Hashing: Hash Functions

Michael Levin

Department of Computer Science and Engineering
University of California, San Diego

Data Structures Fundamentals
Algorithms and Data Structures

Outline

1. Phone Book Data Structure
2. Universal Family
3. Hashing Phone Numbers
4. Hashing Names
5. Analysis of Polynomial Hashing

Phone Book

Design a data structure to store your contacts: names of people along with their phone numbers. The following operations should be fast:

- Add and delete contacts,
- Call person by name,
- Determine who is calling given their phone number.

- We need two Maps:
  (phone number $\rightarrow$ name) and
  (name $\rightarrow$ phone number)

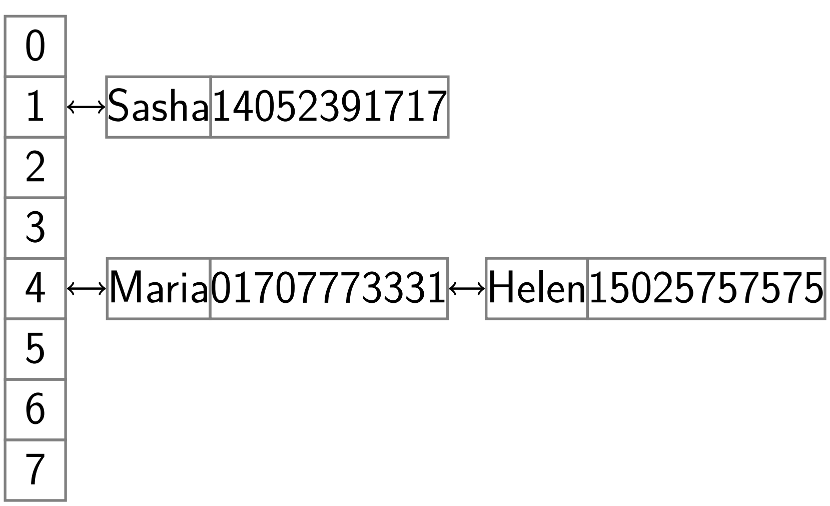
- We need two Maps:
  (phone number $\rightarrow$ name) and
  (name $\rightarrow$ phone number)
- Implement these Maps as hash tables

- We need two Maps:
  (phone number $\rightarrow$ name) and
  (name $\rightarrow$ phone number)
- Implement these Maps as hash tables
- First, we will focus on the Map from phone numbers to names

Chaining for Phone Book

- Select hash function h of cardinality m
- Create array `Chains` of size m
- Each element of `Chains` is a list of pairs $(\texttt{name}, \texttt{phoneNumber})$, called *chain*
- Pair $(\texttt{name}, \texttt{phoneNumber})$ goes into chain at position $h(\texttt{ConvertToInt}(\texttt{phoneNumber}))$ in the array `Chains`

Chaining for Phone Book



Parameters

- n contacts stored

Parameters

- n contacts stored
- m — cardinality of the hash function

Parameters

- n contacts stored
- m — cardinality of the hash function
- Note: hash table size for chaining should be equal to m

Parameters

- n contacts stored
- m — cardinality of the hash function
- Note: hash table size for chaining should be equal to m
- c — length of the longest chain

Parameters

- n contacts stored
- m — cardinality of the hash function
- Note: hash table size for chaining should be equal to m
- c — length of the longest chain
- $\Theta(n + m)$ memory is used

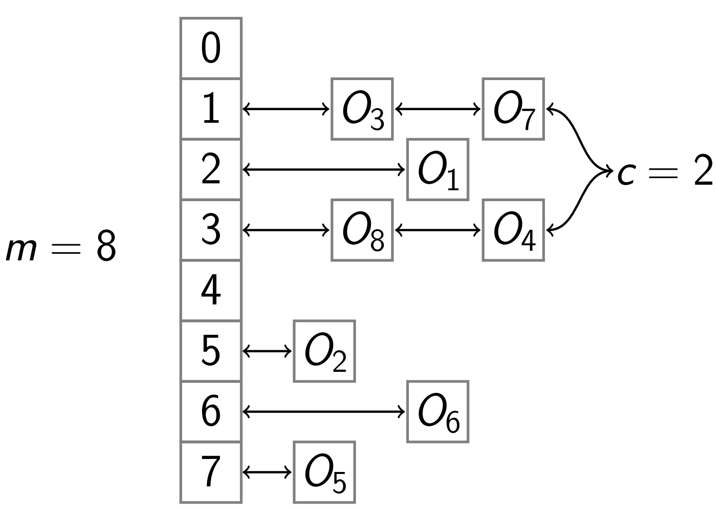
Parameters

- n contacts stored
- m — cardinality of the hash function
- Note: hash table size for chaining should be equal to m
- c — length of the longest chain
- $\Theta(n+m)$ memory is used
- Operations run in time $\Theta(c+1)$

