MATH 357 HW 2

Yellow group

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0.1

1. $x^4 + 10x + 5$ is irreducible by Eisenstein's criterion, $p = 5$
2. Eisenstein's criterion again applies with $p = 3$
3. Observe that $f(x) = x^{p-1} + ... + x + 1$ is a finite geometric series with common ratio x and principle term 1. Then we can write $f(x) = \frac{1-x^p}{1-x} = \frac{x^p-1}{x-1}$. Put $y = x - 1$. Then $f(x)$ is irreducible if and only if $f(y+1)$ is irreducible. We have,

$$f(y+1) = \frac{(y+1)^p - 1}{y} = \frac{1}{y}\left(\sum_{j=0}^{p} \binom{p}{j} y^j\right) - \frac{1}{y}$$

$$= \binom{p}{p} y^{p-1} + \binom{p}{p-1} y^{p-2} + ... + \cancel{\binom{p}{0} y^{-1} - \frac{1}{y}} \rightarrow 1$$

Which is irreducible by Eisenstein's criterion using prime p.

0.2 Goodman 8.1.1

Let R be a ring, and let M be a left R-module.

(a) Let $r \in R$ be given. By the distributive property, $\varphi_r : M \to M$ given by $\varphi_r(m) = rm$ is a group endomorphism. This implies that $\varphi_r(0) = r0 = 0$ for all r. By the same reasoning, for any $m \in M$, $\psi_m : R \to M$ given by $\psi_m(r) = rm$ is a group homomorphism. This implies that $\psi_m(0) = 0m = 0$ for all m.

(b) Using the same notation as in part (a) and exploiting properties of group homomorphisms, we have $r(-m) = \varphi_r(-m) = -\varphi_r(m) = -(rm)$. Similarly, $(-r)m = \psi_m(-r) = -r\psi_m(r) = -(rm)$.

(c) Suppose R is a ring with identity and that M is unital. We have,

$$(-1)m + m = (-1)m + (1)m = (-1+1)m = 0m$$

By part (a), $0m = 0$. This implies that $(-1)m$ is the additive inverse of m, so $(-1)m = -m$. Alternatively, one can use the group homomorphisms φ_r and ψ_m defined in part (a), and use the fact that group homomorphisms take inverses into inverses.

0.3

Let R be a ring, and let M be a left R-module.

(a) We know that $N = \bigcap_{i \in I} N_i$ is an abelian group, since the intersection of any family of abelian subgroups of a group is also an abelian subgroup. It is also clear that the axioms in definition 8.1.2 (Goodman) all hold in N, since they hold for all elements in N_j and $N \subset N_j$. (j is any element of I) It remains to be seen then that N is closed under left multiplication by elements in R. Let $n \in n$ and $r \in R$ be given. Then $n \in N_i$ for each i, and since each N_i is an R-module, we also have that $rn \in N_i$. This implies that $rn \in \bigcap_{i \in I} N_i = N$, and we are done.

Part (b) is trivially false. Let $R = \mathbb{R}$, and let $M = \mathbb{R}^2$. Then the lines $L_1 = \{(x, y) : y = 0\}$ and $L_2 = \{(x, y) : x = 0\}$ are left R-modules. (They are of course vector spaces, a stronger condition). However, $L_1 \cup L_2$ is not a module – it is not even a group with the usual vector addition, ex, $(1, 0) + (0, 1) = (1, 1) \notin L_1 \cup L_2$.

0.4 Advanced Problem A

Put $f_n(x) = (x-1)(x-2)...(x-n)-1, n \in \mathbb{N}$. Suppose, for the sake of a contradiction, that $f_n(x)$ decomposes into $f_n(x) = g(x) \cdot h(x)$, where $g, h \in \mathbb{Z}[x]$ each have degree strictly less than n. For each $1 \leq j \leq n$, we have that $-1 = f_n(j) = g(j)h(j)$. Since f_n, g, h are integer-valued for each input j, we have that

$$g(j) \cdot h(j) = -1 \quad \Rightarrow \quad g(j) = -h(j) : \;\; 1 \leq j \leq n$$

In particular, we have that $g(j) + h(j) = 0$ for each j. This implies that $g + h$ has at least n roots, so $\deg(g + h) \geq n$. But $\deg(g) < n, \deg(h) < n$, so $\deg(g + h) = \max\{\deg(g), \deg(h)\} < n$. Then $\deg(g+h) < n \leq \deg(g+h)$; this contradiction completes the proof.

0.5 Advanced Problem B: DF 10.1-15

The statement is not true in general for $\mathbb{Q}$-actions. Put $M = C_n = \{0, 1, 2, ...n - 1\}$, the cyclic group with n elements. Of course, for each $m \in M$, we have that $nm = 0$, by definition of the clyclic group. Suppose, for the sake of a contradiction, we can extend the action of $\mathbb{Z}$ on M into a $\mathbb{Q}$-module. Then for each $m \in M$, we have

$$m = 1m = (n \cdot \frac{1}{n})m = n \cdot (\frac{1}{n}m) = 0$$

hence $m = 0$. But clearly, M contains nonzero elements; a contradiction.

0.6 Advanced Problem C: DF 10.1-8

Let M be an R-module, and put

$$T = \operatorname{tor}(M) = \{m \in M : \exists r \in R : rm = 0, r \neq 0\}$$

(a) It is clear that the axioms in definition 8.1.2 (Goodman) all hold in T, since they hold for all elements in M and $T \subset M$. We simply need to verify that T is an abelian group which is closed under left multiplication by elements in R.

First, we check that T is an abelian group.

