## Theorem. Iterated Expectation

If $X$ and $Y$ are RVs then

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]].$$

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**Recall:** $\mathbb{E}[X|Y=y]$ here $y \in \mathbb{R}$

$$\underbrace{= \int_{\mathbb{R}} x f(x|y)\,dx}_{\mathbb{R}}$$

So for each $y \in \mathbb{R}$ I get $\mathbb{E}[X|Y=y] \in \mathbb{R}$

This defines a fn;

$$g : \mathbb{R} \to \mathbb{R} \text{ by } g(y) = \mathbb{E}[X|Y=y]$$

same function

Could plug $Y$ into $g$ to get $g(Y) = \mathbb{E}[X|\cancel{Y=y}\,Y]$

$$\boxed{= \mathbb{E}[X|Y]}$$

notation

this is a RV (a fn of $Y$)

1. $\mathbb{E}[X|Y=y]$ is a number
2. $\mathbb{E}[X|Y]$ is a RV.

(2) $E[X|Y]$ is a RV.

**Theorem:**

$$E[X] = E_Y[\underbrace{E[X|Y]}_{\text{is a RV}}]$$

pf. (Cts case)

Facts: $f(x) = \int f(x,y)\,dy$

$$f(x|y) = \frac{f(x,y)}{f(y)} \iff f(x,y) = f(x|y)\,f(y)$$

$$E[X|Y=y] = \int x f(x|y)\,dx$$

$$E[X] = \int x f(x)\,dx = \int x \int f(x,y)\,dy\,dx$$

$$= \int x \int f(x|y) f(y)\,dy\,dx$$

$$= \iint x f(x|y) f(y)\,dx\,dy$$

$$= \int \left[\int x f(x|y)\,dx\right] f(y)\,dy$$

$$= \int E[X|Y=y]\,f(y)\,dy$$

$$= \int g(y)\,f(y)\,dy$$

$$= \mathbb{E}[g(Y)] = \underline{\mathbb{E}_Y[\mathbb{E}[X|Y]]}.$$

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<u>Ex.</u> $X|Y=y \sim \text{Bin}(y, p)$ $p \in [0,1]$

$Y \sim \text{Pois}(\lambda)$

<u>$\mathbb{E}[X]$?</u>

1. $\mathbb{E}[X|Y=y] = yp = g(y)$
2. $\mathbb{E}[X|Y] = g(Y) = Yp$
3. $\mathbb{E}_Y \mathbb{E}[X|Y] = \mathbb{E}[Yp] = p\mathbb{E}[Y]$
$$= p\lambda$$

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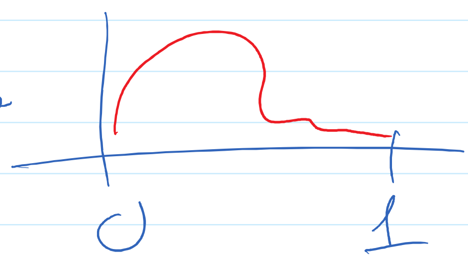
<u>Ex.</u> <u>$X|P=p=$</u> $\sim \text{Bin}(n, p)$ integer

$P \sim \text{Beta}(\alpha, \beta)$




$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|P]]$$
$$= \mathbb{E}[nP]$$
$$= n\mathbb{E}[P] = n\frac{\alpha}{\alpha+\beta}.$$

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<u>Theorem:</u> <u>Law of Total Variance</u>

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\underline{\mathbb{E}[X|Y]})$$

simlar $\mathbb{E}[X|Y]$
(a RV)

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Ex. $X|P=p \sim \mathrm{Bin}(n,p)$
$P \sim \mathrm{Beta}(\alpha,\beta)$.

$\mathrm{Var}(X)$? (1) $\mathbb{E}[X|P=p] = np$
$\mathrm{Var}(X|P=p) = np(1-p)$

(2) $\mathbb{E}[X|P] = nP$
$\mathrm{Var}(X|P) = nP(1-P)$

(3) Law of Tot. Variance

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|P)] + \mathrm{Var}(\mathbb{E}[X|P])$$
$$= \mathbb{E}[nP(1-P)] + \mathrm{Var}(nP)$$
$$= \underbrace{n(\mathbb{E}[P]-\mathbb{E}[P^2])}_{\cdots} + n^2\,\mathrm{Var}(P)$$

