Convex sets

July 10, 2018

1 Abstract convex structures

Axioms for convex sets were introduced by H.M. Stone [**?**], and then studied e.g. in [**?**, **?**]. We mostly follow [**?**].

Definition 1.1. *A convex structure is a set X and a family of binary operations $< \lambda, x, y >$, $\lambda \in [0,1]$, on X such that*

(c1) $< \lambda, p, q > = < 1-\lambda, q, p >$ *(commutativity),*

(c2) $< 0, p, q > = p$ *(endpoint condition),*

(c3) $< \lambda, p, < \mu, q, r >> = < \lambda\mu, < \nu, p, q >, r >$ $(\lambda\mu \neq 1)$*, where* $\nu = \lambda(1-\mu)(1-\lambda\mu)^{-1}$ *(associativity).*

If X is a convex subset of a real linear space, then $< \lambda, p, q >$ corresponds to $(1-\lambda)p + \lambda q$. In [**?**] and [**?**], also the following axiom is given:

(c4) $< \lambda, p, p > = p$ (idempotence).

A convex *prestructure* is a set S with a map $T : [0,1] \times S \times S \to S$ denoted $T(\lambda, p, q) = < \lambda, p, q >$ (no requirements on T). An affine functional is an affine map f from a convex prestucture to the real line $\mathbb{R}$, that is, $f(< \lambda, p, q >) = (1-\lambda)f(p) + \lambda f(q)$ for all $\lambda \in [0,1]$ and $p, q \in S$. We denote by S^* the set of all affine functional on S and say that S^* is *total* (*separating*) if for $p \neq q \in S$ there is $f \in S^*$ such that $f(p) \neq f(q)$.

Theorem 1.2. *A convex prestructure S is isomorphic to a convex set iff S^* is total.*

Proof. Suppose S_0 is a convex set and $F : S \to S_0$ an isomorphism. If S_0 is a convex subset of the linear space V, it is well-known that V^* (the algebraic dual) is total over V. Restricting the elements of V^* to S_0, we get a total set of affine functionals for S_0. If $f \in V^*$, then $f \circ F \in S^*$, so S^* is total.

Conversely, suppose that S^* is total. For $p \in S$ define $J(p) : S^* \to \mathbb{R}$ by $J(p)f = f(p)$. Clearly S^* is a linear space under pointwise operations, and $J(p) \in S^{**}$ so that $J(S) \subseteq S^{**}$. Now $J(S)$ is a convex set - indeed, $(1-\lambda)J(p) + \lambda J(q) =< \lambda, p, q >$ - and $J : S \to S^{**}$ is injective iff S^* is total. Indeed, if S^*

is total and $p \neq q \in S$, then there is $f \in S^*$ with $f(p) \neq f(q)$ so $J(p) \neq J(q)$, and conversely, if J is injective and $p \neq q \in S$, then $J(p) \neq J(q)$ so that there is an $f \in S^*$ such that $f(p) = J(p)f \neq J(q)f = f(q)$. If follows that $J : S \to J(S)$ is an isomorphism. $\square$

The following theorem gives an intrinsic characterization of those convex structures that are convex subsets of a linear space.

Theorem 1.3. *[?, ?]. A convex structure S_1 satisfying axioms* (c1),(c2), (c3) *and in addition* (c4) *embeds into a real vector space iff the following cancellation property holds:*

(c5) $< \lambda, x, y >=< \lambda, x, z >$ with $\lambda \in (0,1) \implies y = z$.

Proof. Clearly, every convex subset of a vector space satisfies this cancellation property.

