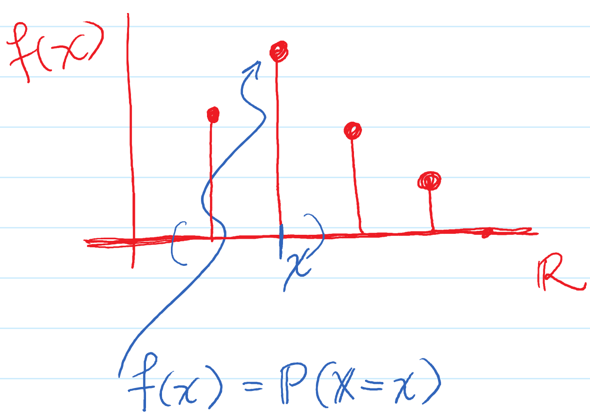
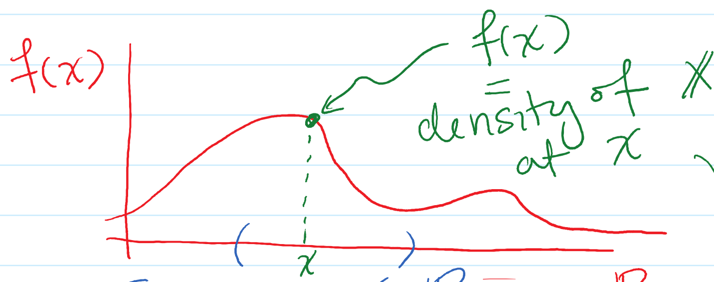
For a r.v. X

discrete : PMF



continuous : PDF



For any $x \in \mathbb{R}$

$P(X=x)=0$.

Note: Not describe ~~P(X=x)~~

Fact: CDF is $F(x) = P(X \le x)$

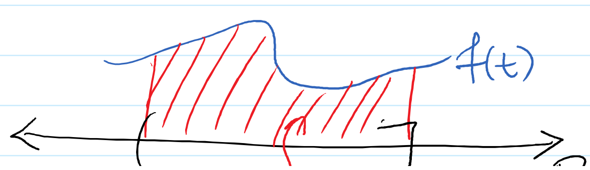
in the continuous case:

$$f(x) = \frac{d}{dx}F(x) \iff F(x) = \int_{-\infty}^{x} f(t)\,dt$$

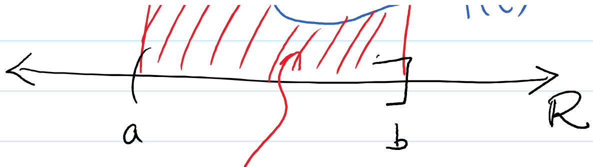
Some properties for $\boxed{\text{cts}}$ RVs

$$P(a < X \le b) = F(b) - F(a)$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$



$= \int^{b}$



$$= \int_a^b f(t)\,dt$$

$$P(a < X \le b) = \int_a^b f(t)\,dt$$

We said: $\boxed{P(X=a) = P(X=b) = 0.}$

$$\left.\begin{aligned} P(a < X \le b) &= P(a \le X \le b) \\ &= P(a \le X < b) \\ &= P(a < X < b) \end{aligned}\right\} \text{For cts ONLY.}$$

More general rule: For $A \subset \mathbb{R}$

For discrete: $\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$ ← PMF

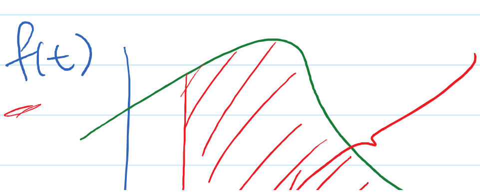
For continuous: $\mathbb{P}(X \in A) = \int_A f(t)\,dt$

For ex: $P(X \in (-\infty, x]) = \int_{(-\infty, x]} f(t)\,dt = \int_{-\infty}^{x} f(t)\,dt$

