Lecture 15 - Transformations

$$\boxed{\underbrace{Y}_{Y: S \to \mathbb{R}} = \underbrace{g}_{g: \mathbb{R} \to \mathbb{R}}(\underbrace{X}_{X: S \to \mathbb{R}})}$$

$g: \mathbb{R} \to \mathbb{R}$

$X: S \to \mathbb{R}$

$Y: S \to \mathbb{R}$

really mean: $Y = g \circ X$

Formalism: $P(Y \in \underbrace{A}_{A \subset \mathbb{R}}) = P(g \circ X \in A)$

$$= P((g \circ X)^{-1}(A))$$

$$= P(X^{-1}(g^{-1}(A)))$$

$$= P(X \in g^{-1}(A))$$

$Y = g(X)$

$$\boxed{P(Y \in A) = P(g(X) \in A) = P(X \in g^{-1}(A))}$$

This lecture:

1. Know something about X
2. What do I know about $Y = g(X)$?

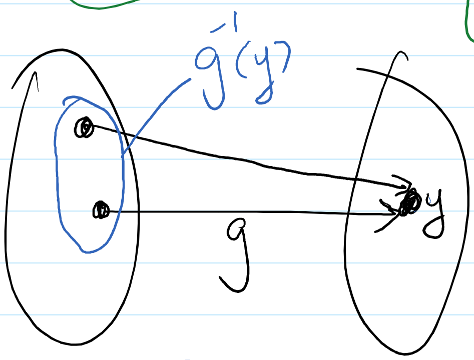
Discrete RVs. X is a discrete r.v.

→ Know f_X ← PMF of X

→ how do I get f_Y ← PMF of $Y = g(X)$?

Notice:

$$\boxed{f_Y(y) = P(Y=y) = P(g(X)=y) = P(X \in g^{-1}(\{y\})) = \sum_{x \in g^{-1}(y)} f_X(x)}$$



For $A \subset \mathbb{R}$,

$$P(X \in A) = \sum_{x \in A} f_X(x)$$

Recall:

g^{-1} = inverse image (typically a set)

Conventions:

① $g^{-1}(y)$ denotes $g^{-1}(\{y\})$ when set contains just y

② if g is invertible — g^{-1} denotes inverse

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① **Theorem:** If X is discrete then if $Y = g(X)$

$$\boxed{f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x)}$$

PMF of Y = sum over all x s.t. $g(x) = y$ of $P(X=x)$

Ex: Let $X \sim Bin(n, p)$ and $Y = n - X$

X: # of successes in n indep. trials w/ prob. p of success on each.

Y: # failures

$y = g(x) = n - x$
$\Updownarrow$
$x = g^{-1}(y) = n - y$

$\Big\}$ aside: get g^{-1}

1 item

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x) = \sum_{x = n-y} f_X(y) = f_X(n-y)$$

$$\boxed{P(Y = y) = f_Y(y) = f_X(n-y) = P(X = n-y)}$$

y failures

$n-y$ successes

$$= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$

why?

$$= \binom{n}{y} p^{n-y} (1-p)^{y}$$

let $q = 1-p$

$$= \underbrace{\binom{n}{y} q^{y} (1-q)^{n-y}}_{\text{PMF of a Bin}(n,\, q = 1-p)} = f_Y(y)$$

Hence $\boxed{Y \sim \text{Bin}(n, 1-p)}$

Move on to continuous RVs.

(2)

Move on to continuous RVs.

