Convex sets

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1 Abstract convex structures

Axioms for convex sets were introduced by H.M. Stone [2], and then studied e.g. in [1, 3]. We mostly follow [1].

Definition 1.1. *A convex structure is a set X and a family of binary operations $< \lambda, x, y >$, $\lambda \in [0,1]$, on X such that*

(c1) $< \lambda, p, q >=< 1-\lambda, q, p >$ *(commutativity),*

(c2) $< 0, p, q >= p$ *(endpoint condition),*

(c3) $< \lambda, p, < \mu, q, r >>=< \lambda\mu, < \nu, p, q >, r >$ $(\lambda\mu \neq 1)$*, where* $\nu = \lambda(1-\mu)(1-\lambda\mu)^{-1}$ *(associativity).*

If X is a convex subset of a real linear space, then $< \lambda, p, q >$ corresponds to $(1-\lambda)p + \lambda q$. In [2] and [3], also the following axiom is given:

(c4) $< \lambda, p, p >= p$ (idempotence).

A convex *prestructure* is a set S with a map $T : [0,1] \times S \times S \to S$ denoted $T(\lambda, p, q) =< \lambda, p, q >$ (no requirements on T). An affine functional is an affine map f from a convex prestucture to the real line $\mathbb{R}$, that is, $f(< \lambda, p, q >) = (1-\lambda)f(p) + \lambda f(q)$ for all $\lambda \in [0,1]$ and $p, q \in S$. We denote by S^* the set of all affine functional on S and say that S^* is *total* (*separating*) if for $p \neq q \in S$ there is $f \in S^*$ such that $f(p) \neq f(q)$.

Theorem 1.2. *A convex prestructure S is isomorphic to a convex set iff S^* is total.*

Proof. Suppose S_0 is a convex set and $F : S \to S_0$ an isomorphism. If S_0 is a convex subset of the linear space V, it is well-known that V^* (the algebraic dual) is total over V. Restricting the elements of V^* to S_0, we get a total set of affine functionals for S_0. If $f \in V^*$, then $f \circ F \in S^*$, so S^* is total.

Conversely, suppose that S^* is total. For $p \in S$ define $J(p) : S^* \to \mathbb{R}$ by $J(p)f = f(p)$. Clearly S^* is a linear space under pointwise operations, and $J(p) \in S^{**}$ so that $J(S) \subseteq S^{**}$. Now $J(S)$ is a convex set - indeed, $(1-\lambda)J(p) + \lambda J(q) =< \lambda, p, q >$ - and $J : S \to S^{**}$ is injective iff S^* is total. Indeed, if S^*

is total and $p \neq q \in S$, then there is $f \in S^*$ with $f(p) \neq f(q)$ so $J(p) \neq J(q)$, and conversely, if J is injective and $p \neq q \in S$, then $J(p) \neq J(q)$ so that there is an $f \in S^*$ such that $f(p) = J(p)f \neq J(q)f = f(q)$. If follows that $J : S \to J(S)$ is an isomorphism. □

The following theorem gives an intrinsic characterization of those convex structures that are convex subsets of a linear space.

Theorem 1.3. *[2, 3]. A convex structure S_1 satisfying axioms* (c1),(c2), (c3) and in addition (c4) embeds into a real vector space iff the following cancellation property holds:

(c5) $< \lambda, x, y >=< \lambda, x, z >$ with $\lambda \in (0,1) \implies y = z$.

Proof. Clearly, every convex subset of a vector space satisfies this cancellation property.

