

① The categories

Aff : Category of affine spaces

Object : vector spaces

Morphisms :

$$X \xrightarrow{F} Y$$

$$f(a_1 x_1 + \dots + a_n x_n) = a_1 f(x_1) + \dots + a_n f(x_n)$$

for all $x_i \in X$ $a_i \in \mathbb{R}$ such that

$$a_1 + \dots + a_n = 1$$

Vect / \mathbb{R}

Objects : linear functionals . $V \xrightarrow{g} \mathbb{R}$

Morphisms : commutative triangles

$$\begin{array}{ccc} V & \xrightarrow{F} & V' \\ g \downarrow & & \downarrow g' \\ \mathbb{R} & & \mathbb{R} \end{array}$$

② Affine spaces as a subcategory of Vect / \mathbb{R}

A functor $J: \text{Aff} \rightarrow \text{Vect} / \mathbb{R}$

On objects : $J(X) = (X \times \mathbb{R} \xrightarrow{p_{\mathbb{R}}^X} \mathbb{R})$

projection onto second coordinate

On morphisms: $X \xrightarrow{F} Y$ - a morphism in aff

$$X \times \mathbb{R} \xrightarrow{J(F)} Y \times \mathbb{R}$$

is given by the rule

$$J(F)(x, r) = (F(x) + (r-1)F(0), r)$$

Claim: $J(F)$ is a linear mapping

ADDITIVITY:

$$J(F)((x_1, r_1) + (x_2, r_2)) =$$

$$J(F)(x_1 + x_2, r_1 + r_2) =$$

$$F(x_1 + x_2) + (r_1 + r_2 - 1) \cdot F(0), r_1 + r_2)$$

$$J(F)(x_1, r_1) + J(F)(x_2, r_2) =$$

$$(F(x_1) + (r_1 - 1) \cdot F(0) + F(x_2) + (r_2 - 1)F(0), r_1 + r_2)$$

Second coordinate: OK

First coordinate:

$$F(x_1) + r_1 F(0) + F(x_2) + r_2 F(0) - F(0) - F(0) =$$

$$\uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow$$
$$1 + 1 - 1 = 1; F \text{ is an affine map}$$

$$= F(x_1 + x_2 - 0) + r_1 F(0) + r_2 F(0) - F(0) =$$

$$= F(x_1 + x_2) + (r_1 + r_2 - 1) \cdot F(0) \quad \checkmark$$

MULTIPLICATION BY A SCALAR

$$\mathcal{J}(F)(\Delta(x, n)) = \mathcal{J}(F)(\Delta x, \Delta n) = \\ = (F(\Delta x) + (\Delta n - 1) \cdot F(0), \Delta n)$$

$$\Delta \cdot \mathcal{J}(F)(x, n) = (\Delta f(x) + \Delta(n-1) \cdot F(0), \Delta n)$$

$$\Delta f(x) + \Delta(n-1) \cdot F(0) =$$

$$= \Delta f(x) + \Delta n F(0) - \Delta f(0) =$$

$$= \underline{\Delta} f(x) + \underline{\Delta n} F(0) - \underline{\Delta} f(0) - \underline{(\Delta n - 1)} F(0) + (\Delta n - 1) F(0) =$$

$$\Delta + \Delta n - \Delta - (\Delta n - 1) = 1$$

$$= f(\Delta x + \Delta n \cdot 0 - \Delta \cdot 0 - (\Delta n - 1) \cdot 0) + (\Delta n - 1) \cdot f(0) =$$

$$= f(\Delta x) + (\Delta n - 1) \cdot f(0)$$

FUNCTORIALITY:

$$\mathcal{J}(\text{id})(x, n) = (x + (n-1) \cdot 0, n) = (x, n)$$

$$\mathcal{J}(F_1 \circ F_2)(x, n) = (F_1(F_2(x)) + (n-1)F_1(F_2(0)), n)$$

$$(\mathcal{J}(F_1) \circ \mathcal{J}(F_2))(x, n) = \mathcal{J}(F_1)(\mathcal{J}(F_2)(x, n)) =$$

