

Equally Likely Outcomes

If I have a sample space

$$S = \{s_1, s_2, \dots, s_n\}$$

and assume

$$P(\{s_i\}) = P(\{s_j\}) \quad \forall i, j.$$

$\frac{1}{n}$

(Rationale: $P(S) = 1$)

$$S = \underbrace{\{s_1\} \cup \{s_2\} \cup \dots \cup \{s_n\}}_{\text{disjoint (partition } S)}$$

$$1 = P(S) = \sum_{i=1}^n P(\{s_i\})$$

each is equal $p \in [0, 1]$

$$1 = \sum_{i=1}^n p = np \Rightarrow p = \frac{1}{n} \quad \text{note: } n = |S|$$

More generally: If $E \subset S$ then

$$P(E) = \frac{\# \text{ outcomes in } E}{\# \text{ outcomes in } S} = \frac{|E|}{|S|}$$

Ex. Roll a six-sided (fair) die.

$$S = \{1, 2, 3, \dots, 6\}$$

if all outcomes are equally likely then

$$E = \{2, 6\}$$

then

$$P(E) = \frac{|E|}{|S|} = \frac{2}{6} = \frac{1}{3}.$$

note:

$$P(E) = \frac{|E|}{|S|}$$

← need to count # of
outcomes in E

← need to count # in S

Canting

Theorem : Fundamental Theorem of Canting (FTC)

If a task or job consists of k subtasks
and the i^{th} subtask can be done in $[n_i]$ ways,
then overall the number of ways to do
the task is

$$N = n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot \dots \cdot n_k$$
$$= \prod_{i=1}^k n_i$$

Ex. An experiment consists of 3 factors :

~ ~ ~ . . .

∴ An experiment consists of 3 factors :

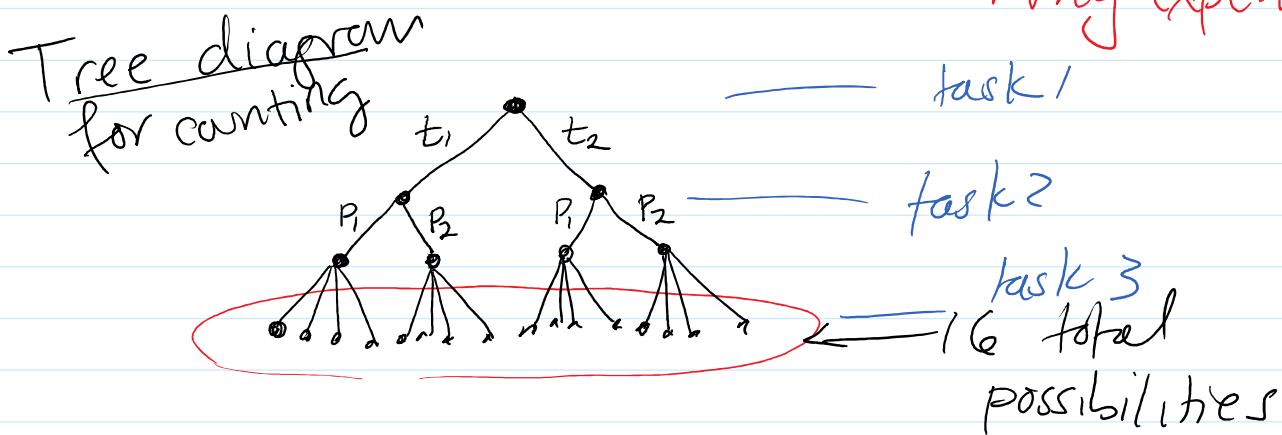
- { ① 2 temperature settings ←
- ② 2 pressure settings ←
- ③ 4 humidity setting ←

Q: How many ways can I run the experiment?

$$k = 3, n_1 = 2, n_2 = 2, n_3 = 4$$

overall I have

$$N = 2 \cdot 2 \cdot 4 = 16 \text{ ways to choose my experiment}$$



Ex. A man has [5 shirts] [2 pairs of pants]
[2 pairs] of shoes.