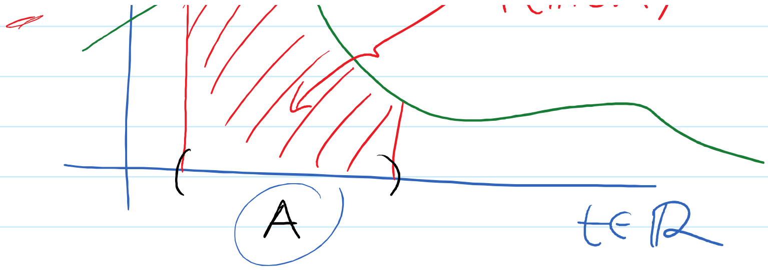
$\mathbb{P}(X \le x) = F(x)$

Ex. $\mathbb{P}(X \in (2,3]) = \int_2^3 f(t)\,dt$

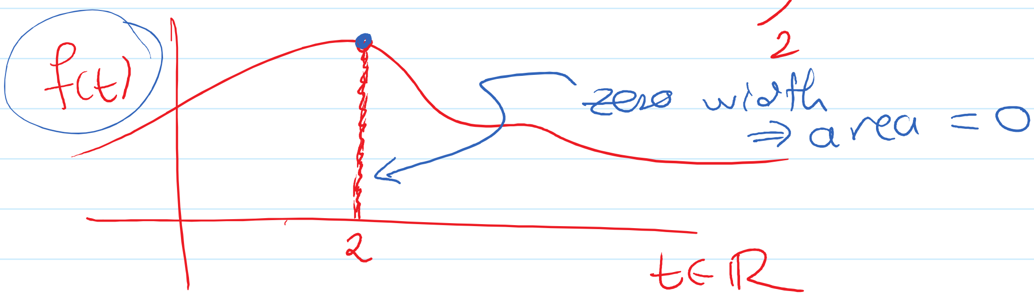
PDF



$P(X \in A)$



Ex. $P(X \in \{2\}) = P(X=2) = \int_2^2 f(t)\,dt = 0$



$f(2) > 0$
but $P(X=2) = 0$.

Ex. $F(x) = \dfrac{1}{1+e^{-x}}$ ← CDF

Fact: $F(x) = \int_{-\infty}^{x} f(t)\,dt$

what is the assoc. PDF?

$$f(x) = \frac{dF}{dx} = \dots = \frac{e^{-x}}{(1+e^{-x})^2}.$$

Ex. Continuous Uniform Distribution (on $[0,1]$).

denoted: $X \sim U(0,1)$

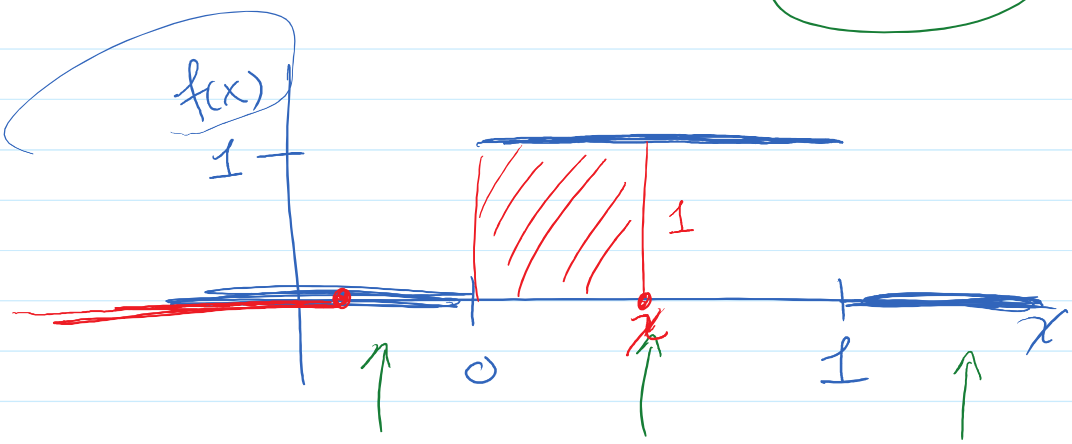
means

$$f(x) \begin{cases} 1 & \text{if } 0 \le x \le 1 \end{cases}$$

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$= 1 \quad \text{for } 0 \le x \le 1$$

Support



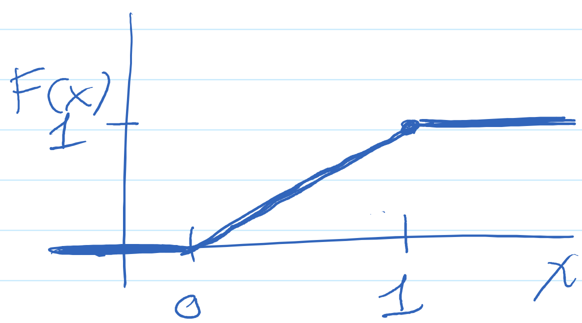
Q: What is the CDF of X?

