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Calculus of Variations and Gradient Flows

Based on seminar at RTG - Analysis of PDEs

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Chapter 0

Introduction

These lecture notes are based on a week-long seminar class taught by Dr. Francesco Maggi at UT Austin at the summer RTG Analysis and PDEs seminar. These cover introductory topics related to calculus of variations, such as first and second variation formulae, gradient flows, and related arguments.

Chapter 1

Euler-Lagrange Equations

Calculus of variations refers to an array of variational techniques that are utilized in optimization problems. In particular, we will be learning how to find minimizers of functionals. That is, we will consider a particular functional (real-valued function that takes in functions), and apply various techniques to find the minimal function that fits our conditions.

First, we need to discuss the class of objects we will be working with. Let $\mathcal{F}$ be a functional, $\Omega \subset \mathbb{R}^n$ an open set, and $u_0 : \partial\Omega \to \mathbb{R}$ is our constraint on the boundary. Then our goal is to find

$$\inf_u \{\mathcal{F}(u) \mid u = u_0 \text{ on } \partial\Omega\}$$

where $u : \Omega \to \mathbb{R}$ is $C^1(\overline{\Omega})$ (differentiable and ∇u is continuous up to $\partial\Omega$). We will choose our functional to be

$$\mathcal{F}(u) = \int_\Omega f(\nabla u(x))\, dx = \int_\Omega f(\nabla u)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary scalar-valued function. In essence, we are looking at a class of functionals that take on this form for some f. It turns out that this form is very natural. Choosing

$$f(\nabla u) = \frac{1}{2}|\nabla u|^2$$

corresponds to $\mathcal{F}$ being the Dirichlet energy, and choosing

$$f(\nabla u) = \sqrt{1 + |\nabla u|^2}$$

corresponds to $\mathcal{F}$ being an area functional. Finally, we might simply have that $\mathcal{F}(u)$ is an n-dimensional representation of a graph of u over Ω:

$$\inf_{\int_\Omega g(u) = M} \mathcal{F}(u) \quad g : \mathbb{R} \to \mathbb{R}$$
$$\inf \left\{ \mathcal{F}(u) \mid \int_\Omega g(u) = M, \quad u = u_0 \partial\Omega \right\}$$

1.1 Gradient Flows

The first step in finding the minimizers is to look at gradient flows.

Definition 1.1.1: Gradient Flow

A **gradient flow** is a system of ordinary differential equations written in the form:

$$x'(t) = -\nabla F(x(t))$$
$$x(0) = x_0$$

where t is a non-negative real variable, $x : [0, \infty) \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}$. We refer to these systems as gradient flows because $-\nabla F(x(t))$ points in the direction of steepest descent of $F : \mathbb{R}^n \to \mathbb{R}$.

If x_0 is a critical point of F, then $\nabla F(x_0) = 0$. This implies $x(t) \equiv x_0$ (and hence is trivial. So we can assume from here on out that it is non-trivial.

We can generalize the notion of gradient flows to functionals. Let $\mathcal{F} : C^1(\overline{\Omega}) \to \mathbb{R}$ be a functional such as $\mathcal{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$, or $\mathcal{F}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}$. A gradient flow of $\mathcal{F}$ is a function $u(x,t) : \Omega \times [0,\infty) \to \mathbb{R}$ that satisfies

$$u_{(t)} = \frac{\partial u}{\partial t} = -\mathrm{grad}\ \mathcal{F}(u(x,t))$$

We need to be a bit more careful with these– need to discuss what the gradient of a functional even looks like. This formulation is very useful, though, as there are many PDEs can be written in this form and hence are expressible as gradient flows.

1.2 Applying Variational Techniques

Our goal is to develop variational techniques that apply to

$$\inf_{u \in C^1(\overline{\Omega})} \{\mathcal{F}(u) \mid u = u_0 \partial\Omega\} \quad (P)$$

We need a few necessary conditions for minimality. We say that $u \in C^1(\overline{\Omega})$ is a minimizer in (P) if

$$\mathcal{F}(u) \le \mathcal{F}(v) \quad \forall v \text{ s.t. } v = u_0 \partial\Omega.$$

Definition 1.2.1: Variations

Fix $\varphi \in C_c^\infty(\Omega)$ and take

$$u + t\Omega \in \mathcal{A} \quad \forall t \in \mathbb{R}$$

(This $\mathcal{A}$ will be our "competition class") We call these the **variations** of u. This gives us that

$$\mathcal{F}(u) \le \mathcal{F}(u + t\Phi) \quad \forall t \in \mathbb{R}$$
$$\delta\mathcal{F}_u(\varphi) = \frac{d}{dt}\ |_{t=0}\ \mathcal{F}(u + t\varphi) = 0$$
$$\delta^2\mathcal{F}_u(\Phi) = \frac{d^2}{dt^2}\ |_{t=0}\ \mathcal{F}(u + t\varphi) \ge 0.$$

We call $\delta\mathcal{F}_u : C_c^\infty(\Omega) \to \mathbb{R}$ the **first variation**, and likewise $\delta^2\mathcal{F}_u : C_c^\infty(\Omega) \to \mathbb{R}$ the **second variation**.

Then we can state the former more simply: if u is a minimizer in P, then

$$\delta\mathcal{F}_u \equiv 0$$
$$\delta^2\mathcal{F}_u \ge 0$$

Why is this the case? We know that

$$\mathcal{F}(u + t\varphi) = \int_\Omega f(\nabla u + t\nabla\varphi) = \int_\Omega f(\nabla u(x) + t\nabla\varphi(x))\, dx$$
$$f(z + tw) = f(z) + t\nabla f(z) \cdot w + \frac{t^2}{2} w \cdot \left[\nabla^2 f(z) w\right] + o(t^2) \quad z, w \in \mathbb{R}^n,\ t \in \mathbb{R}$$

Take $z = \nabla u(x)$ and $w = \nabla\varphi(x)$. Then we write

$$\mathcal{F}(u + t\varphi) = \mathcal{F}(u) + t\int_\Omega \nabla f(\nabla u) \cdot \nabla\varphi + \frac{t^2}{2}\int_\Omega \nabla\varphi \cdot \left[\nabla^2 f(\nabla u)\nabla\varphi\right] + o(t^2)$$

Now see that we have

$$\frac{d}{dt}\ |_{t=0}\ \mathcal{F}(u + t\varphi) = \int_\Omega \nabla f(\nabla u) \cdot \nabla\varphi$$
$$\frac{d^2}{dt^2}\ |_{t=0}\ \mathcal{F}(u + t\varphi) = \int_\Omega \nabla\varphi \cdot \left[\nabla^2 f(\nabla u)\nabla\varphi\right].$$

Remark 1.2.1

The leading term $o(t^2)$ implies that $o(t^2) = h(z, w, t)$for some h such that $\lim_{t\to 0} \sup_{|w|\leq 1} \frac{h(z,w,t)}{t^2} = 0$.

