① The categories

Aff : Category of affine spaces

   Object : vector spaces

   Morphisms :

$$X \xrightarrow{\;F\;} Y$$

$$f(a_1 x_1 + \cdots + a_n x_n) = a_1 f(x_1) + \cdots + a_n f(x_n)$$

for all $x_i \in X$ $a_i \in \mathbb{R}$ such that

$$a_1 + \cdots + a_n = 1$$

$\text{Vect}/\mathbb{R}$

   Objects : linear functionals . $V \xrightarrow{\;g\;} \mathbb{R}$

   Morphisms : commutative triangles

$$V \xrightarrow{\;F\;} V'$$
$$g \searrow \quad \swarrow g'$$
$$\mathbb{R}$$

② Affine spaces as a subcategory of

   $\text{Vect}/\mathbb{R}$

a functor $J : \text{Aff} \longrightarrow \text{Vect}/\mathbb{R}$

on objects : $J(x) = \left(X \times \mathbb{R} \xrightarrow[P_{\mathbb{R}}^{X}]{} \mathbb{R} \right)$

↖ projection onto second coordinate

On morphisms: $X \xrightarrow{\ F\ } Y$ – a morphism in aff

$$X \times \mathbb{R} \xrightarrow{\ J(F)\ } Y \times \mathbb{R}$$

is given by the rule

$$J(F)(x, r) = (F(x) + (r-1)F(0), r)$$

Claim: $J(F)$ is a linear mapping

ADDITIVITY:

$$J(F)((x_1, r_1) + (x_2, r_2)) =$$

$$J(F)(x_1 + x_2, r_1 + r_2) =$$

$$F(x_1 + x_2) + (r_1 + r_2 - 1) \cdot F(0), \ r_1 + r_2)$$

$$J(F)(x_1, r_1) + J(F)(x_2, r_2) =$$

$$\left(F(x_1) + (r_1 - 1) \cdot F(0) + F(x_2) + (r_2 - 1)F(0), \ r_1 + r_2 \right)$$

Second coordinate: OK
First coordinate:

$$F(x_1) + r_1 F(0) + F(x_2) + r_2 F(0) - F(0) - F(0) =$$

↑ ↑ ↑ ↑

$1 + 1 - 1 = 1$; F is an affine map

$$= F(x_1 + x_2 - 0) + r_1 F(0) + r_2 F(0) - F(0) =$$

$$= F(x_1 + x_2) + (r_1 + r_2 - 1) \cdot F(0) \checkmark$$

MULTIPLICATION BY A SCALAR

$$\Im(F)(\delta(x,\lambda)) = \Im(F)(\delta.x, \delta\lambda) =$$

$$= \Big(F(\delta x) + (\delta\lambda - 1).F(0), \delta\lambda\Big)$$

$$\delta. \Im(F)(x,\lambda) = \Big(\delta F(x) + \delta(\lambda-1).F(0), \delta\lambda\Big)$$

$$\delta F(x) + \delta(\lambda-1).F(0) =$$

$$= \delta F(x) + \delta\lambda\, F(0) - \delta f(0) =$$

$$= \delta F(x) + \delta\lambda\, F(0) - \delta F(0) - (\delta\lambda-1)F(0) + (\delta\lambda-1)F(0) =$$

$$\delta + \delta\lambda - \delta - (\delta\lambda - 1) = 1$$

$$= F\Big(\delta.x + \delta\lambda.0 - \delta.0 - (\delta\lambda-1).0\Big) + (\delta\lambda-1).F(0) =$$

$$= F(\delta x) + (\delta\lambda-1).f(0)$$

FUNCTORIALITY:

$$\Im(id)(x,\lambda) = \big(x + (\lambda-1).0,\ \lambda\big) = (x,\lambda)$$

$$\Im(F_1 \circ F_2)(x,\lambda) = \Big(F_1(F_2(x)) + (\lambda-1)F_1(F_2(0)), \lambda\Big)$$

$$\big(\Im(F_1) \circ \Im(F_2)\big)(x,\lambda) = \Im(F_1)\big(\Im(F_2)(x,\lambda)\big) =$$