Let X be a convex structure satisfying (c4) and (c5). Let V_X be the real vector space generated by X, so that V_X has a base $(e_x)_{x\in X}$. Let $U_X \subseteq V_X$ be a subspace generated by the vectors of the form

$$e_{<\lambda,x,y>} - (1-\lambda)e_x - \lambda e_y, x, y \in X, \lambda \in [0,1].$$

Let $W_X = V_X/U_X$ and let $\tilde{e_x}$ be the image of e_x. Then the mapping $X \to W_X$, $x \mapsto \tilde{e_x}$ preserves convex combinations. Vectors in U_X have the form

$$\sum_{i=1}^{m} \alpha_i(e_{<\lambda_i,a_i,b_i>} - (1-\lambda_i)e_{a_i} - \lambda_i e_{b_i}) - \sum_{i=1}^{m} \beta_i e_{<\mu_i,c_i,d_i>} - (1-\mu_i)e_{c_i} - \mu_i e_{d_i})$$

with $\alpha_i, \beta_i \geq 0$, and $a_i, b_i, c_i, d_i \in X$ and $\lambda_i, \mu_i \in [0,1]$. We split this into positive and negative terms as follows:

$$\begin{array}{c} \sum_{i=1}^{m}(\alpha_i e_{<\lambda_i,a_i,b_i>} + \beta_i(1-\mu_i)e_{c_i} + \beta_i\mu_i e_{d_i}) \\ -\sum_{i=1}^{m}(\beta_i e_{<\mu_i,c_i,d_i>} + \alpha_i(1-\lambda_i)e_{a_i} + \alpha_i\lambda_i e_{b_i}) \end{array}$$

and observe that the sum of the coefficients of all negative terms equals to the sum of coefficients of all positive terms, namely $\sum_i(\alpha_i + \beta_i)$. Without loss of generality we may assume this sum to be 1, then both the sums are convex combinations. Interpreting these as convex combinations in X, these sums moreover define the same point in X.

We show the injectivity by proving that $\tilde{e_x} = \tilde{e_y}$ implies $x = y$, $x, y \in X$. The equation $\tilde{e_x} = \tilde{e_y}$ holds whenever $e_x - e_y$ lies in U_X. If this is the case, then the first sum contains the term κe_x, $\kappa > 0$, and the second sum contains the term κe_y for the same κ, and all other terms cancel. Then both the sums define convex combinations of the same points with the same weights, except that the first contains the point x with weight κ, while the second contains the point y with weight κ. Applying the cancellation property, we obtain $x = y$. □

2 Intrinsic metrics

Let S_1 be a convex structure. For $p, q \in S_1$ define

$$\sigma(p,q) := \inf\{0 \leq \lambda \leq 1 :< \lambda, p, p_1 >=< \lambda, q, q_1 >, p_1, q_1 \in S_1\}$$

Since $< \frac{1}{2}, p, q >=< \frac{1}{2}, q, p >$, we have $0 \leq \sigma(p,q) \leq \frac{1}{2}$.

$$\rho(p,q) = \frac{\sigma(p,q)}{1-\sigma(p,q)}, \quad \text{then } 0 \leq \rho(p,q) \leq 1.$$

Theorem 2.1. *([1]) On any convex structure S_1, σ and ρ are semimetrics.*

Proof. Clearly, σ and ρ are nonnegative and symmetric. We have to prove triangle inequality. If $p = s$ or $q = s$, then $\sigma(p,q) \leq \sigma(p,s) + \sigma(s,q)$. Assume $p \neq s, q \neq s$. Assume

$\lambda_1 \in \{0 < \lambda < 1 :< \lambda, p, p_1 >=< \lambda, s, s_1 >, p_1, s_1 \in S_1\}$;
$\lambda_2 \in \{0 < \lambda < 1 :< \lambda, s, s_2 >=< \lambda, q, q_1 > s_2, q_1 \in S_1\}$,
$\lambda_3 := \lambda_1 + \lambda_2 - 2\lambda_1\lambda_2$;
$p_2 :=< \lambda_2(1-\lambda_1)\lambda_3^{-1}, p_1, s_2 >$;
$q_2 :=< \lambda_2(1-\lambda_1)\lambda_3^{-1}, s_1, q_1 >$;
$\lambda_0 := \lambda_3(1-\lambda_1\lambda_2)^{-1}$. Then

