THE DUALITY OF PARTIALLY ORDERED NORMED LINEAR SPACES

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1. *Introduction*

We are concerned with two ways in which a partial ordering of a linear space can induce a semi-norm or norm topology, namely when the cone is generating and has either an order unit or a base. (For definitions see §2.) It is a direct consequence of a theorem of Krein that the dual space of a normed space with an order unit norm has a base norm for the dual partial ordering. This theorem is also true with the terms "base" and "order unit" interchanged.

This duality was carried further by Edwards in [**7**; Theorem 4], in which he studied conditions on a base in a partially ordered linear space in order that it induce a norm which makes the space a Banach dual space. Under these conditions he proved that the subdual Banach space has an order unit norm. The main result of §3 is the dual case of the result just mentioned. In the same way that Theorem 6 dualises [**7**; Theorem 4], it is possible to dualise certain other theorems of [**7**]; however, these results will not be given here.

Theorem 6 can be generalised to give a dual result (Theorem 7) to a theorem of Grosberg and Krein, which links generating and normal cones in dual Banach spaces. This result then gives some information about the duality of AL and AM spaces.

Section 4 considers when the annihilator of an order ideal, in a partially ordered vector space with an order unit, intersects the base in the order dual space in an extremal set. A theorem of Bonsall [**2**] is generalised.

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2. *Partial orderings*

Throughout the paper X will denote a real vector space. A non-empty subset K, of X, is called a *cone* if, whenever $x, y \in K$ and $\lambda \geqslant 0$, then $x+y \in K$, $\lambda x \in K$ and $-x \notin K$ unless $x=0$. A *partial ordering of* X, relative to a cone K, is a relation defined by $x \geqslant y$, or $y \leqslant x$, if and only if $x-y \in K$. A partial ordering of X is called *archimedean* if $x \leqslant \lambda y$, for some $y \geqslant 0$ and all $\lambda > 0$, implies that $x \leqslant 0$; it is called *almost archimedean* if $-\lambda y \leqslant x \leqslant \lambda y$, for some $y \geqslant 0$ and all $\lambda > 0$, implies that $x = 0$. A

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cone K is *generating* in X if $K-K=X$, and $e \in K$ is an *order unit* for K if for each $x \in X$ there is some $\lambda > 0$ such that $-\lambda e \leqslant x \leqslant \lambda e$. A non-empty, convex subset of B, of K, is called a *base* for K if every $x \in K$, $x \neq 0$, has a unique representation $x=\lambda b$, where $b \in B$ and $\lambda > 0$. This uniqueness condition is equivalent to the condition that the least affine extension of B does not contain 0. A linear functional f, on X, is *positive* if $f(x) \geqslant 0$ for each $x \in K$, and it is *strictly positive* if $f(x) > 0$ for each $x \in K$, $x \neq 0$. We write $\{x \in X : a \leqslant x \leqslant b\} = [a, b]$. The following lemma is obvious.

LEMMA 1. *If K has an order unit e, then the function*

$$\|x\| = \inf\{\lambda > 0 : -\lambda e \leqslant x \leqslant \lambda e\}$$

is a norm in X if and only if X is almost archimedean ordered. If X is archimedean ordered, then $[-e, e]$ is the unit sphere for this norm.

LEMMA 2. *K has a base if and only if there exists a strictly positive linear functional on X.*

Proof. If h is a strictly positive linear functional on X, then $B=\{x : h(x)=1,\ x \in K\}$ is a base for K. Conversely, if K has a base B, then a Zorn's Lemma argument shows the existence of a maximal affine set containing B but not 0. This maximal set is clearly a hyperplane, and its associated linear functional is strictly positive.

LEMMA 3. *Suppose that K has a base B, and let p be the Minkowski functional of S, where $S=\mathrm{co}(B \cup -B)$. Then $B=\{x \in K : p(x)=1\}$, and p is a norm in X if and only if K is generating and S is linearly bounded. If S is linearly compact, then $S=\{x \in K-K : p(x) \leqslant 1\}$.*

Proof. S is clearly convex, balanced and absorbing in $K-K$, and hence p is a semi-norm defined on $K-K$. If $x \in B$, then $p(x) \leqslant 1$. Suppose that $x \in B$, and $x=\lambda b-\lambda' b'$, where $\lambda, \lambda' \geqslant 0$, $b, b' \in B$. Then

$$(1+\lambda')^{-1}(x+\lambda' b') = \lambda(1+\lambda')^{-1} b,$$

and since B is a base it is clear that $\lambda = 1+\lambda'$. Therefore $1=\lambda-\lambda'$, so that $\lambda+\lambda' \geqslant 1$, and it follows that $B=\{x \in K : p(x)=1\}$.

The remaining parts of the lemma are proved by standard arguments.

The norms defined in Lemmas 2 and 3 are called *order unit norms* and *base norms*, respectively. A partially ordered vector space is a *vector lattice* if each pair of elements has a least upper bound. A (semi-)norm, defined on a vector lattice X, is called a *lattice (semi-)norm* if $0 \leqslant x \leqslant y$ implies that $\|x\| \leqslant \|y\|$, and if $\|z\| = \||z|\|$ for all $z \in X$. (We employ the usual notation for lattice operations; see, for example, Day [**3**].)

A Banach space, which is also a vector lattice and has a lattice norm, is called: (i) an *AM-space* if $x, y \geqslant 0$ implies that $\|x \vee y\| = \|x\| \vee \|y\|$; (ii) an *AL*-space if $x, y \geqslant 0$ implies that $\|x+y\| = \|x\|+\|y\|$.

THEOREM 1. *Let X be a partially ordered vector space with a generating cone K which has a base B. Then p, the Minkowski functional of $S = \operatorname{co}(B \cup -B)$, is a semi-norm on X which is additive on K. If X is also a vector lattice, then p is a lattice norm.*

For a proof of this theorem see [**7**; Theorem 2].

The following lemma is a simple consequence of the definitions and of Theorem 1.

