Definition 0.0.1: Holomorphic

Let $M \subset \mathbb{C}$ be open. A complex-valued function $f : M \to \mathbb{C}$ is **holomorphic** (or **regular**, **complex differentiable**) if for every $z \in M$, the limit exists:

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} := f'(z)$$

f' is called the **complex derivative** of f.

Note that h is a non-zero complex number in M. Hence the derivative has to exist and be equal for any possible direction that h may approach zero.

Example 0.0.1: Holomorphic and non-holomorphic functions

Any polynomial $p \in \mathbb{C}[z]$ defined by

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n$$

is holomorphic in the entire complex plane with

$$p'(z) = a_1 + \ldots + n a_n z^{n-1}.$$

However, $f(z) = 1/z$ is only holomorphic on open sets that do not contain the origin.

For an example of a function that is never holomorphic, consider the involuntary transformation

$$f(z) = \overline{z}.$$

One can see that

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{h}}{h}$$

which has no limit as $h \to 0$, as if h approaches zero on the real axis, then $\frac{\overline{h}}{h} = 1$, and if it approaches zero on the imaginary axis, then $\frac{\overline{h}}{h} = -1$.

It is useful to review the definition of real differentiation in $\mathbb{R}^2$. At first, it seems there should be no reason to view complex and real differentiation differently, but we will start to see some subtle and important differences soon.

Definition 0.0.2: Real Differentiation

Let $M \subset \mathbb{R}^2$ be open. Let $F : M \to \mathbb{R}^2$. Then F is real-differentiable if there exists a linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for every $z \in M$

$$\lim_{|h| \to 0} \frac{|F(z+h) - F(z) - J_F(h)|}{|h|} = 0$$

where $h \in \mathbb{R}^2$. J_F is called the Jacobian and is exactly the 2×2 real matrix of partial derivatives

$$J = J_F(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

When F is a complex function, there is a special relation between the entries of the Jacobian, if it is holomorphic.

Proposition 0.0.1: Cauchy-Riemann Equations

Let f be a complex function with $f = u + iv$ for u, v real-valued functions. If f is holomorphic, then f satisfies the **Cauchy-Riemann equations**:

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \implies$$
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where (x, y) is a complex variable.

Construction of Cauchy-Riemann Equations. Recall that any complex-valued function f can be parametrized by some mapping

$$f = F(x, y) = (u(x, y), v(x, y))$$

where x, y represent the real and imaginary coordinate respectively, and u, v are real-valued functions. If F is real-differentiable, then the partial derivatives of u, v exist and hence

$$J_F(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

satisfies the necessary properties as $h \to 0$. However, there is an implicit relation imposed, as we utilize the parametrization $h = (h_1, h_2)$. Observe that we can treat x or y as fixed when approaching from the imaginary or real axes respectively, and get

$$f'(z) = \lim_{h_1 \to 0} \frac{f(x + h_1, y) - f(x, y)}{h_1} = \frac{\partial f}{\partial x}(z)$$
$$f'(z) = \lim_{h_2 \to 0} \frac{f(x, y + h) - f(x, y)}{ih_2} = \frac{1}{i}\frac{\partial f}{\partial y}(z)$$

Hence, if f is holomorphic, these limits must be equal and thus

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$$

Substituting $f = u + iv$, we get the relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

∎

Theorem 0.0.1

Let $M \subset \mathbb{C}$ be open. Let $f : M \to \mathbb{C}$. Express $z = x + yi$ and $f = u + vi$ in the usual way. Then f is complex-differentiable if and only if f is real-differentiable and the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This follows directly from the work above.

Proposition 0.0.2: Properties of Complex Differentiation

Let $f, g : M \to \mathbb{C}$ be holomorphic complex-valued functions. Then

$$\begin{aligned}(f+g)' &= f' + g' \\ (fg)' &= f'g + fg' \\ \left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} \text{ at all } g(z) \neq 0 \\ (f \circ g)' &= (f' \circ g) \cdot g' \text{ at all } g(z) \in M\end{aligned}$$

and hence all of these compositions are holomorphic functions.

Exercise 0.0.1

Prove 0.0.2.