Algebra II: Homework 11

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Professor Walton

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Collaborated with the Yellow group.

Problem 1

Claim. Determine the Galois group of $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 5)$. Determine all the subfields of the splitting field of this polynomial. Write down the corresponding lattice of Galois groups.

Proof. The roots of f are $\left\{\pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{5}\right\}$. There are no repeated roots and so f is separable, hence the splitting field over f is a Galois extension. Thus the Galois group is

$$\mathrm{Aut}(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q}).$$

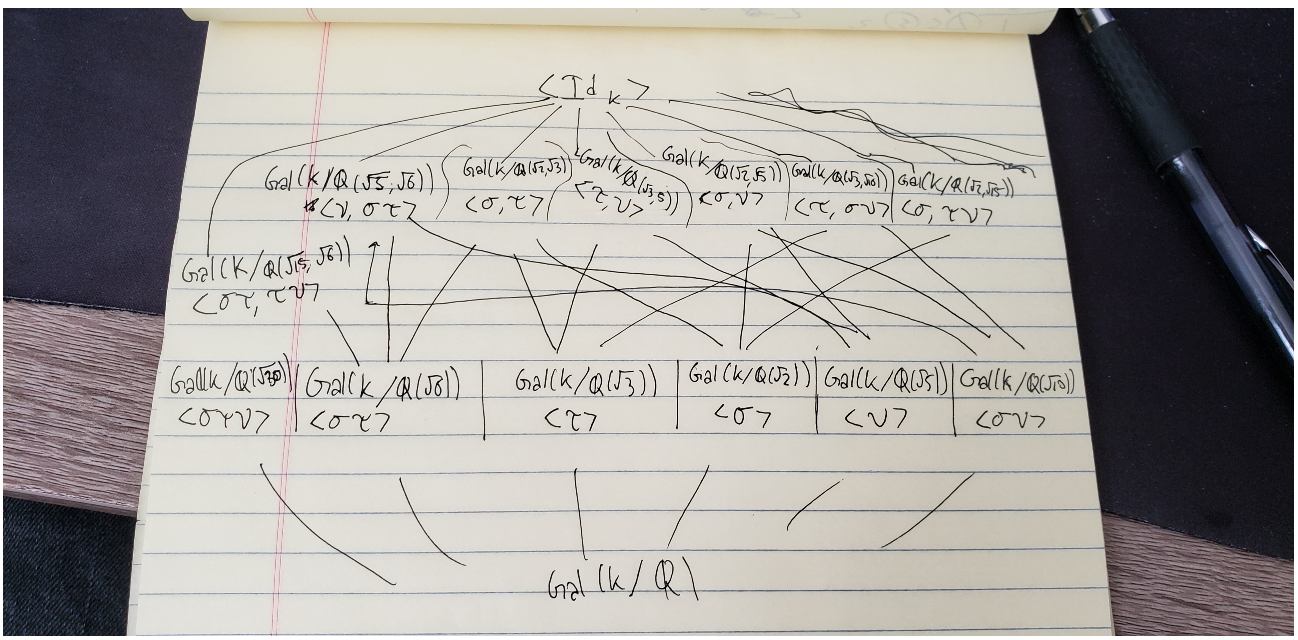
Of course the automorphisms must send each root to either itself or its corresponding negative root. Hence the Galois group is given simply by

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The subfields of the splitting field are then

$$\mathbb{Q},\ \mathbb{Q}(\sqrt{2}),\ \mathbb{Q}(\sqrt{3}),\ \mathbb{Q}(\sqrt{5}),\ \mathbb{Q}(\sqrt{2}, \sqrt{3}),\ \mathbb{Q}(\sqrt{2}, \sqrt{5}),\ \mathbb{Q}(\sqrt{3}, \sqrt{5}).$$

The lattice is given below:



□

Problem 2

Claim. Let $K = \mathbb{Q}(\sqrt[8]{2}, i)$ and let $F_1 = \mathbb{Q}(i), F_2 = \mathbb{Q}(\sqrt{2}), F_3 = \mathbb{Q}(\sqrt{-2})$. Prove that $\mathrm{Gal}(K/F_1) \cong \mathbb{Z}_8$, $\mathrm{Gal}(K/F_2) \cong D_8$, $\mathrm{Gal}(K/F_3) \cong Q_8$.

