Modelling of Complex Systems

Lévy Flights

- Lévy flights are random walks whose jump lengths are not constant, but instead are drawn from a probability distribution with diverging variance, such as distributions with power-law tails $P(l) \propto l^{-\mu}$, with $1 < \mu < 3$.

- The motivation for the study of this kind of random walks is to explain the cases where $\langle|\vec{r}|^2\rangle \propto t^{\gamma}$ with $\gamma \neq 1$.

Lévy Flights in the real world

In physics:

- Fluid dynamics
- Turbulent diffusion
- Chaotic phase diffusion in Josephson junctions
- Slow relaxation in glassy materials

Lévy Flights in the real world

In biology:

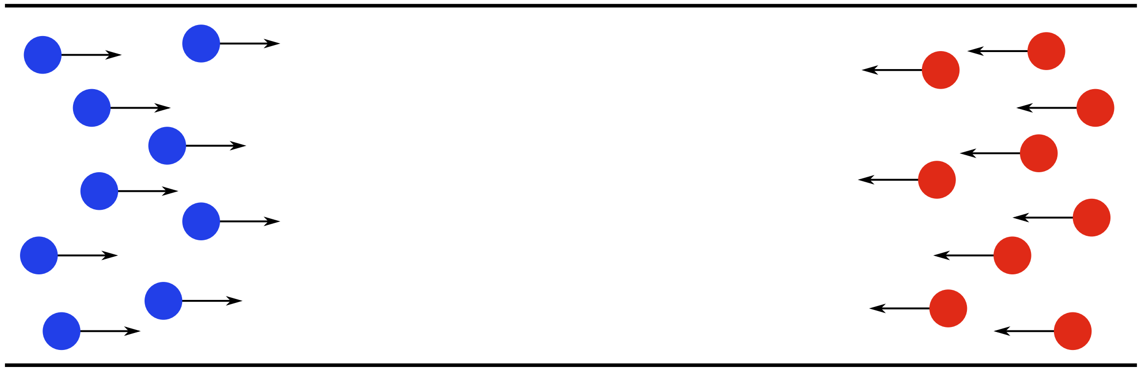
In nature, it has been observed that Lévy flights are an optimal search strategy. Predators use this search strategy when prey is sparsely and unpredictably distributed.

Electronic tags attached to open-see predators, like sharks, tuna fishes, or birds like albatrosses, have shown them to use this strategy when foraging for food.

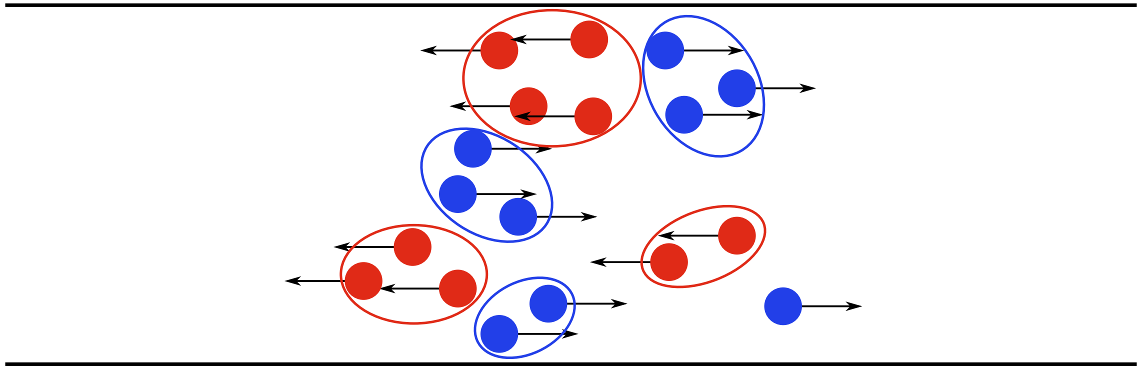
Power-laws distributions are also observed in the group sizes (number of individuals in a group) of many species of animals, like certain fish, buffaloes, mussels, etc.

