1 Convex sets, effects and states

There is a functor $V_b : BConv \to Ban$, mapping each K to the generated base norm space. Ban is the category of Banach spaces and contractions.

2 Convex effect algebras and ordered vector spaces

Let $V : CEA \to OVSu$ be the functor defined in [**?**]. Here $OVSu$ is the category of ordered vector spaces with an order unit and positive unital linear maps. We say that A is archimedean if VA is an order unit space and A is called complete archimedean if VA is an order unit Banach space.

We will now introduce a monad on EA such that the corresponding algebras are precisely the complete archimedean effect algebras. Let $CConv$ be the category whose objects are compact convex subsets in some Hausdorff topological vector spaces and morphisms are continuous affine maps. Note that for any $A \in EA$, $\Sigma(A)$ is a convex subset in $[0,1]^A \subset \mathbb{R}^A$, closed in the product topology, hence compact. Moreover, for any $f : A \to A'$, $\Sigma(f) : \Sigma(A') \to \Sigma(A)$ is continuous. It follows that Σ defines a functor $EA \to CConv^{op}$. To avoid confusion, the functor in this case will be denoted by Σ_c. Let also $E_c : CConv^{op} \to EA$ be defined similarly as E, but now we also require the effect to be continuous. Put $T_c = E_c \circ \Sigma_c$.

It is well known that for any $K \in CConv$, $\Sigma_c E_c(K) \simeq K$, this induces a natural isomorphism $\mu^c : T_c^2 \implies T_c$. Together with the unit $\eta : id \to T_c$, given by the evaluation map, (T_c, η_c, μ_c) defines a monad. Since μ is a natural isomorphism, this monad is idempotent and consequently, all algebras are isomorphisms. Hence if $A \in EA^{T_c}$, $A \simeq T_c(A)$, so that A is a complete archimedean. Conversely, if A is complete archimedean, then $a \mapsto ev_a$ establishes an isomorphism $A \simeq T_c(A)$.

Let us return to $\Sigma : EA \to BConv^{op}$, $E : BConv^{op} \to EA$.

Lemma 1. *Let $K \in BConv$. Then VEK is an order unit Banach space with predual V_bK and for any $f : K_1 \to K_2$, $VEf = f^*$, where f^* is the adjoint map of the extension $f : V_bK_1 \to V_bK_2$.*

Proof. Note that VEK is the space $A_b(K)$ of bounded affine functions $K \to \mathbb{R}$. This is clearly an order unit Banach space, where the order unit norm is given by $\|f\| = \sup_K |f(x)|$. It is clear that any $f \in A_b(K)$ extends to a linear functional on $V_b K$ and for $v = \lambda x - \mu y$ we have

$$|f(v)| \leq \lambda |f(x)| + \mu |f(y)| \leq (\lambda + \mu)\|f\|.$$

Taking the infimum over all expressions for v we obtain that $f \in V_b(K)$. Conversely, any $\varphi \in V_b(K)^*$ defines a bounded affine map over K.

$\square$

3 Monadicity

We want to prove that the adjunction is monadic, applying the monadicity theorem, see [**?**]. For this, we have to draw some diagrams.

Let $K, L \in BConv$ and let $f, g : K \to L$ be an E-absolute coequalizer pair in $BCOnv^{op}$. This means that there is some $A \in EA$ and an arrow $q : E(L) \to A$ such that

$$\begin{tikzcd} E(K) \arrow[r, shift left, "Ef"] \arrow[r, shift right, "Eg"'] & E(L) \arrow[r, "q"] & A \end{tikzcd} \tag{*}$$

is an absolute coequalizer diagram. That is, applying any functor $F : EA \to \mathcal{C}$ to (*) yields a coequalizer diagram in $\mathcal{C}$. We have to show that

(a) there is some $e : L \to L'$ in $BConv^{op}$ such that

$$\begin{tikzcd} E(K) \arrow[r, shift left, "Ef"] \arrow[r, shift right, "Eg"'] & E(L) \arrow[r, "Ee"] & E(L') \end{tikzcd} \tag{1}$$

is a coequalizer in EA

(b) each e as in (a) is a coequalizer of f and g in $BConv^{op}$.

Note first that since $E(K)$ and $E(L)$ are complete archimedean, we have

$$\begin{tikzcd} E(K) \arrow[r, shift left, "Ef"] \arrow[r, shift right, "Eg"'] \arrow[d, leftrightarrow, "\simeq"] & E(L) \arrow[r, "q"] \arrow[d, leftrightarrow, "\simeq"] & A \\ T_c E(K) \arrow[r, shift left, "T_c E f"] \arrow[r, shift right, "T_c E g"'] & T_c E(L) \arrow[r, "T_c q"] & T_c(A) \end{tikzcd}$$

and since there is a coequalizer diagram in both lines, we obtain an isomorphism $T_c(A) \simeq A$. This implies that A is complete archimedean as well.

Let us now apply the functor $V : EA \to OVSu$ and obtain the absolute coequalizer diagram

$$VE(K) \underset{g^*}{\overset{f^*}{\rightrightarrows}} VE(L) \xrightarrow{Vq} VA \qquad (**)$$

where f^*, g^* are as in Lemma 1. Note that $VE(K)$, $VE(L)$ and VA are order unit Banach spaces and f^*, g^* and Vq are bounded linear maps. Since (**) is an absolute coequalizer diagram, applying the forgetful functor $U : OVSu \to Vect$ we obtain a coequalizer diagram in $Vect$. It follows that there is an isomorphism $VA \simeq VE(L)|_{R(f^*-g^*)}$ in $Vect$ such that the diagram

$$\begin{tikzcd} VE(L) \arrow[r, "Vq"] \arrow[dr, "q'"'] & VA \arrow[d, "\simeq"] \\ & VE(L)|_{R(f^*-g^*)} \end{tikzcd}$$

commutes (in $Vect$), here q' is the quotient map. It follows that

$$R((f-g)^*) = R(f^* - g^*) = (Vq)^{-1}(0)$$

and since Vq is continuous, $R((f-g)^*)$ is closed (in the order unit norm topology of $VE(L)$). By the closed range theorem, it follows that

$$R((f-g)^*) = N(f-g)^{\perp}$$

where $N(f-g) = \{x \in V_b(L), (f-g)(x) = 0\}$. and

$$\begin{tikzcd} VE(L) \arrow[r, "Vq"] \arrow[dr, "q'"] \arrow[ddr, "e^*=VEe"'] & VA \arrow[d, "\simeq"] \\ & VE(L)|_{Ker(f-g)^{\perp}} \arrow[d, "\simeq"] \\ & {[Ker(f-g)]^*} \end{tikzcd}$$

where $e : Ker(f-g) \to V_b(L)$ is the embedding. We need to show that $Ker(f-g)$ is positively generated, that is, $Ker(f-g) = V_b(L')$, where

$$L' := Ker(f-g) \cap L = \{x \in L, f(x) = g(x)\}.$$