \in H} \psi(x) \leq \psi(y)$. This implies that ψ is bounded below on P , so that we must have $\psi \in P'$. Normalizing, we may assume that $\psi \in \mathcal{S}(X, P, u)$. Then

$$a = \psi(au) \leq \sup_{x \in H} \psi(x) \leq \psi(y).$$

Since we may take either x_0 or $-x_0$ for y , we have arrived at a contradiction. \square

Theorem 1. *Let (X, P) be an ordered vector space with an order unit u and let $K = \mathcal{S}(X, P, u)$. Then $X^* = P' - P'$ and (X^*, P', K) is a base-normed space, with $\|\cdot\|_K = \|\cdot\|_u^*$.*

Proof. [3] By Lemma 4 any $\varphi \in P' - P'$ is $\|\cdot\|_u$ -bounded. For the converse, let $Y = X \times X$ be ordered by the wedge $Q = P \times P$, then (u, u) is an order unit in (Y, Q) . Let

$$Z = \{t(u, u) - (x, -x), \ t \in \mathbb{R}, x \in X\},$$

then Z is a linear subspace in Y containing the order unit. Let $\varphi \in X'$ be $\|\cdot\|_u$ -bounded and put

$$F_\varphi(z) = t\|\varphi\|_u^* - \varphi(x), \quad z = t(u, u) - (x, -x) \in Z$$

This defines a linear functional on Z . Moreover, note that $z = t(u, u) - (x, -x) \in Q$ iff $\|x\|_u \leq t$ and then $F_\varphi(z) \geq (t - \|x\|_u)\|\varphi\|_u^* \geq 0$. Since Z contains the order unit, F_φ extends to a positive linear functional on Y ([4, Corollary 1.6.2]). Put

$$\psi_1(x) = F_\varphi(x, 0), \quad \psi_2(x) = F_\varphi(0, x), \quad x \in X.$$

Then $\psi_1, \psi_2 \in P'$ and $\varphi = \psi_2 - \psi_1$, so that $\varphi \in P' - P'$. We have

$$\|\varphi\|_u^* = F_\varphi(u, u) = \psi_1(u) + \psi_2(u) \geq \|\varphi\|_K$$

On the other hand, let $\varphi = a\varphi_1 - b\varphi_2$ with $a, b \geq 0$, $\varphi_1, \varphi_2 \in K$, then $\|\varphi\|_u^* \leq a + b$, this shows that $\|\varphi\|_u^* = \|\varphi\|_K$. To finish the proof, we have to show that $\|\cdot\|_K$ is a norm. So let $\|\varphi\|_K = 0$, then φ is zero over the absorbing set $[-u, u]$, so that $\varphi = 0$. □

Corollary 1. *The norm dual of an ordered vector space with order unit norm is a base-normed space, with base formed by the set of states.*

Corollary 2. *Let (X, P) be an ordered vector space with an order unit u .*

- (i) *P is almost Archimedean iff the set of states is separating.*
- (ii) *P is Archimedean iff the set of states is order-determining.*

Proof. (i) is immediate from Lemma 4 (i). For (ii), assume that P is an Archimedean wedge and let $x \in X$ be such that $\varphi(x) \geq 0$ for all states φ . Assume $x \notin P$, then, since P is algebraically closed and $\text{aint}(P) \neq \emptyset$, there exists a functional $\psi \in X'$ such that $\psi(x) < c \leq \inf_{p \in P} \psi(p)$ ([5, 17.5.(2)]). This implies that $\psi(p) \geq 0$ for all $p \in P$, so that $\psi \in P'$. By normalization, we obtain a state with $\psi(x) < 0$, a contradiction.

Conversely, let $L = \{x_t\}$ be any line in X , then $x_t \in P$ iff $\varphi(x_t) \geq 0$ for all states φ , so that $L \cap P$ is a closed subset in L , hence P is algebraically closed. □

The dual of an ordered vector space with a based cone

Let (X, P) be an ordered vector space and let $K \subset P$ be a base of P . By Proposition 4, there is a strictly positive linear functional $u \in X'$, such that $K = \{p \in P, u(p) = 1\}$. Again, we study the subspace of $\|\cdot\|_K$ -bounded functionals

$$X^* = \{\varphi \in X', \|\varphi\|_K^* := \sup_{x \in P-P, \|x\|_K \leq 1} |\varphi(x)| < \infty\}.$$

Note that here the seminorm is defined only on the subspace $P - P$, so that $\|\cdot\|_K^*$ is in general only a seminorm.

