Patrick Yee

March 19, 2021

0.1

Show that the degree 2 subrepresentation U of S_3 given by – is irreducible:

$$U = (e_1 - e_2, e_2 - e_3)$$

Let (U', ρ') be a subrepresentation of (U, ρ). Suppose that U' is proper, then $\deg U' < \deg U$, so $\deg U'$ is either 1 or 0. Since U' is G-invariant, for every $g \in S^3$ and $u' \in U'$, we have that $\rho'(g)(u') \in U'$. Put $g = (12) = [2, 1, 3]$. Then we have

$$\rho'(g) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And

$$\rho'(g) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\rho'(g) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Note $e_1 - e_2 = (1, -1, 0)^T$ and $e_2 - e_3 = (0, 1, -1)^T$.

0.2

(2) [Wed] (a) Let K be a field. Given a K-vector space V and a K-linear operator T on V with $T^2 = T$, show that
$$V = \ker T \oplus \operatorname{im} T$$
as K-vector spaces.

(b) Find a group G, a representation (V, ρ) of G, along with a linear operator T on V that intertwines with the G-action, so that
$$V \neq \ker T \oplus \operatorname{im} T$$
as K-vector spaces. Recall that by T intertwining with the G-action, we mean that $T(\rho_g(v)) = \rho_g(T(v))$ for all $g \in G$, $v \in V$.

a) Let us consider some $v \in V$ and note that $T^2 = T$, we have that $Tv = T^2v$ which implies that $T(v - Tv) = 0$, and thus we have that $v - Tv$ is an element of $Ker(T)$, which we will call p. Thus, we get that $v - Tv = p$ which implies $v = Tv + p$ which shows us that $V = Im(T) + Ker(T)$. Now we must show that the intersection of $Im(T)$ and $Ker(T)$ is equal to 0. To do this, suppose $i \in \operatorname{im}T \cap \ker T$ so that we have $i = Tv$ for some $v \in V$. Multiplying by T again yields $0 = Ti = T^2v = Tv = i$ which shows us that the intersection of $Im(T)$ and $Ker(T)$ is $\{0\}$, and so we have that $V = Im(T) \oplus Ker(T)$, as desired.

b) **I think this doesn't work since T is not invertible?** Let $V = \mathbb{R}^2$, and put
$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
and let G be the group with one element, that is,
$$\rho(G) = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$$
It is clear then that T intertwines with G. Furthermore, we have that
$$\ker T = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$$
and
$$\operatorname{im}T = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$$
so $\ker T \oplus\ \operatorname{im}T \neq V = \mathbb{R}^2$.

0.3 Advanced problem A

Let ρ, ρ' be equivalent representations of G, that is, there is a linear map $\varphi : V \to V'$ which is a vector space isomorphism. Suppose that ρ is reducible. Then there is a proper subspace $W \subset V$ so that $\rho|_W(g) := \rho(g)|_W$ is a representation of G, and denote $W' = \varphi(W)$. Since φ is a vector space isomorphism, there is an induced isomorphism $\Phi : GL(V) \to GL(V')$ so that $\Phi(\rho(g)) = (\rho'(g))$. Of course, this implies that $\Phi(GL(W)) = GL(W')$. Since ρ, Φ are group homomorphisms, the composition $\rho'|_{W'} = \Phi \circ \rho|_W$ is a group homomorphism. Furthermore, $\rho'|_{W'}$ intertwines with ρ':

$$\begin{aligned}\rho'|_{W'}(g) &= \Phi \circ \rho|_W(g) = \Phi \circ \rho(g)|_W \\ &= \Phi(\rho(g)|_W) = \rho'(g)|_{\varphi(W)=W'}\end{aligned}$$

We conclude that $\rho'|_{W'}$ is a subrepresentation of ρ'.

Logically, we have that

$$\rho \text{ reducible} \Rightarrow \rho' \text{ reducible}$$

applying this result in the reverse direction also gives

$$\rho \text{ reducible} \Leftarrow \rho' \text{ reducible}$$

so

$$\begin{aligned}\rho \text{ reducible} &\iff \rho' \text{ reducible} \\ \rho \text{ irreducible} &\iff \rho' \text{ irreducible}\end{aligned}$$

as desired.

0.4 Advanced problem B

Let $(\rho, GL(\mathbb{C}^n))$ be a representation of G of degree n, and define $\rho^* : G \to GL(\mathbb{C}^n)$ by

$$\rho^*(g) = (\rho(g^{-1}))^T$$

We claim that ρ^* is a group representation of degree n. Let $g, h \in G$, then

$$\begin{aligned}\rho^*(gh) &= \rho(h^{-1}g^{-1})^T = (\rho(h^{-1})\rho(g^{-1}))^T \\ &= \rho(g^{-1})^T\rho(h^{-1})^T = \rho^*(g)\rho^*(h)\end{aligned}$$

so ρ^* is a group homomorphism into $GL(\mathbb{C}^n)$, and hence a representation.

0.5 Advanced problem C

0.6 Advanced problem D