1 Ordered vector spaces

Overall reference: [3]

1.1 Basic definitions

Let X be a real vector space. A subset $A \subseteq X$ is

- algebraically open (closed) if the intersection of any line with A is an open (closed) subset of the line
- linearly bounded if the intersection of A with any line is a bounded subset of the line

We say that $a \in A$ is an algebraic interior point of A if it is an interior point of the intersection of any line with A, that is, for any $x \in X$ there is some $\delta > 0$ such that $a + sx \in A$ for all $|s| \le \delta$. The set of all such points is called the algebraic interior of A and is denoted by $aint(A)$. The algebraic closure of A is $acl(A) := X \setminus aint(X \setminus A)$. If A is convex, then

$$acl(A) = \{x \in X,\ \exists y \in X,\ x + \lambda y \in A,\ \forall \lambda \in (0,1)\}.$$

A is algebraically open iff $A = aint(A)$ and algebraically closed iff $A = acl(A)$. If A is convex, then both $aint(A)$ and $acl(A)$ are convex as well.

Remark 1. (cf. [4, §16]) If A is convex, then $aint(A)$ is algebraically open, but in general $aint(aint(A)) \subsetneq aint(A)$. The algebraic closure is not necessarily algebraically closed even if A is convex. The counterexample is as follows. Let X be an infinite dimensional vector space with algebraic basis $\{x_\alpha\}$. Put

$$A = \{x = \sum_\alpha c_\alpha x_\alpha,\ c_\alpha \ge 0\ \forall \alpha,\ \sum_\alpha c_\alpha \ge \frac{1}{n(x)}\}$$

where $n(x) = \#\{\alpha, x = \sum_\alpha c_\alpha x_\alpha,\ c_\alpha \neq 0\}$. Then A is convex, $acl(A) = \{x = \sum_\alpha c_\alpha x_\alpha,\ c_\alpha \ge 0\ \forall \alpha,\ x \neq 0\}$ and $acl(acl(A))$ contains 0, so that $acl(A) \subsetneq acl(acl(A))$. On the other hand, if A is convex and $aint(A) \neq \emptyset$, then $acl(acl(A)) = acl(A)$.

Wedges, cones and orderings

A subset $P \subseteq X$ is called a wedge if $P + P \subseteq P$ and $\lambda P \subseteq P$ for any $\lambda \geq 0$. The preorder $x \leq y$ if $x - y \in P$ is compatible with the linear structure, such a preorder is called an ordering in X. Conversely, for any ordering, the set of positive elements is a wedge.

The pair (X, P) where P is a wedge is called an ordered vector space. The corresponding ordering is a partial order iff $P \cap -P = \{0\}$, in this case P is called a cone. X with this ordering is directed iff P is generating, that is, $X = P - P$.

Positive maps

Let (X, P) and (Y, Q) be ordered vector spaces. A linear map $F : X \to Y$ is called positive if $F(P) \subseteq Q$. Let (P, Q) denote the set of positive maps, then (P, Q) is a wedge in the vector space $L(X, Y)$ of all linear maps $X \to Y$. We have

Lemma 1. (P, Q) *is a cone if and only if* P *is generating and* Q *is a cone.*

Archimedean and almost Archimedean orderings

Let (X, P) be an ordered vector space. We say that the ordering (or P) is Archimedean if $x \leq \lambda y$ for some $y \in X$ and all $\lambda > 0$ implies that $x \leq 0$.

Proposition 1. *The following are equivalent.*

(i) *the ordering is Archimedean.*

(ii) $\exists y \in X, \epsilon > 0$ *such that* $x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x \leq 0$.

(iii) $\exists y \in P, \epsilon > 0$ *such that* $x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x \leq 0$.

(iv) $P = acl(P)$.

The ordering (or P) is almost Archimedean if $-\lambda y \leq x \leq \lambda y$ for some $y \in X$ and all $\lambda > 0$ implies $x = 0$.