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & x \le 0 & (1) \\ x & x \in [0,1] & (2) \\ 1 & x \ge 1 & (3) \end{cases}$$

Case 1: $\int_{-\infty}^{x} f(t)\,dt = 0$

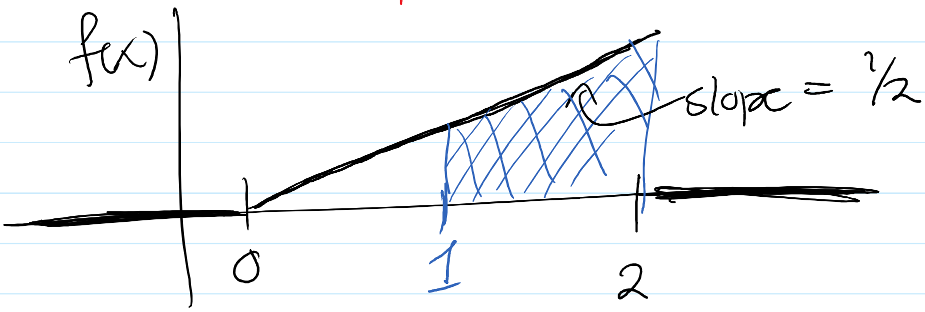
Case 2: $\int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 1\,dt = x$

Case 3: $\int_{-\infty}^{x} f(t)\,dt = \int_{0}^{1} 1\,dt = 1$



Ex. lets say

$$f(x) = \begin{cases} x/2 & 0 < x < 2 \\ 0 & \text{else} \end{cases}$$



$P(X > 1)$

Recall: $P(X \in A) = \int_A f(t)\,dt$

$$= \int_1^{\infty} f(t)\,dt = \int_1^2 t/2 \, dt = \left[\frac{t^2}{4}\right]_1^2 = \frac{4}{4} - \frac{1}{4} = 3/4.$$

Ex. Let

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x > 0 \\ 0 & \text{else} \end{cases}$$

Q. $P(1 < X < 2)$

π

Q: $P(1 < X < 2)$

Theorem: $P(a < X < b) = F(b) - F(b)$

$$P(1 < X < 2) = F(2) - F(1)$$

$$= (1 - e^{-2}) - (1 - e^{-1})$$

$$\boxed{= e^{-1} - e^{-2}}$$

Alternative:

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}(1 - e^{-x}) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{else} \end{cases}$$

PDF

$$P(1 < X < 2) = \int_1^2 f(t)\,dt = \int_1^2 e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_1^2 = (-e^{-2}) - (-e^{-1})$$

$$= \boxed{e^{-1} - e^{-2}}.$$

Theorem: PMF/PDF characterization

A function f is a "valid" pmf/pdf

($\exists$ a r.v. X w/ f as its pmf/pdf)

iff

① $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

② (discrete PMF) $\sum_{x \in \mathbb{R}} f(x) = 1$

(continuous PDF) $\int_{\mathbb{R}} f(t)\,dt = 1$

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aside: discrete: $P(X \in A) = \sum_{x \in A} f(x) \geq 0$

cts: $P(X \in A) = \int_A f(t)\,dt \geq 0$

$$1 = P(S) = P(X \in \mathbb{R}) = \begin{cases} \sum_{x \in \mathbb{R}} f(x) \\ \int_{\mathbb{R}} f(x) \end{cases}$$

Fact: If I have some fn $g(x)$ and

$\boxed{g(x) \geq 0}$

cts: $\int_{\mathbb{R}} g(x)\,dx = c$ then $f(x) = \frac{1}{c} g(x)$ is a valid PDF.