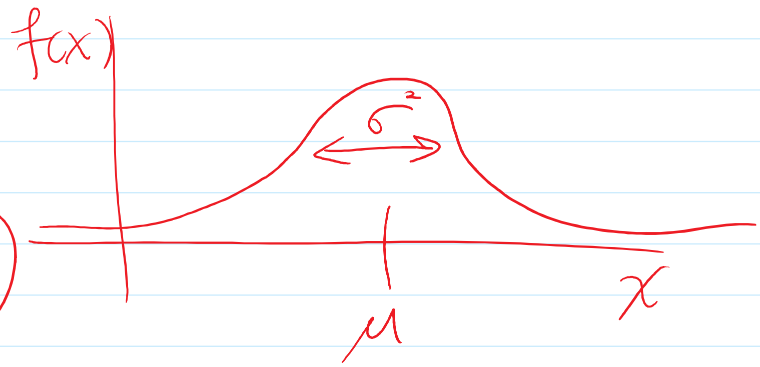
$$\boxed{= n\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)} + n^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

# Bivariate Normal

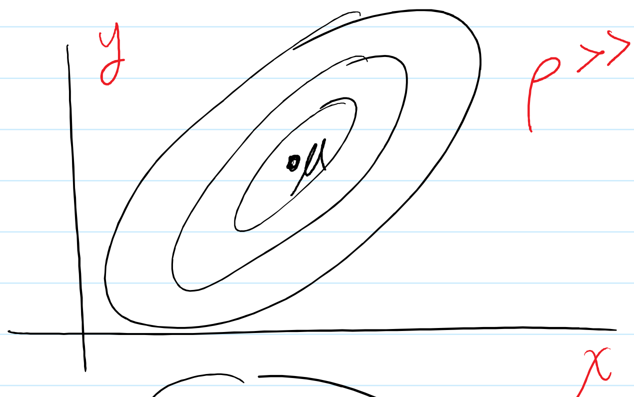
## Univariate Normal

$N(\mu, \sigma^2)$

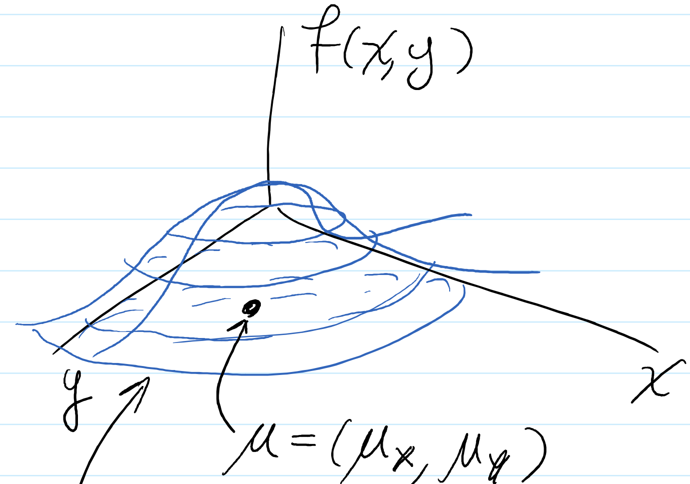
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$



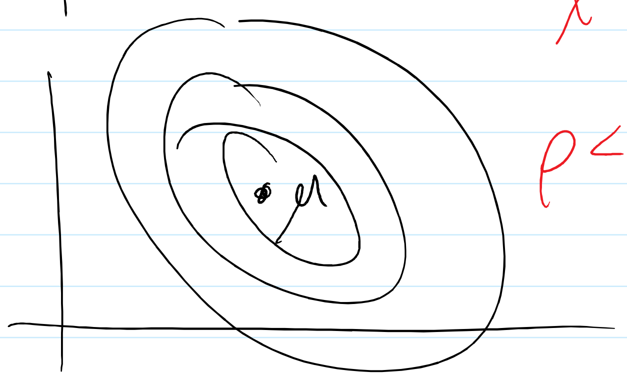

## Bivariate Normal



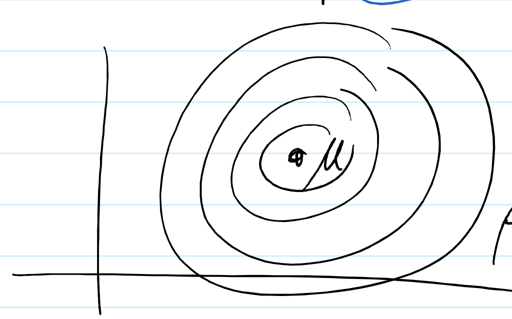

$\rho >> 0$




$\sigma_x^2, \sigma_y^2, \rho$ correlation




$\rho << 0$




$\rho \approx 0$

PDF: $(X, Y) \sim BivN(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\Big(-\frac{1}{2}\frac{1}{1-\rho^2}\Big[\Big(\frac{x-\mu_x}{\sigma_x}\Big)^2 + \Big(\frac{y-\mu_y}{\sigma_y}\Big)^2$$

$$-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\Big]\Big)$$

Simpler: $\mu = (\mu_x, \mu_y)$

Covariance mtx $\to$ $$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{bmatrix}$$