Let X be a convex structure satisfying (c4) and (c5). Let V_X be the real vector space generated by X, so that V_X has a base $(e_x)_{x\in X}$. Let $U_X \subseteq V_X$ be a subspace generated by the vectors of the form

$$e_{<\lambda,x,y>} - (1-\lambda)e_x - \lambda e_y, x, y \in X, \lambda \in [0,1].$$

Let $W_X = V_X/U_X$ and let $\tilde{e_x}$ be the image of e_x. Then the mapping $X \to W_X$, $x \mapsto \tilde{e_x}$ preserves convex combinations. Vectors in U_X have the form

$$\sum_{i=1}^{m} \alpha_i(e_{<\lambda_i,a_i,b_i>} - (1-\lambda_i)e_{a_i} - \lambda_i e_{b_i}) - \sum_{i=1}^{m} \beta_i e_{<\mu_i,c_i,d_i>} - (1-\mu_i)e_{c_i} - \mu_i e_{d_i})$$

with $\alpha_i, \beta_i \geq 0$, and $a_i, b_i, c_i, d_i \in X$ and $\lambda_i, \mu_i \in [0,1]$. We split this into positive and negative terms as follows:

$$\begin{array}{c}\sum_{i=1}^{m}(\alpha_i e_{<\lambda_i,a_i,b_i>} + \beta_i(1-\mu_i)e_{c_i} + \beta_i\mu_i e_{d_i}) \\ -\sum_{i=1}^{m}(\beta_i e_{<\mu_i,c_i,d_i>} + \alpha_i(1-\lambda_i)e_{a_i} + \alpha_i\lambda_i e_{b_i})\end{array}$$

and observe that the sum of the coefficients of all negative terms equals to the sum of coefficients of all positive terms, namely $\sum_i(\alpha_i + \beta_i)$. Without loss of generality we may assume this sum to be 1, then both the sums are convex combinations. Interpreting these as convex combinations in X, these sums moreover define the same point in X.

We show the injectivity by proving that $\tilde{e_x} = \tilde{e_y}$ implies $x = y$, $x, y \in X$. The equation $\tilde{e_x} = \tilde{e_y}$ holds whenever $e_x - e_y$ lies in U_X. If this is the case, then the first sum contains the term κe_x, $\kappa > 0$, and the second sum contains the term κe_y for the same κ, and all other terms cancel. Then both the sums define convex combinations of the same points with the same weights, except that the first contains the point x with weight κ, while the second contains the point y with weight κ. Applying the cancellation property, we obtain $x = y$. $\square$

2 Intrinsic metrics

Let S_1 be a convex structure. For $p, q \in S_1$ define

$$\sigma(p, q) := \inf\{0 \leq \lambda \leq 1 :< \lambda, p, p_1 >=< \lambda, q, q_1 >, p_1, q_1 \in S_1\}$$

Since $< \frac{1}{2}, p, q >=< \frac{1}{2}, q, p >$, we have $0 \leq \sigma(p, q) \leq \frac{1}{2}$.

$$\rho(p, q) = \frac{\sigma(p, q)}{1 - \sigma(p, q)}, \quad \text{then } 0 \leq \rho(p, q) \leq 1.$$

Theorem 2.1. *([?]) On any convex structure S_1, σ and ρ are semimetrics.*

Proof. Clearly, σ and ρ are nonnegative and symmetric. We have to prove triangle inequality. If $p = s$ or $q = s$, then $\sigma(p, q) \leq \sigma(p, s) + \sigma(s, q)$. Assume $p \neq s, q \neq s$. Assume

$\lambda_1 \in \{0 < \lambda < 1 :< \lambda, p, p_1 >=< \lambda, s, s_1 >, p_1, s_1 \in S_1\}$;
$\lambda_2 \in \{0 < \lambda < 1 :< \lambda, s, s_2 >=< \lambda, q, q_1 > s_2, q_1 \in S_1\}$,
$\lambda_3 := \lambda_1 + \lambda_2 - 2\lambda_1\lambda_2$;
$p_2 :=< \lambda_2(1 - \lambda_1)\lambda_3^{-1}, p_1, s_2 >$;
$q_2 :=< \lambda_2(1 - \lambda_1)\lambda_3^{-1}, s_1, q_1 >$;
$\lambda_0 := \lambda_3(1 - \lambda_1\lambda_2)^{-1}$. Then