$$= \Im(F_1)\big(f_2(x) + (\lambda-1)f_2(0), \lambda\big) = \text{skip } \lambda \text{ from now on...}$$

$$= f_1\big(f_2(x) + (\lambda-1)f_2(0)\big) + (\lambda-1)f_1(0) =$$

$$= f_1 \big(1 \cdot \big(F_2(x) + (n-1) F_2(0) \big) \big) + (n-1) f_1(0) +$$
$$+ (1-n) F_1(F_2(0)) + (n-1) F_1(F_2(0)) =$$

$$1 + (n-1) + (1-n) = 0$$

$$= f_1 \big(f_2(x) + (n-1) f_2(0) + (n-1) \cdot 0 +$$
$$+ (1-n) f_2(0) \big) + (n-1) f_1(f_2(0)) =$$

$$= f_1(f_2(x)) + (n-1) f_1(f_2(0))$$

J IS A FAITHFUL FUNCTOR

$$J(f_1) = J(f_2)$$

$$\Downarrow ?$$

$$f_1 = f_2$$

$$f_1, f_2 : X \longrightarrow Y$$

$$\forall (x, n) \in X \times \mathbb{R}$$
$$f_1(x) + (n-1) f_1(0) = f_2(x) + (n-1) f_2(0)$$

PUT $n = 1 \Rightarrow$ DONE

Suppose $X \times \mathbb{R} \xrightarrow{\ \ F\ \ } Y \times \mathbb{R}$ commutes
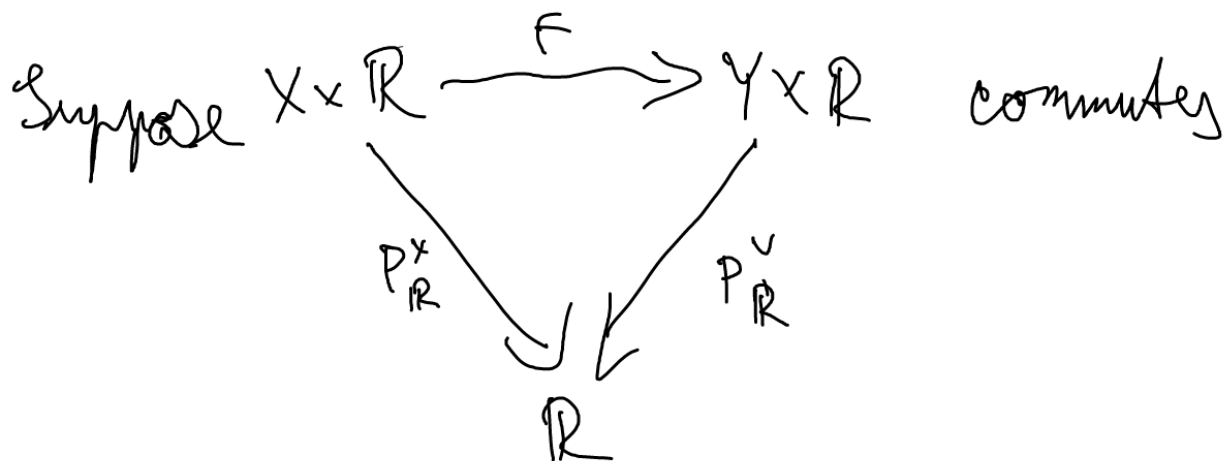
$$F \stackrel{\sim}{=} \langle F_Y, F_{\mathbb{R}} \rangle \qquad F(x, r) = (f_Y(x, r), f_{\mathbb{R}}(x, r))$$

$$f_{\mathbb{R}}(x, r) \stackrel{\sim}{=} r \quad (\text{because the diagram} \atop \text{commutes...})$$

