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Notes on Fractal Geometry

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May 25, 2021

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Chapter 0

Introduction

These notes are personal notes I have created in the process of studying geometric measure theory and contain a wide variety of definitions and techniques that appear often in the field. The notes were created primarily from a reading course taken with Dr. Gregory Chambers at Rice University in Spring 2021 that followed Dr. Kenneth Falconer's textbook *Fractal Geometry*. One will notice that the proofs of major results are either lacking or not included in these notes. This is because this document is primarily intended as a reference– if the reader is looking for a deeper insight as to why these results are true, I would highly recommend one read the details in *Fractal Geometry*– if the proof isn't in there, then I have listed the source separately with the statement.

The results in this book assume a basic understanding of measure theory, and so one should already know the definition of a measure, σ-algebra, Borel set, and so on. Introductory notes on this topic will be provided in the LibreMath repository (soon).

Chapter 1

Box-counting Dimension

The premise behind the box-counting dimension is to count the minimal number of sets of diameter $\leq \delta$ that can cover a subset F of Euclidean space. Denoting this number by $N_\delta(F)$, we observe if there exists a power law so that

$$N_\delta(F) \cong c\delta^{-s}$$

for some positive constants c, s. If this exists, we say that F has box dimension s. Note that we can rewrite this as

$$s = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Definition 1.0.1

Of course, this may not be well-defined. To formalize this concept, we introduce **lower** and **upper box-counting dimensions of** F :

$$\underline{\dim}_B F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$
$$\overline{\dim}_B F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

If they are equal, then we say it is the **box-counting dimension** as above:

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Note that in order to make sure this is well-defined we typically only apply box dimension to non-empty bounded sets.

What makes this definition so interesting is that it can be modified in many ways to get the same result. Using cube, or largest number of disjoint balls still yields the same numbers. In other words, we can use the same statement but redefine $N_\delta(F)$ to be

1. the smallest number of sets of diameter at most δ that cover F
2. the smallest number of closed balls of radius δ that cover F
3. the smallest number of cubes of side δ that cover F
4. the number of δ-mesh cubes that intersect F
5. the largest number of disjoint balls of radius δ with centers in F

and get the same results.

An equivalent definition of a different form is useful. Define the δ-neighborhood to be

$$F_\delta = \{x \in \mathbb{R}^n \mid |x - y| \leq \delta \text{ for some } y \in F\}.$$

We can consider the n-dimensional Lebesgue measure of this object. In fact, if for some $0 < c < \infty$ we have

$$\lim_{\delta \to 0} \left(\mathcal{L}^n(F_\delta)/\delta^{n-s}\right) = c$$

for some $s > 0$, then we can regard F as s-dimensional. In this case, we call c the s-dimensional Minkowski content of F.

Proposition 1.0.1

If F is a subset of $\mathbb{R}^n$, then

$$\underline{\dim}_B F = n - \limsup_{\delta \to 0} \frac{\log \mathcal{L}^n(F_\delta)}{\log \delta}$$
$$\overline{\dim}_B F = n - \liminf_{\delta \to 0} \frac{\log \mathcal{L}^n(F_\delta)}{\log \delta}$$

This proposition is why the box-dimension is also commonly referred to as the Minkowski dimension.

One notable fact is that, if we are considering the box dimension of a compact subset of $\mathbb{R}$, we can instead consider the complementary intervals in order to determine the dimension. Further detail in later chapters.

1.1 Properties of Box-counting Dimension

1. Monotonicity– if $E \subset F$, then $\dim_B E \leq \dim_B F$, along with their supremum and infimum versions.
2. For $F \subset \mathbb{R}^n$ non-empty and bounded,
$$0 \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq n.$$
3. Finite stability–
$$\overline{\dim}_B (E \cup F) = \max \left\{ \overline{\dim}_B E, \overline{\dim}_B F \right\}.$$
4. If $F \subset \mathbb{R}^n$ is open, then $\dim_B F = n$.
5. If F is non-empty and finite, then $\dim_B F = 0$.
6. If F is a smooth bounded m-dimensional submanifold of $\mathbb{R}^n$, then $\dim_H F = m$.

We also have a preservation of dimension under Lipschitz transformations.

