A lot of geometric theorems can be visualized in the complex plane, in which the complex operations are immensely useful for proving theorems.

It is important to establish that the complex plane indeed inherits the metric properties of $\mathbb{R}^2$.

**Definition 0.0.1: Scalar Product and Angles**

Let $z, w \in \mathbb{C}$. The **scalar product** of $z$ and $w$ is defined by

$$\langle z, w \rangle := \mathrm{Re}(z\overline{w}).$$

We define the **angle between** $z$ **and** $w$ to be

$$\theta(z, w) := \cos^{-1}\left(\frac{\langle z, w \rangle}{|z|\,|w|}\right).$$

The angle formula is chosen so that

$$\cos\theta(z, w) = \frac{\langle z, w \rangle}{|z|\,|w|}$$
$$\sin\theta(z, w) = \frac{\langle z, -iw \rangle}{|z|\,|w|}.$$

**Exercise 0.0.1**

Show that the scalar product defined above indeed satisfies the necessary properties for a general scalar product.

## Linear Transformations

**Definition 0.0.2: Linear Fractional Transformation**

A **linear fractional transformation** is a linear transformation of the complex plane given by

$$S(z) := \frac{az + b}{cz + d}$$

for complex numbers $a, b, c, d \in \mathbb{C}$ with one of $c, d$ non-zero.

One can choose to extend this definition via

$$S(\infty) = \frac{a}{c}$$
$$S(-\frac{d}{c}) = \infty$$

or merely define the limits to be these values. This extends the mapping so that $S$ is a topological mapping of the extended plane onto itself, with the topology of distances on the Riemannian sphere.

This is the generalized form of all linear transformations on $\mathbb{C}$– translations, reflections, rotations, and more can be represented as linear fractional transformations.

**Proposition 0.0.1**

Let a linear fractional translation be given

$$S(z) = \frac{az+b}{cz+d}.$$

If $ad - bc \neq 0$, then there exists an inverse

$$S^{-1}(w) = \frac{dw-b}{-cw+b}$$

that satisfies all the expected properties of an inverse.

If $ad - bc = 1$, then we say the linear fractional transformation is **normalized**.

Every linear fractional transformation admits a normalized form. In fact, there are exactly two, which are obtained from each other by changing the sign of the coefficients.

**Matrix Formulation of Linear Fractional Transformations**

Let $z = \frac{z_1}{z_2}$ be an arbitrary complex number and

$$S(z) = \frac{az+b}{cz+d}.$$

Then we can express this linear fractional translation by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

where $w := \frac{w_1}{w_2} = S(z)$. This representation as matrices is useful as it satisfies the standard matrix operations when it comes to addition, composition, inversion, and so on.

Hence, the set of linear fractional transformations forms a (matrix) group. The identity transformation is given by the identity matrix. The rest of the group properties we leave to be checked by the reader.

In fact, if we restrict ourselves to normalized representations, this matrix group is isomorphic to $SL(2, \mathbb{C})$ provided we form an equivalence class on the two equivalent normalized linear transformations. Without the normalized condition, the linear group is isomorphic to the one-dimensional projective group over the complex numbers.

**Definition 0.0.3: Important Classes of Linear Transformations**

The linear fractional transformations of the form

$$T_\alpha(z) := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

for $\alpha \in \mathbb{C}$ are called **parallel translations**, as they correspond to the transformation $S(z) = z + \alpha$.

The linear fractional transformations of the form

$$R_\alpha(z) := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

are referred to as **rotations** if $|\alpha| = 1$, or a **homothetic transformation** if $\alpha$ is real with $\alpha > 0$. For arbitrary complex $\alpha \neq 0$, we can write

$$\alpha = |\alpha| \frac{\alpha}{|\alpha|}$$

and hence the general form can be viewed as a composition of a homothetic transformation with a rotation.

The linear fractional transformation

$$(z)^{-1} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is referred to as an **inversion**, as it represents $S(z) = \frac{1}{z}$.

In some sense, we can view these classes of translations as the fundamental linear transformations. If $S(z)$ is given by

$$\frac{az+b}{cz+d}$$

with $c \neq 0$, then we can write

$$\begin{aligned} S(z) &= \frac{az+b}{cz+d} = \frac{bc-ad}{c^2(z+\frac{d}{c})} + \frac{a}{c} \\ &= T_{a/c} \circ R_{\frac{bc-ad}{c^2}} \circ (\cdot)^{-1} \circ T_{d/c} \end{aligned}$$

In the simpler case where $c = 0$, we have

$$\begin{aligned} S(z) &= \frac{az+b}{d} \\ &= T_{b/d} \circ R_{a/d} \end{aligned}$$

**Theorem 0.0.1: Pythagorem Theorem**

In a right triangle, the square of the hypotenuse is equal to the sum of squares of the legs.By plotting the triangle in the complex plane, we can restate it as $|x+yi|^2 = x^2 + y^2$. Writing $x + yi$ as the polar form, and writing $x, y$ as $r^2$ which is the product of complex conjugates, everything checks out.

**Theorem 0.0.2**

The blue triangle in 0.1 is an equilateral triangle.

Let $a, b, c$ of the original triangle, and let $a', b', c'$ be the vertices of the blue triangle. Then our goal is to do the mult. and get the side lengths we expect.