LEMMA 4. *Let X be a Banach space and a vector lattice. Then,*

(i) *if the norm in X is an order unit norm, X is an AM-space;*

(ii) *if the norm in X is a base norm, X is an AL-space.*

A vector subspace I, of a partially ordered vector space X, is an *order ideal* if $0 \leqslant x \leqslant y \in I$ implies that $x \in I$. If X is a vector lattice, then an order ideal in X is called a *lattice ideal* if it is also a vector sublattice of X.

LEMMA 5. *If I is an order ideal in X, then X/I is a partially ordered vector space with cone K/I. If X is lattice ordered and I is a lattice ideal in X, then X/I is a vector lattice with cone K/I.*

Now let X be a separated locally convex space with dual space X^*, and let K, K' be cones in X, X^* respectively. Then we write

$$K^* = \{f \in X^* : f(x) \geqslant 0,\ \forall\, x \in K\},$$

and

$$K_*' = \{x \in X : f(x) \geqslant 0,\ \forall f \in K'\}.$$

If K^* is a cone, then it is called the *dual cone* of K; and if K_*' is a cone, then it is called the *subdual cone* of K'. When K^* is a cone, K and K^* are said to be *compatible* if $K_*{}^* = K$. (We write $K_*{}^* = (K^*)_*$.)

LEMMA 6. *Let K be a cone in X and K' a cone in X^*. Then,*

(i) *K^* is a cone in X^* if and only if $K-K$ is dense in X;*

(ii) *K and K' are compatible cones if and only if either $K' = K^*$ and K is closed, or $K = K_*'$ and K' is w^*-closed;*

(iii) *if K is a closed cone, then K^*-K^* is w^*-dense in X^*.*

Proof. Part (i) is not difficult to verify, and part (iii) is a simple consequence of parts (i) and (ii). We prove the first half of (ii).

Suppose that K is closed, that $K' = K^*$, and that there exists $x \in K^*$ such that $x \notin K$. Then there is an $f \in X^*$ such that $f(x) \leqslant f(y) - \epsilon$, for all $y \in K$ and for some $\epsilon > 0$. Therefore $f(y) \geqslant 0$ for all $y \in K$, and $f(x) < 0$. This is a contradiction, and hence $K = K_*'$.

The second half of (ii) is proved similarly.

A cone in X is said to be *C-normal* if there exists a topologising family of semi-norms $\{p_\alpha\}$ and a constant $C > 0$ such that, if $z \leqslant x \leqslant y$, then $p_\alpha(x) \leqslant C \max\{p_\alpha(y), p_\alpha(z)\}$, for each p_α.

A cone in a normed space is said to be *C-generating* if there exists a constant $C > 0$, such that each element x has a decomposition $x = x_1 - x_2$, where $x_1, x_2 \geqslant 0$, and such that $\|x_1\| + \|x_2\| \leqslant C\|x\|$. It is clear that the smallest possible value for the constants C is 1.

The following is a statement of the fundamental extension theorem for positive linear functionals, in a form which will be directly applicable later. This theorem, and also a more general form of the theorem, are given in Day [**3**].

THE MONOTONE EXTENSION THEOREM. *Let X be a partially ordered vector space with an order unit e, and let f be a positive linear functional defined on Y, where Y is a vector subspace of X which contains e. Then f can be extended to a positive linear functional defined on X.*

Proof. The theorem is a direct consequence of the Hahn-Banach theorem, where the sublinear functional involved is

$$p(x) = \inf\{f(y) : y \in Y, \text{ and } y \geqslant x\}.$$

3. *Dual partially ordered Banach spaces*

The first theorem gives a criterion for a Banach space to be the dual of a normed space, and this is essential for the work in this section.

THEOREM 2 (Dixmier). *Let X be a Banach space with dual space X^*, and with unit sphere S. Then X is a Banach dual space if and only if there exists a total vector subspace V, of X^*, such that S is $\sigma(X, V)$-compact. If such a V exists, then X is the dual space of V for the natural pairing.*

(*Note.* A *total* subset of X^* is a subset which separates the points of X.)

The proof of this theorem was given by Dixmier in [**4**].

The following theorem is a simple consequence of a theorem of Edwards [**7**; Theorem 4].

THEOREM 3 (Edwards). *Let X be a Banach space with dual space X^*, and let K, K^* be compatible cones in X, X^* respectively. Then, if X^* has a base norm with w^*-compact base, X has an order unit norm.*

The converse of the above result follows from a well-known theorem of Krein [**1**]. The proof given below is an argument which was used by Krein in [**10**].

THEOREM 4 (Krein). *Let X be a partially ordered normed space with an order unit norm, and let X^* be its dual space. Then each $f \in X^*$ has a decomposition $f = g - h$, where $g, h \geqslant 0$ and $\|f\| = \|g\| + \|h\|$.*

Proof. Let $Y = X \times X$, and let Y be partially ordered by the cone $K \times K$, where K is the cone in X. Then it is clear that (e, e) is an order unit for Y. Now let $Z = \{z \in Y : z = t(e, e) - (x, -x)$, for some real number

t and some $x \in X\}$. Then define a function F, on Z, by $F(z) = t\|f\| - f(x)$, where f is an arbitrary, but fixed, element of X^*. It is clear that F is linear; F is also positive, since if $z \geqslant 0$, then $-te \leqslant x \leqslant te$, and so $\|x\| \leqslant t$ and hence $f(x) \leqslant t\|f\|$. Therefore, by the monotone extension theorem, we may extend F to a positive linear functional defined on Y. Let $g(x) = F(x, 0)$ and $h(x) = F(0, x)$. Then g and h are positive linear functionals on X such that $f = g - h$. Finally, it is evident that g, $h \in X^*$ and that $\|g\| + \|h\| = g(e) + h(e) = F(e, e) = \|f\|$.