$z = (x,y)$

$$\boxed{f(z) = \frac{1}{2\pi}\frac{1}{\sqrt{\det\Sigma}}\exp\left(-\frac{1}{2}z^T\Sigma^{-1}z\right)}$$

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Facts: ① $X \sim N(\mu_x, \sigma_x^2)$

$Y \sim N(\mu_y, \sigma_y^2)$

② $Cor(X,Y) = \rho$

③ $aX + bY \sim N(a\mu_x + b\mu_y, \underline{a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_x\sigma_y\rho})$

④ A characterization of BivN

$(X,Y) \sim BivN \iff \forall a,b \quad aX+bY \sim N(\dots)$

⑤ Prev: If $X \perp\!\!\!\perp Y$ then $Cor(X,Y) = 0$

## Partial Converse:

If $(X, Y) \sim \text{BivN}$ and $\underline{\text{Cor}(X,Y) = 0}$

$\underline{\text{then}}$ $X \perp\!\!\!\perp Y$.

Pf. $\rho = 0$

$$f(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\cancel{\rho^2}}} \exp\left(-\frac{1}{2}\frac{1}{\cancel{1-\rho^2}}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 + \cancel{2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)}\right]\right)$$

$$= \underbrace{\frac{1}{2\pi\sigma_X\sigma_Y} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right)}_{h(x)} \underbrace{\exp\left(-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)}_{g(y)}$$

So $f(x,y) = h(x)g(y)$ so $X \perp\!\!\!\perp Y$.

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## Bivariate Transformations

Univariate: $g: \mathbb{R} \to \mathbb{R}$, $g(X)$?

Bivariate: $g: \mathbb{R}^2 \to \mathbb{R}^2$, $g(X, Y)$?

Notation: $(X, Y) \xrightarrow{g} (U, V)$

Ex. $(U, V) = (\underbrace{X^2 Y}_{g_1(X,Y)}, \underbrace{\log(Y)}_{g_2(X,Y)})$

$g_1 : \mathbb{R}^2 \to \mathbb{R}$
$g_2 : \mathbb{R}^2 \to \mathbb{R}$

$(U, V) = (-X/Y, XY)$

## Discrete:

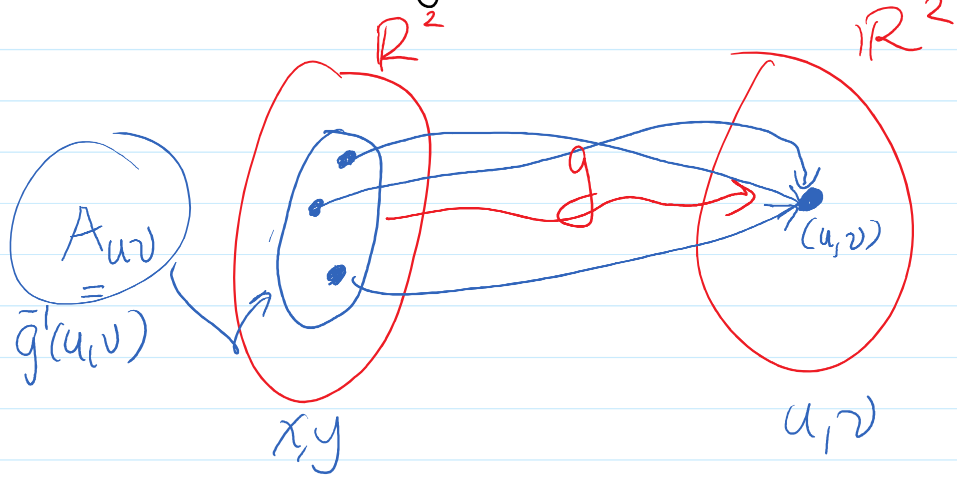
$(U, V) = (g_1(X, Y), g_2(X, Y))$

Assume $X, Y$ are discrete.