Proposition 2. *The following are equivalent.*

(i) *the ordering is almost Archimedean.*

(ii) $\exists y \in X, \epsilon > 0$ *such that* $-\lambda y \leq x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x = 0$.

(iii) $\exists y \in P, \epsilon > 0$ *such that* $-\lambda y \leq x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x = 0$.

(iv) $acl(P)$ *is a cone.*

Remark 2. Note that an almost Archimedean wedge must be a cone. An Archimedean wedge is almost Archimedean iff it is a cone.

1.2 Order units and bases

Order units and seminorms

An element $u \in X$ is an order unit in (X, P) if for any $x \in X$, there is some $\lambda \in \mathbb{R}^+$ such that $x \leq \lambda u$. This is equivalent to $u \in aint(P)$. If $aint(P) \neq \emptyset$, P is generating.

If u is an order unit, then P is (almost) Archimedean iff u is (almost) Archimedean: $x \leq \lambda u$ for all $\lambda > 0$ implies $x \leq 0$ (resp. $-\lambda u \leq x \leq \lambda u$ for all $\lambda > 0$ implies $x = 0$).

For an order unit u, put

$$\|x\|_u = \inf\{\lambda > 0,\ -\lambda u \leq x \leq \lambda u\}.$$

Then $\|\cdot\|_u$ is a seminorm in X. It is a norm iff u is almost Archimedean.

Remark 3. If $u_1, u_2 \in aint(P)$, the associated seminorms $\|\cdot\|_{u_1}$ and $\|\cdot\|_{u_2}$ are equivalent. The corresponding topology is thus a property of the ordering rather than the order unit. In fact, this topology is the finest locally convex topology making all order intervals bounded.

Lemma 2. *Let u be Archimedean. Then $[-u, u] = \{x, \in X,\ \|x\|_u \leq 1\}$ and the wedge P is closed in the topology given by $\|\cdot\|_u$.*

Proof. Let $x \in [-u, u]$, then clearly $\|x\|_u \leq 1$. Conversely, assume that $\|x\|_u \leq 1$, then $-(1+\epsilon)u \leq x \leq (1+\epsilon)u$ for all $\epsilon > 0$. This implies that $\pm x - u \leq \epsilon u$ for all $\epsilon > 0$ and since u is Archimedean, this implies $\pm x \leq u$, that is, $x \in [-u, u]$.

For the second statement, let $x \in \bar{P}$ (the closure of P w.r. to $\|\cdot\|_u$). Then for all $n \in \mathbb{N}$, there is some $p_n \in P$ such that $\|x - p_n\|_u \leq \frac{1}{n}$. This implies that $-x \leq p_n - x \leq \frac{1}{n}u$ for all n and since u is Archimedean, $-x \leq 0$, so that $x \in P$.

□

Let (X_i, P_i), $i = 1, 1$ be ordered vector spaces and let $u_i \in X_i$ be order units. A map $f : X_1 \to X_2$ is called unital if $f(u_1) = u_2$. The following is immediate.

Proposition 3. *Let (X_i, P_i, u_i), $i = 1, 2$ be order unit spaces. Any positive unital map $f : X_1 \to X_2$ is a contraction with respect to the seminorms $\| \cdot \|_{u_1}$ and $\| \cdot \|_{u_2}$.*

Bases and seminorms

Let (X, P) be an ordered vector space. A convex subset $K \subset P$ is called a base of P if for any nonzero $p \in P$ there is a unique $\lambda > 0$ such that $\lambda p \in K$.

Lemma 3. *Any wedge with a base is a cone.*

Proof. Let K be a base of a wedge P, and let $0 \neq x \in P \cap -P$. Then there are $\lambda, \mu > 0$ such that $\lambda x = x_1 \in K$ and $-\mu x = x_2 \in K$. It follows that $\lambda^{-1} x_1 = -\mu^{-1} x_2$ and then $\frac{\mu}{\lambda+\mu} x_1 + \frac{\lambda}{\lambda+\mu} x_2 = 0$. Since K is convex, we obtain $0 \in K$, but then for any $p \in K$, $\lambda p \in K$ for all $\lambda \in [0, 1]$. Hence P must be a cone.

