① The categories

Aff : Category of affine spaces

Object : vector spaces
Morphisms :

$$X \xrightarrow{f} Y$$

$$f(a_1x_1 + \dots + a_nx_n) = a_1 f(x_1) + \dots + a_n f(x_n)$$
for all $x_i \in X$ $a_i \in \mathbb{R}$ such that

$$a_1 + \dots + a_n = 1$$

Vect / $\mathbb{R}$

Objects: linear functionals . $V \xrightarrow{g} \mathbb{R}$
Morphisms: commutative triangles

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "g"'] & V' \arrow[d, "g'"] \\ & \mathbb{R} \end{tikzcd}$$

② Affine spaces as a subcategory of Vect / $\mathbb{R}$

a functor $J : \text{Aff} \longrightarrow \text{Vect}/\mathbb{R}$

On objects : $J(X) = \left(X \times \mathbb{R} \xrightarrow[p^X_{\mathbb{R}}]{} \mathbb{R} \right)$

projection onto second coordinate

On morphisms: $X \xrightarrow{F} Y$ – a morphism in Aff

$$X \times \mathbb{R} \xrightarrow{J(F)} Y \times \mathbb{R}$$

is given by the rule

$$J(F)(x, r) = (F(x) + (r-1)F(0), r)$$

Claim: $J(F)$ is a linear mapping

ADDITIVITY:

$$J(F)((x_1, r_1) + (x_2, r_2)) =$$

$$J(F)(x_1 + x_2, r_1 + r_2) =$$

$$F(x_1 + x_2) + (r_1 + r_2 - 1) \cdot F(0), r_1 + r_2)$$

$$J(F)(x_1, r_1) + J(F)(x_2, r_2) =$$

$$(F(x_1) + (r_1 - 1) \cdot F(0) + F(x_2) + (r_2 - 1)F(0), r_1 + r_2).$$

Second coordinate: OK

First coordinate:

$$F(x_1) + r_1 F(0) + F(x_2) + r_2 F(0) - F(0) - F(0) =$$

$1 + 1 - 1 = 1$; F is an affine map

$$= F(x_1 + x_2 - 0) + r_1 F(0) + r_2 F(0) - F(0) =$$

$$= F(x_1 + x_2) + (r_1 + r_2 - 1) \cdot F(0) \checkmark$$

MULTIPLICATION BY A SCALAR

$$\mathcal{J}(f)(s(x,r)) = \mathcal{J}(f)(s\cdot x, sr) =$$
$$= \left(f(sx) + (sr-1).f(0), sr\right)$$

$$s.\mathcal{J}(f)(x,r) = (sf(x) + s(r-1).f(0), sr)$$

$$sf(x) + s(r-1).f(0) =$$
$$= sf(x) + srf(0) - sf(0) =$$
$$= \underset{\sim}{s}f(x) + \underset{\sim}{sr}f(0) - \underline{s}f(0) - \underline{(sr-1)}f(0) + (sr-1)f(0) =$$

$$s + sr - s - (sr-1) = 1$$

$$= f(s.x + sr.0 - s.0 - (sr-1).0) + (sr-1).f(0) =$$
$$= f(sx) + (sr-1).f(0)$$

FUNCTORIALITY:

$$\mathcal{J}(id)(x,r) = (x + (r-1).0, r) = (x,r)$$
$$\mathcal{J}(f_1 \circ f_2)(x,r) = (f_1(f_2(x)) + (r-1)f_1(f_2(0)), r)$$
$$(\mathcal{J}(f_1) \circ \mathcal{J}(f_2))(x,r) = \mathcal{J}(f_1)(\mathcal{J}(f_2)(x,r)) =$$
$$= \mathcal{J}(f_1)(f_2(x) + (r-1)f_2(0), r) =$$ skip r from now on...
$$= f_1(f_2(x) + (r-1)f_2(0)) + (r-1)f_1(0) =$$