$$= \mathcal{J}(F_1)(F_2(x) + (n-1)F_2(0), n) = \text{skip } n \text{ from now on...}$$

$$= F_1(F_2(x) + (n-1)F_2(0)) + (n-1)F_1(0) =$$

$$= f_1(1 \cdot (f_2(x) + (1-1)f_2(0)) + (1-1)f_1(0) + \\ + (1-1)f_1(f_2(0)) + (1-1)f_1(f_2(0))) =$$

$$\overset{1 + (1-1) + (1-1) = 0}{=} f_1(f_2(x) + (1-1)f_2(0) + (1-1) \cdot 0 + \\ + (1-1)f_2(0)) + (1-1)f_1(f_2(0)) = \\ = f_1(f_2(x)) + (1-1)f_1(f_2(0))$$

J IS A FAITHFUL FUNCTOR

$$J(f_1) = J(f_2)$$

$$f_1, f_2: X \rightarrow Y$$

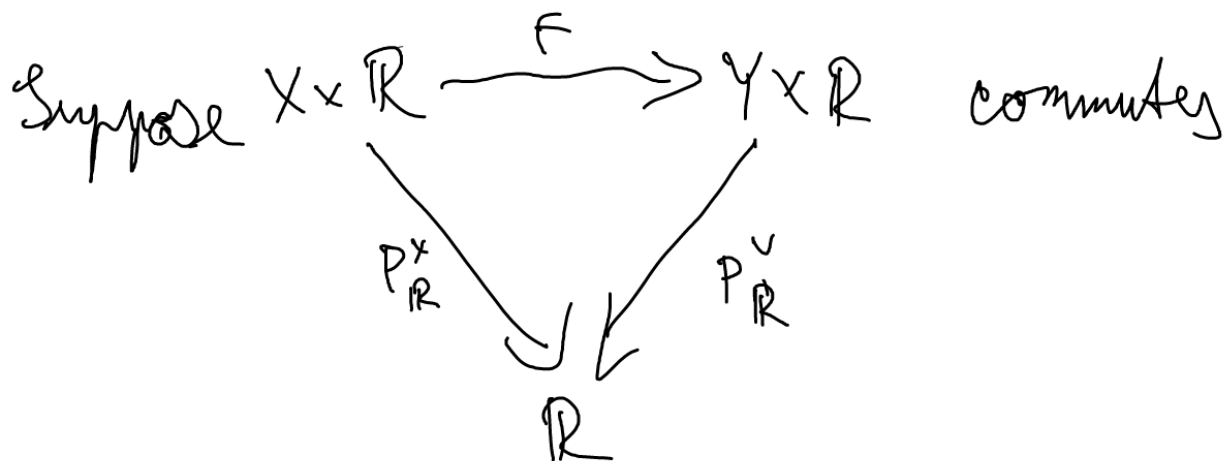
$$\Downarrow ? \\ f_1 = f_2$$

$$\forall (x, \alpha) \in X \times \mathbb{R}$$

$$f_1(x) + (\alpha - 1)f_1(0) = f_2(x) + (\alpha - 1)f_2(0)$$

PUT $\alpha = 1 \Rightarrow \text{DONE}$

F IS A FULL FUNCTOR



$$F \cong \langle F_Y, F_R \rangle \quad f(x, r) = (f_Y(x, r), f_R(x, r))$$

$$f_R(x, r) = r \quad (\text{because the diagram commutes...})$$

We need to find an affine map $F': X \rightarrow Y$ such that

$$J(F') = F, \text{ that means,}$$

$$f(x, r) = (F'(x) + (r-1)F'(0), r)$$

$$\text{But } F'(x) = f_Y(x, 1)$$

$$F'(x) + (r-1)F'(0) =$$

$$= f_Y(x, 1) + (r-1)f_Y(0, 1) =$$

$$= f_Y(x, 1) + f_Y(0, r-1) =$$

$$= f_Y(x, r)$$

It remains to prove that $f': X \rightarrow Y$ is an affine map

$$f'(\lambda_1 x_1 + \dots + \lambda_n x_n) = \lambda_1 f'(x_1) + \dots + \lambda_n f'(x_n)$$