COROLLARY. *Let X be a normed space with an order unit norm, and with dual space X^*. Then, for the dual partial ordering, X^* has a w^*-compact base and a base norm.*

Proof. Let $B = \{f \in X^* : \ f(e) = 1 \ \text{and} \ f(x) \geqslant 0, \ \forall x \geqslant 0\}$. Then, clearly, B is a w^*-compact base in X^*. It is now immediate, from the theorem, that B defines the norm in X^*.

It is not difficult to prove a similar result to the above corollary, but in which the roles of base norm and order unit norm are interchanged.

THEOREM 5. *Let X be a normed space with a base norm, and let X^* be its dual space. Then, for the dual partial ordering, X^* has an order unit norm.*

Proof. Let B be the base in X, and let f be the strictly positive linear functional on X such that $f(b) = 1$, for all $b \in B$ (*cf.* Lemma 2). Then $f \in X^*$, and it is an order unit for the dual partial ordering. Let $g \in X^*$; then

$$\begin{aligned}\|g\| &= \sup\{|g(x)| : x \in \operatorname{co}(B \cup -B)\} \\ &= \sup\{|g(b)| : b \in B\} = \inf\{\lambda > 0 : -\lambda f(b) \leqslant g(b) \leqslant \lambda f(b), \ \forall b \in B\} \\ &= \inf\{\lambda > 0 : -\lambda f \leqslant g \leqslant \lambda f\}.\end{aligned}$$

Therefore, it is clear that the unit sphere of X^* is $[-f, f]$.

The main result in this section is Theorem 6, and this dualises a result of Edwards (*cf.* [**7**; Theorem 4]). A corollary of Theorem 6 will prove the converse of Theorem 5.

Tukey proved in [**12**] that if A and B are closed convex sets in a Banach space, and if A is bounded, then $A - B$ contains every open sphere in which it is dense. We shall require a similar result, Lemma 7, which was stated, without proof, by Klee in [**9**]. The proof which is given below is based on the proof of Tukey's result. The condition, stated in Lemma 7, that B is bounded may be removed by using a slight modification of the proof given; however, this stronger form of the lemma will not be required here.

LEMMA 7. *Let A and B be bounded, closed, convex sets in a Banach space X, and let $C = \operatorname{co}(A \cup B)$. Then C contains every open sphere in which it is dense.*

Proof. It will be sufficient to consider the case when C is dense in the unit sphere S, and then to prove that $0 \in C$, since all other cases follow from this by translation and scalar multiplication. We shall also assume that $0 \notin A$, $0 \notin B$.

First choose $c_1 \in C$ such that $\|c_1\| \leqslant \frac{1}{2}$, $c_1 = \mu_1 a_1 + (1-\mu_1) b_1$, $1 \geqslant \mu_1 > 0$, $a_1 \in A$, $b_1 \in B$. Then there exists $d_1 \in C \cap S$ such that $\|2c_1 + d_1\| \leqslant \frac{1}{2}$, and we have $d_1 = \lambda_1 a_1' + (1-\lambda_1) b_1'$ with $a_1' \in A$, $b_1' \in B$ and $1 \geqslant \lambda_1 \geqslant 0$.

Suppose now that c_n, a_n, b_n, μ_n, d_n, λ_n, a_n', b_n' have been chosen so that $c_n \in C$, $\|c_n\| \leqslant 1/2^n$, $c_n = \mu_n a_n + (1-\mu_n) b_n$, $a_n \in A$, $b_n \in B$, $1 \geqslant \mu_n > 0$, $d_n \in C \cap S$, $\|2^n c_n + d_n\| \leqslant \frac{1}{2}$, $d_n = \lambda_n a_n' + (1-\lambda_n) b_n'$, $a_n' \in A$, $b_n' \in B$, $1 \geqslant \lambda_n \geqslant 0$.

Take
$$c_{n+1} = (2^n+1)^{-1}(2^n c_n + d_n),$$
$$a_{n+1} = (2^n \mu_n + \lambda_n)^{-1}(2^n \mu_n a_n + \lambda_n a_n'),$$
$$b_{n+1} = \{2^n(1-\mu_n) + (1-\lambda_n)\}^{-1}\{2^n(1-\mu_n) b_n + (1-\lambda_n) b_n'\}$$
and
$$\mu_{n+1} = (2^n+1)^{-1}(2^n \mu_n + \lambda_n).$$

Then $1 \geqslant \mu_{n+1} > 0$, $c_{n+1} = \mu_{n+1} a_{n+1} + (1-\mu_{n+1}) b_{n+1}$, $a_{n+1} \in A$, $b_{n+1} \in B$, $c_{n+1} \in C$ and $\|c_{n+1}\| \leqslant 1/2^{n+1}$. Thus d_{n+1} can be chosen so that $d_{n+1} \in C \cap S$, $\|2^{n+1} c_{n+1} + d_{n+1}\| \leqslant \frac{1}{2}$ and $d_{n+1} = \lambda_{n+1} a'_{n+1} + (1-\lambda_{n+1}) b'_{n+1}$, where $a'_{n+1} \in A$, $b'_{n+1} \in B$ and $1 \geqslant \lambda_{n+1} \geqslant 0$.

We prove that $\{a_n\}$ is a Cauchy sequence. We observe firstly that
$$\mu_{n+1} = (2^n+1)^{-1} . (2^n \mu_n + \lambda_n) \geqslant (2^n+1)^{-1} . (2^n \mu_n),$$
and hence, for each n,
$$\mu_n \geqslant \left\{ \prod_{j=1}^{\infty} [2^j/(2^j+1)] \right\} \mu_1 = \alpha \mu_1,$$
where $\alpha > 0$. However, for each n,
$$\|a_{n+1} - a_n\| = (2^n \mu_n + \lambda_n)^{-1} . (\|\lambda_n a_n - \lambda_n a_n'\|) \leqslant 2M(2^n \mu_n)^{-1},$$
where $M = \sup\{\|a\| : a \in A\}$, and so
$$\|a_m - a_n\| \leqslant \sum_{p=1}^{\infty} \|a_{n+p} - a_{n+p-1}\| \leqslant K/2^n,$$
for all $m \geqslant n$, and for some constant K which is independent of n. Therefore $\{a_n\}$ is a Cauchy sequence, and since A is closed and X complete, $a_n \to a \in A$. We observe that $c_n \to 0$, and that we may choose a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\mu_{n_k} \to \mu > 0$.

