Modelling of Complex Systems

Complex networks

- Scale-free networks
- Erdős–Rényi networks
- Diameter of small-world networks
- Clustering coefficient
- Degree-degree correlations

Complex networks

Small-world networks

- Milgram's experiment discussed in the previous lecture.
- Watts-Strogatz model discussed in the previous lecture.
- Preferential attachment (rich become richer) produces power-law degree distributions observed in many real networks.
  - **Scale-free networks** $P(q) \propto q^{-\gamma}$, $(2 < \gamma \leq 3)$.
  - Short distances between nodes (boosting synchronization and spreading processes).
  - Robustness against damage.
  - Fast propagation of disease and information.

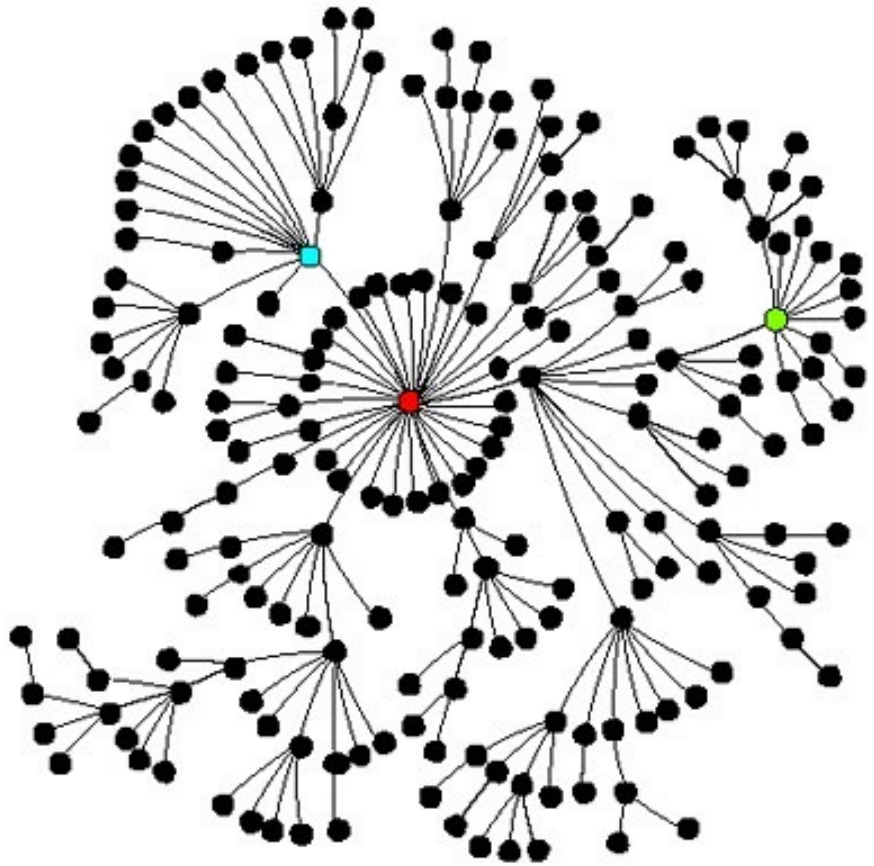
Producing scale-free networks with the Barabási-Albert model:

New nodes arriving to the network connect to an older node i of degree q_i with probability proportional to q_i (rich become richer).

Complex networks

Real scale-free networks

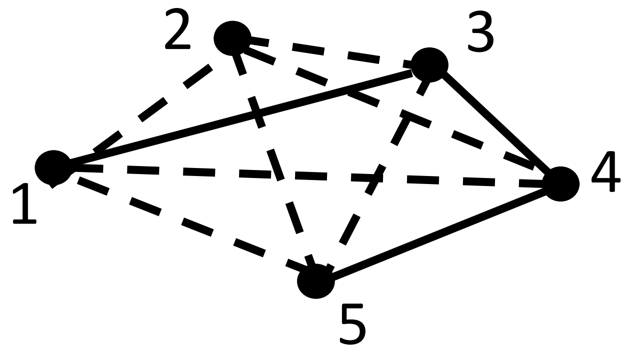
- WWW (2022):
  Size $N \sim 2\times10^9$ webpages,
  Number of links $L \sim 10^{11}$.
  Exponent $\gamma = 2.1 \sim 2.45$; diameter $l \approx 18$.
- Internet (2015):
  Size $N \sim 10^8$ servers,
  Number of connections $L \sim 10^9$.
  Exponent $\gamma \approx 2.2$; diameter $l \approx 5$.
- Brain:
  Size $N \sim 10^{11}$ neurons,
  Number of connections $L \sim 10^{14}$.
  Exponent $\gamma = 2.0 \sim 2.2$; diameter $l \approx 3$.