- *Identity.* $0 \in T$, since $r0 = 0$ for every nonzero $r \in R$.
- *Closure.* Let $m_1, m_2 \in T$. Then there are nonzero elements $r_1, r_2 \in R$ with $r_1m_1 = r_2m_2 = 0$. Then $r_1r_2(m_1 + m_2) = r_1r_2m_1 + r_1r_2m_2 = r_2r_1m_1 + r_1r_2m_2 = r_20 + r_10 = 0$. Since R is an integral domain with $r_1, r_2 \neq 0$, we have that $r_1r_2 \neq 0$, so $(m_1 + m_2) \in T$.
- *Inverses.* If $m \in T$ with $rm = 0, r \neq 0$ we can use the fact that M is an R-module to get $r(-m) = -(rm) = -(0) = 0$. This implies that $-m \in T$.

- *Associativity, commutativity.* Follows from the fact that M is an R-module, and hence an abelian group.

Now that we have verified that T is an abelian group, we simply need to check that it is closed under left multiplication by elements in R. Let $m \in T$. Then there is a nonzero $r \in R$ with $rm = 0$. Let $s \in R$ be given. Then by the commutative and associative properties, $r(sm) = (rs)m = (sr)m = s(rm) = s(0) = 0$. This implies that $sm \in T$, and we are done.

(b) We put $R = \mathbb{Z}/6\mathbb{Z}$, a simple ring that is not an integral domain. Of course, R is an R-module, acting on itself in the usual way. The torsion elements in $R = \{0, 1, 2, 3, 4, 5\}$ are $T = \{0, 2, 3, 4\}$, however, this set is not closed under addition, since $2 + 3 = 5 \notin T$.

(c) (In this proble, I assume $0 \in R$ is excluded from the definition of zero divisors – the statement is false otherwise). Let R be a ring, and suppose $x \neq 0 \in R$ is a 0-divisor, that is, there is a nonzero element $r \in R$ with $rx = 0$. Let M be an R-module, and let $m \neq 0 \in M$ be given. If $xm = 0$, then m is a nonzero torsion element and we are done. Otherwise, if $xm \neq 0$, then $r(xm) = (rx)m = 0m = 0$, so xm is a nonzero torsion element, and we are done.

0.7 Advanced Problem D: Goodman 8.2.1

(a) Let $\varphi \in \hom_R(M, N)$. Put $K = \ker(\varphi)$ and $L = \varphi(M)$. It is clear that the axioms in definition 8.1.2 (Goodman) all hold in K, L, since they hold for all elements in M, N and $K \subset M, L \subset N$. Furthermore, we know that K, L abelian subgroups of M, N since kernels and ranges of group homomorphisms between abelian groups are abelian subgroups. All that is left to do is verify that K, L are closed under left multiplication by elements in R. Let $k \in K, l \in L, r \in R$. Exploiting properties of module homomorphisms, we have,

$$\varphi(rk) = r\varphi(k) = r0 = 0$$

so $rk \in K$. On the other hand, since $l \in L$, there is an element $m \in M$ with $l = \varphi(m)$. Then

$$rl = r\varphi(m) = \varphi(rm)$$

where $rm \in M$. This implies that $rl \in L$, as desired.

(b) Let $\varphi : M \to N$ and $\psi : N \to P$ be module homomorphisms. Of course, $\psi \circ \varphi : M \to P$ is a group homomorphism, since the composition of group homomorphisms are also group homomorphisms. Now, let $m \in M$ and $r \in R$ be given. We have,

$$\psi \circ \varphi(rm) = \psi(r(\varphi(m)) = r(\psi(\varphi(m))) = r(\psi \circ \varphi(m))$$

and we are done.

Patrick's solutions end here: I haven't looked at/proofread anything below this point, reader beware. Also, I'm submitting everything above this as is so change it up a little if you don't want to get honor coded.

0.8

Part (b) is trivially false. It is clear that vector spaces are modules, hence the lines $L : y = 0$ and $J : x = 0$ are modules, as they are vector subspaces of $\mathbb{R}^2$. The union $L \cup J$ is not a module, however, as it is not closed under addition:

$$(1, 0) \in L + (0, 1) \in J = (1, 1) \notin L \cup J$$

0.9 Goodman 8.1.1

Let M be a module over the ring R. Then for all $r \in R$ and $m \in M$,

1. $0m = r0 = 0$.
2. $r(-m) = -(rm) = (-r)m$.
3. If R has a multiplicative identity and M is unital, then $(-1)m = m$.

Let R be a ring, and let M be a left R-module. Take $\{N_i\}_{i \in I}$ to be a nonempty collection of (left R-)submodules of M.

1. Show that $\bigcap_{i \in I} N_i$ is a submodule of M.
2. Is $\bigcup_{i \in I} N_i$ a submodule of M? Prove this statement or provide a counterexample.

1. Now consider if $\bigcap_{i\in I} N_i \neq 0$. Then consider

$$n + rn' \tag{1}$$

for $n, n' \in \bigcap_{i\in I} N_i$. Recall that for any $i \in I$, $n, n' \in N_i$. Then because N_i is a submodule, we know that $n + rn' \in N_i$, and therefore $n + rn' \in \bigcap_{i\in I} N_i$.

2. The statement is false. Counterexample: $N_1 = \{(x, 0) \mid x \in \mathbb{R}\}$ and $N_2 = \{(0, y) \mid y \in \mathbb{R}\}$. These are submodules of $\mathbb{R}$-modules over $\mathbb{R}^2$, as they are vector spaces. However, their union is the x and y-axes in $\mathbb{R}^2$. But this is not closed under the operations of the submodule, as $(x, 0) + (0, y) = (x, y) \notin X, Y$ if $x, y \neq 0$.