Algebra II: Homework 6

Due on March 17, 2021

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Problem 1

Claim. Show that $x^3 + x + 1$ is irreducible over $\mathbb{F}_2$ and let θ be a root. Compute the powers of θ in $\mathbb{F}_2(\theta)$.

Proof. □

Problem 2

Claim. Determine the minimal polynomial over $\mathbb{Q}$ for the element $1 + i$.

Proof. □

Problem 3

Claim. Let $\mathbb{F}$ be a finite field of characteristic p. Prove that $|\mathbb{F}| = p^n$ for some positive integer n.

Proof. □

Problem 4

Claim. Determine the degree over $\mathbb{Q}$ of $2 + \sqrt{3}$ and of $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

Proof. □

Problem 5

Claim. Prove that $x^5 - ax - 1 \in Z[x]$ is irreducible unless $a = 0, 2, -1$. The first two correspond to linear factors, the third corresponds to the factorization $(x^2 - x + 1)(x^3 + x^2 - 1)$.

Proof. □

Problem 6

Claim. Prove that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Conclude that $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : Q] = 4$. Find an irreducible polynomial satisfied by $\sqrt{2} + \sqrt{3}$.

Proof. □

Problem 6 continued on next page...

Problem 7

Claim. Suppose the degree of the extension K/F is a prime p. Show that any subfield E of K containing F is either K or F.

Proof. □

Problem 8

Claim. Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

Proof. □

Problem 8 continued on next page...