Complex networks

Erdős–Rényi networks

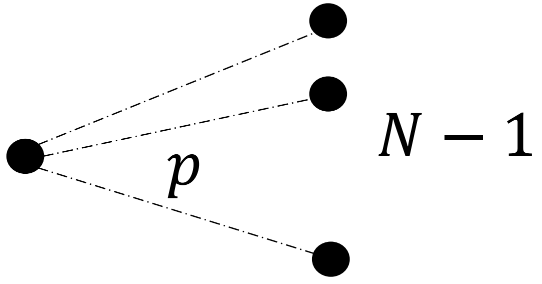
There are N nodes and L edges. We start from N isolated nodes, and insert these L edges one by one between two randomly chosen nodes.



Mean degree is $\langle q \rangle = 2L/N$.

Classical random network

There are N nodes. We make an edge between any of the $\frac{N(N-1)}{2}$ pairs of nodes independently, with probability $p = \frac{\langle q \rangle}{N-1}$, where $\langle q \rangle$ is a parameter.



Probability that a node has q edges is given by the binomial function $P(q) = C_q^{N-1} p^q (1-p)^{N-1-q}$.

Complex networks

For large networks, i.e., $N \gg 1$, and $\langle q \rangle \ll N$ the two models are *equivalent* (networks have the same properties) as long as $p(N-1) = 2L/N$.

In these limits, the binomial distribution approaches the Poisson distribution

$$P(q) = \frac{[(N-1)p]^q}{q!} e^{-(N-1)p}$$

Taking into account that the mean number of edges is

$$\langle q \rangle = p(N-1),$$

we find the degree distribution function of the Erdős–Rényi network:

$$P(q) = \frac{\langle q \rangle^q}{q!} e^{-\langle q \rangle}$$

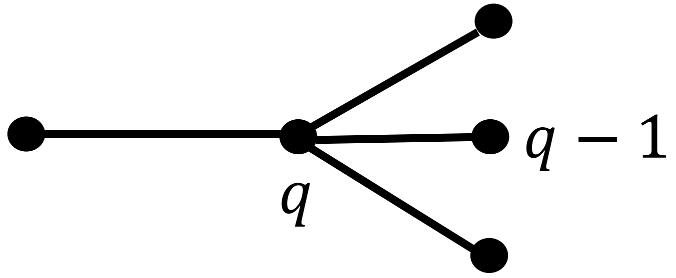
This function is very different from a scale-free distribution $P(q) \propto q^{-\gamma}$ observed in real networks (WWW, Internet, brain networks, etc).

In light of this, what can we say about the connection rule in real networks?
It is better described by preferential attachment than by random connections.

Complex networks

Branching number

The branching coefficient B is the mean number of edges emanating from a nearest neighbour



$$B = \sum_{q=0}^{\infty} \frac{qP(q)}{\langle q \rangle}(q-1) = \frac{\langle q(q-1)\rangle}{\langle q \rangle} = \frac{\langle q^2 \rangle}{\langle q \rangle} - 1$$

