In the statement of the Weierstrass Product Theorem, we implicitly assumed that

- there exists a sequence $\{k_n\} \subset \mathbb{N}$ so that

$$\sum_{n=1}^{\infty} \left(\frac{a}{|z_n|} \right)^{k_n}$$

  converges
- within the set of sequences that give convergence, there exists a "minimal" sequence.

We proved the first point in the proof of the theorem– one can see that choosing $k_n = n$ will guarantee convergence in all cases. As for the question of minimality, we explore in this section what a minimal sequence might be like, and how it reflects the underlying structure of our entire function.

Definition 0.0.1: Order of Entire Function

Let f be an entire function. We say that f is of **order** $\leq \rho$ for real $\rho > 0$ if for all $\varepsilon > 0$, there exists a constant C_ε and a corresponding $R_{C_\varepsilon} > 0$ such that, for all $R > R_{C_\varepsilon}$,

$$\sup_{z \in D_R} |f(z)| \leq C_\varepsilon^{R^{\rho+\varepsilon}} .$$

We say that f is of **strict order** $\leq \rho$ if the inequality holds without the ε:

$$\sup_{z \in D_R} |f(z)| \leq C_\varepsilon^{R^{\rho}} .$$

Finally, the function f is of **order** ρ or **strict order** ρ if ρ is the greatest lower bound to make the inequality valid.

Example 0.0.1

The function e^z is strict order 1 because

$$|e^z| = e^x \leq e^{|z|}.$$

For a more involved example, if we have a canonical Weierstrass product with $\sup_n \{k_n\} = k$, then for all ρ satisfying $k - 1 < \rho < k$, the Weierstrass product is of order $\leq \rho$. The following lemma allows us to obtain the converse.

Lemma 0.0.1

Let f be an entire function of strict order $\leq \rho$. Then for R sufficiently large,

$$\frac{1}{2\pi i} \int_{\partial D_R} \frac{f'}{f} \, dt \leq R^\rho.$$

Of course because f is entire, the left-hand side represents the number of zeros within the disc of radius R.

The following corollary immediately follows:

Corollary 0.0.1

Let f have strict order $\leq \rho$, and let $\{z_n\} \subset \mathbb{C} \setminus \{0\}$ be the zeros of f ordered by increasing modulus. Then for every $\varepsilon > 0$, the series

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\rho+\delta}}$$

converges.

This corollary then gives us the converse– every entire function of strict order $\leq \rho$ has a Weierstrass product form with $\rho < \sup_n k_n < \rho + 1$. We summarize these results into the minimum modulus theorem.

Theorem 0.0.1: Minimum Modulus Theorem

Let f be an entire function of order $\leq \rho$, and let $\{z_n\} \subset \mathbb{C}$ be its sequence of zeros ordered by increasing modulus. Then for all $s > \rho$, $\varepsilon > 0$, there is a corresponding R_0 so that, for all $R > R_0$ and $z \in \mathbb{C} \setminus \overline{D_s}$:

$$|f(z)| \geq e^{-R^{\rho+\varepsilon}}$$

This ensures the construction we gave in Hadamard's Theorem is valid and unique.