Lévy Flights in the real world

Lévy walk processes also describe emergent self-organization in pedestrian crowds. Two groups of people walking in opposite directions in a corridor:

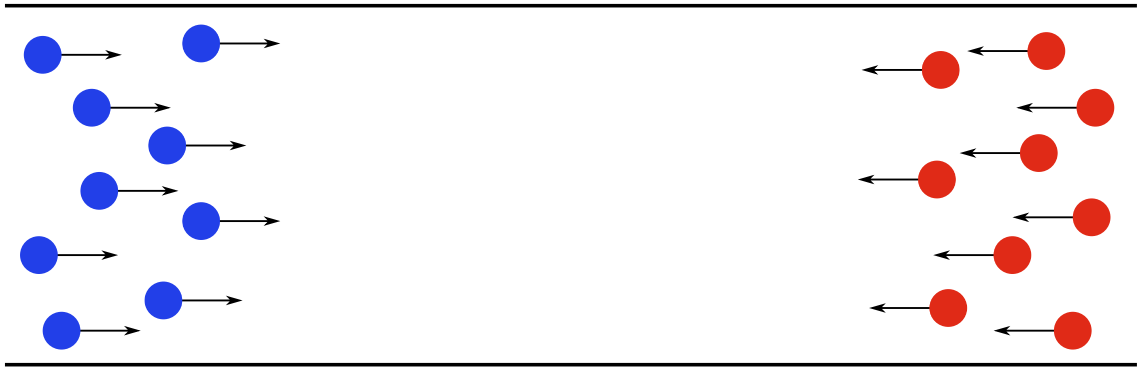


Formation of clusters of various sizes:

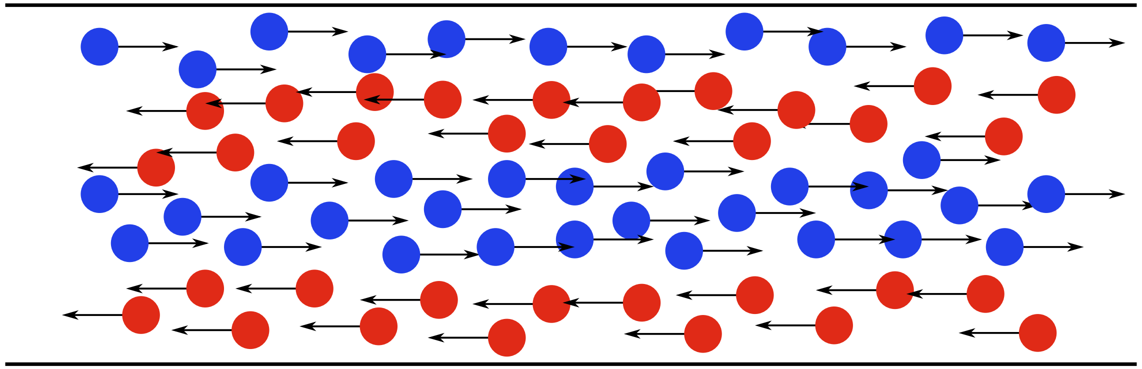


Lévy Flights in the real world

Lévy walk processes also describe emergent self-organization in pedestrian crowds. Two groups of people walking in opposite directions in a corridor:

