Modelling of Complex Systems

Complex networks

Internet

World Wide Web

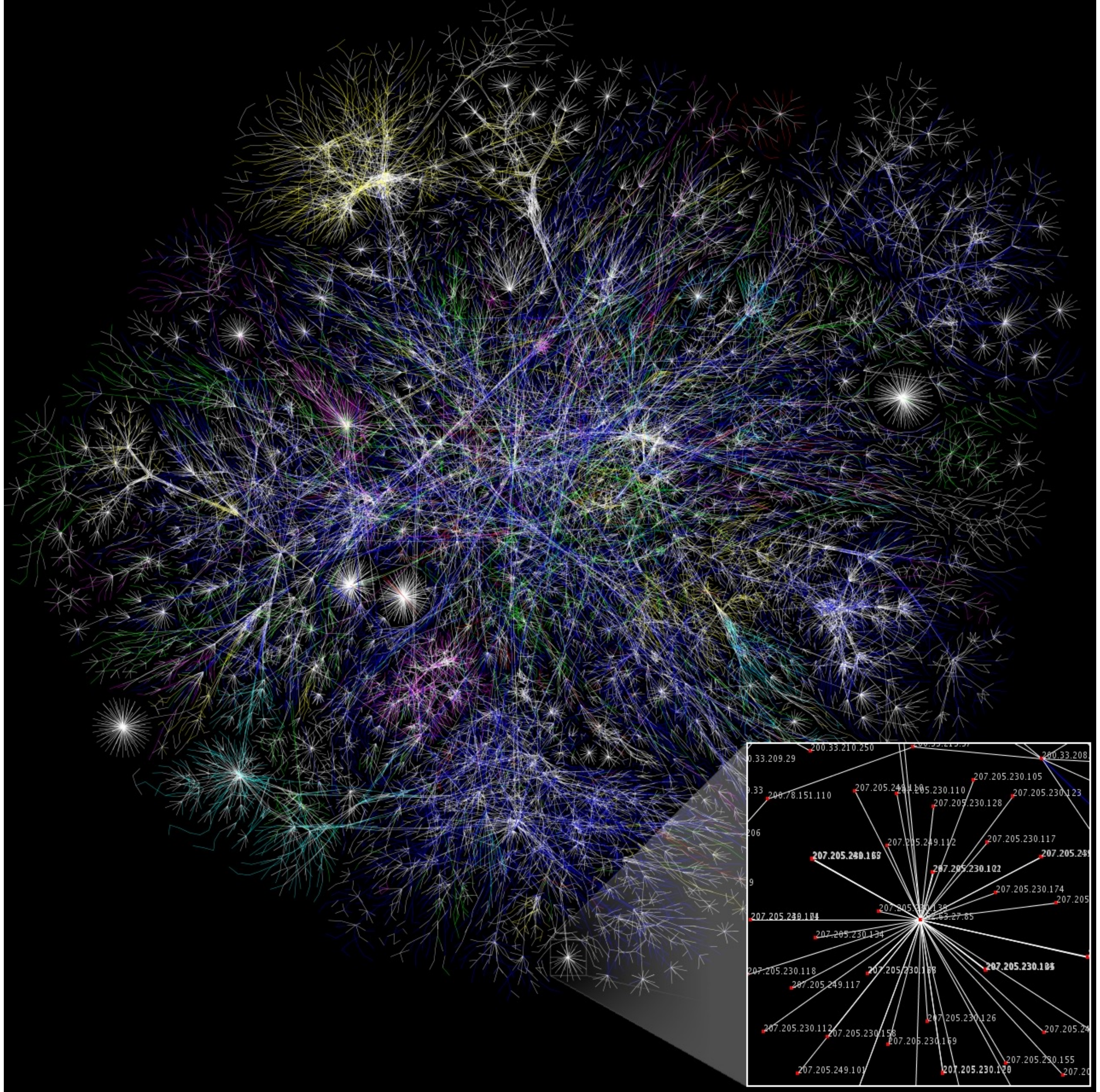
Brain

Biological networks

Transportation networks

Social networks (facebook, twitter, ...)

