Irregularity of non-integer s-sets

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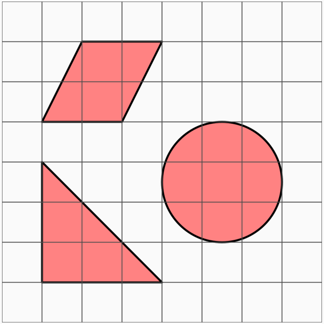
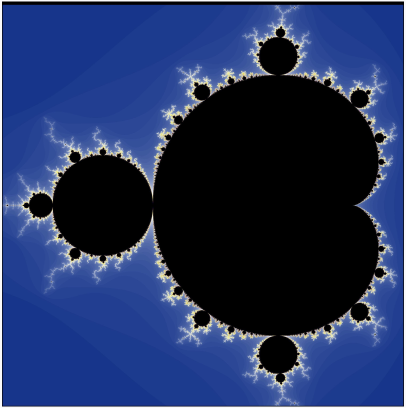
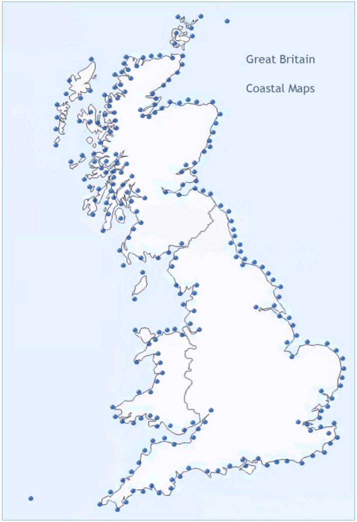
What is Geometric Measure Theory?





How long is the coastline of Britain?



Background Definitions Pt 1

Definition (Hausdorff Measure)
Let F be a subset of $\mathbb{R}^n$ and $s \geq 0$. For each $\delta > 0$, define

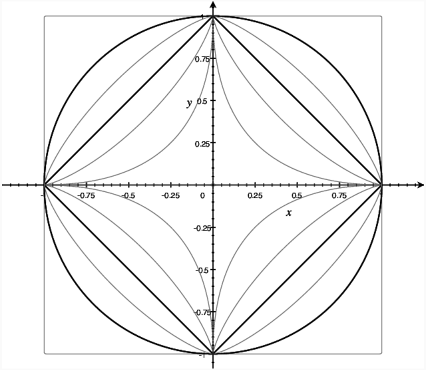
$$\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s \mid \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

The Hausdorff dimension looks at all covers of F of a certain dimension, and minimizes the s-th power of the diameters of the covering set. Notice that as δ decreases, the class of permissible covers in F is reduced, and so the infimum increases. This gives us

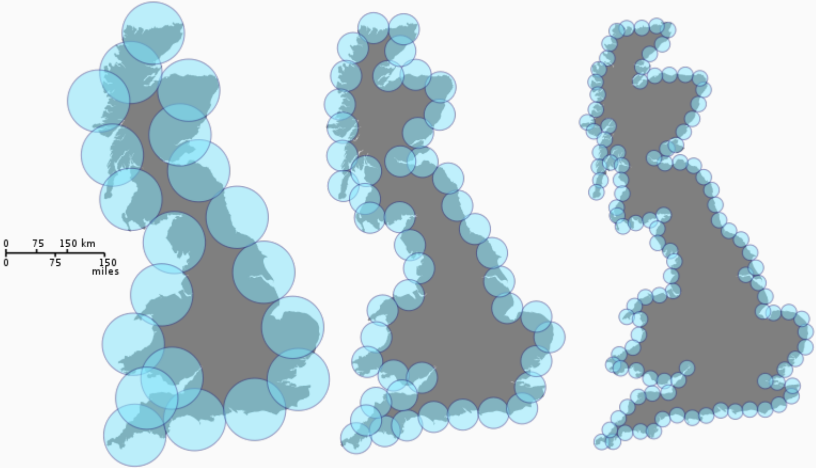
$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$$

which we define the s-**dimensional Hausdorff measure of** F. This limit exists for any subset F, but it can and will usually be 0 or ∞.

