Independence $\rightsquigarrow$ laymen's understanding:

→ things don't affect each other

→ events are independent if the occurance (or not) of one doesn't affect the prob. of another.

<u>Defn: Independence (of Events)</u>

If $A, B \subset S$ we say "A is independent of B" denoted $A \perp\!\!\!\perp B$, if

$$P(AB) = P(A)P(B).$$

→ Kind of distributive law

→ hint at notation for intersection

<u>Theorem</u> (intuition that hopefully is true)

If $A \perp\!\!\!\perp B$ then

$$P(A|B) = P(A).$$

pf.

$$P(A|B) \overset{\text{defn cond.}}{=} \frac{P(AB)}{P(B)} \overset{\text{defn independence}}{=} \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A).$$

<u>Ex.</u> Consider rolling two dice (independently)

$\mathbb{P}(\text{at least one } 6)$

$$\longrightarrow \quad = 1 - \mathbb{P}(\text{"at least one 6"}^c)$$

$$= 1 - \mathbb{P}(\underbrace{\text{no } 6}_{A_1 A_2})$$

$A_1 = \text{no } 6 \text{ on } 1^{st}$

$A_2 = \text{no } 6 \text{ on } 2^{nd}$

independence

$$= 1 - P(A_1 A_2)$$

$$= 1 - P(A_1)P(A_2)$$

$$= 1 - (5/6)(5/6)$$

$$= 1 - 25/36 = \boxed{11/36}$$

Counting perspective:

S { Sampling from $\{1, \ldots, 6\}$ $(n=6)$ two times $(r=2)$ w/ replacement

Ordered: $|S| = n^r = 6^2 = 36$

$E = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5)\}$

$|E| = 11$

$$\boxed{P(E) = 11/36}$$

Unordered: $|S| = \binom{n+r-1}{r} = \binom{7}{2} = 21$

$E = \{\{1,6\}, \ldots, \{5,6\}, \{6,6\}\}$

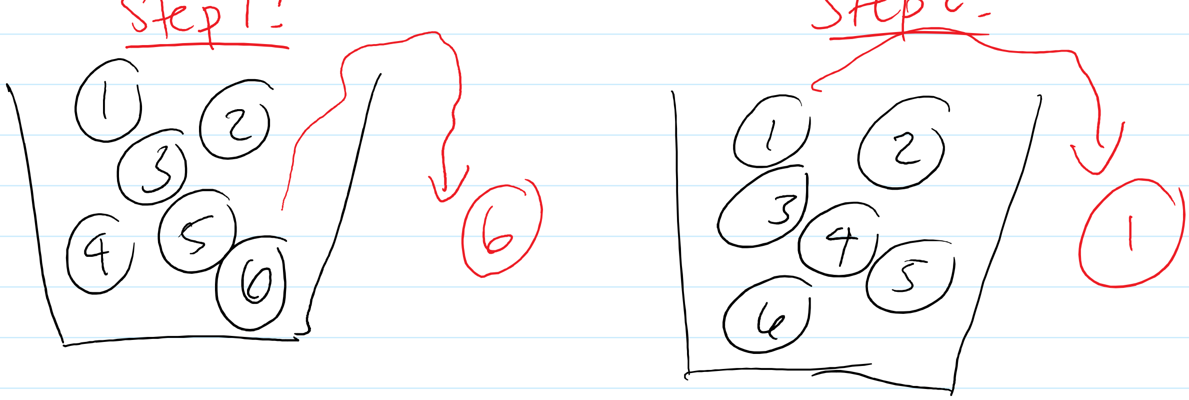
$|E| = 6$

So $\boxed{\mathbb{P}(E) = 6/21}$

Ex. Roll two dice
independent

$E = \{\underline{1} \text{ or } \underline{2} \text{ on } 1^{st} \text{ and } \underline{3}, \underline{4} \text{ or } \underline{5} \text{ on } 2^{nd}\}$.

Step 1: Step 2:



$\mathbb{P}(\{(6,1)\}) = 1/6 \cdot 1/6 = 1/36$

count in ordered way:

$|S| = n^r = n^2 = n \cdot n = 6 \cdot 6 = 36$

$E = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

$= \{1,2\} \times \{3,4,5\}$

$|E| = 6 = |\{1,2\} \times \{3,4,5\}| = |\{1,2\}| \, |\{3,4,5\}|$

$= 2 \cdot 3$

unordered count has multiplicative structure

So $\mathbb{P}(E) = \dfrac{2 \cdot 3}{6 \cdot 6} = \left(\dfrac{2}{6}\right)\left(\dfrac{3}{6}\right)$

So $\mathbb{P}(E) = \frac{2 \cdot 3}{6 \cdot 6} = \left(\frac{2}{6}\right)\left(\frac{3}{6}\right)$ ← multiplicative structure

prob. of 1 or 2, on 1st

prob. 3, 4, 5 on 2nd

Theorem: Complementary Independence

If $A \perp\!\!\!\perp B$ then

1. $A \perp\!\!\!\perp B^c$
2. $B \perp\!\!\!\perp A^c$
3. $A^c \perp\!\!\!\perp B$
4. $A^c \perp\!\!\!\perp B^c$

Pf. Case 1: $A \perp\!\!\!\perp B^c$

$$\mathbb{P}(AB^c) = \mathbb{P}(A) - \mathbb{P}(AB)$$
$$= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)$$
$$= \mathbb{P}(A)(1 - \mathbb{P}(B))$$
$$= \mathbb{P}(A)\mathbb{P}(B^c)$$

