Defn: Sample Space

The sample space S is the set of possible outcome.

Ex. Flip a coin.

$$S = \{H, T\}$$

Ex. Roll a six-sided die.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Ex. Roll two dice.

$$S = \{(1,1), (1,2), (1,3), \ldots$$
$$(2,1), (2,2), \ldots\}$$
$$= \{(i,j) \mid 1 \leq i,j \leq 6\}$$

Ex. Waiting for a bus to arrive.

$$S = [0, \infty)$$

Ex. Look at number of customers arriving in my restaurant.

$$S = \mathbb{N}_0 = \{0, 1, 2, 3, 4, \ldots\}$$

Types of sample spaces:

1. finite $(|S| < \infty)$
2. infinite $(|S| \geq \infty)$
   - i. countable (e.g. $\mathbb{N}_0$)
   - ii. uncountable (e.g. $[0, \infty)$)

Defn: Outcome:

We call elements $s \in S$ "outcomes"

outcome (s) — sample space (S)

Ex. $S = \{1, 2, 3, 4, 5, 6\}$

then $1 \in S$ so "1" is an outcome.

Defn: Event

An event E is a subset of the sample space S.

$$(E \subset S)$$

Ex. Roll a die: $S = \{1,2,3,4,5,6\}$ and

so $E = \{1,2\} \subset S$

rolling a 1 or a 2.

→ Say an event "happens" if the observed outcome of the experiment is an element of E.

Ex. $S \subset S$, here S is an event.

event that something happens/occurs

Ex. $\emptyset \subset S$, so $\emptyset$ is an event.

??? nothing happens?

Axiomatic Probability

Given: an experiment (a sample space S)

want: for any event ($E \subset S$) want to assign some measure of the liklihood that E occurs.

probability

Mathematically:

For each $E \subset S$ assign a probability $\mathbb{P}(E)$

prob. fn.

For each $E \subset S$ assign a probability $\mathbb{P}(E)$

prob. of E occuring.

What are the rules for building $\mathbb{P}$?

1. mathematically consistent
2. preserve/encode (some) of our intuitions about probability.

Defn: Probability Function $\mathbb{P}$

Given a sample space S a prob. function $\mathbb{P}$ is a function

$$\mathbb{P} : \mathcal{P}(S) \longrightarrow \mathbb{R}$$

domain co-domain

that satisfies Kolmogorov Axioms

1. non-negativity

   $\mathbb{P}(E) \geq 0 \quad \forall E \subset S$

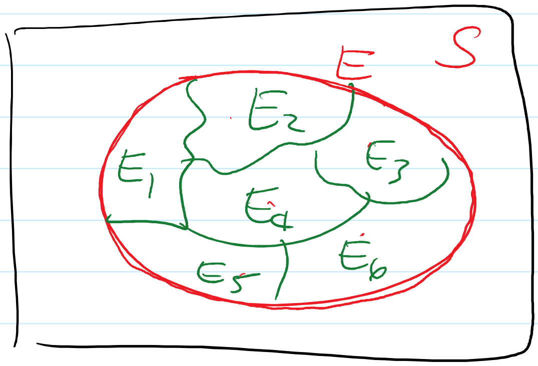
2. unit-measure

   $\mathbb{P}(S) = 1.$

③ Countable additivity

If $(E_i)_{i=1}^{\infty}$ is a partition of E

$$\left(E_iE_j = \emptyset \quad \forall i \neq j \qquad \boxed{\bigcup_{i=1}^{\infty} E_i = E} \right)$$



$$P(E) = \sum_{i=1}^{\infty} \mathbb{P}(E_i).$$

Countable additivity gives us distributive law → for disjoint events.

$$\mathbb{P}(E) = P\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$= \sum_{i=1}^{\infty} P(E_i).$$

Area of E = sum of areas of E_i

Ex. Flip a coin.

$$\boxed{S = \{H, T\}.}$$

What is a valid prob. fn on S?

$$P(\{H\}) = \tfrac{1}{2} \qquad \boxed{P(\underbrace{\{H, T\}}_{S}) = 1.}$$

$$P(\{T\}) = \tfrac{1}{2} \qquad P(\emptyset) = 0.$$

Check if this is a valid prob. fn.

