1 Ordered vector spaces

Overall reference: [4]

1.1 Basic definitions

Let X be a real vector space. A subset $A \subseteq X$ is

- algebraically open (closed) if the intersection of any line with A is an open (closed) subset of the line
- linearly bounded if the intersection of A with any line is a bounded subset of the line

We say that $a \in A$ is an algebraic interior point of A if it is an interior point of the intersection of any line with A, that is, for any $x \in X$ there is some $\delta > 0$ such that $a + sx \in A$ for all $|s| \leq \delta$. The set of all such points is called the algebraic interior of A and is denoted by $aint(A)$. The algebraic closure of A is $acl(A) := X \setminus aint(X \setminus A)$. If A is convex, then

$$acl(A) = \{x \in X,\ \exists y \in X,\ x + \lambda y \in A,\ \forall \lambda \in (0,1)\}.$$

A is algebraically open iff $A = aint(A)$ and algebraically closed iff $A = acl(A)$. If A is convex, then both $aint(A)$ and $acl(A)$ are convex as well.

Remark 1. (cf. [5, §16]) If A is convex, then $aint(A)$ is algebraically open, but in general $aint(aint(A)) \subsetneq aint(A)$. The algebraic closure is not necessarily algebraically closed even if A is convex. The counterexample is as follows. Let X be an infinite dimensional vector space with algebraic basis $\{x_\alpha\}$. Put

$$A = \{x = \sum_\alpha c_\alpha x_\alpha,\ c_\alpha \geq 0\ \forall \alpha,\ \sum_\alpha c_\alpha \geq \frac{1}{n(x)}\}$$

where $n(x) = \#\{\alpha, x = \sum_\alpha c_\alpha x_\alpha,\ c_\alpha \neq 0\}$. Then A is convex, $acl(A) = \{x = \sum_\alpha c_\alpha x_\alpha,\ c_\alpha \geq 0\ \forall \alpha,\ x \neq 0\}$ and $acl(acl(A))$ contains 0, so that $acl(A) \subsetneq acl(acl(A))$. On the other hand, if A is convex and $aint(A) \neq \emptyset$, then $acl(acl(A)) = acl(A)$.

Wedges, cones and orderings

A subset $P \subseteq X$ is called a wedge if $P + P \subseteq P$ and $\lambda P \subseteq P$ for any $\lambda \geq 0$. The preorder $x \leq y$ if $x - y \in P$ is compatible with the linear structure, such a preorder is called an ordering in X. Conversely, for any ordering, the set of positive elements is a wedge.

The pair (X, P) where P is a wedge is called an ordered vector space. The corresponding ordering is a partial order iff $P \cap -P = \{0\}$, in this case P is called a cone. X with this ordering is directed iff P is generating, that is, $X = P - P$.

Positive maps

Let (X, P) and (Y, Q) be ordered vector spaces. A linear map $F : X \to Y$ is called positive if $F(P) \subseteq Q$. If F is invertible with positive inverse, we say that F is an order isomorphism.

Let (P, Q) denote the set of positive maps, then (P, Q) is a wedge in the vector space $L(X, Y)$ of all linear maps $X \to Y$. We have

Lemma 1. *(P, Q) is a cone if and only if P is generating and Q is a cone.*

Archimedean and almost Archimedean orderings

Let (X, P) be an ordered vector space. We say that the ordering (or P) is Archimedean if $x \leq \lambda y$ for some $y \in X$ and all $\lambda > 0$ implies that $x \leq 0$.

Proposition 1. *The following are equivalent.*

(i) the ordering is Archimedean.

(ii) $\exists y \in X, \epsilon > 0$ *such that* $x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x \leq 0$.

(iii) $\exists y \in P, \epsilon > 0$ *such that* $x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x \leq 0$.

(iv) $P = acl(P)$.

The ordering (or P) is almost Archimedean if $-\lambda y \leq x \leq \lambda y$ for some $y \in X$ and all $\lambda > 0$ implies $x = 0$.

Proposition 2. *The following are equivalent.*

(i) the ordering is almost Archimedean.

(ii) $\exists y \in X, \epsilon > 0$ *such that* $-\lambda y \leq x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x = 0$.