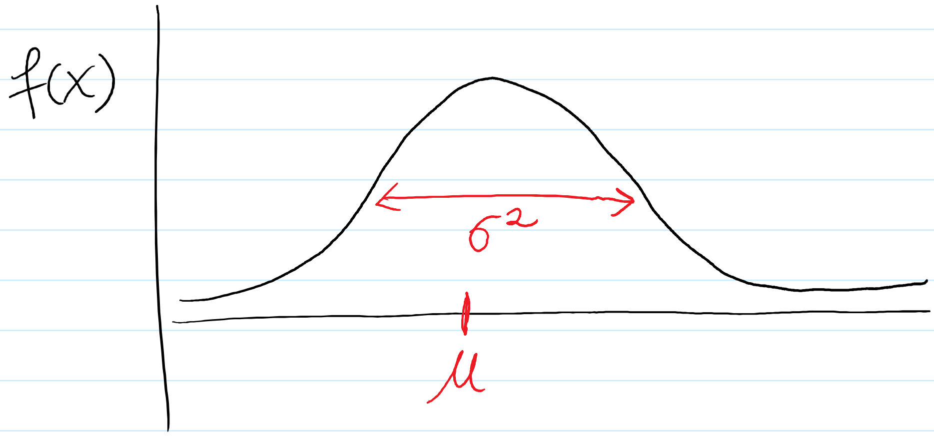
$$\int_{\mathbb{R}} f(x) = \frac{1}{c} \int g(x)\,dx = \frac{1}{c} c = 1$$

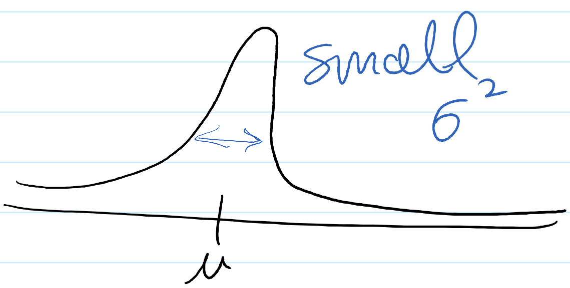
Ex. Normal Distribution / Gaussian

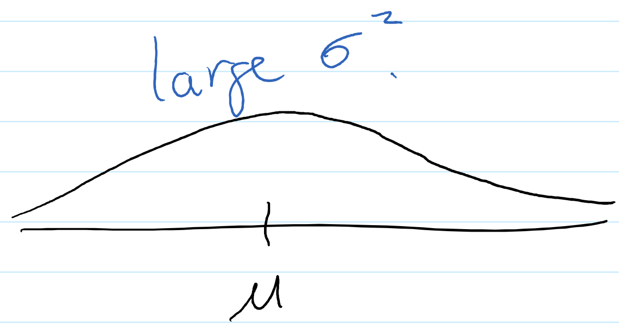
notation: $X \sim N(\mu, \sigma^2)$

mean: $\mathbb{R}$

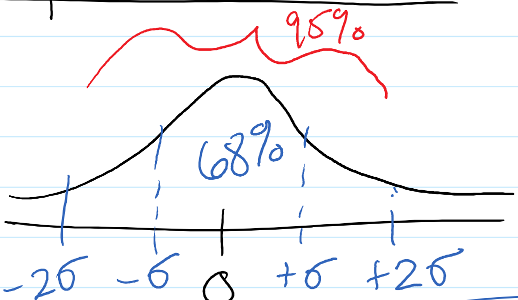
variance $\sigma^2 > 0$







Special Case: Standard Normal



$X \sim N(0, 1)$

$\mu = 0$ $\qquad \sigma^2 = 1$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)} \quad \forall x \in \mathbb{R}$$

≥ 0 $\qquad \geq 0$

$\exp(x) = e^x$

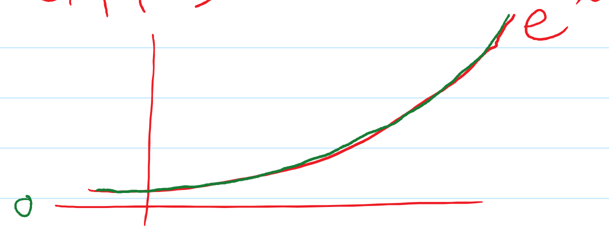
Q: Valid PDF?

Q: Valid PDF? ≥ 0 ✓

① $f(x) \geq 0$?