Suppose that $\mu < 1$. Then we have
$$b_{n_k} = (1-\mu_{n_k})^{-1} . (c_{n_k} - \mu_{n_k} a_{n_k}) \to (\mu - 1)^{-1} . (\mu a) = b \in B,$$
since B is closed, and in this case $\mu a + (1-\mu) b = 0 \in C$. Suppose that $\mu = 1$. Then, since B is bounded, $(1-\mu_{n_k}) b_{n_k} \to 0$, and so $a = 0 \in C$. Therefore, in either case, the lemma is proved.

THEOREM 6. *Let X be a real, partially ordered vector space, with a separated locally convex topology τ, and with an order unit e such that $[0, e]$ is τ-compact. Then X has an order unit norm which makes it a Banach space. For this norm, X is the dual space of the Banach space V, where V is the set of all linear functionals on X which are τ-continuous on norm bounded sets. For its subdual partial ordering, V has a base B such that the Minkowski functional of $S = \mathrm{co}(B \cup -B)$ is the norm in V. The topology $\sigma(X, V)$ agrees with τ on norm bounded sets.*

Remark. This method of constructing the subdual space is precisely that which was used by Edwards in [**7**; Theorem 4]. The main problem here, therefore, is to determine the order structure of V for this particular situation.

Proof of Theorem 6. Because e is an order unit for X, and $[0, e]$ is τ-compact, it follows from the linear boundedness of $[0, e]$ that X is archimedean ordered. Therefore, we can define an order unit norm in X such that the unit sphere, $\Sigma = [-e, e]$, is τ-compact. It is not difficult to verify that the τ-compactness of Σ implies that X is a Banach space, for this norm. Moreover, since Σ is τ-compact, each τ-continuous semi-norm is bounded on Σ, and hence the norm topology of X is finer than τ. It now follows that V is a closed vector subspace of X^*, the dual space of X, such that the $\sigma(X, V)$ topology agrees with τ on norm bounded sets. Therefore Σ is $\sigma(X, V)$-compact, and hence, by Theorem 2, X is the dual space of the Banach space V.

If K is the cone in X, then $n \geqslant 0$, $K \cap n\Sigma = [0, ne]$, which is $\sigma(X, V)$-compact. It follows that K is $\sigma(X, V)$-closed (*cf.* [**6**; V.5.7]). Therefore, if C is the subdual cone of K, then Lemma 6 shows that C and K are compatible.

Let $B = \{f \in C : f(e) = 1\}$; then it is evident that B is a base for C, and that $B = \{f \in C : \|f\| = 1\}$. Therefore $S = \mathrm{co}\,(B \cup -B)$ is a subset of the unit sphere, E, of V. If $x \in X$, then we have

$$\begin{aligned}\|x\| &= \inf\{\lambda > 0 : -\lambda e \leqslant x \leqslant \lambda e\} \\ &= \inf\{\lambda > 0 : -\lambda e(f) \leqslant x(f) \leqslant \lambda e(f),\ \forall f \in C\} \\ &= \sup\{|f(x)| : f \in B\}.\end{aligned}$$

Now, suppose that S is not dense in E. Then there exists $f \in E$, $f \notin S$, and $x \in X$ such that

$$x(f) > \sup\{x(g) : g \in S\} \geqslant \sup\{|x(g)| : g \in B\} = \|x\|,$$

which is a contradiction.

Finally, since B is convex, closed and bounded, Lemma 7 implies that S contains the open unit sphere of V: that is

$$\{f : \|f\| < 1\} \subset S \subset \{f : \|f\| \leqslant 1\}.$$

Therefore C is a generating cone, and the Minkowski functional of S is the norm in V.

COROLLARY. *Let X be a Banach space with dual space X^*, and let K, K^* be compatible cones in X, X^* respectively. Then, if X^* has an order unit norm, X has a base norm.*

Proof. Since K^* is the dual cone of K, the $\sigma(X^*, X)$-topology satisfies the conditions on the τ-topology in the preceding theorem. The result now follows from the fact that, if X is a Banach space, then every linear functional which is w^*-continuous on norm bounded subsets of X^* belongs to X.

Using the result of Theorem 4, Grosberg and Krein obtained the following theorem. (See [**8**].)

THEOREM 7 (Grosberg-Krein). *Let X be a normed space with a cone K, and let X^* be the dual space with the dual cone K^*. Then K has the C-normality property if and only if K^* has the C-generating property.*

The method of generalisation, which was developed by Grosberg and Krein to prove Theorem 7, enables us to generalise Theorem 6, and to prove a dual result of Theorem 7. For our situation, however, the duality problem is more complicated.

THEOREM 8. *Let X be a Banach space with dual space X^*, and let K, K^* be compatible cones in X, X^* respectively, Then K^* has the C-normality property if and only if K has the $(C+\epsilon)$-generating property for each $\epsilon > 0$.*

Proof. Firstly, suppose that K has the C'-generating property. Then, for each $x \in X$, there exist y_x, $z_x \in K$ such that $x = y_x - z_x$ and $\|y_x\| + \|z_x\| \leqslant C'\|x\|$. Then, if $f \in X^*$,

$$\begin{aligned}\|f\| &= \sup\{|f(x)| : \|x\| \leqslant 1\} = \sup\{|f(y_x - z_x)| : \|x\| \leqslant 1\} \\ &\leqslant \sup\{\|y_x\| \sup\{|f(y)| : \|y\| = 1, y \in K\} \\ &\qquad\qquad + \|z_x\| \sup\{|f(z)| : \|z\| = 1, z \in K\} : \|x\| \leqslant 1\} \\ &\leqslant C' \sup\{|f(x)| : \|x\| = 1, x \in K\}.\end{aligned}$$