Internet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207.205.230.118
207.205.249.117
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207.205.230.169
207.205.230.155
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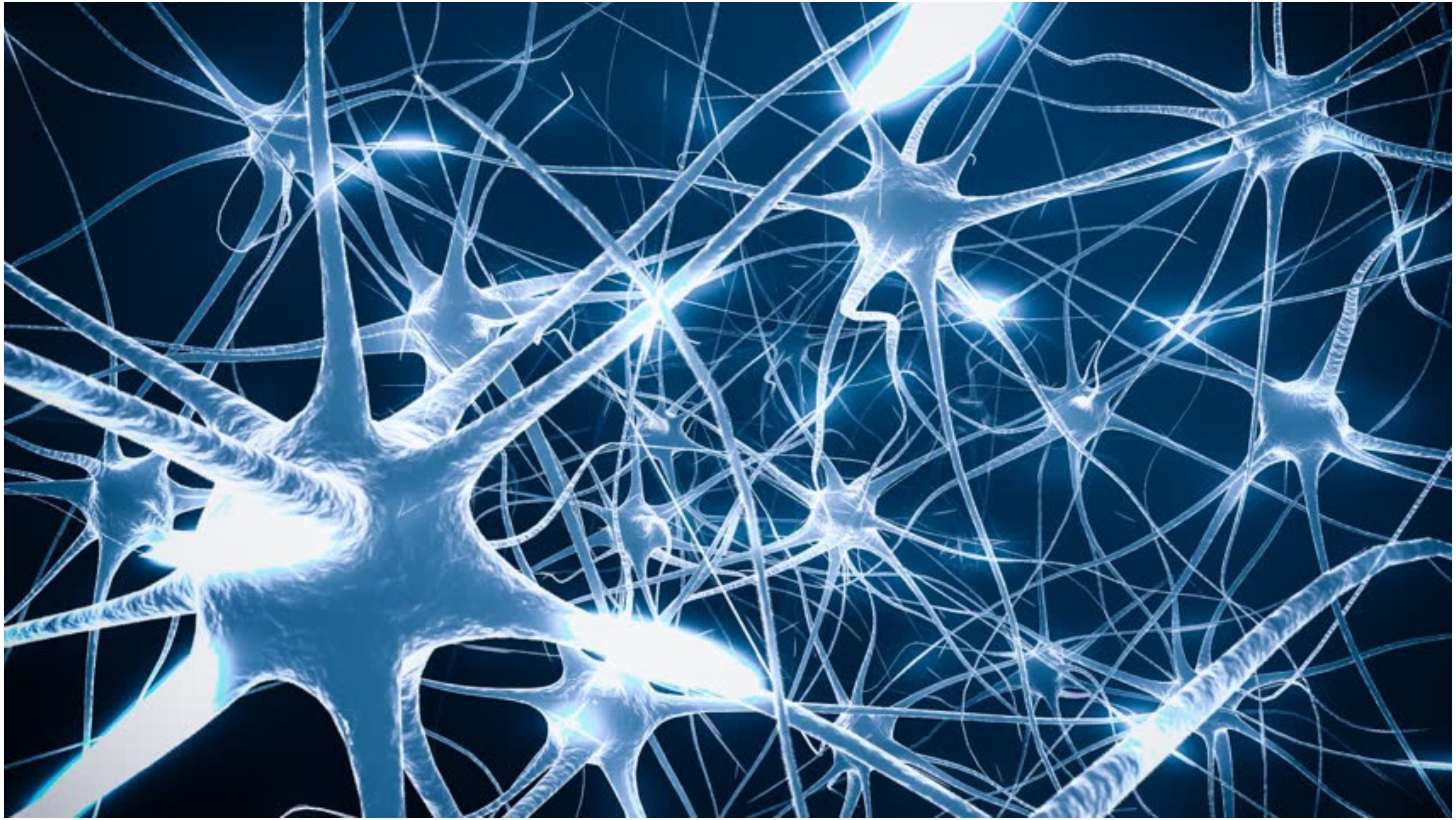
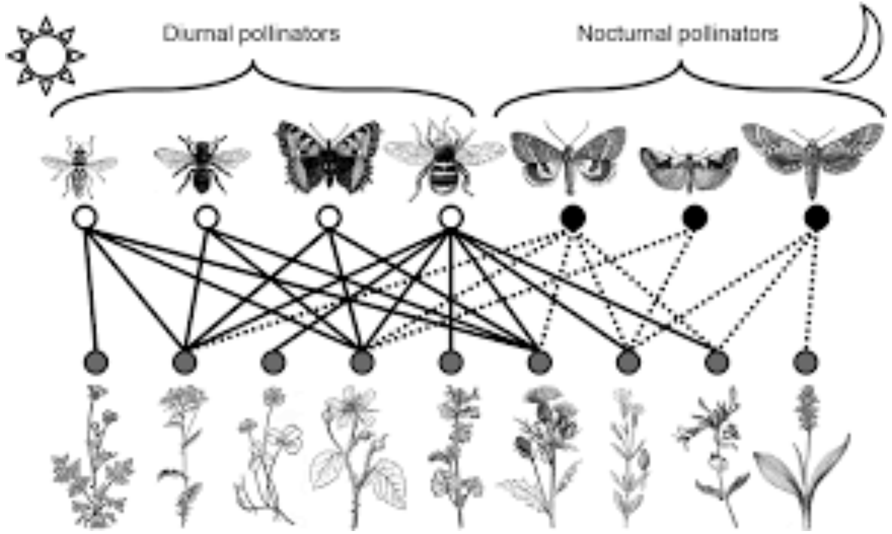
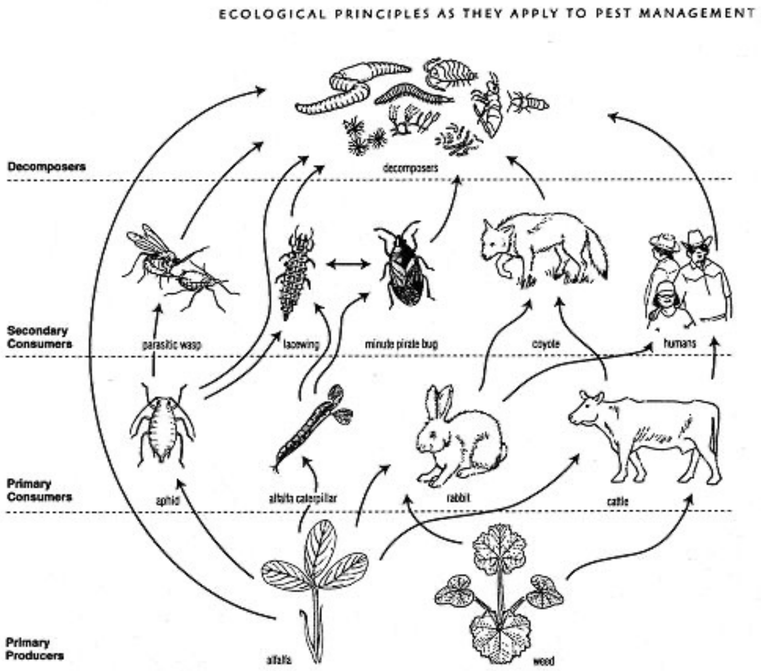
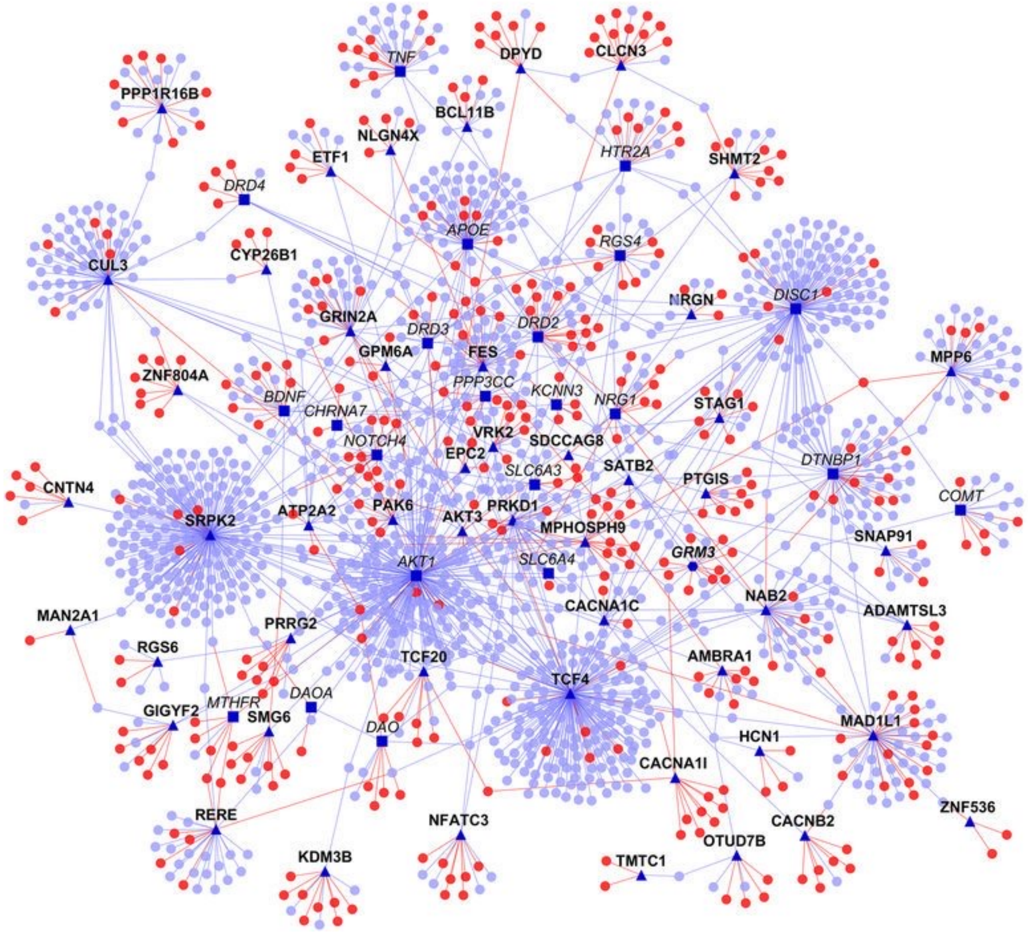
Brain neuronal networks



