Modelling of Complex Systems

Asymmetric One-dimensional Random Walks

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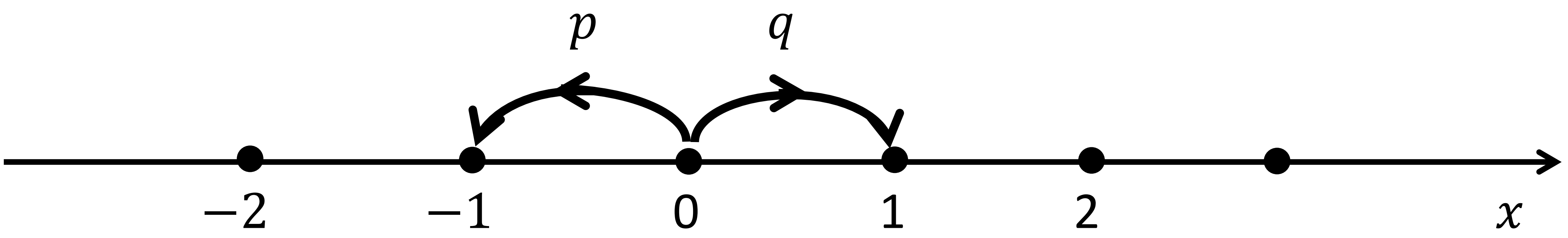
We consider the following process of random *jumps* (displacements) on a one-dimensional periodic lattice:



A particle jumps with probability p on the left and with probability q on the right:

$$p + q = 1$$

Asymmetric jumps can be caused by an external field:

$$p = \frac{1}{2} - \delta$$

$$q = \frac{1}{2} + \delta$$

After t jumps the particle jumped n_+ times to the right and n_- times to the left. Then, we have equalities:

$$\begin{cases} t = n_+ + n_- \\ x = n_+ - n_- \end{cases} \qquad \begin{cases} n_+ = \frac{t+x}{2} \\ n_- = \frac{t-x}{2} \end{cases}$$

The probability that the particle is at point x after t jumps is

$$P(x,t) = C^t_{n_+} p^{n_-} q^{n_+} = C^t_{n_+} \left(\frac{1}{2} - \delta\right)^{n_-} \left(\frac{1}{2} + \delta\right)^{n_+},$$

where:

$p^{n_-} q^{n_+}$ is the probability one trajectory with n_- jumps to the left and n_+ jumps on the right,

$C^t_{n_+} = C^t_{n_-} = \frac{t!}{n_+!\, n_-!}$ is the number of trajectories with n_- jumps to the left and n_+ jumps on the right.

We can write $\left(\frac{1}{2} \pm \delta\right) = \frac{1}{2}(1 \pm 2\delta)$. Therefore,

$$P(x,t) = C_{n_+}^t \left(\frac{1}{2} - \delta\right)^{n_-} \left(\frac{1}{2} + \delta\right)^{n_+} = C_{n_+}^t \frac{1}{2^t}(1-2\delta)^{n_-}(1+2\delta)^{n_+}$$

where we used the fact that $2^{n_-}2^{n_+} = 2^{n_-+n_+} = 2^t$.

This product is the probability $P(x,t)$ for symmetric jumps ($\delta = 0$), as was shown in lecture 3:

$$\frac{C_{n_+}^t}{2^t} \approx \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

We deal with the extra factors $(1 \pm 2\delta)^{n_\pm}$ as follows:

$$(1+2\delta)^{n_+} = \exp[n_+ \ln(1+2\delta)]$$
$$\approx \exp\left[n_+\left(2\delta - \frac{1}{2}(2\delta)^2\right)\right] = \exp[2n_+(\delta-\delta^2)],$$

where we used the Taylor expansion $\ln(1+x) \approx x - \frac{1}{2}x^2$ (assuming $\delta \ll 1$).