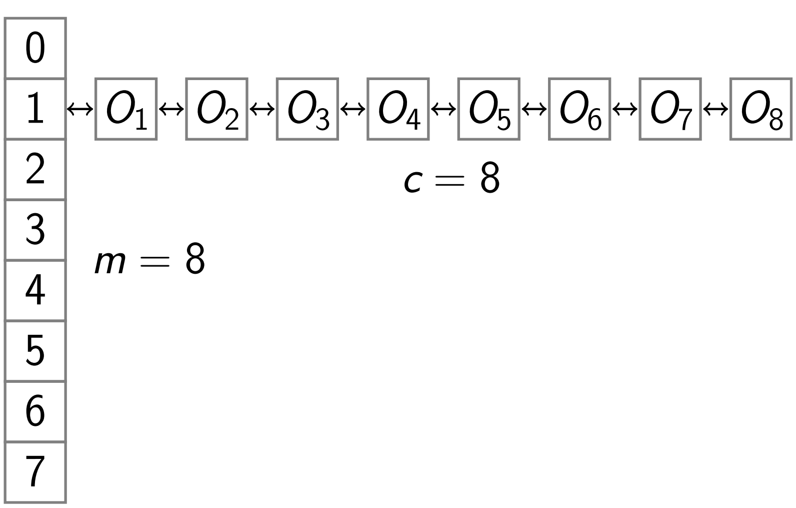
Parameters

- n contacts stored
- m — cardinality of the hash function
- Note: hash table size for chaining should be equal to m
- c — length of the longest chain
- $\Theta(n + m)$ memory is used
- Operations run in time $\Theta(c + 1)$
- You want small m and c! (but $c \geq \frac{n}{m}$)

Good Example



Bad Example



First Digits

- For the map from phone numbers to names, select $m = 1000$

First Digits

- For the map from phone numbers to names, select $m = 1000$
- Hash function: take first three digits

First Digits

- For the map from phone numbers to names, select $m = 1000$
- Hash function: take first three digits
- $h(\texttt{800-123-45-67}) = 800$

First Digits

- For the map from phone numbers to names, select $m = 1000$
- Hash function: take first three digits
- $h(\texttt{800-123-45-67}) = 800$
- Problem: area code

First Digits

- For the map from phone numbers to names, select $m = 1000$
- Hash function: take first three digits
- $h(\texttt{800-123-45-67}) = 800$
- Problem: area code
- $h(\texttt{425-234-55-67}) =$
  $h(\texttt{425-123-45-67}) =$
  $h(\texttt{425-223-23-23}) = \cdots = 425$

Last Digits

- Select $m = 1000$

Last Digits

- Select $m = 1000$
- Hash function: take last three digits

Last Digits

- Select $m = 1000$
- Hash function: take last three digits
- $h(\texttt{800-123-45-67}) = 567$

Last Digits

- Select $m = 1000$
- Hash function: take last three digits
- $h(\texttt{800-123-45-67}) = 567$
- Problem if many phone numbers end with three zeros

Random Value

- Select $m = 1000$

Random Value

- Select $m = 1000$
- Hash function: random number between 0 and 999

Random Value

- Select $m = 1000$
- Hash function: random number between 0 and 999
- Uniform distribution of hash values

Random Value

- Select $m = 1000$
- Hash function: random number between 0 and 999
- Uniform distribution of hash values
- Different value when hash function called again — we won't be able to find anything!

Random Value

- Select $m = 1000$
- Hash function: random number between 0 and 999
- Uniform distribution of hash values
- Different value when hash function called again — we won't be able to find anything!
- Hash function must be deterministic

