Inverse and Open Mapping Theorems

Definition 0.0.1

Let f be a holomorphic function on an open set $X \subset \mathbb{C}$, and let $f(X) = Y$. If Y is open and there exists a holomorphic function $g : Y \to X$ with

$$f \circ g = \mathrm{Id}_Y$$
$$g \circ f = \mathrm{Id}_X$$

then f and g are **analytic isomorphisms**.

If instead there is a point $z_0 \in X$ such that f is an analytic isomorphism for some open neighborhood $U_{z_0} \subset X$, then we say f is a **local analytic isomorphism** or **locally invertible** at z_0.

Note that the word "inverse" here always refers to the composition inverse. When the multiplicative and composition inverse differ, if we wish to refer to the multiplicative inverse, we will explicitly say "multiplicative inverse".

Proposition 0.0.1

The inverse function g of an analytic isomorphism is unique.

Theorem 0.0.1: Complex Inverse Function Theorem

Let f be a holomorphic function on an open set $\Omega \subset \mathbb{C}$. Suppose that $f'(z_0) \neq 0$ for some $z_0 \in \Omega$. Then f is a local analytic isomorphism at z_0.

This can be proven via formal power series. In fact, this can also be proven the standard route fron real analysis. If one assumes the real analysis version of the theorem, then this can be proven by simply decomposing $f = u + iv$, in which case it applies to u, v.

Recall from topology that f is an **open mapping** if for every open subset $U \subset X$, $f(U)$ is open.

Theorem 0.0.2

Let f be holomorphic on an open set $X \subset C$. If for every $z_0 \in X$, f is non-constant on every neighborhood $U_{z_0} \subset X$, then f is an open mapping.

Proof. ■

Theorem 0.0.3: Change of Coordinates

Let f be holomorphic at a point z_0, with $f(z_0) \neq 0$. Then there exists a local analytic isomorphism φ at 0 such that

$$f(z) = f(z_0) + (\varphi(z - z_0))^m$$

where m is the smallest value (greater than 0) for which $f^{(m)}(z_0) \neq 0$.

Theorem 0.0.4

Let f be holomorphic on an open set $\Omega \subset \mathbb{C}$. If f is injective, then

$$f : \Omega \to f(\Omega)$$

is a holomorphic isomorphism, and hence $f'(z) \neq 0$ for all $z \in \Omega$.

Local Maximum Modulus Principle

Definition 0.0.2: Locally Constant

A function f is **locally constant** at a point z_0 if there exists an open neighborhood U_{z_0} such that f is constant on U_{z_0}.

Theorem 0.0.5

Let f be holomorphic on an open set Ω. Suppose that $z_0 \in \Omega$ is a maximum for $|f|$. Then f is locally constant at z_0.

Corollary 0.0.1

Let f be holomorphic on an open set Ω, and suppose that $z_0 \in \Omega$ is a maximum for $\mathrm{Re} f$, that is,

$$\mathrm{Re} f(z_0) \geq \mathrm{Re} f(z)$$

for all $z \in \Omega$. Then f is locally constant at z_0.

This gives us a lot of powerful applications.

Theorem 0.0.6

Let

$$f(z) = \sum_{n=0}^{k} a_n z^n$$

be a non-constant complex polynomial with $a_k \neq 0$. Then there exists $z_0 \in \mathbb{C}$ so that $f(z_0) = 0$.