$$\lambda_1 + \dots + \lambda_n = 1$$

$$f'(\lambda_1 x_1 + \dots + \lambda_n x_n) =$$

$$= f_Y(\lambda_1 x_1 + \dots + \lambda_n x_n, 1) =$$

$$= f_Y(\lambda_1 x_1 + \dots + \lambda_n x_n, \lambda_1 + \dots + \lambda_n) =$$

$$= f_Y((\lambda_1 x_1, \lambda_1) + \dots + (\lambda_n x_n, \lambda_n)) =$$

$$= f_Y(\lambda_1(x_1, 1) + \dots + \lambda_n(x_n, 1)) =$$

$$= \lambda_1 f_Y(x_1, 1) + \dots + \lambda_n f_Y(x_n, 1) =$$

$$= \lambda_1 f'(x_1) + \dots + \lambda_n f'(x_n) \quad \checkmark$$

ESSENTIALLY SURJECTIVE
(not surjective functionals in $\mathbb{K}^n / \mathbb{R}$)

$X \xrightarrow{\gamma} \mathbb{R}$ surjective \Rightarrow there is

$X' \in \text{Aff}$ such that

there is an iso j



But $X' = \text{Ker } g$; $\text{Ker } g$ has codimension 1,
so let $y \in X \setminus \text{Ker } g$ be such that $g(y) = 1$

Let $j: X' \times \mathbb{R} \longrightarrow X$ be given by

$$j(x', r) = x + ry$$

$$\begin{aligned} \text{as } g(j(x', r)) &= g(x + ry) = g(x) + r \cdot g(y) = \\ &= 0 + r \cdot 1 = r = p_2(x', r) \quad \checkmark \end{aligned}$$

(3) The adjunction $\text{Vect}/\mathbb{R} \xrightleftharpoons[\underset{G}{\underset{\text{1}}{\text{F}}}]{\text{F}} \text{Vect}$

For $X \in \text{Vect}$

$$G(X) = (X \times \mathbb{R} \xrightarrow{P_{\mathbb{R}}^X} \mathbb{R})$$

$$G(X \xrightarrow{f} Y) =$$

$$\begin{array}{ccc} X \times \mathbb{R} & \xrightarrow{f \times \text{id}} & Y \times \mathbb{R} \\ & \searrow P_{\mathbb{R}}^X & \swarrow P_{\mathbb{R}}^Y \\ & \mathbb{R} & \end{array}$$

For $g \in \text{Vect}/\mathbb{R}$

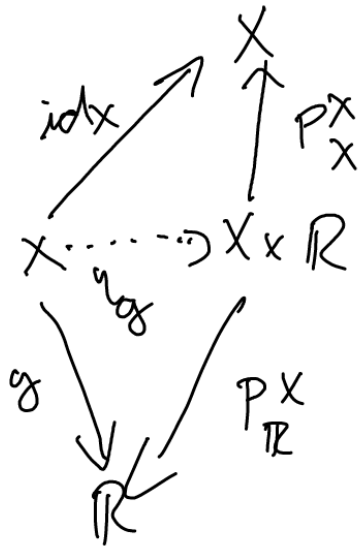
$$F(X \xrightarrow{g} \mathbb{R}) = X$$

$$F\left(\begin{array}{ccc} X & & \mathbb{R} \\ \downarrow f & \searrow g & \\ X' & \nearrow g' & \mathbb{R} \end{array}\right) = (X \xrightarrow{F} X')$$