$$
\begin{aligned}
< \lambda_0, p, p_2 > &= < \lambda_3(1-\lambda_1\lambda_2)^{-1}, p, < \lambda_2(1-\lambda_1)\lambda_3^{-1}, p_1, s_2 >> \\
&= < \lambda_2(1-\lambda_1(1-\lambda_1\lambda_2)^{-1}, < \lambda_1, p, p_1 >, s_2 > \\
&= < \lambda_2(1-\lambda_1)(1-\lambda_1\lambda_2)^{-1}, < \lambda_1, s, s_1 >, s_2 > \\
&= < (1-\lambda_2)(1-\lambda_1\lambda_2)^{-1}, s_2, < \lambda_1, s, s_1 >> \\
&= < \lambda_1(1-\lambda_2)(1-\lambda_1\lambda_2)^{-1}, < 1-\lambda_2, s_2, s > s_1 > \\
&= < (1-\lambda_1)(1-\lambda_1\lambda_2)^{-1}, s_1, < \lambda_2, q, q_1 >> \\
&= < (1-\lambda_1)(1-\lambda_2)(1-\lambda_1\lambda_2)^{-1}, < \lambda_2(1-\lambda_1)\lambda_3^{-1}, s_1, q_1 >, q > \\
&= < \lambda_3(1-\lambda_1\lambda_2)^{1}, q, q_2 > \\
&= < \lambda_0, q, q_2 >
\end{aligned}
$$

so $\lambda_0 \in \{0 < \lambda < 1 :< \lambda, p, p_2 >=< \lambda, q, q_2 >, p_2, q_2 \in S_1\}$. Now since

$$\lambda_0(1-\lambda_0)^{-1} = \lambda_1(1-\lambda_1)^{-1} + \lambda_2(1-\lambda_2)^{-1},$$

we get

$$
\begin{aligned}
\rho(p,q) &= \sigma(p,q)[1-\sigma(p,q)]^{-1} \leq \sigma(p,s)[1-\sigma(p,s)]^{-1} + \sigma(s,q)[1-\sigma(s,q)]^{-1} \\
&= \rho(p,s) + \rho(s,q).
\end{aligned}
$$

The triangle inequality for σ follows similarly from $\lambda_0 \leq \lambda_1 + \lambda_2$.

□

Theorem 2.2. *A necessary and sufficient condition for ρ, σ to be metrics is that whenever there are sequences $\lambda_i \in [0,1]$, $p_i, q_i \in S_1$which satisfy $\lim_{i\to\infty} \lambda_i = 0$, $< \lambda_i, p, p_i >=< \lambda_i, q, q_i >$ then $p = q$.*

Proof. Clearly ρ is a metric iff σ is. If σ is a metric, since $\sigma(p,q) \leq \lambda_i$ $\forall i$ we have $p = q$. Conversely if $\sigma(p,q) = 0$ then $V = \{0 \leq \lambda_i \leq 1 :< \lambda, p, p_1 >=< \lambda, q, q_1 >, p_1, q_1 \in S_1\}$ either contains 0 or 0 is a limit point. In the former case $p =< 0, p, p_1 >=< 0, q, q_1 >= q$. In the latter case there exist $\lambda_i \in V$ with $\lambda_i \to 0$ so again $p = q$. □

Corollary 2.3. *Let S_0 be a convex set in a real vector space X. If there is a topology on X that makes X a Hausdorff topological vector space in which S_0 is bounded, then ρ is a metric.*

Proof. Suppose there are sequences $\lambda_i \in [0,1]$, $\lim \lambda_i = 0, p_i, q_i \in S_0$ such that $(1-\lambda_i)p + \lambda_i p_i = (1-\lambda_i)q + \lambda_i q_i$. Then $p - q = \lambda_i(p-q) + \lambda_i(q_i - p_i)$. Let Λ be a neighborhood of 0. Then there is a neighborhood W of 0 such that $W + W + W \subseteq \Lambda$. Now for sufficiently large i, $\lambda_i(p-q) \in W$. Since S_0 is bounded, there is $\mu > 0$ such that $\lambda S_0 \subseteq W$ for $|\lambda| \leq \mu$. For i sufficiently large, $\lambda_i q_i - \lambda_i p_i \in W + W$. Hence $p - q \in W + W + W \subseteq \Lambda$ for i sufficiently large, and since X is Hausdorff, $p - q = 0$. □

The converse holds only in finite dimensional spaces.

