Let A and C be arbitrary groups.. One question worth exploring is if there exists a group B such that $A/B \cong C$– that is, B is an extension of C by A. The tools we develop to understand this question are exact sequences. If A is isomorphic to a subgroup of B, there is an injective homomorphism from A to B. And if C is isomorphic to the quotient, then there is a surjective homomorphism from B to C. This will give us a chain

$$A \to B \to C$$

where the homomorphisms are compatible with. We formalize this idea via exact sequences.

Definition 0.0.1: Exact Sequences

Let α, β be homomorphisms so that

$$X \to^{\alpha} Y \to^{\beta} Z.$$

If $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$, then we say the pair of homomorphisms are **exact**.

A sequence of homomorphisms

$$\ldots \to X_{n-1} \to X_n \to X_{n+1} \to \ldots$$

is said to be an **exact sequence** if it is exact at every X_n between a pair of homomorphisms.

Hence, our goal is to see whether we can form an exact sequence $A \to B \to C$. Our notions of injectivity and surjectivity correspond exactly to the notions of exactness.

Proposition 0.0.1

Let A, B, C be groups. Then the sequence

$$0 \to A \to^{\psi} B$$

is exact at A if and only if ψ is injective. Likewise, the sequence

$$B \to^{\varphi} \to C \to 0$$

is exact at C if and only if φ is surjective.

Combining the two ideas, the sequence

$$0 \to A \to^{\psi} B \to^{\varphi} C \to 0$$

is exact if and only if ψ is injective, φ is surjective, and $\mathrm{Im}(\psi) = \mathrm{Ker}(\varphi)$.

Definition 0.0.2

An exact sequence of the form

$$0 \to A \to^{\psi} B \to^{\varphi} C \to 0$$

is called an **short exact sequence**.

Our goal then is to determine if two groups admit a short exact sequence, and if so, how many.

Notice that any exact sequence can be written as a succession of short exact sequences. For example, if

$$X \to^{\alpha} Y \to^{\beta} Z$$

is exact at Y, then equivalently

$$0 \to \alpha(X) \to Y \to Y/\mathrm{Ker}(\beta) \to 0$$

is a short exact sequence.

Example 0.0.1

For fixed A, C, there can be many extensions of C by A. Hence, we need to determine a notion of a homomorphism to distinguish exact sequences.

Definition 0.0.3: Homomorphism of Short Exact Sequences

Let

$$0 \to A \to B \to C \to 0$$
$$0 \to A' \to B' \to C' \to 0$$

be two short exact sequences of groups. A **homomorphism of short exact sequences** is a collection of group homomorphisms α, β, γ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

This is an **isomorphism of short exact sequences** if α, β, γ are isomorphisms in which case the extensions B, B' are **isomorphic extensions**.

The two exact sequences are called **equivalent** if $A = A'$, $C = C'$, and there is an isomorphism between them where α, γ are identity. In this case B and B' are **equivalent extensions**.

Equivalency by extensions is strongesomorphisms between B and B'– it tells us that there is an isomorphism between B and B' that restricts to an isomorphism from A to A' and induces an isomorphism on the quotients by C and C'.

Example 0.0.2

Proposition 0.0.2: Short Five Lemma

Let α, β, γ be a homomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

- If α, γ are injective then so is β
- If α, γ are surjective then so is β
- If α, γ are isomorphisms then so is β

Proof of Short Five Lemma. ∎

There is always at least one extension of a group C by A given by $B = A \rtimes C$.

Definition 0.0.4

If

$$1 \to A \to^{\psi} B \to^{\varphi} C \to 1$$

is a short exact sequence of groups, then the sequence is **split** if there is a sugroup complement to $\psi(A)$ in B. Then up to isomorphism $B = A \rtimes C$ up to isomorphism by

$$B = \psi(A) \rtimes C'$$

for some subgroup C', which satisfies $\varphi(C') \cong C$.

We say B is a **split extension of** C **by** A.

This is really just the question of existence of a complement to $\psi(A)$ in B that is isomorphic by φ to C.

Proposition 0.0.3

The short exact sequence of groups

$$1 \to A \to^{\psi} B \to^{\varphi} C \to 0$$

of groups is split if and only if there is a group homomorphism $\mu : C \to B$ such that $\varphi \circ \mu \cong \mathrm{Id}_C$.

Any set map $\mu : C \to B$ such that $\varphi \circ \mu = \mathrm{Id}_C$ is called a **section** of φ. If μ is a homomorphism, then μ is called a **splitting homomorphism** for the sequence.

A section of φ is merely a choice of coset representative in B for $B/\mathrm{Ker}\varphi \cong C$. A section is a splitting homomorphism if this set of coset representatives forms a subgroup, in which case this subgroup gives a complement to $\psi(A)$ in B.

Example 0.0.3

Proposition 0.0.4

Let

$$0 \to A \to^{\psi} B \to^{\varphi} C \to 0$$

be a short exact sequence of groups. Then $B = \psi(A) \rtimes C'$ for some subgroup C' of B with $\varphi(C') \cong C$ if and only if there is a homomorphism $\lambda : B \to A$ such that $\lambda \circ \psi = \mathrm{Id}_A$.

This is stronger than the previous proposition. The existence of a splitting homomorphism on the left end of the sequence gives that the extension group is a direct product (instead of a semidirect product).