(iii) $\exists y \in P, \epsilon > 0$ *such that* $-\lambda y \leq x \leq \lambda y$ *for all* $\epsilon \geq \lambda > 0 \implies x = 0$.

(iv) $acl(P)$ *is a cone.*

Remark 2. Note that an almost Archimedean wedge must be a cone. An Archimedean wedge is almost Archimedean iff it is a cone.

1.2 Order units and bases

Order units and seminorms

An element $u \in X$ is an order unit in (X, P) if for any $x \in X$, there is some $\lambda \in \mathbb{R}^+$ such that $x \leq \lambda u$. This is equivalent to $u \in aint(P)$. If $aint(P) \neq \emptyset$, P is generating.

If u is an order unit, then P is (almost) Archimedean iff u is (almost) Archimedean: $x \leq \lambda u$ for all $\lambda > 0$ implies $x \leq 0$ (resp. $-\lambda u \leq x \leq \lambda u$ for all $\lambda > 0$ implies $x = 0$).

For an order unit u, put

$$\|x\|_u = \inf\{\lambda > 0,\ -\lambda u \leq x \leq \lambda u\}.$$

Then $\|\cdot\|_u$ is a seminorm in X. It is a norm iff u is almost Archimedean.

Remark 3. If $u_1, u_2 \in aint(P)$, the associated seminorms $\|\cdot\|_{u_1}$ and $\|\cdot\|_{u_2}$ are equivalent. The corresponding topology is thus a property of the ordering rather than the order unit. In fact, this topology is the finest locally convex topology making all order intervals bounded.

Lemma 2. *Let u be Archimedean. Then $[-u, u] = \{x, \in X,\ \|x\|_u \leq 1\}$ and the wedge P is closed in the topology given by $\|\cdot\|_u$.*

Proof. Let $x \in [-u, u]$, then clearly $\|x\|_u \leq 1$. Conversely, assume that $\|x\|_u \leq 1$, then $-(1+\epsilon)u \leq x \leq (1+\epsilon)u$ for all $\epsilon > 0$. This implies that $\pm x - u \leq \epsilon u$ for all $\epsilon > 0$ and since u is Archimedean, this implies $\pm x \leq u$, that is, $x \in [-u, u]$.

For the second statement, let $x \in \bar{P}$ (the closure of P w.r. to $\|\cdot\|_u$). Then for all $n \in \mathbb{N}$, there is some $p_n \in P$ such that $\|x - p_n\|_u \leq \frac{1}{n}$. This implies that $-x \leq p_n - x \leq \frac{1}{n} u$ for all n and since u is Archimedean, $-x \leq 0$, so that $x \in P$.

□

Let (X_i, P_i), $i = 1, 1$ be ordered vector spaces and let $u_i \in X_i$ be order units. A map $f : X_1 \to X_2$ is called unital if $f(u_1) = u_2$. The following is immediate.

Proposition 3. *Let (X_i, P_i, u_i), $i = 1, 2$ be order unit spaces. Any positive unital map $f : X_1 \to X_2$ is a contraction with respect to the seminorms $\| \cdot \|_{u_1}$ and $\| \cdot \|_{u_2}$.*

A triple (X, P, u) where X is a vector space, $P \subseteq X$ an Archimedean cone and $u \in aint(P)$ is called an order unit space. To summarize, in this case, $\| \cdot \|_u$ is a norm in X, $[-u, u]$ is the corresponding closed unit ball and P is norm closed. The category of order unit spaces with positive unital maps as morphisms will be denoted by **AOUS**.

Bases and seminorms

Let (X, P) be an ordered vector space. A convex subset $K \subset P$ is called a base of P if for any nonzero $p \in P$ there is a unique $\lambda > 0$ such that $\lambda p \in K$.

Lemma 3. *Any wedge with a base is a cone.*

Proof. Let K be a base of a wedge P, and let $0 \neq x \in P \cap -P$. Then there are $\lambda, \mu > 0$ such that $\lambda x = x_1 \in K$ and $-\mu x = x_2 \in K$. It follows that $\lambda^{-1} x_1 = -\mu^{-1} x_2$ and then $\frac{\mu}{\lambda+\mu} x_1 + \frac{\lambda}{\lambda+\mu} x_2 = 0$. Since K is convex, we obtain $0 \in K$, but then for any $p \in K$, $\lambda p \in K$ for all $\lambda \in [0, 1]$. Hence P must be a cone.

