TD 3 + 4 – Wavelets

The following exercises are meant to illustrate chapter 3 of Gabriel Peyré's ***Mathematics of Data*** *course (`mathematical-tours.github.io`). Some of them are adapted from the past years' exams.*

Please refer to Geert-Jan Huizing (`huizing@ens.fr`) for questions regarding these exercises and their solutions.

Notes We will be playing with different wavelet bases and and with Fast Wavelet Transform (FWT).

Multiresolution approximations of $\mathrm{L}^2(\mathbb{R})$ are families $(V_j)_{j\in\mathbb{Z}}$ of nested closed subspaces of $\mathrm{L}^2(\mathbb{R})$, s.t.

$$\mathrm{L}^2(\mathbb{R}) \supset ... \supset V_{j-1} \supset V_j \supset V_{j+1} \supset ... \supset \{0\},$$

$$f \in V_j \iff f(\cdot/2) \in V_{j+1} \quad \text{and} \quad f \in V_j \iff \forall n \in \mathbb{Z}, f(\cdot + n2^j) \in V_j,$$

$$\text{and } \exists\ \varphi \in \mathrm{L}^2(\mathbb{R}) \text{ so that } \{\varphi(\cdot - n)\}_n \text{ is an Hilbertian orthonormal basis of } V_0.$$

This implies that $\varphi_{j,n}$ form an orthonormal basis of $\mathrm{L}^2(\mathbb{R})$:

$$\left\{\varphi_{j,n}(t) = \frac{1}{\sqrt{2^j}}\varphi\left(\frac{t-2^j n}{2^j}\right)\right\}_{(j,n)\in\mathbb{Z}^2}.$$

Exercises

[$\star$] Multi-resolution Approximation Spaces

Show that these families $(V_j)_j$ are multiresolution approximations of $\mathrm{L}^2(\mathbb{R})$, with scaling functions φ.

1. *(Piecewise constant functions)*

$$V_j = \left\{f \in \mathrm{L}^2(\mathbb{R}); \forall n, f \text{ is constant on } [2^j n, 2^j(n+1)[\right\} \text{ and } \varphi(t) = \mathbf{1}_{[0,1[}$$

2. *(Shannon approximations)*

$$V_j = \left\{f \in \mathrm{L}^2(\mathbb{R}); \forall n, \hat{f} \text{ is zero outside of } [-2^{-j}\pi, 2^{-j}\pi]\right\} \text{ and } \varphi(t) = \frac{\sin \pi t}{\pi t}$$

[$\star$] Haar Wavelets

For $j \in \mathbb{Z}$, we define the space $V_j \subset \mathrm{L}^2(\mathbb{R})$ of functions which are constant on each interval $I_{j,k}$, where

$$\forall\ k \in \mathbb{Z},\ I_{j,k} \overset{\text{def.}}{=} \left[2^j k, 2^j(k+1)\right[$$

We also define the functions

$$\forall x \in \mathbb{R}, \varphi(x) \overset{\text{def.}}{=} \begin{cases} 1 & \text{if } x \in [0,1[\\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad \psi(x) \overset{\text{def.}}{=} \begin{cases} 1 & \text{if } x \in [0,1/2[\\ -1 & \text{if } x \in [1/2,1[\\ 0 & \text{otherwise.} \end{cases}$$

Their dilated and translated versions are defined as

$$\forall (j,k) \in \mathbb{Z}, \quad \psi_{j,k}(t) \overset{\text{def.}}{=} \frac{1}{\sqrt{2^j}}\psi\left(2^{-j}t - k\right) \quad \text{and} \quad \varphi_{j,k}(t) \overset{\text{def.}}{=} \frac{1}{\sqrt{2^j}}\varphi\left(2^{-j}t - k\right).$$

The functions $\psi_{j,k}$ are the so-called "Haar wavelets".

1. Draw the graphs of the functions $\varphi_{0,0}$, $\varphi_{1,0}$ and $\varphi_{2,2}$. Do the same for $\psi_{0,0}$, $\psi_{1,0}$ and $\psi_{2,2}$.
2. Show that $\forall j \in \mathbb{Z}$, there exists a space $W_j \subset \mathrm{L}^2(\mathbb{R})$ such that
$$W_j \perp V_j \quad \text{and} \quad V_{j-1} = V_j \oplus W_j,$$
3. Show that
$$\mathcal{B}_j^\varphi \overset{\text{def.}}{=} \{\varphi_{j,k} \ ; \ k \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{B}_j^\psi \overset{\text{def.}}{=} \{\psi_{j,k} \ ; \ k \in \mathbb{Z}\}$$
are ortho-bases of respectively V_j and W_j.

[$\star$] Up and down sampling (from exam 2020)

We consider $\ell^2(\mathbb{Z})$ the set of sequences $(x_i)_{i\in\mathbb{Z}}$ with $\sum_i x_i^2 < +\infty$ with the inner product $\langle x, y\rangle = \sum_i x_i y_i$.