Defn: Mutual Independence

Generalize independence to multiple events

If $(A_i)_{i=1}^n$ are events, we say they are (mutually) independent if

for any subsequence $\underbrace{A_{i_1}, A_{i_2}, \ldots, A_{i_k}}_{\text{subseq of length } k}$,

$$\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} \mathbb{P}(A_{i_j}).$$

$$\mathbb{P}\left(\bigcap_{j=1} A_{i_j}\right) = \prod_{j=1} \mathbb{P}(A_{i_j}).$$

Ex.

$$\mathbb{P}(A_1 A_3 A_4) = \mathbb{P}(A_1)\mathbb{P}(A_3)\mathbb{P}(A_4)$$

$$\mathbb{P}(A_2 A_7 A_{11} A_{12}) = \mathbb{P}(A_2)\mathbb{P}(A_7)\mathbb{P}(A_{11})\mathbb{P}(A_{12}).$$

etc.

Q: Do we need to check all subsequences?

Can I just check

$$\cancel{\mathbb{P}(A_1 A_2 A_3 \cdots A_n)} = \mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_n)?$$

Nope.

Ex. Roll two dice.

$|A|=6$ $\quad A =$ "doubles" $= \{(1,1),(2,2),\ldots,(6,6)\}$

$B =$ "sum is between 7 and 10"

$= \{(2,5),(1,6),(3,4),(4,3),(5,2),(6,1),$
$(2,6),(3,5),(4,4),(5,3),(6,2)$
$(3,6),(4,5),(5,4),(6,3),$
$(6,4),(5,5),(4,6)\}$

$|B|=18$

$C =$ "sum is 2, 7, 8"

$= \{(1,1), \quad \}$

$|C| = 12$

Q: Mutually independent?

✓ $\mathbb{P}(ABC) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$ ⟵

$\{(4,4)\}$

$\frac{1}{36} = \frac{6}{36} \cdot \frac{18}{36} \cdot \frac{12}{36} = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{36}$

Consider: BC = "Sum is 7 or 8" ⟵

$|BC| = 11$

$\mathbb{P}(BC) = \frac{11}{36} = \underset{(\frac{1}{2})}{\mathbb{P}(B)}\,\underset{(\frac{1}{3})}{\mathbb{P}(C)} = \frac{1}{6}$

check fails for B, C.

A, B, C Not mutually independent?

Defn: Pairwise Independence

(A_i) are pairwise independent if

$$\mathbb{P}(A_i A_j) = \mathbb{P}(A_i)\,\mathbb{P}(A_j) \text{ for } i \neq j.$$

Note: Can $A \perp\!\!\!\perp A$?

$\mathbb{P}(A) = \mathbb{P}(AA) = \mathbb{P}(A)\,\mathbb{P}(A) = \mathbb{P}(A)^2$

$\mathbb{P}(A) = \mathbb{P}(A)^2$ solutions in $[0,1]$

Says $P(A) = 0$ or 1.

Q: Pairwise ~~independence~~ $=$ mutual ?

No.

Ex.

$S = \{$"aaa", "bbb", "ccc", "abc", "bca", "acb", "cba", "bac", "cab"$\}$

Assume all outcomes in S are equally likely

$|S| = 9$.

$A_i = \{$ the i^{th} place in triplet is an "a" $\}$

$A_1 = \{aaa, abc, acb\}$

$A_2 = \{bac, cab, aaa\}$

$A_3 = \{aaa, bca, cba\}$

$P(A_i) = 3/9 = 1/3$

Furthermore $A_i A_j = \{aaa\}$

So $\underbrace{P(A_i A_j)}_{1/9} = \underbrace{P(A_i)}_{1/3} \underbrace{P(A_j)}_{1/3}$ ✓

pairs are independent.

However $P(A_1 A_2 A_3) = P(\{aaa\}) = 1/9$

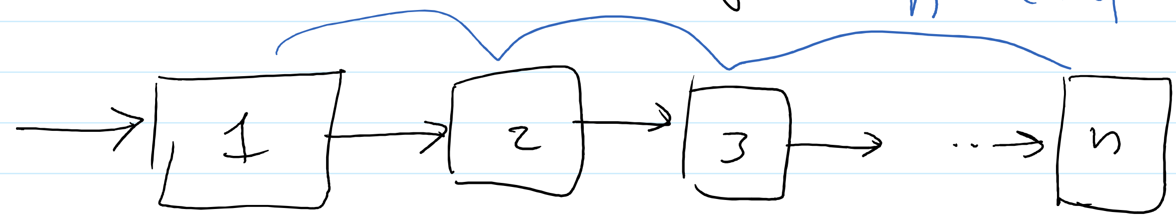
$\neq \underbrace{P(A_1) P(A_2) P(A_3)}_{1/[illegible]}$

1/27

See fails for 3 events.
$\Rightarrow$ not mutually independent.

Ex. Failure in a serial system.

n - components



fail prob. P_1 P_2 P_3 P_n

System works only if all components work.
Assume failure of component is $\boxed{\text{independent}}$.

Let W_i = i^{th} comp. works
$\mathbb{P}(W_i^c) = p_i$

$$\begin{aligned}
\mathbb{P}(\text{system works}) &= \mathbb{P}\left(\bigcap_{i=1}^{n} W_i\right) \\
&= \prod_{i=1}^{n} \mathbb{P}(W_i) \\
&= \prod_{i=1}^{n} \left(1 - \mathbb{P}(W_i^c)\right) \\
&= \prod_{i=1}^{n} (1 - p_i) = (1-p_1)(1-p_2)(1-p_3)\cdots(1-p_n)
\end{aligned}$$