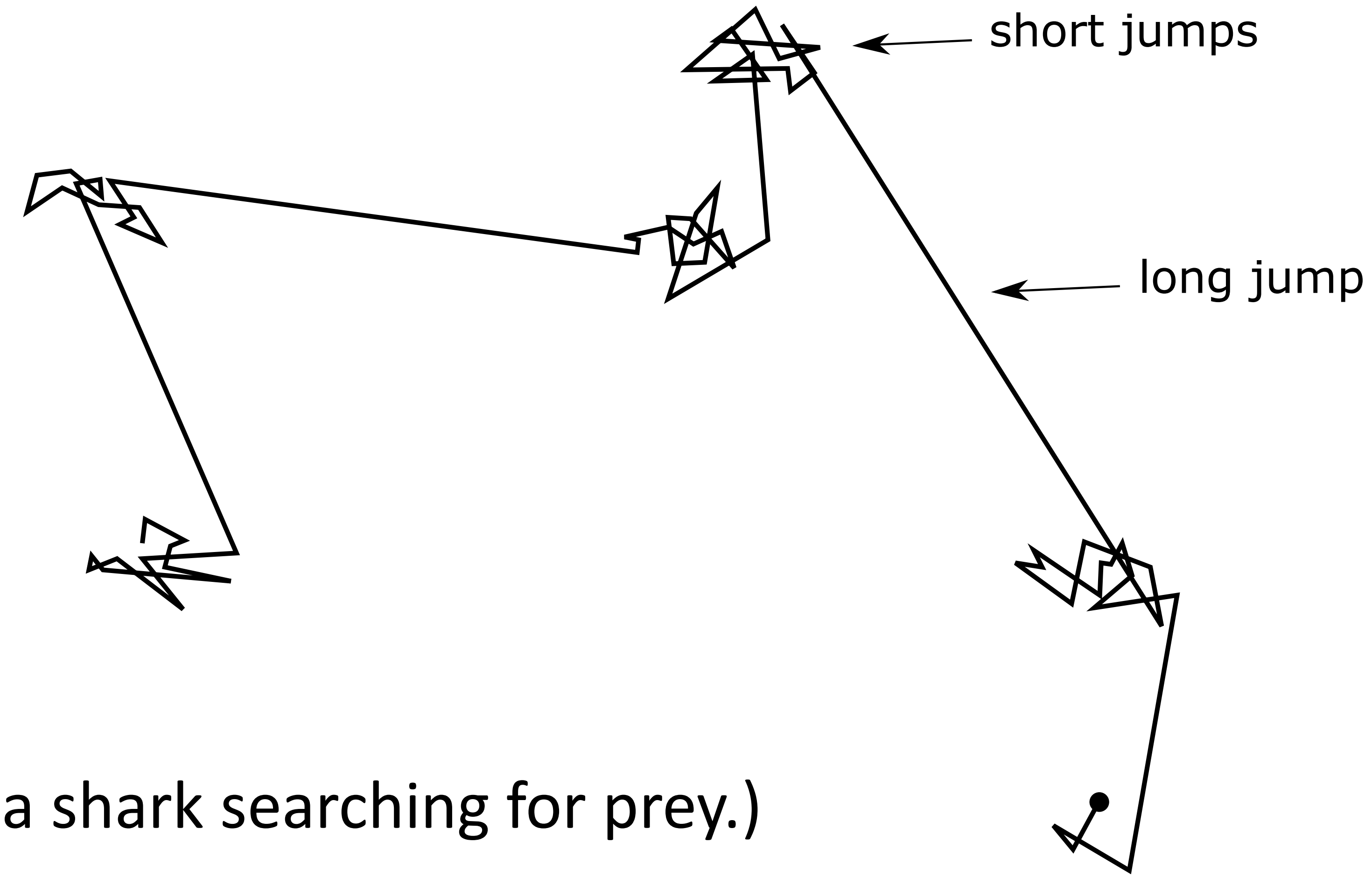


Spontaneous formation of lanes:



Lévy Flights in the real world

An illustration of a Lévy flight trajectory:



(For example, a shark searching for prey.)

Generating Lévy Flights

How to simulate Lévy flights:

We generate random numbers with distribution whose tail is $P(l) \propto l^{-\mu}$ for the jump length. (For the jump direction let us use uniform random number between 0 and 2π.)

Notice that when $\mu > 3$ the length distribution has a finite second moment $\langle l^2 \rangle$, the random walk is essentially brownian motion with $\langle |\vec{r}|^2 \rangle \propto t$.

The limit $\mu \to 1$ is refers to ballistic motion, i.e., $\langle |\vec{r}|^2 \rangle \propto t^2$.

The optimal exponent is typically $\mu = 2$.

Generating Lévy Flights

Let us use a distribution $P(l) = Cl^{-\mu}$ for $l_{min} < l < l_{max}$. The constant C may is determined by the normalization condition:

$$C\int_{l_{min}}^{l_{max}} l^{-\mu} dl = 1$$

For simplicity let $l_{min} = 1$, then

$$C = (\mu - 1)/\left(1 - {l_{max}}^{1-\mu}\right)$$

How to get random variable with a density distribution $P(l)$ from a uniform random variable x?

With **inverse transform sampling**, that is, we find a function $l(x)$ such that l has the desired distribution.

Generating Lévy Flights

$$P(l)dl = Q(x)dx$$

Starting with numbers from a uniform distribution $Q(x) = 1$, for $0 < x < 1$ we get:

$$\frac{dx}{dl} = P(l)$$

For a power-law $P(l)$ we have

$$\frac{dx}{dl} = Cl^{-\mu}$$

Integrating both sides, and imposing the condition $l(x = 0) = l_{min} = 1$ we find

$$x = \frac{C(1 - l^{1-\mu})}{\mu - 1}$$

Generating Lévy Flights

$$x = \frac{C\left(1 - l^{1-\mu}\right)}{\mu - 1}$$

Now all there is left is to invert this expression

$$l(x) = \left[1 - x\,\frac{\mu - 1}{C}\right]^{1/_{1-\mu}}$$

and replace $\frac{\mu-1}{C} = \left(1 - {l_{max}}^{1-\mu}\right)$, to finally get the function:

$$l(x) = \left[1 - x\left(1 - {l_{max}}^{1-\mu}\right)\right]^{1/_{1-\mu}}.$$