Fractional Diameters?



Britain coastline revisited



Background Definitions Pt 2

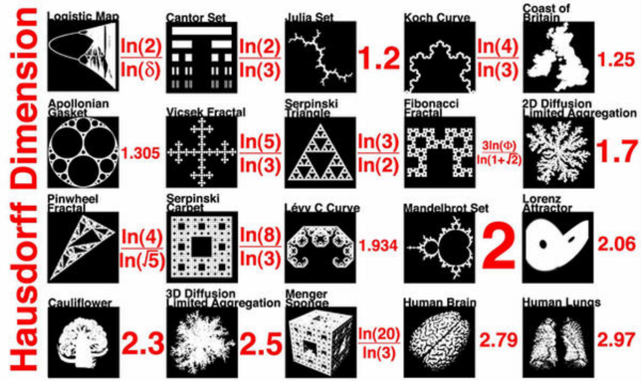
Definition (Hausdorff Dimension)
Let $F \subset \mathbb{R}^n$. Then the **Hausdorff dimension** of F is

$$\dim_H F := \inf \left\{ s \geq 0 \mid \mathcal{H}^s(F) = 0 \right\} = \sup \left\{ s \mid \mathcal{H}^s(F) = \infty \right\}.$$

This immediately gives

$$\mathcal{H}^s(F) = \begin{cases} \infty & 0 \leq s < \dim_H F \\ 0 & s > \dim_H F \end{cases}$$

Note that for $s = \dim_H F$, $\mathcal{H}^s(F)$ can be zero, infinite, or finite. A Borel set that $\mathcal{H}^s$ as finite is called an s-set.



Hausdorff Dimension
Logistic Map
ln(2)/ln(δ)
Cantor Set
ln(2)/ln(3)
Julia Set
1.2
Koch Curve
ln(4)/ln(3)
Coast of Britain
1.25
Apollonian Gasket
1.305
Vicsek Fractal
ln(5)/ln(3)
Serpinski Triangle
ln(3)/ln(2)
Fibonacci Fractal
3ln(Φ)/ln(1+√2)
2D Diffusion Limited Aggregation
1.7
Pinwheel Fractal
ln(4)/ln(√5)
Serpinski Carpet
ln(8)/ln(3)
Lévy C Curve
1.934
Mandelbrot Set
2
Lorenz Attractor
2.06
Cauliflower
2.3
3D Diffusion Limited Aggregation
2.5
Menger Sponge
ln(20)/ln(3)
Human Brain
2.79
Human Lungs
2.97

Background Definitions Pt 3

Definition (Upper and lower densities)
Let $x \in \mathbb{R}^n$ and F be an s-set. The **lower** and **upper density** is given by

$$\underline{D}^s(F,x) = \underline{\lim}_{r \to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s}$$
$$\overline{D}^s(F,x) = \overline{\lim}_{r \to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s}.$$

If they both agree, then the density of F at x exists and is that value.

Definition (Regular points)
If $\underline{D}^s(F,x) = \overline{D}^s(F,x) = 1$, then x is a **regular** point of F, otherwise it is an **irregular** point.
An s-set is called **regular** if, except on a set of $\mathcal{H}^s$-measure, all of its points are regular. If instead all of its points (except on a set of $\mathcal{H}^s$) are irregular, then the s-set is **irregular**.

