84. Verify that the examples in Handout 8 are groups indeed and exhibit some further examples among various sets of numbers, residues, matrices, polynomials, etc. Find groups in fields, rings, and vector spaces.

85. How many elements are in the symmetry group of a (a) rhombus; (b) circle; (c) brick; (d) cube? Which of these groups are commutative?

86. Which congruences of the plane form a group under composition and which of these are commutative? (a) All translations; (b) all rotations; (c) all rotations and translations.

87. *"Old" permutations.* In elementary combinatorics, permutations mean all possible orders $i_1, i_2, \dots, i_n$ of the numbers $1, 2, \dots, n$. Two elements in a permutation are in *inversion*, if the bigger number comes earlier. E.g. there are 5 inversions in $3, 2, 5, 1, 4$, namely 3–2, 3–1, 2–1, 5–1, and 5–4.

(a) How many inversions are in the following permutations:
(a1) $1, 3, 5, \dots, 49, 51, 50, 48, \dots, 4, 2$; (a2) $26, 27, 25, 28, 24, \dots, 51, 1$?

(b) How does the number of inversions change if we transpose
(b1) two adjacent elements; (b2) two arbitrary elements?

(c) Find ALL values k which occur as a number of inversions in a permutation of n elements?

(d) A permutation is *even* or *odd* according to the number of inversions being even or odd, resp. Show that for $n \geq 2$ there are as many even permutations as odd ones.

88. *"New" permutations.* We can consider permutations also as bijections of the set $\{1, 2, \dots, n\}$ onto itself. These form the group S_n under composition.

(a) Represent the following permutations as products of disjoint cycles:

$$\text{(a1)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 2 & 7 & 3 & 6 & 1 & 4 \end{pmatrix}; \qquad \text{(a2)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 7 & 1 & 2 & 8 & 3 & 4 \end{pmatrix}.$$

(b) Verify that an "old" permutation is even iff the corresponding "new" permutation is the product of an even number of transpositions. As an important consequence, we get that representing a permutation as a product of transpositions, the parity of the number of factors is unique.

89. (a) Determine the orders of elements in the groups 85.
(b) Which elements have finite order in $\mathbf{C}^*$?
(c) Assume $o(g) < \infty$. Prove: (c1) $o(g^k) \mid o(g)$;
(c2) $o(g^k) = o(g) \iff (o(g), k) = 1$; (c3) $o(g^k) = o(g)/(o(g), k)$.

90. (a) If $gh = hg$, then
(a1) $o(gh) \mid [o(g), o(h)]$; (a2) $o(gh) = o(g)o(h) \iff (o(g), o(h)) = 1$.
(b) Exhibit an example for $o(g) = o(h) = 2$ whereas $o(gh) = \infty$.

91. Verify that the order of a permutation is the least common multiple of the lengths of its disjoint cycles.

92. Which of the following groups are cyclic: (a) $(\mathbf{Z}, +)$; (b) $(\mathbf{Z}_n, +)$; (c) The non-zero elements of $\mathbf{Z}_{101}$ under multiplication; (d) $\{1, 3, 5, 7\} \subseteq \mathbf{Z}_8$ under multiplication; (e) The nth complex roots of unity under multiplication; (f) All complex roots of unity. (g) D_n; (h); S_n; (i) The symmetry group of a rhombus.

93. True or false:

(a) If $|G| = 81$ and $g^2 \neq g^{29}$ for some $g \in G$, then G is cyclic.

(b) If $|G| = 54$ and $g^2 \neq g^{29}$ for some $g \in G$, then G is cyclic.

(c) If $|G| = 81$ and $g^2 = g^{29}$ for some $g \in G$, then G is not cyclic.

94. Let $H \leq G$. Verify:

(a) $e_H = e_G$ and the same holds for h^{-1}.

(b) $\emptyset \neq H \leq G \iff (a, b \in H \Rightarrow ab^{-1} \in H)$.

95. Find all subgroups in (a) D_4; (b) S_3; (c) $\mathbf{Z}_{20}$; (d) $\mathbf{Z}$; (e) K.

96. Assume $j \leq n$.

(a) Show that S_n has a subgroup isomorphic to (a1) S_j; (a2) D_j.

(b) When does D_n have a subgroup isomorphic to D_j?

97. (a) Prove that the intersection of arbitrarily many subgroups is always a subgroup. (b) When is the union of two subgroups a subgroup?

98. Let H be a non-empty subset in G. True or false:

(a) If H is closed for the operation in G, then $H \leq G$.

(b) If $|H| < \infty$ and H is closed for the operation in G, then $H \leq G$.

(c) If $|H| < \infty$ and H is closed for forming inverses, then $H \leq G$.

99. Let $d(n)$ and $\sigma(n)$ denote the number and sum of divisors of n, resp. Prove:

(a) Every subgroup of a cyclic group is cyclic.

(b) $\mathbf{Z}_n$ has exactly $d(n)$ subgroups.

*(c) D_n has exactly $d(n) + \sigma(n)$ subgroups

100. Exhibit all left and right cosets if $G = D_4$ and H is the cyclic subgroup generated by (a) a reflection; (b) the rotation with 90 degrees.

101. Prove $2020 \mid \varphi(k^{2020} - 1)$ for any $k > 1$ where $\varphi(n)$ is Euler's function counting the integers coprime to n among $1, 2, \ldots, n$.

102. Let $S \subseteq \mathbf{R}$ and $f(s) = s$ for all $s \in S$. Is it true that f is the sum of two periodic functions if S is (a) $\mathbf{Q}$; (b) $\{a + b\sqrt{2} \mid a, b, \in \mathbf{Q}\}$; (c) $\mathbf{R}$?

freud@caesar.elte.hu