So if u is a minimizer in $\inf_u \{\mathcal{F}(u) \mid u = u_0 \text{ on } \partial\Omega\}$ then we must have

$$\delta\mathcal{F}_u(\varphi) = \int_\Omega \mathcal{F}_u(\nabla u) \cdot \nabla\varphi = 0 \quad \forall\varphi \in C_c^\infty(\Omega)$$
$$\delta^2\mathcal{F}_u(\varphi) = \int_\Omega \nabla\varphi \cdot \left[\nabla^2 f(\nabla u)\nabla\varphi\right] \geq 0 \quad \forall\varphi \in C_c^\infty(\Omega)$$

Our goal is to utilize these variations to derive the Euler-Lagrangian equations.

Remark 1.2.2

When $f(z)$ is convex, then $A = \nabla^2 f(z) \geq 0$, and $w \cdot (Aw) \geq 0$ for all $w \in \mathbb{R}^n$, so the second condition is always satisfied if f is convex.

We have two critical tools when applying these variational techniques. The first is called the fundamental lemma of the calculus of variations.

Lemma 1.2.1: Fundamental Lemma of Calculus of Variations

If $u \in L^1_{\mathrm{loc}}(\Omega)$[1]and $\int_\Omega u\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then $u = 0$ almost everywhere in Ω.

That is, there exists a set $S \subset \Omega$ with $\mathcal{L}^n(S) = 0$ such that $u(x) = 0$ for all $x \in \Omega \setminus S$.

Note that this captures the notion that we don't care as much about u at a single point, but more about integrating u over a small ball centered at a point.

The other critical tool is the divergence theorem.

Theorem 1.2.1: Divergence Theorem

If $X \in C^1(\overline{\Omega}, \mathbb{R}^n)$, then $\mathrm{div}X$[2]satisfies

$$\int_\Omega \mathrm{div}X = \int_{\partial\Omega} X \cdot \nu_\Omega$$

where ν_Ω is the outer unit normal.

If u is a minimizer, then

$$0 = \int_\Omega \nabla f(\nabla u) \cdot \nabla\varphi$$

Take $X = \nabla f(\nabla u)$ and observe that

$$\mathrm{div}(\varphi X) = X \cdot \nabla\varphi + \varphi\mathrm{div}X.$$

[1] Recall that

$$L^1_{\mathrm{loc}}(\Omega) = \left\{u : \Omega \to \mathbb{R} \mid u \text{ measurable}, \int_K |u| < \infty \quad \forall K \Subset \Omega \text{ compact}\right\}$$

[2] Recall that

$$\mathrm{div}X = \sum_{i=1}^n \frac{\partial}{\partial X_i}\left[X^{(i)}\right], \quad X = (X^{(1)}, X^{(2)}, \ldots, X^{(n)}).$$

Then observe that

$$\begin{aligned}\int_\Omega \nabla f(\nabla u)\cdot\nabla\varphi &= \int_\Omega X\cdot\nabla\varphi\\ &= \int_\Omega \operatorname{div}(\varphi X) - \int_\Omega \varphi \operatorname{div} X\\ &= \int_{\partial\Omega} (\varphi X)\cdot\nu_\Omega - \int_\Omega \varphi\operatorname{div}X.\end{aligned}$$

Observe that because $\varphi\in C_c^\infty(\Omega)$, the first term is zero! So

$$0 = -\int_\Omega \varphi\operatorname{div}X \quad \forall\varphi\in C_c^\infty(\Omega)$$

By the fundamental lemma of calculus of variations, this implies that $\operatorname{div}X = 0$ almost everywhere in Ω, and because it is continuous, it means that $\operatorname{div}X = 0$ everywhere in Ω.

In conclusion:

$$\begin{aligned}\operatorname{div}\left[\nabla f(\nabla u)\right] = 0 &\quad \text{in } \Omega\\ u = u_0 &\quad \text{in } \partial\Omega\end{aligned}$$

are the **Euler-Lagrange equations** of $\inf_u \{\mathcal{F}(u) \mid u = u_0 \text{ on } \partial\Omega\}$.

Example 1.2.1: Dirichlet problem for the Laplace equation

Take $f(z) = \dfrac{|z|^2}{2}$. Then $\nabla f(z) = z$ and so the Euler-Lagrange equations tell us that under the normal assumptions, u satisfies

$$\begin{cases}\Delta u = 0 & \Omega\\ u = u_0 & \partial\Omega\end{cases}$$

Example 1.2.2: Minimal surfaces equation

If we take $f(z) = \sqrt{1+|z|^2}$, then we have

$$\nabla f(z) = \frac{z}{\sqrt{1+|z|^2}} \implies$$

$$\begin{cases}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \Omega\\ u = u_0 & \partial\Omega\end{cases}$$

Now let us consider the Euler-Lagrange equations of a graph:

$$\inf_u \left\{\int_\Omega f(\nabla u) \mid \int_\Omega g(u) = M\right\} \tag{P_1}$$

$$\inf_u \left\{\int_\Omega f(\nabla u) \mid \int_\Omega g(u) = M;\ u = u_0\ \partial\Omega\right\} \tag{P_2}$$

Example 1.2.3

Consider $f(z) = \sqrt{1+|z|^2}$ and $g(s) = |s|$, $u_0 = 0$. We are looking for minimizers that satisfy $\int_\Omega |u| = M$. If $u \geq 0$, then this integral is exactly the area below the graph of u.