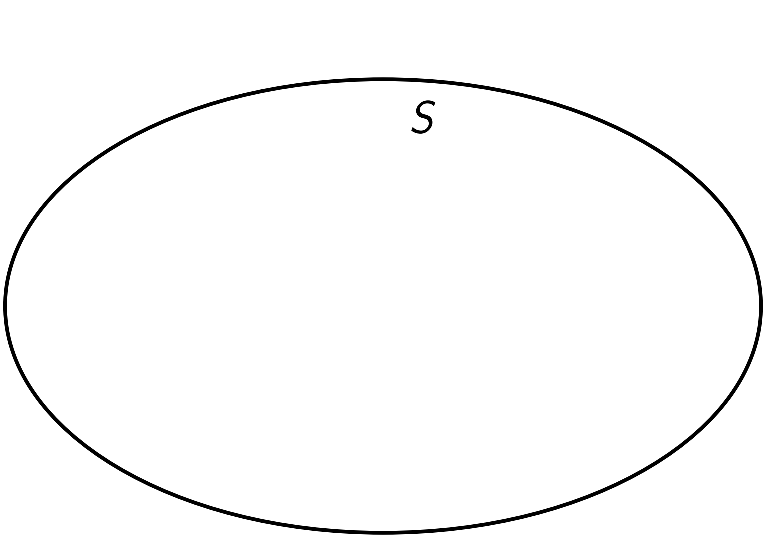
Good Hash Functions

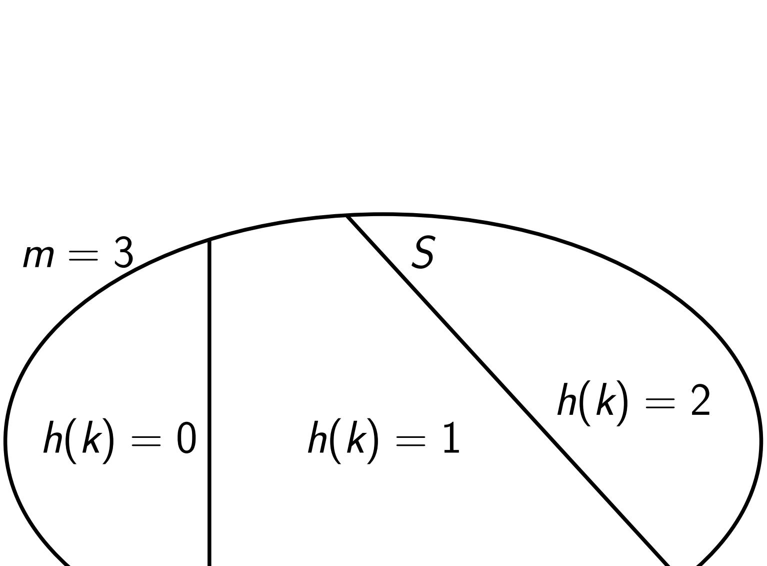
- Deterministic
- Fast to compute
- Distributes keys well into different cells
- Few collisions

No Universal Hash Function

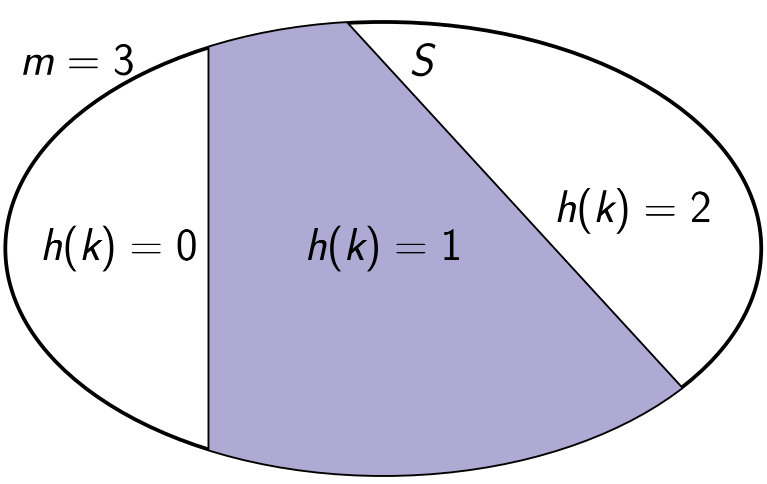
Lemma

If the number of possible keys is big ($|S| \gg m$), for any hash function h there is a bad input resulting in many collisions.

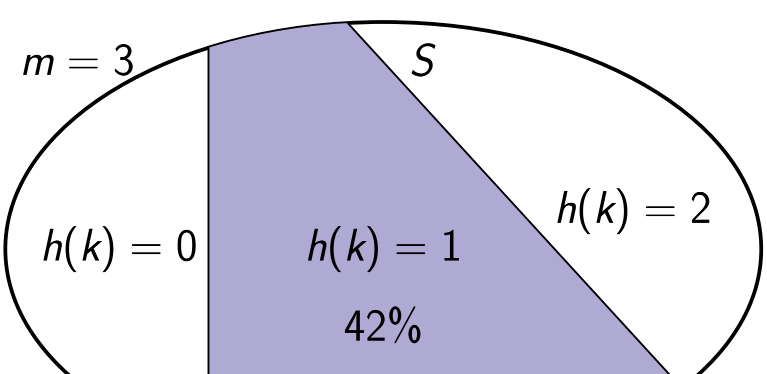

S



$m = 3$
S
$h(k) = 0$
$h(k) = 1$
$h(k) = 2$



$m = 3$
S
$h(k) = 0$
$h(k) = 1$
$h(k) = 2$



$m = 3$
S
$h(k) = 0$
$h(k) = 1$
42%
$h(k) = 2$

Outline

1. Phone Book Data Structure
2. Universal Family
3. Hashing Phone Numbers
4. Hashing Names
5. Analysis of Polynomial Hashing

Idea

- Remember QuickSort?