□

Proposition 4. *A wedge P has a base if and only if there exists a linear functional ξ on X which is strictly positive on P. In this case, we may put $K = \{p \in P, \xi(p) = 1\}$.*

Proof. Let K be a base of P. For $p \in P$, let $\xi(p)$ be the unique positive number such that $\xi(p)^{-1} p \in K$. Then clearly $\xi(sp) = s\xi(p)$. Further, let $p, q \in P$ and let $\alpha = \xi(p) + \xi(q)$, then

$$\alpha^{-1}(p + q) = \frac{\xi(p)}{\alpha} \xi(p)^{-1} p + \frac{\xi(q)}{\alpha} \xi(q)^{-1} q \in K,$$

so that $p \mapsto \xi(p)$ is an additive function $\xi : P \to \mathbb{R}^+$. The function ξ easily extends to $P - P$ and has an extension to all of X by Hahn-Banach theorem. This extension is obviously positive and $K = \{p \in P,\ \xi(p) = 1\}$.

Conversely, let $\xi : X \to \mathbb{R}$ be strictly positive, then $K = \{p \in P, \xi(p) = 1\}$ is a convex subset of P and $\xi(p)^{-1} p \in K$ for any $p \in P$. Uniqueness is obvious.

□

Proposition 5. *([2]) Let P be a generating cone in a vector space X and let K be a base of P. For $x \in X$, put*

$$\|x\|_K := \inf\{a+b,\ x = ap - bq,\ a, b \in \mathbb{R}^+, p, q \in K\}.$$

This defines a seminorm in X, which is a norm if and only if $S := co(K \cup -K)$ is linearly bounded.

Proof. It can be checked easily that $\|\cdot\|_K$ is a seminorm. Note also that $x \in S$ implies $\|x\|_K \le 1$. Indeed, any $x \in S$ has the form $x = \lambda p - (1-\lambda)q$ for some $\lambda \in [0,1]$, $p, q \in K$ and then $\|x\|_K \le \lambda + (1-\lambda) = 1$. Assume that $\|\cdot\|_K$ is a norm and let $x_t := x + ty$ be a line in X. Then $\|y\|_K > 0$ and $x_t \in S$ implies that $1 \ge \|x_t\|_K \ge |\|x\|_K - |t|\|y\|_K|$, so that $|t| \le \frac{1+\|x\|_K}{\|y\|_K}$. Conversely, assume that S is linearly bounded and let $\|x\|_K = 0$. This implies $tx \in S$ for all $t \in \mathbb{R}$, hence we must have $x = 0$.

□

The (semi)norm in the above proposition is called the base (semi)norm in X.

Remark 4. Note that $\|\cdot\|_K$ is the Minkowski functional of S, that is

$$\|x\|_K = \inf\{\lambda > 0, x \in \lambda S\}.$$

To see this, observe that $S = \{sp - (1-s)q,\ s \in [0,1],\ p, q \in K\}$. Denote the Minkowski functional by p_S. If $x = ap - bq$ for some $a, b \in \mathbb{R}^+$ and $p, q \in K$, then if $a + b = 0$, we must have $x = 0$ and the equality obviously holds. Otherwise,

$$x = +b)(\frac{a}{a+b}p - \frac{b}{a+b}q) \in (a+b)S,$$

so that $p_S(x) \le \|x\|_K$. On the other hand, let $x \in \lambda S$ for some $\lambda > 0$. Then $x = \lambda(sp - (1-s)q)$ for some $s \in [0,1]$ and $p, q \in K$, so that

$$\|x\|_K \le \lambda s + (1-\lambda)(1-s) = \lambda,$$

hence $\|x\|_K \le p_S(x)$.

