R denotes a ring.

27. A subset S of R is a *subring* if it is a ring under the (restrictions of) the operations in R. Notation: $S \leq R$.

(a) How can we reformulate the first question asked in Problem 21 using subrings concerning parts 21b, c, f, and g?

(b) Prove that a non-empty subset S is a subring in R iff

(b1) it is closed under addition, multiplication, and forming the negatives of the elements, i.e. $(a, b \in S \Rightarrow a + b, ab, -a \in S)$; or

(b2) it is closed under subtraction and multiplication, i.e. $(a, b \in S \Rightarrow a - b, ab \in S)$.

(c) Verify that the zero element of S is necessarily the same as the zero element of R, and the analog holds for the negatives of the elements, as well.

28. Let $S \neq \{0\}$ be a subring in R. True or false:

(a) If R is commutative, then so is S. (b) If S is commutative, then so is R.

(c) If R is zero-divisor free, then so is S. (d) If S is zero-divisor free, then so is R.

(e) If R is a field, then so is S. (f) If S is a field, then so is R.

(g) If R has an identity, then so does S. (h) If S has an identity, then so does R.

(i) If both R and S have identities, then these are equal.

(j) If R is zero-divisor free and both R and S have identities, then these are equal.

(k) If S is zero-divisor-free and both R and S have identities, then these are equal.

29. (a) Prove that the intersection of arbitrarily many subrings is a subring again.

(b) Find a (simple) necessary and sufficient condition for the union of two subrings to be a subring.

30. Assume that R has an identity. Show that the smallest subring containing the identity is (essentially) $\mathbf{Z}$ or $\mathbf{Z}_n$ for some n.

31. A subset I in R is an *ideal* if it is a subring and is closed under multiplication with any element of R, i.e. $i \in I, r \in R \Rightarrow ri \in I, ir \in I$. Notation: $I \triangleleft R$.

(a) Find the number of subrings and ideals in (a1) $\mathbf{Z}$; (a2) $\mathbf{R}$; (a3) $\mathbf{Q}[x]$; (a4) $\mathbf{Z}_{20}$.

(b) Which ideals of $\mathbf{Z}$ contain both 18 and 45?

32. Verify the following propositions.

(a) A field has exactly two ideals.

(b) Also $\mathbf{R}^{n\times n}$ has exactly two ideals.

(c) If a commutative ring with identity has exactly two ideals, then it is a field.

33. Let H_1 and H_2 subsets in R, and define $H_1 + H_2 = \{h_1 + h_2 \mid h_i \in H_i\}$. True or false:

(a) If H_1 and H_2 are subrings, then so is also $H_1 + H_2$.

(b) If H_1 and H_2 are ideals, then so is also $H_1 + H_2$.

34. (a) Prove that every subring is an ideal in $\mathbf{Z}$ and $\mathbf{Z}_n$.

(b) If R has an identity, then also the converse of (a) is true.

In Problems 35 and 36, we assume that R is a commutative ring with identity.

35. The *principal ideal* generated by $c \in R$ is the set (c) of all multiples of c, i.e. $(c) = \{rc \mid r \in R\}$.

(a) Prove that (c) is an ideal, $c \in (c)$, and if an ideal I contains c, then $I \supseteq (c)$, i.e. (c) is the *smallest* (or tightest) ideal containing c.

(b) Show that all ideals in $\mathbf{Z}$ and $\mathbf{Z}_n$ are principal ideals.

(c) Verify $a \mid b \iff (b) \subseteq (a)$ in $\mathbf{Z}$.

36. The ideal generated by $c_1, c_2, \ldots, c_k$ is $(c_1, c_2, \ldots, c_k) = \{r_1c_1 + r_2c_2 + \ldots + r_kc_k \mid r_i \in R\}$.

(a) Verify that $(c_1, \ldots, c_k)$ is the smallest ideal containing $c_1, \ldots, c_k$.

(b) Describe the ideal $(21, 35)$ in $\mathbf{Z}$. Generalize the observation.

(c) Show that the ideal $(2, x)$ is not a principal ideal in $\mathbf{Z}[x]$, but it is a principal ideal in $\mathbf{Q}[x]$.