TD 2 – Signal processing: Fourier, filtering, FFT

*The following exercises are meant to illustrate chapter 2 of Gabriel Peyré's **Mathematics of Data** course (`mathematical-tours.github.io`). Some of them are adapted from the past years' exams.*

Please refer to Geert-Jan Huizing (`huizing@ens.fr`) for questions regarding these exercises and their solutions.

Exercises

- We denote as $\Sigma_n \overset{\text{def.}}{=} \{a \in \mathbb{R}_+^n; \sum_i a_i = 1\}$ the probability simplex.
- We identify the indices $\{1, \ldots, n\}$ with the elements of $\mathbb{Z}/n\mathbb{Z}$ (i.e. we consider periodic indices).
- We define the convolution between $f \in \mathbb{C}^n$ and $g \in \mathbb{C}^n$ by $(f \star g)_i = \sum_j f_{i-j} g_j$
- For all $f \in \mathbb{C}^n$, we recall that the discrete Fourier transform is defined as

$$\forall k \in \mathbb{Z}/n\mathbb{Z}, \hat{f}_k \overset{\text{def.}}{=} \sum_{p=0}^{n-1} f_p e^{-2i\pi kp/n}.$$

[$\star$] Common Fourier Transforms

Compute the discrete Fourier transforms of the following signals $f \in \mathbb{C}^n$.

1. *Frequency shift.* $f_p = x_p e^{2i\pi p\tau/n}$ (use $\hat{x}_k$ and τ to express $\hat{f}_k$).
2. *Geometric progression.* $f_p = a^p$.
3. *Window.* f is a window of size $W = 2K + 1$ centered around 0:

$$\begin{aligned} f_p &= 1/W \text{ if } \min(p, n-p) \leq K \\ &= 0 \text{ otherwise.} \end{aligned}$$

[$\star\star$] Total Variation (from exam 2020)

1. Let $f \in \mathbb{C}^n$. Bound $|\hat{f}_k|$ using $\|f\|_1$ (the l^1 norm of f).
2. We define the total variation of f as

$$\|f\|_V \overset{\text{def.}}{=} \sum_{i=0}^{n-1} |f_{i+1} - f_i|.$$

   Determine the set of f such that $\|f\|_V = 0$. Show that $\|\cdot\|_V$ statisfies the triangular inequality (it is a semi-norm).

3. Write $\|f\|_V$ as $\|f \star h\|_1$, where $h \in \mathbb{C}^n$.
4. Compute $\hat{h}$ the discrete Fourier transform of h, and compute $|\hat{h}_k|$.
5. Bound $|\hat{f}_k|$ using $\|f\|_V$, n, and k.

[⋆⋆] Markov Chains (from exam 2018)

1. Show that a matrix $P \in \mathbb{R}_+^{n\times n}$ such that $P^\top \mathbb{1}_n = \mathbb{1}_n$ defines a linear map $a \in \Sigma_n \mapsto Pa \in \Sigma_n$ from the simplex to itself.

2. We denote $T_\tau : a \mapsto (a_{i-\tau})_i$ the translation operator for $\tau \in \mathbb{Z}/n\mathbb{Z}$. Show that P commutes with T_τ for all τ if and only if there exists $h \in \Sigma_n$ such that $Pa = h \star a$.

3. Starting from some $a^{(0)} \in \Sigma_n$, we define $a^{(l+1)} \overset{\text{def.}}{=} Pa^{(l)}$. Show that if $\forall k \neq 0, |\hat{h}_k| < 1$, then $a^{(l)} \to a^\star$ as $l \to +\infty$. What is $a^\star$?

[⋆ ⋆ ⋆] Hadamard-Walsh Transform (from exam 2017)

We denote $G = (\mathbb{Z}/2\mathbb{Z})^p$, which is a group with $n \overset{\text{def.}}{=} 2^p$ elements. An element $x \in G$ is written as $x = (x_i)_{i=1}^p$ with $x_i \in \{0, 1\}$ and is equivalently represented as $0 \leq x < n$ where the x_i is the binary writing of x in base 2. We denote $\mathbb{R}[G]$ the vector space of functions $f : G \to \mathbb{R}$, endowed with the canonical inner product $\langle f, g\rangle \overset{\text{def.}}{=} \sum_{x\in G} f(x)g(x)$.

1. For $\omega \in G$, we denote $\psi_\omega(x) \overset{\text{def.}}{=} (-1)^{\sum_i x_i\omega_i}$. Show that $(\psi_\omega)_{\omega\in G}$ is an orthogonal basis of $\mathbb{R}[G]$.

