MATH 354: PROBLEM SET 11

RICE UNIVERSITY, FALL 2019

Due date: Friday, December 6th, by 5pm in my office (you can slide it under the door). You are welcome to turn in your work during lecture on Monday.

Parts A, and B should be handed in **separately**. They will be graded by different TAs.

Please staple your homework!

Reminder from the syllabus: "The homework is not pledged and you can collaborate with other students in the class. In fact, you are very much *encouraged* to do so. However, you are not allowed to look up solutions in any written form; in particular, you are not allowed to look up solutions online. **Students caught violating this rule will be reported to the Honor Council.** You should write up your solutions individually."

1. Part A

Hand in the following exercises from Chapter 7 of Axler's book:

7C: 6, 7. 7D: 5, 6, 13, 18.

2. Part B

Hand in solutions to the following exercises:

(1) **Quadratic forms and Sylvester's Law of Inertia.** A quadratic form $q(x_1, \dots, x_n)$ in n variables over a field $\mathbf{F}$ is a homogeneous polynomial of degree 2 with coefficients in $\mathbf{F}$ (homogeneous just means that every term in the polynomial has the same total degree). For example, if $n = 3$ and $\mathbf{F} = \mathbb{R}$, then

$$q(x_1, x_2, x_3) = x_1^2 - \sqrt{2}x_1x_3 + 7x_2^2 + x_2x_3 - 2x_3^2$$

is a quadratic form over $\mathbb{R}$ in three variables.

Throughout this problem we assume that $\mathbf{F}$ is a field where $2 \neq 0$ (so, for example, $\mathbf{F}$ is not the field with only 2 elements).

(a) Let $q(x_1, \dots, x_n)$ be a quadratic form over a field $\mathbf{F}$. Show there exists a matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

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with $A = A^t$, such that

$$q(x_1, \ldots, x_n) = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \cdot A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

(b) Let $q_A(x_1, \ldots, x_n)$ and $q_B(x_1, \ldots, x_n)$ be two quadratic forms over $\mathbf{F}$, with respective symmetric matrices A and B (as in part (a)). Suppose there exists an invertible $n \times n$ matrix Q such that $B = Q^t A Q$. Let $\alpha \in \mathbf{F}$. Prove: there exist $x_1, \ldots, x_n \in \mathbf{F}$ such that $q_A(x_1, \ldots, x_n) = \alpha$ if and only if there exist $y_1, \ldots, y_n \in \mathbf{F}$ such that $q_B(y_1, \ldots, y_n) = \alpha$.

Quadratic forms q_A and q_B as above are said to represent the same elements of $\mathbf{F}$ and are called equivalent.

(c) Now we specialize to $\mathbf{F} = \mathbb{R}$. Show that a quadratic form $q(x_1, \ldots, x_n)$ is equivalent to a quadratic form with the following shape:

$$q'(x_1, \ldots, x_n) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2,$$

where $r + s \leq n$. Show furthermore that r is the number of positive eigenvalues of the symmetric matrix A associated to q, and s is the number of negative eigenvalues of A. The pair (r, s) is called the signature of q.

(d) Show that if two quadratic forms over $\mathbb{R}$ in the same number of variables have the same signature, then they are equivalent.

The converse statement is also true (you don't have to prove the converse), and together, these two statements are known as **Sylvester's Law of Inertia**. Namely: two quadratic forms in the same number of variables over $\mathbb{R}$ are equivalent if and only if they have the same signature.

(e) Let

$$q(x_1, x_2, x_3) = x_1^2 - x_1x_2 + +x_1x_3 + 5x_2^2 + 4x_2x_3 + 3x_3^2.$$

Do there exist real numbers x_1, x_2, x_3, not all zero, such that $q(x_1, x_2, x_3) = 0$?

(f) A quadratic form over $\mathbb{R}$ is positive (semi-definite) if $q(x_1, \ldots, x_n) \geq 0$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Show that a quadratic form is positive if and only if its associated symmetric matrix A defines a positive operator $\mathbb{R}^n \to \mathbb{R}^n$ (with respect to the standard basis).

(2) Using the method outlined in lecture, compute a singular value decomposition for

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$$

3. Comments

(1) For problem 7C.6: I suggest you divide the problem up into two cases: k is an even integer, and k is an odd integer.

(2) In problem 7D.6: be careful. You have to use an orthonormal basis to compute $\mathcal{M}(D^*)$ from $\mathcal{M}(D)$. The standard basis is not orthonormal, but we computed an orthonormal basis for this space and inner product! See Example 6.33 of Axler.

(3) Problem 7D.18 looks harder than it is. Just use the singular value decomposition theorem, and remember that if $e_1, \dots, e_n$ is an orthonormal basis of an inner product space, then $||a_1 e_1 + \cdots a_n e_n||^2 = |a_1|^2 + \cdots + |a_n|^2$.