MATH 354: PROBLEM SET 5

RICE UNIVERSITY, FALL 2019

Due date: Friday, October 11th, by 5pm in my office (you can slide it under the door). You are welcome to turn in your work during lecture on Friday.

Parts A, and B should be handed in **separately**. They will be graded by different TAs.

Please staple your homework!

Reminder from the syllabus: "The homework is not pledged and you can collaborate with other students in the class. In fact, you are very much *encouraged* to do so. However, you are not allowed to look up solutions in any written form; in particular, you are not allowed to look up solutions online. **Students caught violating this rule will be reported to the Honor Council.** You should write up your solutions individually."

1. Part A

Hand in the following exercises from Chapter 3 of Axler's book:

3B: 4, 6, 14, 18, 22.

Also, write up a solution to the following problem:

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$(x, y, z) \mapsto (2x - y + z, x - z, 3y + 7z).$$

Using the basis $v_1 = (1, 0, 1)$, $v_2 = (1, 2, 0)$, $v_3 = (0, 3, 1)$ for the domain of T, and the standard basis $w_1 = (1, 0, 0)$, $w_2 = (0, 1, 0)$, $w_3 = (0, 0, 1)$ for the codomain of T, write down the matrix $\mathcal{M}(T, (v_1, v_2, v_3), (w_1, w_2, w_3))$ for the linear map T.

2. Part B

Hand in the following exercises from Chapter 3 of Axler's book:

3C: 2, 3, 13. 3D: 6, 9.

Also, write up a solution to the following problem:

Date: October 4th, 2019.

The purpose of this problem is to make the construction you did in Problem 3D.6 explicit in a concrete case. Let $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^3$ and $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^3$ be linear operators specified by the matrices

$$\mathcal{M}(T_1) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} \quad \text{and} \quad \mathcal{M}(T_2) = \begin{pmatrix} 0 & 0 \\ -1 & -2 \\ 3 & 6 \end{pmatrix}$$

Find invertible operators $R \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S \colon \mathbb{R}^3 \to \mathbb{R}^3$, specified by a 2×2 and a 3×3 matrix, respectively, such that $T_1 = ST_2R$.

3. Comments

(1) Many of the exercises in 3B can be solved using Rank-Nullity.

(2) Although they are not assigned problems, please read the statements of 3B.17 and 3B.23. They are twin problems to 3B.18 and 3B.22, respectively.

(3) Problem 3B.22 is a doozy. Perhaps the easiest way to approach it is to define a map $R \colon \ker ST \to V$ by $R(u) = T(u)$, and apply Rank-Nullity to this map. Yes, you did read that correctly: the domain of R is the kernel of the map ST, which is a subspace of U.

(4) Problem 3C.13 can be done by brute force, but I don't recommend this. Rather, it's easier to do the style of proof that we did in class to show that matrix multiplication is associative, i.e., by rewriting the matrix properties in terms of linear maps, and taking advantage of the distributive properties of linear maps (Axler 3.9).

(5) Feel free to assume the result of 3D.4 to solve exercise 3D.6.

(6) In exercises 3D.6 and 3D.9, recall that "invertible operator $T \in \mathcal{L}(V)$" just means "invertible linear map $T \colon V \to V$."

(7) Exercise 3D.6 is very important, though perhaps not the easiest problem ever. Please remember its statement for future reference. If you think of invertible operators $R \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$ as a way to "scramble" bases of V and W, then the exercise says that two seemingly different linear maps $T_1, T_2 \in \mathcal{L}(V, W)$ can be made to look equal by changing bases of V and W if and only if $\dim \ker T_1 = \dim \ker T_2$. The extra problem in Part B is supposed to help you see this in a concrete case: the operators $R \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S \colon \mathbb{R}^3 \to \mathbb{R}^3$ should be thought of as changing the bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ to make the operators T_1 and T_2 coincide.

(8) Problem 3D.9 has a matrix-theoretic version: The product of two matrices is invertible if and only if each of the matrices is invertible. In particular, each matrix has to be a square matrix (why?). So, for example, you can't ever hope to multiply a 4×6 matrix and a 6×4 matrix and get an invertible 4×4 matrix!