

TD 3 + 4 – Wavelets

The following exercises are meant to illustrate chapter 3 of Gabriel Peyré's **Mathematics of Data** course (mathematical-tours.github.io). Some of them are adapted from the past years' exams.

Please refer to Geert-Jan Huizing (huizing@ens.fr) for questions regarding these exercises and their solutions.

Notes We will be playing with different wavelet bases and with Fast Wavelet Transform (FWT).

Multiresolution approximations of $L^2(\mathbb{R})$ are families $(V_j)_{j \in \mathbb{Z}}$ of nested closed subspaces of $L^2(\mathbb{R})$, s.t.

$$L^2(\mathbb{R}) \supset \dots \supset V_{j-1} \supset V_j \supset V_{j+1} \supset \dots \supset \{0\},$$

$$f \in V_j \iff f(\cdot/2) \in V_{j+1} \quad \text{and} \quad f \in V_j \iff \forall n \in \mathbb{Z}, f(\cdot + n2^j) \in V_j,$$

and $\exists \varphi \in L^2(\mathbb{R})$ so that $\{\varphi(\cdot - n)\}_n$ is an Hilbertian orthonormal basis of V_0 .

This implies that $\varphi_{j,n}$ form an orthonormal basis of $L^2(\mathbb{R})$:

$$\left\{ \varphi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \varphi\left(\frac{t - 2^j n}{2^j}\right) \right\}_{(j,n) \in \mathbb{Z}^2}.$$

Exercises

[★] Multi-resolution Approximation Spaces

Show that these families $(V_j)_j$ are multiresolution approximations of $L^2(\mathbb{R})$, with scaling functions φ .

1. (Piecewise constant functions)

$$V_j = \{f \in L^2(\mathbb{R}); \forall n, f \text{ is constant on } [2^j n, 2^j(n+1)[\} \text{ and } \varphi(t) = \mathbf{1}_{[0,1[}$$

2. (Shannon approximations)

$$V_j = \left\{ f \in L^2(\mathbb{R}); \forall n, \hat{f} \text{ is zero outside of } [-2^{-j}\pi, 2^{-j}\pi] \right\} \text{ and } \varphi(t) = \frac{\sin \pi t}{\pi t}$$

[★] Haar Wavelets

For $j \in \mathbb{Z}$, we define the space $V_j \subset L^2(\mathbb{R})$ of functions which are constant on each interval $I_{j,k}$, where

$$\forall k \in \mathbb{Z}, I_{j,k} \stackrel{\text{def.}}{=} [2^j k, 2^j(k+1)[$$

We also define the functions

$$\forall x \in \mathbb{R}, \varphi(x) \stackrel{\text{def.}}{=} \begin{cases} 1 & \text{if } x \in [0, 1[\\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad \psi(x) \stackrel{\text{def.}}{=} \begin{cases} 1 & \text{if } x \in [0, 1/2[\\ -1 & \text{if } x \in [1/2, 1[\\ 0 & \text{otherwise.} \end{cases}$$

Their dilated and translated versions are defined as

$$\forall (j, k) \in \mathbb{Z}, \quad \psi_{j,k}(t) \stackrel{\text{def.}}{=} \frac{1}{\sqrt{2^j}} \psi(2^{-j}t - k) \quad \text{and} \quad \varphi_{j,k}(t) \stackrel{\text{def.}}{=} \frac{1}{\sqrt{2^j}} \varphi(2^{-j}t - k).$$

The functions $\psi_{j,k}$ are the so-called “Haar wavelets”.

1. Draw the graphs of the functions $\varphi_{0,0}$, $\varphi_{1,0}$ and $\varphi_{2,2}$. Do the same for $\psi_{0,0}$, $\psi_{1,0}$ and $\psi_{2,2}$.
2. Show that $\forall j \in \mathbb{Z}$, there exists a space $W_j \subset L^2(\mathbb{R})$ such that

$$W_j \perp V_j \quad \text{and} \quad V_{j-1} = V_j \oplus W_j,$$

3. Show that

$$\mathcal{B}_j^\varphi \stackrel{\text{def.}}{=} \{\varphi_{j,k} ; k \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{B}_j^\psi \stackrel{\text{def.}}{=} \{\psi_{j,k} ; k \in \mathbb{Z}\}$$

are ortho-bases of respectively V_j and W_j .

