Categories of convex sets

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1 The categories Conv and GConv

Let **Conv** denote the category whose objects are convex structures satisfying (c1)-(c4), with affine maps as morphisms. This is the Eilenberg-Moore category for the distribution monad.

For $X \in \mathbf{Conv}$, the elements of $\mathbf{Conv}(X, \mathbb{R})$ are called *functionals*. Note that $\mathbf{Conv}(X, \mathbb{R})$ can be given a structure of a vector space, which we denote by $A(X)$, with an ordering defined by the wedge $A(X)^+$ of positive affine maps. Clearly, $A(X)^+$ is an Archimedean cone, but there is no order unit in general.

Example 1.1. *Let $X = \mathbb{R}$, with usual affine structure. Any affine map $f : \mathbb{R} \to \mathbb{R}$ has the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. It follows that the only elements in $A(X)^+$ are positive constants, none of which can be an order unit.*

Let $A_b(X)$ denote the vector subspace of bounded functionals, $A_b(X)^+$ the set of positive bounded functionals and let 1_K denote the constant $1_K(x) \equiv 1$. Then $(A_b(X), A_b(X)^+, 1_X)$ is an order unit Banach space, with order unit norm satisfying

$$\|f\|_{1_X} = \sup_{x \in X} |f(x)|.$$

Let also $E(X) := \mathbf{Conv}(X, [0, 1])$, then $E(X)$ is the interval between 0 and 1_X in $(A_b(X), A_b(X)^+)$. Functionals in $E(X)$ will be called *effects*.

A convex structure X is called *geometric* if it is isomorphic to a convex subset of a vector space. Any such isomorphism will be called a geometric representation of X. The category **GConv** of geometric convex sets is a full subcategory of **Conv**. [6, Thm. 1.3] gives an intrinsic characterization of geometric convex sets. Further, by [6, Thm. 1.2], X is geometric iff it is separated by elements of $A(X)$. In this case, the map $\phi : X \to A(X)'$, given by

$$\phi(x)(f) = f(x), \qquad f \in A(X),\ x \in X,$$

is a geometric representation of X. We will identify X with its image $\phi(X)$ in $A(X)'$. Note that this image lies in the hyperplane $\{\varphi \in A(X)',\ \varphi(1_X) = 1\}$, which does not contain 0. Put $V(X) := \operatorname{span}\{X\} \subseteq A(X)'$, $V(X)^+ := \cup_{\lambda\geq 0}\lambda X \subseteq (A(X)^+)'$. Let $u_X \in V(X)'$ be given by the restriction of the functional $1_X \in A(X) \subseteq A(X)''$.

Proposition 1.2. *(i) $V(X)^+$ is a generating cone in $V(X)$, with base X.*

(ii) $(A(X), A(X)^+) \simeq (V(X)', (V(X)^+)')$, in the category **OVS**.

(iii) $(A_b(X), A_b(X)^+, 1_X) \simeq (V(X)^*, (V(X)^+)^*, u_X)$, in the category **OUS**, *where $V(X)^*$ is the space of functionals bounded with respect to the base seminorm and $(V(X)^+)^* = V(X)^* \cap (V(X)^+)'$, see [5].*

Proof. $V(X)^+$ is a generating wedge in $V(X)$ by definition. Let $v \in V(X)^+ \cap -V(X)^+$, so that there are some $a, b \in \mathbb{R}^+$ and $x, y \in X$ such that $v = ax = -by$. Assume $a + b > 0$, then by convexity

$$0 = \frac{a}{a+b}x + \frac{b}{a+b}y \in X,$$

which is impossible. Hence $a = b = 0$ and $v = 0$. To show that X is a base of $V(K)^+$, it suffices to observe that $X = \{v \in V(X)^+, u_X(v) = 1\}$. This proves (i).