Idea

- Remember QuickSort?
- Choosing random pivot helped

Idea

- Remember QuickSort?
- Choosing random pivot helped
- Use randomization!

Idea

- Remember QuickSort?
- Choosing random pivot helped
- Use randomization!
- Define a family (set) of hash functions

Idea

- Remember QuickSort?
- Choosing random pivot helped
- Use randomization!
- Define a family (set) of hash functions
- Choose random function from the family

Universal Family

Definition

Let U be the universe — the set of all possible keys.

Universal Family

Definition

Let U be the universe — the set of all possible keys. A set of hash functions

$$\mathcal{H} = \{h : U \to \{0, 1, 2, \ldots, m-1\}\}$$

Universal Family

Definition

Let U be the universe — the set of all possible keys. A set of hash functions

$$\mathcal{H} = \{h : U \to \{0, 1, 2, \ldots, m - 1\}\}$$

is called a universal family if

Universal Family

Definition

Let U be the universe — the set of all possible keys. A set of hash functions

$$\mathcal{H} = \{h : U \to \{0, 1, 2, \ldots, m-1\}\}$$

is called a universal family if for any two keys $x, y \in U, x \neq y$ the probability of collision

$$\Pr[h(x) = h(y)] \leq \frac{1}{m}$$

Universal Family

$$\Pr[h(x) = h(y)] \leq \frac{1}{m}$$

means that a collision $h(x) = h(y)$ for any fixed pair of different keys x and y happens for no more than $\frac{1}{m}$ of all hash functions $h \in \mathcal{H}$.

How Randomization Works

- $h(x) = \texttt{random}(\{0, 1, 2, \ldots, m-1\})$
  gives probability of collision exactly $\frac{1}{m}$.

How Randomization Works

- $h(x) = \texttt{random}(\{0, 1, 2, \ldots, m-1\})$
  gives probability of collision exactly $\frac{1}{m}$.
- It is not deterministic — can't use it.

How Randomization Works

- $h(x) = \texttt{random}(\{0, 1, 2, \ldots, m-1\})$ gives probability of collision exactly $\frac{1}{m}$.
- It is not deterministic — can't use it.
- All hash functions in $\mathcal{H}$ are deterministic

How Randomization Works

- $h(x) = \texttt{random}(\{0, 1, 2, \ldots, m-1\})$ gives probability of collision exactly $\frac{1}{m}$.
- It is not deterministic — can't use it.
- All hash functions in $\mathcal{H}$ are deterministic
- Select a random function h from $\mathcal{H}$

How Randomization Works

- $h(x) = \texttt{random}(\{0, 1, 2, \ldots, m-1\})$ gives probability of collision exactly $\frac{1}{m}$.
- It is not deterministic — can't use it.
- All hash functions in $\mathcal{H}$ are deterministic
- Select a random function h from $\mathcal{H}$
- Fixed h is used throughout the algorithm

Load Factor

Definition

The ratio $\alpha = \frac{n}{m}$ between number of objects n stored in the hash table and the size of the hash table m is called load factor.

Running Time

Lemma

If h is chosen randomly from a universal family, the average length of the longest chain c is $O(1 + \alpha)$, where $\alpha = \frac{n}{m}$ is the load factor of the hash table.

Corollary

If h is from universal family, operations with hash table run on average in time $O(1 + \alpha)$.

Choosing Hash Table Size

- Control amount of memory used with m

Choosing Hash Table Size

- Control amount of memory used with m
- Ideally, load factor $0.5 < \alpha < 1$

Choosing Hash Table Size

- Control amount of memory used with m
- Ideally, load factor $0.5 < \alpha < 1$
- Use $O(m) = O(\frac{n}{\alpha}) = O(n)$ memory to store n keys

Choosing Hash Table Size

- Control amount of memory used with m
- Ideally, load factor $0.5 < \alpha < 1$
- Use $O(m) = O(\frac{n}{\alpha}) = O(n)$ memory to store n keys
- Operations run in time $O(1 + \alpha) = O(1)$ on average

Dynamic Hash Tables

- What if number of keys n is unknown in advance?

