Recall that a holomorphic isomorphism is a holomorphic map

$$f : U \to V$$

with a holomorphic inverse

$$g : V \to U$$

so that

$$f \circ g = \mathrm{Id}_V$$
$$g \circ f = \mathrm{Id}_U$$

We say that f is an **automorphism** if $U = V$, and refer to the set of automorphisms of U by $\mathrm{Aut}(U)$.

0.0.1 Automorphisms of the Unit Disc

Theorem 0.0.1: Schwarz Lemma

Let $f : D_1 \to D_1$ be a holomorphic automorphism of the unit disc with $f(0) = 0$. Then

$$|f(z)| \leq |z|$$

for all $z \in D_1$. If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$, then

$$f(z) = e^{i\theta} z$$

for some $\theta \in \mathbb{R}$.

In particular, $|f'(0)| \leq 1$. If equality holds, then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

In other words, an automorphism of the unit disc that fixes the origin has an implicit "contraction" requisite. If it isn't a contraction, then it must be a rotation.

Proof. ■

We can apply Schwarz Lemma to classify all the automorphisms of the unit disc. Because every automorphism must send exactly one point to zero, we can transform the general case to the origin.

Theorem 0.0.2

Let $f : D_1 \to D_1$ be a holomorphic automorphism of the unit disc, and let $z_0 \in D_1$ be the point for which $f(z_0) = 0$. Then for some $\theta \in \mathbb{R}$

$$f(z) = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z}$$

0.0.2 Isomorphism from the Upper Half Plane

We began this chapter with automorphisms of the unit disc because it turns out that we can obtain an isomorphism between larger parts of the complex plane to the disc, hence characterizing their automorphisms.

We will now show that the entire upper half complex plane is isomorphic to the unit disc.

Theorem 0.0.3

Let $\mathbb{H} \subset \mathbb{C}$ be the upper half plane. The map

$$f : \mathbb{H} \to D_1$$
$$f : z \mapsto \frac{z - i}{z + i}$$

is an isomorphism.

The inverse map for f is given by

$$g : D_1 \to \mathbb{H}$$
$$g : z \mapsto -i\frac{z + 1}{z - 1}$$

We leave the verification as an exercise to the reader.

Corollary 0.0.1: Automorphisms of Upper Half Plane

$$\mathrm{Aut}(\mathbb{H}) = f^{-1}\mathrm{Aut}(D)f$$

Of course, it remains to actually calculate what automorphisms of the upper half plane look like.

Theorem 0.0.4

Every automorphism of $\mathbb{H}$ is a linear fractional translation of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} S_A(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ with $\det(A) \neq 0$.

Two automorphisms S_A and $S_{A'}$ are equal if and only if $A' = \pm A$.

Hence, the automorphism group $\mathrm{Aut}(\mathbb{H})$ is isomorphic to the projective special linear group– that is,

$$\mathrm{Aut}(\mathbb{H}) \cong PSL_2(\mathbb{R})$$

0.0.3 Isomorphisms Between Boundaries of Closed Curves

We note that so far, many results have been shown by utilizing the behavior of the function on the boundary of an open set. This is no coincidence– we formalize this notion via the following theorems.

Theorem 0.0.5

Let Ω be a bounded connected open set. Suppose f is a non-constant holomorphic function on Ω that is continuous on $\partial\Omega$, which maps $\partial\Omega$ into ∂D_1, that is,

$$|f(z)| = 1$$

for all $z \in \partial\Omega$. Then

$$f(\Omega) = D_1.$$

If we further assume that f is injective, then it follows that f is an isomorphism.

Proof. ■

We can further generalize this via interiors of curves, but first we prove a lemma that will be essential:

Lemma 0.0.1

Let γ be a piecewise smooth closed curve in an open set $\Omega \subset \mathbb{C}$. Suppose that γ has an interior denoted by $\mathrm{Int}(\gamma)$. Then

$$\mathrm{Int}(\gamma) \cup \gamma$$

is compact.

This follows from the compactness of γ combined with the continuity of the winding number within the interior.

Theorem 0.0.6

Let f be a non-constant holomorphic function on some open connected set $\Omega \subset \mathbb{C}$, and let γ be a piecewise-smooth closed curve in Ω homologous to 0.

If γ and $f \circ \gamma$ have interiors so that

$$f \circ \gamma \cap f(\mathrm{Int}\gamma) = \emptyset$$

then f is injective on $\mathrm{Int}(\gamma)$ and hence $f : \mathrm{Int}(\gamma) \to f(\mathrm{Int}\gamma)$ is an isomorphism. If $\mathrm{Int}(f \circ \gamma)$ is connected, then

$$f(\mathrm{Int}\gamma) = \mathrm{Int}(f \circ \gamma).$$

Proof. ■

Example 0.0.1

Example 0.0.2