To show (ii), let $\varphi \in V(X)'$, then clearly $\varphi|_X \in A(X)$ and $\varphi \in (V(X)^+)'$ iff $\varphi|_X \in A(X)^+$. Conversely, any $f \in A(X)$ extends to an element $\varphi_f \in V(X)'$, which is unique, since X is generating. To define the extension, put $\varphi_f(0) := 0$ and $\varphi_f(v) := af(x) - bf(y)$ for $v = ax - by$ with $a, b \geq 0$ and $x, y \in X$. To show that this extension is well defined, assume that $v = ax - by = cx' - dy'$ for $a, b, c, d \in \mathbb{R}^+$ and $x, x', y, y' \in X$. Then $ax + dy' = cx' + by$ and applying u_X implies that $a + d = c + b$. If $a + d = 0$, then $v = 0$ and $\varphi_f(v) = 0 = af(x) - bf(y)$. Otherwise, we obtain

$$\frac{a}{a+d}x + \frac{d}{a+d}y' = \frac{c}{c+b}x' + \frac{b}{c+b}y$$

and since f is affine, we get back to $af(x) - bf(y) = cf(x') - df(y')$. This shows that $\varphi \mapsto \varphi|_X$ defines an order isomorphism of $(V(X)', (V(X)^+)')$ and $(A(X), A(X)^+)$.

(iii) follows directly by [5, Theorem 2 (iii)].

□

Remark 1.3. *Let $\psi : X \to V$ be any geometric representation. Let $\tilde{\psi} : X \to V \oplus \mathbb{R}$ be defined by $\tilde{\psi}(x) = (\phi(x), 1)$, then the image $\tilde{\psi}(X)$ lies in the hyperplane $\{(v, a) \in V \oplus \mathbb{R},\ u(x, a) := a = 1\}$. In all these constructions, we may replace ϕ with the representation $\tilde{\psi}$ and 1_X by the functional u. It is easy to see that all the resulting structures will be isomorphic.*

2 BConv and CConv

A convex structure X is called *bounded* if X is geometric and $co(X \cup -X)$ is linearly bounded in $V(X)$. The full subcategory of bounded convex structures will be denoted by **Bconv**.

Proposition 2.1. **Bconv** *and* **BN** *are equivalent categories.*

Proof. For $X \in$ **Bconv**, let $F(X) = (V(X), V(X)^+, X)$ and for an affine map $f : X \to Y$, define $F(f) : V(X) \to V(Y)$ as the unique extension of f (existence an uniqueness is proved similarly as in the proof of Prop. 1.2). By [5, Prop. 5], F is a functor **BConv** $\to$ **BN**. Since any $(V, P, K) \in$ **BN** is isomorphic to $F(K)$, F is surjective on objects and it is easy to see that it is also full and faithful. Hence F yields an equivalence of the two categories.

□

We have the following characterizations of objects in **Bconv**.

Proposition 2.2. *Let X be a convex structure. Then the following conditions are equivalent.*

(i) X is geometric and the intrinsic semimetric ρ in X is a metric.

(ii) (c5) holds and if for any $\epsilon \in (0, 1]$, there are $p_\epsilon, q_\epsilon \in X$ with $<\epsilon, p, p_\epsilon> = <\epsilon, q, q_\epsilon>$, then $p = q$.

(iii) X is separated by $A_b(X)$.

(iv) X is separated by $E(X)$.

(v) X is bounded.

Proof. The equivalence of (i) and (ii) follows essentially by [6, Thms. 1.3 and 2.2], since the second condition in (ii) is equivalent to the condition in [6, Thm. 2.2]. Indeed, assume that (ii) holds and let $\lambda_i \in [0,1]$, $p_i, q_i \in X$ be such that $\lambda_i \to 0$ and $< \lambda_i, p, p_i >=< \lambda_i, q, q_i >$. Then for any $\epsilon \in [0,1]$ we can find some i such that $\lambda_i \leq \epsilon$. Put $p_\epsilon =< \lambda_i/\epsilon, p, p_i >$, $q_\epsilon =< \lambda_i/\epsilon, q, q_i >$. By conditions (c3) and (c4), we obtain

$$< \epsilon, p, p_\epsilon >=< \lambda_i, p, p_i >=< \lambda_i, q, q_i >=< \epsilon, q, q_\epsilon >,$$

so that $p = q$. Since the converse is quite obvious, this proves the equivalence (i) $\iff$ (ii). Moreover, the equivalence (i) $\iff$ (v) follows by [6, Thm. 2.5] and [1] ([5, Prop. 5]). Since $E(X)$ contains an order unit, $A_b(X)$ is spanned by $E(X)$, so that (iii) and (iv) are equivalent.