Proposition 1.1.1

(a). If $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ is a Lipschitz transformation, that is,

$$|f(x) - f(y)| \leq c\,|x - y| \quad \forall x, y \in F$$

then

$$\underline{\dim}_B f(F) \leq \underline{\dim}_B F$$
$$\overline{\dim}_B f(F) \leq \overline{\dim}_B F$$

(b). If $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ is a bi-Lipschitz transformation, that is,

$$c_1\,|x - y| \leq |f(x) - f(y)| \leq c\,|x - y| \quad \forall x, y \in F$$

with $0 < c_1 \leq c < \infty$, then we have equality instead.

This applies to a lot of transformations, such as affine translations. We also begin to get more information about projections– for example, we can bound the dimension of the projection above by the original dimension.

Proposition 1.1.2

$$\underline{\dim}_B \overline{F} = \underline{\dim}_B F$$
$$\overline{\dim}_B \overline{F} = \overline{\dim}_B F.$$

This is concerning and has some undesired consequences. FOr example, if F is a dense subset of an open region in $\mathbb{R}^n$, its dimension must be n. This implies that countable sets, which are very small compared to $\mathbb{R}$, can have non-zero box dimension. Worse yet, the box dimension of individual rationals is clearly zero, yet their union has dimension 1. In other words, we cannot assert any sort of closure under unions.

1.2 Modified Box-counting Dimension

We can resolve some of these issues, but obviously it comes at a cost (of being more difficult to apply directly).

Definition 1.2.1

We define the **lower** and **upper modified** box-counting dimension by:

$$\underline{\dim}_{MB} F = \inf \left\{ \sup_i \underline{\dim}_B F_i \mid F \subset \bigcup_{i=1}^{\infty} F_i \right\}$$
$$\overline{\dim}_{MB} F = \inf \left\{ \sup_i \overline{\dim}_B F_i \mid F \subset \bigcup_{i=1}^{\infty} F_i \right\}.$$

This is bounded above by the original definition, but has the added benefit that countable sets have dimension zero.

This carries all the properties that we had earlier with the box-counting dimension, but with some other nice properties, such as **countably stable**:

$$\underline{\dim}_{MB} \left(\bigcup_{i=1}^{\infty} F_i \right) = \sup_i \{\underline{\dim}_{MB} F_i\}$$

for any countable sequence of sets $\{F_i\}$ (and of course for the upper modified box dimension). We have a nice property that allows us to determine if this version is equivalent to the standard box dimension.

Proposition 1.2.1

Let $F \subset \mathbb{R}^n$ be compact. Suppose that

$$\overline{\dim}_B (F \cap V) = \overline{\dim}_B F$$

for all open sets V that intersect F. Then $\overline{\dim}_B F = \overline{\dim}_{MB} F$, and likewise for lower box-counting dimensions.

This can be proven by application of Baire's category theorem.

Recall that we say a set $F \subset \mathbb{R}^n$ is of **second category** in $\mathbb{R}^n$ if it cannot be expressed as a countable union of nowhere dense sets (closure has empty interior). Equivalently, if $F \subset \bigcup_{i=1}^{\infty} F_i$, there is some F_i and non-empty open set V such that $V \subset \overline{F \cap F_i}$.

Proposition 1.2.2

Let $F \subset \mathbb{R}^n$ be of second category. Then $\underline{\dim}_{MB} F = \overline{\dim}_{MB} F = n$.

Chapter 2

Hausdorff and Packing Measures and Dimensions

Definition 2.0.1

Let F be a subset of $\mathbb{R}^n$ and $s \geq 0$. For each $\delta > 0$, define

$$\mathcal{H}_\delta^s(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s \mid \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}.$$

The Hausdorff dimension looks at all covers of F of a certain dimension, and minimizes the s-th power of the diameters of the covering set. Notice that as δ decreases, the class of permissible covers in F is reduced, and so the infimum increases. This gives us

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F)$$

which we define the **s-dimensional Hausdorff measure of** F. This limit exists for any subset F, but it can and will usually be 0 or ∞.