Let S_0 be a convex set in a real linear space V, ρ the intrinsic metric on S_0. Some terminology: S_0 is

- *absorbing* iff $\forall x \in V \; \exists \, \delta(x) > 0$: $\lambda x \in S_0 \; \forall \lambda$ with $|\lambda| \leq \delta(x)$.
- *balanced* iff $x \in S_0$, $|\lambda| \leq 1 \implies \lambda x \in S_0$.
- *radial* iff $x \in S_0$, $0 \leq \lambda \leq 1 \implies \lambda x \in S_0$.
- *normalized* iff $\in S_0$, $\alpha \neq 1 \implies \alpha x \notin S_0$.

Define $P := \{\alpha S_0 : \alpha \geq 0\}$, then P is a wedge.

Definition 2.4. $x \in X$, $\|x\| := \inf\{c + d : x = cp - dq; c, d \geq 0; p, q \in S_0\}$.

Theorem 2.5. *If S_0 is normalized or radial then $\|p-q\| = 2\rho(p,q)$, $p, q \in S_0$. Moreover, $\|.\|$ is a norm iff ρ is a metric.*

Proof. Assume S_0 is normalized. For $p, q \in S_0$, if $p - q = cp_1 - dq_1$, $p_1, q_1 \in S_0$, $c, d \geq 0$, then $p + dq_1 = q + cp_1$. which implies

$$(1+d)(\frac{p}{1+d} + \frac{dq_1}{1+d}) = (1+c)(\frac{1}{1+c}q + \frac{c}{1+c}p_1).$$

Then $q_2 := \frac{1}{1+c}q + \frac{c}{1+c}p_1 \in S_0$, and $\frac{1+c}{1+d}q_2 = \frac{p}{1+d} + \frac{dq_1}{1+d} \in S_0 \implies \frac{1+c}{1+d} = 1 \implies c = d$.

So all representations of $p - q$ are of the form $c(p_1 - q_1), c \geq 0, p_1, q_1 \in S_0$.

We then have

$$\begin{aligned}
\sigma(p,q) &= \inf\{0 \le \lambda < 1 : (1-\lambda)p + \lambda p_1 = (1-\lambda)q + \lambda q_1, p_1, q_1 \in S_0\} \\
&= \inf\{0 \le \lambda < 1 : p - q = \frac{\lambda}{1-\lambda}(q_1 - p_1), p_1, q_1 \in S_0\} \\
&= \inf\{\frac{c}{c+1}, c \ge 0 : p - q = c(q_1 - p_1), p_1, q_1 \in S_0\} \\
&= [\inf\{c \ge 0 : p - q = c(q_1 - p_1)][\inf\{c \ge 0 : p - q = c(q_1 - p_1)| + 1]^{-1]} \\
&= \frac{\frac{1}{2}\inf\{2c : p - q = cp_1 - cq_1\}}{\frac{1}{2}\inf\{2c : p - q = cp_1 - cq_1\} + 1} \\
&= \frac{\frac{1}{2}\|p-q\|}{\frac{1}{2}\|p-q\| + 1}.
\end{aligned}$$

From this we get $\frac{1}{2}\|p-q\| = \frac{\sigma(p,q)}{1-\sigma(p,q)} = \rho(p,q)$.