✓ (1) $\mathbb{P}(E) \geq 0 \quad \forall E \subset S$

✓ (2) $\mathbb{P}(S) = 1$

(3) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$ for any partition $(E_i)_{i=1}^{\infty}$ of any event E

$$\left.\begin{array}{l} E_1 = \{H\} \\ E_2 = \{T\} \end{array}\right) \text{disjoint}$$

S is partioned into $E_1 \cup E_2$

$$1 = \mathbb{P}(S) = \mathbb{P}(E_1 \cup E_2) = \underbrace{\mathbb{P}(E_1)}_{1/2} + \underbrace{\mathbb{P}(E_2)}_{1/2}$$

Ex. Other ways of defining $\mathbb{P}$ on S?

$\alpha \in [0, 1]$

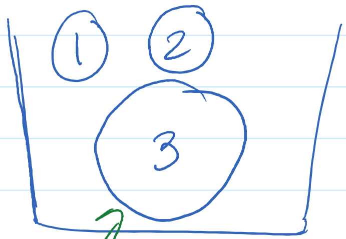
$$\mathbb{P}(S) = 1 \qquad \mathbb{P}(\{H\}) = \alpha\ )$$

$$\mathbb{P}(\emptyset) = 0 \qquad \mathbb{P}(\{T\}) = 1 - \alpha\ )$$

Valid? Yes. Why?

Ex.

|| (1) (2) / $S = \{1, 2, 3\}$



2x as likely to choose ball 3

$S = \{1, 2, 3\}$

$P_1 = \frac{1}{4}$ $P_2 = \frac{1}{4}$ $P_3 = \frac{1}{2}$

$\mathbb{P}(\{1,2\}) = \frac{1}{4} + \frac{1}{4}$

$= P_1 + P_2$

$\mathbb{P}(\{1,3\}) = P_1 + P_3 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$

Theorem: Finite Sample Space Theorem

If $S = \{s_1, s_2, s_3, \dots, s_n\}$, $|S| = n < \infty$

to build up $\mathbb{P}$ choose corresponding

$P_1, P_2, P_3, \dots, P_n$

where $\boxed{① \; P_i \geq 0}$ and $\boxed{② \; \sum_{i=1}^{n} P_i = 1}$

then for $E \subseteq S$,

$$\boxed{\mathbb{P}(E) = \sum_{i: s_i \in E} P_i .}$$

valid!

Ex. $\mathbb{P}(\{s_1, s_5\}) = P_1 + P_5$

$\mathbb{P}(\{s_2, s_7, s_{11}\}) = P_2 + P_7 + P_{11}$

E

Then $\mathbb{P}$ is a valid prob. fn.

<u>Pf.</u> Check that $\mathbb{P}$ satisfies the Komogorov Axioms.

1. $\boxed{\mathbb{P}(E) \geq 0 \quad \forall E \subset S}$

$$\mathbb{P}(E) = \sum \overbrace{p_i}^{\geq 0} \geq 0$$

2. $\boxed{\mathbb{P}(S) = 1}$

$$\mathbb{P}(S) = \sum_{i : s_i \in S} p_i = \sum_{i=1}^{n} p_i = 1$$

3. If $E \subset S$ and (E_i) partition E ((E_i) disjoint and $\bigcup_i E_i = E$) then $\mathbb{P}(E) = \mathbb{P}\left(\bigcup_i E_i\right) = \sum_i \mathbb{P}(E_i)$

WLOG assume (E_i) is countably infinite.

Then $E = \bigcup_{i=1}^{\infty} E_i$

So if $s_j \in S$ and $s_j \in E$ then

$\boxed{s_j \in \bigcup_{i=1}^{\infty} E_i} \iff s_j$ is in

$$\Rightarrow \boxed{s_j \in \bigcup_{i=1}^{\infty} E_i} \Longleftrightarrow s_j \text{ is in } \cancel{\text{at least}}\ \text{exactly one } E_i$$

$$P(E) = P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$$= \sum_{j : s_j \in \bigcup_{i=1}^{\infty} E_i} p_j$$

$$= \sum_{i=1}^{\infty} \underbrace{\sum_{j : s_j \in E_i} p_j}_{P(E_i)}$$

$$= \sum_{i=1}^{\infty} P(E_i),$$

Def'n:

$$P(E) = \sum_{i : s_i \in E} p_i$$

Ex, $p_1 + p_3 + p_5 + p_6 + p_7 + p_{11}$

$s_1 \quad s_3 \quad s_5 \quad s_6 \quad s_7 \quad s_{11}$ etc.

$P(E_1) + P(E_2) + P(E_3)$

$p_1 + p_5 + p_3 + p_6 + p_7 + p_{11}$