Q: How many outfit does the man have?

$$N = 5 \cdot 2 \cdot 2 = 20 \text{ outfits.}$$

Ex. I have a deck of 52 cards.

denominations: A, 2, 3, ..., 10, J, Q, K (13)

suits: C, D, H, S (4)

I shuffle the cards so that each ordering of the cards is equally likely

Q: What is the prob. that I get the cards "in order" (A-K, C, D, H, S)

E = "in order"

S = all possible orderings of 52 cards

then $P(E) = \frac{|E|}{|S|}$

$|E| = 1$
 $|S| = ??$

$k = 52$ subtasks to create an ordering of the cards

task 1: choose card 1 \rightarrow

task 2: " 2 \rightarrow

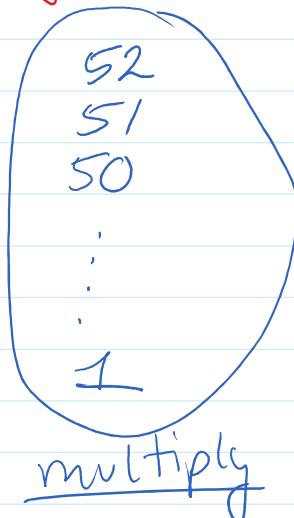
task 3: " 3 \rightarrow

:

:

:

task 52: " 52 \rightarrow



$$|S| = 52 \cdot 51 \cdot 50 \cdot 49 \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

hence

$$P(E) = \frac{1}{|S|} = \frac{1}{52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1}$$
$$\approx 10^{-68}$$

Defn: Factorial

For any non-negative integer n we define
 n -factorial (denoted $n!$) as

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$
$$= \prod_{i=1}^n i$$

Note: $0! = 1$.

Ex. Prev. example:

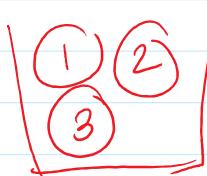
$$P(E) = \frac{1}{52!}$$

Sampling w/ or w/o Ordering and Replacement

Ordering: order matters (or not)

do I care about the order of my sample?

do I care about the order of my sample?



draw 1:
1 3 2

draw 2:
2 1 3

different?

Replacement: Can I draw the same item more than once?



w/
replacement: possible to draw
1 1 2

w/o
replacement: not possible to draw
1 1 2

This yields 4 scenarios

| | w/o replacement | w/ replacement |
|------------|---------------------|----------------|
| ordered | $\frac{n!}{(n-r)!}$ | 2 |
| un-ordered | 4 | 3 |

Permutation:

A permutation is an ordering of a collection of objects.

Ex. objects $\{A_1, A_2, A_3\}$

permutations

3 items

$3!$

$A_1 A_2 A_3, A_1 A_3 A_2, A_2 A_1 A_3 \quad \} \quad 6 \text{ possible}$
 $A_2 A_3 A_1, A_3 A_1 A_2, A_3 A_2 A_1 \quad \} \quad \text{permutations}$

Theorem: Permutation Counting

The number of ways to permute n items
is $n!$

Pf. Use FTC.

$k = n$ subtasks

| task # | task | # ways |
|--------|---------------|--------|
| 1 | choose item 1 | n |
| 2 | " 2 | $n-1$ |
| 3 | " 3 | $n-2$ |
| : | : | |
| n | " n | 1 |

multiply

this gives a total # ways

$$n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

Theorem: Ordered Sampling w/o Replacement

If I have n items and I draw a sample of size r (note: $r \leq n$)

and I draw my sample w/o replacement
and I care about the order of my draws

I can do this (# of possible samples) is

$$\boxed{(n)_r = \frac{n!}{(n-r)!}}$$

pf. $k=r$ subtasks

| task # | task | # ways |
|--------|----------------------|---------|
| 1 | draw 1 st | n |
| 2 | draw 2 nd | $n-1$ |
| 3 | draw 3 rd | $n-2$ |
| : | : | : |
| i | ; | |
| r | draw r^{th} | $n-r+1$ |

multiply

total # ways = $\boxed{n(n-1)(n-2) \cdots (n-r+1)}$

$$\frac{n!}{(n-r)!} = \frac{n(n-1) \cdots (n-r+2)(n-r+1)(n-r)}{(n-r)(n-r-1)(n-r-2) \cdots 3 \cdot 2 \cdot 1}$$

Ex. 10 students want to form a committee
consists of 3 members: ... it be. 0

Ex. 10 students want to form a committee consisting of 3 members: \leftarrow must be distinct President, VP, treasurer

How many possible different committees can I form?