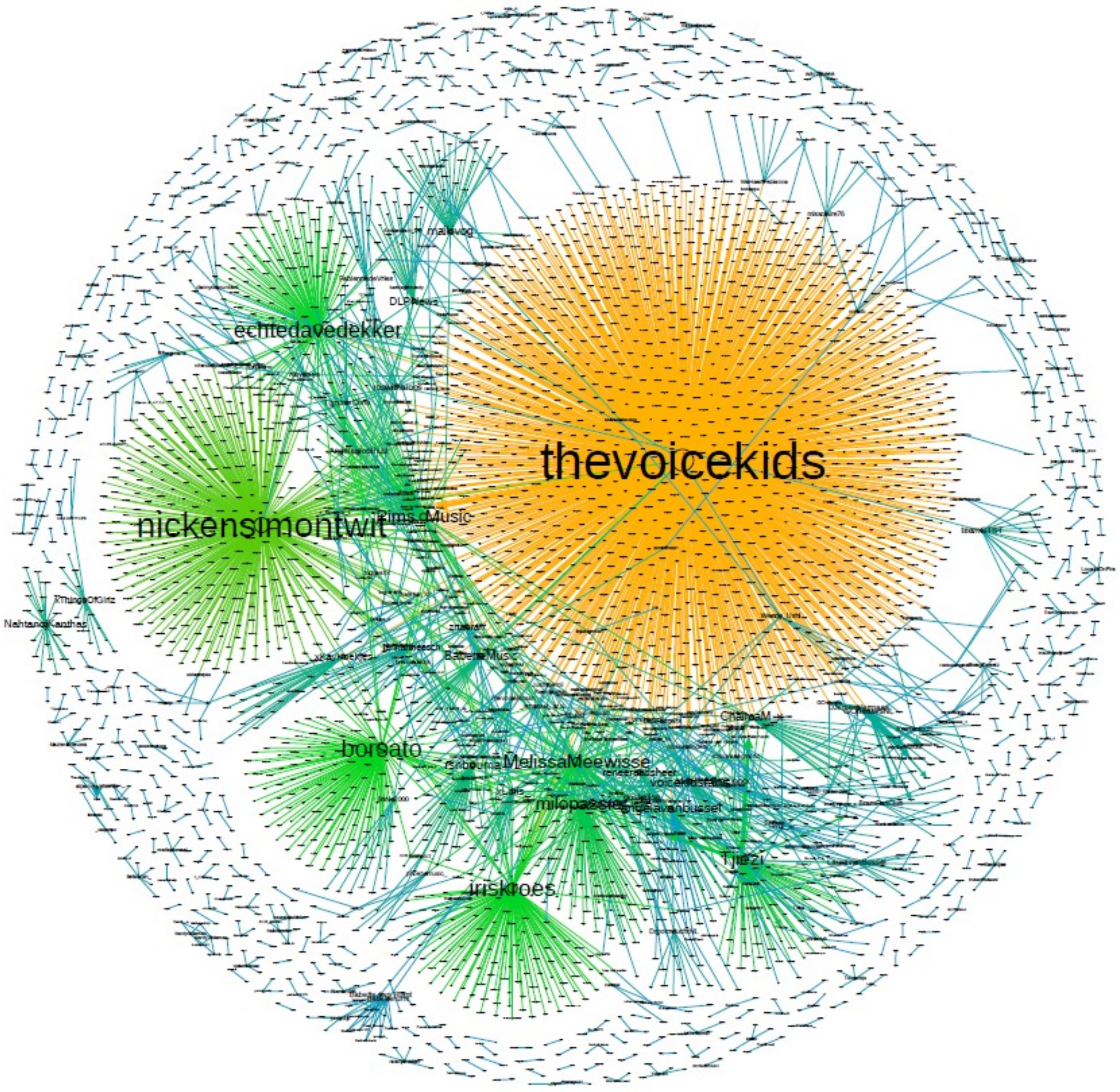
Biological networks







Twitter

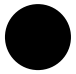


thevoicekids
nickensimontwit
echtedavedekker
borsato
MelissaMeewisse
irisproes
Tjiezi

Complex networks

Some essentials

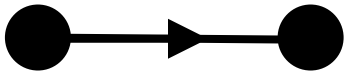
- Node (or vertex):