□

Proposition 4. *A wedge P has a base if and only if there exists a linear functional ξ on X which is strictly positive on P. In this case, we may put $K = \{p \in P, \xi(p) = 1\}$.*

Proof. Let K be a base of P. For $p \in P$, let $\xi(p)$ be the unique positive number such that $\xi(p)^{-1} p \in K$. Then clearly $\xi(sp) = s\xi(p)$. Further, let $p, q \in P$ and let $\alpha = \xi(p) + \xi(q)$, then

$$\alpha^{-1}(p + q) = \frac{\xi(p)}{\alpha} \xi(p)^{-1} p + \frac{\xi(q)}{\alpha} \xi(q)^{-1} q \in K,$$

so that $p \mapsto \xi(p)$ is an additive function $\xi : P \to \mathbb{R}^+$. The function ξ easily extends to $P - P$ and has an extension to all of X by Hahn-Banach theorem. This extension is obviously positive and $K = \{p \in P,\ \xi(p) = 1\}$.

Conversely, let $\xi : X \to \mathbb{R}$ be strictly positive, then $K = \{p \in P, \xi(p) = 1\}$ is a convex subset of P and $\xi(p)^{-1}p \in K$ for any $p \in P$. Uniqueness is obvious.

$\square$

Proposition 5. *([3]) Let P be a generating cone in a vector space X and let K be a base of P. For $x \in X$, put*

$$\|x\|_K := \inf\{a + b,\ x = ap - bq,\ a, b \in \mathbb{R}^+, p, q \in K\}.$$

This defines a seminorm in X, which is a norm if and only if $S := co(K \cup -K)$ is linearly bounded.

Proof. It can be checked easily that $\|\cdot\|_K$ is a seminorm. Note also that $x \in S$ implies $\|x\|_K \leq 1$. Indeed, any $x \in S$ has the form $x = \lambda p - (1 - \lambda)q$ for some $\lambda \in [0, 1]$, $p, q \in K$ and then $\|x\|_K \leq \lambda + (1 - \lambda) = 1$. Assume that $\|\cdot\|_K$ is a norm and let $x_t := x + ty$ be a line in X. Then $\|y\|_K > 0$ and $x_t \in S$ implies that $1 \geq \|x_t\|_K \geq |\|x\|_K - |t|\|y\|_K|$, so that $|t| \leq \frac{1+\|x\|_K}{\|y\|_K}$. Conversely, assume that S is linearly bounded and let $\|x\|_K = 0$. This implies $tx \in S$ for all $t \in \mathbb{R}$, hence we must have $x = 0$.

$\square$

The (semi)norm in the above proposition is called the base (semi)norm in X.

Remark 4. Note that $\|\cdot\|_K$ is the Minkowski functional of S, that is

$$\|x\|_K = \inf\{\lambda > 0, x \in \lambda S\}.$$

To see this, observe that $S = \{sp - (1 - s)q,\ s \in [0, 1],\ p, q \in K\}$. Denote the Minkowski functional by p_S. If $x = ap - bq$ for some $a, b \in \mathbb{R}^+$ and $p, q \in K$, then if $a + b = 0$, we must have $x = 0$ and the equality obviously holds. Otherwise,

$$x = +b)(\frac{a}{a+b}p - \frac{b}{a+b}q) \in (a + b)S,$$

so that $p_S(x) \leq \|x\|_K$. On the other hand, let $x \in \lambda S$ for some $\lambda > 0$. Then $x = \lambda(sp - (1 - s)q)$ for some $s \in [0, 1]$ and $p, q \in K$, so that

$$\|x\|_K \leq \lambda s + (1 - \lambda)(1 - s) = \lambda,$$

hence $\|x\|_K \leq p_S(x)$.

Remark 5. Linear boundedness of K is in general not enough. There are some weird infinite dimensional examples such that K is linearly bounded but $co(K \cup -K)$ is not.