2. We denote $\hat{f}(\omega) \overset{\text{def.}}{=} \langle f, \psi_\omega\rangle$ the Hadamard-Walsh transform of f. We denote $f_0, f_1 : (\mathbb{Z}/2\mathbb{Z})^{p-1} \to \mathbb{R}$ defined as

$$f_0(x_1, \ldots, x_{p-1}) \overset{\text{def.}}{=} f(x_1, \ldots, x_{p-1}, 0) \quad \text{and} \quad f_1(x_1, \ldots, x_{p-1}) \overset{\text{def.}}{=} f(x_1, \ldots, x_{p-1}, 1)$$

   Find a relation between $\hat{f}$ and $(\hat{f}_0, \hat{f}_1)$. Write in pseudo-code a fast recursive algorithm to compute $\hat{f}$ from f. What is the complexity of this algorithm ?

Solutions

Common Fourier Transforms

1. $\hat{f}_k = \hat{x}_{k-\tau}$
2. Using the geometric progression formula, we get: $\hat{f}_k = n$ if $a = e^{2ik\pi/n}$, else
$$\hat{f}_k = \frac{1-a^n}{1-ae^{-2ik\pi/n}}$$
3. $$\hat{f}_k = \frac{1}{W}(-1 + \sum_{p\leq K} e^{-2ik\pi p/n} + \sum_{p\leq K} e^{2ik\pi p/n})$$
Using the geometric progression formula, and then trigonometry formulas, we get
$$\hat{f}_k = \frac{\sin(k\pi W/n)}{W\sin(k\pi/n)}$$

Total Variation

1. $|\hat{f}_k| = |\sum_p f_p e^{-2i\pi kp/n}| \leq \sum_p |f_p e^{-2i\pi kp/n}| = \sum_p |f_p| = \|f\|_1$.
2. $\|f\|_V = 0 \iff f$ constant. And
$$\|f+g\|_V = \sum_i |f_{i+1} + g_{i+1} - f_i - g_i| \leq \sum_i |f_{i+1} - f_i| + \sum_i |g_{i+1} - g_i| = \|f\|_V + \|g\|_V.$$
3. $h_0 = -1, h_1 = 1$ and 0 elsewhere.
4. $\hat{h}_k = -1 + e^{-2i\pi k/n}$. And $|\hat{h}_k| = 2 \times |\sin(k\pi/n)|$
5. Replacing f by $f \star h$ in the first question, we get $|\hat{f}_k \hat{h}_k| \leq \|f \star h\|_1 = \|f\|_V$. Hence
$$|\hat{f}_k| \leq \frac{\|f\|_V}{2\times|\sin(k\pi/n)|}.$$

Markov Chains

1. $\sum_i (Pa)_i = \sum_i \sum_j P_{ij} a_j = \sum_j a_j (\sum_i P_{ij}) = \sum_j a_i = 1$
2. $(i) \implies (ii)$: We denote $\delta_i = (0,\ldots,0,1,0,\ldots,0)^\top$, and P_j the j-th column of P.
$$T_\tau(P\delta_0) = T_\tau(P_0) \quad \text{and} \quad PT_\tau(\delta_0) = P\delta_\tau = P_\tau,$$
So $\forall\tau, \forall i, P_{i,\tau} = P_{i-\tau,0} \stackrel{\text{def.}}{=} h(i-\tau)$, and
$$(Pa)_i = \sum_j P_{ij} a_j = \sum_j h(i-j) a_j = (h \star a)_i$$
$(ii) \implies (i)$:
$$\begin{aligned} T_\tau(Pa)_i = T_\tau(h\star a)_i = (h\star a)_{i-\tau} &= \sum_j h_{i-\tau-j} a_j \\ &= \sum_{j'} h_{i-j'} a_{j'-\tau} = (h \star T_\tau(a))_i = (PT_\tau(a))_i \end{aligned}$$
3. We have for $k > 0$,
$$\hat{a}_k^{(l+1)} = \hat{h}_k \times \hat{a}_k^{(l)} = \ldots = (\hat{h}_k)^{(l+1)} \times \hat{a}_k^{(0)},$$
which tends to 0 if $|\hat{h}_k| < 1$. On the other hand, $\hat{a}_0^{(l+1)} = \hat{a}_0^{(0)}$ because $h \in \Sigma_n$.
By linearity ($\implies$ continuity) of the inverse Fourier transform,
$$a^{(l+1)} \to a^{(\infty)}$$
where $\hat{a}^{(\infty)} = (1,0,\ldots,0)^\top$, hence $a^{(\infty)} = \mathbb{1}_n/n$.