MATH 354: PROBLEM SET 10

RICE UNIVERSITY, FALL 2019

Due date: Monday, November 25th, by 5pm in my office (you can slide it under the door). You are welcome to turn in your work during lecture on Monday.

Parts A, and B should be handed in **separately**. They will be graded by different TAs.

Please staple your homework!

Reminder from the syllabus: "The homework is not pledged and you can collaborate with other students in the class. In fact, you are very much *encouraged* to do so. However, you are not allowed to look up solutions in any written form; in particular, you are not allowed to look up solutions online. **Students caught violating this rule will be reported to the Honor Council.** You should write up your solutions individually."

1. Part A

Hand in the following exercises from Chapters 6 and 7 of Axler's book:

6B: 14.   7A: 4, 5, 6.

2. Part B

Hand in the following exercises from Chapter 7 of Axler's book:

7B: 2, 4, 6.

In addition, hand in solutions to the following problems.

(1) This problem is meant to help you find diagonalizations of a matrix (it is **not** about the spectral theorem). Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and let $A = \mathcal{M}(T)$ be the matrix of T with respect to the standard basis. Suppose that the operator T is diagonalizable. Recall that by 5.41 in Axler, this means there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of T.

  (a) Let P be an $n \times n$ matrix whose columns consist of n linearly independent eigenvectors of T. Show that $D := P^{-1}AP$ is a diagonal matrix.

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(b) Suppose that $n = 3$ and

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -4 & -3 \end{pmatrix}$$

Find a matrix P such that $D = P^{-1}AP$ is diagonal. Use this matrix to compute A^{15} *without* just computing powers of A until you drop dead. Hint: $2^{15} = 32,768$ and $3^{15} = 14,348,907$.

(2) Let A be an $n \times n$ matrix with entries in $\mathbb{R}$. We say that A is symmetric if $A = A^t$. Prove that if A is symmetric, then there exists an $n \times n$ matrix Q such that Q^tAQ is a diagonal matrix. [Hint: View A as the matrix of a self-adjoint operator of $\mathbb{R}^n$. Construct the columns of Q from the basis of $\mathbb{R}^n$ furnished by the spectral theorem (using the usual dot product as your inner product), and show that $Q^{-1} = Q^t$.]

In the final problem set, we will use this exercise to investigate quadratic forms, and relate the positivity of an operator to the positivity of the set of values a quadratic form takes.

3. Comments

(1) Problem 6B.14 requires both the triangle inequality and Cauchy-Schwarz, judiciously applied. A good place to start is: suppose that $a_1v_1 + \cdots + a_nv_n = 0$. We want to show that $a_1 = \cdots = a_n = 0$. Now, using our assumption, we note that

$$a_1e_1 + \cdots a_ne_n = a_1(e_1 - v_1) + \cdots + a_n(e_n - v_n).$$

Start using the triangle inequality now, as well as the hypothesis of the problem and then Cauchy-Schwarz.

(2) Problems 7A.4 and 7A.5 require a tiny bit about orthogonal complements, which I did not cover in class. So let me give you a blueprint of a few quick things you should read to answer these questions. Read Definition 6.45 and the blue boxes 6.47 and 6.50 (you can skip the proofs, but your life may feel a little empty if you do so). Then look at blue box 7.7. Now you have all the tools you need to solve the problems. The actual solutions are not long.

(3) The problems from section 7B can all be done using the spectral theorem.