In fact, $\mathcal{H}^s$ is a measure. Hausdorff measure generlizes the typical ideas of length, area, volumne, and in fact for subsets of $\mathbb{R}^n$, n-dimensional Hausdorff measure is within a constant multiple of n-dimensional Lebesgue measure. In particular, if F is a Borel subset of $\mathbb{R}^n$, then

$$\mathcal{H}^n(F) = c_n^{-1}\mathrm{vol}^n(F)$$

where c_n is the volume of an n-dimensional ball of diameter 1.

Proposition 2.0.1

Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a Holder mapping– that is, it satisfies

$$|f(x) - f(y)| \leq c\,|x - y|^\alpha \quad \forall x, y \in F$$

for constants $\alpha > 0$ and $c > 0$. Then for each s,

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/a}\mathcal{H}^s(F).$$

In particular, if f is a Lipschitz mapping, then

$$\mathcal{H}^s(f(F)) \leq c^s\mathcal{H}^s(F).$$

Proposition 2.0.2: Scaling Property

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity transformation of scale factor $\lambda > 0$. If $F \subset \mathbb{R}^n$, then

$$\mathcal{H}^s(f(F)) = \lambda^s\mathcal{H}^s(F).$$

2.1 Hausdorff Dimension

Using the robustness of the Hausdorff dimension, we can better construct a definition of dimension.

Observe that for $\delta < 1$, $\mathcal{H}^s_\delta(F)$ is non-increasing with s. In particular, for $t > s$

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F).$$

This seems to imply that there is some critical value of s at which $\mathcal{H}^s(F)$ jumps from ∞ to 0. This critical value is what we call the Hausdorff dimension.

Definition 2.1.1

Let $F \subset \mathbb{R}^n$. Then the **Hausdorff dimension** of F is

$$\dim_H F := \inf\{s \geq 0 \mid \mathcal{H}^s(F) = 0\} = \sup\{s \mid \mathcal{H}^s(F) = \infty\}.$$

This immediately gives

$$\mathcal{H}^s(F) = \begin{cases} \infty & 0 \leq s < \dim_H F \\ 0 & s > \dim_H F \end{cases}$$

Note that for $s = \dim_H F$, $\mathcal{H}^s(F)$ can be zero, infinite, or finite. A Borel set that $\mathcal{H}^s$ as finite is called an s-set.

Fortunately, $\mathcal{H}^s$ satisfies many of the same properties as the box-counting dimension. Furthermore, it satisfies the Holder condition exactly as expected:

Proposition 2.1.1

Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ satisfies the Holder condition

$$|f(x) - f(y)| \leq c\,|x - y|^\alpha \quad \forall x, y \in F.$$

Then $\dim_H f(F) \leq (1/\alpha)\dim_H F$. If f is instead bi-Lipschitz, then we have equality instead.

There is also a clear relationship between Hausdorff dimension and box-counting dimension.

Proposition 2.1.2

For every non-empty bounded $F \subset \mathbb{R}^n$,

$$\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F.$$

Note that so far, bi-Lipschitz mappings preserve all our notions of dimension. So similar to homeomorphisms, one can regard two sets as equivalent if there is a bi-Lipschitz mapping between them. This allows us to begin distinguishing topological properties from dimension.

Proposition 2.1.3

Every set $F \subset \mathbb{R}^n$ with $\dim_H F < 1$ is totally disconnected.

Proof. Let x and y be distinct points of F. We define a mapping $f(z) = |z - x|$. The reverse triangle inequality gives us

$$|f(z) - f(w)| \leq |z - w|,$$

so that f is Lipschitz and so $\dim_H f(F) < 1$. This implies that $f(F)$ is a subset of $\mathbb{R}$ of $\mathcal{H}^1$-measure zero– which implies it has a dense complement. Choosing r with $r \notin f(F)$ and $0 < r < f(y)$, it follows that

$$F = \{z \in F \mid |z - x| < r\} \cup \{z \in F \mid |z - x| > r\}.$$

That is, F is contained in two disjoint open sets with x in one set and y in the other– and so x, y lie in different connected components of F. ∎

Example 2.1.1: Middle third Cantor set

The Cantor set F splits into a left part $F_L = F \cap \left[0, \frac{1}{3}\right]$ and a right part $F_R = F \cap \left[\frac{2}{3}, 1\right]$. Both parts are geometrically similar to F but simply scaled by a ratio of $\frac{1}{3}$. Furthermore, $F = F_L \sqcup F_R$. Thus, for any s,

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_L) + \mathcal{H}^s(F_R) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(F) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(F)$$

by the scaling property of Hausdorff measures. Assuming that at the critical value $s = \dim_H F$, we have that the Hausdorff measure is finite (a nontrivial assumption), then we can divide both sides by $\mathcal{H}^s(F)$ to get $1 = 2\left(\frac{1}{3}\right)^s$ which then gives $s = \log 2/\log 3$.