We need to find an affine map $F': X \longrightarrow Y$
such that

$$J(f') = F, \text{ that means,}$$

$$f(x, r) = (f'(x) + (r-1) \cdot f'(0), r)$$

Put $f'(x) = f_Y(x, 1)$

$$f'(x) + (r-1) f'(0) =$$

$$= f_Y(x, 1) + (r-1) f_Y(0, 1) =$$

$$= f_Y(x, 1) + f_Y(0, r-1) =$$

$$= f_Y(x, r)$$

It remains to prove that $f': X \longrightarrow Y$ is an affine map

$$f'(s_1 x_1 + \ldots + s_n x_n) = s_1 f'(x_1) + \ldots + s_n f'(x_n)$$

$$s_1 + \ldots + s_n = 1$$

$$f'(s_1 x_1 + \ldots + s_n x_n) =$$

$$= f_y(s_1 x_1 + \ldots + s_n x_n, 1) =$$

$$= f_y(s_1 x_1 + \ldots + s_n x_n, s_1 + \ldots + s_n) =$$

$$= f_y((s_1 x_1, s_1) + \ldots + (s_n x_n, s_n)) =$$

$$= f_y(s_1(x_1, 1) + \ldots + s_n(x_n, 1)) =$$

$$= s_1 f_y(x_1, 1) + \ldots + s_n f_y(x_n, 1) =$$

$$= s_1 f'(x_1) + \ldots + s_n f'(x_n) \checkmark$$

ESSENTIALLY SURJECTIVE
(wrt <u>surjective functionals</u> in $\mathscr{V}\!ect/\mathbb{R}$)

$x \xrightarrow{\;\gamma\;} \mathbb{R}$ surjective $\implies$ there is

$x' \in \mathscr{Aff}$ such that

there is an iso $\hat{\gamma}$

$$\begin{array}{ccc} & & X \xrightarrow{\;\gamma\;} \mathbb{R} \\ \hat{\gamma} & \Big\uparrow{\scriptstyle\cong} & \nearrow_{P_2} \\ & X' \times \mathbb{R} & \end{array}$$

via range of γ

Put $X' = \text{Ker } g$; $\text{Ker } g$ has codimension 1,
so let $y \in X \setminus \text{Ker } g$ be such that $g(y) = 1$
Let $j: X' \times \mathbb{R} \longrightarrow X$ be given by

$$j(x', r) = x + ry$$

as $g(j(x', r)) = g(x + ry) = g(x) + r \cdot g(y) =$
$$= 0 + r \cdot 1 = r = P_2(x', r) \checkmark$$

③ The adjunction $\text{Vect}/\mathbb{R} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\rightleftarrows}}}} \text{Vect}$

For $X \in \text{Vect}$

$$G(X) = (X \times \mathbb{R} \xrightarrow{P^X_{\mathbb{R}}} \mathbb{R})$$
$$G(X \xrightarrow{f} Y) =$$

$$X \times \mathbb{R} \xrightarrow{f \times id} Y \times \mathbb{R}$$

$P^X_{\mathbb{R}} \searrow \quad \swarrow P^Y_{\mathbb{R}}$

$$\mathbb{R}$$

For $g \in \text{Vect}/\mathbb{R}$

$$F(X \xrightarrow{g} \mathbb{R}) = X$$

$$F\left(\begin{array}{c} X \\ f \downarrow \quad \searrow^g \\ X' \xrightarrow[g']{} \mathbb{R} \end{array}\right) = (X \xrightarrow{f} X')$$