Dynamic Hash Tables

- What if number of keys n is unknown in advance?
- Start with very big hash table?

Dynamic Hash Tables

- What if number of keys n is unknown in advance?
- Start with very big hash table?
- You will waste a lot of memory

Dynamic Hash Tables

- What if number of keys n is unknown in advance?
- Start with very big hash table?
- You will waste a lot of memory
- Copy the idea of dynamic arrays!

Dynamic Hash Tables

- What if number of keys n is unknown in advance?
- Start with very big hash table?
- You will waste a lot of memory
- Copy the idea of dynamic arrays!
- Resize the hash table when α becomes too large

Dynamic Hash Tables

- What if number of keys n is unknown in advance?
- Start with very big hash table?
- You will waste a lot of memory
- Copy the idea of dynamic arrays!
- Resize the hash table when α becomes too large
- Choose new hash function and rehash all the objects

Rehashing

Keep load factor below 0.9:

`Rehash(`T`)`

$loadFactor \leftarrow \frac{T.\texttt{numberOfKeys}}{T.\texttt{size}}$
`if` $loadFactor > 0.9$:
  `Create` T_{new} `of size` $2 \times T.\texttt{size}$
  `Choose` h_{new} `with cardinality` $T_{new}.\texttt{size}$
  `For each object in` T:
    `Insert object in` T_{new} `using` h_{new}
  $T \leftarrow T_{new}, h \leftarrow h_{new}$

Rehashing

Keep load factor below 0.9:

`Rehash(`T`)`

$loadFactor \leftarrow \frac{T.\mathtt{numberOfKeys}}{T.\mathtt{size}}$

```
if loadFactor > 0.9:
  Create T_new of size 2 × T.size
  Choose h_new with cardinality T_new.size
  For each object in T:
    Insert object in T_new using h_new
  T ← T_new, h ← h_new
```

Rehashing

Keep load factor below 0.9:

Rehash(T)

$loadFactor \leftarrow \frac{T.\texttt{numberOfKeys}}{T.\texttt{size}}$
`if` $loadFactor > 0.9$:
  `Create` T_{new} `of size` $2 \times T.\texttt{size}$
  `Choose` h_{new} `with cardinality` $T_{new}.\texttt{size}$
  `For each object in` T:
    `Insert object in` T_{new} `using` h_{new}
  $T \leftarrow T_{new}, h \leftarrow h_{new}$

Rehashing

Keep load factor below 0.9:

`Rehash(`T`)`

$loadFactor \leftarrow \frac{T.\texttt{numberOfKeys}}{T.\texttt{size}}$
`if` $loadFactor > 0.9$:
  `Create` T_{new} `of size` $2 \times T.\texttt{size}$
  `Choose` h_{new} `with cardinality` $T_{new}.\texttt{size}$
  `For each object in` T:
    `Insert object in` T_{new} `using` h_{new}
  $T \leftarrow T_{new}, h \leftarrow h_{new}$

Rehashing

Keep load factor below 0.9:

Rehash(T)

$loadFactor \leftarrow \frac{T.\texttt{numberOfKeys}}{T.\texttt{size}}$

```
if loadFactor > 0.9:
  Create T_new of size 2 × T.size
  Choose h_new with cardinality T_new.size
  For each object in T:
    Insert object in T_new using h_new
  T ← T_new, h ← h_new
```

Rehashing

Keep load factor below 0.9:

Rehash(T)

$loadFactor \leftarrow \frac{T.\texttt{numberOfKeys}}{T.\texttt{size}}$
if $loadFactor > 0.9$:
  Create T_{new} of size $2 \times T.\text{size}$
  Choose h_{new} with cardinality $T_{new}.\text{size}$
  For each object in T:
    Insert object in T_{new} using h_{new}
  $T \leftarrow T_{new}, h \leftarrow h_{new}$

Rehashing

Keep load factor below 0.9:

Rehash(T)

$loadFactor \leftarrow \frac{T.\texttt{numberOfKeys}}{T.\texttt{size}}$
if $loadFactor > 0.9$:
  Create T_{new} of size $2 \times T.\texttt{size}$
  Choose h_{new} with cardinality $T_{new}.\texttt{size}$
  For each object in T:
    Insert object in T_{new} using h_{new}
  $T \leftarrow T_{new}, h \leftarrow h_{new}$

Rehashing

Keep load factor below 0.9:

Rehash(T)

