VE281

Data Structures and Algorithms

Hashing

Announcement

- Programming Assignment Two posted
  - Due time: midnight, Oct. 23rd

Outline

- Deterministic selection algorithm

- Review of Dictionary
- Hashing Basics
- Hash Function
- Collision Resolution: Separate Chaining

Review: Deterministic Selection Algorithm

```
Dselect(int A[], int n, int i) {
// find i-th smallest item of array A of size n
  if(n == 1) return A[1];
  Break A into groups of 5, sort each group;
  C = n/5 medians;
  p = Dselect(C, n/5, n/10);              ChoosePivot
  Partition A using pivot p;
  Let j be the index of p;
  if(j == i) return p;
  if(j > i) return Dselect(1st part of A, j-1, i);
  else return Dselect(2nd part of A, n-j, i-j);
}
```

The function has two recursive calls

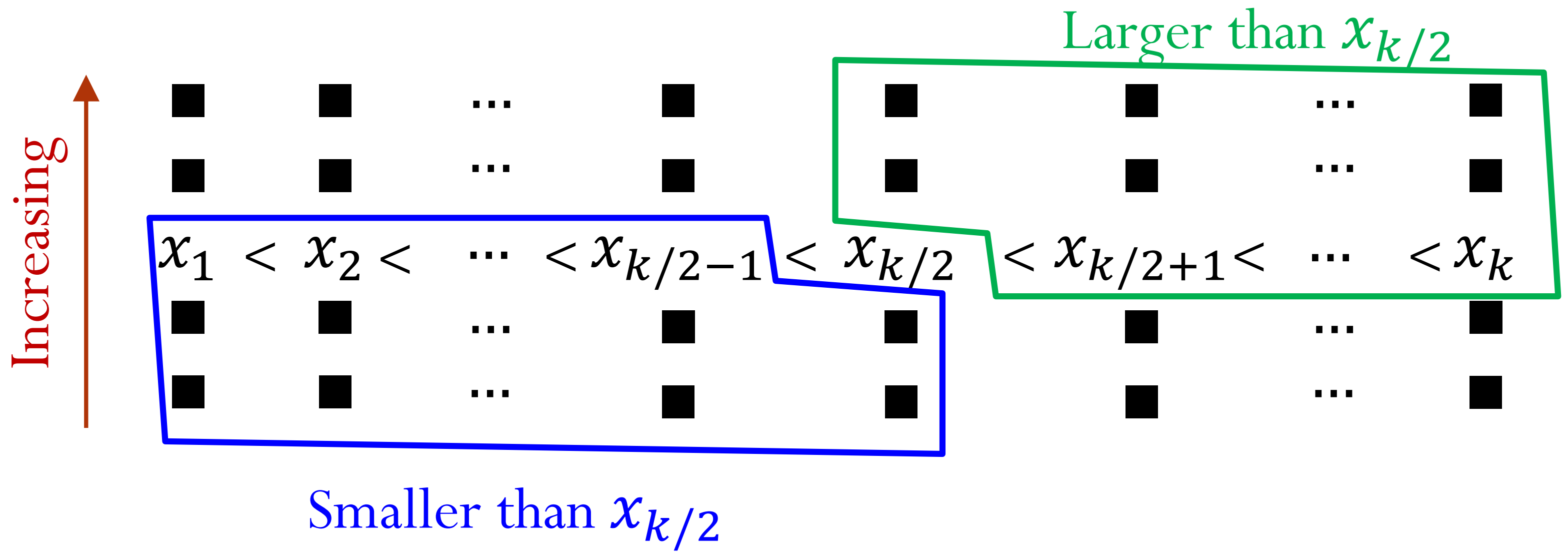
Review: Runtime of Dselect

- There exists a positive constant c such that
  - $T(1) \leq c$
  - $T(n) \leq cn + T\left(\frac{n}{5}\right) + T(?)$

- The next question is what is the size of the array of the second recursive call
  - <u>Lemma</u>: 2nd recursive call guaranteed to be on an array of size $\leq 0.7n$ (roughly)

