**Defn: Expectation** For a r.v. $X$ then

$$E[X] = \begin{cases} \int_{\mathbb{R}} x f(x)\,dx & X \text{ continuous} \\ \sum_{x \in \mathbb{R}} x f(x) & X \text{ discrete} \end{cases}$$

**Defn: Variance**

$$Var(X) = E[(X - E[X])^2]$$

**theorem:** $= \mathbb{E}[X^2] - \mathbb{E}[X]^2$

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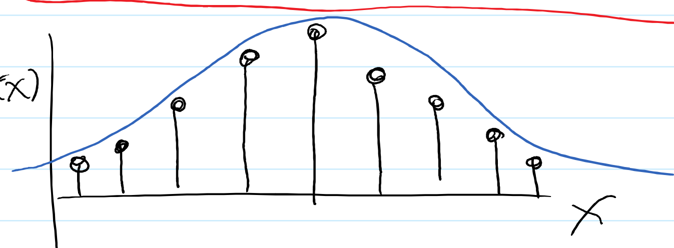
**Ex.** $X \sim Bin(n, p)$ — $p \in [0,1]$; $n$ integer $> 0$

canonical experiment: do $n$ independent coin flips each w/ a prob $p$ of H, then $X = \#\text{ heads} \sim Bin(n,p)$

**recall!** $E[X] = np$

$$E[X^2] = \sum_{x=0}^{n} x^2 f(x)$$

$$= \sum_{x=\cancel{0}1}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$



$$= \sum_{x=1}^{n} x\, n \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

① $x\binom{n}{x} = n\binom{n-1}{x-1}$

$$= \sum_{y=0}^{n-1} (y+1)\, n \binom{n-1}{y} p^{y+1} (1-p)^{n-y-1}$$

② change of variables $y = x-1$

$$= \sum^{n-1} (y+1) \binom{n-1}{y} p^y (1-p)^{(n-1)-y}$$

$x = y+1$

$$= np \sum_{y=0}^{n-1} (y+1) \binom{n-1}{y} p^y (1-p)^{(n-1)-y} \qquad y = x-1, \quad x = y+1$$

$$= np \left( \underbrace{\sum_{y=0}^{n-1} y \underbrace{\binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{\text{pmf}}}_{E[\text{Bin}(n-1,p)] = (n-1)p} + \underbrace{\sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{\text{Sum of Bin}(n-1,p)\text{ pmf} = 1} \right)$$

$$= \boxed{np\big((n-1)p + 1\big) = E[X^2].}$$

$$\text{Var}(X) = \boxed{E[X^2] - E[X]^2}$$

$$= np\big((n-1)p+1\big) - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np - np^2 \boxed{= np(1-p)}$$

Intro Stats: $\sqrt{\frac{p(1-p)}{n}}$

$$\text{Sd}(X) = \sqrt{\text{Var}(X)} = \sqrt{np(1-p)}$$

Defn: Moments of a RV

If $r$ is a pos. integer we define the $r^{th}$ moment of $X$ as

$$\mu_r = \mathbb{E}[X^r].$$

Ex. $\mu_1 = \mu = \mathbb{E}[X]$

$\mu_2 = \mathbb{E}[X^2]$

$\mu_3 = \mathbb{E}[X^3]$ ...

Ex. $X \sim Bin(n,p)$

$\mu_1 = \mathbb{E}[X] = np$

$\mu_2 = \mathbb{E}[X^2] = np((n-1)p+1)$

Defn: Moment Generating Function (MGF).

If $X$ is a r.v. then the MGF of $X$ is a function

$$M : \mathbb{R} \longrightarrow \mathbb{R}$$

defined for $t \in \mathbb{R}$ as

$$M(t) = \mathbb{E}[e^{tX}].$$

for some neighborhood around 0

$0$

For discrete

$$M(t) = \mathbb{E}[e^{tX}] = \sum_x e^{tx} f(x)$$

continuous

$$M(t) = \int e^{tx} f(x)\,dx$$

$$M(t) = \int_{\mathbb{R}} e$$

---

Ex. $X \sim \mathrm{Exp}(\lambda)$ ; $\lambda > 0$

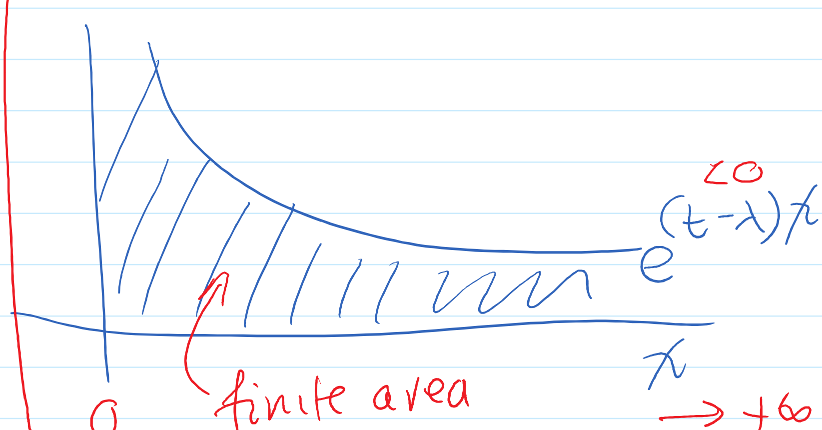
$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