$$\begin{aligned}
<\lambda_0, p, p_2> &= <\lambda_3(1-\lambda_1\lambda_2)^{-1}, p, <\lambda_2(1-\lambda_1)\lambda_3^{-1}, p_1, s_2>> \\
&= <\lambda_2(1-\lambda_1(1-\lambda_1\lambda_2)^{-1}, <\lambda_1, p, p_1>, s_2> \\
&= <\lambda_2(1-\lambda_1)(1-\lambda_1\lambda_2)^{-1}, <\lambda_1, s, s_1>, s_2> \\
&= <(1-\lambda_2)(1-\lambda_1\lambda_2)^{-1}, s_2, <\lambda_1, s, s_1>> \\
&= <\lambda_1(1-\lambda_2)(1-\lambda_1\lambda_2)^{-1}, <1-\lambda_2, s_2, s> s_1> \\
&= <(1-\lambda_1)(1-\lambda_1\lambda_2)^{-1}, s_1, <\lambda_2, q, q_1>> \\
&= <(1-\lambda_1)(1-\lambda_2)(1-\lambda_1\lambda_2)^{-1}, <\lambda_2(1-\lambda_1)\lambda_3^{-1}, s_1, q_1>, q> \\
&= <\lambda_3(1-\lambda_1\lambda_2)^{1}, q, q_2> \\
&= <\lambda_0, q, q_2>
\end{aligned}$$

so $\lambda_0 \in \{0 < \lambda < 1 :< \lambda, p, p_2 >=< \lambda, q, q_2 >, p_2, q_2 \in S_1\}$. Now since

$$\lambda_0(1-\lambda_0)^{-1} = \lambda_1(1-\lambda_1)^{-1} + \lambda_2(1-\lambda_2)^{-1},$$

we get

$$\begin{aligned}
\rho(p,q) &= \sigma(p,q)[1-\sigma(p,q)]^{-1} \leq \sigma(p,s)[1-\sigma(p,s)]^{-1} + \sigma(s,q)[1-\sigma(s,q)]^{-1} \\
&= \rho(p,s) + \rho(s,q).
\end{aligned}$$

The triangle inequality for σ follows similarly from $\lambda_0 \leq \lambda_1 + \lambda_2$.

□

Theorem 2.2. *A necessary and sufficient condition for ρ, σ to be metrics is that whenever there are sequences $\lambda_i \in [0,1]$, $p_i, q_i \in S_1$which satisfy $\lim_{i\to\infty} \lambda_i = 0$, $<\lambda_i, p, p_i>=<\lambda_i, q, q_i>$ then $p = q$.*

Proof. Clearly ρ is a metric iff σ is. If σ is a metric, since $\sigma(p,q) \leq \lambda_i$ $\forall i$ we have $p = q$. Conversely if $\sigma(p,q) = 0$ then $V = \{0 \leq \lambda_i \leq 1 :< \lambda, p, p_1 >=< \lambda, q, q_1 >, p_1, q_1 \in S_1\}$ either contains 0 or 0 is a limit point. In the former case $p =< 0, p, p_1 >=< 0, q, q_1 >= q$. In the latter case there exist $\lambda_i \in V$ with $\lambda_i \to 0$ so again $p = q$. □

Corollary 2.3. *Let S_0 be a convex set in a real vector space X. If there is a topology on X that makes X a Hausdorff topological vector space in which S_0 is bounded, then ρ is a metric.*

Proof. Suppose there are sequences $\lambda_i \in [0,1]$, $\lim \lambda_i = 0, p_i, q_i \in S_0$ such that $(1-\lambda_i)p + \lambda_i p_i = (1-\lambda_i)q + \lambda_i q_i$. Then $p - q = \lambda_i(p-q) + \lambda_i(q_i - p_i)$. Let Λ be a neighborhood of 0. Then there is a neighborhood W of 0 such that $W + W + W \subseteq \Lambda$. Now for sufficiently large i, $\lambda_i(p-q) \in W$. Since S_0 is bounded, there is $\mu > 0$ such that $\lambda S_0 \subseteq W$ for $|\lambda| \leq \mu$. For i sufficiently large, $\lambda_i q_i - \lambda_i p_i \in W + W$. Hence $p - q \in W + W + W \subseteq \Lambda$ for i sufficiently large, and since X is Hausdorff, $p - q = 0$. $\square$

The converse holds only in finite dimensional spaces.