$g : X \longrightarrow \mathbb{R} \in \mathsf{Vect}/\mathbb{R}$
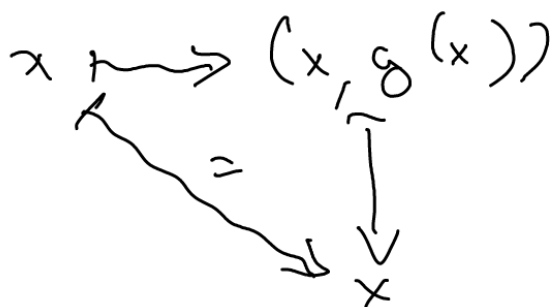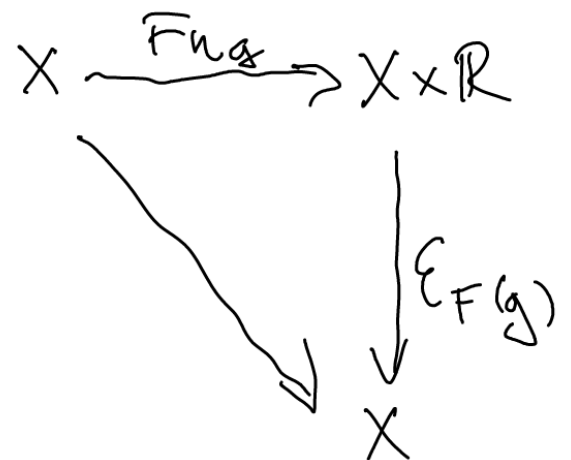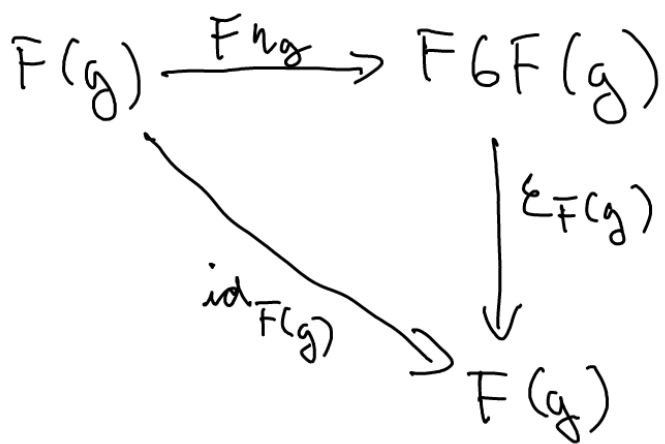
$\eta_g : g \longrightarrow GF(g)$

$X \in \mathsf{Vect}$

$\varepsilon_X : FG(X) \rightsquigarrow X$

$X \times \mathbb{R} \longrightarrow X$

$\varepsilon_X = P_X^X$

Diagram (unit):

X at top; id_X arrow and P_X^X arrow into X; $X \cdots \eta_g \cdots \to X \times \mathbb{R}$; g arrow and $P_\mathbb{R}^X$ arrow into $\mathbb{R}$.

TRIANGLE IDENTITIES

$F(g) \xrightarrow{\ F\eta_g\ } FGF(g)$

$\mathrm{id}_{F(g)}$

$\varepsilon_{F(g)}$

$F(g)$

$X \xrightarrow{\ F\eta_g\ } X \times \mathbb{R}$

$\varepsilon_{F(g)}$

X

$x \longmapsto (x, g(x))$

$=$

X

$$G(x) \xrightarrow{\;\eta_{G(x)}\;} GFG(x)$$

$$\downarrow{G\varepsilon_x}$$

$$id_{G(x)} \searrow \quad G(x)$$

$$(X \times \mathbb{R} \xrightarrow{\;P_x^{\mathbb{R}}\;} \mathbb{R})$$

$$X \times \mathbb{R}$$
$$id \nearrow \quad \nwarrow P_{1,2}$$
$$X \times \mathbb{R} \dashrightarrow[\eta_{G(x)}] X \times \mathbb{R} \times \mathbb{R}$$
$$G(x) = P_x^{\mathbb{R}} \downarrow \quad \swarrow P_3 = GFG(v)$$
$$\mathbb{R} \quad \uparrow$$
$$\eta_{G(x)}(x,r) =$$
$$= (x, r, r)$$

$$\varepsilon_x : X \times \mathbb{R} \xrightarrow{\;P_x\;} X$$

$$G(\varepsilon_x)$$

$$(X \times \mathbb{R}) \times \mathbb{R} \xrightarrow{\;P_x \times id\;} X \times \mathbb{R}$$
$$P_3 \downarrow \quad \swarrow P_2$$
$$\mathbb{R}$$

$$X \times \mathbb{R} \xrightarrow{\;\eta_{\varepsilon(x)}\;} X \times \mathbb{R} \times \mathbb{R} \qquad (x,r) \longmapsto (x,r,r)$$
$$P_2 \downarrow \quad \swarrow P_3 \quad \downarrow P_x \times id \qquad \downarrow \quad \nearrow \quad \downarrow$$
$$\mathbb{R} \xleftarrow{\;P_2\;} X \times \mathbb{R} \qquad r \xleftarrow{\;} (x,r)$$