Let (X_i, P_i) be ordered vector spaces and let K_i be a base of P_i, $i = 1, 2$. A linear map $f : X_1 \to X_2$ is called base-preserving if $f(K_1) \subseteq K_2$

Proposition 6. *Any base-preserving linear map is positive and contractive with respect to the base seminorms.*

A triple (X, P, K), where X is a vector space, P a generating cone and K a base of P such that $co(K \cup -K)$ is linearly bounded is called a base-normed space. The category of base-normed spaces with base-preserving linear maps will be denoted by **BN**.

Some examples

The wedges X and $\{0\}$ are trivial.

1. The only nontrivial wedges in $\mathbb{R}$ are $\mathbb{R}^+$ and $\mathbb{R}^-$. $\mathbb{R}$ with the usual ordering and norm concides with both the order unit space $(\mathbb{R}, \mathbb{R}^+, 1)$ and the base-normed space $(\mathbb{R}, \mathbb{R}^+, \{1\})$.

2. **Function spaces:** Let S be a set, $X = \{f : S \to \mathbb{R}\}$, $P = \{f, f(S) \subseteq \mathbb{R}^+\}$. P is an Archimedean cone, $(X, \leq)$ is a lattice. If S is not finite, $aint(P) = \emptyset$.

3. As 2, but bounded functions. In this case P is an Archimedean cone, $aint(P)$ is the set of strictly positive functions.

4. Let K be a convex set, we will denote the set of all affine functions $K \to \mathbb{R}$ by $A(K)$, the set of bounded affine functions by $A_b(K)$. If K is also a topological space, we denote the set of continuous affine functions by $A_c(K)$ We also denote by $A(K)^+$ ($A_b(K)^+$, $A_c(K)^+$) the set of positive affine (bounded, continuous) functions and 1_K the constant function $1_K(x) \equiv 1$. Then $(A_b(K), A_b^+(K), 1_K)$ is an order unit space, with $\|f\|_{1_K} = \sup_{x \in K} |f(x)|$. If K is also a compact Hausdorff topological space, then the same is true for $(A_c(K), A_c^+(K), 1_K)$.

5. $X = \{f : \mathbb{R} \to \mathbb{R}\}$, with the cone of nondecreasing functions.

6. **Sequence spaces:** X the set of all (or bounded, summable, convergent, converging to 0,...) sequences, with usual positive cone.

7. $\mathbb{R}^2$ with the usual or lexicographic ordering, with $P = \{(x, y),\ x > 0, y > 0\} \cup \{0\}$ or $P = \{(x, y),\ x > 0\} \cup \{0\}$.

Completeness

We give some sufficient conditions for completeness of order unit norms and base norms.

Proposition 7. *[4] Let (X, P) be an ordered vector space with an almost Archimedean order unit u. If every majorized increasing sequence in (X, P) has a supremum, then $(X, \|\cdot\|_u)$ is complete.*

Proof. We first show that any increasing Cauchy sequence has a limit. So let $\{x_n\}$ be such a sequence and let $\epsilon > 0$. Then $\|x_n - x_m\|_u < \epsilon$ for $m, n \geq N$. We then have for all $m \geq N$, $x_m - x_N \leq \epsilon u$, so that $x_m \leq x_N + \epsilon u$. It follows that $\{x_n\}$ is a majorized increasing sequence, so that there is some x_0 such that $x_0 = \sup_n x_n$. For all $m, n \geq N$, we have $x_n \leq x_m + \epsilon u$, hence $x_0 \leq x_m + \epsilon u$ and we have $0 \leq x_0 - x_m \leq \epsilon u$. This implies $\|x_0 - x_m\|_u \leq \epsilon$ for all $m \geq N$, so that $\lim_n x_n = x_0$.

Let now $\{x_n\}$ be any Cauchy sequence. Let $V_n = \{p - q,\ p, q \in [0, 2^{-n}]\}$, then V_n contains the ball with center 0 and radius 2^{-n+1} and is therefore a neighborhood of 0. Hence there is a subsequence such that $x_n - x_{n-1} \in V_n$. Let $a_n, b_n \in [0, 2^{-n}]$ be such that $x_n - x_{n-1} = a_n - b_n$. Then $\{\sum_{k=1}^n a_k\}$ and $\{\sum_{k=1}^n b_k\}$ are increasing Cauchy sequences and hence have a limit by the first part of the proof. Moreover, we have $x_n = \sum_{k=1}^n (a_k - b_k)$, so that x_n converges as well.