1. The down and up-sampling operators are
$$x \downarrow_2 \overset{\text{def.}}{=} (x_{2i})_{i\in\mathbb{Z}} \quad \text{and} \quad (x \uparrow^2)_i \overset{\text{def.}}{=} \begin{cases} x_{i/2} & \text{if } i \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$
Prove that $\downarrow_2$ and $\uparrow^2$ are adjoints i.e. $\langle x \downarrow_2, y\rangle = \langle x, y \uparrow_2\rangle$.
2. What is the adjoint of the linear operator $x \mapsto x \star h$, where $(x \star h)_i \overset{\text{def.}}{=} \sum_{j\in\mathbb{Z}} x_j h_{i-j}$?
3. Prove that $(x \downarrow_2) \star h = (x \star (h \uparrow^2)) \downarrow_2$. Then use this recursively to show
$$f^k(x) = (x \star H^k) \downarrow_{2^k} \ \text{ where } H^k \overset{\text{def.}}{=} h^0 \star h^1 \star \ldots h^{k-1} \text{ and } f : x \mapsto (x \star h) \downarrow_2 .$$
We denote $h^s = h \uparrow^{2^s}$ ($\uparrow^{2^s}$ is defined similarly to $\uparrow^2$ by inserting $2^s - 1$ zeros).
4. What is H^k in the case of a box filter $[h_0, h_1] = [1, 1]/2$ (the other entries of the vector being 0) ?

[$\star\star$] Heisenberg inequality

A wavelet is a filter with mean 0, localized both in the spatial and Fourier domains. How localized can it be ? For a function $f \in \mathrm{L}^2(\mathbb{R})$ such that the following quantities exist, we define:

- Time mean $m : f \mapsto \frac{1}{\|f\|_2^2}\int_{-\infty}^{+\infty} t|f(t)|^2 \mathrm{d}t$ and variance $s^2 : f \mapsto \frac{1}{\|f\|_2^2}\int_{-\infty}^{+\infty} (t - m(f))^2 |f(t)|^2 \mathrm{d}t$
- Fourier mean $\mu : f \mapsto \frac{1}{2\pi\|f\|_2^2}\int_{-\infty}^{+\infty} \omega|\hat{f}(\omega)|^2 \mathrm{d}\omega$ and variance $\sigma^2 : f \mapsto \frac{1}{2\pi\|f\|_2^2}\int_{-\infty}^{+\infty} (\omega - \mu(f))^2 |\hat{f}(\omega)|^2 \mathrm{d}\omega$

1. Let us consider a scaling factor $\gamma \in \mathbb{R}_+^*$ and $f_\gamma(t) = \frac{1}{\sqrt{\gamma}} f(\frac{t}{\gamma})$.
   (a) Show that $\|f_\gamma\|_2^2 = \|f\|_2^2$
   (b) Derive the time mean and variance of f_γ as a function of those of f.
   (c) Derive the Fourier mean and variance of f_γ as a function of those of f.
   (d) How does scaling affect time and Fourier resolution ?
2. Let's assume $m = \mu = 0$ (this amounts to replacing f with $g : t \mapsto f(t-m)e^{i\mu t}$). Using the Plancherel identity ($2\pi \|u\|_2^2 = \|\hat{u}\|_2^2$), prove that
$$s^2\sigma^2 = \frac{1}{\|f\|^4}\int_{-\infty}^{+\infty} |tf(t)|^2 \mathrm{d}t \int_{-\infty}^{+\infty} |f'(t)|^2 \mathrm{d}t.$$
3. Prove that
$$s^2\sigma^2 \geq \frac{1}{\|f\|^4}\left(\int_{-\infty}^{+\infty} tf(t)\overline{f'(t)}\mathrm{d}t\right)^2.$$
4. Assuming $\lim_{|t|\to+\infty} \sqrt{t}f(t) = 0$, prove the Heisenberg inequality:
$$s^2\sigma^2 \geq 1/4$$
5. Find f such that this is an equality. You should recover what is called the *Morlet wavelet.*

Solutions

Multi-resolution Approximation Spaces

1. Since $f \in V_j$ is constant on intervals of length 2^j, clearly $V_{j+1} \subset V_j$.
   $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = \mathrm{L}^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (because non-zero constant functions of $\mathbb{R}$ are not in $\mathrm{L}^2(\mathbb{R})$).
   Finally, $f = \sum_{n\in\mathbb{Z}} f(n)\mathbf{1}_{[n,n+1[} = \sum_{n\in\mathbb{Z}} \langle f, \varphi(\cdot - n)\rangle \varphi(\cdot - n)$

2. It clearly follows from definition that $V_{j+1} \in V_j$.
   $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = \mathrm{L}^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, because $\mathcal{F}^{-1}(\mathrm{L}^2(\mathbb{R})) = \mathrm{L}^2(\mathbb{R})$ and $\mathcal{F}^{-1}(0) = 0$.
   The Nyquist-Shannon theorem yields $f(t) = \sum_{n\in\mathbb{Z}} f(n)\varphi(t-n)$. And $f(n) = \mathcal{F}^{-1}(\hat{f}\mathbf{1}_{[-\pi,\pi[})(n) = \mathcal{F}^{-1}(\hat{f}\hat{\varphi})(n) = \mathcal{F}^{-1}(\widehat{f \star \varphi})(n) = (f \star \varphi)(n) = \langle f, \varphi(\cdot - n)\rangle$, hence the result.