$$= f_1\left(1\cdot\left(f_2(x) + (r-1)f_2(0)\right)\right) + (r-1)f_1(0) +$$

$$+ (1-r)f_1(f_2(0)) + (r-1)f_1(f_2(0)) =$$

$1 + (r-1) + (1-r) = 0$

$$= f_1\big(f_2(x) + (r-1)f_2(0) + (r-1)\cdot 0 +$$

$$+ (1-r)f_2(0)\big) + (r-1)f_1(f_2(0)) =$$

$$= f_1(f_2(x)) + (r-1)f_1(f_2(0))$$

J IS A FAITHFUL FUNCTOR

$J(f_1) = J(f_2)$ $\qquad$ $f_1, f_2 : X \longrightarrow Y$

$\Downarrow ?$

$f_1 = f_2$

$\forall (x, r) \in X \times \mathbb{R}$

$$f_1(x) + (r-1)f_1(0) = f_2(x) + (r-1)f_2(0)$$

PUT $r = 1 \Rightarrow$ DONE

F IS A FULL FUNCTOR

Suppose
$$\begin{tikzcd} X \times \mathbb{R} \arrow[rr, "f"] \arrow[dr, "p^X_{\mathbb{R}}"'] & & Y \times \mathbb{R} \arrow[dl, "p^Y_{\mathbb{R}}"] \\ & \mathbb{R} & \end{tikzcd}$$
commutes

$f \cong \langle f_Y, f_{\mathbb{R}} \rangle \qquad f(x,r) = (f_Y(x,r), f_{\mathbb{R}}(x,r))$

$f_{\mathbb{R}}(x,r) = r$ (because the diagram commutes...)

We need to find an affine map $f': X \longrightarrow Y$ such that

$J(f') = f$, that means,

$f(x,r) = (f'(x) + (r-1).f'(0), r)$

Put $f'(x) = f_Y(x,1)$

$f'(x) + (r-1)f'(0) =$

$= f_Y(x,1) + (r-1)f_Y(0,1) =$

$= f_Y(x,1) + f_Y(0,r-1) =$

$= f_Y(x,r)$

It remains to prove that $f' : X \longrightarrow Y$ is an affine map

$$f'(s_1 x_1 + \dots + s_n x_n) = s_1 f'(x_1) + \dots + s_n f'(x_n)$$

$$s_1 + \dots + s_n = 1$$

$$\begin{aligned}
f'(s_1 x_1 + \dots + s_n x_n) &= \\
&= f_Y(s_1 x_1 + \dots + s_n x_n, 1) = \\
&= f_Y(s_1 x_1 + \dots + s_n x_n, s_1 + \dots + s_n) = \\
&= f_Y((s_1 x_1, s_1) + \dots + (s_n x_n, s_n)) = \\
&= f_Y(s_1 (x_1, 1) + \dots + s_n (x_n, 1)) = \\
&= s_1 f_Y(x_1, 1) + \dots + s_n f_Y(x_n, 1) = \\
&= s_1 f'(x_1) + \dots + s_n f'(x_n) \checkmark
\end{aligned}$$

ESSENTIALLY SURJECTIVE
(wrt surjective functionals in Vect/$\mathbb{R}$)

$X \xrightarrow{g} \mathbb{R}$ surjective $\Rightarrow$ there is

$X' \in \mathrm{Aff}$ such that

there is an iso j

$$\begin{tikzcd}
X \arrow[r, "g"] & \mathbb{R} \\
X' \times \mathbb{R} \arrow[u, "j", "\cong"'] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

in range of g

Put $X' = \operatorname{Ker} g$; $\operatorname{Ker} g$ has codimension 1,
so let $y \in X \setminus \operatorname{Ker} g$ be such that $g(y) = 1$
Let $j : X' \times \mathbb{R} \longrightarrow X$ be given by

$j(x', r) = x + ry$

as $g(j(x', r)) = g(x + ry) = g(x) + r.g(y) =$
$= 0 + r.1 = r = p_2(x', r)$ ✓

(3) The adjunction $\mathrm{Vect}/\mathbb{R} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Vect}$ $\perp$

