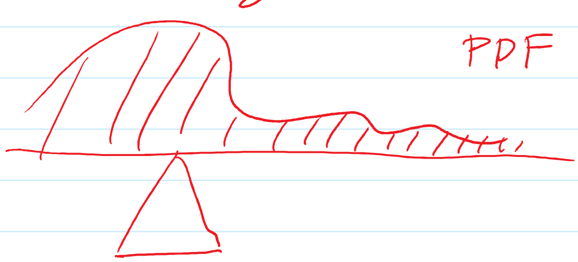
Expectation:

If X is a r.v. then the **expected value** or **mean** is defined as

① **discrete:** $\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x f(x)$ ← PMF

② **continuous:** $\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\, dx$ ← PDF

interpret: balancing pt.



Ex. Let $X \sim \text{Exp}(\lambda)$

$\lambda > 0$ parameter (rate)

exponential distribution (continuous)

means

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$

Q: What is the expectation of X?

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\infty} x f(x)\,dx = \int_{0}^{\infty} x \lambda e^{-\lambda x}\,dx$$

Integration by Parts

$u = x \qquad v = -e^{-\lambda x}$

$du = dx \qquad dv = \lambda e^{-\lambda x}\,dx$

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^\infty x\lambda e^{-\lambda x}\,dx = \int_0^\infty u\,dv = [uv]_0^\infty - \int_0^\infty v\,du$$

$$= \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$

$$= \left[-xe^{-\lambda x}\right]_0^\infty - \left[\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty$$

$$[0 - 0] - \left[0 - \frac{1}{\lambda}\right] = \boxed{1/\lambda} = \mathbb{E}[X].$$

$x \to \infty$ $\quad$ L'Hopital's Rule

$e^{-\lambda x} \to 0 \rightsquigarrow xe^{-\lambda x} \to 0$

Ex. $X \sim Bin(n, p)$

integer > 0

probability $p \in [0, 1]$

Binomial distribution
(discrete distribution)

Canonical experiment

If I flip a coin n times (independently) and the prob. of getting a H on any flip is p then

X = # heads among my n flips

then $X \sim Bin(n,p)$

Saw previously:

prob. of getting exactly x out of n heads

$$\rightarrow f(x) = \mathbb{P}(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\boxed{\text{Justify: } \sum_{x=0}^{n} f(x) = 1}$$

Binomial Theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

hint: $(p+(1-p))^n$

$x=0$
$xf(x)=0$

$$E[X] = \sum_{x=\cancel{0}1}^{n} x f(x) = \sum_{x=1}^{n} x f(x)$$

$$\boxed{x\binom{n}{x}} = \cancel{x}\frac{n!}{\cancel{x!}(n-x)!} \quad (x-1)!$$

$$= \frac{n!}{(x-1)!(n-x)!}$$

$$= \sum_{x=1}^{n} x\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} n\binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$= \sum^{n-1} n\binom{n-1}{y} p^{y+1} (1-p)^{n-(y+1)} \quad y = x-1$$

$x = y+1$

$$= \sum_{y=0}^{n-1} n \binom{n-1}{y} p^{y+1} (1-p)^{n-(y+1)} \quad y = x-1, \; x = y+1$$

$$= n \frac{(n-1)!}{(x-1)!((n-1)-(x-1))!}$$

$$\boxed{= n \binom{n-1}{x-1}}$$

$$= np \boxed{\sum_{y=0}^{n-1} \binom{n-1}{y} p^{y} (1-p)^{(n-1)-y}}$$

Sum of PMF of $Bin(n-1, p) \to 1$

$Y \sim Bin(n-1, p)$

$$\sum_{y=0}^{n-1} \binom{n-1}{y} p^{y} (1-p)^{(n-1)-y} = 1$$

$$\boxed{= np} = E[X]$$

flips

prob. of H on any flip

General trick! PMF/PDF trick

Often recognize in a calculation either

$$\sum_{x \in \mathbb{R}} f(x) = 1 \quad \text{or} \quad \int_{\mathbb{R}} f(x)\,dx = 1$$

Functions of RVs

Note: a function g applied to a RV X to form $\boxed{g(X)}$ is a RV.