Proof. We use the subfield lattice described in the book. This shows that F_1 corresponds to $\langle \sigma \rangle$, F_2 corresponds to $\langle \sigma^2, \tau \rangle$, and F_3 corresponds to $\langle \sigma^2, \tau\sigma^3 \rangle$. We have from the book that:

$$\sigma = \begin{cases} \theta \mapsto \zeta\theta \\ i \mapsto i \\ \zeta \mapsto \zeta^5 \end{cases}, \sigma^2 = \begin{cases} \theta \mapsto \zeta^6\theta \\ i \mapsto i \\ \zeta \mapsto \zeta \end{cases},$$
$$\tau = \begin{cases} \theta \mapsto \theta \\ i \mapsto -i \\ \zeta \mapsto \zeta^7 \end{cases}, \tau\sigma^3 = \begin{cases} \theta \mapsto \zeta\theta \\ i \mapsto -i \\ \zeta \mapsto \zeta^3 \end{cases} \text{ Now we check that}$$

$$\sigma^8(\theta) = \sigma^7(\zeta\theta) = \sigma^6(\zeta^6\theta) = \sigma^5(\zeta^7\theta) = \sigma^4(\zeta^4\theta) = \sigma^3(\zeta^5\theta) = \sigma^2(\zeta^2\theta) = \sigma(\zeta^3\theta) = \theta$$

So θ generates every element in the group and it is then isomorphic to $\mathbb{Z}_8$.

Then we check that

$$(\sigma^2)^4(\theta) = (\sigma^2)^3(\zeta^6\theta) = (\sigma^2)^2(\zeta^4\theta) = \sigma^2(\zeta^2\theta) = \theta$$

And so we can see from this that σ^2 corresponds exactly to D_8.

Finally we see that

$$\begin{aligned} (\sigma^2)^4 &= e \\ (\sigma^2)^2 = \sigma^4 &= (\tau\sigma^3)^2 \\ \tau\sigma^3\sigma^2 = \tau\sigma^5 &= (\sigma^2)^{-1}\tau\sigma^3 \end{aligned}$$

And these elements exactly correspond to $a^4 = e$, $a^2 = b^2$, $ba = a^{-1}b$, as expected of the group Q_8.

□

Problem 3

Claim. Determine all the subfields of the splitting field of $x^8 - 2$ which are Galois over $\mathbb{Q}$.

Proof. Let K be the splitting field of $x^8 - 2$. K is then $\mathbb{Q}(\sqrt[8]{2}, i)$. By a claim from class, we are interested only in subgroups of K which are normal. Referencing the lattice on page 580 of DF, we simply check which groups are fixed under conjugation by σ, τ. A presentation for K is $\langle \sigma, \tau : \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$, which allows us to check:

$$\begin{aligned} \sigma(\tau\sigma^{2k})\sigma^{-1} = \sigma\tau\sigma^{2k-1} &= \tau\sigma^{2k+2} \\ &\neq \tau\sigma^{2k} \end{aligned}$$

so all subgroups of G of degree 2 are not normal. Next, we check normality of subgroups of G of degree 4:

- $\langle \sigma^4, \tau\sigma^6 \rangle$ Conjugating by σ yields

$$\begin{aligned} \sigma\langle \sigma^4, \tau\sigma^6 \rangle \sigma^{-1} &= \langle \sigma^4, \sigma\tau\sigma^5 \rangle \\ &= \langle \sigma^4, \sigma^2\tau\sigma^2 \rangle \end{aligned}$$

- $\langle \sigma^4, \tau \rangle$ Conjugating by σ yields

$$\sigma \langle \sigma^4, \tau \rangle \sigma^{-1} = \langle \sigma^4, \sigma\tau\sigma^{-1} \rangle = \langle \sigma^4, \tau\sigma^2 \rangle$$

...