For $X \in \mathrm{Vect}$

$$G(X) = \left(X \times \mathbb{R} \xrightarrow{p^X_{\mathbb{R}}} \mathbb{R}\right)$$

$$G(X \xrightarrow{f} Y) =$$

$$\begin{tikzcd} X \times \mathbb{R} \arrow[rr, "f \times id"] \arrow[dr, "p^X_{\mathbb{R}}"'] & & Y \times \mathbb{R} \arrow[dl, "p^Y_{\mathbb{R}}"] \\ & \mathbb{R} & \end{tikzcd}$$

For $g \in \mathrm{Vect}/\mathbb{R}$

$$F(X \xrightarrow{g} \mathbb{R}) = X$$

$$F\left(\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] & \\ X' \arrow[r, "g'"'] & \mathbb{R} \end{tikzcd}\right) = (X \xrightarrow{f} X')$$

UNIT

$$g : X \longrightarrow \mathbb{R} \in \mathrm{Vect}/\mathbb{R}$$

$$\eta_g : g \longrightarrow GF(g)$$

$$\begin{tikzcd}
 & X & \\
X \arrow[ur, "id_X"] \arrow[rr, dotted, "\eta_g"'] \arrow[dr, "g"'] & & X \times \mathbb{R} \arrow[ul, "p^X_X"'] \arrow[dl, "p^X_{\mathbb{R}}"] \\
 & \mathbb{R} &
\end{tikzcd}$$

COUNIT

$$X \in \mathrm{Vect}$$

$$\varepsilon_X : FG(X) \longrightarrow X$$

$$X \times \mathbb{R} \xrightarrow[\varepsilon_X = p^X_X]{} X$$

TRIANGLE IDENTITIES

$$\begin{tikzcd}
F(g) \arrow[r, "F\eta_g"] \arrow[dr, "id_{F(g)}"'] & FGF(g) \arrow[d, "\varepsilon_{F(g)}"] \\
 & F(g)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "F\eta_g"] \arrow[dr] & X \times \mathbb{R} \arrow[d, "\varepsilon_{F(g)}"] \\
 & X
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dr, "="'] & (x, g(x)) \arrow[d] \\
 & x
\end{tikzcd}$$

$$\begin{tikzcd}
G(X) \arrow[r, "\eta_{G(X)}"] \arrow[dr, squiggly, "\mathrm{id}_{G(X)}"'] & GFG(X) \arrow[d, "G\varepsilon_X"] \\
& G(X)
\end{tikzcd}$$

$$\left(X \times \mathbb{R} \xrightarrow{p^{\mathbb{R}}_X} \mathbb{R} \right)$$

$$\begin{tikzcd}
& X \times \mathbb{R} \\
X \times \mathbb{R} \arrow[ur, "\mathrm{id}"] \arrow[r, dashed, "\eta_{G(X)}"'] \arrow[d, "G(X) = p^{\mathbb{R}}_X"'] & X \times \mathbb{R} \times \mathbb{R} \arrow[u, "p_{1,2}"'] \arrow[dl, "p_3 = GFG(X)"] \\
\mathbb{R}
\end{tikzcd}$$

$\eta_{G(X)}(x, r) =$

$= (x, r, r)$

$$\varepsilon_X : X \times \mathbb{R} \xrightarrow[p_X]{} X$$

$G(\varepsilon_X)$

$$\begin{tikzcd}
(X \times \mathbb{R}) \times \mathbb{R} \arrow[r, "p_X \times \mathrm{id}"] \arrow[d, "p_3"'] & X \times \mathbb{R} \arrow[dl, "p_2"] \\
\mathbb{R}
\end{tikzcd}$$

$$\begin{tikzcd}
X \times \mathbb{R} \arrow[r, "\eta_{G(X)}"] \arrow[d, "p_2"'] & X \times \mathbb{R} \times \mathbb{R} \arrow[dl, "p_3"] \arrow[d, "p_X \times \mathrm{id}"] \\
\mathbb{R} & X \times \mathbb{R} \arrow[l, "p_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(x, r) \arrow[r, mapsto] \arrow[d, mapsto] & (x, r, r) \arrow[dl, mapsto] \arrow[d, mapsto] \\
r & (x, r) \arrow[l, mapsto]
\end{tikzcd}$$