Ex. If X is # of H, X^2 = # heads sqared

$\log(X)$ = log of # heads

Any fn of a RV is also a RV.

$$S \xrightarrow{X} \mathbb{R} \qquad g \circ X$$

$$S \xrightarrow{g(X)} \mathbb{R} \quad \text{b/c} \quad S \xrightarrow{X} \mathbb{R} \xrightarrow{g} \mathbb{R}$$

Theorem: The Law of the Unconscious Statistician

If $g: \mathbb{R} \to \mathbb{R}$ and X is a r.v. then

$$E[g(X)] = \begin{cases} \sum_x g(x) f(x) & \text{(discrete)} \\ \int g(x) f(x)\, dx & \text{(continuous)} \end{cases}$$

Ex. $X \sim \text{Exp}(\lambda)$; $\lambda > 0$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

Recall: $E[X] = 1/\lambda$.

$$\int_{\mathbb{R}} f(x)\, dx = 1 \quad \text{so} \quad \int \lambda e^{-\lambda x} dx = 1.$$

Q: $\mathbb{E}[X^2]$? here: $g(x) = x^2$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx$$

<u>by parts</u>: $u = x^2 \qquad v = -e^{-\lambda x}$

$du = 2x\,dx \qquad dv = \lambda e^{-\lambda x}\,dx$

$$= \int_0^\infty u\,dv = uv - \int v\,du$$

$$= \left[-x^2 e^{-\lambda x}\right]_0^\infty + \int e^{-\lambda x}\, 2x\,dx$$

$$= 0 + \frac{2}{\lambda} \underbrace{\int_0^\infty x \underbrace{\lambda e^{-\lambda x}}_{f(x)} dx}_{\mathbb{E}[X]}$$

$$= \frac{2}{\lambda}\frac{1}{\lambda}$$

$$\boxed{= 2/\lambda^2 = \mathbb{E}[X^2]}$$

<u>Notice</u>:

$$\mathbb{E}[X^2] = \frac{2}{\lambda^2} \neq \frac{1}{\lambda^2} = \mathbb{E}[X]^2$$

<u>Aside</u>: recall improper integrals

<u>ex.</u> $\int_0^\infty \frac{1}{x^2}\,dx$ has a finite answer

like $\sum_{k=1}^\infty 1/k^2 < \infty$ converges

however $\int_0^\infty$ ... ∞

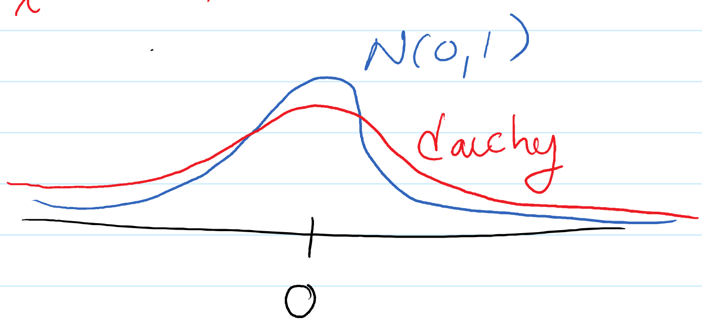
however $\boxed{\int_0^\infty \frac{1}{x}\,dx}$ doesn't converge $\iff \sum_{k=1}^{\infty} \frac{1}{k}$ doesn't converge.

Does expected value always exist?
(have a finite well-behaved value?)

No.

Ex. Cauchy Distribution

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$



$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\,dx$$

looks like (asymptotically)

$$\frac{x}{1+x^2} \sim \frac{1}{x} \quad \text{behaves like } \int_{-\infty}^{\infty} \frac{1}{x}\,dx$$

doesn't converge.

No expected value.

Theorem: Properties of Expectation.

① Expectation is linear:

$$\boxed{E[aX+b] = aE[X]+b.}$$

pf. (cts)

integration is linear

$$\int (ax+b)f(x)\,dx = a\int x f(x)\,dx + b\underbrace{\int f(x)\,dx}_{1}$$

$$= aE[X]+b$$

$\boxed{② \text{ If } X \geq 0 \text{ then } E[X] \geq 0}$

support of X is non-neg

pf. (cts)

