MATH 354: PROBLEM SET 2

RICE UNIVERSITY, FALL 2019

Due date: Friday, September 13th, by 5pm in my office (you can slide it under the door). You are welcome to turn in your work during lecture on Friday.

Parts A, and B should be handed in **separately**. They will be graded by different TAs.

Please staple your homework!

Reminder from the syllabus: "The homework is not pledged and you can collaborate with other students in the class. In fact, you are very much *encouraged* to do so. However, you are not allowed to look up solutions in any written form; in particular, you are not allowed to look up solutions online. **Students caught violating this rule will be reported to the Honor Council.** You should write up your solutions individually."

1. Part A

Hand in the following exercises from Chapter 1 of Axler's book:

1C: 11, 12, 19, 21, 22.

2. Part B

Hand in the following exercises from Chapters 1 and 2 of Axler's book:

1C: 24.   2A: 1, 3, 5, 12.

In addition, please hand in a solution to the following problem on induction:

(1) The *Fibonacci numbers* are a famous sequence of integers $\{F_n\}_{n=0}^{\infty}$ defined by a recurrence relation. We set $F_1 = F_2 = 1$, and from then on the next term of the sequence is obtained by adding the two previous terms of the sequence:

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for all } n \geq 2.$$

Thus, the first few terms of this sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

Date: September 5th, 2019.

Use induction to prove that

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

3. Comments

(1) Problem 1C.11 looks similar to last week's 1C.10. But there is a key difference. Last week, you were asked to prove that the intersection of *two* subspaces is a subspace. Now you're being asked to show that the intersection of an *arbitrary collection* of subspaces is a subspace. Let me help you set this up. The idea is to label the subspaces in our collection using a set. So let I be a set, and suppose that we have a collection of subspaces $U_\alpha \subset V$, one for each $\alpha \in I$. What we want to prove is that $\bigcap_{\alpha\in I} U_\alpha$ is a subspace of V. Last week's problem is the special case $I = \{1, 2\}$, because in that case $\bigcap_{\alpha\in I} U_\alpha$ is just $U_1 \cap U_2$. But now I could be infinite, or it might not even consist of whole numbers! For example, V might be $\mathbb{R}^2$, and I might be $\mathbb{R}$, and for each $\alpha \in I$, U_α could be the line of slope α through the origin!

(2) For problem 1C.12, part of what we must learn to do in this course is to translate statements like the one in the problem into more formal mathematical statements. The formality is there to help organize your thoughts. So the first thing I would do in this problem is to give names to the two subspaces of V. Let's call then U_1 and U_2. The question is now asking us to prove the following: show that if $U_1 \cup U_2$ is a subspace of V, then either $U_1 \subset U_2$, or $U_2 \subset U_1$. Much better!

Perhaps the easiest way of solving this problem is to do a proof by contraction. Suppose that neither $U_1 \subset U_2$ nor $U_2 \subset U_1$. That means there must be a $u_1 \in U_1$ that is not in U_2 and a $u_2 \in U_2$ that is not in U_1. Now we have something to work with!

(3) Problem 1C.24 is quite neat when you unravel what the conclusion is trying to tell you: it says that every function $f\colon \mathbb{R} \to \mathbb{R}$ can be expressed in exactly one was as the sum of an even function and an odd function!

(4) The induction problem is fascinating. Where does the formula for F_n come from? There are square roots of 5 everywhere in it, and yet every time you plug in an integer n into it, all of those square roots somehow have to harmonize together to give you an integer! Later in the course we will use linear algebra to see how one might come up with this formula for Fibonacci numbers.