Problems of this sort are referred to as being of **isoperimetric type**. We have a fixed perimeter, and we want to minimize a function which is constrained along it. It does not always admit a solution, however.

To see this, consider $\Omega = B_R(0)$. The minimizer will be a spherical cap described by $u(x) = \sqrt{S^2 - |x|^2} - \sqrt{S^2 - R^2}$ for some $S \geq R$. If $M > \frac{|B_R|}{2}$ then there is no graph which can attain M within the constraints!

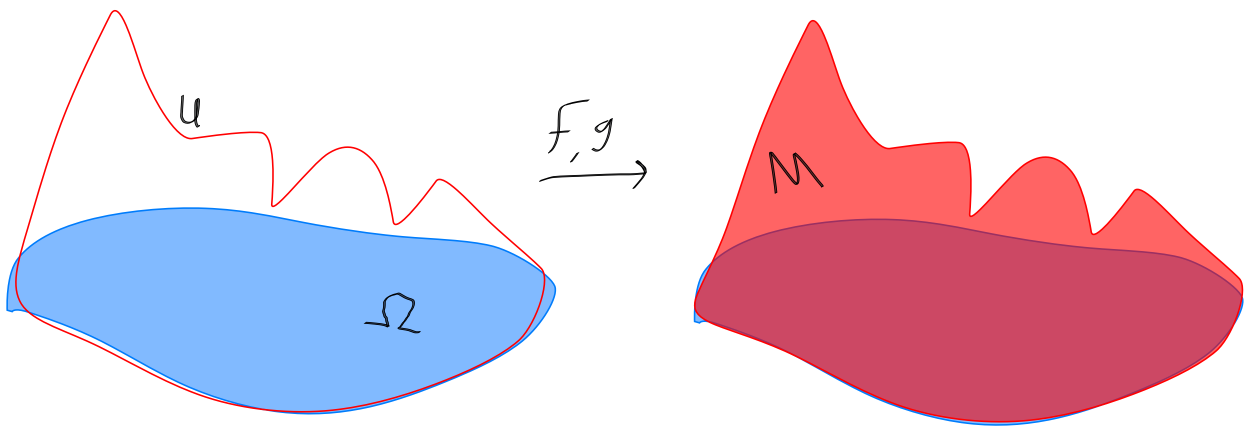

Figure 1.1: Area graph with $f(z) = \sqrt{1 + |z|^2}$ and $g(s) = |s|$

Example 1.2.4

Consider $f(\nabla u) = \frac{|\nabla u|^2}{2}$, $g(u) = \frac{u^2}{2}$ and $u_0 = 0$.

In this case, minimizers *always* exist, and are eigenfunctions of $-\Delta$ that satisfy the zero Dirichlet condition.

1.3 Linear Transformations by Smooth Functions

We can learn more about minimizers on the graph family of Euler-Lagrange equations by combining them with nicely behaved functions. Consider

$$\int_\Omega g(u + t\varphi) = M$$

for all non-negative real t. What constraints must φ have to satisfy this? We can see that

$$g(u + t\varphi) = g(u) + tg'(u)\varphi + \frac{t^2}{2} g''(u)\varphi^2 + o(t^3) \implies$$
$$M = M + t\int_\Omega g'(u)\varphi + \frac{t^2}{2}\int_\Omega g''(u)\varphi^2 + o(t^3)$$

for all t. We can actually infer from this that $\int_\Omega g'(u)\varphi = 0$ by the first-order conservation of $\int g(u)$![3] Then the sum $u + t\varphi$ only differs from u by $o(t^2)$, and so for small t is relatively close.

[3] To check this, apply integration by parts and see what you get.

In the context of differential geometry, we can interpret this new family as the functions in

$$T_u\mathcal{M} = \left\{\varphi \mid \int g'(u)\varphi = 0\right\}.$$

This is because if $\varphi : \int_\Omega g'(u)\varphi = 0$, then there should be an $O(t^2)$ correction such that

$$u + t\varphi + O(t^2) \in \mathcal{M}.$$

Let us discuss these families more formally. Let $\varphi, \xi \in C^\infty(\overline{\Omega})$ such that

$$\int_\Omega g'(u)\varphi = 0 \quad \int_\Omega g'(u)\xi = 1$$

Consider $u + t\varphi + s\xi \in C^\infty(\Omega)$ with two parameters (t, s). The implicit function theorem tells us that there exists $\varepsilon > 0$ and $s(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that, if $u(t) = u + t\varphi + s(t)\xi$, then

$$\begin{aligned}\int_\Omega g(u(t)) = M \quad \forall\, |t| < \varepsilon \\ s(0) = 0\end{aligned}$$

This $s(t)$ is in essence our $O(t^2)$ correction that allows the families φ to still satisfy our condition. Let us look in more detail how $s(t)$ interacts with our current formulations:

$$\begin{aligned}\int_\Omega g(u) &= \int_\Omega g(u + t\varphi + s(t)\xi) \\ &= \int_\Omega g(u) + t\int_\Omega g'(u)\left[\varphi + s'(0)\xi\right] + \frac{t^2}{2}\int_\Omega g''(u)\left[\varphi + s'(0)\xi\right]^2 + g'(u)\xi s''(0) + o(t^2) \\ &\implies 0 = \int_\Omega g'(u)\left[\varphi + s'(0)\xi\right] = s'(0)\int_\Omega g'(u)\xi = s'(0) \\ &\implies 0 = \int_\Omega g''(u)\varphi^2 + s''(0)\int_\Omega g'(u)\xi\end{aligned}$$