□

Proposition 8. *[1] Let K be a base of P and asume that K is compact with respect to some Hausdorff topology τ, compatible with the linear structure of X. Then X is $\|\cdot\|_K$-complete.*

Proof. Note that $S = co(K \cup -K)$ is also τ-compact, hence must be linearly bounded. It follows that $\|\cdot\|_K$ is a norm and it is easy to verify that S is the closed unit ball. Let $\{x_n\}$ be a Cauchy sequence, then it is norm-bounded, so we may assume that $\{x_n\} \subset S$. Let $y \in S$ be a τ-accumulation point of $\{x_n\}$.

For $\epsilon > 0$, $\|x_n - x_m\|_K < \epsilon$ for $n, m \geq N$. This implies that $x_n \in x_N + \epsilon S$ for $n \geq N$. Since S is τ-closed, $y \in x_N + \epsilon S$. It follows that

$$\|y - x_n\|_K \leq \|y - x_N\|_K + \|x_N - x_n\|_k \leq 2\epsilon,$$

this finishes the proof.

□

1.3 Duality

Positive functionals

Let (X, P) be an ordered vector space and let X' denote the algebraic dual of X. The dual wedge of P is defined as

$$P' := \{\varphi \in X', \varphi(p) \geq 0, \forall p \in P\}$$

Note that $P' = (P, \mathbb{R}^+)$ and it follows by Lemma 1 that P' is a cone iff P is generating.

Remark 6. To see the above duality in this specific case, note that P is a generating wedge in the subspace $P - P$, whose algebraic dual can be identified with the quotient space $X'|_{(P-P)^\perp}$. Here

$$(P - P)^\perp = \{\varphi \in X',\ \varphi(x) = 0,\ \forall x \in P - P\} = P' \cap -P'.$$

If $P - P = X$, then $P' \cap -P' = (P - P)^\perp = \{0\}$, so P' is a cone. Conversely, if P' is a cone, then

$$P - P = (P - P)^{\perp\perp} = \{0\}^\perp = X,$$

this holds since any subspace $E \subseteq X$ satisfies $E^{\perp\perp} = E$. However, this is no longer true for subspaces in X' ([5]), so a dual statement does not hold. More precisely, it is easily checked that $P \cap -P \subseteq (P' - P')^\perp$, so that if P' is generating, P must be a cone. The converse is not true in general: we only have $P' - P' \subseteq (P' - P')^{\perp\perp} = (P \cap -P)^\perp$, so P' may be not generating even if P is a cone (there are indeed counterexamples).

The dual of a vector space with an order unit

Let (X, P) be an ordered vector space with an order unit u. Positive unital linear functionals are called states, the set of all states will be denoted by $\mathcal{S}(X, P, u)$.

Lemma 4. *(i) Any $\varphi \in P'$ is $\|\cdot\|_u$-bounded, with*

$$\|\varphi\|_u^* := \sup_{\|x\|_u \leq 1} |\varphi(x)| = \varphi(u)$$

(ii) $\mathcal{S}(X, P, u)$ is a base of P'.

(iii) If $\varphi \in X'$ is such that $\|\varphi\|_u^* = \varphi(u)$, then $\varphi \in P'$.

(iv) For $x \in X$, $\|x\|_u = \sup_{\varphi \in \mathcal{S}(X,P,u)} |\varphi(x)|$.