Remark 5. Linear boundedness of K is in general not enough. There are some weird infinite dimensional examples such that K is linearly bounded but $co(K \cup -K)$ is not.

Proposition 6. *Let (X_i, P_i), $i = 1, 2$ be ordered vector spaces and let $K_i \subset P_i$ be a base of P_i. Any base-preserving linear map $f : X_1 \to X_2$ is a positive contraction with respect to the base seminorms.*

Some examples

The wedges X and $\{0\}$ are trivial.

1. The only nontrivial wedges in $\mathbb{R}$ are $\mathbb{R}^+$ and $\mathbb{R}^-$.

2. **Function spaces:** Let S be a set, $X = \{f : S \to \mathbb{R}\}$, $P = \{f, f(S) \subseteq \mathbb{R}^+\}$. P is an Archimedean cone, $(X, \leq)$ is a lattice. If S is not finite, $aint(P) = \emptyset$.

3. As 2, but bounded functions. In this case P is an Archimedean cone, $aint(P)$ is the set of strictly positive functions.

4. If S is a topological (linear, convex,...) space, we may take spaces as in 2, 3, but restricting to continuous (linear, affine,...) functions.

5. $X = \{f : \mathbb{R} \to \mathbb{R}\}$, with the cone of nondecreasing functions.

6. **Sequence spaces:** X the set of all (or bounded, summable, convergent, converging to 0,...) sequences, with usual positive cone.

7. $\mathbb{R}^2$ with the usual or lexicographic ordering, with $P = \{(x, y),\ x > 0, y > 0\} \cup \{0\}$ or $P = \{(x, y),\ x > 0\} \cup \{0\}$.

Completeness

We give some sufficient conditions for completeness of order unit norms and base norms.

Proposition 7. *[3] Let (X, P) be an ordered vector space with an almost Archimedean order unit u. If every majorized increasing sequence in (X, P) has a supremum, then $(X, \|\cdot\|_u)$ is complete.*

Proof. We first show that any increasing Cauchy sequence has a limit. So let $\{x_n\}$ be such a sequence and let $\epsilon > 0$. Then $\|x_n - x_m\|_u < \epsilon$ for $m, n \geq N$. We then have for all $m \geq N$, $x_m - x_N \leq \epsilon u$, so that $x_m \leq x_N + \epsilon u$. It follows that $\{x_n\}$ is a majorized increasing sequence, so that there is some x_0 such that $x_0 = \sup_n x_n$. For all $m, n \geq N$, we have $x_n \leq x_m + \epsilon u$, hence $x_0 \leq x_m + \epsilon u$ and we have $0 \leq x_0 - x_m \leq \epsilon u$. This implies $\|x_0 - x_m\|_u \leq \epsilon$ for all $m \geq N$, so that $\lim_n x_n = x_0$.

Let now $\{x_n\}$ be any Cauchy sequence. Let $V_n = \{p - q,\ p, q \in [0, 2^{-n}]\}$, then V_n contains the ball with center 0 and radius 2^{-n+1} and is therefore a neighborhood of 0. Hence there is a subsequence such that $x_n - x_{n-1} \in V_n$. Let $a_n, b_n \in [0, 2^{-n}]$ be such that $x_n - x_{n-1} = a_n - b_n$. Then $\{\sum_{k=1}^n a_k\}$ and $\{\sum_{k=1}^n b_k\}$ are increasing Cauchy sequences and hence have a limit by the first part of the proof. Moreover, we have $x_n = \sum_{k=1}^n (a_k - b_k)$, so that x_n converges as well.

□

1.3 Duality

The order dual

Let (X, P) be an ordered vector space and let X' denote the algebraic dual of X. Then the dual wedge of P is defined as

$$P' := \{\varphi \in X', \varphi(p) \geq 0, \forall p \in P\}$$

Then (X', P') is an ordered vector space: the order dual of X. Note that $P' = (P, \mathbb{R}^+)$ and it follows by Lemma 1 that P' is a cone iff P is generating. Further, note that $p \in P \cap -P$ implies that $\varphi(p) = 0$ for all $\varphi \in P'$, hence if P' is generating, P must be a cone. The converse is not true in general.