UNIT

$$g: X \longrightarrow R \in \mathbf{Vect}/\mathbb{R}$$

$$\eta_g: g \longrightarrow GF(g)$$



COUNIT

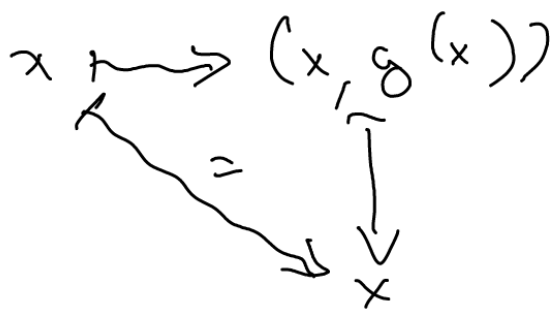
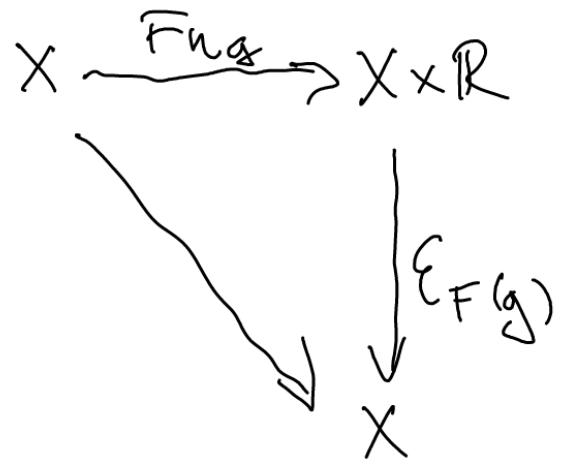
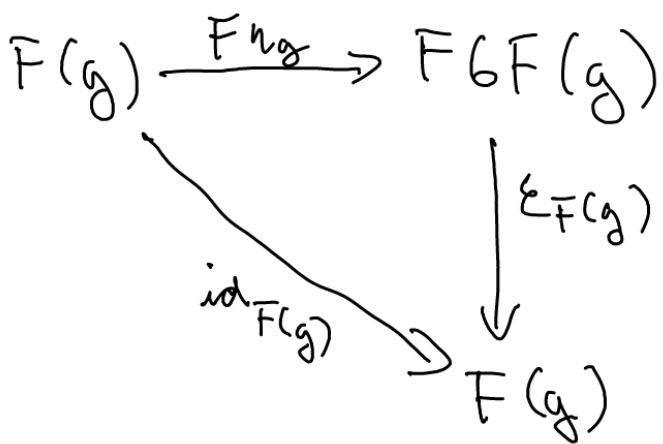
$$X \in \mathbf{Vect}$$

$$\epsilon_X: F(G(X)) \longrightarrow X$$

$$X \times \mathbb{R} \longrightarrow X$$

$$\epsilon_X = p_X^X$$

TRIANGLE IDENTITIES



$$G(X) \xrightarrow{\eta_{G(X)}} GF G(X)$$

$$\searrow \text{id}_{G(X)} \quad \downarrow G\epsilon_X$$

$$G(X)$$

$$(X \times \mathbb{R} \xrightarrow{P_X^R} \mathbb{R})$$

$$\begin{array}{ccc}
 & X \times \mathbb{R} & \\
 \text{id} \nearrow & & \nearrow P_{1,2} \\
 X \times \mathbb{R} & \xrightarrow{\eta_{G(X)}} & X \times \mathbb{R} \times \mathbb{R} \\
 \downarrow P_X^R & \nearrow P_3 = GF G(v) & \\
 \mathbb{R} & &
 \end{array}$$

$G(X) = P_X^R \downarrow$
 $\eta_{G(X)}(x, r) = (x, r, r)$

$$\epsilon_X: X \times \mathbb{R} \xrightarrow{P_X} X$$

$G(\epsilon_X)$

$$(X \times \mathbb{R} \times \mathbb{R} \xrightarrow{P_X \times \text{id}} X \times \mathbb{R})$$

$$\begin{array}{ccc}
 & & \\
 P_3 \downarrow & & \nearrow P_2 \\
 \mathbb{R} & &
 \end{array}$$

$$X \times \mathbb{R} \xrightarrow{\eta_{G(X)}} X \times \mathbb{R} \times \mathbb{R}$$

$$\begin{array}{ccc}
 P_2 \downarrow & & \downarrow P_X \times \text{id} \\
 \mathbb{R} & \xleftarrow{P_3} & X \times \mathbb{R} \\
 & \xleftarrow{P_2} &
 \end{array}$$

$$(x, r) \mapsto (x, r, r)$$

$$\begin{array}{ccc}
 \downarrow & & \downarrow \\
 \mathbb{R} & \xleftarrow{\quad} & (x, r) \\
 & \xleftarrow{\quad} &
 \end{array}$$