④ $\mathrm{Vect}/\mathbb{R}$ as coalgebras

The adjunction
$$\begin{tikzcd} \mathrm{Vect}/\mathbb{R} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Vect} \arrow[l, shift left, "G"] \end{tikzcd}$$
induces a comonad on Vect

$$X \xleftarrow{\varepsilon_X} X \times \mathbb{R} \xrightarrow{\sigma_X} X \times \mathbb{R} \times \mathbb{R}$$

$$\sigma_X(x, r) = (x, r, r)$$

$\mathrm{Vect}/\mathbb{R}$ is isomorphic to the category of coalgebras for this comonad

$X \xrightarrow{\gamma} X \times \mathbb{R}$ is a coalgebra iff

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[dr, "id"'] & X \times \mathbb{R} \arrow[d, "\varepsilon_X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & X \times \mathbb{R} \arrow[d, "\sigma_X"] \\ X \times \mathbb{R} \arrow[r, "\gamma \times id_{\mathbb{R}}"'] & X \times \mathbb{R} \times \mathbb{R} \end{tikzcd}$$

$$\downarrow \qquad\qquad\qquad \searrow$$

$\gamma(x) = (x, g(x))$
for some $g : X \to \mathbb{R}$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & (x, g(x)) \arrow[d, mapsto] \\ (x, g(x)) \arrow[r, mapsto] & (x, g(x), g(x)) \end{tikzcd}$$

commutes trivially.

Morphism of coalgebras:

$$x \xmapsto{\gamma_X} (x, g_X(x))$$
$$y \xmapsto{\gamma_Y} (y, g_Y(y))$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma_X"'] & Y \arrow[d, "\gamma_Y"] \\ X \otimes \mathbb{R} \arrow[r, "f \otimes id"'] & Y \otimes \mathbb{R} \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ & (f(x), g_Y(f(x))) \arrow[d, phantom, "\shortparallel"] \\ (x, g_X(x)) \arrow[r, mapsto] & (f(x), g_X(x)) \end{tikzcd}$$

So the square commutes iff

$$\forall x : g_Y(f(x)) = g_X(x)$$

meaning

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g_X"'] & Y \arrow[dl, "g_Y"] \\ \mathbb{R} & \end{tikzcd}$$

commutes.

(5) Monoidal structure on $Vect/\mathbb{R}$

This structure comes from the fact that $\mathbb{R}$ is a (multiplicative) monoid in $Vect$.

Tensor product

$$\left(\begin{array}{c} X_1 \\ \downarrow g_1 \\ \mathbb{R} \end{array}\right) \otimes \left(\begin{array}{c} X_2 \\ \downarrow g_2 \\ \mathbb{R} \end{array}\right) = \begin{tikzcd} X_1 \otimes X_2 \arrow[d] \\ \mathbb{R}\otimes\mathbb{R} \arrow[d, "\cdot"] \\ \mathbb{R} \end{tikzcd}$$

Unit object

$$\begin{tikzcd} \mathbb{R} \arrow[d, "id"'] \\ \mathbb{R} \end{tikzcd}$$

Unit object is „neutral"

$$\left(\begin{array}{c} \mathbb{R} \\ id \downarrow \\ \mathbb{R} \end{array}\right) \otimes \left(\begin{array}{c} X \\ \downarrow g \\ \mathbb{R} \end{array}\right) = \begin{tikzcd} \mathbb{R}\otimes X \arrow[r, "\lambda_X"] \arrow[d, "id\otimes g"] & X \arrow[ddl, "g"] \\ \mathbb{R}\otimes\mathbb{R} \arrow[d, "\cdot"] & \\ \mathbb{R} & \end{tikzcd}$$

this triangle is λ_g in the „lifted monoidal structure"

$$\begin{tikzcd} r\otimes x \arrow[r, mapsto] \arrow[d, mapsto] & rx \arrow[dd, mapsto] \\ r\otimes g(x) \arrow[d, mapsto] & \\ r\cdot g(x) & \end{tikzcd}$$

Claim: If we restrict to the full subcategory of $\mathrm{Vect}/\mathbb{R}$ that is spanned by surjective functionals, this tensor product comes from the „normal" tensor product of affine spaces.