Assume S_0 is radial, and $p - q = cp_1 - dq_1, p, q, p_1, q_1 \in S_0$, $c, d \ge 0$. Then

$$p - q = (c+d)(\frac{c}{c+d}p_1 - \frac{d}{c+d}q_1) = (c+d)(p_2 - q_2), p_2, q_2 \in S_0.$$

That is, $p - q = b(p_2 - q_2), b > 0, p_2, q_2 \in S_0$. From this we get $\|p - q\| = 2\inf\{c > 0, p - q = c(p_1 - q_1), p_1, q_1 \in S_0\}$ and similarly as in the previous case we obtain $\|p - q\| = 2\rho(p,q)$.

Clearly, if $\|.\|$ is a norm, then ρ is a metric. Suppose ρ is a metric, $\|x - y\| = 0, x, y \in X$.

1. S_0 is radial.

$\|x-y\| = 0 \implies \exists p, q \in S_0, 0 \le c, d \le 1 : x-y = cp-dq = p_1-q_1, p_1, q_1 \in S_0$. Then $2\rho(p_1, q_1) = \|p_1 - q_1\| = 0 \implies p_1 = q_1 \implies x = y$.

2. S_0 is normalized. We show first that $\|p\| = 1 \forall p \in S_0$.

If $p = cp_1 - dq_1, p_1, q_1 \in S_0$, $c, d \ge 0$, then

$$\frac{1}{1+d}p + \frac{d}{1+d}q_1 = c\frac{1}{1+d}p_1 \implies c = 1 + d \ge 1,$$

so that $\|p\| = \inf\{c + d : p = cp_1 - dq_1, c, d \ge 0, p_1, q_1 \in S_0\} \ge 1$. But also $p = p - 0q \implies \|p\| = 1$.

Let $0 = \|x - y\| = \|cp - dq\| \ge \|c\|p\| - d\|q\|\| = |c - d|$, so $c = d$. Hence $0 = \|x - y\| = c\|p - q\| = 2c\rho(p,q)$. If $c \ne 0$ then $\rho(p,q) = 0 \implies p = q$, hence $x = y$. If $c = 0$, then again $x = y$. $\square$

3 The categories Conv, GConv and Bconv

Let **Conv** denote the category whose objects are convex structures satisfying (c1)-(c4), with affine maps as morphisms. This is the Eilenberg-Moore category for the distribution monad.

For $X \in \mathbf{Conv}$, the elements of $\mathbf{Conv}(X, \mathbb{R})$ are called *functionals*. Note that $\mathbf{Conv}(X, \mathbb{R})$ can be given a structure of a vector space, which we denote by $A(X)$, with an ordering defined by the wedge $A(X)^+$ of positive affine maps. Clearly, $A(X)^+$ is an Archimedean cone, but there is no order unit in general.

Example 3.1. *Let $X = \mathbb{R}$, with usual affine structure. Any affine map $f : \mathbb{R} \to \mathbb{R}$ has the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. It follows that the only elements in $A(X)^+$ are positive constants, none of which can be an order unit.*

Let $A_b(X)$ denote the vector subspace of bounded functionals, $A_b(X)^+$ the set of positive bounded functionals and let 1_K denote the constant $1_K(x) \equiv 1$. Then $(A_b(X), A_b(X)^+, 1_X)$ is an order unit Banach space, with order unit norm satisfying

$$\|f\|_{1_X} = \sup_{x \in X} |f(x)|.$$

Let also $E(X) := \mathbf{Conv}(X, [0, 1])$, then $E(X)$ is the interval between 0 and 1_X in $(A_b(X), A_b(X)^+)$. Functionals in $E(X)$ will be called *effects*.

A convex structure X is called *geometric* if it is isomorphic to a convex subset of a vector space. Any such isomorphism will be called a geometric representation of X. The category **GConv** of geometric convex sets is a full subcategory of **Conv**. Theorem 1.3 gives an intrinsic characterization of geometric convex sets. Further, by Theorem 1.2, X is geometric iff it is separated by elements of $A(X)$. In this case, the map $\phi : X \to A(X)'$, given by

$$\phi(x)(f) = f(x), \qquad f \in A(X), \ x \in X,$$

is a geometric representation of X. We will identify X with its image $\phi(X)$ in $A(X)'$. Note that this image lies in the hyperplane $\{\varphi \in A(X)', \ \varphi(1_X) = 1\}$, which does not contain 0. Put $V(X) := \text{span}\{X\} \subseteq A(X)'$, $V(X)^+ := \cup_{\lambda \geq 0} \lambda X \subseteq (A(X)^+)'$. Let $u_X \in V(X)'$ be given by the restriction of the functional $1_X \in A(X) \subseteq A(X)''$.