Theorem 2. (i) Let $\varphi \in X'$, then $\varphi \in X^*$ iff φ is bounded over K and

$$\|\varphi\|_K^* = \sup_{x \in K} |\varphi(x)|.$$

(ii) Let $P^* = P' \cap X^*$, then u is an Archimedean order unit in (X^*, P^*) and $\|\cdot\|_u = \|\cdot\|_K^*$.

(iii) If $X = P - P$, then (X^*, P^*, u) is an order unit space, isomorphic to $(A_b(K), A_b(K)^+, 1_K)$ (see Example 4) (we mean a unital order isomorphism).

Proof. Let $\varphi \in X'$ and let $x = ax_1 - bx_2$ with $a, b \geq 0$ and $x_1, x_2 \in K$. Then

$$|\varphi(x)| \leq a|\varphi(x_1)| + b|\varphi(x_2)| \leq (a+b) \sup_{x \in K} |\varphi(x)|.$$

It follows that $\varphi \in X^*$ if it is bounded over K , and also that $\|\varphi\|_K^* \leq \sup_{x \in K} |\varphi(x)|$. The rest of (i) is easy.

Since the dual wedge is Archimedean, P^* is Archimedean as well and $u \in P^*$ by (i). We also obtain that $\|\varphi\|_K^* \leq M$ iff $-Mu(x) \leq \varphi(x) \leq Mu(x)$ for all $x \in K$ and since K is a base of P , this is equivalent to $-Mu \leq \varphi \leq Mu$, so that u is an order unit in (X^*, P^*) , with $\|\varphi\|_u = \|\varphi\|_K^*$, this proves (ii).

For (iii), assume that $X = P - P$. Then P' , and hence also P^* , is an Archimedean cone, so that (X^*, P^*, u) is an order unit space. Further, any element $\varphi \in X'$ restricts to a function $\varphi|_K \in A(K)$ and conversely, any function in $A(K)$ extends to a unique linear functional in X' . It is easy to see from (i) and (ii) that this defines a unital order isomorphism between (X^*, P^*, u) and $(A_b(K), A_b(K)^+, 1_K)$. □

Corollary 3. *The norm dual of a base-normed space is an order unit space.*

Preduals

We next discuss the Banach space preduals of order unit and base-normed spaces. Below, $(X, \|\cdot\|)$ is a Banach space and $(X^*, \|\cdot\|^*)$ its norm dual. If $P \in X$ is a wedge, we will denote

$$P^* := \{\varphi \in X^*, \varphi(p) \geq 0, \forall p \in P\} = P' \cap X^*.$$

Similarly, if Q is a wedge in X^* , we will denote

$$Q_* := \{x \in X, q(x) \geq 0, \forall q \in Q\} = Q' \cap X.$$

It is clear that P^* and Q_* are wedges. Moreover, $(P^*)_* = \bar{P}$ is the norm=weak closure of P and $(Q_*)^*$ is the weak*-closure of Q .

Theorem 3. [3, 2] *If X^* is an order unit space, then X is base-normed. More precisely, if there is an Archimedean cone $Q \subset X^*$ with an order unit u such that $\|\cdot\|^* = \|\cdot\|_u$, then $Q_* \subset X$ has a base $K = \{p \in Q_*, u(p) = 1\}$ and (X, Q_*, K) is a base-normed space with $\|\cdot\| = \|\cdot\|_K$.*

Proof. Note first that the cone Q^* is weak*-closed. Indeed, by Lemma 2, $[-u, u]$ is the closed unit ball B_1^* in $(X^*, \|\cdot\|^*)$ and is therefore weak*-compact by Banach-Alaoglu theorem. Hence also $u + [-u, u] = [0, 2u]$ is weak*-compact, so that this is true for $Q \cap \rho B_1^* = [0, \rho u]$ for any $\rho \geq 0$. By Krein-Smulyan theorem, this implies that Q is weak*-closed.