Now, let f, g, $h \in X^*$ be such that $g \leqslant f \leqslant h$ and $\|g\|$, $\|h\| \leqslant 1$. Then, for $x \in K$, $g(x) \leqslant f(x) \leqslant h(x)$, and hence

$$\begin{aligned}\|f\| &\leqslant C' \sup\{|f(x)| : \|x\| = 1, x \in K\} \\ &\leqslant C' \sup\{\max[|g(x)|, |h(x)|] : \|x\| = 1, x \in K\} \leqslant C'.\end{aligned}$$

Therefore $\|f\| \leqslant C + \epsilon$, for each $\epsilon > 0$, and so $\|f\| \leqslant C$. Thus, K^* has the C-normality property.

Conversely, suppose that X^* has the C-normality property, and let $Y = X^* \times R$, where R denotes the real numbers. We define a partial ordering of Y as follows: $(f, t) \geqslant 0$ if and only if $t \geqslant 0$ and there exists $g \in X^*$ such that $\|g\| \leqslant 1$ and $f+tg \geqslant 0$. It is easily verified that this partial ordering of Y is almost archimedean, and that $(0, 1)$ is an order unit. Thus, we may define an order unit norm topology in Y, which can be shown to be equivalent to the product topology. Therefore X^* is a Banach subspace of Y for the order unit norm $\|.\|_C$, and also, if H is the cone in Y, $H \cap X^* = K^*$. If $f \in X^*$, then $f+\|f\|\{(-\|f\|)^{-1}.f\} = 0$, and so $\|f\| \geqslant \|f\|_C$. On the other hand, if $(0, -\lambda) \leqslant (f, 0) \leqslant (0, \lambda)$, then there exist $g, h \in X^*$ such that $\|g\|, \|h\| \leqslant 1$ and $-\lambda h \leqslant f \leqslant \lambda g$. Therefore, because K^* has the C-normality property, $\|f\| \leqslant C\|f\|_C$, and hence

$$\|f\|_C \leqslant \|f\| \leqslant C\|f\|_C. \tag{i}$$

We define a new norm in $X \times R$ by setting, for each $\bar{x} \in X \times R$,

$$\|\bar{x}\|_C = \sup \{|f(\bar{x})| : \|f\|_C \leqslant 1, \ f \in Y\}.$$

This definition is valid because Y is the dual space of $X \times R$, when both spaces have the product topology. It follows immediately from (i) that, for each $x \in X$,

$$\|x\| \leqslant \|x\|_C \leqslant C\|x\|. \tag{ii}$$

We prove that Y is the Banach dual space of $X \times R$, when both spaces have the norms $\|.\|_C$.

Firstly, H is an archimedean cone. For suppose that $(f, t) \in Y$, and that $(f, t) \leqslant \frac{1}{n}(0, 1)$, for $n = 1, 2, \ldots$. Then, for each n, there exists $g_n \in X^*$ such that $-f+\left(\frac{1}{n}-t\right) g_n \in K^*$, and $\|g_n\| \leqslant 1$. Therefore, since the unit sphere in X^* is $\sigma(X^*, X)$-compact, we may choose a $\sigma(X^*, X)$-convergent subnet $\{g_{n_\alpha}\}$ of $\{g_n\}$, such that $g_{n_\alpha} \to g$, where $\|g\| \leqslant 1$. However, K^* is $\sigma(X^*, X)$-closed, and so $-f-tg \in K^*$, that is $(f, t) \leqslant 0$. Thus, H is archimedean.

Secondly, S_Y, the unit sphere of Y for the norm $\|.\|_C$, is $\sigma(Y, X \times R)$-closed. For suppose that $y_\alpha \in S_Y$, $y_\alpha = (f_\alpha, t_\alpha)$, and that $y_\alpha \to y = (f, t)$ for the topology $\sigma(Y, X \times R)$. Then, it is obvious that $f_\alpha \to f$ for $\sigma(X^*, X)$, and that $t_\alpha \to t$. Now $\|y_\alpha\|_C \leqslant 1$, and therefore, since H is archimedean, $(0, -1) \leqslant (f_\alpha, t_\alpha) \leqslant (0, 1)$, and so there exist $g_\alpha, h_\alpha \in X^*$ satisfying $\|g_\alpha\|$, $\|h_\alpha\| \leqslant 1$, $-f_\alpha+(1-t_\alpha)g_\alpha \in K^*$, $f_\alpha+(1+t_\alpha)h_\alpha \in K^*$. Again, because of the $\sigma(X^*, X)$-compactness of S^*, the unit sphere of X^*, we may choose a subnet $\{f_\beta\}$, of $\{f_\alpha\}$, such that $g_\beta \to g \in S^*$ and $h_\beta \to h \in S^*$ for the topology $\sigma(X^*, X)$. Therefore, since K^* is $\sigma(X^*, X)$-closed, we have

$$-f+(1-t)g \in K^*, \quad f+(1+t)h \in K^* \quad \text{and} \quad -1 \leqslant t \leqslant 1,$$

and hence $(0, -1) \leqslant (f, t) \leqslant (0, 1)$ and so $y \in S_Y$.

We have thus proved that S_Y is $\sigma(Y, X\times R)$-compact. Therefore, by Theorem 2, Y is the Banach dual space of $X\times R$, when both spaces have the norms $\|.\|_C$.

