In certain native South-American tribes, a boy and a girl could marry only if their signs obtained at birth from the wizard were the same. There were finitely many possible signs, and the sign depended only on the sex of the child and on the (common) sign of the parents.

Let us investigate, which principles govern the distribution of the signs.

Let the signs be $1, 2, \ldots, n$, and the signs obtained by the boys and girls be $B(1), B(2), \ldots, B(n)$, and $G(1), G(2), \ldots, G(n)$, resp. Clearly, $B, G \in S_n$, otherwise some signs would disappear after a generation.

The following requirements are natural assumptions:

R1 Brothers and sisters must not marry.

R2 A child must not marry her father/mother (though this possibility did not occur at the tribes).

R3 Allowing or forbidding the marriage between relatives should depend only on the relationship and should not depend on the eventual signs of the participants.

R4 The tribe should not split into disjoint castes, so any two people should have the chance that some of their descendants can marry each other.

What is the mathematical interpretation of these assumptions?

R1 For every i, we have $B(i) \neq G(i)$, i.e. $G^{-1}B(i) \neq i$, thus $G^{-1}B$ has no fixed points.

R2 For every i, we have $B(i) \neq i$ and $G(i) \neq i$, i.e. B and G have no fixed points.

R3 As an example, consider the case of first cousins where the grandparents are the same, the grandson is on the male line and the granddaughter is on the female line, i.e. the father of the boy and the mother of the girl are siblings. If the sign of the grandparents was i, then the grandson's sign is $BB(i)$, and the granddaughter's sign is $GG(i)$. If we allow the marriage of such a relationship, then for *every* i we must have $BB(i) = GG(i)$. If we forbid the marriage, then for *every* i we must have $BB(i) \neq GG(i)$. Thus, $G^{-2}B^2(i) = i$ holds either for every i, or for no i. In other words, either $G^{-2}B^2 = I$, or $G^{-2}B^2$ has no fixed points. Since every kinship can be expressed similarly as a product of B, G, and their inverses, the above consideration implies that the subgroup $H = \langle B, G \rangle$ generated by B and G is free of fixed points, i.e. no permutation in H has fixed points except the identity. In the light of this fact, R1 and R2 can be simplified to $B \neq G$ and that none of B and G is the identity.

R4 The descendants of a person of sign i have signs $\pi(i)$ where $\pi \in H = \langle B, G \rangle$. Therefore, to every i and j there must exist some $\pi, \varrho \in H$ satisfying $\pi(i) = \varrho(j)$, i.e. $\varrho^{-1}\pi(i) = j$. This means that to every i, j there must exist some permutation in H which maps i to j, i.e. H must be a *transitive* subgroup in S_n.

Summarizing the above, we need a subgroup $H \leq S_n$ which is transitive and free of fixed points, and we have to find two distinct generators B and G in H, where none of them is the identity.

Transitivity implies $|H| \geq n$ and the lack of fixed points implies $|H| \leq n$, therefore $|H| = n$. We can find suitable subgroups H by the representation in Cayley's Theorem: Let $G = \{g_1, \ldots, g_n\}$ be any group of size n, and to each $g \in G$ we assign the permutation π_g of the set G which maps g_i to gg_i. These permutations form a subgroup H in S_n, where H is transitive and free of fixed points and $H \cong G$.

Consider e.g. $n = 4$. Then $G \cong \mathbf{Z}_4$ or K. In the first case the Cayley representation yields an H generated by a 4 element cycle, e.g. $\pi = (1234)$. Here any two elements can be chosen for B and G except the identity. For the Klein group, $H = \{e, (12)(34), (13)(24), (14)(23)\}$, and again, any two elements can be chosen for generators except the identity.