A more rigorous approach can be shown to calculate this value, but this heuristic is particularly useful for self-similar sets.

2.2 Equivalent definitions of Hausdorff dimension

It is useful to have equivalent definitions, as some definitions will be easier to compute for certain classes of sets. One simple variation is done via covering by spherical balls: let

$$\mathcal{B}^s_\delta(F) = \inf \left\{ \sum_i |B_i|^s \mid \{B_i\} \text{ is a } \delta\text{-cover of } F \text{ by balls} \right\}$$

and consider the measure $\mathcal{B}^s(F) = \lim_{\delta\to 0} \mathcal{B}^s_\delta(F)$. Once again, we obtain a dimension when $\mathcal{B}^s(F)$ jumps from ∞ to zero. One can verify that this bounds the Hausdorff measure on both sides by a constant, and so the value of s where the jumps occur must be the same.

Of course, we can further restrict by using covers by only open sets, or closed sets. If F is compact, we can even consider finite subcovers of open covers.

One important variant is the net measure. For now, consider the cases when F is a subset of $[0, 1]$. Recall that a binary interval is an interval of the form $[r_2^{-k}, (r+1)2^{-k}]$ where $k = 0, 1, \dots$ and $r = 0, 1, \dots, 2^k - 1$. Then

$$\mathcal{M}^s_\delta(F) = \inf \left\{ \sum |U_i|^s \mid \{U_i\} \text{ is a } \delta\text{-cover of } F \text{ by binary intervals} \right\}$$

which leads to the net measures

$$\mathcal{M}^s(F) = \lim_{\delta\to 0} \mathcal{M}^s_\delta(F).$$

This form can be more convenient, as two binary intervals are either disjoint or contained in one another, allowing any cover of a set by binary intervals to become a cover by disjoint binary intervals.n

2.3 Packing Measure

For $s \geq$ and $\delta > 0$, we define

$$\mathcal{P}^s_\delta(F) = \sup \left\{ \sum_{i=1}^\infty |B_i|^s \mid \{B_i\} \text{ is a collection of disjoint balls of radii at most } \delta \text{ with centers in } F \right\}.$$

The limit

$$\mathcal{P}^s_0(F) = \lim_{\delta\to 0} \mathcal{P}^s_\delta(F)$$

exists. However, it is not a measure– so we modify the definition by decomposing F into a countable collection of sets and define

$$\mathcal{P}^s(F) = \inf \left\{ \sum_{i=1}^\infty \mathcal{P}^s_0(F_i) \mid F \subset \bigcup_{i=1}^\infty F_i \right\}$$

This is now a measure on $\mathbb{R}^n$ known as the s-dimensional packing measure. Similarly, the packing dimension is the jump value given by

$$\dim_P F = \sup \{s \geq 0 \mid \mathcal{P}^s(F) = \infty\} = \inf \{s \mid \mathcal{P}^s(F) = 0\}.$$

In fact, this definition is the same as the modified upper box dimension.

Lemma 2.3.1

For F a non-empty bounded subset of $\mathbb{R}^n$,

$$\dim_P F \leq \overline{\dim}_B F.$$

And hence

Proposition 2.3.1

If $F \subset \mathbb{R}^n$, then $\dim_P F = \overline{\dim}_{MB} F$.

This gives us the relations

$$\dim_H F \leq \underline{\dim}_{MB} F \leq \overline{\dim}_{MB} F = \dim_P F \leq \overline{\dim}_B F$$

This connection greatly opens the options in computing the geometry of fractals, however it is difficult to calculate. The following corollary strengthens the connection between the modified box dimension and the packing dimension.