Let S_0 be a convex set in a real linear space V, ρ the intrinsic metric on S_0. Some terminology: S_0 is

- *absorbing* iff $\forall x \in V \; \exists \; \delta(x) > 0$: $\lambda x \in S_0 \; \forall \lambda$ with $|\lambda| \leq \delta(x)$.
- *balanced* iff $x \in S_0$, $|\lambda| \leq 1 \implies \lambda x \in S_0$.
- *radial* iff $x \in S_0$, $0 \leq \lambda \leq 1 \implies \lambda x \in S_0$.
- *normalized* iff $x \in S_0$, $\alpha \neq 1 \implies \alpha x \notin S_0$.
- *positive* iff $x \in S_0$, $\alpha < 0 \implies \alpha x \notin S_0$.

Define $P := \{\alpha S_0 : \alpha \geq 0\}$, then P is a wedge.

Definition 2.4. $x \in X$, $\|x\| := \inf\{c + d : x = cp - dq; c, d \geq 0; p, q \in S_0\}$.

Theorem 2.5. *If S_0 is normalized or radial then $\|p-q\| = 2\rho(p,q)$, $p, q \in S_0$. Moreover, $\|.\|$ is a norm iff ρ is a metric.*

Proof. Assume S_0 is normalized. For $p, q \in S_0$, if $p - q = cp_1 - dq_1$, $p_1, q_1 \in S_0$, $c, d \geq 0$, then $p + dq_1 = q + cp_1$. which implies

$$(1+d)\left(\frac{p}{1+d} + \frac{dq_1}{1+d}\right) = (1+c)\left(\frac{1}{1+c}q + \frac{c}{1+c}p_1\right).$$

Then $q_2 := \frac{1}{1+c}q + \frac{c}{1+c}p_1 \in S_0$, and $\frac{1+c}{1+d}q_2 = \frac{p}{1+d} + \frac{dq_1}{1+d} \in S_0 \implies \frac{1+c}{1+d} = 1 \implies c = d$.

So all representations of $p - q$ are of the form $c(p_1 - q_1), c \geq 0, p_1, q_1 \in S_0$.

We then have

$$\begin{aligned}
\sigma(p,q) &= \inf\{0 \le \lambda < 1 : (1-\lambda)p + \lambda p_1 = (1-\lambda)q + \lambda q_1, p_1, q_1 \in S_0\} \\
&= \inf\{0 \le \lambda < 1 : p - q = \frac{\lambda}{1-\lambda}(q_1 - p_1), p_1, q_1 \in S_0\} \\
&= \inf\{\frac{c}{c+1}, c \ge 0 : p - q = c(q_1 - p_1), p_1, q_1 \in S_0\} \\
&= [\inf\{c \ge 0 : p - q = c(q_1 - p_1)][\inf\{c \ge 0 : p - q = c(q_1 - p_1)| + 1]^{-1]} \\
&= \frac{\frac{1}{2}\inf\{2c : p - q = cp_1 - cq_1\}}{\frac{1}{2}\inf\{2c : p - q = cp_1 - cq_1\} + 1} \\
&= \frac{\frac{1}{2}\|p-q\|}{\frac{1}{2}\|p-q\| + 1}.
\end{aligned}$$

From this we get $\frac{1}{2}\|p-q\| = \frac{\sigma(p,q)}{1-\sigma(p,q)} = \rho(p,q)$.

Assume S_0 is radial, and $p - q = cp_1 - dq_1, p, q, p_1, q_1 \in S_0$, $c, d \ge 0$. Then

$$p - q = (c+d)(\frac{c}{c+d}p_1 - \frac{d}{c+d}q_1) = (c+d)(p_2 - q_2), p_2, q_2 \in S_0.$$

That is, $p - q = b(p_2 - q_2), b > 0, p_2, q_2 \in S_0$. From this we get $\|p - q\| = 2\inf\{c > 0, p - q = c(p_1 - q_1), p_1, q_1 \in S_0\}$ and similarly as in the previous case we obtain $\|p - q\| = 2\rho(p,q)$.

Clearly, if $\|.\|$ is a norm, then ρ is a metric. Suppose ρ is a metric, $\|x - y\| = 0, x, y \in X$.