② **Theorem:** If X is a continuous random variable, then

① If g is an increasing function and $Y = g(X)$

then

(*) $$\boxed{F_Y(y) = F_X(g^{-1}(y))}$$

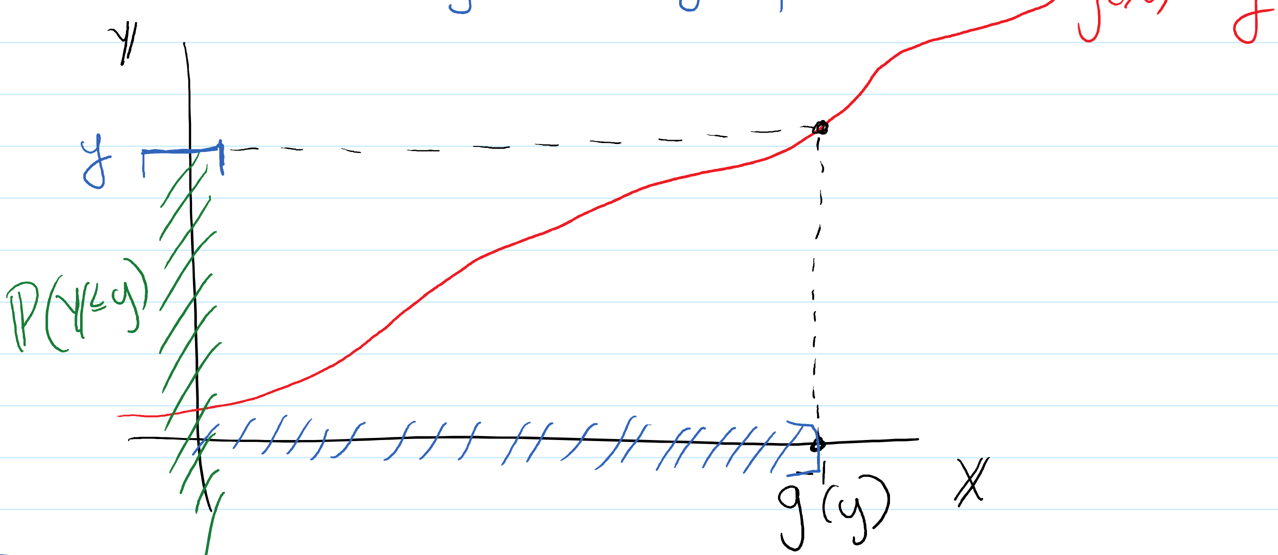
if g strictly increasing then it is invertible

② If g is a decreasing function and $Y = g(X)$

then

$$\boxed{F_Y(y) = 1 - F_X(g^{-1}(y)).}$$

Pf. Case 1: strictly increasing fn.

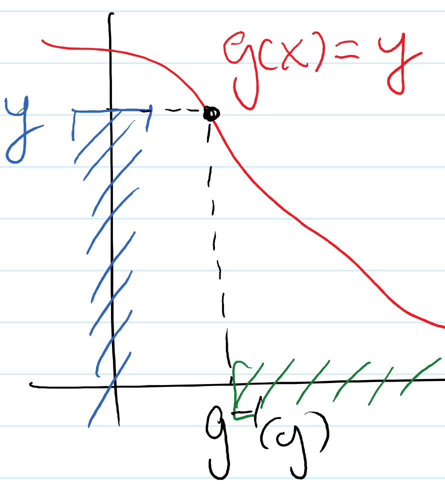


$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y))$$

$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y) = \mathbb{P}(X \le g^{-1}(y))$

$= F_X(g^{-1}(y))$

Case 2: Decreasing g



$F_Y(y) = \mathbb{P}(Y \le y)$

$= \mathbb{P}(g(X) \le y)$

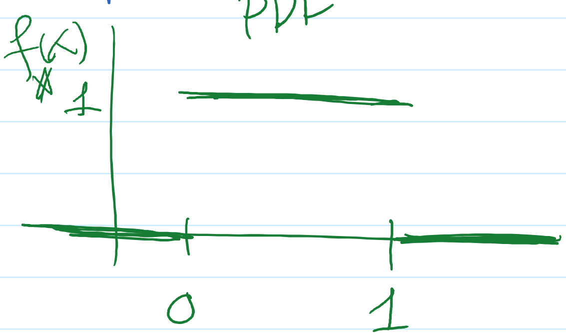
$= \mathbb{P}(X \ge g^{-1}(y))$ ← req. X continuous