④ Vect / ℝ as coalgebras

The adjunction $\text{Vect} / \mathbb{R} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{I}}}} \text{Vect}$

induces a comonad on Vect

$$X \xleftarrow{\varepsilon_X} X \times \mathbb{R} \xrightarrow{\sigma_X} X \times \mathbb{R} \times \mathbb{R}$$

$$\sigma_X(x, v) = (x, v, v)$$

Vect / ℝ is isomorphic to the category of coalgebras for this comonad

$$X \xrightarrow{\gamma} X \times \mathbb{R} \quad \text{is a coalgebra iff}$$

$$X \xrightarrow{\gamma} X \times \mathbb{R} \qquad X \xrightarrow{\gamma} X \times \mathbb{R}$$

with id from X down to X, and ε_X from $X \times \mathbb{R}$ down to X; and γ downward, σ_X downward, with $\gamma \times id_\mathbb{R}$ from $X \times \mathbb{R} \to X \times \mathbb{R} \times \mathbb{R}$

$$\gamma(x) = (x, g(x))$$
for some $g : X \longrightarrow \mathbb{R}$

$$x \longmapsto (x, g(x))$$

$$(x, g(x)) \longmapsto (x, g(x), g(x))$$

commutes trivially.

Morphism of coalgebras:

$$x \xrightarrow{\ \gamma_x\ } (x, g_x(x))$$
$$y \xrightarrow{\ \gamma_y\ } (y, g_y(y))$$

$$
\begin{array}{ccc}
X & \xrightarrow{\ F\ } & Y \\
{\scriptstyle \gamma_x}\downarrow & & \downarrow{\scriptstyle \gamma_y} \\
X \otimes R & \xrightarrow{F \otimes id} & Y \otimes R
\end{array}
$$

$$
\begin{array}{ccc}
x & \longmapsto & F(x) \\
\downarrow & & \downarrow \\
(x, g_x(x)) & \longmapsto & (F(x), g_y(F(x))) \\
& & \shortparallel \\
& & (F(x), g_x(x))
\end{array}
$$

So the square commutes iff

$$\forall x : \quad g_y(F(x)) = g_x(x)$$

meaning

$$
\begin{array}{ccc}
X & \xrightarrow{\ F\ } & Y \\
 & g_x \searrow \quad \swarrow g_y & \\
 & R &
\end{array}
$$

commutes.

⑤ Monoidal structure on Vect/$\mathbb{R}$

This structure comes from the fact
that $\mathbb{R}$ is a (multiplicative) monoid
in Vect.

Tensor
product

$$X_1 \quad X_2 \qquad\qquad X_1 \otimes X_2$$
$$g_1 \downarrow \otimes \downarrow g_2 \quad = \qquad\qquad \downarrow$$
$$\mathbb{R} \quad \mathbb{R} \qquad\qquad \mathbb{R} \otimes \mathbb{R}$$
$$\qquad\qquad\qquad\qquad \downarrow \cdot$$
$$\qquad\qquad\qquad\qquad \mathbb{R}$$

Unit object

$$\mathbb{R}$$
$$id \downarrow$$
$$\mathbb{R}$$

Unit object is „neutral"

$$\mathbb{R} \quad X \qquad\qquad \mathbb{R} \otimes X \xrightarrow{\;\lambda_X\;} X$$
$$id \downarrow \otimes \downarrow g \quad = \qquad\qquad \downarrow id \otimes g$$
$$\mathbb{R} \quad \mathbb{R} \qquad\qquad \mathbb{R} \otimes \mathbb{R}$$
$$\qquad\qquad\qquad\qquad \downarrow \cdot$$
$$\qquad\qquad\qquad\qquad \mathbb{R}$$

g

this triangle is λ_g
in the „lifted monoidal
structure"

$$r \otimes x \longmapsto rx$$
$$\downarrow$$
$$r \otimes g(x)$$
$$\downarrow$$
$$r \cdot g(x)$$

Claim: If we restrict to the full subcategory of $\text{Vect}/\mathbb{R}$ that is spanned by surjective functionals, this tensor product comes from the "normal" tensor product of affine spaces.

The proper formulation of this claim follows:

Claim: J is a strict monoidal functor.

Step 1: Construct an isomorphism

$$J(A \otimes B) \longrightarrow J(A) \otimes J(B).$$