Proposition 3.2. *(i) $V(X)^+$ is a generating cone in $V(X)$, with base X.*

(ii) $(A(X), A(X)^+) \simeq (V(X)', (V(X)^+)')$*, in the category* **OVS***.*

(iii) $(A_b(X), A_b(X)^+, 1_X) \simeq (V(X)^*, (V(X)^+)^*, u_X)$*, in the category* **OUS***, where* $V(X)^*$ *is the space of functionals bounded with respect to the base seminorm and* $(V(X)^+)^* = V(X)^* \cap (V(X)^+)'$*, see [4].*

Proof. $V(X)^+$ is a generating wedge in $V(X)$ by definition. Let $v \in V(X)^+ \cap -V(X)^+$, so that there are some $a, b \in \mathbb{R}^+$ and $x, y \in X$ such that $v = ax = -by$. Assume $a + b > 0$, then by convexity

$$0 = \frac{a}{a+b}x + \frac{b}{a+b}y \in X,$$

which is impossible. Hence $a = b = 0$ and $v = 0$. To show that X is a base of $V(K)^+$, it suffices to observe that $X = \{v \in V(X)^+, u_X(v) = 1\}$. This proves (i).

To show (ii), let $\varphi \in V(X)'$, then clearly $\varphi|_X \in A(X)$ and $\varphi \in (V(X)^+)'$ iff $\varphi|_X \in A(X)^+$. Conversely, any $f \in A(X)$ extends to an element $\varphi_f \in V(X)'$, which is unique, since X is generating. To define the extension, put $\varphi_f(0) := 0$ and $\varphi_f(v) := af(x) - bf(y)$ for $v = ax - by$ with $a, b \geq 0$ and $x, y \in X$. To show that this extension is well defined, assume that $v = ax - by = cx' - dy'$ for $a, b, c, d \in \mathbb{R}^+$ and $x, x', y, y' \in X$. Then $ax + dy' = cx' + by$ and applying u_X implies that $a + d = c + b$. If $a + d = 0$, then $v = 0$ and $\varphi_f(v) = 0 = af(x) - bf(y)$. Otherwise, we obtain

$$\frac{a}{a+d}x + \frac{d}{a+d}y' = \frac{c}{c+b}x' + \frac{b}{c+b}y$$

and since f is affine, we get back to $af(x) - bf(y) = cf(x') - df(y')$. This shows that $\varphi \mapsto \varphi|_X$ defines an order isomorphism of $(V(X)', (V(X)^+)')$ and $(A(X), A(X)^+)$.

(iii) follows directly by [4, Theorem 2 (iii)].

□

Remark 3.3. *Let* $\psi : X \to V$ *be any geometric representation. Let* $\tilde{\psi} : X \to V \oplus \mathbb{R}$ *be defined by* $\tilde{\psi}(x) = (\phi(x), 1)$*, then the image* $\tilde{\psi}(X)$ *lies in the hyperplane* $\{(v, a) \in V \oplus \mathbb{R},\ u(x, a) := a = 1\}$*. In all these constructions, we may replace* ϕ *with the representation* $\tilde{\psi}$ *and* 1_X *by the functional* u*. It is easy to see that all the resulting structures will be isomorphic.*

A convex structure X is called *bounded* if X is geometric and $co(X \cup -X)$ is linearly bounded in $V(X)$. The full subcategory of bounded convex structures will be denoted by **Bconv**.

