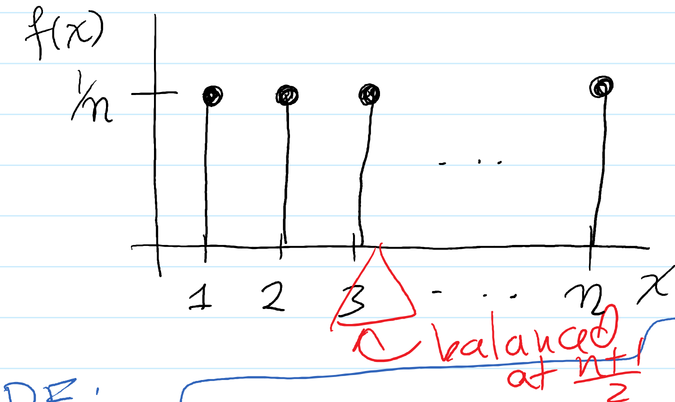
Discrete Uniform

$X \sim \text{Unif}\{1, \dots, n\}$



PMF:

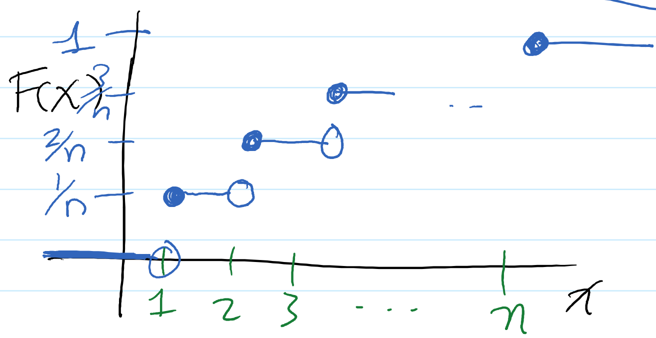
$$\boxed{f(x) = 1/n \text{ for } x = 1, 2, 3, \dots, n}$$

Note:

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} 1/n = n/n = 1$$

CDF:

$$\boxed{F(x) = \sum_{i \le x} f(x) = \begin{cases} 0 & x < 1 \\ 1/n & 1 \le x < 2 \\ 2/n & 2 \le x < 3 \\ \vdots & \\ 1 & x \ge n \end{cases} = \begin{cases} \frac{\lfloor x \rfloor}{n} & 0 < x < n \\ 0 & x < 0 \\ 1 & x > 1 \end{cases}}$$



Check: (1) non-decreasing
(2) right cts
(3) $\lim_{x \to \infty} F(x) = 1$
$\lim_{x \to -\infty} F(x) = 0$

Expectation

$$E[X] = \sum_{i=1}^{n} i f(i) = \sum_{i=1}^{n} i \, 1/n = \frac{1}{n}\left(\sum_{i=1}^{n} i\right)$$

Known formula

$$= \frac{1}{n}\left[\frac{n(n+1)}{2}\right]$$

$$\boxed{E[X] = \frac{n+1}{2}}$$

Known formula

$$\mathbb{E}[X^2] = \cdots = \sum_{i=1}^{n} i^2 \, 1/n = \frac{1}{n}\left(\sum_{i=1}^{n} i^2\right) = \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$\boxed{Var(X)} = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \cdots \text{algebra} = \boxed{\frac{n^2-1}{12}}$$

Moment Gen. Fn (MGF)

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{i=1}^{n} e^{ti} \, 1/n = \frac{1}{n} \sum_{i=1}^{n} (e^t)^i$$

recall: partial sum of a Geometric series

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r} \quad \text{for } |r|<1$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} (e^t)^{i+1} = \frac{e^t}{n} \sum_{i=0}^{n-1} (e^t)^i$$

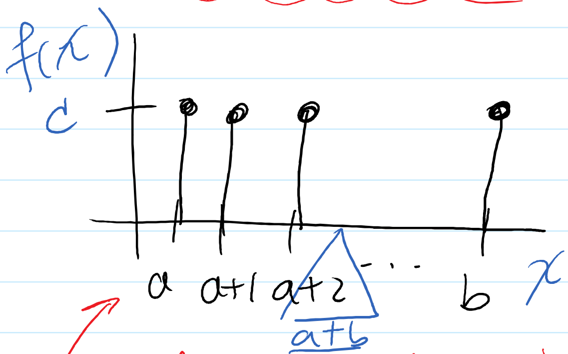
$r = e^t$ and $|e^t| < 1$

$$= \frac{e^t}{n} \frac{1-(e^t)^n}{1-e^t}$$

$$\rightarrow \quad M(t) = e^t - e^{t(n+1)} \quad \text{for } e^t < 1$$

$$\boxed{M(t) = \frac{e^t - e^{t(n+1)}}{n(1-e^t)}} \quad \text{for } e^t < 1$$

Consider: $X \sim U\{a, \dots, b\}$



total: $b-a+1$ pts in support

$$\sum_{i=a}^{b} f(i) = 1$$

$c(b-a+1)$ so $c = \dfrac{1}{b-a+1}$

$$\boxed{f(x) = \frac{1}{b-a+1} = \frac{1}{n}} \quad \text{for } x = a, a+1, \dots, b$$

