Now let us consider the Euler-Lagrange equations of a graph:

$$\inf_u \left\{ \int_\Omega f(\nabla u) \mid \int_\Omega g(u) = M \right\} \qquad (P_1)$$

$$\inf_u \left\{ \int_\Omega f(\nabla u) \mid \int_\Omega g(u) = M;\ u = u_0\ \partial\Omega \right\} \qquad (P_2)$$

Example 0.0.1

Consider $f(z) = \sqrt{1+|z|^2}$ and $g(s) = |s|$, $u_0 = 0$. We are looking for minimizers that satisfy $\int_\Omega |u| = M$. If $u \geq 0$, then this integral is exactly the area below the graph of u.

Problems of this sort are referred to as being of **isoperimetric type**. We have a fixed perimeter, and we want to minimize a function which is constrained along it. It does not always admit a solution, however.

To see this, consider $\Omega = B_R(0)$. The minimizer will be a spherical cap described by $u(x) = \sqrt{S^2 - |x|^2} - \sqrt{S^2 - R^2}$ for some $S \geq R$. If $M > \frac{|B_R|}{2}$ then there is no graph which can attain M within the constraints!

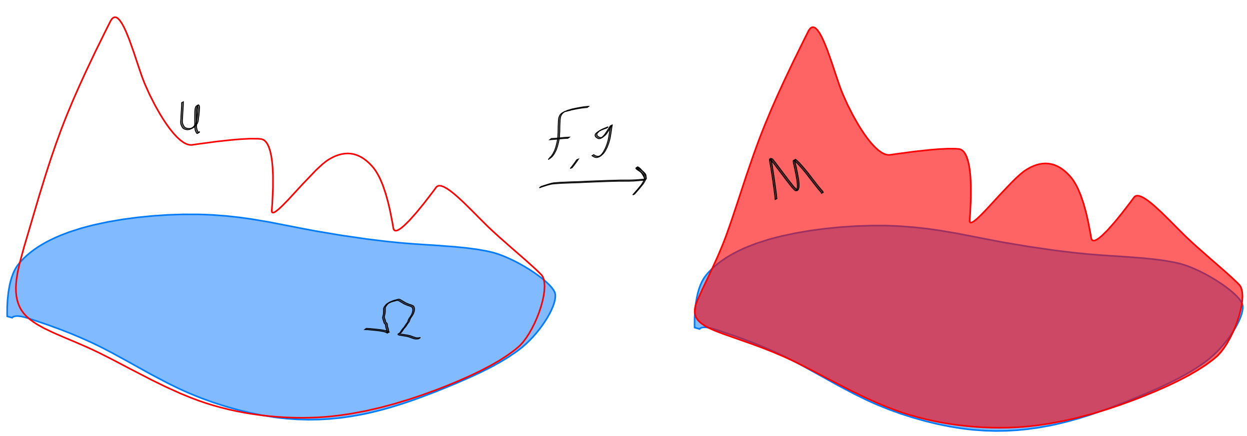

Figure 0.1: Area graph with $f(z) = \sqrt{1+|z|^2}$ and $g(s) = |s|$

Example 0.0.2

Consider $f(\nabla u) = \frac{|\nabla u|^2}{2}$, $g(u) = \frac{u^2}{2}$ and $u_0 = 0$.

In this case, minimizers *always* exist, and are eigenfunctions of $-\Delta$ that satisfy the zero Dirichlet condition.

0.1 Linear Transformations by Smooth Functions

We can learn more about minimizers on the graph family of Euler-Lagrange equations by combining them with nicely behaved functions. Consider

$$\int_\Omega g(u + t\varphi) = M$$

for all non-negative real t. What constraints must φ have to satisfy this? We can see that

$$g(u + t\varphi) = g(u) + tg'(u)\varphi + \frac{t^2}{2}g''(u)\varphi^2 + o(t^3) \implies$$
$$M = M + t\int_\Omega g'(u)\varphi + \frac{t^2}{2}\int_\Omega g''(u)\varphi^2 + o(t^3)$$

for all t. We can actually infer from this that $\int_\Omega g'(u)\varphi = 0$ by the first-order conservation of $\int g(u)$![1] Then the sum $u + t\varphi$ only differs from u by $o(t^2)$, and so for small t is relatively close.