1. S_0 is radial.

$\|x-y\| = 0 \implies \exists p, q \in S_0, 0 \le c, d \le 1 : x-y = cp-dq = p_1-q_1, p_1, q_1 \in S_0$. Then $2\rho(p_1, q_1) = \|p_1 - q_1\| = 0 \implies p_1 = q_1 \implies x = y$.

2. S_0 is normalized. We show first that $\|p\| = 1 \forall p \in S_0$.

If $p = cp_1 - dq_1, p_1, q_1 \in S_0$, $c, d \ge 0$, then

$$\frac{1}{1+d}p + \frac{d}{1+d}q_1 = c\frac{1}{1+d}p_1 \implies c = 1 + d \ge 1,$$

so that $\|p\| = \inf\{c + d : p = cp_1 - dq_1, c, d \ge 0, p_1, q_1 \in S_0\} \ge 1$. But also $p = p - 0q \implies \|p\| = 1$.

Let $0 = \|x - y\| = \|cp - dq\| \ge \|c\|p\| - d\|q\|\| = |c - d|$, so $c = d$. Hence $0 = \|x - y\| = c\|p - q\| = 2c\rho(p,q)$. If $c \ne 0$ then $\rho(p,q) = 0 \implies p = q$, hence $x = y$. If $c = 0$, then again $x = y$. $\square$

Theorem 2.6. *Let S_0 be normalized or positive and radial convex set in a real vector space V and let $\|.\|$ be the induced seminorm on $X = P - P \subseteq V$. If S_0 is normalized (positive radial) and $T : S_0 \to S_0$ is an affine (and homogeneous) map then T has a unique extension $\hat{T}$ to X and $\|T\| \leq 1$ (i.e., $\|Tx\| \leq \|x\|$ for all $x \in X$). If T is a bijection, then $\hat{T}$ is an isometry.*

Proof. For all $x \in X$, $x = cp - dq$, $c, d \geq 0, p, q \in S_0$. Define $\hat{T}x := cTp - dTq$. $\hat{T}$ is well defined: Suppose that also $x = c_1 p_1 - d_1 q_1$, $c_1, d_1 \geq 0, p_1, q_1 \in S_0$. First suppose that S_0 is normalized and T is affine. From $cp - dq = c_1 p_1 - d_1 q_1$ we have $c(c + d_1)^{-1} p + d_1 (c + d_1)^{-1} q_1 = (c_1 + d)(c + d_1)^{-1}.(c_1 (_1 + d)^{-1} p_1 + d(c_1 + d)^{-1} q)$, so $c_1 + d = c + d_1$ and hence

$$c(c + d_1)^{-1} Tp + d_1 (c + d_1)^{-1} Tq_1 = c_1 (c + d_1)^{-1} Tp_1 + d(c + d_1)^{-1} Tq.$$

It follows that $cTp - dTq = c_1 Tp_1 - d_1 Tq_1$.

Next suppose S_0 is positive, radial and T is affine and homogeneous. Positivity implies that $c_1 + d, c + d_1 > 0$. Either $c_1 + d \leq c + d_1$ or $c_1 + d \geq c + d_1$. Assume the former. Then

$$\begin{aligned} &c(c + d_1)^{-1} p + d_1 (c + d_1)^{-1} q_1 \\ &= (c + d)(c + d_1)^{-1} (c_1 (c_1 + d)^{-1} p_1 + d(c_1 + d)^{-1} q) \in S_0. \end{aligned}$$

From the facts that S_0 is radial and T is affine, homogeneous, we get $cTp - dTq = c_1 Tp_1 - dTq_1$. Thus $\hat{T}$ is well defined and it is easy to show that $\hat{T}$ is a linear operator on X.