Let $p \in Q_*$ be such that $u(p) = 0$, then for any $\varphi \in Q$,

$$0 \leq \varphi(p) \leq \|\varphi\|_u u(p) = 0.$$

Since $X^* = Q - Q$ separates points in X , we obtain $p = 0$. Hence u defines a strictly positive linear functional on (X, Q_*) and K is a base of Q_* . For $p \in Q_*$, we have

$$\|p\| = \sup_{\varphi \in [-u, u]} |\varphi(p)| = u(p).$$

It follows that $S = \text{co}(K \cup -K)$ is a subset of the closed unit ball B_1 of X , so that $\|\cdot\| \leq \|\cdot\|_K$ (since $\|\cdot\|_K$ is the Minkowski functional of S and $\|\cdot\|$ is the Minkowski functional of B_1). We next show that S is dense in B_1 . Since $Q = (Q_*)^*$, we have for $\varphi \in X^*$:

$$\begin{aligned} \|\varphi\|_u &= \inf\{\lambda > 0, \lambda u \pm \varphi \in Q\} = \inf\{\lambda > 0, (\lambda u \pm \varphi)(p) \geq 0, \forall p \in Q_*\} \\ &= \inf\{\lambda > 0, |\varphi(p)| \leq \lambda, \forall p \in K\} = \sup_{p \in K} |\varphi(p)|. \end{aligned}$$

Assume that $x_0 \in B_1 \setminus \bar{S}$, then by Hahn-Banach separation theorem, there is some $\varphi \in X^*$ such that

$$\|\varphi\|_u = \sup_{p \in K} |\varphi(p)| \leq \sup_{x \in S} \varphi(x) < \varphi(x_0) \leq \|\varphi\|^* = \|\varphi\|_u,$$

a contradiction. It follows that $\bar{S} = B_1$.

Choose any $\alpha > 1$ and let $\alpha_n > 0$ be a sequence such that $1 + \sum_n \alpha_n < \alpha$. For $x_0 \in B_1 = \bar{S}$ there is some element $x_1 \in S$ such that $\|x_0 - x_1\| < \alpha_1$, that is, $x_0 - x_1 \in \alpha_1 B_1$. But then similarly there is some $x_2 \in \alpha_1 S$ such that $\|x_0 - x_1 - x_2\| < \alpha_2$. Continuing by induction, we obtain a sequence $\{x_n\}$ in X such that $\|x_n\|_K \leq \alpha_{n-1}$ and $\|x_0 - \sum_n x_n\| < \alpha_n \rightarrow 0$. Hence

$$\|x_0\|_K = \left\| \sum_n x_n \right\|_K \leq \sum_n \|x_n\|_K \leq 1 + \sum_n \alpha_n < \alpha,$$

so that $x_0 \in \alpha S$. Since the above inequality holds for all $\alpha > 1$, we have $1 = \|x_0\| \leq \|x_0\|_K \leq 1$. It also follows that $B_1 \subset \alpha S$ for any $\alpha > 1$ and consequently $X = Q_* - Q_*$. □

Theorem 4. *Let $(X^*, \|\cdot\|^*)$ be a base-normed space with a positive cone Q having a weak*-compact base K . Let u be a strictly positive functional corresponding to K . Then $u \in X$ and (X, Q_*, u) is an order unit space isomorphic to $(A_c(K), A_c(K)^+, 1_K)$, with $\|\cdot\|_u = \|\cdot\|$.*

Proof. By Theorem 2, (X^{**}, Q^*, u) is an order unit space isomorphic to $(A_b(K), A_b(K)^+, 1_K)$ and $\|\cdot\|^{**} = \|\cdot\|_u$. Since X can be identified with the subspace of weak*-continuous functionals in X^{**} , it is enough to prove that $\phi \in X^{**}$ is weak*-continuous iff $\phi|_K \in A_c(K)$.

So assume the latter. It is enough to show that $\phi^{-1}(0)$ is weak*-closed in X^* . By Krein-Smulian theorem, this is equivalent to

$$A = \{\varphi \in X^*, \phi(\varphi) = 0, \|\varphi\|_K \leq 1\}$$

being weak*-closed. Clearly, $\varphi \in A$ iff $\varphi = a(\psi_1 - \psi_2)$, with $\psi_1, \psi_2 \in K$ and $0 \leq a \leq 1/2$. Let $\psi \in K$ be any element, then $\varphi = 1/2(2a\psi_1 + (1 - 2a)\psi - (2a\psi_2 + (1 - 2a)\psi))$, it follows that $A = 1/2(K - K)$ is weak*-closed and hence $\phi \in X$. The converse is obvious. □

Corollary 4. *Any order unit space (X, P, u) is isomorphic to a subspace of $(A_c(K), A_c(K)^+, 1_K)$, where $K = \mathcal{S}(X, P, u)$.*

References

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