In the context of differential geometry, we can interpret this new family as the functions in

$$T_u\mathcal{M} = \left\{\varphi \mid \int g'(u)\varphi = 0\right\}.$$

This is because if $\varphi : \int_\Omega g'(u)\varphi = 0$, then there should be an $O(t^2)$ correction such that

$$u + t\varphi + O(t^2) \in \mathcal{M}.$$

Let us discuss these families more formally. Let $\varphi, \xi \in C^\infty(\overline{\Omega})$ such that

$$\int_\Omega g'(u)\varphi = 0 \qquad \int_\Omega g'(u)\xi = 1$$

Consider $u + t\varphi + s\xi \in C^\infty(\Omega)$ with two parameters (t, s). The implicit function theorem tells us that there exists $\varepsilon > 0$ and $s(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that, if $u(t) = u + t\varphi + s(t)\xi$, then

$$\int_\Omega g(u(t)) = M \quad \forall\, |t| < \varepsilon$$
$$s(0) = 0$$

This $s(t)$ is in essence our $O(t^2)$ correction that allows the families φ to still satisfy our condition. Let us look in more detail how $s(t)$ interacts with our current formulations:

$$\begin{aligned}
\int_\Omega g(u) &= \int_\Omega g(u + t\varphi + s(t)\xi) \\
&= \int_\Omega g(u) + t\int_\Omega g'(u)\left[\varphi + s'(0)\xi\right] + \frac{t^2}{2}\int_\Omega g''(u)\left[\varphi + s'(0)\xi\right]^2 + g'(u)\xi s''(0) + o(t^2) \\
&\implies 0 = \int_\Omega g'(u)\left[\varphi + s'(0)\xi\right] = s'(0)\int_\Omega g'(u)\xi = s'(0) \\
&\implies 0 = \int_\Omega g''(u)\varphi^2 + s''(0)\int_\Omega g'(u)\xi
\end{aligned}$$

But notice that $\int_\Omega g'(u)\xi = 1$, so that

$$s''(0) = -\int_\Omega g''(u)\varphi^2$$

This classifies our correction factor up to low orders, so we have that

$$s \approx -\left[\int_\Omega g''(u)\varphi^2\right]\cdot\frac{t^2}{2}$$

[1] To check this, apply integration by parts and see what you get.

Now we can plug this back into the Euler-Lagrange equations of the problem

$$\inf_u \left\{ \int_\Omega f(\nabla u) \mid \int_\Omega g(u) = M \right\}.$$

to get

$$\mathcal{F}(u + t\varphi + s(t)\xi) \geq \mathcal{F}(u) \quad \forall |t| < \varepsilon$$

and our variations on

$$f(\nabla u + t\nabla\varphi + s(t)\nabla\xi)$$

are given by

$$\begin{aligned}
&\frac{d}{dt} f(\nabla u + t\nabla\varphi + s(t)\nabla\xi) = \nabla f(\nabla u + \ldots) \cdot (\nabla\varphi + s'(t)\nabla\xi) \\
&= \nabla f(\nabla u) \cdot \nabla\varphi \\
&\frac{d^2}{dt^2} f(\nabla u + t\nabla\varphi + s(t)\nabla\xi) = (\nabla\varphi + s'(t)\nabla\xi) \cdot \nabla^2 f(\nabla u + \ldots)(\nabla\varphi + s'(t)\nabla\xi) + \nabla f(\nabla u + \ldots) \cdot \nabla\xi s''(t) \\
&= \nabla\varphi \cdot (\nabla^2 f(\nabla u)\nabla\varphi) + \nabla f(\nabla u) \cdot \nabla\xi s''(0)
\end{aligned}$$

Now let $\psi \in C^\infty(\overline{\Omega})$ and choose

$$\varphi = \psi - \left[\frac{\int g'(u)\psi}{\int g'(u)^2}\right] g'(u) = \int_\Omega g'(u)\varphi = 0$$

Or in other words, we are choosing φ to be ψ subtracted by its projection along $g'(u)$. We are choosing φ this way so that $\langle g'(u), \varphi \rangle = 0$. Note that there is an implicit assumption here that g satisfies $\int g'(u)^2 > 0$. Lastly, choose $\xi = \dfrac{g'(u)}{\int_\Omega g'(u)^2}$ so that $\langle g'(u), \xi \rangle = 1$.