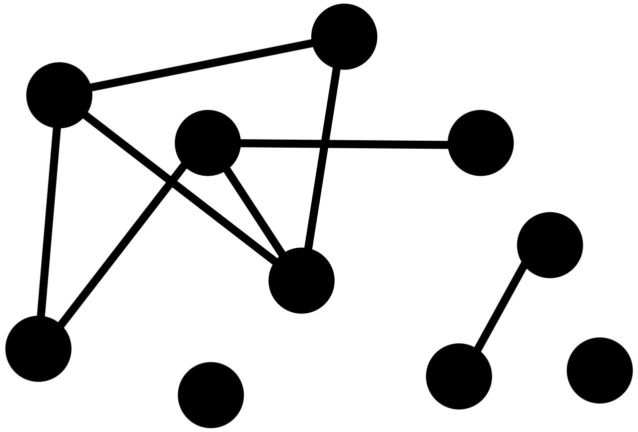
- Links (or edges):

undirected

directed



- Network (or graph):



Distance between two nodes is defined as the number of links of the shortest path.

Complex networks

Real Networks

- WWW (2022):
  Size $N \sim 2 \times 10^9$ webpages,
  Number of links $L \sim 10^{11}$.

- Internet (2015):
  Size $N \sim 10^8$ servers,
  Number of connections $L \sim 10^9$.

- Brain:
  Size $N \sim 10^{11}$ neurons,
  Number of connections $L \sim 10^{14}$.

Complex networks

Small-world experiment

- In the 1960’s, Stanley Milgram and Jeffrey Travers designed an experiment based on Pool and Kochen’s work:
    - A single “target” in Boston.
    - 300 initial “senders” in Omaha and Wichita.
    - Each sender asked to forward a packet to a friend who was “closer” to the target.
    - The friends got the same instruction.
- Out of 300 “letter chains”, 64 reached the target.
- Found that typical chain length was 6.
- Led to the famous phrase “six-degrees of separation”.

Complex networks

How “Small” is the World?

“Six degrees of separation between us and everyone else on this planet”
– John Guare, 1990

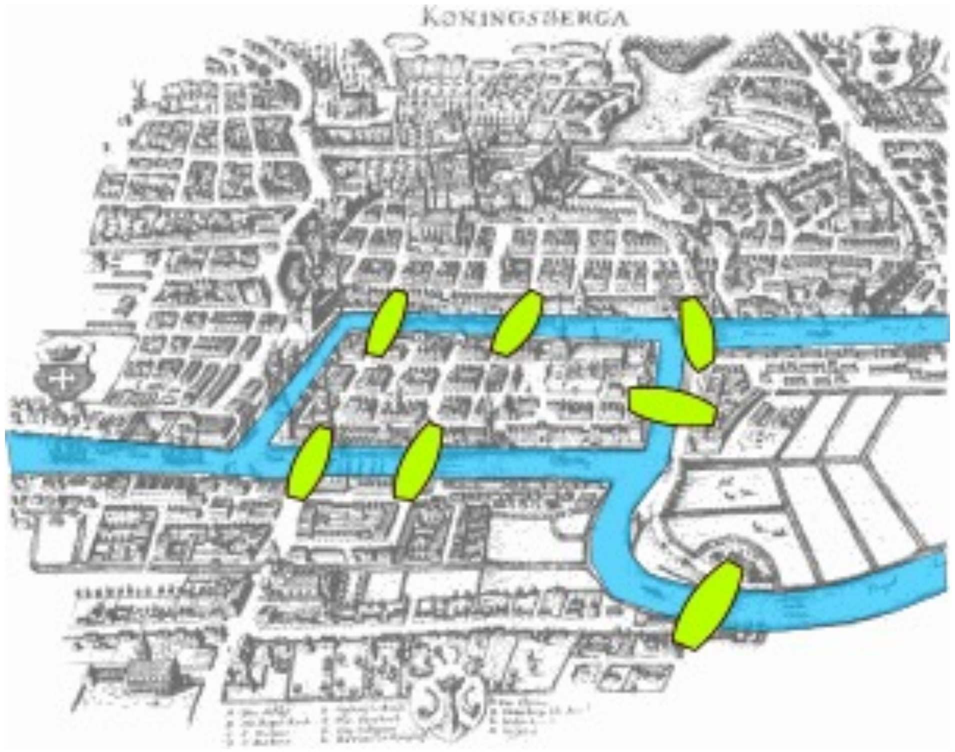
First mentioned in 1920’s by Frigyes Karinthy.

1950’s Pool and Kochen first posed it as a math problem involving network structure.

First became famous in 1960’s as a result of Milgram’s ingenious experiment.

