Algebra II: Homework 5

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Collaborated with the Yellow group.

Problem 1

Claim. Take a G-homomorphism $\varphi : V \to W$. Show that

(a). $\text{Ker}(\varphi)$ is a subrepresentation of V, and

(b). $\text{Im}(\varphi)$ is a subrepresentation of W.

Proof. (a). First, observe that $\text{Ker}(\varphi)$ is a vector subspace of V. To verify that it is a subrepresentation, we apply the g-action on an arbitrary element. Let $v \in \text{Ker}(\varphi)$:

$$\varphi(\rho_g(v)) = \rho_g(\varphi(v)) = \rho_g(0) = 0.$$

And hence it is closed under the g-action.

(b). Liekwise, we know that $\text{Im}(\varphi)$ is a vector subpace of V. Now we verify it is a subrepresentation:

$$\rho_g(w) = \rho_g(\varphi(v)) = \varphi(\rho_g(v))$$

and hence $\varphi_g(v)$ is in the image of φ as desired.

□

Problem 2

Claim. Let (V, ρ) be a representation of G, and take $g \in G$. Let X_V represent the character table of G. Prove:

1. $X_V(e) = \text{dim}V$
2. $X_{V \oplus W}(g) = X_V(g) + X_W(g)$
3. $X_V(g^{-1}) = \overline{X_V(g)}$
4. $\overline{X_V}$ is a character of G

Proof. □

Problem 3

Claim. Prove directly that the map $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an isomorphism of $\mathbb{Q}(\sqrt{2})$ with itself.

Proof. □

Problem 3 continued on next page. . .

Problem 4

Claim. Given a finite abelian group G, describe its irreducible complex representations, up to equivalence. Illustrate this for the Klein-four group $G = C_2 \times C_2$.

Proof. □

Problem 5

Claim. Let V and W be irreducible complex representations of a group G, and take $\varphi \in \mathrm{Hom}_G(V, W)$. Show that

(a). If $V \ncong W$, then φ is the zero map.

(b). If $V \cong W$ and $\varphi \neq 0$, then φ is a G-isomorphism.

Proof. □

Problem 6

Claim. There is a group G of order 8 that has five conjugacy classes of elements; suppose that the representations are $g_1, \ldots, g_5$. Given irreducible characters $\chi_1, \ldots, \chi_4$, complete the character table below for the row corresponding to the final irreducible character χ_5. Make sure to verify that χ_5 is indeed irreducible and pairwise inequivalent to $\chi_1, \ldots, \chi_4$ by using its values $\chi_5(g_j)$ for $j = 1, \ldots, 5$.

g_i	g_1	g_2	g_3	g_4	g_5
$\lvert C_G(g_i)\rvert$	8	8	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	?	?	?	?	?

Proof. □

Problem 7

Claim. Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of a finite group G. Show that

$$Z(G) = \left\{ g \in G \mid \sum_{i=1}^{r} \overline{\chi_i(g)}\chi_i(g) = |G| \right\}.$$

Here, $Z(G)$ is the center of G, which consist of elements $g \in G$ so that $gh = hg$ for each $h \in G$.

Proof. □

Problem 7 continued on next page. . .