Proof of Lemma

- Imagine we layout elements of A in a 2-D grid



- At least ~ (3/5)*(1/2) = 30% elements smaller than $x_{k/2}$
- At least ~ 30% elements larger than $x_{k/2}$
- <u>Result</u>: Number of elements < $x_{k/2}$ is in between 30% and 70%. The same for number of elements > $x_{k/2}$

Example

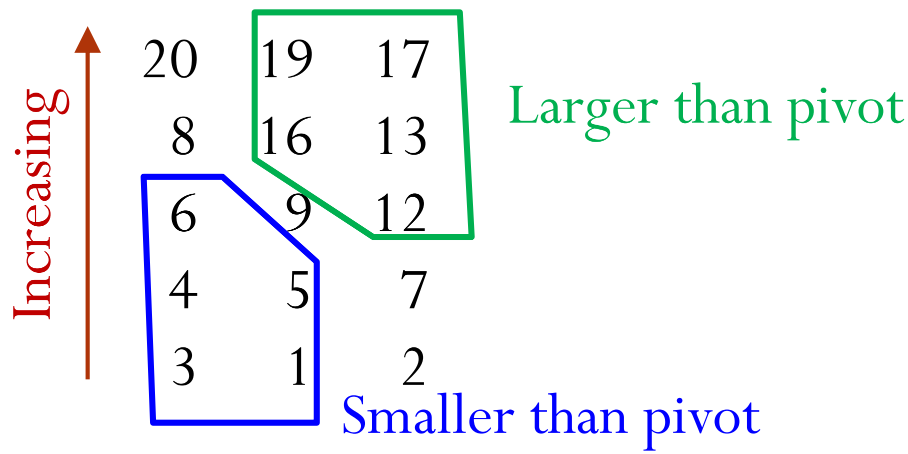
- Input:

  7,2,17,12,13,8,20,4,6,3,19,1,9,5,16

- After sorting each group of 5 elements

  2,7,12,13,17, 3,4,6,8,20, 1,5,9,16,19

pivot



Recurrence

- There exists a positive constant c such that
  - $T(1) \leq c$
  - $T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$
- Note: different-sized sub-problems. Cannot use master method!
- How can we solve this?
  - Strategy: Hope and check
- Hope: there is a constant a (independent of n) such that $T(n) \leq an$ for all $n > 1$
  - Then $T(n) = O(n)$
- We choose $a = 10c$

Proof $T(n) = O(n)$

- Claim: suppose there exists a positive constant c such that
  1. $T(1) \leq c$
  2. $T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$

  Then $T(n) \leq 10cn$
- Proof by induction
  - Base case: $T(1) \leq 10c$
  - Inductive step: inductive hypothesis $T(k) \leq 10ck, \forall k < n$. Then

$$T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) \leq cn + 2cn + 7cn = 10cn$$

Dselect runs in linear time

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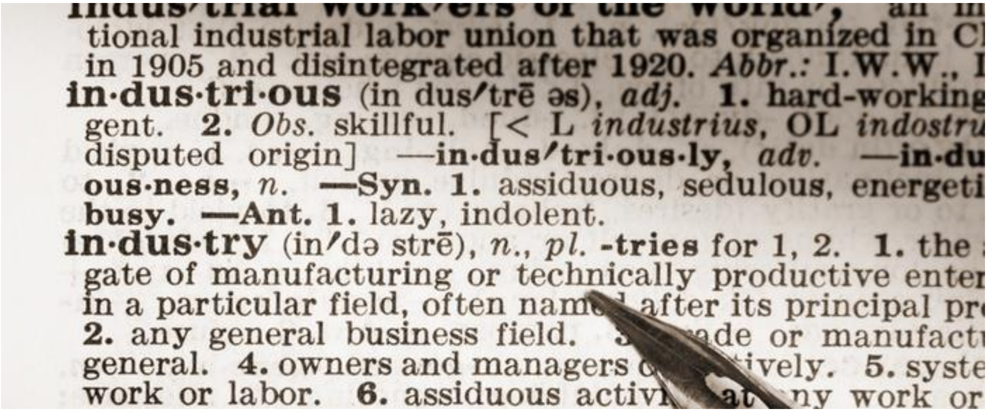
Dictionary

- How do you use a dictionary?
  - Look up a "word" and find its meaning.
- We also have an abstract data type called dictionary.
  - It is a collection of pairs, each containing a **key** and an **value**

    `(key, value)`

  - **Important**: Different pairs have different keys.