Proof. (i) is quite easy. This also implies that u is strictly positive over P', hence the set of states forms a base of P' by Proposition 4. For (iii), we may assume $\varphi(u) = 1$. Let $x \in P$ and let $\lambda > 0$ be such that $0 \leq x \leq \lambda u$. Then $\|x - \lambda u\|_u \leq \lambda$ and we have

$$|\varphi(x) - \lambda| = |\varphi(x - \lambda u)| \leq \|\varphi\|_u^* \|x - \lambda u\|_u \leq \lambda.$$

This implies $\varphi(x) \geq 0$. For (iv), let $x \in X$ be such that $-\lambda u \leq x \leq \lambda u$, then $|\varphi(x)| \leq \lambda$ for any $\varphi \in \mathcal{S}(X, P, u)$, so that $\sup_{\varphi \in \mathcal{S}(X,P,u)} |\varphi(x)| \leq \|x\|_u$. Assume that this inequality is strict for some x_0, we may put $\|x_0\|_u = 1$. Then there is some $0 < a < 1$ such that $|\varphi(x_0)| \leq a$ for any state φ. Let $H = \{x \in X, x \leq au\}$, then we have either $x_0 \notin H$ or $-x_0 \notin H$. Note that $H = au - P$ is a convex set such that $aint(H) \neq \emptyset$. If $y \notin H$, then by a separation theorem by Edelheit [4, 0.2.4], there is some nonzero $\psi \in X'$ such that $\sup_{x \in H} \psi(x) \leq \psi(y)$. This implies that ψ is bounded below on P, so that we must have $\psi \in P'$. Normalizing, we may assume that $\psi \in \mathcal{S}(X, P, u)$. Then

$$a = \psi(au) \leq \sup_{x \in H} \psi(x) \leq \psi(y).$$

Since we may take either x_0 or $-x_0$ for y, we have arrived at a contradiction. □

Theorem 1. *Let (X, P) be an ordered vector space with an order unit u and let $K = \mathcal{S}(X, P, u)$. Then $P' - P'$ is the space of $\|\cdot\|_u$-bounded functionals and $(P' - P', P', K)$ is a base-normed space, with $\|\cdot\|_K = \|\cdot\|_u^*$.*

Proof. [3] By Lemma 4 any $\varphi \in P' - P'$ is $\|\cdot\|_u$-bounded. For the converse, let $Y = X \times X$ be ordered by the wedge $Q = P \times P$, then (u,u) is an order unit in (Y,Q). Let

$$Z = \{t(u,u) - (x,-x),\ t \in \mathbb{R}, x \in X\},$$

then Z is a linear subspace in Y containing the order unit. Let $\varphi \in X'$ be $\|\cdot\|_u$-bounded and put

$$F_\varphi(z) = t\|\varphi\|_u^* - \varphi(x), \qquad z = t(u,u) - (x,-x) \in Z$$

This defines a linear functional on Z. Moreover, note that $z = t(u,u) - (x,-x) \in Q$ iff $\|x\|_u \leq t$ and then $F_\varphi(z) \geq (t - \|x\|_u)\|\varphi\|_u^* \geq 0$. Since Z contains the order unit, F_φ extends to a positive linear functional on Y ([4, Corollary 1.6.2]). Put

$$\psi_1(x) = F_\varphi(x,0), \quad \psi_2(x) = F_\varphi(0,x), \qquad x \in X.$$

Then $\psi_1, \psi_2 \in P'$ and $\varphi = \psi_2 - \psi_1$, so that $\varphi \in P' - P'$. We have

$$\|\varphi\|_u^* = F_\varphi(u,u) = \psi_1(u) + \psi_2(u) \geq \|\varphi\|_K$$

On the other hand, let $\varphi = a\varphi_1 - b\varphi_2$ with $a,b \geq 0$, $\varphi_1, \varphi_2 \in K$, then $\|\varphi\|_u^* \leq a + b$, this shows that $\|\varphi\|_u^* = \|\varphi\|_K$. To finish the proof, we have to show that $\|\cdot\|_K$ is a norm. So let $\|\varphi\|_K = 0$, then φ is zero over the absorbing set $[-u,u]$, so that $\varphi = 0$.

□

Corollary 1. *The norm dual of an ordered vector space with order unit norm is a base-normed space, with base formed by the set of states.*

Corollary 2. *Let (X,P) be an ordered vector space xith an order unit u.*

(i) P is almost Archimedean iff the set of states is separating.

(ii) P is Archimedean iff the set of states is order-determining.