Corollary 2.3.1

Let $F \subset \mathbb{R}^n$ be compact and assume that

$$\overline{\dim}_B(F \cap V) = \overline{\dim}_B F$$

for all open sets V that intersect F. Then $\dim_P F = \overline{\dim}_B F$.

If $F \subset \mathbb{R}^n$ is of second category, then $\dim_P F = n$. That is, this holds if F is or contains a dense G_δ set.

Chapter 3

Calculating Dimensions

3.1 Basic Methods

Typically, we upper-bound Hausdorff measure/dimension by finding effective coverings by small sets. We obtain lower bounds by putting measures on the set.

Proposition 3.1.1

Suppose F can be covered by n_k sets of diameter at most δ_k with $\delta_k \to 0$ as $k \to \infty$. Then

$$\dim_H F \leq \underline{\dim}_B F \leq \underline{\lim}_{k\to\infty} \frac{\log n_k}{-\log \delta_k}$$

Moreover, if $n_k \delta_k^s$ remains bounded as $k \to \infty$, then $\mathcal{H}^s(F) < \infty$. If $\delta_k \to \infty$ but $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, then

$$\overline{dim}_B F \leq \overline{\lim_{k\to\infty}} \frac{\log n_k}{-\log \delta_k}.$$

Typically, the upper bound for Hausdorff dimension is the actual value. We can obtain that by evaluating sums of coverings of sets. However, there are many such covers, and so while upper bounds might be easily boundable, lower bounds are difficult to obtain. As a result, we instead show that no individual set U covers the whole of F. We do this by mass distribution– recall that a **mass distribution** on F is a measure with support contained in F such that $0 < \mu(F) < \infty$.

Theorem 3.1.1: Mass distribution principle

Let μ be a mass distribution on F and suppose that for some $s > 0$, there are numbers $c > 0$ and $\varepsilon > 0$ such that

$$\mu(U) \geq c\,|U|^s$$

for all sets U with $|U| \leq \varepsilon$. Then $\mathcal{H}^s(F) \geq \mu(F)/c$ and

$$s \leq \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F.$$

Lemma 3.1.1: Vitali Covering Lemma

Let C be a family of balls contained in some bounded region of $\mathbb{R}^n$. Then there is a (finite or countable) disjoint subcollection $\{B_i\}$ such that

$$\bigcup_{B\in\mathcal{C}} B \subset \bigcup_i \tilde{B}_i$$

where $\tilde{B}_i$ is the closed ball concentric with B_i and of four times the radius.

Proposition 3.1.2

Let μ be a mass distribution on $\mathbb{R}^n$, let $F \subset \mathbb{R}^n$ be a Borel set and let $0 < c < \infty$ be a constant.

1. If $\overline{\lim}_{r\to 0}\mu(B(x,r))/r^s < c$ for all $x \in F$, then $\mathcal{H}^{\int}(\mathcal{F}) \geq \preceq(\mathcal{F})/\rfloor$
2. If $\overline{\lim}_{r\to 0}\mu(B(x,r))/r^s > c$ for all $x \in F$, then $\mathcal{H}^s(F) \leq 2^s\mu(\mathbb{R}^n)/c$.

We will briefly discuss subsets of finite measure. It is important as sets with infinite measure can be unwieldly, so reducing them to sets of positive finite measure can be helpful.

Theorem 3.1.2

Let F be a Borel subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(F) \leq \infty$. Then there is a compact set $E \subset F$ such that $0 < \mathcal{H}^s(E) < \infty$.

Proving this is very difficult. The sketch of the proof focuses on the case with F a compact subset of $[0, 1)$. In general, one applies the net measures to an inductively defined decreasing sequence $E_0 \supset E_1 \supset \ldots$ of compact subsets of F. By defining this inductively sequence carefully, we can ensure that the net measure is finite and continuous on a limiting sequence that converges to E.

Many results apply only to s-sets, and so one way to approach s-dimensional sets with $\mathcal{H}^s(F) = \infty$ is to use the above theorem to extract a subset of positive finite measure, study the subset, then interpret the larger set F based on these properties.