The norm dual of a vector space with an order unit norm

Let (X, P) be an ordered vector space with an order unit u. Positive unital linear functionals are called states, the set of all states will be denoted by $\mathcal{S}(X, P, u)$.

Lemma 4. *If $\mathcal{S}(X, P, u)$ separates the points of X, then $\|\cdot\|_u$ is a norm.*

Proof. Let $x \in X$, $-\lambda u \leq x \leq \lambda u$ and let $\varphi \in \mathcal{S}(X, P, u)$. Then $|\varphi(x)| \leq \lambda$. It follows that $\sup_{\varphi \in \mathcal{S}(X,P,u)} |\varphi(x)| \leq \|x\|_u$. Let $x \neq 0$ and let $\varphi \in \mathcal{S}(X, P, u)$ be such that $\varphi(x) \neq 0$, then $0 < |\varphi(x)| \leq \|x\|_u$, so that $\|\cdot\|_u$ is a norm.

□

Assume now that $\|\cdot\|_u$ is a norm. In this case, P is an almost Archimedean generating cone. We do not assume that (X, P, u) is an order unit space, so u does not have to be Archimedean. Let X^* be the normed space dual of $(X, \|\cdot\|_u)$ and let $\|\cdot\|_u^*$ be the norm in X^*.

Lemma 5. *(i) Any $\varphi \in P'$ is bounded, with $\|\varphi\|_u^* = \varphi(u)$.*

(ii) If $\varphi \in X^*$ is such that $\|\varphi\|_u^* = \varphi(u)$, then $\varphi \in P'$.

Proof. (i) is quite easy. For (ii), we may assume $\varphi(u) = 1$. Let $x \in P$ and let $\lambda > 0$ be such that $0 \leq x \leq \lambda u$. Then $\|x - \lambda u\|_u \leq \lambda$ and we have

$$|\varphi(x) - \lambda| = |\varphi(x - \lambda u)| \leq \|\varphi\|_u^* \|x - \lambda u\|_u \leq \lambda.$$

This implies $\varphi(x) \geq 0$.

□

Theorem 1. *Let (X, P) be an ordered vector space wwith an order unit norm $\|\cdot\|_u$. Then P' has a w^*-compact base K such that (X^*, P', K) is a base-normed space and $\|\cdot\|_K = \|\cdot\|_u^*$.*

Proof. [2] The set $K = \varphi \in \mathcal{S}(X, P, u)$ is a w^*-compact base of P'. We will show that the base seminorm $\|\cdot\|_K$ equals to the dual norm in X^* and hence is itself a norm.

Let $Y = X \times X$ be ordered by the wedge $Q = P \times P$, then (u, u) is an order unit in (Y, Q). Let

$$Z = \{t(u, u) - (x, -x),\ t \in \mathbb{R}, x \in X\},$$

then Z is a linear subspace in Y containing the order unit. For $\varphi \in X^*$, put

$$F_\varphi(z) = t\|\varphi\|_u^* - \varphi(x), \qquad z = t(u, u) - (x, -x) \in Z$$

This defines a linear functional on Z. Moreover, note that $z = t(u, u) - (x, -x) \in Q$ iff $\|x\|_u \leq t$ and then $F_\varphi(z) \geq (t - \|x\|_u)\|\varphi\|_u^* \geq 0$. Since Z contains the order unit, F_φ extends to a positive linear functional on Y (e.g. Krein's theorem). Put

$$\psi_1(x) = F_\varphi(x, 0), \quad \psi_2(x) = F_\varphi(0, x), \qquad x \in X.$$

Then $\psi_1, \psi_2 \in P'$ and $\varphi = \psi_2 - \psi_1$, this shows that P' is generating in X^*. Moreover, $F_\varphi(u, u) = \|\varphi\|_u^*$

$$\|\varphi\|_u^* = F_\varphi(u, u) = \psi_1(u) + \psi_2(u) \geq \|\varphi\|_K$$

On the other hand, let $\varphi = a\varphi_1 - b\varphi_2$ with $a, b \geq 0$, $\varphi_1, \varphi_2 \in K$, then $\|\varphi\|_u \leq a + b$, this shows the opposite inequality.