Examples of Regularity

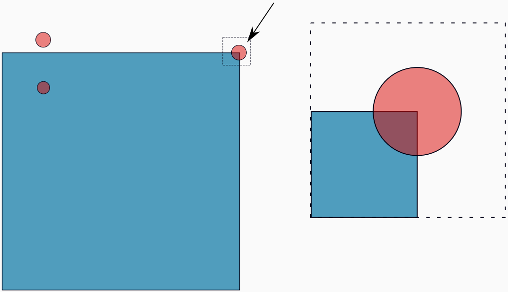
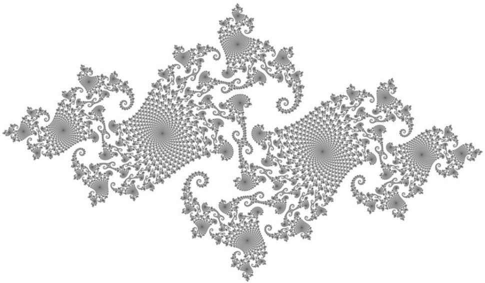


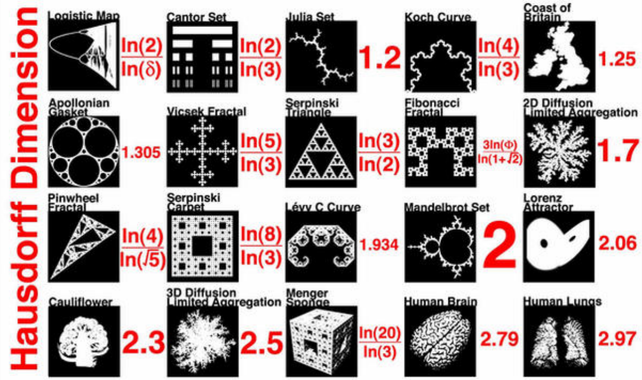
Figure 1: Square

Examples of Irregularity



Theorem

Falconer 5.2 Let F be an s-set in $\mathbb{R}^2$. Then F is irregular unless s is an integer.



Proof - Outline

We will only show here the $0 < s < 1$ case.

Idea
Show that the density $D^s(F, x)$ fails to exist almost everywhere in F, by contradiction.

We assume for a contradiction that there is a set $F_1 \subset F$ of positive measure where $\underline{D}^s(F, x) = \overline{D}^s(F, x)$.

Egoroff's Theorem
Tells us that there is $r_0 > 0$, and Borel set $E \subset F_1$ with $\mathcal{H}^s(E) > 0$ such that

$$D^s(F, x) = \frac{\mathcal{H}^s(F \cap B(x, r))}{(2r)^s} > \frac{1}{2}$$

for all $x \in E$ and $r < r_0$.

Proof - Annulus

Let $y \in E$ be a point with other points of E arbitrarily close. Let η satisfy $0 < \eta < 1$, and consider the annulus given below:

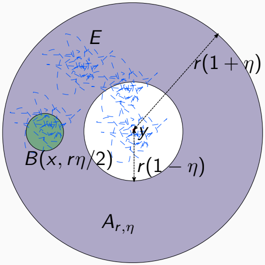


Figure 2: Annulus

Observe that

$$\lim_{r \to 0} \frac{\mathcal{H}^s(F \cap A_{r,\eta})}{(2r)^s} = D^s(F, y)((1+\eta)^s - (1-\eta)^s)$$

Final Contradiction

Because y is a cluster point, there is always an $x \in E$ with $|x - y| = r$, and so by construction $B(x, r\eta/2) \subset A_{r,\eta}$, yielding

$$\frac{\eta^s}{2} < \frac{\mathcal{H}^s(F \cap B(x, r\eta/2))}{r^s} \leq \mathcal{H}^s(F \cap A_{r,\eta})$$

which combined with:

$$\begin{aligned} 2^{-s-1}\eta^s &\leq D^s(F, y)((1+\eta)^s - (1-\eta)^s) \\ &= D^s(F, y)(2s\eta + \text{terms in } \eta^2) \end{aligned}$$

gives us a contradiction.

Credit

Huge thanks to Dr. Gregory Chambers for supporting and directing my work. Thanks to RUSP as well for contributing funding that enables our research.

Questions?