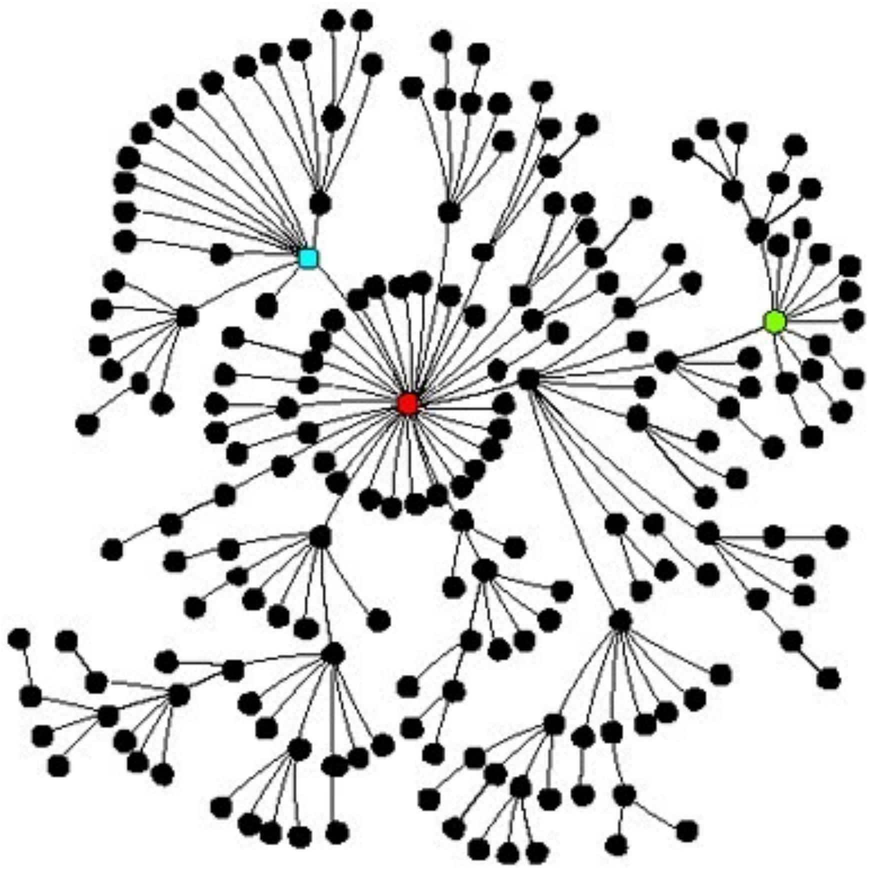
This shows why scale-free networks have many ‘special’ properties:
Their branching number is infinite: $P(q) \propto q^{-\gamma}$, $(2 < \gamma \leq 3)$ leads to finite $\langle q \rangle$ and infinite $\langle q^2 \rangle$.
On average, each node has a finite number of first-neighbours and an infinite number of second-neighbours.

Complex networks

Diameter

Let us estimate the diameter of a complex network.

For this purpose we assume that each node has $\langle q \rangle$ nearest neighbours and the branching number $\boldsymbol{B}$ is finite and equal to all links. Let us find the number of nodes at distance l from an arbitrary chosen node. We assume that the network is **locally tree-like**.



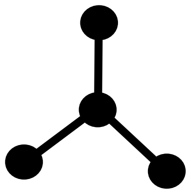
Complex networks

Diameter

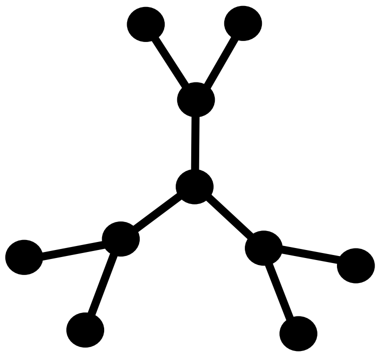
We start from a node and visit all nearest neighbours recursively.

number of nodes reached

$$1$$



$$1 + \langle q \rangle$$



$$1 + \langle q \rangle + \langle q \rangle B$$

After l steps we have:

$$N = 1 + \langle q \rangle + \langle q \rangle B + \cdots \langle q \rangle B^{l-1} = 1 + \langle q \rangle \frac{B^l - 1}{B - 1}$$

Complex networks

Diameter

Assuming that after l steps we reach all nodes and $N, B, l \gg 1$, we get an estimation of the diameter l by equating the number visited nodes with N:

$$N \approx \langle q \rangle B^{l-1} \Rightarrow l = 1 + \frac{\ln[N/\langle q \rangle]}{\ln B} \approx \frac{\ln N}{\ln B}$$

For example, the ER network has $B = \langle q \rangle$. Thus, for large N we get

$$l_{ER} = \frac{\ln N}{\ln \langle q \rangle}$$

In contrast to a D-dimensional system with the same size $N = Al^D$, then

$$l(D) = \left(\frac{N}{A}\right)^{1/D}$$

For a given size N, the diameter of any D-dimensional system is much larger than the diameter in a network of the same size, that is $l(D) \gg l(network)$, because $N^{1/D} \gg \ln N$. **Complex networks are small worlds.**

Complex networks

Diameter

The relation between the diameter l and network size N can be viewed in a different perspective.