④ Vect/\mathbb{R} as coalgebras

The adjunction $\text{Vect}/\mathbb{R} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} \text{Vect}$ induces a comonad on Vect

$$X \xleftarrow{\epsilon_X} X \times \mathbb{R} \xrightarrow{\sigma_X} X \times \mathbb{R} \times \mathbb{R}$$

$$\sigma_X(x, r) = (x, r, r)$$

Vect/\mathbb{R} is isomorphic to the category of coalgebras for this comonad

$X \xrightarrow{\gamma} X \times \mathbb{R}$ is a coalgebra iff

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & X \times \mathbb{R} \\ \searrow \text{id} & & \downarrow \epsilon_X \\ & & X \\ \downarrow & & \\ \gamma(x) = (x, g(x)) & & \end{array}$$

$$\text{for some } g: X \rightarrow \mathbb{R}$$

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & X \times \mathbb{R} \\ \downarrow \gamma & & \downarrow \sigma_X \\ X \times \mathbb{R} & \xrightarrow{\gamma \times \text{id}_{\mathbb{R}}} & X \times \mathbb{R} \times \mathbb{R} \\ & & \downarrow \\ x \mapsto (x, g(x)) & & \\ \downarrow & & \downarrow \\ (x, g(x)) \mapsto (x, g(x), g(x)) & & \end{array}$$

commutes trivially.

Morphism of coalgebras:

$$\begin{aligned} x &\xrightarrow{g_x} (x, g_x(x)) \\ y &\xrightarrow{g_y} (y, g_y(y)) \end{aligned}$$

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow g_x & & \downarrow g_y \\ X \otimes R & \xrightarrow{F \otimes \text{id}} & Y \otimes R \end{array}$$

$$\begin{array}{ccc} x & \xrightarrow{\quad} & F(x) \\ \downarrow & & \downarrow \\ (x, g_x(x)) & \xrightarrow{\quad} & (F(x), g_y(F(x))) \\ & & \uparrow \\ & & (F(x), g_x(x)) \end{array}$$

So the square commutes iff

$$\forall x: g_y(F(x)) = g_x(x)$$

meaning

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ g_x \downarrow & & \swarrow g_y \\ & & R \end{array}$$

commutes.

⑤ Monoidal structure on Vect/\mathbb{R}

This structure comes from the fact that \mathbb{R} is a (multiplicative) monoid in Vect .

Tensor product

$$\begin{array}{ccc} X_1 & X_2 & X_1 \otimes X_2 \\ g_1 \downarrow & \otimes \downarrow g_2 & \downarrow \\ \mathbb{R} & \mathbb{R} & \mathbb{R} \otimes \mathbb{R} \\ & & \downarrow \\ & & \mathbb{R} \end{array} =$$

Unit object

$$\begin{array}{c} \mathbb{R} \\ \text{id} \downarrow \\ \mathbb{R} \end{array}$$

Unit object is "neutral"

$$\begin{array}{ccc} \mathbb{R} & X & \mathbb{R} \otimes X \xrightarrow{\lambda_X} X \\ \text{id} \downarrow \otimes \downarrow g & \downarrow g & \downarrow \text{id} \otimes g \\ \mathbb{R} & \mathbb{R} & \mathbb{R} \otimes \mathbb{R} \\ & & \downarrow \cdot \\ & & \mathbb{R} \end{array}$$

$\nearrow g$

this triangle is λ_g
in the "lifted monoidal structure"

$$\begin{array}{ccc} r \otimes x & \xrightarrow{\quad} & rx \\ \downarrow & & \uparrow \\ r \otimes g(x) & & \\ \downarrow & & \nwarrow \\ r \cdot g(x) & & \end{array}$$

Claim: If we restrict to the full subcategory of Vect/\mathbb{R} that is spanned by surjective functionals, this tensor product comes from the "normal" tensor product of affine spaces.