$= 1 - \mathbb{P}(X \le g^{-1}(y))$

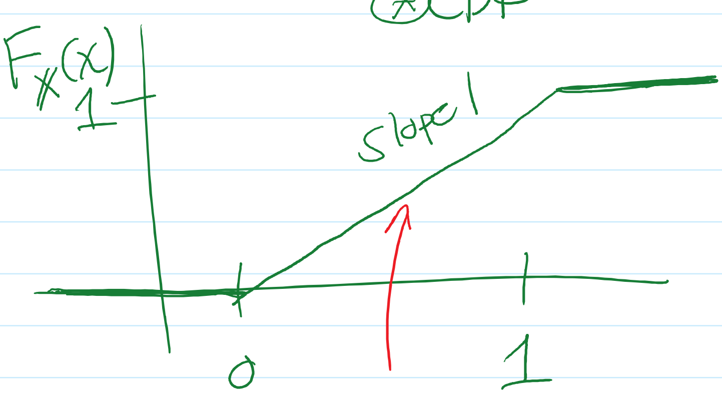
$= 1 - F_X(g^{-1}(y))$

Ex. $\boxed{X \sim U(0,1)}$

PDF

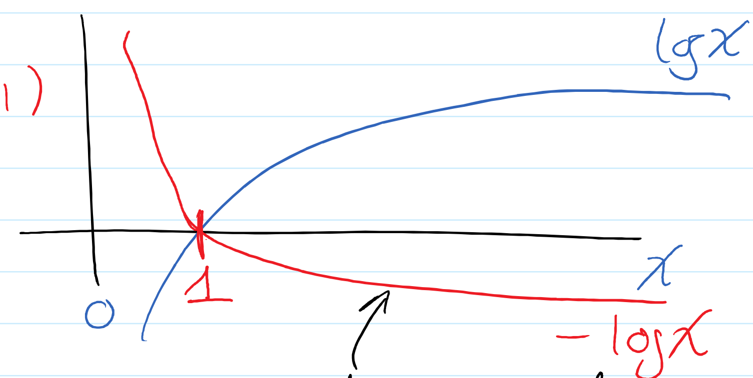


CDF



Let $Y = -\log X$ ≥ 0 ← $U(0,1)$

CDF of Y?



decreasing fn

$g(x) = -\log x$

$F_Y(y) = 1 - F_X(g^{-1}(y))$

$y(x) = -\log x$

$= 1 - F_X(e^{-y})$ $\qquad g^{-1}(y) = e^{-y}$

$= 1 - e^{-y}$ $\qquad \boxed{0 < e^{-y} < 1}$ for $y > 0$

Show $\qquad F_X(x) = x, \ 0 < x < 1$

$F_Y(y) = 1 - e^{-y}$

is the CDF of $Exp(\lambda = 1)$

hence $\boxed{Y \sim Exp(\lambda = 1)}$

What about PDFs?

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3 Theorem: If X is

1 a continuous RV.

and $Y = g(X)$, and

2 g is continuous, monotone (dec or inc.)

3 g^{-1} is differentiable

then,

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$

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pf. Case 1: g increasing then $\frac{dg^{-1}}{dy} > 0$

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$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

Chain-rule

$$= f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Case 2: g decreasing so $\frac{dg^{-1}}{dy} < 0$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y))\right]$$

$$= - f_X(g^{-1}(y)) \underbrace{\frac{dg^{-1}}{dy}}_{<0}$$

$$= f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Ex, $X \sim \text{Gamma}(\alpha = n, \lambda = 1/\beta)$

integer

some number > 0

$\Gamma(n) = (n-1)!$

means:

$$\rightarrow f_X(x) = \frac{1}{(n-1)!\, \beta^n} x^{n-1} e^{-x/\beta}, \quad \text{for } x > 0$$

Let $Y = 1/X$ equiv. $g(x) = 1/x$.

What is the PDF of Y?

1. X continuous
2. g decreasing
3. $g^{-1}(y) = 1/y$

Hence

Hence

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

③ $g^{-1}(y) = 1/y$ is differentiable, $\frac{dg^{-1}}{dy} = -1/y^2$.

$$= f_X(1/y) \, 1/y^2$$

$$= \frac{1}{(n-1)!\,\beta^n} \left(\frac{1}{y}\right)^{n-1} e^{-\frac{1}{y\beta}} \, 1/y^2$$

$$\boxed{= \frac{1}{(n-1)!\,\beta^n} \left(\frac{1}{y}\right)^{n+1} e^{-\frac{1}{y\beta}} \quad \text{for } y>0}$$

$Y \sim \text{Inverse Gamma}(n, \beta)$

What about non-monotone transformations?

OK so long as piecewise-monotone.