But notice that $\int_\Omega g'(u)\xi = 1$, so that

$$s''(0) = -\int_\Omega g''(u)\varphi^2$$

This classifies our correction factor up to low orders, so we have that

$$s \approx -\left[\int_\Omega g''(u)\varphi^2\right] \cdot \frac{t^2}{2}$$

Now we can plug this back into the Euler-Lagrange equations of the problem

$$\inf_u \left\{\int_\Omega f(\nabla u) \mid \int_\Omega g(u) = M\right\}.$$

to get

$$\mathcal{F}(u + t\varphi + s(t)\xi) \geq \mathcal{F}(u) \quad \forall\, |t| < \varepsilon$$

and our variations on

$$f(\nabla u + t\nabla\varphi + s(t)\nabla\xi)$$

are given by

$$\begin{aligned}&\frac{d}{dt} f(\nabla u + t\nabla\varphi + s(t)\nabla\xi) = \nabla f(\nabla u + \ldots) \cdot (\nabla\varphi + s'(t)\nabla\xi) \\ &= \nabla f(\nabla u) \cdot \nabla\varphi \\ &\frac{d^2}{dt^2} f(\nabla u + t\nabla\varphi + s(t)\nabla\xi) = (\nabla\varphi + s'(t)\nabla\xi) \cdot \nabla^2 f(\nabla u + \ldots)(\nabla\varphi + s'(t)\nabla\xi) + \nabla f(\nabla u + \ldots) \cdot \nabla\xi s''(t) \\ &= \nabla\varphi \cdot (\nabla^2 f(\nabla u)\nabla\varphi) + \nabla f(\nabla u) \cdot \nabla\xi s''(0)\end{aligned}$$

Now let $\psi \in C^\infty(\overline{\Omega})$ and choose

$$\varphi = \psi - \left[\frac{\int g'(u)\psi}{\int g'(u)^2}\right] g'(u) = \int_\Omega g'(u)\varphi = 0$$

Or in other words, we are choosing φ to be ψ subtracted by its projection along $g'(u)$. We are choosing φ this way so that $\langle g'(u), \varphi\rangle = 0$. Note that there is an implicit assumption here that g satisfies $\int g'(u)^2 > 0$. Lastly, choose $\xi = \dfrac{g'(u)}{\int_\Omega g'(u)^2}$ so that $\langle g'(u), \xi\rangle = 1$.

Now let's reconsider the Euler-Lagrange equations, but with the choices of φ and ξ as above. That is, for all φ and ξ of the form above:

$$\int_\Omega \nabla f(\nabla u) \cdot \nabla \varphi = 0$$
$$\int_\Omega \nabla \varphi \cdot (\nabla^2 f(\nabla u)\nabla\varphi) + s''(0)\nabla f(\nabla u)\cdot \nabla \xi \geq 0$$

Now we substitute in the projection form for φ :

$$\begin{aligned}\int_\Omega \nabla f(\nabla u) \cdot \left[\nabla\psi - \left[\frac{\int g'(u)\psi}{\int g'(u)^2}\right] g''(u)\nabla u\right] = 0 \\ \implies \int_\Omega \nabla f(\nabla u)\cdot\nabla\psi - \lambda(u)\int_\Omega g'(u)\psi = 0\end{aligned}$$

and so

$$\lambda(u) = \frac{\int_\Omega g''(u)\left[\nabla u \cdot \nabla f(\nabla u)\right]}{\int_\Omega g'(u)^2}$$

This $\lambda(u)$ is what we refer to by a **Lagrange multiplier**.

Now denote $X \cong \nabla f(\nabla u)$ and observe:

$$\begin{aligned}\int_\Omega X\cdot\nabla\psi &= \int_\Omega dw(\psi X) - \int_\Omega \psi \mathrm{div} X \\ &= \int_{\partial\Omega}\psi(X\cdot\nu_\Omega) - \int_\Omega \psi\mathrm{div}X.\end{aligned}$$

$$0 = \int_{\partial\Omega}\psi\nu_\Omega\cdot\nabla f(\nabla u) + \int_\Omega \psi\left[-dw(\nabla f(\nabla u)) - \lambda g'(u)\psi\right] \quad \forall\psi\in C^\infty(\overline{\Omega})$$

Testing on $\psi = 0$ on $\partial\Omega$, but arbitrary otherwise, we get on Ω :

$$-dw(\nabla f(\nabla u)) = \lambda g'(u)$$

Once we know this we get

$$\begin{cases} -dw(\nabla f(\nabla u)) = \lambda g'(u) & \Omega \\ \nu_\Omega\cdot\nabla f(\nabla u) = 0 & \partial\Omega\end{cases}$$

Example 1.3.1

Choosing $f(z) = \dfrac{|z|^2}{2}$ and $g(u) = \dfrac{u^2}{2}$, the above becomes

$$\begin{aligned}-\Delta u = \lambda u \quad \Omega \\ \frac{\partial u}{\partial \nu_\Omega} = \nabla u\cdot\nu_\Omega = 0 \quad \partial\Omega\end{aligned}$$

This is the Neumann eigenfunctions of the Laplacian Ω.

Example 1.3.2

Consider the space $\Omega = (0, \pi)$, and let $u_k(x) = \cos(kx)$ for $k \in \mathbb{N}$. These are all solutions to

$$\begin{aligned} -u_k'' &= \lambda u_k \quad (0, \pi) \\ u_k' \mid_{0,\pi} &= 0 \end{aligned}$$

$\lambda = k^2$ and $(u_k) \cong k^2$. This is an example of a variational problem with many critical points.

Notice that if $f(z) = \sqrt{1 + |z|^2}$ then $\nabla f(\nabla u) \cdot \nu_\Omega = \frac{\nabla u \cdot \nu_\Omega}{\sqrt{1+|\nabla u|^2}} = 0$.

1.4 Existence of minimizers by direct method

We will focus on finding the existence of minimizers to problems of ther form

$$\begin{aligned} \inf_{u \in C(\overline{\Omega})} &\{\mathcal{F}(u) \mid u = u_0 \text{ on } \partial\Omega\} \\ \inf_{i} &\{\mathcal{F}(u) \mid g\} \end{aligned}$$

For classes of the form

$$\inf_{u \in \mathcal{A}} \mathcal{F}(u)$$

where $\mathcal{A}$ is some competition class, we start by picking a minimizing sequence (implicitly we are taking $\mathcal{A} \neq \emptyset$). We will pick the minimizing sequence so that:

$$\begin{aligned} \lim_{j \to \infty} \mathcal{F}(u_j) = \inf_{\mathcal{A}} &\mathcal{F}(u) < \infty \\ \{u_j\}_j &\subset \mathcal{A} \end{aligned}$$

Then by sequential compactness, there exists a subsequence $\{\hat{u}_j\}_j$ for which $\lim_{j\to\infty} \{\tilde{u}_j\}_j \to u$ for some u. Note that while u may not be in our space, the energy of u will be in our space.

