Modelling of Complex Systems

First Passage Processes

- First Passage Time probability distribution
- Survival probability

First Passage Times

Previously, we showed that the probability of finding a (symmetric) random walker at point $\vec{r} = (x, y, z, \ldots)$ after t jumps is

$$P(\vec{r}, t) = \left(\frac{1}{\pi 4Dt}\right)^{d/2} e^{-\frac{|r|^2}{4Dt}},$$

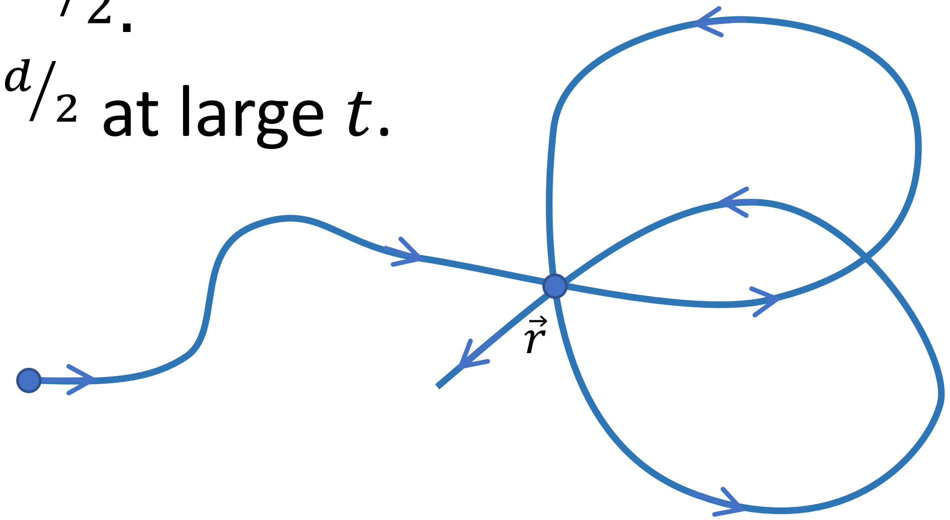
where D is the diffusion coefficient.

The probability that the particle is back at the starting point after t jumps is

$$P(0; t) = (\pi 4Dt)^{-d/2}.$$

For a generic point $\vec{r}$ we still have $P(\vec{r}; t) \propto t^{-d/2}$ at large t.

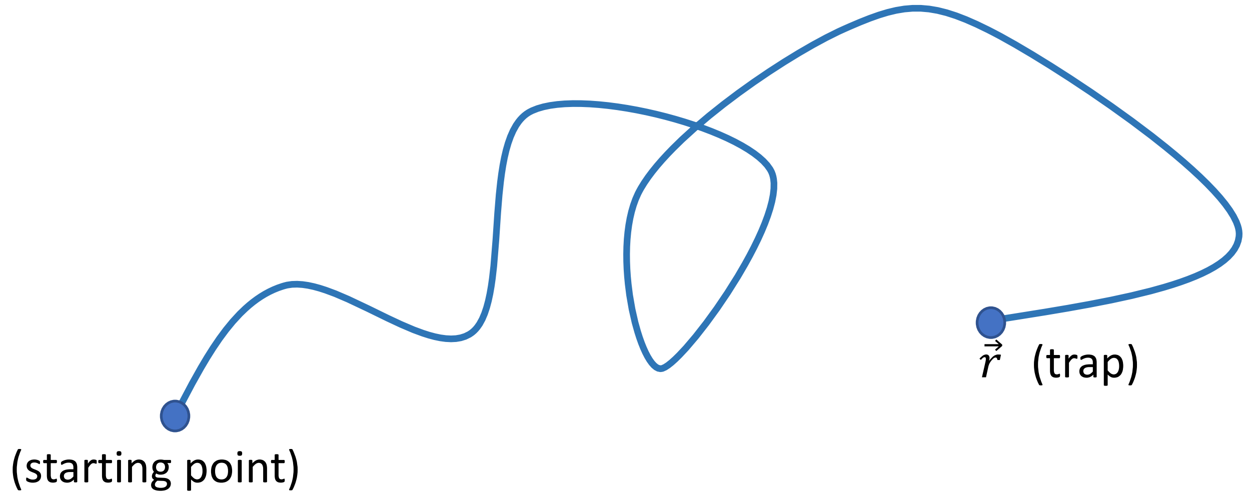
These probability include the cases in which the particle visits that point $\vec{r}$ multiple times.



First Passage Times

Now, let's consider the number of steps that takes for the particle to **visit point $\vec{r}$ for the first time.**

We can think of this problem as a random walk with an absorbing trap at a specific point $\vec{r}$.



If the walk stops when the particle reaches $\vec{r}$, what is the probability of the walk having t jumps?

First Passage Times

Definitions:

- The **first passage times probability** $F(\vec{r},t)\Delta t$ is the probability that the particle hits a specific point $\vec{r}$ (e.g. the trap) for the first time during the time interval $[t, t+\Delta t)$.
  The function $F(\vec{r},t)$ is the probability density distribution of first passage times.

$$F(\vec{r},t) = \frac{\begin{array}{c}\text{number of tragectories that reach point } \vec{r} \\ \text{for the first time during interval } [t, t+\Delta t]\end{array}}{\text{total number of trajectories}} \frac{1}{\Delta t} = \frac{N_{\mathrm{fpt}}(\vec{r},t,t+\Delta t)}{N\Delta t}$$

- The **survival probability** $S(t)$ is the probability that the particle does not hit the absorbing trap in the interval $[0,t]$.

