Ex. $U = X + Y$

$V = X - Y$

$$\boxed{f_{U,V}(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)\frac{1}{2}}$$

Assume $X, Y \overset{iid}{\sim} N(0,1)$

independent and identically dist.

$X \sim N(0,1)$, $Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$\boxed{f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u+v}{2}\right)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-v}{2}\right)^2} \frac{1}{2}}$$

$$\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2$$

$$= \frac{1}{4}(u^2 + 2uv - v^2) + \frac{1}{4}(u^2 - 2uv + v^2)$$

$$= \frac{1}{2}(u^2 + v^2)$$

$$= \frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2}\left(\frac{1}{2}u^2 + v^2\right)} \frac{1}{\sqrt{2\pi \cdot 2}}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2}\frac{1}{2}u^2}}_{\substack{h(u) \\ N(0,2)}} \underbrace{\frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2}\frac{1}{2}v^2}}_{\substack{g(v) \\ N(0,2)}}$$

So $U \perp\!\!\!\perp V$ and $U, V \sim N(0,2)$.

Recap: $X, Y \overset{iid}{\sim} N(0,1)$

$X \pm Y \overset{iid}{\sim} N(0,2)$

Theorem: If $X \perp\!\!\!\perp Y$ and $X \sim N(\mu, \sigma^2)$

$Y \sim N(\lambda, \tau^2)$

then $X \pm Y$ are independent

$X \pm Y \sim N(\mu \pm \lambda, \sigma^2 + \tau^2)$

Theorem: Independence and Trasformations

If $X \perp\!\!\!\perp Y$ and g, h are fns $\mathbb{R} \to \mathbb{R}$

and

$U = g(X) \longleftarrow$ U fn of X only

$V = h(Y) \longleftarrow$ V fn of Y only

then $U \perp\!\!\!\perp V$.

Ex. $U = X^2$ and $V = -\log Y$

then if $X \perp\!\!\!\perp Y$, $U \perp\!\!\!\perp V$.

---

pf. $F_{U,V}(u,v) = P(U \le u, V \le v)$

recall: $U = g(X)$
$V = h(Y)$

$$= P(g(X) \le u,\ h(Y) \le v)$$

$$= P(X \in g^{-1}((-\infty, u]),\ Y \in h^{-1}((-\infty, v]))$$

if $X \perp\!\!\!\perp Y$

$$= P(X \in g^{-1}((-\infty, u]))\, P(Y \in h^{-1}((-\infty, v]))$$

$$= P(g(X) \le u)\, P(h(Y) \le v)$$

$$= P(U \le u)\, P(V \le v)$$

$$= F_U(u)\, F_V(v).$$

So $U \perp\!\!\!\perp V$.

Ex. $X \sim \text{Beta}(\alpha, \beta)$ and $Y \sim \text{Beta}(\alpha+\beta, \gamma)$

and $X \perp\!\!\!\perp Y$.

Consider: $\boxed{U = XY}$ and $V = X$

what is the joint PDF of U and V?

notice: $0 \le X, Y \le 1$ and so $0 \le U \le V \le 1$

to get the inverse transformation

$$u = xy \quad \text{and} \quad \boxed{v = x}$$

$$\frac{y}{v} = \frac{u}{x} = y \Rightarrow \boxed{y = u/v}$$

$x = g_1^{-1}(u,v) = v$ and $y = g_2^{-1}(u,v) = u/v$.