□

Corollary 1. *Let (X,P) be an ordered vector space with an order unit u. Then u is almost Archimedean iff $K=\mathcal{S}(X,P,u)$ separates the points of X.*

Proof. Assume u is almost Archimedean, then $\|\cdot\|_u$ is a norm. Let $X \ni x \neq 0$. By Theorem 1,

$$\|x\|_u = \sup_{\|\varphi\|_u^* \leq 1} |\varphi(x)| = \sup_{\varphi \in S} |\varphi(x)| = \sup_{\varphi \in K} |\varphi(x)|,$$

so that we must have $\varphi(x) \neq 0$ for some state φ. The converse is Lemma 4. □

The norm dual of a base-normed space

Let (X,P,K) be a base-normed space and let X^* be the normed space dual of $(X,\|\cdot\|_K)$. Let $P^* = P' \cap X^*$.

Theorem 2. *There is an order unit $u \in X^*$ such that (X^*,P^*,u) is an order unit space.*

Proof. Let (X,P,K) be a base-normed space. Note first that for any $\varphi \in X'$, we have

$$\|\varphi\|_K^* = \sup_{x \in S} |\varphi(x)| = \sup_{x \in K} |\varphi(x)|,$$

where $S = co(K \cup -K)$. There is a strictly positive functional $u \in X'$ such that $K = \{p \in P, u(p) = 1\}$. Note that u is a base-preserving linear map into the base-normed space $(\mathbb{R}, \mathbb{R}^+, 1)$, hence is a positive contraction. Moreover, for $\varphi \in X^*$ and $x \in K$, we have $-\|\varphi\|_K \leq \varphi(x) \leq \|\varphi\|_K$, so that $-\|\varphi\|_K u \leq \varphi \leq \|\varphi\|_K u$, it follows that u is an order unit in $(X^*, P' \cap X^*)$ and $\|\varphi\|_u \leq \|\varphi\|_K^*$. Conversely, $-\lambda u \leq \varphi \leq \lambda u$ implies that $\sup_{x\in K} |\varphi(x)| \leq \lambda$, so that $\|\varphi\|_u = \|\varphi\|_K^*$. To show that u is Archimedean, let $\varphi \leq \lambda u$ for all $\lambda > 0$. Then for $x \in K$, $\varphi(x) \leq \lambda$ for any $\lambda > 0$, hence $\varphi(x) \leq 0$. □

Preduals

We next discuss the Banach space preduals of order unit and base-normed spaces. Here $(X,\|\cdot\|)$ is a Banach space and $(X^*,\|\cdot\|^*)$ the dual space. If $P \in X$ is a wedge, we will denote

$$P^* := \{\varphi \in X^*,\ \varphi(p) \geq 0,\ \forall p \in P\} = P' \cap X^*.$$

Similarly, if Q is a wedge in X^*, we will denote

$$Q_* := \{x \in X,\ q(x) \geq 0,\ \forall q \in Q\} = Q' \cap X.$$

It is clear that P^* and Q_* are wedges. Moreover, $(P^*)_* = \bar{P}$ and $(Q_*)^*$ is the weak*-closure of Q.