First Passage Times

We have N particles (or make N realizations). At $t = 0$ the particles start at $\vec{r} = 0$, then they make random walks. When a particle hits the trap, it disappears. After time t we count the number of surviving particles $N_S(t)$.
Then the survival probability is

$$S(t) = \lim_{N \to \infty} \frac{N_S(t)}{N}.$$

There is a relation between $S(t)$ and $F(t)$ that is a consequence of the conservation of the number of particles:

$$N_S(t) + N_h(t) = N$$

where $N_h(t) = \int_0^t NF(t')dt'$ is the number of particles that hit the trap before time t.

First Passage Times

$$N_S(t) + N_h(t) = N_S(t) + \int_0^t NF(t')dt' = N$$

Dividing all terms by N we get

$$\Rightarrow \quad S(t) = 1 - \int_0^t F(t')dt',$$

where the integral is equal to the probability of hitting the trap before time t.

Equivalently, we can write this relation as

$$\Rightarrow \quad \frac{\partial S(t)}{\partial t} = -F(t).$$

First Passage Times

Theoretical calculations show that at large times, more precisely, for $t \gg |r|^2$, we have:

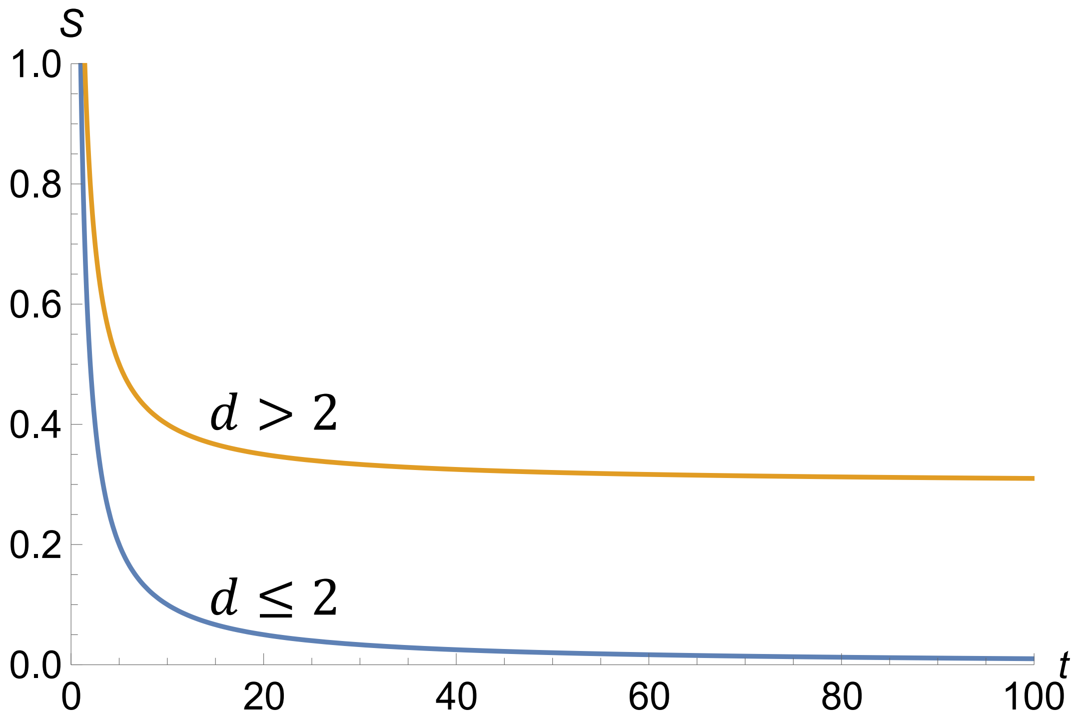
$$F(t) \propto \begin{cases} t^{-3/2}, & \text{for } d = 1 \\ \dfrac{1}{t(\ln t)^2}, & \text{for } d = 2 \\ t^{-d/2}, & \text{for } d \geq 3 \end{cases}$$

$$S(t) \propto \begin{cases} t^{-1/2}, & \text{for } d = 1 \\ \dfrac{1}{\ln t}, & \text{for } d = 2 \\ \text{const} + At^{1-d/2}, & \text{for } d \geq 3 \end{cases}$$

This constant is the probability of the particle never hitting the trap.

First Passage Times

A random walker has a finite probability of surviving forever in a system with a trap if the dimensionality of the system is larger than 2.



The drunk man, randomly walking in a town will reach sooner or later reach his home, because he is moving in a 2D system.

First Passage Times

We can give a simple physical basis for this effect:

At time t, the particle explores a roughly spherical domain of radius $\sqrt{Dt}$. The number of sites visited during this exploration is proportional to t. Then, the density of visited sites is inside this exploration sphere $\rho \propto t/t^{d/2} \propto t^{1-d/2}$.

One one hand, for $d < 2$, $t^{1-d/2}$ diverges as $t \to \infty$, which means that the particle visits each site inside the sphere infinite times.

On the other hand, for $d > 2$ the density of explored sites approaches 0 as $t \to \infty$, which means that the walk misses most of the lattice sites.

Interestingly, although every site is visited for $d < 2$, the expected first passage time is infinite, because $\langle t \rangle = \int_0^\infty tF(t)dt$ diverges!