G, G_i denote groups.

103. Find all normal subgroups in (a) D_4; (b) S_3; (c) $\mathbf{Z}_{24}$.

104. Let $H \le G$ and assume $bH = Hb$ for some $b \in G \setminus H$. Does this imply $H \lhd G$ if $|G : H|$ is (a) 3; (b) 4; (c) 5?

105. (a) Prove that the center $Z(G) \lhd G$, moreover all its subgroups are normal in G.
(b) Determine the center of (b1) $\mathbf{Z}_n$; (b2) D_n; (b3) S_n.

106. (a) Describe all factor groups of the groups in Problem 103.
(b) Find a relation between the order of g in G and the order of gN in G/N.
(c) Prove: If $|G : N| < \infty$, then $g^{|G:N|} \in N$ for every $g \in G$.

107. True or false? (N, N_i are non-trivial normal subgroups in G.)

(a) If G is commutative, then so is N.
(b) If N is commutative, then so is G.
(c) If G is commutative, then so is G/N.
(d) If G/N is commutative, then so is G.
(e) If N is commutative, then so is G/N.
(f) If G/N is commutative, then so is N.
(g) If G is cyclic, then so is G/N.
(h) If G/N is cyclic, then so is G.
(i) If N is cyclic, then so is G/N.
(j) If G/N is cyclic, then so is N.
(k) If $N_1 \cong N_2$, then $G/N_1 \cong G/N_2$.
(l) If $G/N_1 \cong G/N_2$, then $N_1 \cong N_2$.

108. Let G be a finite non-abelian group. Prove:
(a) $G/Z(G)$ is not cyclic.
(b) $|Z(G)| \le |G|/4$ and equality can hold.
*(c) (Some measure of commutativity.) Let $\alpha(G)$ be the ratio of the pairs of elements which commute, i.e. $\alpha = \dfrac{|\{(a,b) \mid ab = ba\}|}{|G|^2}$. Then $\alpha \le 5/8$ and equality can hold.

109. Let $\varphi : G_1 \to G_2$ be a group homomorphism, $g \in G_1$. Prove:
(a) If $o(g) = \infty$, then $o(\varphi(g))$ can be any integer or ∞.
(d) If $o(g) < \infty$, then $o(\varphi(g)) \mid o(g)$.

110. For which finite groups G is $\varphi(g) = g^2$ a $G \to G$ isomorphism?

111. If $|G_1| < \infty$ and $\varphi : G_1 \to G_2$ is a homomorphism, then $|G_1| = |\operatorname{Ker}\varphi| \cdot |\operatorname{Im}\varphi|$.

112. For which integers n does there exist a homomorphism $\varphi : D_n \to G_2$ such that $\operatorname{Im}\varphi$ is a Klein group?

113. (a) $G_1 \times G_2$ is commutative iff both G_1 and G_2 are commutative;

(b) $o\big((a,b)\big) = \begin{cases} [o(a), o(b)], & \text{if } o(a), o(b) < \infty; \\ \infty & \text{otherwise.} \end{cases}$

114. Find the number of subgroups in $\mathbf{Z}_p \times \mathbf{Z}_p$ where p is a prime.

115. (a) Represent the following groups as a direct product of cyclic groups $\mathbf{Z}_{p^k}$ where p is a prime:
(a1) K(=Klein group); (a2) $\mathbf{Z}_{3000}$; (a3) the symmetry group of a brick.
(b) Determine the number of pairwise non-isomorphic Abelian groups having 144 elements.
(c) Let $A(k)$ be the number of pairwise non-isomorphic Abelian groups. Show that if k and m are coprime, then $A(km) = A(k)A(m)$.