Modelling of Complex Systems

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Project 3

Two-dimensional random walks
First passage time and survival probability
Lévy flights

Part 1 — Two-dimensional Random Walks

Perform simulations of symmetric random walks on a regular square lattice. The probabilities of jumps to the left, to the right, up and down are all equal to 1/4.

Task 1.1. Plot any 3 trajectories after 100 jumps.

Task 1.2. Compute and plot the probability $P(x, y; t)$ to find this particle after time t (i.e., after t jumps) at site (x, y). At $t = 0$ the particle is at site $(x, y) = (0,0)$. Use $t = 50\,000, 50\,001$, and then average over even and odd t and find the averaged probabilities $\bar{P}(x, y; t)$. Check the normalization, averages $\langle x \rangle = \langle y \rangle = 0$, and variance of the distance to the starting point $\langle x^2 + y^2 \rangle = t$. Compare between your simulations and the theoretical probability density distribution function,

$$P(x, y; t) = \frac{1}{\pi t} e^{-\frac{x^2+y^2}{t}}$$

Algorithm for random walks on a regular square lattice:

1. Start at point $(x, y) = (0,0)$.
2. Generate uniformly at random integer numbers $A = 1,2,3$, and 4. If $A = 1$ then the particle jumps up, if $A = 2$ it jumps to the right, if $A = 3$ it jumps down, and if $A = 4$ it jumps to the left.
3. Update the point (x, y).
4. Repeat t times the steps 2 and 3 to get a trajectory. Simulate a large number, N, of trajectories.
5. Calculate the number $N(x, y; t)$ of times that the end point of the trajectory is at site (x, y). Calculate the probability

$$P(x, y; t) = \frac{N(x, y; t)}{N}.$$

6. Repeat for even and odd times, and average:

$$\bar{P}(x, y; t) = \frac{1}{2}[P(x, y; t) + P(x, y; t + 1)]$$

7. Check the normalization $\sum_{x,y} \bar{P}(x,y;t) = 1$, the average $\sum_{x,y} x\bar{P}(x,y;t) = \sum_{x,y} y\bar{P}(x,y;t) = 0$, and the variance $\sum_{x,y}(x^2+y^2)\bar{P}(x,y;t) = t$.
8. Plot the averaged distribution function $\bar{P}(x,y;t)$ in the (x,y)-plane.
9. In the same plot, draw the theoretical distribution function

$$P(x,y;t) = \frac{1}{\pi t} e^{-\frac{x^2+y^2}{t}}.$$

Part 2 — First passage time and survival probability

For this part insert an absorbing boundary in the square lattice, along the vertical line at $x_b = -30$, i.e., particles walk in the semi-infinite plane with $x \geq x_b$. The starting point is at $(x_0, y_0) = (0,0)$. When the particle hits this boundary the walk stops. Repeat these random walks $N = 50\,000$ times. (In other words, there are $N = 50\,000$ particles.)

Task 2.1. Calculate the number $N_{\text{fpt}}(t)$ of particles that hit the boundary for the first time during the time interval $(t, t+\Delta t]$ with $\Delta t = 10$ (i.e., take a binning of the domain of t with a bin width of 10). Plot the first passage time distribution $F(t) = N_{\text{fpt}}(t)/(N\Delta t)$, the probability to hit the boundary for the first time at time interval $(t, t+\Delta t]$ (the first passage time probability density). Analyse behaviour of $F(t)$ at large t (let the maximum time be 50 000 jumps). For this purpose plot the function $\ln F(t)$ versus $\ln t$ (log-log scale).

Calculate the survival probability $S(t)$. For this purpose, find the number $N_S(t)$ of particles which are not trapped, i.e., survive, at time t. The definition of the survival probability is $S(t) = N_S(t)/N$. Plot the function $S(t)$.

Compare between simulation results and the theoretical predictions:

$$S(t) = \operatorname{erf}\left(\frac{|x_b - x_0|}{2\sqrt{Dt}}\right),$$

$$F(t) = \frac{|x_b - x_0|}{2\sqrt{\pi D t^3}} \exp\left(-\frac{(x_b - x_0)^2}{4Dt}\right),$$

$$F(t) \approx \frac{|x_b - x_0|}{2\sqrt{\pi D t^3}} \quad \text{and} \quad S(t) \approx \frac{|x_b - x_0|}{\sqrt{\pi D t}},$$

Here, $x_0 = 0$ is the starting point, $x_b = -30$ is the boundary position, and $D = 1/4$ is the diffusion coefficient.

Can you propose a qualitative explanation of the position of the maximum of $F(t)$?

Imaging that the position of the absorbing boundary is unknown [it can be placed either to the left, or to the right, or above, or below from the point $(x,y) = (0,0)$]. You stay at this point, (0,0), and measure particles that can come to you from different directions. Can you propose a method to find on which side is the boundary?

Part 3 — Lévy flights

This part is aimed to study 2D-random walks with variable length of jumps. The probability $P(l)$ that a jump has the length l is determined by the Lévy distribution,

$$P(l) = Cl^{-\mu},$$

where C is the normalization constant, for $l_{min} < l < l_{max}$. If the minimum length of the jumps is $l_{min} = 1$, respectively, then the normalization constant C is

$$C = \frac{\mu - 1}{1 - l_{max}^{1-\mu}}$$

After the n'th jump the particle will be at a point with coordinates $(x_n, y_n) = (x_{n-1}, y_{n-1}) + (l\cos\varphi, l\,\sin\varphi)$. Jumps are isotropic, which means that the probability to jump at an angle φ does not depend on the angle, i.e., $p(\varphi) = 1/2\pi$.

Task 3.1. For three values of the exponent $\mu = 1.6, 2$, and 2.6 and $l_{max} = 1000$, generate trajectories with $N = 1000$ random jumps. Plot these three trajectories and compare with isotropic 2D-random walks having a fixed length of jumps, $l = 1$. Analyse qualitatively the trajectories.

Task 3.2. Show numerically that, if x is a random number generated uniformly at random in the interval [0,1], then the random numbers

$$l(x) = \left[1 - x(1 - l_{max}^{1-\mu})\right]^{\frac{1}{1-\mu}}$$

are distributed according the Lévy flights distribution, $P(l) = Cl^{-\mu}$.

Task 3.3 Imagine you are a hungry shark. What strategy to forage for food will you choose in the following cases: (1) fishes are distributed homogeneously in the see; (2) fishes are forming flocks somewhere randomly in the sea. Explain the reasons of your choices

Algorithm to generate the Lévy flights:

1. Start the random walk at point $(x_0, y_0) = (0,0)$.
2. Generate a random number $r \in [0,1)$ with the uniform probability.
3. Calculate the length of jump using the function
$$l(r) = \left[1 - r\left(1 - l_{max}^{1-\mu}\right)\right]^{\frac{1}{1-\mu}}.$$
4. Generate the angle of the jump φ from a uniform probability distribution. Namely, generate a new random number r in the interval $[0,1)$, and then get the angle as $\varphi = 2\pi r$.
5. Update the position of the particle after the jump: $(x_n, y_n) = (x_{n-1}, y_{n-1}) + (l\cos\varphi, l\,\sin\varphi)$.
6. Repeat N times the steps 2-5 (N jumps).
7. Plot the trajectory.