Now we restrict our functions to be lower semicontinuous. Then we can show that if $v_j \to v$, then $\mathcal{F}(v) \leq \liminf_{j\to\infty} \mathcal{F}(v_j)$.

Now we need to show that the limit to which we converge to is within our competition class– that is, $u \in \mathcal{A}$. Once we have this, then we have that

$$\begin{aligned} \inf_{\mathcal{A}} \mathcal{F} \geq \mathcal{F}(u) \leq \liminf_{j\to\infty} \mathcal{F}(\tilde{u}_j) = \lim_{j\to\infty} \mathcal{F}(\tilde{u}_j) &= \inf_{\mathcal{A}} \mathcal{F} \\ \implies \mathcal{F}(u) &= \inf_{\mathcal{A}} \mathcal{F} \end{aligned}$$

Example 1.4.1

Let $\mathcal{F}(u) = \int_0^1 u^2 + ((u')^2 - 1)$ and choose our competition class to be

$$\mathcal{A} = \left\{u \in C^1([0,1]) \mid u(0) = u(1) = 0\right\}$$

First, observe that $\mathcal{F}(u) > 0$ for all $u \in \mathcal{A}$. Positivity of the integral tells us that if $\mathcal{F}(u) = 0$ then $u = 0$, but $\mathcal{F}(0_{\mathcal{A}}) = 1$.

However, we can see that $\inf_{\mathcal{A}} \mathcal{F} = 0$. Choose $u_j \in \mathrm{Lip}(0,1), u_j' = \pm 1, 0 \leq u_j \leq \frac{1}{2j}$. This satisfies $u_j(0) = u_j(1) = 0$, and

$$F(u_j) = \int_0^1 u_j^2 + \left((\pm 1)^2 - 1\right)^2 = \int_0^1 (u_j)^2 \leq \frac{1}{(4j)^2}$$

giving us the infimum.

Example 1.4.2

Let

$$\mathcal{F}(u) = \int_{\mathbb{R}^n} |\nabla u|^2$$
$$\mathcal{A} = \left\{ u \in C^1(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |u|^q = 1 \right\} \quad 1 \leq q < \infty$$

We can see that $\mathcal{F}(u) > 0$ for all $u \in \mathcal{A}$ (check). However, $\inf_{\mathcal{A}} \mathcal{F} = 0$ ONLY when $n \geq 3$ AND $q = \dfrac{2n}{n-2}$. We can see this by picking any $u_0 \in \mathcal{A}$ and setting $u_\lambda(x) = \lambda^{\frac{n}{q}} u(\lambda x)$:

$$\int_{\mathbb{R}^n} |u_\lambda|^q = \int_{\mathbb{R}^n} |u_0|^q = 1 \quad \forall \lambda > 0$$
$$\int_{\mathbb{R}^n} |\nabla u_\lambda|^2 = \int_{\mathbb{R}^n} \left| \lambda^{\frac{n}{q}+1} \nabla u(\lambda x) \right|^2 dx$$
$$= \lambda^{2\left(\frac{n}{q}+1\right)} \int_{\mathbb{R}^n} |\nabla u(\lambda x)|^2 \, dx = \lambda^{2\left(\frac{n}{q}+1\right)-n} \int_{\mathbb{R}^n} |\nabla u_0|^2$$

in which case, unless $2\left(\frac{n}{q}+1\right) - n = 0$, we get that the above integral goes to zero.

Example 1.4.3: Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let

$$\mathcal{F} = \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 \mid \right\}$$
$$\mathcal{A} = \left\{ u \in C^1(\overline{\Omega}) \mid u = u_0 \text{ on } \partial\Omega \right\}$$

It follows immediately that $\mathcal{A} \neq \emptyset$ and $\inf_{\mathcal{A}} \mathcal{F} < \infty$.

Now consider a sequence of minimizers $\{u_j\}_j \subset C^1(\overline{\Omega})$ that converge to $\inf_{\mathcal{A}} \mathcal{F} < \infty$.

Remark 1.4.1: Properties of $L^2(\Omega$

We define

$$L^2(\Omega) := \left\{ h : \Omega \to \mathbb{R} \mid h \text{ is measurable and } \int_\Omega |h|^2 < \infty \right\}.$$

Note that we say that two functions are equivalent if they agree Lebesgue almost-everywhere on Ω.

$L^2(\Omega)$ is a complete separable metric space with a metric given by

$$d(h_1, h_2) = \|h_1 - h_2\|_{L^2(\Omega)} = \left(\int_\Omega |h_1 - h_2|^2 \right)^{\frac{1}{2}}$$
$$\|h\|_{L^2(\Omega)} = \sqrt{\langle h \mid h \rangle_{L^2(\Omega)}}$$
$$\langle h \mid g \rangle_{L^2(\Omega)} = \int_\Omega fg$$

That is, $L^2(\Omega)$ is a Hilbert space. There exists an orthonormal basis $\{h_j\}_j \subset L^2(\Omega)$ such that

$$\langle h_i \mid h_j \rangle_{L^2} = \delta_{ij}$$

$$\lim_{n\to\infty} \|h - \sum_{j=1}^{n} \langle h \mid h_j \rangle h_j\|_{L^2} = 0$$

We also have the **Riesz representation theorem**. If $\ell : L^2(\Omega) \to \mathbb{R}$ is a (continous) linear functional, then there exists $h \in L^2(\Omega)$ such that

$$\ell(g) = \langle h \mid g \rangle_{L^2(\Omega)} \quad \forall g \in L^2(\Omega)$$

This only works if and only if there exists a C such that

$$|\ell(g)| \leq C\|g\|_{L^2} \quad \forall g \in L^2$$

This is just another way of saying that linear functionals on bounded spaces are always continuous.