Therefore

$$(1-2\delta)^{n_-}(1+2\delta)^{n_+} \approx \exp[2n_-(-\delta-\delta^2) + 2n_+(\delta-\delta^2)]$$
$$= \exp(2(n_+ - n_-)\delta - 2(n_+ + n_-)\delta^2) = \exp(2x\delta - 2t\delta^2)$$

Then, the probability distribution at large t, and $x \ll t$ is:

$$P(x,t) = C_{n_+}^t \frac{1}{2^t}(1-2\delta)^{n_-}(1+\delta)^{n_+} \approx \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} e^{2x\delta - 2t\delta^2}$$

We can write

$$-\frac{x^2}{2t} + 2x\delta - 2t\delta^2 = -\frac{1}{2t}(x-2\delta t)^2$$

Thus, in the case of asymmetric jumps, the **probability density** is

$$P(x,t) \approx \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} e^{2x\delta - 2t\delta^2} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-2\delta t)^2}$$

Let us introduce

$$v_d = 2\delta,$$

which is the *drift velocity* of the maximum of $P(x,t)$.
Then

$$P(x,t) \approx \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-v_d t)^2}$$

At a given time t, probability density of finding the particle at the position where $P(x,t)$ is maximum, i.e., at $x = \langle x \rangle(t) = v_d t$, is

$$P(x = v_d t, t) = \frac{1}{\sqrt{2\pi t}}$$

The normalization:

$$\int_{-\infty}^{\infty} P(x,t)dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{2t}(x-v_d t)^2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = 1$$

where $y = (x - v_d t)/\sqrt{t}$.

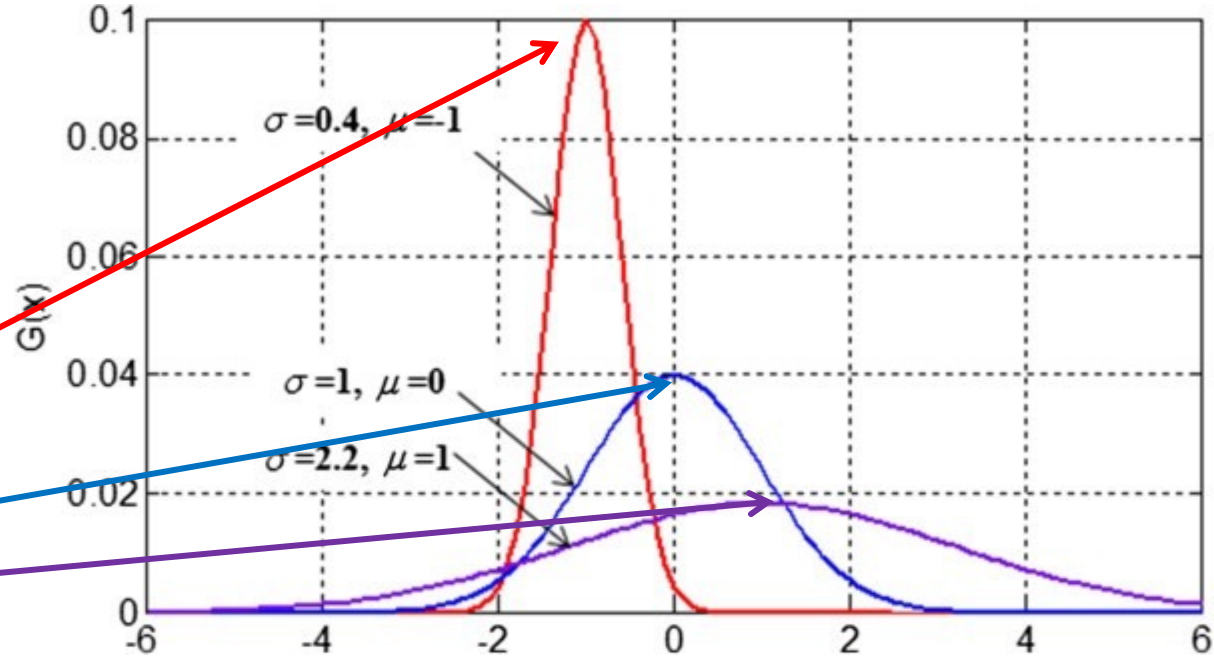
Probability to observe the particle at $x = v_d t$ is equal to $P(v_d t, t) = \frac{1}{\sqrt{2\pi t}}$.