Assume (iv), then by Theorem [6, Thm. 1.2], X is geometric. By Proposition 1.2, any $f \in E(X)$ extends uniquely to a linear functional φ_f on $V(X)$. Let $S = co(X \cup -X)$ and let $v_t := v + tw$ for $v, w \in V(X)$, $w \neq 0$ and $t \in \mathbb{R}$. Note that there must be some $g \in E(X)$ such that $\varphi_g(w) \neq 0$. Indeed, we have $w = ax - by$ for $a, b \in \mathbb{R}^+$ and $x, y \in X$. If $\varphi_f(w) = 0$ for all $f \in E(X)$, then also $a - b = \varphi_{1_X}(w) = 0$, hence $a = b$ and $w = a(x - y)$. From $\varphi_f(w) = a(f(x) - f(y)) = 0$ for all $f \in E(X)$, it follows that either $a = 0$ or $x = y$, but in both cases $w = 0$.

If t is such that $v_t \in S$, then

$$\varphi_g(v_t) = \varphi_g(v) + t\varphi_g(w) \in g(S) = co(g(X) \cup -g(X)) \subseteq [-1, 1],$$

and since $\varphi_f(w) \neq 0$, this implies that t must be in a bounded interval. Hence (v) holds.

Finally, if (v) is true, then $(V(X), V(X)^+, X)$ is a base-normed space. By Proposition 1.2 (iii), the dual Banach space $V(X)^*$ is isomorphic to $A_b(X)$ and since the elements of $V(K)^*$ separate points of $V(K)$, this implies (iii). $\square$

Let $X \in \mathbf{BConv}$ and let $\tilde{V}$ be the completion of $V(X)$ with respect to the base norm $\|\cdot\|_X$. Then $V(X)$ is isometrically isomorphic to a norm-dense subspace in $\tilde{V}$ and hence $\tilde{V}^* \simeq V(X)^* \simeq A_b(X)$. By [5, Theorem] and its proof, $\tilde{V}$ has a structure of a base normed space $(\tilde{V}, \tilde{V}^+, \tilde{K})$, with $\tilde{V}^+ = \{v \in \tilde{V},\ \langle f, v\rangle \geq 0,\ \forall f \in A_b(X)^+\}$ and $\tilde{K} = \{v \in \tilde{V}^+,\ \langle v, 1_X\rangle = 1\}$, moreover, $\|\cdot\|_{\tilde{K}} = \|\cdot\|_X$ on $V(X)$.

Let $x \in \tilde{K}$. Since $V(X)$ is dense in $\tilde{V}$, there is a sequence $v_n \in V(X)$ such that $\|v_n - x\|_{\tilde{K}} \to 0$, in particular, $\|v_n\|_X = \|v_n\|_{\tilde{K}} \to \|x\|_{\tilde{K}} = 1$. For any $n \in \mathbb{N}$, we have

$$v_n = \lambda_n x_n - \mu_n y_n, \quad x_n, y_n \in X,\ \lambda_n, \mu_n \geq 0,\ \lambda_n + \mu_n \leq \|v_n\|_X + \frac{1}{n},$$

hence $\lambda_n + \mu_n \to 1$. On the other hand,

$$\lambda_n - \mu_n = \langle 1_X, v_n \rangle \to \langle 1_X, x \rangle = 1$$

so that $\lambda_n \to 1$ and $\mu_n \to 0$. It follows that

$$\lim_n \lambda_n^{-1} x_n = \lim_n v_n = x$$

so that x is a limit of elements in X, hence $\tilde{K}$ is the norm closure of X in $\tilde{V}$.

Assume now that X is complete in the intrinsic metric ρ. Since $\rho(x, y) = \|x - y\|_X = \|x - y\|_{\tilde{K}}$, it follows that we must have $\tilde{K} = X$ and hence also $V(X) = \tilde{V}$ is a Banach space. This proves the following, see also [2] ([6, Thm. 2.7]).

Theorem 2.3. *Let $X \in$ **BConv** be such that (X, ρ) is a complete metric space. Then $(V(X), V(X)^+, X)$ is a base-normed Banach space.*

The full subcategory of bounded convex structures that are complete in ρ will be denoted by **CConv**.

Proposition 2.4. **CConv** *is equivalent to the category* **BNB** *of base-normed Banach spaces with a closed base.*

3 Limits and colimits in Conv

Since **Conv** is a category of algebras for a monad over **Set**, it is complete a cocomplete for general reasons.

