MATH 357 hw 5

Yellow Group

March 19, 2021

0.1

Take a G-homomorphism $\varphi : V \to W$. Show that

1. $\text{Ker}(\varphi)$ is a subrepresentation of V, and
2. $\text{Im}(\varphi)$ is a subrepresentation of W.

Let $\varphi : V \to W$ be a homomorphism of representations, and put $K = \ker \varphi$ and $I = \text{im}\varphi = \varphi(V)$. First, we verify that K is a subrepresentation of V. It is clear that K is a vector subspace of V. Now, let

We will use the relationship $\phi((p_V)_g(v)) = (p_W)_g(\phi(v))$ as seen in the notes often so let us keep it in mind. To see that $Ker(\phi)$ is a subrepresentation, we must show that the g action on any element brings it back to $Ker(\phi)$. Thus, let us take any $v \in Ker(\phi)$ and note that we get $\phi((p_V)_g(v)) = (p_W)_g(\phi(v)) = (p_W)_g(0)$ since $\phi(v) = 0$ by definition. Since we must also have that $(p_W)_g$ must map 0 to 0 in V, this becomes $\phi((p_W)_g(v)) = 0$, showing that $(p_W)_g(v) \in Ker(\phi)$, as desired.

For the image of ϕ, let us take a $w \in W$ such that $w \in Im(\phi)$, which implies that there exists a $v \in V$ such that $w = \phi(v)$. Then we see that $(p_W)_g(w) = (p_W)_g(\phi(v)) = \phi((p_V)_g(v))$, showing that $(p_W)_g(w) \in Im(\phi)$, as desired.

0.2

<u>More facts about characters</u>

<u>Proposition</u>: Let (V, ρ) be a representation of G, and take $g \in G$. Then

(1) $\chi_V(e) = \dim V$

(2) $\chi_V(g)$ is a sum of roots of unity

(3) $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$ [Here, $(\chi_V + \chi_W)(g) = \chi_V(g) + \chi_W(g)$]

(4) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

(5) $\overline{\chi_V}$ is a character of G; here, $\overline{\chi_V}(g) = \overline{\chi_V(g)}$

- (1) Since ρ is a group homomorphism, we have that $\rho(e) = I_{n \times n}$. It is clear then that $\chi_v(e) = \text{trace } I_{n \times x} = n \cdot 1 = n$, where n is the dimension of V.
- (3) $\rho_{v \oplus w}(g)x = (\rho_v(g)x, \rho_w(g)x)$. (This means we are appending the two vectors together)

0.3

Exercise 4: Prove directly that the map from $\varphi : a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an isomorphism of $Q(\sqrt{2})$ with itself.
First we note that for the identity element 1, we have that $\varphi(1 + 0\sqrt{2}) = 1 - 0\sqrt{2} = 1$. Next we note that $\varphi((a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2})) = \varphi(a_1a_2 + 2b_1b_2 + (a_1b_2 + a_2b_1)\sqrt{2}) = a_1a_2 + 2b_1b_2 - a_1b_2\sqrt{2} - a_2b_1\sqrt{2} = (a_1 - b_1\sqrt{2})(a_2 - b_2\sqrt{2}) = \varphi(a_1 + b_1\sqrt{2})\varphi(a_2 + b_2\sqrt{2})$, as desired. Finally, we have that $\varphi(a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2}) = \varphi(a_1 + a_2 + (b_1 + b_2)\sqrt{2}) = a_1 + a_2 - (b_1 + b_2)\sqrt{2} = (a_1 - b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = \varphi(a_1 + b_1\sqrt{2}) + \varphi(a_2 + b_2\sqrt{2})$, showing that φ is a ring homomorphism, as desired.

Since for each $x = a + b\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$, we have that $x = \varphi(a - b\sqrt{2})$, so φ is onto. Furthermore, if $a + bi \neq c + di$, then either $a \neq c$ or $b \neq d$. In either case, $\varphi(a + b\sqrt{2}) = a - b\sqrt{2} \neq c - d\sqrt{2} = \varphi(c + d\sqrt{2})$, so φ is one to one.

0.4

0.5 A

Given a finite abelian group G, describe its irreducible complex representations, up to equivalence. Illustrate this for the Klein-four group $G = C_2 \times C_2$.