$loadFactor \leftarrow \frac{T.\texttt{numberOfKeys}}{T.\texttt{size}}$
`if` $loadFactor > 0.9$:
  `Create` T_{new} `of size` $2 \times T.\texttt{size}$
  `Choose` h_{new} `with cardinality` $T_{new}.\texttt{size}$
  `For each object in` T:
    `Insert object in` T_{new} `using` h_{new}
  $T \leftarrow T_{new}, h \leftarrow h_{new}$

Rehash Running Time

You should call `Rehash` after each operation with the hash table

Similarly to dynamic arrays, single rehashing takes $O(n)$ time, but amortized running time of each operation with hash table is still $O(1)$ on average, because rehashing will be rare

Outline

Hashing Phone Numbers

- We can convert phone numbers to integers, for example 148-25-67 $\to$ 1482567

Hashing Phone Numbers

- We can convert phone numbers to integers, for example 148-25-67 $\to$ 1482567
- As a result, any phone number will be converted to an integer less than 10^{15}

Hashing Phone Numbers

- We can convert phone numbers to integers, for example 148-25-67 $\to$ 1482567
- As a result, any phone number will be converted to an integer less than 10^{15}
- If we come up with a universal family for integers up to 10^{15}, we will be able to map phone numbers to names efficiently using chaining

Hashing Integers

Lemma

$\mathcal{H}_p = \left\{ h_p^{a,b}(x) = ((ax + b) \bmod p) \bmod m \right\}$ for all $a, b : 1 \leq a \leq p - 1, 0 \leq b \leq p - 1$ is a universal family for the set of integers between 0 and $p - 1$, for any prime p.

Hashing Phone Numbers

Example

Select $a = 34$, $b = 2$, so $h = h_p^{34,2}$ and consider $x = 1\ 482\ 567$ corresponding to phone number 148-25-67. $p = 10\ 000\ 019$.

Hashing Phone Numbers

Example

Select $a = 34$, $b = 2$, so $h = h_p^{34,2}$ and consider $x = 1\ 482\ 567$ corresponding to phone number 148-25-67. $p = 10\ 000\ 019$.

$(34 \times 1482567 + 2) \bmod 10000019 = 407185$

Hashing Phone Numbers

Example

Select $a = 34$, $b = 2$, so $h = h_p^{34,2}$ and consider $x = 1\ 482\ 567$ corresponding to phone number 148-25-67. $p = 10\ 000\ 019$.

$(34 \times 1482567 + 2) \bmod 10000019 = 407185$

$407185 \bmod 1000 = 185$

Hashing Phone Numbers

Example

Select $a = 34$, $b = 2$, so $h = h_p^{34,2}$ and consider $x = 1\ 482\ 567$ corresponding to phone number 148-25-67. $p = 10\ 000\ 019$.

$(34 \times 1482567 + 2) \bmod 10000019 = 407185$

$407185 \bmod 1000 = 185$

$h(x) = 185$

Proof Ideas

- For any pair of different keys (x, y), any two of the $p(p-1)$ hash functions in $\mathcal{H}_p$ hash them into different pairs (r, s) of different remainders modulo p

Proof Ideas

- For any pair of different keys (x, y), any two of the $p(p-1)$ hash functions in $\mathcal{H}_p$ hash them into different pairs (r, s) of different remainders modulo p
- Thus any pair (r, s) of different remainders modulo p has equal probability $\frac{1}{p(p-1)}$

Proof Ideas

- For any pair of different keys (x, y), any two of the $p(p-1)$ hash functions in $\mathcal{H}_p$ hash them into different pairs (r, s) of different remainders modulo p
- Thus any pair (r, s) of different remainders modulo p has equal probability $\frac{1}{p(p-1)}$
- The ratio of pairs (r, s) of different remainders modulo p such that $r \equiv s \pmod{m}$ is less than $\frac{1}{m}$ $\square$

General Case

- Define maximum length L of a phone number

General Case

- Define maximum length L of a phone number
- Convert phone numbers to integers from 0 to $10^L - 1$

General Case

- Define maximum length L of a phone number
- Convert phone numbers to integers from 0 to $10^L - 1$
- Choose prime number $p > 10^L$

General Case

- Define maximum length L of a phone number
- Convert phone numbers to integers from 0 to $10^L - 1$
- Choose prime number $p > 10^L$
- Choose hash table size m

General Case

- Define maximum length L of a phone number
- Convert phone numbers to integers from 0 to $10^L - 1$
- Choose prime number $p > 10^L$
- Choose hash table size m
- Choose random hash function from universal family $\mathcal{H}_p$ (choose random $a \in [1, p-1]$ and $b \in [0, p-1]$)

Outline

Lookup Phone Numbers by Name

- Time to implement Map from names to phone numbers

Lookup Phone Numbers by Name

- Time to implement Map from names to phone numbers
- Use chaining again

Lookup Phone Numbers by Name

- Time to implement Map from names to phone numbers
- Use chaining again
- Need a hash function for names

Lookup Phone Numbers by Name

- Time to implement Map from names to phone numbers
- Use chaining again
- Need a hash function for names
- Hash arbitrary strings of bounded length

Lookup Phone Numbers by Name

- Time to implement Map from names to phone numbers
- Use chaining again
- Need a hash function for names
- Hash arbitrary strings of bounded length
- You will learn how string hashing is implemented in Java!