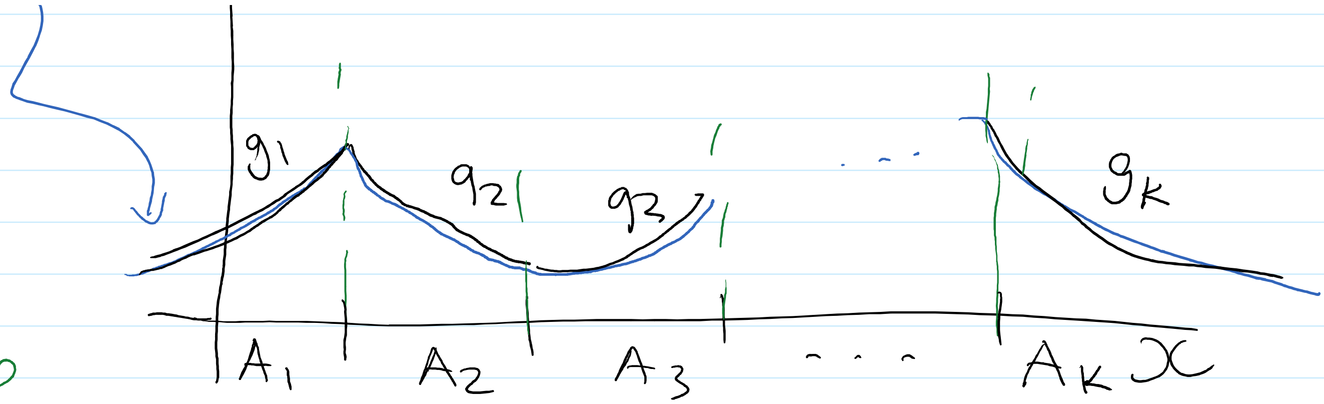
④ Theorem

Let X be continuous w/ support $\mathcal{X}$

and for $i = 1, \ldots, K$ let $\{A_i\}_{i=1}^K$ partition $\mathcal{X}$,

g monotone on each part of partition

let g_i be g on A_i



If

① the prev. theorem is satisfied for each g_i on A_i

② the image of A_i under g_i is the same $\forall i$

then

$$\boxed{\text{if } Y = g(X) \text{ then } \quad f_Y(y) = \sum_{i=1}^{K} \underbrace{f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}}{dy} \right|}_{\text{prev. theorem applied to } g_i}}$$

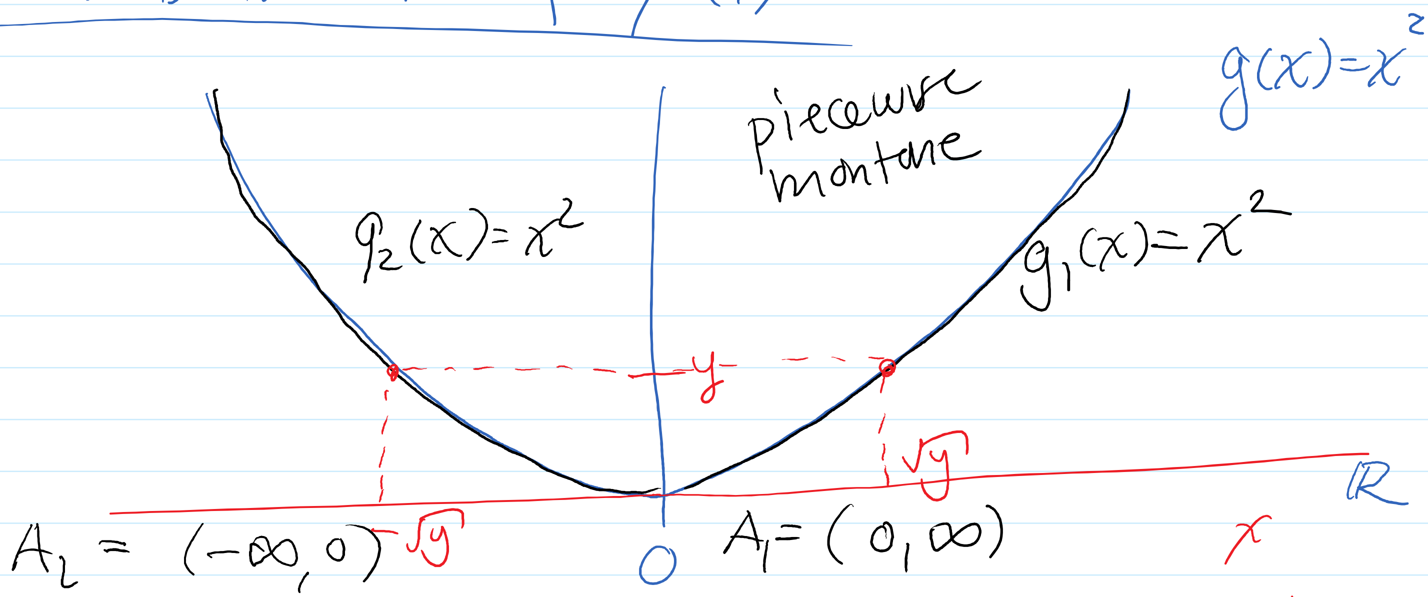
Ex. Chi-Squared Distribution.