We may now apply the corollary to Theorem 6 to show that the subdual cone H^*, in $X\times R$, is generating, and that, given $\bar{x}\in X\times R$ and $\delta>0$, there exist $\bar{y}, \bar{z}\in H^*$ such that $\bar{x}=\bar{y}-\bar{z}$ and $\|\bar{y}\|_C+\|\bar{z}\|_C\leqslant(1+\delta)\|\bar{x}\|_C$. If $x\in X$, then x is a continuous linear functional on X^*, and hence, by the Hahn-Banach theorem, it may be extended to a continuous linear functional $\bar{x}$, defined on Y, such that $\|\bar{x}\|_C=\|x\|_C$. Hence there is some real number λ such that $\bar{x}=(x, \lambda)$. Now, for a given $\delta>0$, let $\bar{y}, \bar{z}$ be defined as above, and let y, z be their restrictions to X^*, when considered as linear functionals on Y. Then, since K is the subdual cone of K^*, $y, z\in K$, $x=y-z$, and hence K is generating in X.

Finally, we have

$$\|y\|_C\leqslant\|\bar{y}\|_C, \quad \|z\|_C\leqslant\|\bar{z}\|_C,$$

and so

$$\|y\|_C+\|z\|_C\leqslant\|\bar{y}\|_C+\|\bar{z}\|_C\leqslant(1+\delta)\|\bar{x}\|_C=(1+\delta)\|x\|_C.$$

Therefore, by (ii), $\|y\|+\|z\|\leqslant(1+\delta)\,C\|x\|$, and hence K is $(C+\epsilon)$-generating for each $\epsilon>0$.

Theorem 8 throws some light onto some problems concerning lattice orderings of dual Banach spaces. We shall require the following well-known theorem of Kakutani.

THEOREM 9 (Kakutani). *Let X be a partially ordered Banach space with dual space X^*, endowed with the dual partial ordering. Then,*

(i) *X^* is an AL-space if X is an AM-space;*

(ii) *X^* is an AM-space, with order unit norm, if X is an AL-space.*

For a proof of this theorem see, for example, Day [**3**; VI.1.Theorem 2].

THEOREM 10. *Let X be a Banach space, and let X^*, X^{**} be its first, second dual spaces, respectively. If X, X^* and X^{**} have compatible cones, and if X^{**} is an AL-space, then X is an AL-space and X^* is an AM-space with an order unit norm.*

Proof. If X^{**} is an AL-space, then it has the 1-generating property. Therefore, by Theorem 7, X^* has the 1-normality property, and so, by Theorem 8, X has the $(1+\epsilon)$-generating property for each $\epsilon>0$.

Therefore, if $x\in X$ and $n>0$, we can find $y_n, z_n\in X$ such that $y_n, z_n\geqslant 0$, $x=y_n-z_n$ and

$$\|y_n\|+\|z_n\|\leqslant\left(1+\frac{1}{n}\right)\|x\|. \tag{i}$$

Now let $X \to \hat{X} \subset X^{**}$ be the canonical embedding map. Then $\hat{x} = x^+ - x^-$, where x^+, $x^- \in X^{**}$, $x^+ = \hat{x} \vee 0$, and $x^- = (-\hat{x}) \vee 0$, and also

$$\|x^+\| + \|x^-\| = \|\hat{x}\| = \|x\|.$$

Therefore $\hat{y}_n \geqslant x^+$, $\hat{z}_n \geqslant x^-$, for $n = 1, 2, \ldots$, and hence, because the norm in X^{**} is additive on the cone,

$$\|\hat{y}_n\| = \|x^+\| + \|\hat{y}_n - x^+\| \text{ and } \|\hat{z}_n\| = \|x^-\| + \|\hat{z}_n - x^-\|. \qquad \text{(ii)}$$

Putting (i) and (ii) together, we get

$$\|\hat{y}_n - x^+\| + \|\hat{z}_n - x^-\| + \|x\| \leqslant \left(1 + \frac{1}{n}\right) \|x\|,$$

and it is clear that $\hat{y}_n \to x^+$ and $\hat{z}_n \to x^-$. However, $\hat{X}$ is closed in X^{**}, and so x^+, $x^- \in \hat{X}$, and hence $\hat{X}$ is a vector sublattice of X^{**}. It now follows immediately that the norm in X is a lattice norm which is additive on the cone in X, and therefore X is an AL-space. Theorem 9 then asserts that X^* is an AM-space with an order unit norm.

The following corollary was first obtained by Dixmier in the theory of normal measures. (See [**5**].)

COROLLARY 1 (Dixmier). *Let X be a Banach space with dual space X^*, such that X^* has an order unit norm and is an AM-space. Then X, for the subdual partial ordering, is an AL-space,*

Proof. Continuity of lattice operations in an AM-space implies that the ordering is archimedean. It then follows quite easily that the cone in X^* must be w^*-closed. Therefore the cones in X, X^* and X^{**}, for their respective subdual or dual partial orderings, are compatible. The result now follows from Theorems 9 and 10.

A partially ordered vector space X is said to have the *Riesz decomposition property* if, whenever $x, y, z \in X$ are such that $y, z \geqslant 0$ and $0 \leqslant x \leqslant y+z$ then there exist u, $v \in X$ such that $x = u+v$, $0 \leqslant u \leqslant y$ and $0 \leqslant v \leqslant z$. Any vector lattice has the Riesz decomposition property, but the converse is not true.

COROLLARY 2. *Let X be a Banach space with a closed cone and a base norm, and which has the Riesz decomposition property. Then X is an AL-space.*

Proof. It is a standard result that, since X has the Riesz decomposition property, X^* is a vector lattice for the dual partial ordering. (See Day [**3**; Ch. VI].) Moreover, by Theorem 5, X^* has an order unit norm, and hence it is an AM-space. The result now follows from Corollary 1.

COROLLARY 3. *Let X be a Banach space with a closed cone K, and let $X^{(n)}$ be the n-th dual space of X, for each positive integer n (where we take*

$X^{(0)} = X$). *Suppose that there exists some* $n \geqslant 0$ *such that* $X^{(2n+1)}$ *is an* AM*-space for the dual partial ordering. Then* $X^{(2r)}$ *is an* AL*-space and* $X^{(2r+1)}$ *is an* AM*-space, for every positive integer* r, *for their dual partial orderings.*

Proof. The fact that K is closed and that $X^{(2n+1)}$ is an AM-space for the dual partial ordering, implies that all the dual cones are compatible. Repeated applications of Theorems 9 and 10 now give the result.