The proper formulation of this claim follows:

Claim: $\mathfrak{I}$ is a strict monoidal functor.

Step 1: Construct an isomorphism

$$\mathfrak{I}(\_ \hat{\otimes} \_) \longrightarrow \mathfrak{I}(\_) \otimes \mathfrak{I}(\_)$$

„By Yoneda" this is the same as to construct a bijection of homsets

$$\mathrm{Vect}/\mathbb{R}\left(\mathfrak{I}(A \hat{\otimes} B), X \xrightarrow{g} \mathbb{R}\right) \cong \mathrm{Vect}/\mathbb{R}\left(\mathfrak{I}(A) \otimes \mathfrak{I}(B), X \xrightarrow{g} \mathbb{R}\right)$$

naturally in A, B, g.

$$
\left\{
\begin{tikzcd}
(A\hat{\otimes}B)\times\mathbb{R} \arrow[rr, "F"] \arrow[dr, "p^{AB}_{\mathbb{R}}"'] & & X \arrow[dl, "g"] \\
& \mathbb{R} &
\end{tikzcd}
\right\}
\quad \xleftrightarrow{\text{bijection}} \quad
\left\{
\begin{tikzcd}
(A\times\mathbb{R})\otimes(B\times\mathbb{R}) \arrow[d, "p^A_{\mathbb{R}}\otimes p^B_{\mathbb{R}}"] \arrow[dr, "F"] & \\
\mathbb{R}\otimes\mathbb{R} \arrow[d, "\cdot"] & \cancel{X} \arrow[dl, "g"] \\
\mathbb{R} &
\end{tikzcd}
\right\}
$$

Set of all (on each side)

① For $g = 0$, the set is empty.

② If $g \neq 0$, then g is isomorphic to some $\mathcal{I}(X') = (X'\times\mathbb{R} \xrightarrow[p^{X'}_{\mathbb{R}}]{} \mathbb{R})$.

So we may consider

$$
\begin{tikzcd}
\mathcal{I}(A\hat{\otimes}B) & \mathcal{I}(X') & \\[-20pt]
(A\hat{\otimes}B)\times\mathbb{R} \arrow[r, "\mathcal{I}(F')"] \arrow[rr, bend left, "F"] \arrow[d, "p^{AB}_{\mathbb{R}}"'] & X'\times\mathbb{R} \arrow[r, "\stackrel{\sigma}{\cong}"] \arrow[dl, "p^{X'}_{\mathbb{R}}"] & X \arrow[dll, bend left, "g"] \\
\mathbb{R} & &
\end{tikzcd}
$$

such triangles are in 1-1 correspondence with affine maps $A\hat{\otimes}B \xrightarrow[F']{} X'$

On the right-hand side, we have

$$
\begin{tikzcd}
(A\times\mathbb{R})\otimes(B\times\mathbb{R}) \arrow[r, "F"] \arrow[d, "p^A_{\mathbb{R}}\otimes p^B_{\mathbb{R}}"'] & X'\times\mathbb{R} \arrow[r, "p"] \arrow[ddl, "p^{X'}_{\mathbb{R}}"] & X' \\
\mathbb{R}\otimes\mathbb{R} \arrow[d, "\cdot"'] & & \\
\mathbb{R} & &
\end{tikzcd}
$$

We need to prove that every such F determines an affine bimorphism $A\times B \longrightarrow X'$ and vice versa.