Complex networks

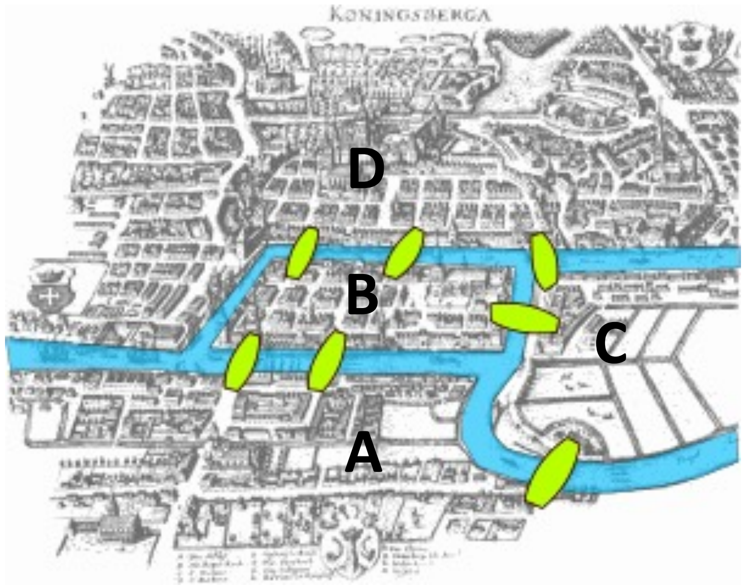
Seven Bridges of Königsberg

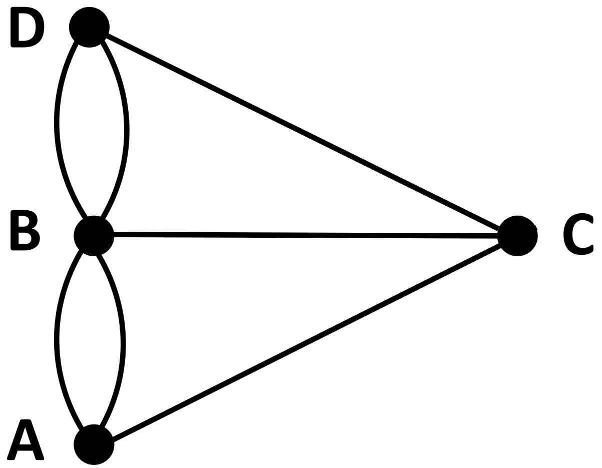


Probably the first problem in graph theory:
Is there a path that crosses each bridge once and only once?

Complex networks

Seven Bridges of Königsberg





Is there a path that crosses each bridge once and only once?

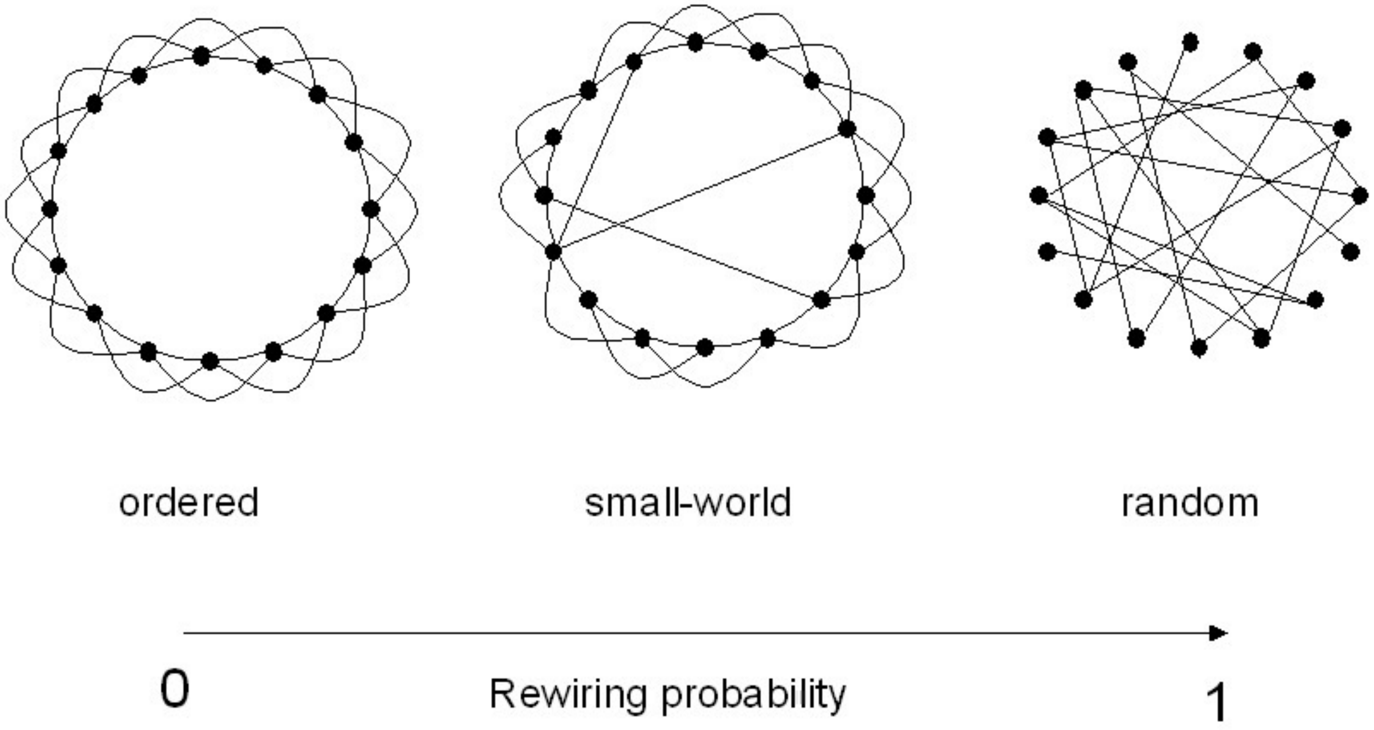
$$q_A = 3$$
$$q_B = 5$$
$$q_C = 3$$
$$q_D = 3$$