Define

$A_{uv}$ = inverse img of $(u, v)$ under $g$

$= \{(x, y) \mid g_1(x, y) = u$ and $g_2(x, y) = v\}$




Want: joint pmf of $U, V$ from joint pmf of $X, Y$

joint pmf of $U, V$

$$f_{U,V}(u, v) = P(U = u, V = v)$$

$$= P((X, Y) \in A_{uv})$$

$$= \sum f_{X,Y}(x, y)$$

$(x,y)\in A_{uv}$

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Ex. Let $X \perp\!\!\!\perp Y$

and $X \sim Pois(\theta)$
$Y \sim Pois(\lambda)$ } discrete

$$f(x,y) = f(x)f(y) = \frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^y e^{-\lambda}}{y!} \leftarrow$$

Let $\underline{U = X+Y}$ and $\underline{V = Y}$

$$(U,V) = g(X,Y)$$

$$g_1(x,y) = x+y \; ; \; g_2(x,y) = y$$

$A_{uv}$: if $\underline{u = x+y}$ and $\boxed{v = y}$
then $\boxed{u - v = x}$

$g$ is invertible

$$A_{uv} = g^{-1}(u,v) = \underline{(u-v, v)}$$

$$f_{U,V}(u,v) = \sum_{(x,y)\in A_{uv}} f_{X,Y}(x,y)$$

$$= f_{X,Y}(u-v, v) = \boxed{\frac{\theta^{u-v} e^{-\theta} \lambda^{v} e^{-\lambda}}{(u-v)! \quad v!}}$$

$$= \frac{1}{\ }\binom{u}{\ } e^{-(\theta+\lambda)} \theta^{u-v} \lambda^{v}$$

$$= \frac{1}{u!}\binom{u}{v} e^{-(\theta+\lambda)} \theta^{u-v} \lambda^{v}$$

What is the marginal of $U$?

$$f_U(u) = \sum_{v=0}^{u} f(u,v) = \frac{1}{u!} e^{-(\theta+\lambda)} \underbrace{\sum_{v=0}^{u} \binom{u}{v} \theta^{u-v} \lambda^{v}}_{\text{Binomial Theorem } (\theta+\lambda)^u}$$

$$= \frac{1}{u!} e^{-(\theta+\lambda)} (\theta+\lambda)^u$$

Claim: $U \sim \text{Poisson}(\theta+\lambda)$

$U = X + Y \sim \text{Poisson}(\theta+\lambda)$

Theorem:

If $X \perp Y$, $X \sim \text{Pois}(\theta)$, $Y \sim \text{Pois}(\lambda)$

$$X + Y \sim \text{Pois}(\theta+\lambda)$$

Continuous Case: $X, Y$ cts RVs,

$$(U,V) = g(X,Y) = (g_1(X,Y), g_2(X,Y)).$$

$$g: \mathbb{R}^2 \to \mathbb{R}^2$$

Assume: → $g$ is invertible
→ $g$ is continuously differentiable (in both coords)

then

$$\boxed{f_{U,V}(u,v) = f_{X,Y}\left(g_1^{-1}(X,Y),\ g_2^{-1}(X,Y)\right) \left| \det\left(\frac{\partial g^{-1}}{\partial(u,v)}\right) \right|} \quad (*)$$

Jacobian

Analogy: $Y = g(X)$ (Univariate)

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}}{\partial y} \right|$$

Generally: $h : \mathbb{R}^2 \to \mathbb{R}^2$

$$h(x,y) = (h_1(x,y),\ h_2(x,y))$$

then the Jacobian is

$$J = \frac{\partial h}{\partial(x,y)} = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix}$$

in our case

$$J = \frac{\partial g^{-1}}{\partial(u,v)} = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix}$$

## Determinants:

2x2 mtx $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ; $\det A = ad - cb$

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$(u, v) = g(x, y)$

Steps: ① get $g^{-1}$

② find $J$ and get determinant

③ plug in formula ⊛

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Ex. $(u, v) = (x + y, x - y)$

$u = g_1(x, y) = x + y$

$v = g_2(x, y) = x - y$

① get $g_1^{-1}$ and $g_2^{-1}$

$\frac{u+v}{2} = x$ and $\frac{u-v}{2} = y$

$g_1^{-1}(x, y) = \frac{u+v}{2}$ and $g_2^{-1}(x, y) = \frac{u-v}{2}$

② Jacobian Determinant

$$J = \frac{\partial g^{-1}}{\partial(u,v)} = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\det(J) = -\frac{1}{4} - \frac{1}{4} = -1/2$$

So $|\det J| = 1/2$

③ Plug into formula

$$f_{U,V}(u,v) = f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v))\,|\det J|$$

$$\boxed{= f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)\frac{1}{2}}$$