If $X \sim N(0,1)$ and $Y = X^2$ $(g(x) = x^2)$

then we say Y has a Chi-Squared distribution

(w/ 1 degree of freedom)

What is the PDF of $\chi^2(1)$?



$$\begin{cases} A_1 = (0,\infty); \; g_1(x) = x^2; \; g_1^{-1}(y) = \sqrt{x}; \; \dfrac{dg_1^{-1}}{dy} = \dfrac{1}{2\sqrt{y}} \\ A_2 = (-\infty, 0); \; g_2(x) = x^2; \; g_2^{-1}(y) = -\sqrt{x}; \; \dfrac{dg_2^{-1}}{dy} = \dfrac{-1}{2\sqrt{y}} \end{cases}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}x^2\right)$$

normal PDF w/ $\mu = 0$, $\sigma^2 = 1$.

$$f_Y(y) = \sum_{i=1,2} f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}}{dy} \right|$$

$$= f_X(g_1^{-1}(y)) \left| \frac{dg_1^{-1}}{dy} \right| + f_X(g_2^{-1}(y)) \left| \frac{dg_2^{-1}}{dy} \right|$$

$$= f_X(\sqrt{y})\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\underbrace{\sqrt{y}^2}_{|y|=y>0}\Big) + \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\underbrace{(-\sqrt{y})^2}_{y}\Big)\frac{1}{2\sqrt{y}}$$

$$\boxed{= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}}\exp\left(-\frac{1}{2}y\right)}$$

PDF of a $\chi^2(1)$.

Probability Integral Transformation

If X is a cts r.v. w/ CDF F_X, then

$$Y = F_X(X) \sim U(0,1).$$

is just some g.

Pf: Assume that F_X is strictly increasing so that F_X is invertible.

Then $Y = F_X(X)$ and so by our CDF theorem

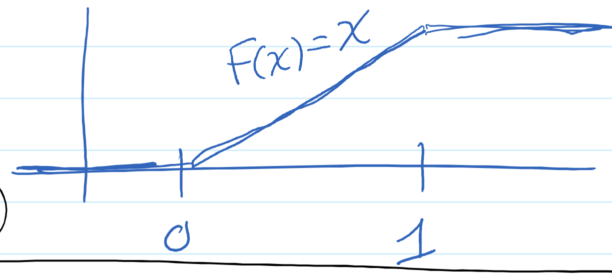
g

$$F_Y(y) = F_X(g^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

So the CDF of Y is the CDF of a $U(0,1)$

hence $Y \sim U(0,1)$.

CDF of a uniform:



IFF $g(X) \sim U(0,1) \iff g = F_X$.

Can use to generate RVs.

idea: Want to generate $X \sim F_X$

1. generate $Y \sim U(0,1)$
2. Let $X = F_X^{-1}(Y)$

then $X \sim F_X$.

why?
$$\begin{aligned} F_X(x) &= \mathbb{P}(X \le x) \\ &= \mathbb{P}(F_X^{-1}(Y) \le x) \\ &= \mathbb{P}(Y \le F_X(x)) \\ &= F_Y(F_X(x)) \\ &= F_X(x) \end{aligned}$$

$U(0,1)$