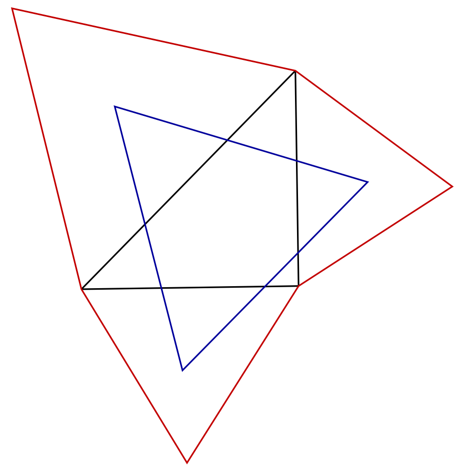

Figure 0.1: Complex Pythagorem Theorem

**Theorem 0.0.3**

Consider a regular $n$-gon in the unit circle. From a vertex, draw a line segment to the other vertices. The product of the lengths of these $n-1$ line segments is $n$.

*Proof.* WLOG, the vertices are the $n$-th roots of unity: $w^k$ $(k = 0, \ldots, n-1\ )$ where $w = e^{2\pi\frac{i}{n}}$. Also WLOG, the vertex we start with is 1. Then we want to show that $\prod_{k=1}^{n-1} |1 - w^k| = n$. Equivalently, $\left|\prod_{k=1}^{n-1}(1 - w^k)\right| = n$. Now we evaluate the product inside. To do this, we substitute in $z$ for 1, so we calculate $\prod_{n=1}^{n-1}(z - w^k)$. We then think of the inside as a polynomial. It has roots for $w^k$, except for 1, so we multiply by $(z-1)$, and now it has roots for $w^k$ including 0. So this is really $z^n - 1$. So our original product is simply $\frac{z^n-1}{z-1} = 1 + z + z^2 + \ldots + z^{n-1}$. Plugging in 1, you get $n$. We are done. ∎

**Theorem 0.0.4**

Let $A_1, \ldots, A_n$ be a regular $n$-gon in the unit circle. Let $p$ be an arbitrary point on the unit circle. Then the maximum of $pA_1 \cdot \ldots \cdot pA_n$ equals 2 (where p varies).

*Proof.* Think of $A_1, \ldots, A_n$ as complex numbers $a_1, \ldots, a_n$. WLOG, these $n$ numbers are the $n$-th roots of $-1$, or $w^{\frac{2k-1}{n}\pi i}$. Relabeling $p$ as $z$, we want to show

$$\max_{|z|=1} \prod_{k=1}^{n} |z - a_k| = 2$$

Equivalently,

$$\max_{|z|=1} \left|\prod_{k=1}^{n}(z - a_k)\right| = 2$$

because the $a_k's$ are roots of the RHS. So the statement reduces to

$$\max_{|z|=1}|z^n + 1| = 2$$

By the triangle inequality, it is clear that this statement holds. Indeed, for any $z$ with $|z| = 1$, $|z^n+1| \leq |z^n|+1 = |z|^n+1 = 2$. ∎

**Theorem 0.0.5**

Let $A_1, \ldots, A_n$ be a non-regular $n$-gon in the unit circle. Then

$$\max_{p \text{ on the unit circle}} pA_1 \cdot \ldots \cdot pA_n > 2$$

*Proof.* We shall think of the vertices as complex numbers $a_1, \ldots, a_n$. and $p$ as a complex number. We want to show that

$$\max_{|z|=1} \left| \prod_{k=1}^{n} (z - a_k) \right| > 2$$

Look at the constant term of the polynomial inside: $(-a_1)(-a_2)\ldots(-a_n) = (-1)^n a_1 \ldots a_n$. Rotating all the $a_k's$ by the same angle does not change the maximum. In other words, for a given $u \in \mathbb{C}$ on the unit circle, we can replace each $a_k$ by $ua_k$, without changing the maximum in question. This changes the constant term to $u^n(-1)^n a_1 \ldots a_n$. Now we will pick $u$ so the constant term is 1. WLOG, this results in

$$\prod_{k=1}^{n} (z - a_k) = z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + 1$$

We will call the terms in the middle (not $z^n$ and not 1) $p(z)$. Then we want to show that

$$\max_{|z|=1} |z^n + 1 + p(z)| > 2$$

Plugging in $n$-th roots of unity, we simply need to show that $p(w^k)$ can be positive. One of the roots must be non-zero, as it is of smaller degree and cannot be identically zero (would be a regular $n$-gon). Moreover, the sum of all the $p(w^k)$ must be zero. We will show this, which will then show that there is a $k$ such that $p(w^k)$ has a positive real part. Now let's prove that statement. Recall that

$$p(z) = \sum_{m=1}^{n-1} c_m z^m$$

So

$$\sum_{k=0}^{n-1} p(w^k) = \sum_{k=0}^{n-1} \sum_{m=1}^{n-1} c_m w^{km} = \sum_{m=1}^{n-1} c)_m \sum_{k=0}^{n-1} w^{km} = \sum_{m=1}^{n-1} \frac{w^{nm} - 1}{w^m - 1} = \sum_{m=1}^{n-1} c_m * 0 = 0$$

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