We know $K/\mathbb{Q}$ is Galois, and by part 3 of the Fundamental Theorem of Galois Theory, the "top level" of extensions (those of order 8) will be Galois as well. Therefore $K/\mathbb{Q}(\sqrt[4]{2}, i)$, $K/\mathbb{Q}(\sqrt[8]{2})$, $K/\mathbb{Q}(\sqrt[8]{2}i)$, $K/\mathbb{Q}(\sqrt[8]{2}\zeta^3)$, $K/\mathbb{Q}(\sqrt[8]{2}\zeta)$ are all Galois. $\square$

Problem 4

Claim. Give an example of fields F_1, F_2, F_3 with $\mathbb{Q} \subset F_1 \subset F_2 \subset F_3$, with $[F_3 : \mathbb{Q}] = 8$ and each field is Galois over all its subfields with the exception that F_2 is not Galois over $\mathbb{Q}$.

Proof. Let $F_1 = \mathbb{Q}(\sqrt{2})$, $F_2 = \mathbb{Q}(\sqrt[4]{2})$, $F_3 = \mathbb{Q}(\sqrt[4]{2}, i)$. It is obvious that $\mathbb{Q} \subset F_1 \subset F_2 \subset F_3$, and immediately clear that $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}] = 8$. The minimal polynomials are:

$$F_1 = x^2 - 2$$
$$F_2 = x^4 - 2$$
$$F_3 = x^8 - 4$$

We have already seen that F_1 is Galois over $\mathbb{Q}$. F_2 is Galois over F_1 because $\sqrt[4]{2} = \sqrt{\sqrt{2}}$, and F_3 is Galois over both because it is the splitting field of $x^8 - 4$. But F_2 is not Galois over $\mathbb{Q}$ as it has duplicate roots in $\mathbb{Q}$. $\square$

Problem 4 continued on next page...

Problem 5

Claim.

(a). Prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$.

(b). Show that the roots of this quartic are

$$\alpha_1 = \sqrt{1+\sqrt{3}} \qquad \alpha_3 = -\sqrt{1+\sqrt{3}}$$
$$\alpha_2 = \sqrt{1-\sqrt{3}} \qquad \alpha_4 = -\sqrt{1-\sqrt{3}}$$

(c). Let $K_1 = \mathbb{Q}(\alpha_1)$ and $K_2 = \mathbb{Q}(\alpha_2)$. Show that $K_1 \neq K_2$ and $K_1 \cap K_2 = \mathbb{Q}(\sqrt{3}) = F$.

(d). Prove that K_1, K_2 and K_1K_2 are Galois over F with $\mathrm{Gal}(K_1K_2/F)$ congruent to the Klien 4-group. Write out the elements of $\mathrm{Gal}(K_1K_2/F)$ explicitly. Determine all the subgroups of the Galois group and give their corresponding fixed subfields of K_1K_2 containing F.

(e). Prove that the splitting field of $x^4 - 2x^2 - 2$ over $\mathbb{Q}$ is of degree 8 with dihedral Galois group.

Proof. (a). We can factor this as $x^4 - 2x^2 - 2 = (x^2 + \sqrt{3} - 1)(x^2 - \sqrt{3} - 1)$ which clearly has irrational roots, and hence must be irreducible in $\mathbb{Q}$.

(b). This is clear based on the factored form of $(x^2 + \sqrt{3} - 1)(x^2 - \sqrt{3} - 1)$

(c).

(d).

(e).

□

Problem 6

Claim. Determine all the Galois group of the splitting field over $\mathbb{Q}$ of $x^4 - 14x^2 + 9$.

Proof. We can expand the polynomial as

$$(x^2 - 7)^2 - 40$$

which gives us roots

$$\begin{aligned} \sqrt{7+2\sqrt{10}} &= \sqrt{5} + \sqrt{2} \\ \sqrt{7-2\sqrt{10}} &= \sqrt{5} - \sqrt{2} \\ -\sqrt{7+2\sqrt{10}} &= -\sqrt{5} - \sqrt{2} \\ -\sqrt{7-2\sqrt{10}} &= -\sqrt{5} + \sqrt{2} \end{aligned}$$

And hence we can see that the splitting field is $\mathbb{Q}(\sqrt{5}, \sqrt{2})$. By previous problems we can see that the Galois group will be exactly $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

Problem 6 continued on next page. . .