$$E[X] = \int_0^\infty \underbrace{x}_{\geq 0}\underbrace{f(x)}_{\geq 0}\,dx$$

integral of a pos fn is ≥ 0

③ If g_1 and g_2 are functions then

→ (i) $E[g_1(X)+g_2(X)] = E[g_1(X)]+E[g_2(X)]$

→ (ii) If $g_1(x) \leq g_2(x)$ then

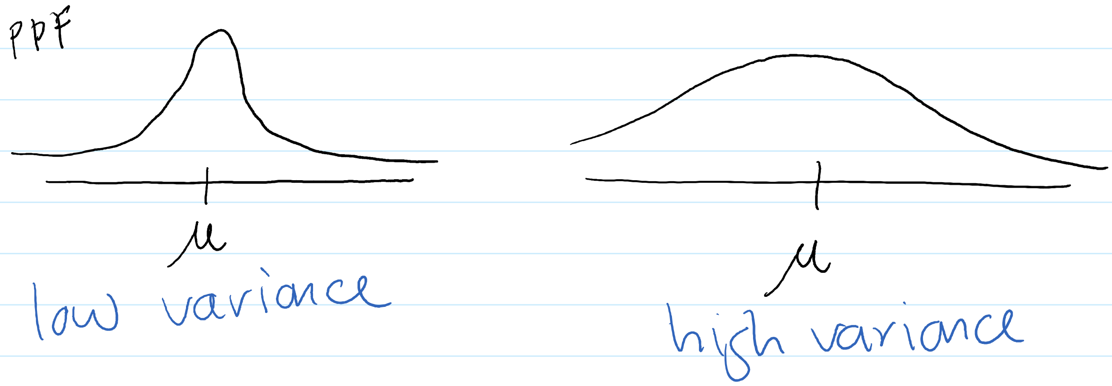
$$E[g_1(X)] \leq E[g_2(X)].$$

pf. applying ①, ②' Law of Unconscious Stat.

④ If $a \le X \le b$ then $a \le \mathbb{E}[X] \le b$.

pf. apply ② twice

Defn: Variance



Variance ≈ how spread dist. is around the mean.

The variance of a r.v. X is defined as

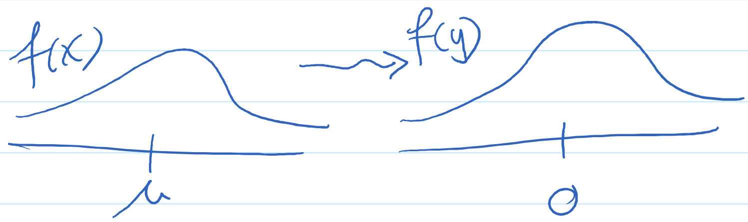
$$\boxed{\mathrm{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right]}$$

expected squared deviation from mean

Other ways:

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \overbrace{\mathbb{E}[X]}^{\mu \in \mathbb{R}})^2\right]$$

Note: $Y = X - \mu$



$$\mathbb{E}[Y] = \mathbb{E}[X-\mu] = \mathbb{E}[X] - \mu = \mu - \mu = 0.$$

Ex. $X \sim \text{Exp}(\lambda)$

Recall: $\underbrace{\mathbb{E}[X]}_{\mu =} = 1/\lambda$ and $\mathbb{E}[X^2] = 2/\lambda^2$.

$$\text{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right] = \int_0^\infty (x-\mu)^2 f(x)\,dx$$

$$= \int_0^\infty (x^2 - 2x\mu + \mu^2) f(x)\,dx$$

$$= \underbrace{\int_0^\infty x^2 f(x)\,dx}_{\mathbb{E}[X^2]} - 2\mu \underbrace{\int_0^\infty x f(x)\,dx}_{\mathbb{E}[X]} + \mu^2 \underbrace{\int_0^\infty f(x)\,dx}_{1}$$

$$= 2/\lambda^2 - 2\frac{1}{\lambda}\frac{1}{\lambda} + \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{2}{\lambda^2} + \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2} = \text{Var}(X)}$$

all we needed was $\mathbb{E}[X^2]$ and $\mathbb{E}[X]$.

This is generally true.

Theorem: Short-Cut formula for Variance.

$$\boxed{Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.}$$

pf.

$$Var(X) = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

linearity of expectation

$$= \mathbb{E}[X^2] - 2\mu\,\mathbb{E}[X] + \mu^2$$

$$= \mathbb{E}[X^2] - 2\,\mathbb{E}[X]^2 + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

exp. of sq. — sq. of exp.

Ex. As above. $X \sim Exp(\lambda)$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= 2/\lambda^2 - \left(\frac{1}{\lambda}\right)^2$$

$$= 2/\lambda^2 - 1/\lambda^2 \boxed{= 1/\lambda^2}$$

Theorem:

Theorem:

$$\boxed{Var(aX+b) = a^2 Var(X)}$$

1. multiply $\rightarrow$ square coeff. $Var(aX) = a^2 Var(X)$
2. add constants $\rightarrow$ no change $Var(X+b) = Var(X)$

pf.

$$Var(aX+b) = \mathbb{E}[(aX+b)^2] - \mathbb{E}[aX+b]^2$$

$$= \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\mathbb{E}[X] + b)^2$$

$$= a^2 E[X^2] + \cancel{2ab E[X]} + \cancel{b^2} - (a^2 E[X]^2 + \cancel{2ab E[X]} + \cancel{b^2})$$

$$= a^2 \mathbb{E}[X^2] - a^2 \mathbb{E}[X]^2$$

$$= a^2 Var(X^2)$$ short-cut.