Remark. It is natural to enquire whether dual results to Corollaries 1 and 2 are valid, in which the roles of AL-space and AM-space are interchanged. Lindenstrauss [**11**] has shown that such results are not possible, by giving an example of a Banach space which is not an AM-space, but which has an order unit norm and the Riesz decomposition property, and hence whose dual space is an AL-space which has a w^*-compact base.

4. *Perfect order ideals*

We now turn our attention to order ideals in a partially ordered vector space with an order unit, and to their annihilators, and we prove a generalisation of a theorem of Bonsall.

Let X be a partially ordered vector space with an order unit e, and let K^* be the set of all positive linear functionals on X, and $X^* = K^* - K^*$. In fact, if we define, in the usual way, an order unit semi-norm topology on X, then X^* is simply the dual space of X for this topology. Let $B = \{f \in K^* : f(e) = 1\}$, that is, B is the base in X^* for the dual partial ordering. Then it is well known that the null spaces of the linear functionals in B form the set of maximal order ideals in X.

Let A be a subset of X, and D a subset of X^*. Then we write

$$A^\perp = \{f \in X^* : f(x) = 0, \ \forall\, x \in A\}, \quad D_\perp = \{x \in X : f(x) = 0, \ \forall f \in D\}$$

and
$$A^* = A^\perp \cap B.$$

$A^\perp$ and $D_\perp$ are called the *annihilators* of A and D, respectively.

An order ideal I, in X, is said to be *perfect* if, for each $x \in I$ and $\epsilon > 0$, there exists $w_\epsilon \in I$ such that $-w_\epsilon - \epsilon e \leqslant x \leqslant w_\epsilon + \epsilon e$.

THEOREM 11 (Bonsall). *Let* X *be a partially ordered vector space with an order unit, and let* I *be an order ideal in* X. *Then* I^* *is an extreme point of* B *if and only if* I *is a perfect maximal ideal.*

The proof of this theorem was given by Bonsall in [**2**].

We aim to characterise those order ideals whose annihilators intersect B in an extremal set. We recall that a non-empty subset E of B is *extremal* in B if, whenever E contains the mid-point of a line segment in B, then the whole line segment is contained in E.

In the future I will always denote an order ideal in X. Let

$$G(I) = \{x \in X: \text{ for each } \epsilon > 0 \text{ there exist } y_\epsilon,\ z_\epsilon \in I \text{ such that } y_\epsilon - \epsilon e \leqslant x \leqslant z_\epsilon + \epsilon e\}.$$

It is easy to verify that $G(I)$ is an order ideal. Let $F(I) = (I^*)_\perp$. Then it is clear that $F(I)$ is the intersection of all the maximal order ideals which contain I, and therefore $F(I)$ is also an order ideal. Moreover, we see that $I \subset G(I) \subset F(I)$, and hence $F\big(G(I)\big) = F(I)$.

LEMMA 8. *X/I is almost archimedean ordered if and only if $I = G(I)$.*

Proof. The lemma is a direct consequence of the definition of $G(I)$.

LEMMA 9. *X/I is almost archimedean ordered if and only if $I = F(I)$. Moreover $F(I) = G(I)$, for any order ideal I.*

Proof. If $I = F(I)$, then $F(I) = G(I) = I$, and so, by Lemma 8, X/I is almost archimedean ordered.

Conversely, suppose that X/I is almost archimedean ordered, and let $x \to \hat{x}$ be the natural mapping $X \to X/I$. Then it follows that I is closed for the order unit semi-norm topology in X, and that X/I has an order unit norm defined by

$$\|\hat{x}\|_1 = \inf\{\lambda > 0: \text{ there exist } y_\lambda,\ z_\lambda \in I, \text{ such that } y_\lambda - \lambda e \leqslant x \leqslant z_\lambda + \lambda e\}.$$

However, X/I has a quotient semi-norm topology, which is defined by

$$\|\hat{x}\| = \inf\{\lambda > 0: \text{ there exists } y_\lambda \in I, \text{ such that } -\lambda e \leqslant x + y_\lambda \leqslant \lambda e\}.$$

Therefore the quotient topology is stronger than the order unit norm topology, and so it must itself be a norm topology.

Now, the dual space of X/I for the norm $\|.\|$ is isomorphic to $I^\perp$, and the dual space of X/I for the norm $\|.\|_1$ is isomorphic to $I^\perp \cap K^* - I^\perp \cap K^*$. It follows from this that $I^\perp \cap K^* - I^\perp \cap K^*$ is $\sigma(X^*, X)$dense in $I^\perp$, and hence that $I = (I^\perp)_\perp = (I^*)_\perp = F(I)$.

Finally, since $G(I)$ is an order ideal and $F\big(G(I)\big) = F(I)$, by Lemma 8 and the above result we have $G(I) = F(I)$ for every order ideal I.

LEMMA 10. *I is perfect if and only if $G(I)$ is perfect.*

Proof. Suppose that I is perfect, and let $x \in G(I)$. Then, for each $\epsilon > 0$, there exist y_ϵ, z_ϵ, w_ϵ, $w_\epsilon' \in I$ such that $y_\epsilon - \epsilon e \leqslant x \leqslant z_\epsilon + \epsilon e$, $-w_\epsilon - \epsilon e \leqslant y_\epsilon \leqslant w_\epsilon + \epsilon e$ and $-w_\epsilon' - \epsilon e \leqslant z_\epsilon \leqslant w_\epsilon' + \epsilon e$. Therefore

$$-(w_\epsilon + w_\epsilon') - 3\epsilon e \leqslant x \leqslant (w_\epsilon + w_\epsilon') + 3\epsilon e,$$

and so $G(I)$ is perfect.

