Math 357: Undergraduate Abstract Algebra 2. Instructor: Dr. C. Walton

Homework #4 on lecture material from ~~M 2/15, F 2/19~~ M 2/22, W 2/24.
(changes due to Winter storm)

Due: Wednesday, March 3, 2021 at noon *sharp* *(better to submit early)*

Include full statements of problems in your solution set.
See syllabus for grading guide and teaching page for writing tips

Please include your full name at the top of your homework set.

Practice Problems (discussed during class time)

(1) [Mon] Show that the degree 2 representation U of S_3 in Lecture 10-Video 4 is irreducible.

(2) [Wed] (a) Let K be a field. Given a K-vector space V and a K-linear operator T on V with $T^2 = T$, show that

$$V = \ker T \oplus \operatorname{im} T$$

as K-vector spaces.

(b) Find a group G, a representation (V, ρ) of G, along with a linear operator T on V that intertwines with the G-action, so that

$$V \neq \ker T \oplus \operatorname{im} T$$

as K-vector spaces. Recall that by T intertwining with the G-action, we mean that $T(\rho_g(v)) = \rho_g(T(v))$ for all $g \in G$, $v \in V$.

Advanced Problems (completed outside of class time, and *can discuss in class if time permits)

(A) [Mon*] Let $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ be equivalent representations of a group G. Show that (V, ρ) is irreducible if and only if (V', ρ') is irreducible.

(B) [Mon*] Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of a group G of degree n. Consider the map:

$$\rho^* : G \to GL_n(\mathbb{C}), \quad g \mapsto (\rho_{g^{-1}})^T.$$

Show that ρ^* defines a degree n representation of G. Here, T stands for transpose.

(C) [Wed*] Let G be a finite group, and let (V, ρ) be a degree 2 representation of G. Assume that the characteristic of the ground field does not divide $|G|$. Suppose that there are elements $g, h \in G$ so that ρ_g and ρ_h do not commute as matrices. Show that (V, ρ) is irreducible.

(D) [Wed*] Consider the degree 2 real representation (V, ρ) of $\mathbb{Z}$, given by

$$\rho_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R}), \quad \forall n \in \mathbb{Z}.$$

Show that (V, ρ) is not completely reducible.