[★] Up and down sampling (from exam 2020)

We consider $\ell^2(\mathbb{Z})$ the set of sequences $(x_i)_{i \in \mathbb{Z}}$ with $\sum_i x_i^2 < +\infty$ with the inner product $\langle x, y \rangle = \sum_i x_i y_i$.

1. The down and up-sampling operators are

$$x \downarrow_2 \stackrel{\text{def.}}{=} (x_{2i})_{i \in \mathbb{Z}} \quad \text{and} \quad (x \uparrow^2)_i \stackrel{\text{def.}}{=} \begin{cases} x_{i/2} & \text{if } i \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that \downarrow_2 and \uparrow^2 are adjoints i.e. $\langle x \downarrow_2, y \rangle = \langle x, y \uparrow_2 \rangle$.

2. What is the adjoint of the linear operator $x \mapsto x \star h$, where $(x \star h)_i \stackrel{\text{def.}}{=} \sum_{j \in \mathbb{Z}} x_j h_{i-j}$?
3. Prove that $(x \downarrow_2) \star h = (x \star (h \uparrow^2)) \downarrow_2$. Then use this recursively to show

$$f^k(x) = (x \star H^k) \downarrow_{2^k} \quad \text{where} \quad H^k \stackrel{\text{def.}}{=} h^0 \star h^1 \star \dots \star h^{k-1} \quad \text{and} \quad f : x \mapsto (x \star h) \downarrow_2.$$

We denote $h^s = h \uparrow^{2^s}$ (\uparrow^{2^s} is defined similarly to \uparrow^2 by inserting $2^s - 1$ zeros).

4. What is H^k in the case of a box filter $[h_0, h_1] = [1, 1]/2$ (the other entries of the vector being 0) ?

[★★] Heisenberg inequality

A wavelet is a filter with mean 0, localized both in the spatial and Fourier domains. How localized can it be ? For a function $f \in L^2(\mathbb{R})$ such that the following quantities exist, we define:

- Time mean $m : f \mapsto \frac{1}{\|f\|_2^2} \int_{-\infty}^{+\infty} t |f(t)|^2 dt$ and variance $s^2 : f \mapsto \frac{1}{\|f\|_2^2} \int_{-\infty}^{+\infty} (t - m(f))^2 |f(t)|^2 dt$
- Fourier mean $\mu : f \mapsto \frac{1}{2\pi\|f\|_2^2} \int_{-\infty}^{+\infty} \omega |\hat{f}(\omega)|^2 d\omega$ and variance $\sigma^2 : f \mapsto \frac{1}{2\pi\|f\|_2^2} \int_{-\infty}^{+\infty} (\omega - \mu(f))^2 |\hat{f}(\omega)|^2 d\omega$

1. Let us consider a scaling factor $\gamma \in \mathbb{R}_+^*$ and $f_\gamma(t) = \frac{1}{\sqrt{\gamma}} f(\frac{t}{\gamma})$.

- (a) Show that $\|f_\gamma\|_2^2 = \|f\|_2^2$
- (b) Derive the time mean and variance of f_γ as a function of those of f .
- (c) Derive the Fourier mean and variance of f_γ as a function of those of f .
- (d) How does scaling affect time and Fourier resolution ?

2. Let's assume $m = \mu = 0$ (this amounts to replacing f with $g : t \mapsto f(t - m)e^{i\mu t}$). Using the Plancherel identity ($2\pi \|u\|_2^2 = \|\hat{u}\|_2^2$), prove that

$$s^2 \sigma^2 = \frac{1}{\|f\|_4^4} \int_{-\infty}^{+\infty} |t f(t)|^2 dt \int_{-\infty}^{+\infty} |f'(t)|^2 dt.$$

3. Prove that

$$s^2 \sigma^2 \geq \frac{1}{\|f\|_4^4} \left(\int_{-\infty}^{+\infty} t f(t) \overline{f'(t)} dt \right)^2.$$

4. Assuming $\lim_{|t| \rightarrow +\infty} \sqrt{t} f(t) = 0$, prove the Heisenberg inequality:

$$s^2 \sigma^2 \geq 1/4$$

5. Find f such that this is an equality. You should recover what is called the *Morlet wavelet*.