Need: $|\det J|$ $\quad \frac{\partial g^{-1}}{\partial(u,v)}$

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{bmatrix}$$

$$|\det(J)| = |1/v| = 1/v$$

$$\boxed{f_{U,V}(u,v) = f_{X,Y}(\overbrace{g_1^{-1}(u,v)}^{v}, \overbrace{g_2^{-1}(u,v)}^{u/v}) \, \overbrace{|\det J|}^{1/v}}$$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \frac{y^{\alpha+\beta-1}(1-y)^{\gamma-1}}{B(\alpha+\beta,\gamma)}$$

$$= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \frac{y^{\alpha+\beta-1}(1-y)^{\gamma-1}}{B(\alpha+\beta,\gamma)}$$

$$\boxed{= \frac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha,\beta)} \frac{(u/v)^{\alpha+\beta-1}(1-u/v)^{\gamma-1}}{B(\alpha+\beta,\gamma)}.}$$

Theorem: Non-Invertible

As long as we can break g into chunks that are invertible, we're ok.

Let $A = \text{Support}(X,Y) \subset \mathbb{R}^2$

and A partitioned into $(A_k)_{k=1}^{K}$

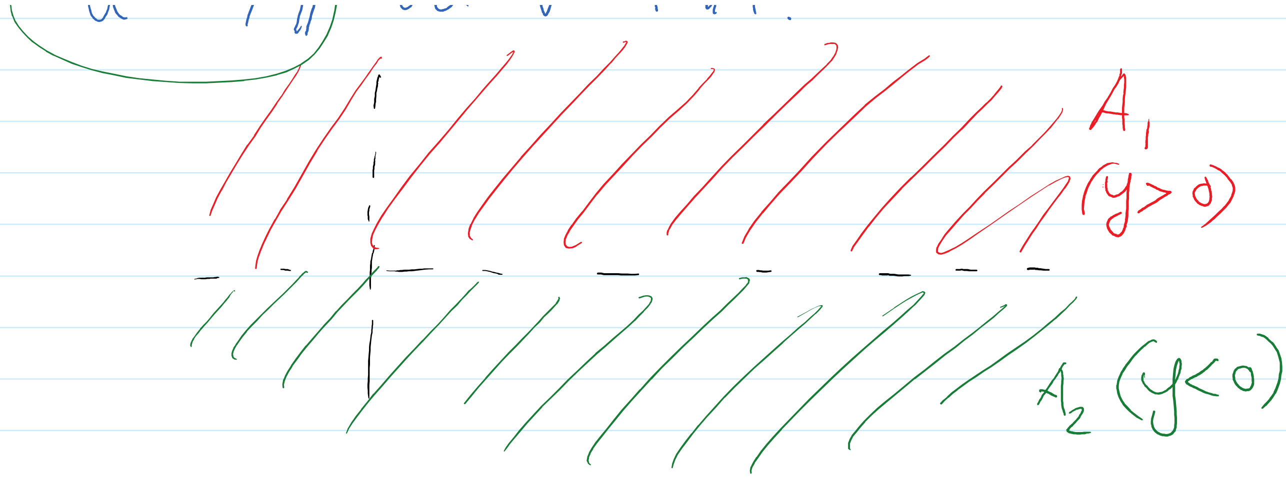
and on A_k $(U,V) = (g_1^{(k)}(X,Y), g_2^{(k)}(X,Y))$

assume $g_1^{(k)}$ and $g_2^{(k)}$ are invertible, then

$$\boxed{f_{U,V}(u,v) = \sum_{k=1}^{K} f_{X,Y}\left(g_1^{(k)^{-1}}(u,v), g_2^{(k)^{-1}}(u,v)\right) \left| \det \frac{\partial g^{(k)^{-1}}}{\partial(u,v)} \right|}$$

Ex. $X, Y \overset{iid}{\sim} N(0,1)$

$U = X/Y$ and $V = |Y|$.