These properties combine to tell us that, if $\{h_j\}_j$ is a bounded sequence in $L^2(\Omega)$, then it converges to some $h \in L^2(\Omega)$ in some subsequence.

However, without boundedness, we might not have traditional convegence– even with subsequences. If $\{h_j\}_j$ is a sequence in $L^2(\Omega)$ such that $\int_\Omega |h_j|^2 = 1$, there does not necessarily exist $h \in L^2(\Omega)$ such that $h_j \to j$ in $L^2(\Omega)$. To see this, consider the following example:

Example 1.4.4

Let $\Omega = (0, \pi)$, $h_j(x) = \frac{2}{\pi}\sin(jx)$ with $j \in \mathbb{Z}_+$. Observe that

$$\int_0^\pi |h_j|^2 = 1 \quad \forall j$$

By our properties from earlier, we have that for all $g \in L^2(0,\pi)$, $g = \lim_{N\to\infty} \sum_{j=1}^{N} \langle g \mid h_j \rangle h_j$ which implies

$$\int_0^\pi g^2 = \sum_{j=1}^{\infty} \langle g \mid h_j \rangle^2 \implies \lim_{j\to\infty} \langle g \mid h_j \rangle = 0.$$

Where the last parts follow from the fact that $\int_0^\pi g(x)\sin(jx)\,dx \to 0$ as j gets sufficiently large. But now we have an issue– for any subsequence $\{\tilde{h}_j\}_j \subset \{h_j\}_j$, there exists an $h \in L^2(0,\pi)$ such that

$$0 = \lim_{j\to\infty} \int_0^\pi \left|h - \tilde{h}_j\right|^2 = \int_0^\pi h^2 + \int_0^\pi \tilde{h}_j^2 - 2\int_0^\pi h\tilde{h}_j = 2$$

and hence cannot exist.

In other words, $L^2(\Omega)$ is not compact. However– it is compact in the weak sense (sequentially compact when using weak convergence).

Proposition 1.4.1

If $\{h_j\}_j \subset L^2(\Omega)$ such that

$$\int_\Omega |h_j|^2 \leq C \quad \forall j$$

then there exists an $h \in L^2(\Omega)$ and a subsequence $\{\tilde{h}_j\}_j$ such that

$$\int_\Omega \tilde{h}_j g \to \int_\Omega hg \quad \forall g \in C^2(\Omega).$$

In other words, we have convergence in the inner product of our subsequence with a fixed test function g.

Proof of weak compactness. Take our sequence

$$\int_\Omega |h_j|^2 \leq C < \infty \quad \forall j$$

Let $Q = \{\varphi_k\}_{k=1}^\infty$ be a countable dense subset of $L^2(\Omega)$. Then for all fixed k, we have

$$\int_\Omega h_j \varphi_k \subset \mathbb{R} \text{ bounded}$$
$$\left| \int_\Omega h_j \varphi_k \right| \subset \|h_j\|_{L^2} \|\varphi_k\|_{L^2} \leq C \|\varphi_k\|_{L^2(\Omega)}$$

In other words, we have our weak convergence on our countable dense subset, as the boundedness gives us a convergent subsequence.

By countable density, we have that there exists a subsequence $\left\{\tilde{h}_j\right\}_j$ such that

$$\lim_{j\to\infty} \int_\Omega \tilde{h}_j \varphi_k \to \ell_k \quad \forall k \in \mathbb{Z}_+$$

and we can take $\ell : Q \to \mathbb{R}$ to be

$$\ell(\varphi_k) = \lim_{j\to\infty} \int_\Omega \tilde{h}_j \varphi_k = \ell_k$$
$$|\ell(\varphi_k)| \leq C \|\varphi_k\|_{L^2(\Omega)}$$

In other words, ℓ is a bounded linear functional. Then we can extend $\ell : L^2(\Omega) \to \mathbb{R}$ by defining

$$\ell(\varphi) = \lim_{k\to\infty} \ell(\tilde{\varphi}_k)$$
$$\overset{L^2}{\lim_{k\to\infty}} \{\tilde{\varphi}_{k\,k}\} = \varphi$$

This is well-posed, and moreover $|\ell\,(\varphi) \leq C\|\varphi\|_{L^2}|$ for all $\varphi \in L^2$ by the density.

We can apply the Riesz representation theorem to get that there exists an $h \in L^2(\Omega)$ such that $\ell(\varphi) = \int_\Omega \varphi h$ because

$$\int_\Omega h\varphi_k = \ell_k = \lim_{j\to\infty} \int_\Omega \tilde{h}_j \varphi_k \implies$$
$$\lim_{j\to\infty} \langle \tilde{h}_j \mid \varphi_k \rangle = \langle h \mid \varphi_k \rangle \quad \forall k \in \mathbb{Z}_+$$

and by the density of $Q = \{\varphi_k\}_k$ in L^2, we have convergence in the whole space. ∎

1.5 More Direct Methods

Let $u_0 \in C^1(\overline{\Omega})$ with Ω some open bounded set. We want to minimize the functional:

$$\inf_u \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 \mid u = u_0 \text{ on } \partial\Omega \right\}$$

Let $\{u_j\}_j$ be a minimizing sequence for $\inf_{\mathcal{A}} \mathcal{F}$ that satisfies the boundary conditions, and satisfying

$$\lim_{j\to\infty} \frac{1}{2} \int_\Omega |\nabla u_j|^2 = \inf_{\mathcal{A}} \mathcal{F}.$$

Because the limit supremum of the sequence is finite, we know that there exists a subsequence $\{\tilde{u}_j\}_j$ and $T \in L^2(\Omega, \mathbb{R}^n)$ such that

$$\int_\Omega T\varphi = \lim_{j\to\infty} \int_\Omega \varphi \nabla \tilde{u}_j \quad \forall \varphi \in L^2(\Omega)$$

Does this subsequence also have a limit of u? And is this T simply ∇u?

Theorem 1.5.1: Trace Inequality for Minimizers

If $u \in C^1(\overline{\Omega})$ and Ω is bounded with smooth boundary, then

$$\int_\Omega u^2 \le C(n, \mathrm{diam}\Omega)\left[\int_{\partial\Omega} u^2 + \int_\Omega |\nabla u|^2\right].$$

This quantifies the restrictions of the boundary conditions $\begin{cases} u = 0 & \partial\Omega \\ \nabla u = 0 & \Omega \end{cases} \implies u \cong 0$ on Ω.