3.1 Limits

Everything we say here about limits follows from the general theory of categories of algebras. An object of **Conv** is a terminal object (that means, the limit of an empty diagram) iff it is a one-element object. The operations in the product $\Pi_{i\in I}X_i$ are defined componentwise, as usual. The equalizer of a pair

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \tag{1}$$

is (the inclusion mapping of) a subalgebra E of X given by $E = \{x \in X \colon f(x) = g(x)\}$.

3.2 Colimits

As far as I know [GJ] there is no general theory of colimits in categories of algebras over a cocomplete category. However, we are dealing here with colimits in a category algebras over **Set**, and that is a well-understood topic, see [**?**]. Nevertheless, we shall describe explicitely colimits in **Conv**.

The coequalizers are easy. Consider a parallel pair (1). Equip Y with a congruence $\sim$ generated by all pairs $(f(x), g(x))$, where $x \in X$. Then the quotient mapping $Y \to Y/\sim$ is a coequalizer of f, g.

The following description of coproducts is due to Jacobs [**?**]. We start with the description of $X + \bullet$, where $\bullet$ is the one-element object. The underlying set of $X + \bullet$ is

$$|X + \bullet| = \{(\lambda, x) \colon \lambda = 1 \Leftrightarrow x = \bullet\}$$

The convex structure on the set $|X + \bullet|$ is given by the rules

$$\langle \rho, (\lambda_1, x_1), (\lambda_2, x_2)\rangle = \begin{cases} \bullet & \text{if } \lambda_1 = \lambda_2 = 1 \\ (\tau, \langle(\frac{\rho(1-\lambda_2)}{1-\tau}, x_1, x_2\rangle & \text{otherwise,} \end{cases}$$

where $\tau = (1-\rho)\lambda_1 + \rho\lambda_2 = \langle \rho, \lambda_1, \lambda_2\rangle$ – the latter expression is to be understood within the convex structure of the real $[0, 1]$ interval.

Let us prove that this is a convex structure.

(c1) If $\lambda_1 = \lambda_2 = 1$,

$$\langle 1-\rho, (\lambda_2, x_2), (\lambda_1, x_1)\rangle = \bullet = \langle \rho, (\lambda_1, x_1), (\lambda_2, x_2)\rangle.$$

So suppose that at least one of λ_1, λ_2 is not equal to 1. Then,

$$\langle 1-\rho, (\lambda_2, x_2), (\lambda_1, x_2)\rangle = (\langle 1-\rho, \lambda_2, \lambda_1\rangle, \langle \frac{(1-\rho)(1-\lambda_1)}{1-\langle 1-\rho, \lambda_2, \lambda_1\rangle}, x_2, x_1\rangle)$$

Note that $\langle 1-\rho, \lambda_2, \lambda_1\rangle = \langle \rho, \lambda_1, \lambda_2\rangle = \tau$ and that

$$\frac{(1-\rho)(1-\lambda_1)}{1-\tau} = 1 - \frac{\rho(1-\lambda_2)}{1-\tau}.$$

Therefore,

$$(\langle 1-\rho, \lambda_2, \lambda_1\rangle, \langle \frac{(1-\rho)(1-\lambda_1)}{1-\langle 1-\rho, \lambda_2, \lambda_1\rangle}, x_2, x_1\rangle) = (\tau, \langle (\frac{\rho(1-\lambda_2)}{1-\tau}, x_1, x_2\rangle) = \langle \rho, (\lambda_1, x_1), (\lambda_2, x_2)\rangle$$

(c2) Trivial.

(c3) TODO.

The coproduct $X+Y$ is then a subset of $(X+\bullet)\times(Y+\bullet)$ given by the rule

$$X+Y = \{((\lambda, x), (\mu, y)) \colon \lambda + \mu = 1\}$$

equipped with the coprojections $j_X : X \to X+Y$ and $j_Y : Y \to X+Y$ given by $j_X(x) = ((0,x),(1,\bullet))$ and $j_Y(y) = ((1,\bullet),(0,y))$. Put

$$u((\lambda, x), (\mu, y)) = \langle \lambda, f_X(x), f_Y(y)\rangle.$$

This is the universal arrow for f_X, f_Y. To see the uniqueness of u, observe that $X+Y$ is generated by the ranges of j_X, j_Y:

$$((\lambda, x), (\mu, y)) = \langle \lambda, j_X(x), j_Y(y)\rangle.$$

It remains to prove that u is a morphism of convex structures.

References

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