On A_1 $\quad u = x/y$ and $V = |y| = y$

$$A_1 = \{(x,y) \mid y > 0\}$$

$u = x/y$ and $\boxed{v = y}$

then $x = uy = uv \Rightarrow \boxed{x = uv}$

$$\boxed{g_2^{(1)-1}(u,v) = v}$$

$$\boxed{g_1^{(1)-1}(u,v) = uv}$$

$$J = \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix} \quad \text{so } |\det J| = v$$

On A_2: $\quad A_2 = \{(x,y) \mid y < 0\}$

Can show: $\quad g_1^{(2)-1}(u,v) = -uv$

$$g_2^{(2)-1}(u,v) = -v$$

and $|\det J| = v$

Combine Chunks:

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

So

$$f_{U,V}(u,v) = f_{X,Y}\left(g_1^{(1)-1}(u,v), g_2^{(1)-1}(u,v)\right) v + f_{X,Y}\left(g_1^{(2)-1}(u,v), g_2^{(2)-1}(u,v)\right) v$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(uv)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}}\, v + \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-uv)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-v)^2}}\, v$$

$$= \frac{\cancel{2}}{\cancel{2}\pi} e^{-\frac{1}{2}(u^2v^2+v^2)} v$$

$$\boxed{= \frac{v}{\pi} \exp\left(-\frac{1}{2}v^2(1+u^2)\right)}$$

What is the marginal of $U = X/Y$.

$$\underline{f(u)} = \int f(u,v)\,dv = \int_0^\infty \frac{v}{\pi} e^{-\frac{1}{2}v^2(1+u^2)} dv$$

$\beta = 1+u^2$

$$= \frac{1}{\pi}\int_0^\infty v e^{-\frac{1}{2}\beta v^2} dv$$

let $w = \frac{1}{2}v^2\beta$

$$= \frac{1}{\;} \int^\infty \frac{1}{\;} e^{-w}\, dw$$

let $w = \frac{1}{2} v \beta$
$dw = \beta v \, dv$

$$= \frac{1}{\pi} \int_0 \frac{1}{\beta} e^{-w} \, dw$$

$$= \frac{1}{\pi \beta} \int_0^{\infty} e^{-w} \, dw$$

$$= \frac{1}{\pi \beta} = \boxed{\frac{1}{\pi} \frac{1}{1+u^2}}$$

Cauchy Distribution

marginal pdf of U

Ex. $X \sim \text{Gamma}(\alpha, \lambda)$ $X \perp Y$
$Y \sim \text{Gamma}(\beta, \lambda)$

$\boxed{U = X + Y}$ and $\boxed{V = \frac{X}{X+Y}}$.

① get inverse

$u = x + y$ and $v = \frac{x}{x+y} = x/u$

$\Rightarrow \boxed{x = uv} = g_1^{-1}(u,v)$

$u = uv + y$

So $\boxed{y = u(1-v)} = g_2^{-1}(u,v)$.

$$J = \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} \Rightarrow |\det J| = |-uv - u(1-v)|$$
$$\boxed{= u}$$

(2) Plug in to joint of X, Y.

$X \sim Gama(\alpha, \lambda)$
$Y \sim Gama(\beta, \lambda)$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$
$$= \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \frac{\lambda^\beta y^{\beta-1} e^{-\lambda y}}{\Gamma(\beta)}$$

$$f(u,v) = f_{X,Y}(uv, u(1-v))\, u$$
$$\boxed{= \frac{\lambda^\alpha (uv)^{\alpha-1} e^{-\lambda(uv)}}{\Gamma(\alpha)} \frac{\lambda^\beta (u(1-v))^{\beta-1} e^{-u(1-v)\lambda}}{\Gamma(\beta)} u}$$

$= \dots$ algebra

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{u^{\alpha+\beta-1} e^{-\lambda u}}_{\text{only } u} \underbrace{v^{\alpha-1}(1-v)^{\beta-1}}_{\text{only } v}$$

prop. to PDF of   prop. to PDF of

prop. to PDF of Gamma$(\alpha+\beta, \lambda)$ prop. to PDF of Beta(α, β)

So

$$\boxed{\begin{array}{l} U \perp\!\!\!\perp V \\ U \sim \text{Gamma}(\alpha+\beta, \lambda) \\ V \sim \text{Beta}(\alpha, \beta) \end{array}}$$

$X+Y$

$\frac{X}{X+Y}$