Let $H \leq S_n$. For $1 \leq x \leq n$, the *orbit* of x is the set of all numbers which are images of x under the permutations in H: $\mathrm{orb}(x) = \{\pi(x) \mid \pi \in H\}$. E.g. if $n = 5$ and $H = \langle (12)(345) \rangle$, then $\mathrm{orb}(1) = \mathrm{orb}(2) = \{1, 2\}$ and $\mathrm{orb}(3) = \mathrm{orb}(4) = \mathrm{orb}(5) = \{3, 4, 5\}$. The *stabilizer* of x is the set of all permutations in H for which x is a fixed point: $\mathrm{Stab}(x) = \{\sigma \in H \mid \sigma(x) = x\}$. In the previous example, $\mathrm{Stab}(1) = \mathrm{Stab}(2) = \{e, (345), (354)\}$ and $\mathrm{Stab}(3) = \mathrm{Stab}(4) = \mathrm{Stab}(5) = \{e, (12)\}$.

Burnside's Lemma: The average number of fixed points is equal to the number of orbits.

Illustration: In the above example there are 2 orbits. The permutations in H are

$$e, (12)(345), \big((12)(345)\big)^2 = (354), \big((12)(345)\big)^3 = (12), \big((12)(345)\big)^4 = (345), \big((12)(345)\big)^5 = (12)(354),$$

and have 5, 0, 2, 3, 2, 0 fixed points, resp. Thus their average is $12/6 = 2$.

Proof: We count in two different ways the number k of pairs (σ, x) for which $\sigma(x) = x$.

Counting by permutations, k is simply the sum of the fixed points of the permutations in H.

Counting by points, (*) $k = \sum_{1 \leq x \leq n} |\mathrm{Stab}(x)|$. We show that (**) $|\mathrm{Stab}(x)| \cdot |\mathrm{orb}(x)| = |H|$. Every $\sigma \in H$ carries x into exactly one $y \in \mathrm{orb}(x)$. We claim that every y will be the image of $|\mathrm{Stab}(x)|$ permutations in H. Indeed, if $\sigma(x) = y$, then $\pi(x) = y \iff (\sigma^{-1}\pi)(x) = x$. This proves (**). Substituting (**) into (*), we obtain $k = \sum_{1 \leq x \leq n} (|H|/|\mathrm{orb}(x)|)$. Adding the terms for all elements in one orbit, we obtain just $|H|$. Hence k is $|H|$ times the number of orbits. Dividing by $|H|$, we get the statement of the lemma.

Example: In how many ways can we color 4 squares red in a 5×5 square grid where two colorings count the same if a rotation or a reflection can transform them into each other?

We permute the four element subsets of the 25 element grid by the symmetries of the (big) square and ask for the number of orbits. Thus, now a "point" is a 4 element subset of the grid, $n = \binom{25}{4}$, and $H = D_4$. By Burnside's Lemma, the number of orbits is the same as the average of the fixed points. Every point is a fixed point of the identity, i.e. e has $\binom{25}{4}$ fixed points. A point is fixed for some reflection, if either all the 4 squares are on the axis, or two are on the axis, and one-one are symmetric to it, or none are on the axis and two-two are symmetric to it. Thus, a reflection has $5 + \binom{5}{2} \cdot 10 + \binom{10}{2} = 150$ fixed points. A rotation of ± 90 degrees has $(25 - 1)/4 = 6$ fixed points, since the center cannot be among the four squares which are just the rotated images of each other. Similarly, the rotation of 180 degrees has $\binom{12}{2}$ fixed points. Therefore, the average number of fixed points is $\dfrac{\binom{25}{4} + 4 \cdot 150 + 2 \cdot 6 + \binom{12}{2}}{8} = 1666$. Thus, the number of colorings is 1666.