If $Y \sim U\{1, \dots, n\}$ $\quad n = b-a+1$

then $X = Y + (a-1)$

and $X \sim U\{a, \dots, b\}$

$$E[X] = E[Y + (a-1)] = E[Y] + a - 1$$

$$= \frac{n+1}{2} + a - 1$$

$$= \frac{b-a+1+1}{2} + a - 1$$

$$= \frac{b-a+2+2a-2}{2}$$

$$= \frac{b+a}{2} = \boxed{\frac{a+b}{2}}$$

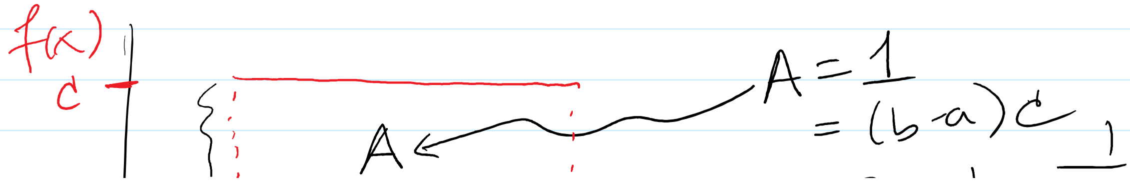
$$\mathrm{Var}(X) = \mathrm{Var}(Y + a - 1)$$
$$= \mathrm{Var}(Y)$$
$$= \frac{n^2 - 1}{12} = \boxed{\frac{(b-a+1)^2 - 1}{12}}$$

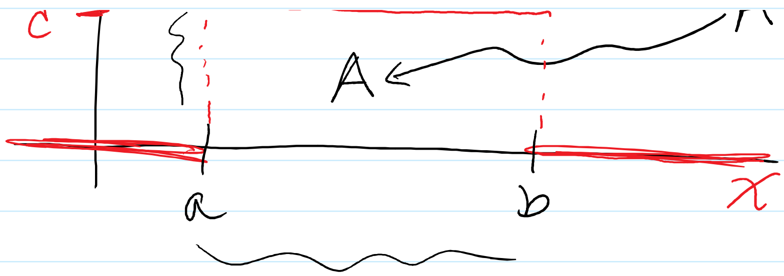
$$X = Y + a - 1$$
$$M_X = e^{(a-1)t} M_Y(t)$$
$$= e^{(a-1)t} \frac{e^t - e^{t(n+1)}}{n(1-e^t)}$$
$$= \frac{e^{at} - e^{t(n+a)}}{n(1-e^t)}$$
$$= \frac{e^{at} - e^{t(b-\cancel{a}+1+\cancel{a})}}{(b-a+1)(1-e^t)} \boxed{= \frac{e^{at} - e^{t(b+1)}}{(b-a+1)(1-e^t)}}$$

Continuous Uniform

$$X \sim U(a, b)$$

means: density is uniform over (a, b)





$= (b-a)c$

so $c = \dfrac{1}{b-a}$

$f(x) = c$ in (a,b)

f must satisfy $\displaystyle\int_{\mathbb{R}} f(x)\,dx = 1$

$$1 = \int_a^b c\,dx = c(b-a) \quad \text{so} \quad c = \frac{1}{b-a}$$

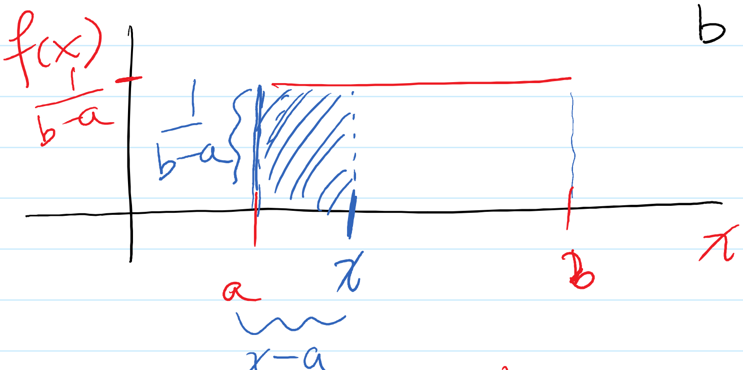
PDF: $\boxed{f(x) = \dfrac{1}{b-a} \text{ for } x \in (a,b)}$

