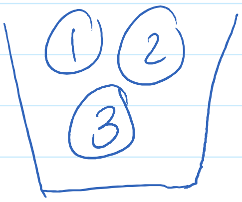
Lecture 5 - Unordered Counting

	w/ repl.	w/o repl.
ordered	n^r	$\frac{n!}{(n-r)!}$
unordered	2	1 $\binom{n}{r}$

Sample r object from n

Ex. Sampling w/o repl. and unordered



draw $r=2$ from $n=3$

If order matters

(1, 2) (1, 3) (2, 3)
(2, 1) (3, 1) (3, 2)

6 possibilities

$$\frac{n!}{(n-r)!} = \frac{3!}{(3-2)!} = 3! = 6$$

If order doesn't matter

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$

3 possibilities

reverse: each unordered sample can be used to create 2 ordered

Samples

$$\{1,2\} \to \left.\begin{matrix}(1,2)\\(2,1)\end{matrix}\right\} \; 2! = 2$$

Ex.

$$\{1,2,3,\dots,r\} \to \left.\begin{matrix}(2,1,\dots,r)\\(3,1,2,\dots,r)\\ \vdots\end{matrix}\right\} \; r!$$

General rule: (w/o repl.)

$$(\#\text{ ordered samples}) = (\#\text{ unordered samples})\, r!$$

$$\to \frac{n!}{(n-r)!} = (\#\text{ unordered})\, r!$$

$$\to (\#\text{ unord samples}) = \boxed{\frac{n!}{(n-r)!\,r!}}$$

Theorem: Sample r items from a total of n w/o replacement and w/o ordering.

This can be done read: "n choose r"

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} \quad \text{ways.}$$

binomial coefficient

(shows up as coefficients $(x+y)^n$)
See: Binomial Theorem.

Ex. I have $10^{=n}$ professors, how many co-equal (unordered) committes of size $4^{=r}$ can I create?

ans: $\binom{10}{4} = \frac{10!}{(10-4)!\,4!} = \frac{10!}{6!\,4!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot \cancel{6!}}{\cancel{6!}\,4!}$

$= \frac{10\cdot 9\cdot 8\cdot 7}{4!}$

$= \frac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}}$

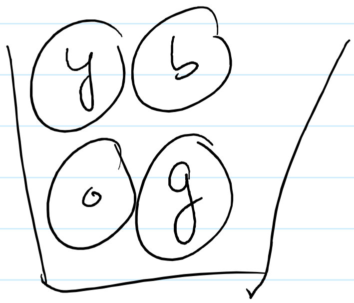
$= 10\cdot 3\cdot 7 = 210.$

Ex. How many 5-card poker hands can I possibly get?
unordered
w/o replacement

Think of as choosing $5^{=r}$ cards from $n=52$.

$\binom{52}{5} = \frac{52!}{5!\,(52-5)!} \approx 2.5$ mill

Ex. I have a Jar w/ a yellow, blue, orange, and green marble.
$n=4$



I choose (w/o replacement) $3 = r$ marbles (w/ all choices being equally likely) what is the prob. I have a yellow and blue in my selection?

$$P(E) = \frac{|E|}{|S|}$$

$S = \{$all 3-samples from 4 w/o repl. or ordering$\}$

$= \{ybo, bgo, gby, yog\}$

$$|S| = \binom{n}{r} = \binom{4}{3} = \frac{4!}{(4-3)!\,3!} = \frac{4!}{1!\,3!} = \frac{4\cdot\cancel{3!}}{\cancel{3!}}$$

$$= 4$$

$E = \{$sample contains y and b$\}$

$= \{ybo, ybg\}$

$|E| = 2$

hence $P(E) = 2/4 = 1/2$.

Last case: sampling $\boxed{\text{w/ replacement}}$ and w/o ordering

Ex, tempation:

$$\cancel{(\#\text{ ordered}) = (\#\text{ unordered})\, r!}$$

Consider: $n=3$, $r=2$

Ordered samples (w/ replacement)

(1) (2) (3)

$(1,1)$ $(2,1)$ $(3,1)$ $(2,3)$ $(3,3)$

$(1,2)$ $(1,3)$ $(2,2)$ $(3,2)$] 9 total

Last time: $n^r = 3^2 = 9$ ✓

$$\frac{9 \text{ total}}{\cancel{r!}} \qquad 2! = 2$$

Unordered: $r! = 2! = 2$

$\{1,1\}$, $\{1,2\}$, $\{1,3\}$, $\{2,2\}$, $\{2,3\}$, $\{3,3\}$] 6

Game of Partitioning

Ex, $n=5$, $r=2$

How many ways can I partition $r = 2$ objects using $n-1 = 4$ walls

1-1 corresp b/c these "partitions" and samples w/ repl. w/o ordering

option 1: | O || O |

0 1 0 1 0 $\Longleftrightarrow \{2,4\}$

opt 2: | O | O ||

0 1 1 0 0 $\Longleftrightarrow \{2,3\}$

Unord. Sample of size $r=2$

opt 2:

0 | 1 | 1 | 0 | 0 $\Leftrightarrow \{2,3\}$

opt 3: || O O ||

0 | 0 | 2 | 0 | 0 $\Leftrightarrow \{3,3\}$

... of size $r=2$ from $n=5$ w/ repl.

