Math 357: Undergraduate Abstract Algebra 2. Instructor: Dr. C. Walton

Homework #5 on lecture material from F 2/26, W 3/3, F 3/5

Due: Wednesday, March 10, 2021 at noon *sharp* *(better to submit early)*

Include full statements of problems in your solution set.
See syllabus for grading guide and teaching page for writing tips

Please include your full name at the top of your homework set.

Practice Problems (discussed during class time)

(1) [Fri] Take a G-homomorphism $\phi : V \to W$. Show that
   (a) $\ker(\phi)$ is a subrepresentation of V, and
   (b) $\mathrm{im}(\phi)$ is a subrepresentation of W.

(2) [Wed] Prove parts (1), (3), (4), (5) of the Proposition in Lecture 15, Video 3.
   Hint for part (5): use Exercise B of HW #4.

(3) [Fri] Dummit-Foote, Section 13.1, Exercise #4

Advanced Problems (completed outside of class time, and *can discuss in class if time permits)

(A) [Fri*] Given a finite abelian group G, describe its irreducible complex representations, up to equivalence. Illustrate this for the Klein-four group $G = C_2 \times C_2$.

(B) [Fri*] Let V and W be irreducible complex representations of a group G, and take $\phi \in \mathrm{Hom}_G(V, W)$. Show that
   (a) If $V \ncong W$, then ϕ is the zero map.
   (b) If ($V \cong W$ and) $\phi \neq 0$, then ϕ is a G-isomorphism.

(C) [Wed*] There is a group G of order 8 that has five conjugacy classes of elements; suppose that the representatives are $g_1, \dots, g_5$. Given irreducible characters $\chi_1, \dots, \chi_4$, complete the character table below for the row corresponding to the final irreducible character χ_5. Make sure to verify that χ_5 is indeed irreducible and pairwise inequivalent to $\chi_1, \dots, \chi_4$ by using its values $\chi_5(g_j)$ for $j = 1, \dots, 5$.

g_i	g_1	g_2	g_3	g_4	g_5
$\lvert C_G(g_i)\rvert$	8	8	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	?	?	?	?	?

(D) [Wed*] Let $\chi_1, \dots, \chi_r$ be the irreducible characters of a finite group G. Show that

$$Z(G) = \left\{ g \in G \ : \ \sum_{i=1}^{r} \overline{\chi_i(g)}\chi_i(g) = |G| \right\}.$$

Here, $Z(G)$ is the center of G, which consist of elements $g \in G$ so that $gh = hg$ for each $h \in G$.

(No advanced problem for Fri 3/5)