String Length Notation

Definition

Denote by $|S|$ the length of string S.

Examples

$|\text{“edx”}| = 3$
$|\text{“ucsd”}| = 4$
$|\text{“chaining”}| = 8$

Hashing Strings

- Given a string S, compute its hash value

Hashing Strings

- Given a string S, compute its hash value
- $S = S[0]S[1] \dots S[|S| - 1]$, where $S[i]$ are individual characters

Hashing Strings

- Given a string S, compute its hash value
- $S = S[0]S[1] \ldots S[|S| - 1]$, where $S[i]$ are individual characters
- We should use all the characters in the hash function

Hashing Strings

- Given a string S, compute its hash value
- $S = S[0]S[1] \dots S[|S| - 1]$, where $S[i]$ are individual characters
- We should use all the characters in the hash function
- Otherwise there will be many collisions

Hashing Strings

- Given a string S, compute its hash value
- $S = S[0]S[1]\dots S[|S|-1]$, where $S[i]$ are individual characters
- We should use all the characters in the hash function
- Otherwise there will be many collisions
- For example, if $S[0]$ is not used, then $h(\text{“aa”}) = h(\text{“ba”}) = \cdots = h(\text{“za”})$

Preparation

- Convert each character $S[i]$ to integer

Preparation

- Convert each character $S[i]$ to integer
- ASCII encoding, Unicode, etc.

Preparation

- Convert each character $S[i]$ to integer
- ASCII encoding, Unicode, etc.
- Choose big prime number p

Polynomial Hashing

Definition

Family of hash functions

$$\mathcal{P}_p = \left\{ h_p^x(S) = \sum_{i=0}^{|S|-1} S[i]x^i \bmod p \right\}$$

with a fixed prime p and all $1 \leq x \leq p-1$ is called polynomial.

PolyHash(S, p, x)

```
hash ← 0
for i from |S| − 1 down to 0:
   hash ← (hash · x + S[i]) mod p
return hash
```

```
PolyHash(S, p, x)

hash ← 0
for i from |S| − 1 down to 0:
   hash ← (hash · x + S[i]) mod p
return hash
```

PolyHash(S, p, x)

```
hash ← 0
for i from |S| − 1 down to 0:
    hash ← (hash · x + S[i]) mod p
return hash
```

```
PolyHash(S, p, x)

hash ← 0
for i from |S| − 1 down to 0:
   hash ← (hash · x + S[i]) mod p
return hash
```

```
PolyHash(S, p, x)

hash ← 0
for i from |S| − 1 down to 0:
   hash ← (hash · x + S[i]) mod p
return hash
```

`PolyHash`(S, p, x)

```
hash ← 0
for i from |S| − 1 down to 0:
   hash ← (hash · x + S[i]) mod p
return hash
```

Example: $|S| = 3$

$\texttt{PolyHash}(S, p, x)$

```
hash ← 0
for i from |S| − 1 down to 0:
    hash ← (hash · x + S[i]) mod p
return hash
```

Example: $|S| = 3$

1. $\texttt{hash} \leftarrow 0$

$\texttt{PolyHash}(S, p, x)$

```
hash ← 0
for i from |S| − 1 down to 0:
   hash ← (hash · x + S[i]) mod p
return hash
```

Example: $|S| = 3$

1. $\texttt{hash} \leftarrow 0$
2. $\texttt{hash} \leftarrow S[2] \bmod p$

`PolyHash`(S, p, x)

```
hash ← 0
for i from |S| − 1 down to 0:
    hash ← (hash · x + S[i]) mod p
return hash
```

Example: $|S| = 3$

1. $\texttt{hash} \leftarrow 0$
2. $\texttt{hash} \leftarrow S[2] \bmod p$
3. $\texttt{hash} \leftarrow S[1] + S[2]x \bmod p$

PolyHash(S, p, x)

```
hash ← 0
for i from |S| − 1 down to 0:
    hash ← (hash · x + S[i]) mod p
return hash
```

Example: $|S| = 3$

1. hash $\leftarrow 0$
2. hash $\leftarrow S[2] \bmod p$
3. hash $\leftarrow S[1] + S[2]x \bmod p$
4. hash $\leftarrow S[0] + S[1]x + S[2]x^2 \bmod p$

Java Implementation

The method `hashCode` of the built-in Java class `String` is very similar to our `PolyHash`, it just uses $x = 31$ and for technical reasons avoids the $\pmod p$ operator.