Dictionary

- Key space is usually more regular/structured than value space, so easier to search.

- Dictionary is optimized to quickly **add** **`(key, value)`** pair and **retrieve value** by key.

Methods

- **`Value find(Key k)`** : Return the value whose key is **`k`**. Return **`Null`** if none.
- **`void insert(Key k, Value v)`** : Insert a pair **`(k, v)`** into the dictionary. If the pair with key as **`k`** already exists, update its value.
- **`Value remove(Key k)`** : Remove the pair with key as **`k`** from the dictionary and return its value. Return **`Null`** if none.

Runtime for Array Implementation

`Pair Array[MAXSIZE]:`

a	b	c	d			

- Unsorted array
  - find() $O(n)$
  - insert() $O(n)$: $O(n)$ to verify duplicate, $O(1)$ to put at the end
  - remove() $O(n)$: $O(n)$ to verify existence, $O(1)$ to exchange the "hole" with the last element
- Sorted array
  - find() $O(\log n)$: binary search
  - insert() $O(n)$: $O(\log n)$ to verify duplicate, $O(n)$ to insert
  - remove() $O(n)$: $O(\log n)$ to verify existence, $O(n)$ to remove

Can we do **`find`**, **`insert`**, and **`remove`** in $O(1)$ time?

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Hashing: High-Level Idea

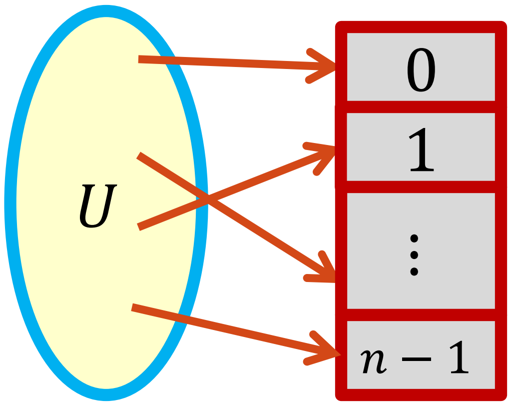
- **Setup**: A universe U of objects
  - E.g., All names, all IP addresses, etc.
  - Generally, very BIG!
- **Goal**: Want to maintain a evolving set $S \subseteq U$
  - E.g., 200 students, 500 IP addresses
  - Generally, of reasonable size.
- Naïve solutions

1. Array-based solution (index by $u \in U$)
   - $\Theta(1)$ operation time, BUT $\Theta(|U|)$ space.
2. Linked list-based solution:
   - $\Theta(|S|)$ space, BUT $\Theta(|S|)$ operation time.

Can we get the best of both solutions?

Hashing: High-Level Idea

- Solution:
  - Pick an array A of n buckets.
    - $n = c|S|$: a small multiple of $|S|$.
  - Choose a hash function $h: U \rightarrow \{0,1,\ldots,n-1\}$
    - h is fast to compute.
    - The same key is always mapped to the **same** location.
  - Store item k in A[h(k)]



- The array is called **hash table**
  - An array of **buckets**, where each bucket contains items as assigned by a hash function.
  - h[k] is called the **home bucket** of key k.

Hashing Example

- Pairs are: (22,a), (33,b), (3,c), (73,d), (85,e)
- Hash table is **`A[0:7]`** and table size is **`M = 8`**
- Hash function is **`h[key] = key/11`**
- Every item with **`key`** is stored in the bucket **`A[h(key)]`**

(3,c)		(22,a)	(33,b)			(73,d)	(85,e)
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]

Question: What is the time complexity for **`find()`**, **`insert()`**, and **`remove()`**?

O(1)

What Can Go Wrong?

(3,c)		(22,a)	(33,b)			(73,d)	(85,e)
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]

- Where does (35, g) go?
- Problem: The home bucket for (35, g) is already occupied!
  - This is a "**collision**".

Collision and Collision Resolution

- Collision occurs when the hash function maps two or more items—all having **different** search keys—into the **same** bucket.

- What to do when there is a collision?
  - **Collision-resolution scheme**: assigns distinct locations in the hash table to items involved in a collision.