Define $\hat{F}: A\times B \longrightarrow X'$ by the rule

$$\hat{F}(x,y) = p\,(F((x,1)\otimes(y,1)))\ ; \ p \text{ is the proj.}$$

We claim that this is a bimorphism of affine spaces.

$$x = s_1x_1 + \ldots + s_nx_n \qquad s_1 + \ldots + s_n = 1$$

$$p(F((x,1)\otimes(y,1))) =$$

$$p\,(F((s_1x_1 + \ldots + s_nx_n, 1)\otimes(y,1))) =$$

$$p\,(F((s_1(x_1,1) + \ldots + s_n(x_n,1))\otimes(y,1))) =$$

$$p\left(f\left(s_1(x_1,1)\otimes(y,1)+\dots+s_n(x_n,1)\otimes(y,1)\right)\right)=$$

$$= s_1\cdot p\left(f\left((x_1,1)\otimes(y,1)\right)\right)+\dots+s_n\cdot p\left(f\left((x_n,1)\otimes(y,1)\right)\right)$$

$$= s_1\hat{F}(x_1,y)+\dots+s_n\hat{F}(x_n,y)$$

So $\hat{F}$ is an affine bimorphism.

For the opposite direction, let $b: A\times B\longrightarrow X'$ be an affine bimorphism.

Let $(x_i)_{i\in H}$ be a basis of the vector space A
$(y_j)_{j\in K}$ be a basis of the vector space B

Then $\{(x_i,0)\}_{i\in H}\cup\{(0,1)\}$ is a basis of $A\times\mathbb{R}$;

clearly,

$\{(x_i,1)\}_{i\in H}\cup\{(0,1)\}$ is another basis of $A\times\mathbb{R}$ and

$\{(y_j,1)\}_{j\in K}\cup\{(0,1)\}$ is a basis of $B\times\mathbb{R}$.

Therefore, there is a basis of

$(A\times\mathbb{R})\otimes(B\times\mathbb{R})$

$\{(x,1)\otimes(y,1) : x \in \{x_i\}_{i\in H} \cup \{0\},\ y \in \{y_j\}_{j\in K} \cup \{0\}\}$;

for the elements of this basis, put

$$f((x,1)\otimes(y,1)) = b(x,y).$$

This defines a linear map

$$f : (A\times\mathbb{R})\otimes(B\times\mathbb{R}) \longrightarrow X'.$$

We need to prove that $\hat{f} = b$, that means, that $f((z,1)\otimes(w,1)) = b(z,w)$ for <u>all</u> $z\in A$, $w \in B$.

$$z = \sum c_i x_i \qquad w = \sum d_j y_j$$

$$(z,1) = \sum_i c_i(x_i,1) + (1-\sum_i c_i)(0,1)$$

$$(w,1) = \sum_j d_j(y_j,1) + (1-\sum_j d_j)(0,1)$$

$$f((z,1)\otimes(w,1)) =$$

$$f\Big(\big(\sum_i c_i(x_i,1) + (1-\sum_i c_i)(0,1)\big)\otimes \big(\sum_j d_j(y_j,1) + (1-\sum_j d_j)(0,1)\big)\Big) =$$

$$= f\Big(\Big[\sum_{i,j} c_i d_j (x_i,1)\otimes(y_j,1)\Big] +$$

$$+\left[\sum_i c_i\left(1-\sum_j d_j\right)(x_i,1)\otimes(0,1)\right]+$$

$$+\left[\sum_j\left(1-\sum_i c_i\right)d_j(0,1)\otimes(y_j,1)\right]+$$

$$+\left[\left(1-\sum_i c_i\right)\left(1-\sum_j d_j\right)(0,1)\otimes(0,1)\right]\Bigg)=$$

$$=\sum_i c_i\left[\sum_j d_j\, b(x_i,y_j)+\left(1-\sum_j d_j\right)b(x_i,0)\right]+$$

$$+\left(1-\sum_i c_i\right)\left[\sum_j d_j\, b(0,y_j)+\left(1-\sum_j d_j\right)b(0,0)\right]$$

$$=\sum_i c_i\left[b(x_i,w)\right]+\left(1-\sum_i c_i\right)b(0,w)=$$

b is an affine bimorphism

$$=b(z,w)$$

again

NATURALITY