Argue: # samples of size r from n w/o ord and w/ replacement ??

= # partions I can draw using r objects and $n-1$ walls.

how to count

I have r objects and $n-1$ walls.

In total I have $\boxed{r+n-1}$ things.

Each "partion" is a permutation of these $r+n-1$ things.

$n=9$ $r=6$

opt 1: ||| OO | OO || O | O | $\leftarrow$ $\{4,4,5,5,7,8\}$

0 | 0 | 0 | 2 | 2 | 0 | 1 | 1 | 0

Some permutation of r objects and $n-1$ walls

opt. 2 $\boxed{| O || O | OO || O | O |}$ $\leftarrow$ encode a r sample from n w/ repl. w/o ord.

How many partions can I draw?

$$(r+n-1)!$$

Can switch any walls $^{(n-1)!}$ or objects $^{r!}$ w/ each other.

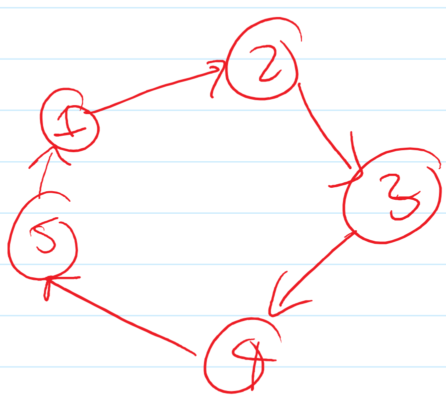
I must divide by $(n-1)!\,r!$

<u>ans:</u> $$\boxed{\frac{(r+n-1)!}{(n-1)!\,r!}} = \binom{r+n-1}{r}.$$

<u>Theorem:</u> The number of ways to draw a sample of r from n w/o ordering and w/ replacement is

$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1}.$$

<u>Ex.</u> 10 passengers on a bus route w/ 5 hotels. The bus driver counts how many get off at each of the hotels. <u>Q:</u> <u>how many</u> possible records are there?



<u>record</u>

hotel	# people
1	0
2	3
3	1
4	2
5	4

$\Longleftrightarrow \{2,2,2,3,4,4,5,5,5,5\}$

Convert to counting w/ repl. w/o ordering — of $r=10$ from $n=5$.

$$\binom{r+n-1}{r} = \binom{10+5-1}{10} = \binom{14}{10} = \dots = 1001$$

Ex. Jar contains yellow, blue, orange, green marbles. $n=4$

Draw a sample of size 3 $=r$ w/ replacement.

Stipulate: all unordered samples w/ repl are equally likely

Q: what is prob. the sample contains y and b?

$S = \{\text{all such samples}\}$

$$|S| = \binom{r+n-1}{r} = \binom{3+4-1}{3} = \binom{6}{3} = 20$$

$E = \{\{y,b,g\}, \{y,b,o\}, \{y,b,b\}, \{y,y,b\}\}$

$|E| = 4$

hence $\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{4}{20} = 1/5.$

Four possibilities

w/ repl. , w/o repl.

Four possibilities

	w/ repl.	w/o repl.
ordered	n^r	$\frac{n!}{(n-r)!}$
unordered	$\binom{r+n-1}{r}$	$\frac{n!}{(n-r)!\,r!}$

The point of counting:

I have S w/ equally likely outcomes sample space

then

$$P(E) = \frac{|E|}{|S|}$$ need to count

The most important fact here is that we assume all outcomes are equally likely.

Q: ordering or not? replacement or not?

need to respect the prev. fact.

Ex. Flip a coin twice.

What is the prob. of getting a H and T.

Option 1: $\boxed{\text{Unordered}}$ Counting.

Order of H/T doesn't matter (coin flips)

$S = \{HH, TT, HT\}$, $|S| = 3$

basically sampling $r=2$ from $n=2$ $\{H, T\}$

w/o ordering
w/ replacement

$$\binom{r+n-1}{r} = \binom{2+2-1}{2} = \binom{3}{2} = 3$$

$E = \{HT\}$, $|E| = 1$

hence $\mathbb{P}(E) = \frac{1}{3}$. Seems wrong...

Another argument:

→ Independence: things don't affect each other.

$\overbrace{H}\overbrace{T}$ or TH

$$\underbrace{\tfrac{1}{2} \cdot \tfrac{1}{2}}_{1/4} + \underbrace{\tfrac{1}{2} \cdot \tfrac{1}{2}}_{1/4} = 1/2.$$

Option 2: $\boxed{\text{Ordered}}$ Sampling

$S = \{HH, TT, HT, TH\}$ $|S| = 4$

$E = \{HT, TH\}$ $|E| = 2$

Sample from $\{H, T\}$

$\mathbb{P}(E) = \frac{2}{4} = 1/2$

$$\mathbb{P}(E) = \frac{2}{4} = 1/2.$$

General rule that mostly works:

If I assume my sampling comes about from a seq. of independent actions.

Then counting w/ $\boxed{\text{ordering}}$ mattering typically gives the correct results.

When sampling $\boxed{\text{w/ replacement}}$ we have to be careful about ordering.

This doesn't come up as much when sampling $\boxed{\text{w/o replacement}}$.

→ ordered/unordered are a $r!$ multiple of each other.

$$\mathbb{P}(E) = \frac{\#}{\#} = \frac{r!\,\#}{r!\,\#}$$