Proof of trace inequality. Choose $X(x) = x$ so that $\mathrm{div}(X) = n$. We will pick $x_0 \in \mathbb{R}^n$ later so that the following is satisfied:

$$\begin{aligned} n\int_\Omega u^2 &= \int_\Omega u^2 \mathrm{div}(x - x_0) \\ &= \int_\Omega \mathrm{div}\left((x-x_0)u^2\right) - \int_\omega 2u\nabla u \cdot (x - x_0) \\ &\le \int_{\partial\Omega} u^2(x - x_0)\cdot \nu_\Omega + \int_\Omega 2\,|u|\,|\nabla u|\,(x - x_0) \end{aligned}$$

Using some standard algebraic inequalities (i.e. $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2$ so that $(\sqrt{\varepsilon}a = \frac{b}{\sqrt{\varepsilon}})^2 \ge 0$), we can put bounds on both terms individually to get

$$\begin{aligned} \int_{\partial\Omega} u^2(x-x_0)\cdot\nu_\Omega + \int_\Omega 2\,|u|\,|\nabla u|\,(x-x_0) &\le 2\mathrm{diam}(\Omega)\left[\int_{\partial\Omega} u^2 + \int_\Omega 2\,|u|\,|\nabla u|\right] \\ &\le 2\mathrm{diam}(\Omega)\int_{\partial\Omega} u^2 + 2\mathrm{diam}(\Omega)\left[\varepsilon\int_\Omega u^2 + \frac{1}{\varepsilon}\int_\Omega |\nabla u|^2\right] \\ \implies (n - 2\varepsilon\mathrm{diam}(\Omega))\int_\Omega u^2 &\le 2\mathrm{diam}(\Omega)\left[\int_{\partial\Omega} u^2 + \frac{1}{\varepsilon}\int_\Omega |\nabla u|^2\right] \end{aligned}$$

Now we simply choose $\varepsilon = \dfrac{n}{4\mathrm{diam}(\Omega)}$ and we have the statement

$$\int_\Omega u^2 \le \frac{4\mathrm{diam}(\Omega)}{n}\left[\int_{\partial\Omega} u^2 + \frac{4\mathrm{diam}(\Omega)}{n}\int_\Omega |\nabla u|^2\right]$$

■

What is the relation between T and u?

Definition 1.5.1

Let $u \in L^1_{\mathrm{loc}}(\Omega)$, and $T \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. We say that T is the unique vector field called the **weak** or **distributional gradient** of u if the following is satisfied:

$$\int_\Omega u\nabla\varphi = -\int_\Omega \varphi T \quad \forall \varphi \in C_c^\infty(\Omega)$$

Earlier, we asked if the limit of the subsequence was equivalent to the u earlier. Now we claim that the weak limit T of $\{\nabla u_j\}_j$ is the weak gradient of

$$u = \overset{w}{\lim_{n\to\infty}} \{u_j\}_j .$$

This follows because

$$\begin{aligned} \int_\omega u\nabla\varphi = \lim_{j\to\infty}\int_\Omega u_j\nabla\varphi &= -\lim_{j\to\infty}\int_\Omega \varphi\nabla u_j \\ &= -\int_\Omega \varphi T \end{aligned}$$

In summary, by weak compactness in L^2 and by Trace inequality, we have

$$\begin{cases} u_j \rightharpoonup u & L^2(\Omega) \quad u \in W^{1,2}(\Omega) \\ \nabla u_j \rightharpoonup \nabla u & L^2(\Omega, \mathbb{R}^n) \end{cases}$$

We need to define a space where these weak objects can live:

Definition 1.5.2: L^2 Sobolev Space

We define the **Sobolev Space in** L^2 to be

$$W^{1,2}(\Omega) := \left\{ u \in L^2(\Omega) \mid \exists \text{ weak gradient } \nabla u \in L^2(\Omega, \mathbb{R}^n) \right\}$$

Now we ask a new question: is it true that

$$\int_\Omega |\nabla u|^2 \leq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2$$

Yes. Pick any $T \in L^2(\Omega, \mathbb{R}^n)$. Then

$$\int_\Omega (\nabla u) \cdot T = \lim_{j\to\infty} \int_\Omega (\nabla u_j) \cdot T$$
$$\leq \lim_{j\to\infty} \left(\int_\Omega |\nabla u_j|^2 \right)^{1/2} \left(\int_\Omega |T|^2 \right)^{1/2}$$

Now choose

$$T = \frac{\nabla u}{\|\nabla u\|_{L^2}}$$

to get the inequality we desire.

Another question. Is $u \in \mathcal{A}$, where

$$\mathcal{A} = \{|\}$$

Example 1.5.1

Consider

$$u_\alpha(x) = \frac{1}{|x|^\alpha}$$
$$\nabla u_\alpha(x) = \frac{-\alpha}{|x|^{\alpha+1}} \hat{x}$$

where $\hat{x} = \dfrac{x}{|x|}$. Then

$$\int_{B_\varepsilon(0)} |\nabla u_\alpha|^2 \approx \int_0^\varepsilon \frac{\rho^{n-1}}{(\rho^{\alpha+1})^2}\, d\rho < \infty$$
$$\iff n - 1 - 2(\alpha+1) > -1 \iff \alpha < \frac{n}{2} - 1$$

Now we ask a seemingly silly question: is ∇u_α the weak gradient of u_α? Yes, but we have to be careful– if $\nabla u = 0$ almost everywhere in $\mathbb{R}^2$, say, u is a step function, then the weak gradient doesn't exist. One can show that

$$\int \varphi \nabla u_\alpha = -\int u_\alpha \nabla \varphi \quad \forall \varphi \in C_c^\infty(B_1(0))$$

via some standards calculations from distribution theory. Because this holds, we have

$$u^{(n)} = \sum_{k=1}^{n} \frac{2^{-k}}{|x - x_k|\,\alpha}$$

where $\{x_k\}_{k=1}^{\infty}$ is countably dense in $B_1(0)$. We have that $u^{(n)} \in W^{1,2}(\Omega)$ if $\alpha < \frac{n}{2} - 1$.In fact, this is a Cauchy sequence in $W^{1,2}(\Omega)$ (check this), and so

$$\exists u \in W^{1,2}(\Omega) \text{ such thatt } u = \lim_{N\to\infty} u^{(N)} = \sum_{k=1}^{\infty} \frac{2^{-k}}{|x - x_k|^{\alpha}}$$