The proper formulation of this claim follows:

Claim: \mathcal{I} is a strict monoidal functor.

Step 1: Construct an isomorphism

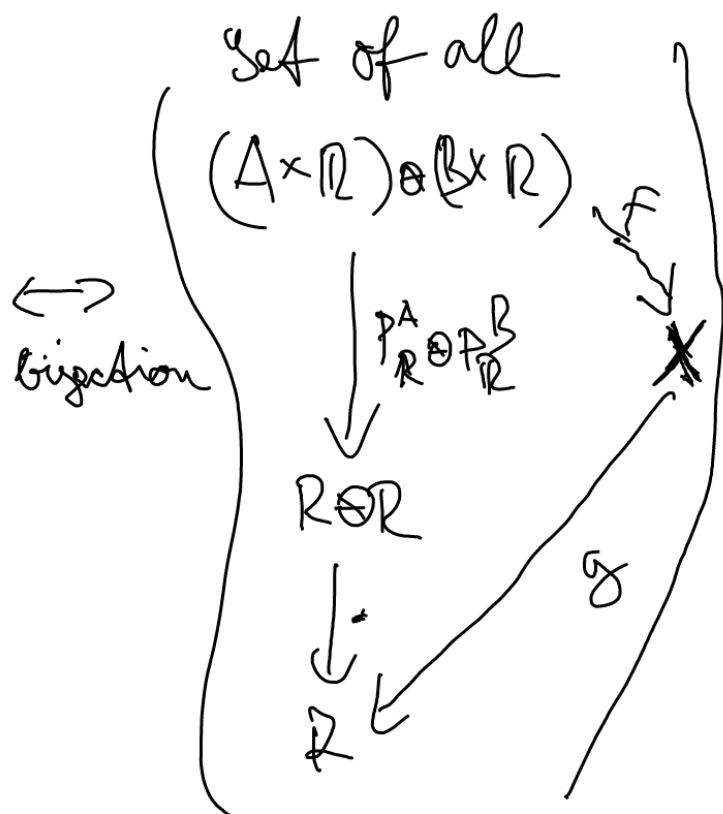
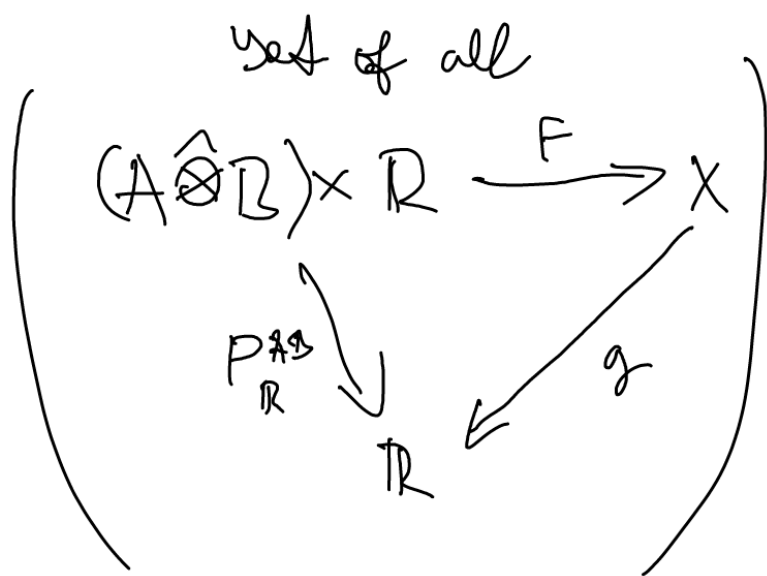
$$\mathcal{I}(- \hat{\otimes} -) \longrightarrow \mathcal{I}(-) \otimes \mathcal{I}(-)$$

