MATH 354: PROBLEM SET 6

RICE UNIVERSITY, FALL 2019

Due date: Friday, October 18th, by 5pm in my office (you can slide it under the door). You are welcome to turn in your work during lecture on Friday.

Parts A, and B should be handed in **separately**. They will be graded by different TAs.

Please staple your homework!

Reminder from the syllabus: "The homework is not pledged and you can collaborate with other students in the class. In fact, you are very much *encouraged* to do so. However, you are not allowed to look up solutions in any written form; in particular, you are not allowed to look up solutions online. **Students caught violating this rule will be reported to the Honor Council.** You should write up your solutions individually."

1. Part A

Hand in the following exercises from Chapter 3 of Axler's book:

3D: 18, 20; 3E: 1.

2. Part B

Hand in the following exercises from Chapter 3 of Axler's book:

3E: 4, 8, 12.

3. Comments

(1) Problem 3D.20 has an important moral, which your colleagues in Math 355 will beat to death: an inhomogeneous system

$$\begin{aligned} A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n &= c_1, \\ A_{2,1}x_1 + A_{2,2}x_2 + \cdots + A_{2,n}x_n &= c_2, \\ &\vdots \\ A_{n,1}x_1 + A_{n,2}x_2 + \cdots + A_{n,n}x_n &= c_n, \end{aligned}$$

Date: October 11th, 2019.

of n linear equations in the n variables $x_1, \dots, x_n$ is guaranteed to have a solution if the corresponding homogeneous system

$$\begin{aligned}
A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n &= 0, \\
A_{2,1}x_1 + A_{2,2}x_2 + \cdots + A_{2,n}x_n &= 0, \\
&\vdots \\
A_{n,1}x_1 + A_{n,2}x_2 + \cdots + A_{n,n}x_n &= 0.
\end{aligned}$$

only has the trivial solution.

(2) Exercise 3E.1 introduces the concept of "graph of a map". This is an important concept across many fields of mathematics. The exercise is mostly book keeping, but I took it as an opportunity to expose you to the idea of a map graph.

(3) Exercise 3E.4 is also book-keeping, but the result gets used, sometimes implicitly, in many applications of linear algebra. Please remember the statement! Perhaps the way to procede that avoids clutter is first do the problem when $m = 2$. Here you have to define maps

$$\Phi\colon \mathcal{L}(V_1 \times V_2, W) \to \mathcal{L}(V_1, W) \times \mathcal{L}(V_2, W)$$

and

$$\Psi\colon \mathcal{L}(V_1, W) \times \mathcal{L}(V_2, W) \to \mathcal{L}(V_1 \times V_2, W)$$

in such a way that the maps are linear (a claim you should justify) and such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identities of their respective domains. The general case can be treated by induction, where you can use the case $m = 2$ in the inductive step.

(4) Exercise 3E.8 says that a nonempty subset of a vector space is affine if and only if it contains all the lines joining any two of its points. This is a useful characterization of affine subsets to keep in your head: it allows you to detect "by inspection" if a subset has no chance of being affine, for example.

(5) Exercise 3E.12 shows that the notation V/U has voodoo-like properties, because it appears to obey the cancellation property $V \simeq U \times V/U$. Be careful though: treating spaces like objects of grade-school arithmetic can lead to dark places.

(6) WARNING: Note that in Exercise 3E.12 the space V is not assumed to be finite dimensional. However, V/U is assumed finite-dimensional, so it's OK to choose a basis for *this* space. We just don't have a tool to extend this to a basis for V (we didn't even define what a basis for an infinite dimensional vector space would look like!)