Proposition 3.1.3

Let F be a Borel set satisfying $0 < \mathcal{H}^s(F) < \infty$. There is a constant b and a compact set $E \subset F$ with $\mathcal{H}^s(E) > 0$ such that

$$\mathcal{H}^s(E \cap B(x,r)) \leq br^s$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

Corollary 3.1.1: Frostman's Lemma

Let F be a Borel subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(F) \leq \infty$. Then there is a compact set $E \subset F$ such that $0 < \mathcal{H}^s(E) < \infty$ and a constant b such that

$$\mathcal{H}^s(E \cap B(x,r)) \leq br^s$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

This is often regarded as aa converse of the mass distribution principle.

3.2 Potential Theoretic Methods

This is one of the most widely used techniques currently. The idea is to use potentials and energy so that integration can yield results on dimension.

Recall that for $s \geq 0$, the s-**potential** at a point x of $\mathbb{R}^n$ resulting from the mass distribution μ on $\mathbb{R}^n$ is defined as

$$\varphi_S(x) = \int \frac{d\mu(y)}{|x-y|^s}.$$

Similarly, the s-**energy** of μ is

$$I_s(\mu) = \int \varphi_s(x)\, d\mu(x) = \int \frac{d\mu(x)\, d\mu(y)}{|x-y|^s}.$$

Theorem 3.2.1

Let F be a subset of $\mathbb{R}^n$.

1. If there is a mass distribution μ on F with $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and $\dim_H F \geq s$.
2. If F is a Borel set with $0 < \mathcal{H}^s(F) \leq \infty$, then for all $0 < t < s$, there exists a mass distribution μ on F with $I_t(\mu) < \infty$.

Chapter 4

Local structure of fractals

In order to analyze the local properties of fractals, we restrict to s-sets, which are Borel sets of Hausdorff dimension s with positive finite $s-$ dimensional Hausdorff measure. This is a first introduction to problems in geometric measure theory.

4.1 Densities

Definition 4.1.1: Density

Let F be a subset of the plane. The **density** of F at x is

$$\lim_{r\to 0} \frac{\mathrm{area}(F\cap \overline{B}(x,r))}{\pi r^2}$$

where the denominator is simply the area of $\overline{B}(x,r)$.

The Lebesgue density theorem tells us that for F Borel, the value is either 0 or 1 depending on whether x is in F.

Now we consider s-dimensional Hausdorff measure on these sets for F with dimension s.

Definition 4.1.2: Upper and lower densities

Let $x\in\mathbb{R}^n$ and F be an s-set. The **lower** and **upper density** is given by

$$\underline{D}^s(F,x) = \underline{\lim}_{r\to 0}\frac{\mathcal{H}^s(F\cap B(x,r))}{(2r)^s}$$
$$\overline{D}^s(F,x) = \overline{\lim}_{r\to 0}\frac{\mathcal{H}^s(F\cap B(x,r))}{(2r)^s}.$$

If they both agree, then the density of F at x exists and is that value.

Definition 4.1.3: Regular points

If $\underline{D}^s(F,x) = \overline{D}^s(F,x) = 1$, then x is a **regular** point of F, otherwise it is an **irregular** point.

An s-set is called **regular** if, except on a set of $\mathcal{H}^s$-measure, all of its points are regular. If instead all of its points (except on a set of $\mathcal{H}^s$) are irregular, then the s-set is **irregular**.

Note that unlike points, an s-set can be not regular, but not irregular.

Despite expectations, the densities of irregular sets do have some conditions.

Proposition 4.1.1

Let F be an s-set in $\mathbb{R}^n$. Then (a). $\underline{D}^s(F,x) = \overline{D}^s(F,x) = 0$ for $\mathcal{H}^s$-almost all $x \notin F$

(b). $2^{-s} \leq \overline{D}^s(F,x) \leq 1$ for $\mathcal{H}^s$-almost all $x \in F$.

Of course, it follows from (b) that the lower density of irregular sets is strictly less than 1 almost everywhere.

It can be shown that if $E \subset F$ is a Borel subset, then E is regular if F is regular and E is irregular if F is irregular. This also gives us that the intersection of a regular and irregular set must have $\mathcal{H}^s$-measure zero.

Theorem 4.1.1

Let F be an s-set in $\mathbb{R}^2$. Then F is irregular unless s is an integer.