Since we expect the size of D-dimensional system to grow with its linear dimension as $N = Al^D$, let us define dimensionality of a system as the limit

$$D = \lim_{N \to \infty} \frac{\ln N}{\ln l}$$

For small-world networks where $N \propto B^l$ we have

$$\ln l \approx \ln \ln N$$

Therefore,

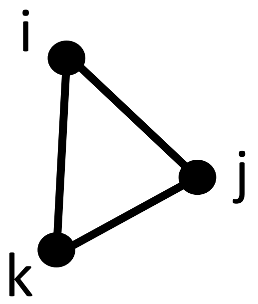
$$D = \lim_{N \to \infty} \frac{\ln N}{\ln \ln N} \to \infty$$

These small-world networks are infinite dimensional systems.

Complex networks

Clustering coefficient

Real networks are not tree-like because there are many loops. A simplest loop is a triangle formed by three inter-connected nodes:



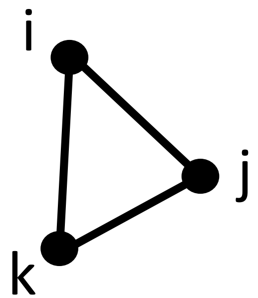
In order to characterize the prevalence of triangles in a network we introduce the so-called “clustering coefficient”

$$C = \frac{N_{tr}}{N_{ptr}}$$

where N_{tr} is the total number of triangles in the network, N_{ptr} is the number of possible triangles.

Complex networks

Clustering coefficient



The number of triangles in the network is

$$N_{tr} = \frac{1}{2}\frac{1}{3}\sum_{i \neq j \neq k} a_{ij}a_{jk}a_{ki} = \frac{1}{6} trA^3$$

The number of possible triangles is defined as

$$N_{ptr} = \frac{1}{6}\sum_{i=1}^{N} q_i(q_i - 1) = \frac{N}{6}\sum_{q} q(q-1)P(q) = \frac{N}{6}\langle q(q-1)\rangle$$

Complex networks

Clustering coefficient

The clustering coefficient is

$$C = \frac{N_{tr}}{N_{ptr}} = \frac{tr\hat{A}^3}{N\langle q(q-1)\rangle}$$

For Erdős–Rényi networks we have

$$\langle q(q-1)\rangle = \langle q\rangle^2$$

We can make the following estimation of trA^3:

$$trA^3 = \sum_{i\neq j\neq k} a_{ij}a_{jk}a_{ki} = \sum_{i\neq j\neq k} p^3 = N(N-1)(N-2)p^3 \approx \langle q\rangle^3$$

where we used the probability $p = \langle q\rangle/(N-1)$.

Therefore, in ER networks the clustering coefficient equals

$$C \approx \frac{\langle q\rangle}{N}$$

Complex networks

Degree-degree correlations

Erdős–Rényi networks are uncorrelated random graphs. There are no correlations between degrees of nodes. In real nodes there are degree-degree correlations. These correlations are quantified by the "Pearson coefficient".

$$r = \frac{\sum_{i \neq j}^{N} a_{ij}(q_i - Q)(q_j - Q)}{\sigma^2 \sum_{i \neq j}^{N} a_{ij}}$$

Here Q is the mean degree of the vertex at the end of a random edge,

$$Q = \frac{\langle q^2 \rangle}{\langle q \rangle}$$

and σ^2 is a normalization factor,

$$\sigma^2 = \frac{\langle q^3 \rangle}{\langle q \rangle} - \frac{\langle q^2 \rangle^2}{\langle q \rangle^2}$$

In Erdős–Rényi networks the Pearson coefficient is $r = 0$.