$n = 10, r = 3$ no repl., care about order

Sq: $1^{\text{st}} = \text{Pres}, 2^{\text{nd}} = \text{VP}, 3^{\text{rd}} = \text{treasurer}$

$$\begin{aligned} \# \text{ ways} &= \frac{n!}{(n-r)!} = \frac{10!}{(10-3)!} = \frac{10!}{7!} \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} \\ &= 10 \cdot 9 \cdot 8 = 720. \end{aligned}$$

Ex. Lotto.

I have a box w/ zigzag balls numbered 1, .., 25



Lotto: draws 4 balls, if you guess the 4 in correct order you win.

my choice: (1)(3)(22)(7)

All possibilities are equally likely

Q: What is the prob. I win?

$$E = \{1, 3, 22, 7\}$$

$S = \{\text{all possible drawings}\}$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{|S|}$$

$$|S| = \frac{n!}{(n-r)!} = \frac{25!}{(25-4)!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21!}{21!}$$

$n=25, r=4$

$$= 25 \cdot 24 \cdot 23 \cdot 22$$

$$P(E) = \frac{1}{25 \cdot 24 \cdot 23 \cdot 22}$$



Theorem: Sampling w/ Replacement and w/ Ordering

The number of ways to draw a sample of size r from n items

- (1) w/ replacement
- (2) w/ ordering

is

$$\boxed{n^r}$$

note: don't need
 $r \leq n$

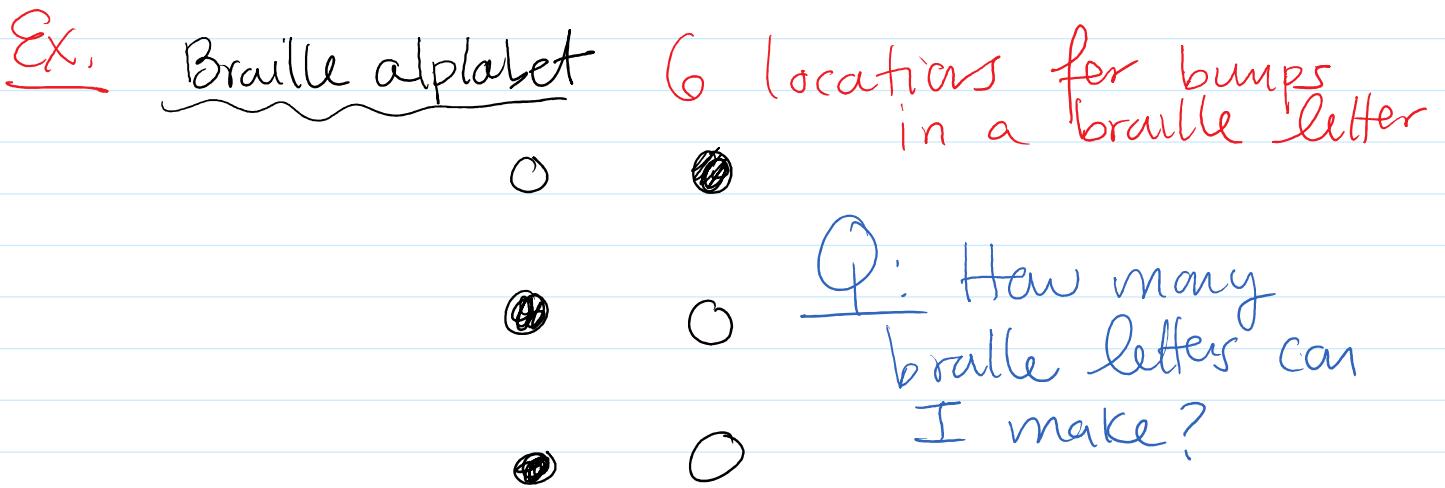
pf. Use FTC. $k=r$

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| task # | task | # ways |
|--------|------------------------|--------|
| 1 | Select 1 st | n |
| 2 | Select 2 nd | n |
| 3 | Select 3 rd | n |
| : | : | : |
| r | Select r th | n |

multiply

total # ways = $\underbrace{n \cdot n \cdot n \cdots n}_{r\text{-times}} = n^r$.



idea: draw 6 samples (bumps) from two possibilities "raised" or "not-raised"

Sample w/ repl. w/o ordering.

$$n=2, r=6$$

total # ways to do this : $2^6 = 64$.