Conversely, suppose that $G(I)$ is perfect, and let $x \in I$. Then, for each $\epsilon > 0$, there exist $w_\epsilon \in G(I)$ and $y_\epsilon \in I$ such that $-w_\epsilon - \epsilon e \leqslant x \leqslant w_\epsilon + \epsilon e$

and $w_\epsilon \leqslant y_\epsilon + \epsilon e$. Therefore

$$-y_\epsilon - 2\epsilon e \leqslant x \leqslant y_\epsilon + 2\epsilon e,$$

and hence I is perfect.

THEOREM 12. *Let X be a partially ordered vector space with an order unit e, and let I be an order ideal in X. Then I^* is an extremal subset of B if and only if I is perfect.*

Proof. Firstly, suppose that I is perfect, and let $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$, where $f \in I^*$, $f_1, f_2 \in B$. Then, if $x \in I$ and $\epsilon > 0$, there exists $w_\epsilon \in I$ such that $-w_\epsilon - \epsilon e \leqslant x \leqslant w_\epsilon + \epsilon e$. Therefore we have

$$f_1(w_\epsilon) + f_2(w_\epsilon) = 2f(w_\epsilon) = 0, \quad f_i(w_\epsilon + \epsilon e) \geqslant 0, \quad i = 1, 2,$$

and hence $$-\epsilon \leqslant f_i(w_\epsilon) \leqslant \epsilon, \quad i = 1, 2.$$

However, for $i = 1, 2$,

$$-f_i(w_\epsilon) - \epsilon \leqslant f_i(x) \leqslant f_i(w_\epsilon) + \epsilon,$$

and so $$-2\epsilon \leqslant f_i(x) \leqslant 2\epsilon.$$

Therefore, since ϵ was arbitrary, $f_1(x) = f_2(x) = 0$ for all $x \in I$. Thus $f_1, f_2 \in I^*$, and so I^* is an extremal subset of B.

Conversely, suppose that I^* is an extremal subset of B, and let $Y = \{x \in X : f(x) = g(x), \forall f, g \in I^*\}$. Then Y is a vector subspace of X which contains $F(I)$ and e, and hence e is an order unit in Y for the relative partial ordering. If $y \in Y$, then there exists a constant α such that $f(y) = \alpha$, for all $f \in I^*$. Therefore $f(y - \alpha e) = 0$, for all $f \in I^*$, and so $y - \alpha e \in F(I)$. Thus $F(I)$ is a maximal vector subspace of Y, and hence it is a maximal order ideal in Y.

Let $K_Y{}^*$ be the set of all positive linear functionals on Y, and let $B_Y = \{f \in K_Y{}^* : f(e) = 1\}$. Now let $f \in B_Y$ be the linear functional associated with $F(I)$, and suppose that $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$, where $f_1, f_2 \in B_Y$. Then, by the monotone extension theorem, f_1 and f_2 may be extended to positive linear functionals, $\bar{f}_1$ and $\bar{f}_2$, defined on X. Let $\bar{f} = \frac{1}{2}\bar{f}_1 + \frac{1}{2}\bar{f}_2$. Then it is clear that $\bar{f} \in I^*$ and $\bar{f}_1, \bar{f}_2 \in B$, and therefore, since I^* is assumed to be an extremal subset of B, we see that $\bar{f}_1, \bar{f}_2 \in I^*$. This implies that $f_1 = f_2 = f$, and hence f is an extreme point of B_Y. Therefore, applying Theorem 11, we see that $F(I)$ is perfect in Y, and hence it is perfect in X.

Finally, Lemmas 9 and 10 show that I is itself perfect, and so the proof is complete.

COROLLARY. *$I^\perp$ is an order ideal in X^*, for the dual partial ordering, if and only if I is perfect.*

Proof. Suppose that I is perfect, and let $x \in I$. Then, for each $\epsilon > 0$, there exists $w_\epsilon \in I$ such that $-w_\epsilon - \epsilon e \leqslant x \leqslant w_\epsilon + \epsilon e$. Therefore, if $0 \leqslant f \leqslant g \in I^\perp$, it follows that $f(x) = 0$, and hence $I^\perp$ is an order ideal in X^*.

Conversely, suppose that $I^{\perp}$ is an order ideal in X^*, and suppose that $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$, where $f \in I^*$, $f_1, f_2 \in B$. Then evidently $0 \leqslant f_1 \leqslant 2f$ and $0 \leqslant f_2 \leqslant 2f$, and hence $f_1, f_2 \in I^{\perp} \cap B = I^*$. Therefore I^* is an extremal subset of B, and so, by Theorem 12, I is perfect.

Note added in proof 27 *July*, 1964.

A partial dual to the Grosberg-Krein theorem was proved recently by T. Andô in his paper *On fundamental properties of a Banach space with a cone*, Pacific J. Math. 12 (1962), 1163–9. The normality and generating constants of the present Theorem 8 are, however, not computed by Andô.

In Theorem 7 it is not necessary that K^* be a true cone. Similarly, in Theorem 8 it is not necessary for K and K^* to be compatible cones, but simply that K^* is a w^*-closed cone in X^* and $K = (K^*)_*$.

In the proof of Theorem 8 the use of nets makes the proof that H is archimedean unduly complicated. The following proof is more direct. If $t > 0$, then $(f, t) \in H$ if and only if $f \in t(S^* + K^*)$. The set $S^* + K^*$ is w^*-closed, convex and absorbing and thus, if p denotes its Minkowski functional, we have $S^* + K^* = \{g \in X^* : p(g) \leqslant 1\}$. Now if $(f, t) \leqslant \frac{1}{n}(0, 1)$ for all $n \geqslant 1$, then $t \leqslant 0$ and $p(-f) \leqslant -t$. Hence if $t \neq 0$, then $-f \in -t(S^* + K^*)$ and so $(f, t) \leqslant 0$. If $t = 0$ then, since K^* is w^*-closed, it is clear that $(f, t) \leqslant 0$.

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