Proposition 3.4. **Bconv** *and* **BN** *are equivalent categories.*

Proof. For $X \in$ **Bconv**, let $F(X) = (V(X), V(X)^+, X)$ and for an affine map $f : X \to Y$, define $F(f) : V(X) \to V(Y)$ as the unique extension of F (existence an uniqueness is proved similarly as in the proof of Prop. 3.2). By [4, Prop. 5], F is a functor **BConv** $\to$ **BN**. Since any $(V, P, K) \in$ **BN** is isomorphic to $F(K)$, F is surjective on objects and it is easy to see that it is also full and faithful. Hence F yields an equivalence of the two categories. □

We have the following characterizations of objects in **Bconv**.

Proposition 3.5. *Let X be a convex structure. Then the following conditions are equivalent.*

(i) The intrinsic semimetric ρ in X is a metric.

(ii) If for any $\epsilon \in [0, 1]$, there are $p_\epsilon, q_\epsilon \in X$ with $<\epsilon, p, p_\epsilon> = <\epsilon, q, q_\epsilon>$, then $p = q$.

(iii) X is separated by $A_b(X)$.

(iv) X is separated by $E(X)$.

(v) X is bounded.

Proof. The equivalence of (i) and (ii) is essentially Theorem 2.2, since the condition (ii) is equivalent to the condition in this theorem. Indeed, assume that (ii) holds and let $\lambda_i \in [0, 1]$, $p_i, q_i \in X$ be such that $\lambda_i \to 0$ and $<\lambda_i, p, p_i> = <\lambda_i, q, q_i>$. Then for any $\epsilon \in [0, 1]$ we can find some i such that $\lambda_i \leq \epsilon$. Put $p_\epsilon = <\lambda_i/\epsilon, p, p_i>$, $q_\epsilon = <\lambda_i/\epsilon, q, q_i>$. By conditions (c3) and (c4), we obtain

$$<\epsilon, p, p_\epsilon> = <\lambda_i, p, p_i> = <\lambda_i, q, q_i> = <\epsilon, q, q_\epsilon>,$$

so that $p = q$. Since the converse is quite obvious, this proves the equivalence (i) $\iff$ (ii). Moreover, the equivalence (i) $\iff$ (v) follows by Theorem

2.5 and [4, Prop. 5]. Since $E(X)$ contains an order unit, $A_b(X)$ is spanned by $E(X)$, so that (iii) and (iv) are equivalent.

Assume (iii), then by Theorem 1.2, X is geometric. By Proposition 3.2, any $f \in E(X)$ extends uniquely to a linear functional φ_f on $V(X)$. Let $S = co(X \cup -X)$ and let $v_t := v + tw$ for $v, w \in V(X)$, $w \neq 0$ and $t \in \mathbb{R}$. Note that there must be some $g \in E(X)$ such that $\varphi_g(w) \neq 0$. Indeed, we have $w = ax - by$ for $a, b \in \mathbb{R}^+$ and $x, y \in X$. If $\varphi_f(w) = 0$ for all $f \in E(X)$, then also $a - b = \varphi_{1_X}(w) = 0$, hence $a = b$ and $w = a(x - y)$. From $\varphi_f(w) = a(f(x) - f(y)) = 0$ for all $f \in E(X)$, it follows that either $a = 0$ or $x = y$, but in both cases $w = 0$.

If t is such that $v_t \in S$, then

$$\varphi_g(v_t) = \varphi_g(v) + t\varphi_g(w) \in g(S) = co(g(X) \cup -g(X)) \subseteq [-1, 1],$$

and since $\varphi_f(w) \neq 0$, this implies that t must be in a bounded interval. Hence (v) holds.

Finally, if (v) is true, then $(V(X), V(X)^+, X)$ is a base-normed space. By Proposition 3.2 (iii), the dual Banach space $V(X)^*$ is isomorphic to $A_b(X)$ and since the elements of $V(K)^*$ separate points of $V(K)$, this implies (iv).

□

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