"By Yoneda" this is the same as

to construct a bijection of homsets

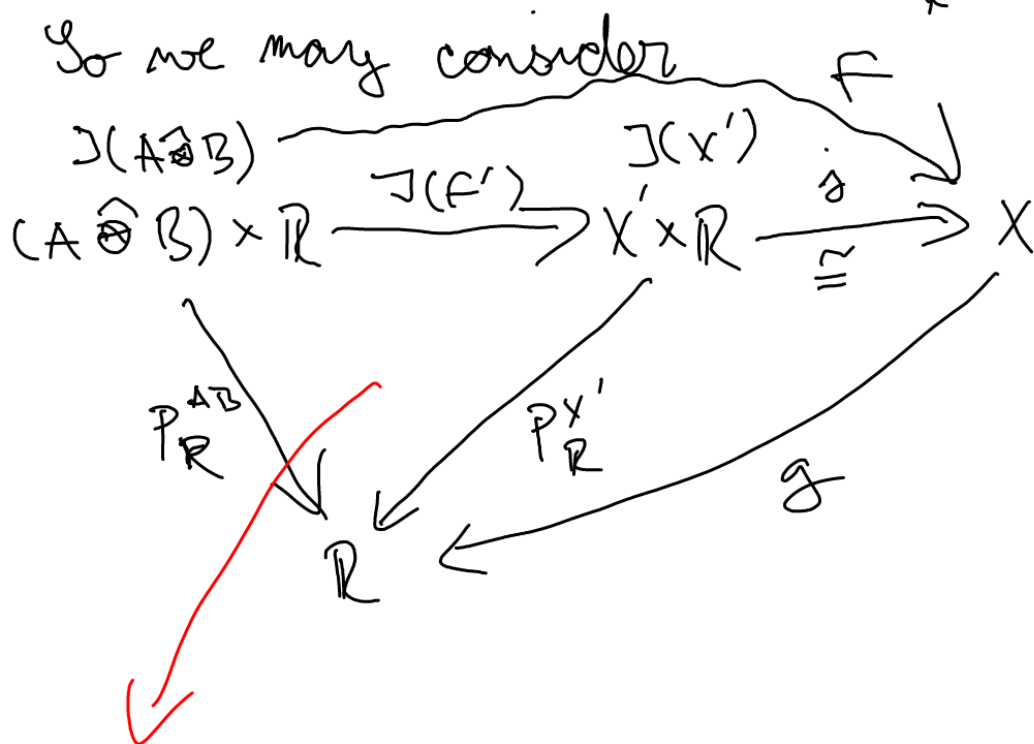
$$\text{Vect}/\mathbb{R}(\mathcal{I}(A \hat{\otimes} B), x \xrightarrow{g} \mathbb{R}) \stackrel{\sim}{=} \text{Vect}/\mathbb{R}(\mathcal{I}(A) \otimes \mathcal{I}(B), x \xrightarrow{g} \mathbb{R})$$

naturally in A, B, g .



① For $g=0$, the set is empty.

② If $g \neq 0$, then g is isomorphic to some $J(X') = (X' \times \mathbb{R} \xrightarrow{P_{\mathbb{R}}^{X'}} \mathbb{R})$.



such triangles are in 1-1 correspondence with affine maps $A \hat{\otimes} B \xrightarrow{f'} X'$

On the right-hand side, we have

$$\begin{array}{ccccc}
 (A \times \mathbb{R}) \otimes (B \times \mathbb{R}) & \xrightarrow{F} & X' \times \mathbb{R} & \xrightarrow{P} & X' \\
 \downarrow P_A^A \otimes P_B^B & & \nearrow P_{\mathbb{R}}^{X'} & & \\
 \mathbb{R} \otimes \mathbb{R} & & & & \\
 \downarrow & & & & \\
 \mathbb{R} & & & &
 \end{array}$$

We need to prove that every such F determines an affine bismorphism $A \times B \rightarrow X'$ and vice versa.