Proof. (i) is immediate from Lemma 4 (i). For (ii), assume that P is an Archimedean cone, then $\|\cdot\|_u$ is a norm and P is norm-closed, hence also weakly closed. Since P' is contained in the norm dual of X, P is equal to its double dual

$$P'' = \{x \in X, \varphi(x) \geq 0,\ \forall \varphi \in P'\}.$$

Since the set of states is a base of P', this implies that it determines the order in (X, P). Conversely, let $L = \{x_t\}$ be any line in X, then $x_t \in P$ iff $\varphi(x_t) \geq 0$ for all states φ, so that $L \cap P$ is a closed subset in L, hence P is algebraically closed.

□

The dual of an ordered vector space with a based cone

Let (X, P) be an ordered vector space and let $K \subset P$ be a base of P. By Proposition 4, there is a strictly positive linear functional $u \in X'$, such that $K = \{p \in P, u(p) = 1\}$.

Theorem 2. *Let $X^* \subseteq X'$ be the set of $\|\cdot\|_K$-bounded linear functionals and let $P^* = X^* \cap P'$. Then (X^*, P^*, u) is is an order unit space, isomorphic to $(A_b(K), A_b(K)^+, 1_K)$ (see Example 4). Moreover,*

$$\|\varphi\|_K^* := \sup_{\|x\|_K \leq 1} |\varphi(x)| = \|\varphi\|_u = \sup_{x \in K} |\varphi(x)|.$$

Proof. It is easy to see that any element $\varphi \in X'$ restricts to a function $\varphi|_K \in A(K)$ and conversely, any function in $A(K)$ extends uniquely to some linear functional in X'. Moreover, $\varphi \in P'$ iff $\varphi|_K \in A(K)^+$ and $u|_K = 1_K$. Let now $\varphi \in X'$ and let $x \in X$, $\|x\|_K \leq 1$. Then for any $\epsilon > 0$, $x = a_\epsilon x_1 - b_\epsilon x_2$, where $x_1, x_2 \in K$ and $a_\epsilon, b_\epsilon \geq 0$ are such that $a_\epsilon + b_\epsilon < 1 + \epsilon$. Then

$$|\varphi(x)| \leq a_\epsilon |\varphi(x_1)| + b_\epsilon |\varphi(x_2)| \leq (1 + \epsilon) \sup_{x \in K} |\varphi(x)|. \tag{1}$$

It follows that $\varphi \in X^*$ iff $\varphi|_K \in A_b(K)$. This establishes a unital order isomorphism between (X^*, P^*, u) and $(A_b(K), A_b(K)^+, 1_K)$. We also have by (1) that $\|\varphi\|_K^* \leq \sup_{x \in K} |\varphi(x)| \leq \|\varphi\|_K^*$.

□

Preduals

We next discuss the Banach space preduals of order unit and base-normed spaces. Below, $(X, \|\cdot\|)$ is a Banach space and $(X^*, \|\cdot\|^*)$ its norm dual. If $P \in X$ is a wedge, we will denote

$$P^* := \{\varphi \in X^*,\ \varphi(p) \geq 0,\ \forall p \in P\} = P' \cap X^*.$$

Similarly, if Q is a wedge in X^*, we will denote

$$Q_* := \{x \in X,\ q(x) \geq 0,\ \forall q \in Q\} = Q' \cap X.$$

It is clear that P^* and Q_* are wedges. Moreover, $(P^*)_* = \bar{P}$ and $(Q_*)^*$ is the weak*-closure of Q.

Theorem 3. *[3, 2] If X^* is an order unit space with weak*-closed positive cone, then X is base-normed. More precisely, if there is an Archimedean weak*-closed cone $Q \subset X^*$ with an order unit u such that $\|\cdot\|^* = \|\cdot\|_u$, then $Q_* \subset X$ has a base $K = \{p \in Q_*, u(p) = 1\}$ and (X, Q_*, K) is a base-normed space with $\|\cdot\| = \|\cdot\|_K$.*