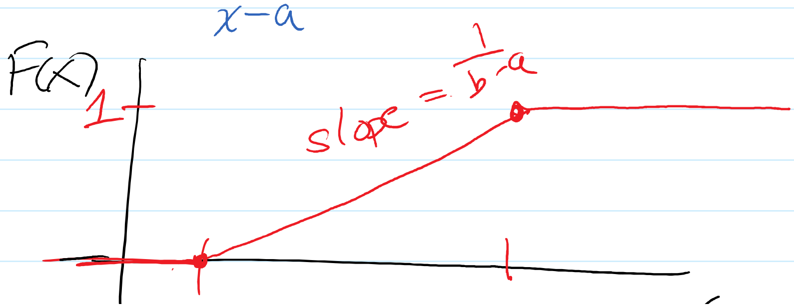
CDF:

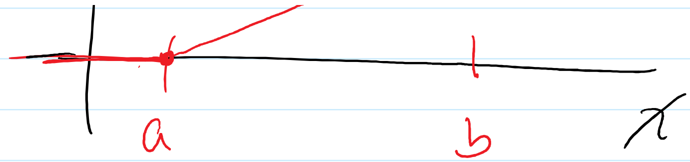
$a < x < b$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_a^x \frac{1}{b-a}\,dt = \frac{1}{b-a}(x-a)$$

$$= \frac{x-a}{b-a}$$







$$\boxed{F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}}$$

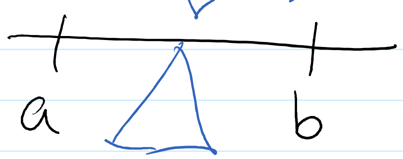
$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$$

balances at $\frac{a+b}{2}$ the midpoint

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2 - a}{2(b-a)}$$

$$= \frac{(a+b)\cancel{(b-a)}}{2\,\cancel{b-a}}$$

$$\boxed{= \frac{a+b}{2}}$$



$$\mathbb{E}[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2+ba+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ba+a^2}{3}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$b^2+ba+a^2 \qquad (a+b)^2$$

$$= \frac{b^2 + ba + a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$= \cdots$ algebra $\cdots$

$$\boxed{= \frac{(b-a)^2}{12}}$$

MGF:

$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$\boxed{= \frac{e^{tb} - e^{ta}}{t(b-a)}}$$

Bernalli Distribution

$\rightarrow$ Discrete

$p \in [0,1]$

$X \sim \text{Bern}(p)$

Canonical experiment:

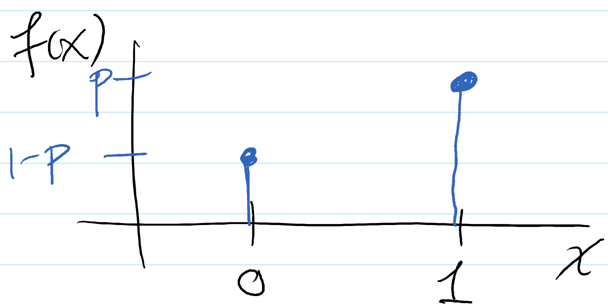
$\rightarrow$ flip a coin w/ prob. p of H

$$\rightarrow X = \begin{cases} 1 & \text{if } H \\ 0 & \text{if } T \end{cases}$$

Then $X \sim \text{Bern}(p)$.

PMF:

Then $X \sim \text{Bern}(p)$.



PMF:

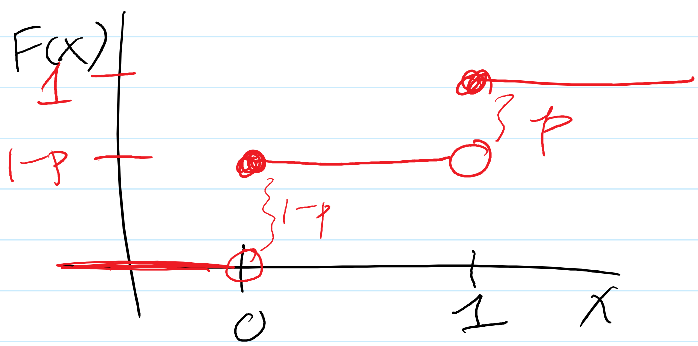
$$f(x) = \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$$

$$= p^x(1-p)^{1-x} \quad \text{for } x=0,1$$

familiar looks like Binomial R.V.

indeed: $\text{Bin}(n,p)$ w/ $n=1$ is a $\text{Bern}(p)$.

