116. Which of the following groups are direct products of two groups of size > 1:
(a) $\mathbf{Z}_{27}$; (b) $\mathbf{Z}_{28}$; (c) D_6; (d) D_3; (e) Q(=quaternion group); (f) $\mathbf{Z}$?

117. The converse of Lagrange's Theorem is false: $n \,\big|\, |G|$ does not imply that G has an element of order n or a subgroup of order n.
(a) A_4 has no subgroup of 6 elements.
(b) To every composite n there exists a group G satisfying $n \,\big|\, |G|$ without elements of order n.
(c) If $|G|$ is even, then G contains an element of order 2.
(d) Cauchy's Theorem: If n is a prime and $n \,\big|\, |G|$, then G contains an element of order n.

Remark: Concerning subgroups, the following hold: (i) If n is a prime power and $n \,\big|\, |G|$, then G has a subgroup of size n (Sylow's First Theorem); (ii) If n is not a prime power, then there exists a group G satisfying $n \,\big|\, |G|$ without subgroups of size n.

118. Prove that if $|G| = 4k + 2$, then G has a non-trivial normal subgroup.

119. $H \le S_n$ is *transitive* if to every $1 \le i, j \le n$ there exists a $\sigma \in H$ satisfying $\sigma(i) = j$.
(a) For which positive integers n is A_n transitive?
(b) Show that if H is transitive, then $|H| \ge n$.
(c) Verify that if $H \le S_3$ and $|H| \ge 3$, then H is transitive.
(d) Find to every $n > 3$ a non-transitive subgroup H of size $\ge n$.
(e) Prove that the Cayley representation creates a transitive subgroup.

120. $H \le S_n$ is free of fixed points if $e \ne \sigma \in H$ implies $\sigma(i) \ne i$ for every $1 \le i \le n$.
(a) For which positive integers is A_n free of fixed points?
(b) Show that if H is free of fixed points, then $|H| \le n$.
(c) Prove that the Cayley representation creates a subgroup free of fixed points.

121. In certain native South-American tribes, a boy and a girl could marry only if their signs obtained at birth from the wizard were the same. There were finitely many possible signs, and the sign depended only on the sex of the child and on the (common) sign of the parents. Find out, which natural principles govern the distribution of the signs and set a mathematical model.

122. In how many ways can we color 4 squares red in a 5×5 square grid where two colorings count the same if a rotation or a reflection can transform them into each other?

123. The 12 numbers on a traditional clock are erased, the places of 4 of them are painted green, the others are painted red. How many such colorings are possible where two colorings count the same if a rotation or a reflection can transform them into each other?

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