Proof. Let $p \in Q_*$ be such that $u(p) = 0$, then for any $\varphi \in Q$,

$$0 \leq \varphi(p) \leq \|\varphi\|_u u(p) = 0.$$

Since $X^* = Q - Q$ separates points in X, we obtain $p = 0$. Hence u defines a strictly positive linear functional on (X, Q_*) and K is a base of Q_*. For $p \in Q_*$, we have

$$\|p\| = \sup_{\varphi \in [-u,u]} |\varphi(p)| = u(p),$$

it follows that $S = co(K \cup -K)$ is a subset of the unit ball X_1 of X, so that $\|\cdot\| \leq \|\cdot\|_K$ (since $\|\cdot\|_K$ is the Minkowski functional of S). We next show that S is dense in X_1. Since $Q = (Q_*)^*$, we have for $\varphi \in X^*$:

$$\begin{aligned}\|\varphi\|_u &= \inf\{\lambda > 0,\ \lambda u \pm \varphi \in Q\} = \inf\{\lambda > 0,\ (\lambda u \pm \varphi)(p) \geq 0,\ \forall p \in Q_*\} \\ &= \inf\{\lambda > 0,\ |\varphi(p)| \leq \lambda,\ \forall p \in K\} = \sup_{p \in K} |\varphi(p)|.\end{aligned}$$

Assume that $x_0 \in X$ is such that $\|x_0\| \leq 1$ and $x_0 \neq \bar{S}$, then by Hahn-Banach separation theorem, there is some $\varphi \in X^*$ such that

$$\|\varphi\|_u = \sup_{p \in K} |\varphi(p)| \leq \sup_{x \in S} \varphi(x) < \varphi(x_0) \leq \|\varphi\|^* = \|\varphi\|_u,$$

a contradiction. It follows that $\bar{S} = X_1$.

Further, choose any $\alpha > 1$ and let $\alpha_n > 0$ be a sequence such that $1 + \sum_n \alpha_n < \alpha$. Since $x_0 \in \bar{S}$, there is some element $x_1 \in S$ such that $\|x_0 - x_1\| < \alpha_1$. Similarly, there is some $x_2 \in \alpha_1 S$ such that $\|x_0 - x_1 - x_2\| <$

α_2. Continuing by induction, we obtain a sequence $\{x_n\}$ in X such that $\|x_n\|_K \le \alpha_{n-1}$ and $\|x_0 - \sum_n x_n\| < \alpha_n \to 0$. Hence

$$\|x_0\|_K = \|\sum_n x_n\|_K \le \sum_n \|x_n\|_K \le 1 + \sum_n \alpha_n < \alpha,$$

so that $x_0 \in \alpha S$. Since the above inequality holds for all $\alpha > 1$, we have $1 = \|x_0\| \le \|x_0\|_K \le 1$. It also follows that $X_1 \subset \alpha S$ for any $\alpha > 1$ and consequently $X = Q_* - Q_*$.

□

Theorem 4. *Let $(X^*, \|\cdot\|^*)$ be a base-normed space with a positive cone Q having a weak*-compact base K. Then (X, Q_*, u) is an order unit space, isomorphic to $(A_c(K), A_c(K)^+, 1_K)$ and $\|\cdot\| = \|\cdot\|_u$.*

Proof. By Theorem 2, (X^{**}, Q^*, u) is an order unit space isomorphic to $(A_b(K), A_b(K)^+, 1_K)$ and $\|\cdot\|^{**} = \|\cdot\|_u$. Since X can be identified with the subspace of weak*-continuous functionals in X^{**}, it is enough to prove that $\phi \in X^{**}$ is weak*-continuous iff $\phi|_K \in A_c(K)$.

So assume the latter. It is enough to show that $\phi^{-1}(0)$ is weak*-closed in X^*. By Krein-Smulian theorem, this is equivalent to

$$A = \{\varphi \in X^*,\ \phi(\varphi) = 0,\ \|\varphi\|_K \le 1\}$$

being weak*-closed. Clearly, $\varphi \in A$ iff $\varphi = a(\psi_1 - \psi_2)$, with $\psi_1, \psi_2 \in K$ and $0 \le a \le 1/2$. Let $\psi \in K$ be any element, then $\varphi = 1/2(2a\psi_1 + (1-2a)\psi - (2a\psi_2 + (1-2a)\psi)$, it follows that $A = 1/2(K - K)$ is weak*-closed and hence $\phi \in X$. The converse is obvious.

□

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