Define $\hat{F}: A \times B \rightarrow X'$ by the rule

$$\hat{F}(x, y) = P(F((x, 1) \otimes (y, 1))) ; P \text{ is the proj.}$$

We claim that this is a bismorphism of affine spaces.

$$X = \Delta_1 x_1 + \dots + \Delta_n x_n \quad \Delta_1 + \dots + \Delta_n = 1$$

$$P(F((x, 1) \otimes (y, 1))) =$$

$$P(F((\Delta_1 x_1 + \dots + \Delta_n x_n, 1) \otimes (y, 1))) =$$

$$P(F(\Delta_1(x_1, 1) + \dots + \Delta_n(x_n, 1)) \otimes (y, 1))) =$$

$$\begin{aligned}
& p\left(f\left(\Delta_1(x_1, 1) \otimes (y, 1) + \dots + \Delta_n(x_n, 1) \otimes (y, 1)\right)\right) \\
&= \Delta_1 \cdot p(f((x_1, 1) \otimes (y, 1))) + \dots + \Delta_n \cdot p(f((x_n, 1) \otimes (y, 1))) \\
&= \Delta_1 \hat{F}(x_1, y) + \dots + \Delta_n \hat{F}(x_n, y)
\end{aligned}$$

So \hat{F} is an affine bismorphism.

For the opposite direction, let $b: A \times B \rightarrow X'$ be an affine bismorphism.

Let $(x_i)_{i \in H}$ be a basis of the vector space A
 $(y_j)_{j \in K}$ be a basis of the vector space B

Then $\{(x_i, 0)\}_{i \in H} \cup \{(0, 1)\}$ is a basis of $A \times \mathbb{R}$;
clearly,

$\{(x_i, 1)\}_{i \in H} \cup \{(0, 1)\}$ is another basis of $A \times \mathbb{R}$ and

$\{(y_j, 1)\}_{j \in K} \cup \{(0, 1)\}$ is a basis of $B \times \mathbb{R}$.

Therefore, there is a basis of
 $(A \times \mathbb{R}) \otimes (B \times \mathbb{R})$

$$\{(x, 1) \otimes (y, 1) : x \in \{x_i\}_{i \in H} \cup \{0\}, y \in \{y_j\}_{j \in K} \cup \{0\}\}$$

for the elements of this basis, put

$$f((x, 1) \otimes (y, 1)) = b(x, y).$$

This defines a linear map

$$F: (A \times \mathbb{R}) \otimes (B \times \mathbb{R}) \longrightarrow X'.$$

We need to prove that $\hat{F} = b$, that means,

that $f((z, 1) \otimes (w, 1)) = b(z, w)$ for all

$z \in A, w \in B$.

$$z = \sum_i c_i x_i \quad w = \sum_j d_j y_j$$

$$(z, 1) = \sum_i c_i (x_i, 1) + (1 - \sum_i c_i)(0, 1)$$

$$(w, 1) = \sum_j d_j (y_j, 1) + (1 - \sum_j d_j)(0, 1)$$

$$f((z, 1) \otimes (w, 1)) =$$

$$\begin{aligned} & f\left(\left(\sum_i c_i (x_i, 1) + (1 - \sum_i c_i)(0, 1)\right) \otimes \right. \\ & \quad \left. \left(\sum_j d_j (y_j, 1) + (1 - \sum_j d_j)(0, 1)\right)\right) = \\ & = f\left(\left[\sum_{i,j} c_i d_j (x_i, 1) \otimes (y_j, 1)\right] + \right. \end{aligned}$$

$$\begin{aligned}
& + \left[\sum_i c_i (1 - \sum_j d_j) (x_{i,1}) \otimes (0,1) \right] + \\
& + \left[\sum_j (1 - \sum_i c_i) d_j (0,1) \otimes (y_{j,1}) \right] + \\
& + \left[(1 - \sum_i c_i) (1 - \sum_j d_j) (0,1) \otimes (0,1) \right] = \\
& = \sum_i c_i \left[\sum_j d_j b(x_i, y_j) + (1 - \sum_j d_j) b(x_i, 0) \right] + \\
& + (1 - \sum_i c_i) \left[\sum_j d_j b(0, y_j) + (1 - \sum_j d_j) b(0, 0) \right] \\
& = \sum_i c_i [b(x_i, w)] + (1 - \sum_i c_i) b(0, w) =
\end{aligned}$$

↓
 this is an affine bismorphism

$$= b(z, w) \quad \text{☺}$$

↓
 again

NATURALITY