Haar Wavelets

1. Notice the translation and the dilation.
2. $$W_j \overset{\text{def.}}{=} \left\{ f \in \mathrm{L}^2(\mathbb{R}) \text{ s.t. } f \text{ constant on } I_{j-1,k} \text{ and } \int_{I_{j,k}} f = 0 \right\}$$
   Orthogonal to V_j because of the zero integral in the segments $I_{j,k}$.
3. Orthogonality should be clear from the drawing.
   $$f \in V_j = \sum_k 2^{j/2}\varphi_{j,k} f(2^j k) = \sum_k \langle \varphi_{j,k}, f\rangle \varphi_{j,k}$$
   $$\forall g \in W_j, \sum_k \langle g, \psi_{j,k}\rangle \psi_{j,k} \propto \sum_j \varphi_{j-1,k} \begin{cases} g(2^{j-1}k) - g(2^{j-1}(k+1)) = 2 \times g(2^{j-1}k) \text{ if k even} \\ g(2^{j-1}k) - g(2^{j-1}(k-1)) = 2 \times g(2^{j-1}k) \text{ if k odd} \end{cases} \propto g$$

Up and down sampling

1. One has
   $$\langle x \downarrow_2, y\rangle = \sum_i x_{2i} y_i = \sum (x_{2i} y_i + x_{2i+1} 0) = \sum_i (x_{2i} (y \uparrow^2)_{2i} + x_{2i+1} (y \uparrow^2)_{2i+1}).$$
2. It is the convolution against h_{-i}.
3. One has
   $$[(x \downarrow_2) \star h]_i = \sum_k x_{2k} h_{i-k} = \sum_k (x_{2k} h_{i-k} + x_{2k+1} 0) = \sum_k (x_{2k} (h \uparrow^2)_{2(i-k)} + x_{2k+1} (h \uparrow^2)_{2(i-k)-1}) = (\downarrow_2 (x \star (h \uparrow^2)))_i$$
   One thus has
   $$[((x \star H^k) \downarrow_{2^k}) \star h] \downarrow_2 = [x \star H^k \star (h \uparrow^{2^k})] \downarrow_{2^{k+1}} = [x \star H^{k+1}] \downarrow_{2^{k+1}}$$
4. One has
   $$H^1 = [1,1]/2 \star [1,0,1,0]/2 = [1,1,1,1]/4$$
   and more generally $H^s = 1_{[0,\ldots,2^{s+1}-1]}/2^{s+1}$

Heisenberg inequality

1. (a) Change of variables
   (b) By change of variables, $m(f_\gamma) = \gamma \times m(f)$ and $s(f_\gamma) = \gamma \times s(f)$
   (c) Since $\hat{f}_\gamma(\omega) = \sqrt{\gamma}\hat{f}(\omega)$, by change of variables, $\mu(f_\gamma) = \mu(f)/\gamma$ and $\sigma(f_\gamma) = \sigma(f)/\gamma$
   (d) Reducing variance in one domain makes it bigger in the other.

2. The product reads
$$s^2\sigma^2 = \frac{1}{2\pi\|f\|^4}\int_{-\infty}^{+\infty}|tf(t)|^2\mathrm{d}t\int_{-\infty}^{+\infty}|\omega\hat{f}(\omega)|^2\mathrm{d}\omega.$$
We notice that since $\omega \mapsto i\omega\hat{f}(\omega)$ is the Fourier transform of $t \mapsto f'(t)$, the Plancherel equality gives
$$s^2\sigma^2 = \frac{1}{\|f\|^4}\int_{-\infty}^{+\infty}|tf(t)|^2\mathrm{d}t\int_{-\infty}^{+\infty}|f'(t)|^2\mathrm{d}t.$$

3. By appyling Cauchy-Schwarz,
$$s^2\sigma^2 \geq \frac{1}{\|f\|^4}\left(\int_{-\infty}^{+\infty}tf(t)\overline{f'(t)}\mathrm{d}t\right)^2.$$

4. We can rewrite this as
$$s^2\sigma^2 \geq \frac{1}{\|f\|^4}\left(\int_{-\infty}^{+\infty}\frac{t}{2}\left(\overline{f'(t)}f(t) + \overline{f(t)}f'(t)\right)\mathrm{d}t\right)^2$$
then
$$s^2\sigma^2 \geq \frac{1}{4\|f\|^4}\left(\int_{-\infty}^{+\infty}t(|f(t)|^2)'\mathrm{d}t\right)^2$$
Finally, integration by parts (since $\lim_{|t|\to+\infty}\sqrt{t}f(t) = 0$) gives
$$\int_{-\infty}^{+\infty}t(|f(t)|^2)'\mathrm{d}t = \int_{-\infty}^{+\infty}f(t)^2\mathrm{d}t,$$
hence
$$s^2\sigma^2 \geq \frac{1}{4}.$$

5. Equality case of Cauchy-Schwarz, $f'(t) = \lambda tf(t)$. So $f(t) \propto e^{\lambda t^2/2}$.