55. Which positive integers can be written as the difference of two squares and in how many ways?

To solve the analogous problem for sums instead of differences, we introduce the *Gaussian integers*: $\alpha = a+bi$, where a and b are integers. The Gaussian integers form an integral domain G, divisibility, unit, gcd, irreducible, and prime are defined as usual. We show that G is a Euclidean ring, hence it is (a PID, and so) a UFD. Therefore, the irreducible elements are the same as the primes, and we shall use the (shorter) name Gaussian primes. Also, we shall characterize all Gaussian primes. An important tool is the *norm*: $N(\alpha) = a^2 + b^2 = |\alpha|^2 = \alpha \cdot \overline{\alpha}$.

56. Prove:
(a) $\alpha \mid N(\alpha)$; (b) $N(\alpha\beta) = N(\alpha)N(\beta)$; (c) $\alpha \mid \gamma \Rightarrow N(\alpha) \mid N(\gamma)$, but the converse is false;
(d) $N(\alpha) = 0 \iff \alpha = 0$; (e) $N(\alpha) = 1 \iff \alpha$ is a unit $\iff \alpha = \pm 1, \pm i$.

57. Which Gaussian integers are divisible by $1 + i$?

58. Prove that G is a Euclidean ring with the norm as a Euclidean function: To every $\beta \neq 0, \alpha \in G$ there exist $\gamma, \varrho \in G$ satisfying $\alpha = \beta\gamma + \varrho$ and $N(\varrho) < N(\beta)$.

59. Verify: (a) $\alpha \mid \gamma \iff \overline{\alpha} \mid \overline{\gamma}$; (b) α is a Gaussian prime iff $\overline{\alpha}$ is a Gaussian prime.

60. Let $p > 2$ be a prime. Prove that the congruence $x^2 \equiv -1$ is solvable iff $p \equiv 1 \pmod 4$.

61. We characterize all Gaussian primes π:
(a) Every π divides exactly one positive prime p;
(b) Every p is either a Gaussian prime, or the product of two conjugate Gaussian primes;
(c) $2 = (1 + i)(1 - i) = -i(1 + i)^2$ provides $\pi = 1 + i$ (and its associates);
(d) Each prime $p \equiv -1 \pmod 4$ is a Gaussian prime;
(e) Each prime $p \equiv 1 \pmod 4$ is the product of two conjugate, non-associate Gaussian primes: $p = \pi \cdot \overline{\pi}$.

62. Factor (a) $35000i$; (b) $270 + 2610i$; (c) $86 + 162i$ into the product of Gaussian primes.

63. True or false ($\alpha = a + bi$):
(a) If α and β are coprime, then also $N(\alpha)$ and $N(\beta)$ are coprime.
(b) If $N(\alpha)$ and $N(\beta)$ are coprime, then also α and β are coprime.
(c) If a and b are coprime, then also α and $\overline{\alpha}$ are coprime.
(d) If α and $\overline{\alpha}$ are coprime, then also a and b are coprime.
(e) If α is a Gaussian prime, then $N(\alpha)$ is a prime number.
(f) If $N(\alpha)$ is a prime number, then α is a Gaussian prime.
(g) If α is the cube of a Gaussian integer, then $N(\alpha)$ is the cube of a non-negative integer.
(h) If $N(\alpha)$ is the cube of a non-negative integer, then α is the cube of a Gaussian integer.

***64.** Which positive integers can be represented and in how many ways as the sum of squares of (a) two integers; (b) two *coprime* integers?

65. Determine the largest r such that there exist infinitely many sequences of r consecutive integers each being the sum or difference of two squares.

***66.** How many representations has a positive integer as the sum of two squares, in average? In a precise formulation, we ask about the approximate behavior of the mean value function

$$\frac{r(1) + r(2) + \ldots + r(n)}{n}$$

for "large" values of n, where $r(n)$ denotes the number of integer solutions of the equation $x^2 + y^2 = n$.

***67.** Find all integer solutions of the equation $x^2 + 4 = y^3$.