This probability decreases with time t.

The width of the peak increases with t.

The mean value

$$\langle x \rangle = \int_{-\infty}^{\infty} xP(x,t)dx = v_d t.$$

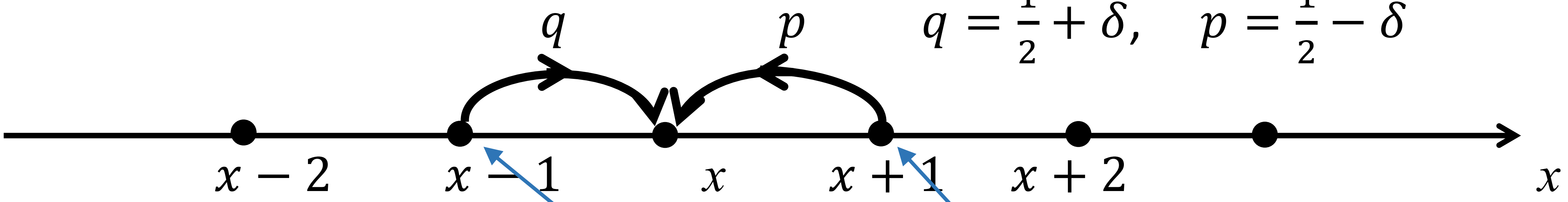


$t_1 < t_2 < t_3$

Master equation approach

We consider asymmetric random walks of a particle on a 1D-lattice

$$q = \frac{1}{2} + \delta, \quad p = \frac{1}{2} - \delta$$



Again, $P(x,t)$ is the probability that at time t the particle is at a point x.

Let us find a relation between $P(x,t)$ and $P(x,t+1)$.

The master equation:

$$P(x,t+1) = qP(x-1,t) + pP(x+1,t).$$

The initial condition is $P(x,t=0) = \delta_{x,0}$. The master equation takes a form

$$P(x,t+1) = \left(\frac{1}{2} + \delta\right)P(x-1,t) + \left(\frac{1}{2} - \delta\right)P(x+1,t)$$

$$= \boxed{\frac{1}{2}P(x-1,t) + \frac{1}{2}P(x+1,t)} - \delta[P(x+1,t) - P(x-1,t)]$$

These terms correspond to the symmetric jumps.

We want to find the distribution function $P(x,t)$ over x at $t \gg 1$. We assume that $P(x,t)$ varies slowly in time and space. We use the Taylor expansion:

$$P(x, t + \Delta t) = P(x,t) + \frac{\partial P(x,t)}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 P(x,t)}{\partial t^2}(\Delta t)^2 + \cdots$$

$$P(x + \Delta x, t) = P(x,t) + \frac{\partial P(x,t)}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 P(x,t)}{\partial x^2}(\Delta x)^2 + \cdots$$

where $\Delta t = \pm 1, \Delta x = \pm 1$.

The Taylor expansion gives

$$P(x+1,t) - P(x-1,t) = 2\frac{\partial P(x,t)}{\partial x}$$

The master equation takes a form

$$P(x,t+1) = \frac{1}{2}P(x-1,t) + \frac{1}{2}P(x+1,t) - \delta[P(x+1,t) - P(x-1,t)]$$

$$= P(x,t) + \frac{1}{2}\frac{\partial^2 P(x,t)}{\partial x^2} - 2\delta\frac{\partial P(x,t)}{\partial x}$$

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 P(x,t)}{\partial x^2} - v_d \frac{\partial P(x,t)}{\partial x}$$

This is the well-known diffusion equation with a drift velocity $v_d = 2\delta$ in a one-dimensional lattice.

In a general case

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} - v_d \frac{\partial P(x,t)}{\partial x}$$

D is the diffusion coefficient. In a regular lattice $D = 1/2$.

The general solution of the diffusion equation with a drift is

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x - v_d t)^2}{4Dt}}$$

which gives the density of particles at point x at time t.