Java Implementation

The method `hashCode` of the built-in Java class `String` is very similar to our `PolyHash`, it just uses $x = 31$ and for technical reasons avoids the $\pmod p$ operator.

You now know the implementation of the function that is used trillions of times a day in many thousands of programs!

Outline

Lemma

For any two different strings s_1 and s_2 of length at most $L + 1$, if you choose h from $\mathcal{P}_p$ at random (by selecting a random $x \in [1, p-1]$), the probability of collision $\Pr[h(s_1) = h(s_2)]$ is at most $\frac{L}{p}$.

Lemma

For any two different strings s_1 and s_2 of length at most $L+1$, if you choose h from $\mathcal{P}_p$ at random (by selecting a random $x \in [1, p-1]$), the probability of collision $\Pr[h(s_1) = h(s_2)]$ is at most $\frac{L}{p}$.

Proof idea

This follows from the fact that the equation $a_0 + a_1 x + a_2 x^2 + \cdots + a_L x^L = 0 \pmod{p}$ for prime p has at most L different solutions x.

Cardinality Fix

For use in a hash table of size m, we need a hash function of cardinality m.

Cardinality Fix

For use in a hash table of size m, we need a hash function of cardinality m.

First, apply random h_x from $\mathcal{P}_p$, and then hash the resulting value again using universal family for integers. Denote the resulting function by h_m.

Cardinality Fix

For use in a hash table of size m, we need a hash function of cardinality m.

First, apply random h_x from $\mathcal{P}_p$, and then hash the resulting value again using universal family for integers. Denote the resulting function by h_m.

$$h_m(S) = h_{a,b}(h_x(S)) \bmod m$$

Cardinality Fix

For use in a hash table of size m, we need a hash function of cardinality m.

First, apply random h_x from $\mathcal{P}_p$, and then hash the resulting value again using universal family for integers. Denote the resulting function by h_m.

$$h_m(S) = h_{a,b}(h_x(S)) \bmod m$$

Cardinality Fix

For use in a hash table of size m, we need a hash function of cardinality m.

First, apply random h_x from $\mathcal{P}_p$, and then hash the resulting value again using universal family for integers. Denote the resulting function by h_m.

$$h_m(S) = h_{a,b}(h_x(S)) \bmod m$$

Cardinality Fix

For use in a hash table of size m, we need a hash function of cardinality m.

First, apply random h_x from $\mathcal{P}_p$, and then hash the resulting value again using universal family for integers. Denote the resulting function by h_m.

$$h_m(S) = h_{a,b}(h_x(S)) \bmod m$$

Lemma

For any two different strings s_1 and s_2 of length at most $L + 1$ and cardinality m, the probability of collision $\Pr[h_m(s_1) = h_m(s_2)]$ is at most $\frac{1}{m} + \frac{L}{p}$.

Polynomial Hashing

Corollary

If $p > mL$, for any two different strings s_1 and s_2 of length at most $L + 1$ the probability of collision $\Pr[h_m(s_1) = h_m(s_2)]$ is $O(\frac{1}{m})$.

Proof

$\frac{1}{m} + \frac{L}{p} < \frac{1}{m} + \frac{L}{mL} = \frac{1}{m} + \frac{1}{m} = \frac{2}{m} = O(\frac{1}{m})$ □

Running Time

- For $p > mL$ we have
  $c = O(1 + \frac{n}{m}) = O(1 + \alpha)$ again

Running Time

- For $p > mL$ we have $c = O(1 + \frac{n}{m}) = O(1 + \alpha)$ again
- Computing $\texttt{PolyHash}(S)$ runs in $O(|S|)$

Running Time

- For $p > mL$ we have $c = O(1 + \frac{n}{m}) = O(1 + \alpha)$ again
- Computing `PolyHash`(S) runs in $O(|S|)$
- If lengths of the names in the phone book are bounded by constant L, computing $h(S)$ takes $O(L) = O(1)$ time

Conclusion

- You learned how to hash integers and strings
- Phone book can be implemented as two hash tables
- Mapping phone numbers to names and back
- Search and modification run in $O(1)$ on average!