Theorem 3. *[2, 1] Let X^* be an order unit space with weak*-closed positive cone. Then X is base-normed. More precisely, if there is an Archimedean weak*-closed cone $Q \subset X^*$ with an order unit u such that $\|\cdot\|^* = \|\cdot\|_u$, then $Q_* \subset X$ has a base $K = \{p \in Q_*, u(p) = 1\}$ and (X, Q_*, K) is a base-normed space with $\|\cdot\| = \|\cdot\|_K$.*

Proof. Let $p \in Q_*$ be such that $u(p) = 0$, then for any $\varphi \in Q$,

$$0 \leq \varphi(p) \leq \|\varphi\|_u \varphi(u) = 0.$$

Since $X^* = Q - Q$ separates points in X, we obtain $p = 0$. Hence u defines a strictly positive linear functional on (X, Q_*) and K is a base of Q_*. For $p \in Q_*$, we have

$$\|p\| = \sup_{\varphi \in [-u,u]} |\varphi(p)| = u(p),$$

it follows that $S = co(K \cup -K)$ is a subset of the unit ball of X. Hence $\|\cdot\| \leq \|\cdot\|_K$ (since $\|\cdot\|_K$ is the Minkowski functional of S). Since $Q = (Q_*)^*$, we have for $\varphi \in X^*$:

$$\begin{aligned}\|\varphi\|_u &= \inf\{\lambda > 0,\ \lambda u \pm \varphi \in Q\} = \inf\{\lambda > 0,\ (\lambda u \pm \varphi)(p) \geq 0,\ \forall p \in Q_*\} \\ &= \inf\{\lambda > 0,\ |\varphi(p)| \leq \lambda,\ \forall p \in K\} = \sup_{p \in K} |\varphi(p)|.\end{aligned}$$

Assume that $x_0 \in X$ is such that $\|x_0\| \leq 1$ and $x_0 \neq \bar{S}$, then by Hahn-Banach separation theorem, there is some $\varphi \in X^*$ such that

$$\|\varphi\|_u = \sup_{p \in K} |\varphi(p)| = \sup_{x \in S} \varphi(x) < \varphi(x_0) \leq \|\varphi\|^* = \|\varphi\|_u.$$

It follows that S is dense in the unit ball X_1 of X. Choose any $\alpha > 1$ and let $\alpha_n > 0$ be a sequence such that $1 + \sum_n \alpha_n < \alpha$. There is some element $x_1 \in S$ such that $\|x_0 - x_1\| < \alpha_1$. Similarly, there is some $x_2 \in \alpha_1 S$ such that $\|x_0 - x_1 - x_2\| < \alpha_2$. Continuing by induction, we obtain a sequence $\{x_n\}$ in X such that $\|x_n\|_K \leq \alpha_{n-1}$ and $\|x_0 - \sum_n x_n\| < \alpha_n \to 0$. Hence

$$\|x_0\|_K = \|\sum_n x_n\|_K \leq \sum_n \|x_n\|_K \leq 1 + \sum_n \alpha_n < \alpha,$$

so that $X_1 \subset \alpha S$ and consequently $X = Q_* - Q_*$. Since the above inequality holds for all $\alpha > 1$, we have $\|\cdot\| = \|\cdot\|_K$.

$\square$

1.4 Categories of ordered vector spaces

Order unit spaces

A triple (X, P, u) where X is a vector space, $P \subseteq X$ an Archimedean cone and $u \in aint(P)$ is called an order unit space. To summarize, in this case, $\|\cdot\|_u$ is a norm in X, $[-u, u]$ is the corresponding closed unit ball and P is norm closed. If (X_i, P_i, u_i), $i = 1, 2$ are order unit spaces, a linear map $f : X_1 \to X_2$ is called unital if $f(u_1) = u_2$.

Base-normed spaces

A triple (X, P, K), where X is a vector space, P a generating cone and K a base of P such that $co(K \cup -K)$ is linearly bounded is called a base-normed space. Let (X_i, P_i, K_i) be base-normed spaces. A linear map $f : X_1 \to X_2$ is called base-preserving if $f(K_1) \subset K_2$.

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