To show that $\hat{T}$ is a contraction we have for $x \in X$,

$$\begin{aligned} &\|\hat{T}x\| = \inf\{c + d : \hat{T}x = cp - dq, c, d \geq 0, p, q \in S_0\} \\ &\leq \inf\{c + d : x = cp - dq, c, d \geq 0, p, q \in S_0\} = \|x\|. \end{aligned}$$

□

Theorem 2.7. *Let S_0 be a normalized or radial convex set in a real vector space V and let X be the generated subspace. Let ρ be the intrinsic semimetric on S_0 and $\|.\|$ the induced seminorm on X. If (S_0, ρ) is complete then so is $(X, \|.\|)$.*

Proof. For $p \in S_0$ since $p = 1.p$ we have $\|p\| \leq 1$. Assume (S_0, ρ) is complete. Let (x_n) be a Cauchy sequence in X. We may assume that $\|x_{n+1} - x_n\| < 2^{-n}$

for $n = 1, 2, \ldots$. We can write $x_{n+1} - x_n = c_n p_n - d_n q_n, 0 \leq c_n, d_n < 2^{-n}$, $p_n, q_n \in S_0$ and we can assume $c_1, d_1 > 0$. Let

$$a_n := \sum_{i=1}^{n} c_i, \ b_n := \sum_{i=1}^{n} d_i.$$

Now

$$\sum_{i=1}^{n} a_n^{-1} c_i p_i, \ i = 1, 2, \ldots$$

and

$$\sum_{i=1}^{n} b_n^{-1} d_i q_i, \ i = 1, 2, \ldots$$

are Cauchy sequences in S_0. Indeed, it is clear that (a_n) is a Cauchy sequence and we have

$$\begin{aligned} 2\rho(\sum_{i=1}^{n+k} a_{n+k}^{-1} c_i p_i &, \quad \sum_{i=1}^{n} a_n^{-1} c_i p_i) \\ &= \|[a_{n+k}^{-1} - a_n^{-1}] \sum_{i=1}^{n} c_i p_i + \sum_{i=n+1}^{n+k} a_{n+k}^{-1} c_i p_i\| \\ &\leq (a_n^{-1} - a_{n+k}^{-1}) \sum_{i=1}^{n} c_i + a_{n+k}^{-1} \sum_{i=n+1}^{n+k} c_i \\ &= (a_n^{-1} - a_{n+k}^{-1}) a_n + a_{n+k}^{-1} (a_{n+k} - a_n) \\ &= 2(1 - a_n a_{n+k}^{-1}), \end{aligned}$$

where the last term approaches to zero and $n, k \to \infty$. Thus there are elements $p, q \in S_0$ such that $\sum_{i=1}^{n} a_n^{-1} c_i p_i \to p$ and $\sum_{i=1}^{n} b_n^{-1} d_i q_i \to q$. Suppose

$a_n \to a$, $b_n \to b$, then $x_n \to x_1 + ap - bq$. Indeed,

$$\begin{aligned}
\|x_{n+1} - x_1 - ap + bq\| &= \|x_{n+1} - x_n + x_n - x_{n-1} + \cdots + x_2 - x_1 - ap + bq\| \\
&\leq |a|\|\sum_{i=1}^{n} a^{-1} c_i p_i - p\| + |b|\|\sum_{i=1}^{n} b^{-1} d_i q_i - q\| \\
&\leq |a|[\|\sum_{i=1}^{n} (a^{-1} c_i - a_n^{-1} c_i) p_i\| + \|\sum_{i=1}^{n} a_n^{-1} c_i p_i - p\|] \\
&+ |b|[\|\sum_{i=1}^{n} (b^{-1} d_i - b_n^{-1} d_i) q_i\| + \|\sum_{i=1}^{n} b_n^{-1} d_i q_i - q\|] \\
&\leq |a|[1 - a^{-1} a_n + \|\sum_{i=1}^{n} a_n^{-1} c_i p_i - p\|] \\
&+ |b|[1 - b^{-1} b_n + \|\sum i = 1^n b_n^{-1} d_i q_i - q\|] \to 0 (n \to \infty).
\end{aligned}$$

$\square$

References

[1] Gudder, S., Convex strustures and operational quantum mechanics, Commun. math. Phys.**29** (1973) 249–264.

[2] M.H. Stone, Postulates for the barycenter calculus, Memoiri di M.H. Stone (Chicago, USA)

[3] Capraro, V., Fritz, T., On the axiomatization of convex sybsets of Banach spaces, arXiv:1105.1270v3[math.MG]20Oct.2015