- Two major schemes:
  - Separate chaining
  - Open addressing

Insight of Collision: Birthday Problem

- Consider n people with random birthdays (i.e., with each day of the year equally likely). What is the smallest n so that there is at least a 50% chance that two people have the same birthday?
  A. 23 ✓
  B. 57
  C. 184
  D. 367

Collision is inevitable!

Hash Table Issues

- Choice of the hash function.
- Collision resolution scheme.
- Size of the hash table and rehashing.

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Hash Function Design Criteria

- Must compute a bucket for every key in the universe.
- Must compute the same bucket for the same key.
- Should be easy and quick to compute.
- Minimizes collision ← The hardest criterion
  - Spread keys out evenly in hash table
  - **Gold standard**: **completely random hashing**
    - The probability that a randomly selected key has bucket i as its home bucket is $1/n$, $0 \leq i < n$.
    - Completely random hashing **minimizes** the likelihood of an collision when keys are selected at random.
    - However, completely random hashing is **<u>infeasible</u>** due to the need to remember the random bucket.

Bad Hash Functions

- Example: keys = phone number in China (11 digits)
  - $|U| = 10^{11}$
  - **Terrible** hash function: $h(key)$ = first 3 digits of key, i.e., area code
    - The keys are not spread out evenly. Buckets 010, 021 may have a lot of keys mapped to them, while some buckets have no keys.
  - **Mediocre** hash function: $h(key)$ = last 3 digits of key.
    - Still vulnerable to patterns in last 3 digits.

Hash Functions

- Hash function ($h(key)$) maps key to buckets in two steps:

1. Convert key into an integer in case the key is not an integer.
   - A function $t(key)$ which returns an integer value, known as **hash code**.
2. **Compression map**: Map an integer (hash code) into a home bucket.
   - A function $c(hashcode)$ which gives an integer in the range $[0, n-1]$, where n is the number of buckets in the table.

- In summary, $h(key) = c(t(key))$, which gives an index in the table.

Map Non-integers into Hash Code

- String: use the ASCII (or UTF-8) encoding of each char and then perform arithmetic on them.

- Floating-point number: treat it as a string of bits.

- Images, (viral) code snippets, (malicious) Web site URLs: in general, treat the representation as a bit-string, using all of it or **extracting** parts of it (i.e., www.**abc**.com.**cn**).

Strings to Integers

- Simple scheme: adds up all the ASCII codes for all the chars in the string.
  - Example: t("He") = 72 + 101 = 173.

- Not good. Why?
  - Consider English words "post", "pots", "spot", "stop", "tops".

Strings to Integers

- A better strategy: Polynomial hash code taking **positional** info into account.

$$t(s[]) = s[0]a^{k-1} + s[1]a^{k-2} + \cdots + s[k-2]a + s[k-1]$$

where a is a constant.

- If $a = 33$, the hash codes for "post" and "stop" are

$$t(\text{post}) = 112 \cdot 33^3 + 111 \cdot 33^2 + 115 \cdot 33 + 116 = 4149734$$
$$t(\text{stop}) = 115 \cdot 33^3 + 116 \cdot 33^2 + 111 \cdot 33 + 112 = 4262854$$

Strings to Integers

$$t(s[]) = s[0]a^{k-1} + s[1]a^{k-2} + \cdots + s[k-2]a + s[k-1]$$

- Good choice of a for English words: 31, 33, 37, 39, 41
  - What does it mean for a to be a **good** choice? Why are these particular values **good**?
  - Answer: according to statistics on 50,000 English words, each of these constants will produce less than 7 collisions.
- In Java, its `string` class has a built-in `hashCode()` function. It takes $a = 31$. Why?
  - Multiplication by 31 can be replaced by a shift and a subtraction for **better performance**: `31*i == (i << 5) - i`

Hash function criteria: Should be easy and quick to compute.

Compression Map

- Map an integer (hash code) into a home bucket.
- The most common method is by **modulo arithmetic**.

  `homeBucket = c(hashcode) = hashcode % n`

  where n is the **number of buckets** in the hash table.

- Example: Pairs are (22,a), (33,b), (3,c), (55,d), (79,e). Hash table size is 7.