Proof. We will only show the case for when $0 < s < 1$, as the other cases are much harder. We will do this by showing that the density $D^s(F,x)$ fails to exist almost everywhere in F. Suppose for the sake of contradiction that there exists $F_1 \subset F$ of positive measure where the density exists and so

$$\frac{1}{2} < 2^{-s} \leq D^s(F,x).$$

By Egoroff's theorem, we may find $r_0 > 0$ and a Borel set $E \subset F_1 \subset F$ with $\mathcal{H}^s(E) > 0$ such that

$$\mathcal{H}^s(F \cap B(x,r)) > \frac{1}{2}(2r)^s$$

for all $x \in E$ and $r < r_0$. Let $y \in E$ be a cluster point of E, and η be a number with $0 < \eta < 1$, and let $A_{r,\eta}$ be the annulus given by $B(y, r(1+\eta)) \setminus B(y, r(1-\eta))$ as can be seen in Figure 4.1. Then

$$\begin{aligned}(2r)^{-s}\mathcal{H}^s(F \cap A_{r,\eta}) &= (2r)^{-s}\mathcal{H}^s(F \cap B(y, r(1+\eta))) - (2r)^{-s}\mathcal{H}^s(F \cap B(y, r(1-\eta))) \\ &\to D^s(F,y)((1+\eta)^s - (1-\eta)^s)\end{aligned}$$

as $r \to 0$. For each term in a sequence of values of r tending to 0, we can find some $x \in E$ with $|x - y| = r$. This tells us that $B(x, r\eta/2) \subset A_{r,\eta}$ and hence

$$\begin{aligned}\frac{1}{2}r^s\eta^s &< \mathcal{H}^s(F \cap B(x, r\eta/2)) \leq \mathcal{H}^s(F \cap A_{r,\eta}) \implies \\ 2^{-s-1}\eta^s &\leq D^s(F,y)((1+\eta)^s - (1-\eta)^s) = D^s(F,y)(2s\eta + \text{higher order terms})\end{aligned}$$

As $\eta \to 0$, this cannot hold for $0 < s < 1$, and so we have a contradiction. ■

4.2 Structure of 1-sets

We cannot generalize integral dimension s-sets as easily, but fortunately we can sometimes obtain decomposition theorems that can allow us to analyze s-sets.

Theorem 4.2.1: Decomposition Theorem

Let F be a 1-set. The set of regular points of F form a regular set, and the set of irregular points forms an irregular set.

Recall the definition of curves:

Definition 4.2.1: Jordan curve

A **Jordan curve** C is the image of a continuous injection $\psi : [a,b] \to \mathbb{R}^2$, where $[a,b] \subset \mathbb{R}$ is a proper closed interval.

By this definition, curves are not self-intersecting, have two ends, and are compact connected subsets of the plane. The length $\mathcal{L}(C)$ of the curve C is given by the approximation

$$\mathcal{L}(C) = \sup \sum_{i=1}^{m} |x_i - x_{i-1}|$$

where the supremum is taken over all partitions. If $\mathcal{L}(C)$ is positive and finite, we call C a **rectifiable curve**. Of course, the length of a curve equals its 1-dimensional Hausdorff measure.

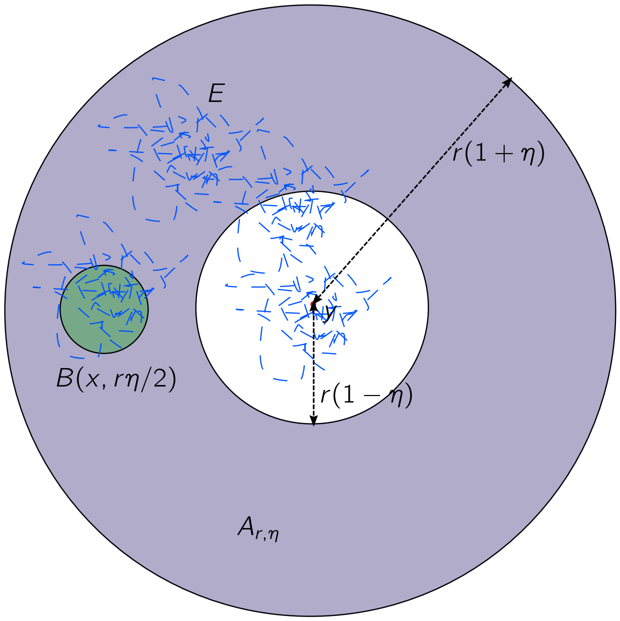

Figure 4.1: Annulus

Lemma 4.2.1

If C is a rectifiable curve, then $\mathcal{H}^1(C) = \mathcal{L}(C)$.

