Algebra II: Homework 7

Due on March 24, 2021

Professor Walton

Gabriel Gress

Last edited March 24, 2021

Collaborated with the Yellow group

Problem 1

Claim. Compute the splitting field of $x^4 - 4x^2 - 5$ over $\mathbb{Q}$, and show that it has degree 4 over $\mathbb{Q}$.

Proof. (a). First we factor as much as possible in $\mathbb{Q}$, then extend the field. So

$$x^4 - 4x^2 - 5 = (x^2 - 5)(x^2 + 1)$$

Thus our polynomial has $\pm\sqrt{5}$ and $\pm i$. Thus the splitting field is $\mathbb{Q}(\sqrt{5}, i)$. Since $[\mathbb{Q} : (\sqrt{5}) : \mathbb{Q}] = [\mathbb{Q}(i) : \mathbb{Q}] = 2$, and the basis for each extension is independent of the other, by the tower theorem we have that $[\mathbb{Q}(\sqrt{5}, i) : \mathbb{Q}] = 4$.

□

Problem 2

Claim. Compute the splitting field of $x^4 - 2$ over the fields $\mathbb{Q}$ and $\mathbb{R}$.

Proof. The roots of $x^4 - 2$ are $\pm\sqrt[4]{2}$ and $\pm i\sqrt[4]{2}$. Thus the splitting field of the polynomial as an extension of $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[4]{2}, i)$. Over $\mathbb{R}$, however, the splitting field is $\mathbb{R}(i)$, as $\sqrt[4]{2}$ is in $\mathbb{R}$. □

Problem 3

Claim. Which of the following is a normal extension of $\mathbb{Q}$?

$$\mathbb{Q}(\sqrt{3})$$
$$\mathbb{Q}(\sqrt[3]{3})$$
$$\mathbb{Q}(\sqrt{5}, i)$$
$$\mathbb{Q}(\sqrt[4]{5})$$

Proof. (a). This is a normal extension, as it is a splitting field of $f = x^2 - 3$.

(b). This is not a normal extension– $x^3 - 3$ is an irreducible polynomial in F that has two non-real roots not in F.

(c). Of course this is the splitting field of $x^4 - 4x^2 - 5$ and so must be a normal extension.

(d). This is not a normal extension. It has two roots in F given by $\pm\sqrt[4]{5}$, but the other two roots are complex and hence not in F.

□

Problem 3 continued on next page...

Problem 4

Claim. Compute the splitting field of $x^6 + x^3 + 1$ over $\mathbb{Q}$

Proof. Observe that $x^6 + x^3 + 1$ has complex roots given below, which can be checked to verify that they indeed result in yielding zero:

$$\begin{aligned} x_1 &= -(-1)^{1/9} \\ x_2 &= (-1)^{2/9} \\ x_3 &= (-1)^{4/9} \\ x_4 &= -(-1)^{5/9} \\ x_5 &= -(-1)^{7/9} \\ x_6 &= (-1)^{8/9} \end{aligned}$$

Of course it is easy to see that these are all 9th roots of unity, and it can visually be seen that they are generated by the principle root $\omega_1 = e^{2\pi i/9}$. Hence, the splitting field is then $\mathbb{Q}(\omega_1)$ □

Problem 5

Claim. Let K_1 and K_2 be finite extensions of F contained in the field K, and assume both are splitting fields over F.

(a). Prove that their composite K_1K_2 is a splitting field over F.

(b). Prove that $K_1 \cap K_2$ is a splitting field over F.

Proof. (a). Let p_1, p_2 be the polynomial over which K_1 and K_2 are splitting fields. Let $a_1, \ldots, a_n$ be roots of p_1 and $b_1, \ldots, b_m$ be roots of p_2. Of course, the extension K_1K_2 is generated by the roots $a_1, \ldots, a_n, b_1, \ldots, b_m$. These are precisely the roots of $p = p_1p_2$, and since K_1K_2 is the smallest field containing K_1, K_2, K_1K_2 is the splitting field of $p = p_1p_2$.

(b). Recall that the intersection of two fields is a field. Suppose that p has a root in $K_1 \cap K_2$. Then we know that p splits completely in K_1 and K_2. Thus, if $a_1, \ldots, a_n$ are the roots of p, they all must lie in K_1 and K_2. Hence p is a splitting polynomial for $K_1 \cap K_2$.

□

Problem 6

Claim. Prove that a finite field extension K over F is normal if and only if K has the following property:

When L is a field extension of K and $\varphi : K \to L$ is a field embedding with $\varphi(f) = f$ for all $f \in F$, we get that $\varphi(K) \subset K$.

Proof. □

Problem 6 continued on next page. . .