$\boxed{② \int_{\mathbb{R}} f(x)\,dx = 1 ?}$

$\overset{\geq 0}{\exp(x)} = e^x$



$$\underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx}_{I \,=\, \text{some real number} \,\geq\, 0} = 1$$

Want: $I = 1 \iff \boxed{I^2 = 1}$

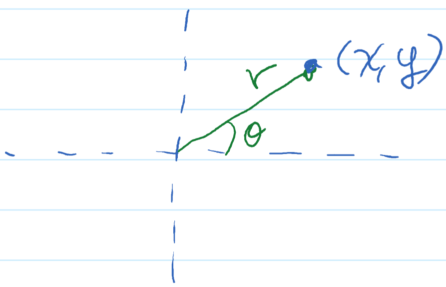
$$\left[\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx\right]\left[\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy\right] = I^2$$

Calc III

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi} \exp\left(-\frac{1}{2}x^2\right) \exp\left(-\frac{1}{2}y^2\right) dx\,dy$$

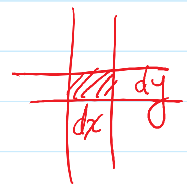
$$\boxed{= \int_{\mathbb{R}^2} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right) dx\,dy}$$

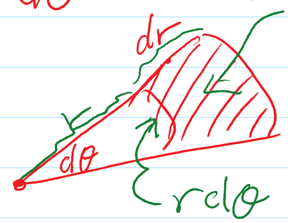
Recall: Polar Coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$$

$$dx\,dy = r\,dr\,d\theta$$





$$\frac{1}{2\pi}\int_0^\infty\int_0^\infty \exp\left(-\tfrac{1}{2}(x^2+y^2)\right)dx\,dy = \frac{1}{2\pi}\int_0^{2\pi}\underbrace{\int_0^\infty \exp\left(-\tfrac{1}{2}r^2\right)r\,dr}\,d\theta$$

u-substitution

$$u = \tfrac{1}{2}r^2$$

$$du = r\,dr$$

$$\int \exp\left(-\tfrac{1}{2}r^2\right)r\,dr = \int e^{-u}\,du = -e^{-u}$$

$$= \left[-e^{-\frac{1}{2}r^2}\right]_0^\infty = \left[0-(-1)\right] = 1$$

$$\frac{1}{2\pi}\int_0^{2\pi} 1\,d\theta = \frac{1}{2\pi}\,2\pi = 1 = I^2$$

hence $I = 1$ ...

Expected Value

Expected Value

If X is a r.v. then the mean or expected value of X,

denoted $\mathbb{E}[X]$ or μ

is defined as

PMF

1 discrete: $\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x f(x)$

$= \sum_{x \in \text{Support}} x f(x)$

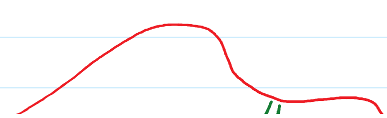
weighted sum of values X can attain weighted by the prob. we see them.

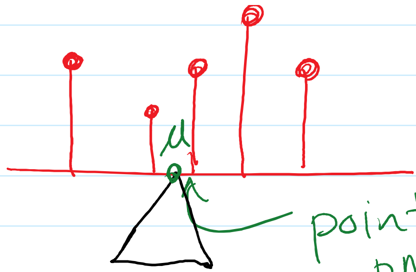
2 continuous: $\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) dx$.

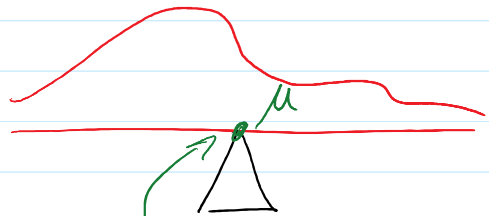
Interpretation: $\mathbb{E}[X]$ = balancing point of distribution of X.

discrete:

continuous:







point to balance PMF is

$\mu = \mathbb{E}[X]$.

discrete r.v.
on
$x_1, x_2, x_3 \dots$ support

$$\mathbb{E}[X] = x_1 \mathbb{P}(X = x_1) + x_2 \mathbb{P}(X = x_2) + x_3 \mathbb{P}(X = x_3)$$

Similarly

$$\mathbb{E}[X] = \int x f(x)\, dx$$

value · density