Rectifiable curves act nicely in the plane.

Lemma 4.2.2

A rectifiable curve is a regular 1-set.

Of course, this also tells us that curve-like structures are also regular. That is,

Proposition 4.2.1

A 1-set contained in a countable union of rectifiable curves is a regular 1-set.

We can also say that a 1-set is curve-free if its intersection with every rectifiable curve has $\mathcal{H}^1$-measure zero,

Proposition 4.2.2

An irregular 1-set is curve-free.

Proposition 4.2.3

Let F be a curve-free 1-set in $\mathbb{R}^2$. Then $\underline{D}^1(F, x) \leq \frac{3}{4}$ at almost all $x \in F$.

Theorem 4.2.2

(a). A 1-set in $\mathbb{R}^2$ is irregular if and only if it is curve-free.

(b). A 1-set in $\mathbb{R}^2$ is regular if and only if it is the union of a curve-like set and a set of $\mathcal{H}^1$-measure zero.

These are remarkable as they classify densities of sets by curves. In fact, this even told us that in any 1-set F, the set of points for which $\frac{3}{4} < \underline{D}^1(F, x) < 1$ has $\mathcal{H}^1$-measure zero.

Some other nice properties we have are total disconnectedness of irregular 1-sets.

4.3 Tangents to s-sets

At first, the concepts of tangents may seem unrelated to our discussion on dimension and local volume. However, the topic is more related than one might expect– if a smooth curve C has a tangent at x, then when one is close to x, the set C is concentrated in two directions that are diametrically opposite. This is a notable property, and one we hope to extend to more generalized s-sets.

Of course, we have to focus locally on sets of positive measure (i.e. almost all points).

Definition 4.3.1: Tangent

An s-set F in $\mathbb{R}^n$ has a **tangent at** x **in direction** θ, where θ is a unit vector, if

$$\overline{D}^s(F, x) > 0$$

and for every angle $\varphi > 0$,

$$\lim_{r \to 0} r^{-s}\mathcal{H}^s(F \cap (B(x, r) \setminus S(x, \theta, \varphi))) = 0$$

where $S(x, \theta, \varphi)$ is the double sector with vector x, consisting of those y such that the line segment $[x, y]$ makes an angle at most φ with θ or $-\theta$.

In other words, a tangent in direction θ requires that (a). a significant part of F lies near x, and (b) a negligible amount close to x lies outside of any double sector near θ.

First we discuss 1-sets for posterity.

Proposition 4.3.1

A rectifiable curve C has a tangent at almost all of its points.

We already know by a previous lemma that the upper density is 1 for almost all $x \in C$. The rest of the proof follows by the fact that the change in length of the curve is a well-defined function that exists as a vector almost everywhere. Of course, by arc length reparametrization, its magnitude is always one. This derivative then is precisely the unit vector θ, and we can constrain via epsilon-delta techniques so that the length derivative is contained in S provided that the parametrization is within ε of the tangent point x. Thus we can force the set outside of the double sector to be empty, and hence it has a tangent at almost all x.

Proposition 4.3.2

A regular 1-set F in $\mathbb{R}^2$ has a tangent at almost all of its points.

This follows because regular 1-sets can be covered a.e. by a countable collection of rectifiable curves.

Proposition 4.3.3

At almost all points of an irregular 1-set, no tangent exists.

This proof depends on the characterisation of irreuglar sets as curve-free sets, which is very involved.

Proposition 4.3.4

If F is an s-set in $\mathbb{R}^2$ with $1 < s < 2$, then at almost all points of F no tangent exists.

These results start to illuminate a much larger picture. For example, it can be shjown that if $s > 1$, almost every line through $\mathcal{H}^s$-a.e. point of an s-set F intersects F in a set of dimension $s - 1$.