TD 5 – Approximation

The following exercises are meant to illustrate chapter 4 of Gabriel Peyré's ***Mathematics of Data*** *course (`mathematical-tours.github.io`). Some of them are adapted from the past years' exams.*

Please refer to Geert-Jan Huizing (`huizing@ens.fr`) for questions these exercises and their solutions.

Notes We consider functions $f \in \mathrm{L}^2([0,1])$ and we define $\langle f, g\rangle \overset{\text{def.}}{=} \int_0^1 fg$ and $\|f\|^2 \overset{\text{def.}}{=} \langle f, f\rangle$.

Exercises

[⋆⋆] Non-linear approximation (from exam 2019)

For an *arbitrary* $y \in [0,1]$, we consider the indicator function $f \overset{\text{def.}}{=} 1_{[y,1]} \in \mathrm{L}^2([0,1])$. Let $M \in \mathbb{N}^\star$.

1. **M-term linear approximation.** For $0 \le k < M$, we denote $\theta_k \overset{\text{def.}}{=} \sqrt{M} 1_{[\frac{k}{M}, \frac{k+1}{M}[}$.

   - Show that $(\theta_k)_k$ is an orthonormal family and give the expression for the linear approximation
   $$f_M \overset{\text{def.}}{=} \sum_k \langle f, \theta_k\rangle \theta_k.$$
   - Bound as sharply as possible $\|f - f_M\|$ as a function of M, independently of y.

2. **Haar wavelets.** We denote $\theta \overset{\text{def.}}{=} 1_{[0,1]}$ and $\psi = 1_{[0,1/2[} - 1_{[1/2,1[}$ the Haar wavelet. We denote, for $j \le 0$, and $0 \le n < 2^{-j}$ the wavelet functions as $\psi_{j,n} \overset{\text{def.}}{=} 2^{-j/2}\psi(2^{-j}x - n)$.

   - For some $j_{\min} < 0$, show that
   $$\{\theta\} \cup \{\psi_{j,n} \ ; \ 0 \ge j \ge j_{\min} \quad \text{and} \quad 0 \le n < 2^{-j}\}$$
   is an orthogonal family. What is the space spanned by this family ?
   - For each j, what is the set Σ_j of index n where $\langle f, \psi_{j,n}\rangle$ is non-zero? For these $n \in \Sigma_j$, show that $|\langle f, \psi_{j,n}\rangle| \le 2^{j/2}$.

3. **M-term non-linear approximation.** For $T > 0$ such that $M = |\{(j,n) \ ; \ |\langle f, \psi_{j,n}\rangle| > T\}|$,
   $$\hat{f}_T \overset{\text{def.}}{=} \langle f, \theta\rangle\theta + \sum_{|\langle f, \psi_{j,n}\rangle| > T} \langle f, \psi_{j,n}\rangle \psi_{j,n}.$$
   - Find a cutoff scale j_0 such that $|\langle f, \psi_{j,n}\rangle| > T \implies j \ge j_0$.
   - We now define an approximation of f using the M first nonzero terms, instead of M biggest:
   $$\tilde{f}_M \overset{\text{def.}}{=} \langle f, \theta\rangle\theta + \sum_{j > -M, \ n \in \Sigma_j} \langle f, \psi_{j,n}\rangle \psi_{j,n}.$$
   Show that $\|f - \hat{f}_T\| \le \|f - \tilde{f}_M\|$.

4. **Linear vs. non-linear approximation.** Bound $\|f - \tilde{f}_M\|$ as a function of M. Use this to compare the decay with M of $\|f - f_M\|$ vs. $\|f - \hat{f}_T\|$.

[$\star\star$] **Convergence of orthogonal projections (from exam 2018)**

For $n \in \mathbb{N}^*$ we denote

$$\forall k \in \{0, \dots, n-1\}, \quad I_{n,k} \overset{\text{def.}}{=} \begin{cases} [\frac{k}{n}, \frac{k+1}{n}[\text{ if } k < n-1, \\ [\frac{k}{n}, \frac{k+1}{n}] \text{ if } k = n-1, \end{cases}$$

We denote $V_n \subset \mathrm{L}([0,1])$ the set of functions which are constant on the intervals $I_{n,k}$.

For $n \in \mathbb{N}^*$ we consider the orthogonal basis $\{\theta_{n,k}\}_{k=0}^{n-1}$ of V_n, with

$$\theta_{n,k}(x) \overset{\text{def.}}{=} \sqrt{n}\mathbb{1}_{I_{n,k}}.$$

For $f \in \mathrm{L}^2([0,1])$ we denote $\mathrm{P}_{V_n}(f)$ the orthogonal projection of f on V_n, which satisfies

$$P_{V_n}(f) = \sum_{k=0}^{n-1} \langle f, \theta_{n,k} \rangle \theta_{n,k}.$$

1. Give the value of $\mathrm{P}_{V_n}(f)$ on each interval $I_{n,k}$.
2. For f continue, show that $\mathrm{P}_{V_n}(f)$ converges uniformly toward f as $n \to +\infty$. Note that since f is continuous on the compact $[0,1]$, f is uniformly continuous on $[0,1]$.
3. Using the question above, show that if $f \in \mathrm{L}^2([0,1])$ then the projection $\mathrm{P}_{V_n}(f)$ converges toward f for the norm $\|\cdot\|$ of $\mathrm{L}^2([0,1])$ as $n \to +\infty$.

Solutions

Non-linear approximation

1.
   - For $x \in [\frac{k}{M}, \frac{k+1}{M}[$, $f_M = M \times \int_{\frac{k}{M}}^{\frac{k+1}{M}} f$.
   - Assume $y \in [\frac{k}{M}, \frac{k+1}{M}[$ for some k, $\|f - f_M\|^2 \leq \frac{1}{4M}$. Indeed, the largest error occurs when y is in the middle of $[\frac{k}{M}, \frac{k+1}{M}[$.

2.
   - The space are functions witch are constant on the intervals $I_{n,k} \stackrel{\text{def.}}{=} [kn, (k+1)n[$ for $n = 2^{-j}$.
   - Let k such that $y \in I_{2^{-j},k}$. Then $\Sigma_j = \{k\}$. Since $|f| \leq 1$ and $|\psi| \leq 1$, $|\langle f, \psi_{j,n}\rangle| \leq 2^{j/2}$

3.
   - $j \leq j_0 \implies 2^{j/2} \leq 2^{j_0/2} \implies |\langle f, \psi_{j,n}\rangle| \leq 2^{j_0/2}$. So if $j_0 = \lfloor \log_2 T^2 \rfloor$ then by contraposition $|\langle f, \psi_{j,n}\rangle| > T \implies j > j_0$
   - Last question implies $M \leq \lceil |\log_2 T^2| \rceil$ and $\|f - \tilde{f}_M\| = \sum_{j \leq j_0, n \in \Sigma_j} |\langle f, \psi_{j,n}\rangle|^2 \leq \sum_{j \leq j_0, n \in \Sigma_j} (2^{j/2})^2 = \mathcal{O}(2^{-M})$. Finally, $\|f - \hat{f}_T\| \leq \|f - \tilde{f}_M\|$ because $\hat{f}_T$ uses bigger coefficients.

4.
   - As shown previously, $\|f - \tilde{f}_M\| = \mathcal{O}(2^{-M})$, and $\|f - f_M\| = \mathcal{O}(1/M)$, so non-linear approximation has a faster decay.

Convergence of orthogonal projections

See solutions of exam 2018