Example 1.5.2

Let $u \notin L^{\omega}_{\mathrm{loc}}(B_1(0))$. For all $B_\varepsilon(x) \subset B_1(0)$, we have

$$\text{essential } \sup_{B_\varepsilon} |u| = \infty$$

Remark 1.5.1

What is the essential supremum? Let $f : \Omega \to \mathbb{R}$. Consider the case when $\tilde{f} : \Omega \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \Omega \setminus Q^n \\ j & x = x_j \in Q^n \cap \Omega \end{cases}$$

So that $f = \tilde{f}$ almost everywhere in Ω and $\sup_j \tilde{f} = \infty$. A traditional supremum is unbounded and so won't work nicely here. Instead, notice that

$$\{x \in \Omega \mid f(x) > t\} = \{f > t\}$$
$$\mathcal{L}^n(\{f > t\}) = \mathcal{L}^n\left(\{\tilde{f} > t\}\right)$$

With this in mind, we define

$$\text{essential } \sup_\Omega f := \inf_t \{t \mid \mathcal{L}^n(\{f > t\}) = 0\}.$$

Example 1.5.3

$u = u_0$ on $\partial\Omega$?

$u_j \to u$ in $L^2(\Omega)$ but we do *not* have convergence of boundary data. To see this, look at the diagram in 1.2.

The moral is that we need more properties to hold: if $u_j = u_0 \partial\Omega$, $\sup_j \int |\nabla u_j|^2 < \infty$ and

$$\begin{cases} u_j \to u \\ \nabla u_j \to \nabla u \end{cases}$$

then we can say that $u = u_0$ on $\partial\Omega$ in the distributional sense.

To summarize the work we've done, if we are working with problems of the form

$$\inf_u \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 \mid u = u_0 \text{ on } \partial\Omega \text{ in distribution} \right\}$$

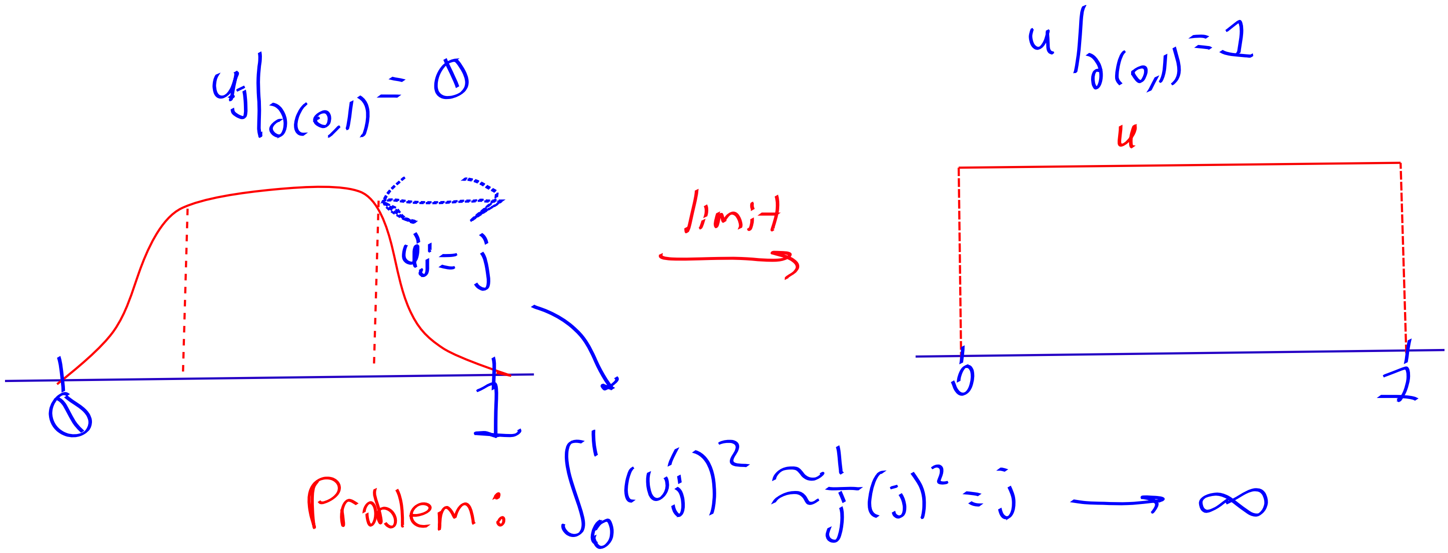

Figure 1.2: Convergence of Boundary Data

where $u \in W^{1,2}(\Omega)$, then the direct method gives you a minimizer. For the restricted problem with $u \in C^1(\overline{\Omega})$:

$$\inf_u \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 \mid u = u_0 \text{ on } \partial\Omega \text{ in classical sense} \right\}$$
$$= \frac{1}{2} \int_\Omega |\nabla u|^2$$

we have the existence of a minimizer $u \in W^{1,2}(\Omega)$.

Some classical results give us some insight:

Theorem 1.5.2: Serrin '61

If $\{u_j\}_j \subset L^1_{\text{loc}}(\Omega)$ and $\lim_{j\to\infty} u_j = u$ in $L^1_{\text{loc}}(\Omega)$ and u_j, u have weak gradient in $L^1(\Omega, \mathbb{R}^n)$, then for all $f : \mathbb{R}^n \to [0, \infty)$ convex, we have

$$\int_\Omega f(\nabla u) \le \liminf_{j\to\infty} \int_\Omega f(\nabla u_j).$$

We have a converse too. If $f : \mathbb{R}^n \to [0, \infty)$ is continuous and the above result holds, then for all u_j, j as above, f is convex.