Now let's reconsider the Euler-Lagrange equations, but with the choices of φ and ξ as above. That is, for all φ and ξ of the form above:

$$\begin{aligned}
\int_\Omega \nabla f(\nabla u) \cdot \nabla\varphi = 0 \\
\int_\Omega \nabla\varphi \cdot (\nabla^2 f(\nabla u)\nabla\varphi) + s''(0)\nabla f(\nabla u) \cdot \nabla\xi \geq 0
\end{aligned}$$

Now we substitute in the projection form for φ :

$$\begin{aligned}
\int_\Omega \nabla f(\nabla u) \cdot \left[\nabla\psi - \left[\frac{\int g'(u)\psi}{\int g'(u)^2}\right] g''(u)\nabla u\right] = 0 \\
\implies \int_\Omega \nabla f(\nabla u) \cdot \nabla\psi - \lambda(u) \int_\Omega g'(u)\psi = 0
\end{aligned}$$

and so

$$\lambda(u) = \frac{\int_\Omega g''(u)\left[\nabla u \cdot \nabla f(\nabla u)\right]}{\int_\Omega g'(u)^2}$$

This $\lambda(u)$ is what we refer to by a **Lagrange multiplier**.

Now denote $X \cong \nabla f(\nabla u)$ and observe:

$$\begin{aligned}
\int_\Omega X \cdot \nabla\psi &= \int_\Omega dw(\psi X) - \int_\Omega \psi \operatorname{div} X \\
&= \int_{\partial\Omega} \psi(X \cdot \nu_\Omega) - \int_\Omega \psi \operatorname{div} X.
\end{aligned}$$

$$0 = \int_{\partial\Omega} \psi\nu_\Omega \cdot \nabla f(\nabla u) + \int_\Omega \psi\left[-dw(\nabla f(\nabla u)) - \lambda g'(u)\psi\right] \quad \forall\psi \in C^\infty(\overline{\Omega})$$

Testing on $\psi = 0$ on $\partial\Omega$, but arbitrary otherwise, we get on Ω :

$$-dw(\nabla f(\nabla u)) = \lambda g'(u)$$

Once we know this we get

$$\begin{cases} -dw(\nabla f(\nabla u)) = \lambda g'(u) & \Omega \\ \nu_\Omega \cdot \nabla f(\nabla u) = 0 & \partial\Omega \end{cases}$$

Example 0.1.1

Choosing $f(z) = \frac{|z|^2}{2}$ and $g(u) = \frac{u^2}{2}$, the above becomes

$$-\Delta u = \lambda u \quad \Omega$$

$$\frac{\partial u}{\partial \nu_\Omega} = \nabla u \cdot \nu_\Omega = 0 \quad \partial\Omega$$

This is the Neumann eigenfunctions of the Laplacian Ω.

Example 0.1.2

Consider the space $\Omega = (0, \pi)$, and let $u_k(x) = \cos(kx)$ for $k \in \mathbb{N}$. These are all solutions to

$$-u_k'' = \lambda u_k \quad (0, \pi)$$

$$u_k' \mid_{0,\pi} = 0$$

$\lambda = k^2$ and $(u_k) \cong k^2$. This is an example of a variational problem with many critical points.

Notice that if $f(z) = \sqrt{1 + |z|^2}$ then $\nabla f(\nabla u) \cdot \nu_\Omega = \frac{\nabla u \cdot \nu_\Omega}{\sqrt{1+|\nabla u|^2}} = 0$.