CDF:



$$\boxed{F(x) = \begin{cases} 0 & x<0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}}$$

Expectation:

$$E[X] = \sum_x x f(x) = \sum_{x=0,1} x f(x) = (0)f(0) + (1)f(1)$$

$$= f(1) = \boxed{p}$$

$$E[X^2] = \sum_{x=0,1} x^2 f(x) = 0^2 f(0) + (1)^2 f(1)$$

$$= f(1) = \boxed{p}$$

indeed, $\boxed{E[X^r] = p.}$

Note: $Y \sim \text{Bin}(n,p)$

$$\boxed{E[X^2] = p.}$$

Note: $Y \sim Bin(n,p)$
$E[Y] = np$
$Var(Y) = np(1-p)$

Variance:

$$Var(X) = E[X^2] - E[X]^2$$
$$= p - p^2$$
$$\boxed{= p(1-p)}$$

MGF:

$$M(t) = E[e^{tX}] = \sum_{x=0,1} e^{tx} f(x)$$
$$= e^{t(0)} f(0) + e^{t(1)} f(1)$$
$$\boxed{= 1 - p + e^t p}$$

recall: $Bin(n,p)$ $M(t) = (1 - p + e^t p)^n$

Normal / Gaussian Distribution

$X \sim N(\mu, \sigma^2)$ $\mu \in \mathbb{R}$ $\sigma^2 > 0$

PDF: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad \forall x \in \mathbb{R}$

special case: $\mu = 0$ and $\sigma^2 = 1$
(standard normal)
$X \sim N(0,1)$

then $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \quad \forall x \in \mathbb{R}$

CDF: No closed form.

Claims: $E[X] = \mu \quad Var(X) = \sigma^2$.

MGF:

$$M(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

$(x-\mu)^2 = x^2 - 2x\mu + \mu^2$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(\underbrace{-2\sigma^2 tx + x^2 - 2x\mu + \mu^2}_{\text{complete the square}}\right)\right) dx$$

$$x^2 - 2x(\sigma^2 t + \mu) + \mu^2$$

$$= (x - (\mu + \sigma^2 t))^2 + \mu^2 - (\mu + \sigma^2 t)^2$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left[(x-(\mu+\sigma^2 t))^2 + \mu^2 - (\mu+\sigma^2 t)^2\right]\right) dx$$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2\right)}_{\substack{\text{pdf of} \\ N(\mu+\sigma^2 t,\ \sigma^2) \\ \Rightarrow \text{ integrates to } 1}} \underbrace{\exp\left(\frac{\mu^2 - (\mu+\sigma^2 t)^2}{-2\sigma^2}\right)}_{\text{doesn't depend on } x} dx$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(\mu^2 - (\mu+\sigma^2 t)^2\right)\right) = M(t)$$

$$= \ \ldots \text{ algebra}$$

$$\boxed{\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) = M(t)}$$

MGF of $N(\mu, \sigma^2)$.

Want to show: $E[X] = \mu$ and $Var(X) = \sigma^2$.

$$\left.\frac{dM}{dt}\right|_{t=0} = \left.(\mu + \sigma^2 t)\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\right|_{t=0} = \boxed{\mu = E[X]}$$

$$\left.\frac{d^2M}{dt^2}\right| = (\sigma^2)\exp\left(\mu t + \sigma^2 t\right) + (\mu + \sigma^2 t)^2 \exp\left(\mu t + \sigma^2 t^2\right)\Big|$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \left(\frac{\sigma^2}{2}\right)\exp\left(\mu t + \frac{\sigma^2 t}{2}\right) + \left(\mu + \sigma^2 t\right)^2 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\Bigg|_{t=0}$$

$$= \sigma^2(1) + \mu^2 = \mu^2 + \sigma^2 = E[X^2].$$

hence $\underline{Var(X)} = E[X^2] - E(X)^2 = \mu^2 + \sigma^2 - \mu^2$

$$\boxed{= \sigma^2.}$$

Theorem: Linear Functions of Normal RVs

Let $X \sim N(\mu, \sigma^2)$ and

$$Y = aX + b.$$

Then $Y \sim N(a\mu + b, a^2\sigma^2)$

intuition: $E[Y] = aE[X] + b = a\mu + b$

$$Var(Y) = a^2 Var(X) = a^2\sigma^2.$$

pf. $M_Y(t) = e^{tb} M_X(at)$

$$= e^{tb} e^{\mu a t + \frac{\sigma^2 a^2 t^2}{2}}$$

$$= e^{(a\mu + b)t + \frac{(a^2\sigma^2)t^2}{2}}$$

Notice that this is the

aside: $M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

Notice that this is the

→ MGF of $N(a\mu+b, a^2\sigma^2)$.