In 1736 Euler proved that there is no such path.
Euler's solution to this problem laid the foundations of graph theory.

Complex networks

Watts and Strogatz's small-world networks (1998)

Three basic network types



Complex networks

Watts and Strogatz's small-world networks (1998)

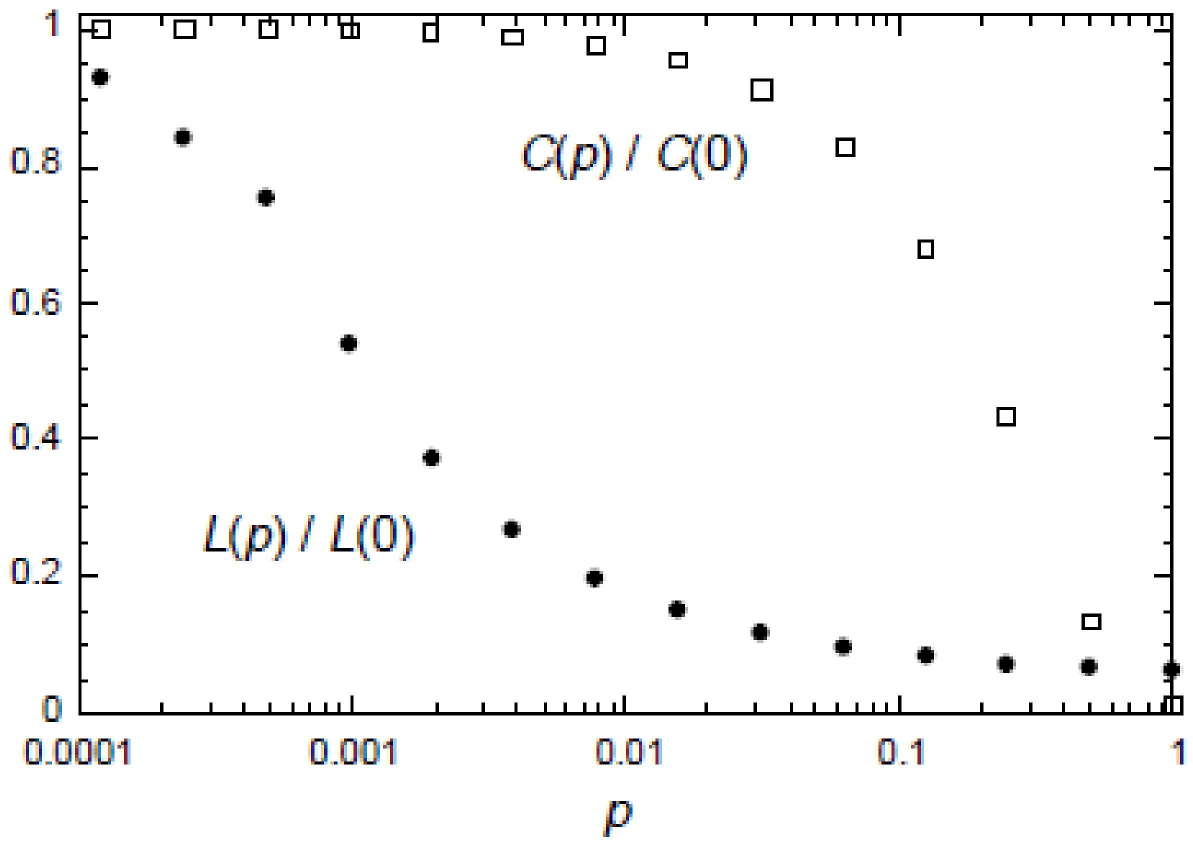
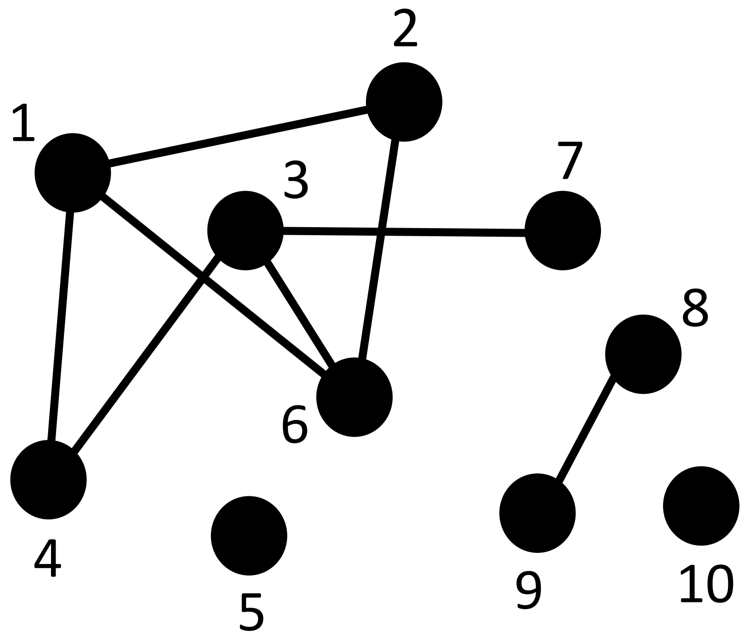


