Subring:

A subset S of a ring R which is a ring under the (restrictions of) the operations in R. Notation: $S \leq R$.

$\emptyset \neq S \leq R \iff (a, b \in S \Rightarrow a+b, ab, -a \in S) \iff (a, b \in S \Rightarrow a-b, ab \in S)$.

$0_S = 0_R$ and $(-a)_S = (-a)_R$, but this does not hold in general for the identities, see Problem 28g-k.

Ideal:

A subring I which is closed also under multiplication with elements of R. Notation: $I \lhd R$.

$I \lhd R \iff (i, j \in I, r \in R \Rightarrow i-j, ri, ir \in I)$.

If R is commutative and has an identity, then the ideal generated by elements $c_1, \ldots, c_k$ is $(c_1, c_2, \ldots, c_k) = \{r_1c_1 + r_2c_2 + \ldots + r_kc_k \mid r_i \in R\}$.

This is the smallest ideal containing $c_1, c_2, \ldots, c_k$.

For $k = 1$, we get the *principal ideal* $(c) = \{rc \mid r \in R\}$ generated by c which consists of all multiples of c.

Factor ring:

Let $I \lhd R$ and define a *coset* as $r+I = \{r+i \mid i \in I\}$. Two such cosets are either equal, or disjoint. We define addition and multiplication for the cosets by $(r+I)+(s+I) = (r+s)+I$ and $(r+I)(s+I) = rs+I$. Then we get the factor ring R/I. E.g. $\mathbf{Z}/(m) = \mathbf{Z}_m$.

Ring homomorphism:

A map from a ring R to a ring S which preserves the operations, i.e. $\varphi : R \to S$ where $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$ and $\varphi(r_1r_2) = \varphi(r_1)\varphi(r_2)$.

$\varphi(0) = 0$ and $\varphi(-a) = -\varphi(a)$, but $\varphi(1)$ is not necessarily an identity in S.

Isomorphism is a bijective homomorphism. If $\varphi : R \to S$ is an isomorphism, then R and S are *isomorphic*(=are "of the same form"), i.e. they are essentially the same, just the elements and operations bear different names. Notation: $R \cong S$. E.g. the ring (b1) in Problem 5 is isomorphic to $\mathbf{Z}_5$; (c5), (d1), (d4) and (d5) are isomorphic to $\mathbf{R}$, etc.

Kernel: $\operatorname{Ker}\varphi = \{r \in R \mid \varphi(r) = 0\}$; Image: $\operatorname{Im}\varphi = \{\varphi(r) \mid r \in R\}$.

$\operatorname{Ker}\varphi \lhd R$, $\operatorname{Im}\varphi \leq S$. φ is an isomorphism $\iff$ ($\operatorname{Ker}\varphi = 0$ and $\operatorname{Im}\varphi = S$).

Homomorphism theorem: $\operatorname{Im}\varphi \cong R/\operatorname{Ker}\varphi$.

Natural homomorphism: $\psi : R \to R/I$ where $\psi(r) = r + I$. Here $\operatorname{Ker}\psi = I$ and $\operatorname{Im}\psi = R/I$.

Homomorphism theorem and natural homomorphism together show that for a ring R, there is a one-to-one correspondence between its ideals (or factor rings) and the homomorphisms from R to some other ring.