	(22,a)	(79,e)	(3,c)		(33,b)	(55,d)
[0]	[1]	[2]	[3]	[4]	[5]	[6]

Hashing by Modulo

- In practice, keys of an application tend to have a specific pattern
  - For example, memory address in computer is multiple of 4.

- The choice of the hash table size n will affect the distribution of home buckets.

Hashing by Modulo

- Suppose the keys of an application are more likely to be mapped into even integers.
  - E.g., memory address is always a multiple of 4.

- When the hash table size n is an **even** number, **even** integers are hashed into **even** home buckets.
  - E.g., n = 14: 20%14 = 6, 30%14 = 2, 8%14 = 8

- The bias in the keys results in a bias toward the **even** home buckets.
  - All **odd** buckets are **guaranteed** to be empty.
  - The distribution of home buckets is not uniform!

Hashing by Modulo

- However, when the hash table size n is **odd**, even (or odd) integers may be hashed into both odd and even home buckets.
  - E.g., n = 15: 20%15 = 5, 30%15 = 0, 8%15 = 8
    15%15 = 0, 3%15 = 3, 23%15 = 8
- The bias in the keys does not result in a bias toward either the odd or even home buckets.
  - Better chance of uniform distribution of home buckets.
- So **do not** use an even hash table size n.

Hashing by Modulo

- Similar **biased** distribution of home buckets happens in practice when the hash table size n is a multiple of small prime numbers.

- The effect of each prime divisor p of n **decreases** as p gets **larger**.

- Ideally, choose the hash table size n as a **large prime number**.

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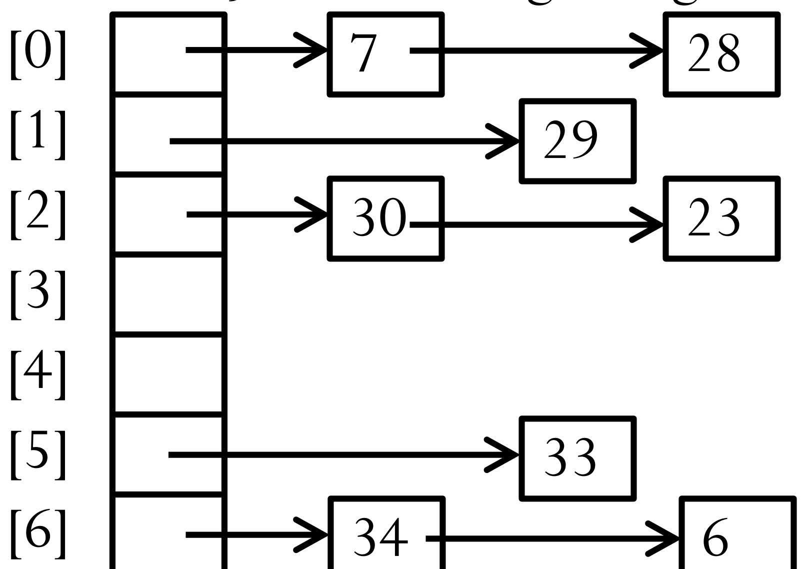
Collision Resolution Scheme

- **Collision-resolution scheme**: assigns distinct locations in the hash table to items involved in a collision.
- Two major scheme:
  - Separate chaining
  - Open addressing

Separate Chaining

- Each bucket keeps a **linked list** of all items whose home buckets are that bucket.
- Example: Put pairs whose keys are 6, 23, 34, 28, 29, 7, 33, 30 into a hash table with $n = 7$ buckets.
  - **`homeBucket = key % 7`**
  - **Note**: we insert object at the beginning of a linked list.



Separate Chaining

- **`Value find(Key key)`**
  - Compute **`k = h(key)`**
  - Search in the linked list located at the k-th bucket (e.g., check every entry) with the key.

- **`void insert(Key key, Value value)`**
  - Compute **`k = h(key)`**
  - Search in the linked list located at the k-th bucket. If found, update its value; otherwise, insert the pair at the beginning of the linked list in O(1) time.

Separate Chaining

- **`Value remove(Key key)`**
  - Compute **`k = h(key)`**
  - Search in the linked list located at the k-th bucket. If found, remove that pair.