Figure 2 Characteristic path length $L(p)$ and clustering coefficient $C(p)$ for the

Complex networks

Some more essentials

Degree q_i of node i is the number of connections to his nearest neighbours.



$$q_1 = 3,\ q_2 = 2,\ q_3 = 3,\ \ q_4 = 2,\ \ q_5 = 0,\ \ q_6 = 3,\ \ q_7 = 1,\ \ q_8 = 1,\ q_9 = 1,\ q_{10} = 0.$$

Complex networks

Some more essentials

Take a network which consists of N nodes of degrees $q_1, q_2, q_3, \dots, q_N$.

Let us denote as $N(q)$ the number of nodes with a given degree q.

We introduce the so-called **degree distribution**: $P(q) = \frac{N(q)}{N}$. This is the probability that a randomly chosen node has degree q.

Normalization:
$$\sum_{q=0}^{q_{max}} P(q) = 1$$

Mean degree:
$$\langle q \rangle = \frac{1}{N}\sum_{i=1}^{N} q_i = \sum_{q=0}^{q_{max}} P(q)q.$$

Complex networks

Some more essentials

Sparse networks:

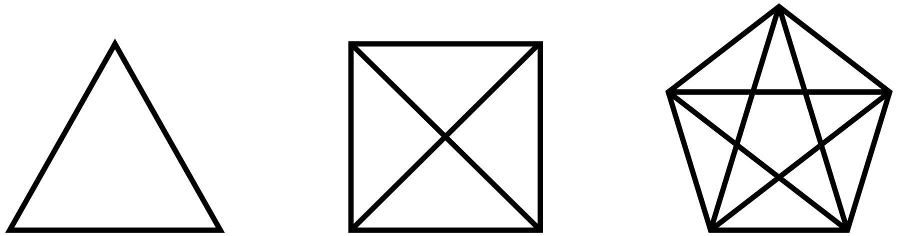
Mean degree $\langle q \rangle$ is finite, so the number of edges is proportional to N.

$$L = \frac{1}{2}\sum_{i=1}^{N} q_i = \frac{N}{2}\frac{1}{N}\sum_{i=1}^{N} q_i = \frac{1}{2}N\langle q \rangle$$

Dense networks:

The number of edges grows super-linearly with N; $\langle q \rangle$ diverges with $N \to \infty$.

For example for complete graphs (all-to-all connections):



$$L = \frac{1}{2}N(N-1) \propto O(N^2)$$

Complex networks

Some more essentials

A simple graph is fully defined by its **adjacency matrix**, whose entries are

$$A_{ij} = \begin{cases} 1, & \text{if there is an edge between } i \text{ and } j, \\ 0, & \text{otherwise.} \end{cases}$$

For undirected graphs the adjacency matrix is symmetric $A_{ij} = A_{ji}$.

By definition, the diagonal entries $A_{ii} = 0$.

$$q_i = \sum_j A_{ij}$$

$$\langle q \rangle = \frac{1}{N} \sum_i q_i = \frac{1}{N} \sum_i \sum_j A_{ij}$$

Taking powers of the adjacency matrix $\boldsymbol{A}^l$ gives the number of paths of length l between each pair of nodes.