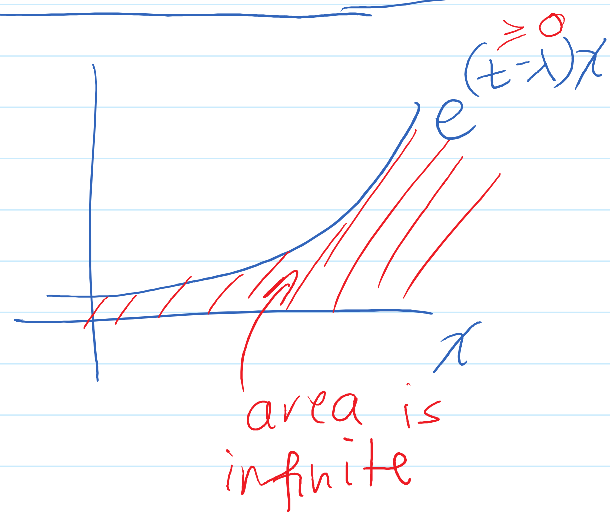
$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} e^{(t-\lambda)x}\,dx$$

$t < \lambda$ then $t - \lambda < 0$

$t \geq \lambda$ then $t - \lambda \geq 0$




$$= \lambda \int_0^{\infty} e^{(t-\lambda)x}\,dx = \lambda \left[ \frac{e^{\overset{<0}{(t-\lambda)}x}}{t-\lambda} \right]_0^{\infty} = \frac{\lambda}{t-\lambda}[0-1]$$

$$M(t) = \frac{\lambda}{\ }$$

$$\boxed{M(t) = \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda}$$

$$\boxed{M(t) = \frac{\lambda}{\lambda-t} \text{ for } t<\lambda}$$

Recall for $Exp(\lambda)$

$\mathbb{E}[X] = 1/\lambda$

$\mathbb{E}[X^2] = 2/\lambda^2$

Consider:

1. $\frac{dM}{dt} = \frac{\lambda}{(\lambda-t)^2}$ and $m'(0) = \left.\frac{dM}{dt}\right|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

$= \mathbb{E}[X]$

2. $\frac{d^2M}{dt^2} = \frac{2\lambda}{(\lambda-t)^3}$ so $M''(0) = \left.\frac{d^2M}{dt^2}\right|_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$

$= \mathbb{E}[X^2]$

**Theorem:** $\left.\frac{d^rM}{dt^r}\right|_{t=0} = M^{(r)}(0) = \mathbb{E}[X^r] = \mu_r.$

pf (continuus case)

$$\frac{d^rM}{dt^r} = \frac{d^r}{dt^r}\mathbb{E}[e^{tX}] = \frac{d^r}{dt^r}\int e^{tx}f(x)\,dx$$

$$\overset{\circledast}{=} \int \frac{d^r}{dt^r}\left(e^{tx}\right)f(x)\,dx$$

$$= \int \frac{d^r}{dt^r}\left(e^{tX}\right) f(x)\,dx$$

$$\frac{d}{dt} e^{tX} = x e^{tX}, \quad \frac{d^2}{dt^2} e^{tX} = \frac{d}{dt}\left(x e^{tX}\right) = x \frac{d}{dt} e^{tX} = x^2 e^{tX}$$

in general $\frac{d^r}{dt^r} e^{tX} = x^r e^{tX}$

$$\Rightarrow \frac{d^r M}{dt^r} = \int x^r e^{tX} f(x)\,dx$$

So $$\left.\frac{d^r M}{dt^r}\right|_{t=0} = \int x^r e^{0x} f(x)\,dx = \int x^r f(x)\,dx$$

$$= \mathbb{E}[X^r] = \mu_r.$$

2 Hand-wavy: $\Rightarrow$ expand function as a power series

$$f(x) = \sum_r a_r x^r$$

$\rightarrow$ analogy: powers of $x$ tell us about a fn

and so $\mathbb{E}[X^r]$ tell us about a r.v.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots$$

$2! \quad 3!$

$$\begin{cases} \dfrac{de^{tX}}{dt} = X + \dfrac{2tX^2}{2!} + \cdots \\ \dfrac{d^2e^{tX}}{dt^2} = X^2 + \cdots \end{cases}$$

$$\Rightarrow \frac{d}{dt}E[e^{tX}] = E\left[\frac{d}{dt}e^{tX}\right]\Big|_{t=0}$$
$$= E\left[X + \cancel{\frac{2(0)X^2}{2!}}^{\,0} + \cdots\right]$$
$$= E[X] = \mu_1$$

$$\Rightarrow \frac{d^2}{dt^2}E[e^{tX}] = E\left[\frac{d^2}{dt^2}e^{tX}\right]\Big|_{t=0}$$
$$= E[X^2 + 0] = \mu_2$$

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Ex. $X \sim Bin(n,p)$

$$M(t) = E[e^{tX}] = \sum_{x=0}^{n} (e^t)^x \binom{n}{x} (p)^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (\overset{a}{pe^t})^x (\overset{b}{1-p})^{n-x}$$

$$= (pe^t + 1 - p)^n$$

Binomial Theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

For binomial $\boxed{M(t) = (pe^t + 1 - p)^n}$

Theorem: recall $E[X] = np$; $E[X^2] = np((n-1)p+1)$
$= n^2p^2 - np^2 + np$

$$\frac{dM}{dt}\Big|_{t=0} = \frac{d}{dt}(pe^t + 1 - p)^n\Big|_{t=0}$$
$$= n(pe^t + 1 - p)^{n-1} pe^t\Big|_{t=0}$$

$$= n(p(1)+1-p)^{n-1}p(1)$$

$$= np = E[X] = \mu_1$$

$$\frac{d^2M}{dt^2} = \frac{d}{dt}\left(n\underbrace{(pe^t+1-p)^{n-1}}pe^t\right)$$

$$= n(n-1)(pe^t+1-p)^{n-2}pe^tpe^t$$

$$+ n(pe^t+1-p)^{n-1}pe^t$$

So

$$\left.\frac{d^2M}{dt}\right|_{t=0} = n(n-1)(\underbrace{p(1)+1-p}_{1})^{n-2}p(1)p(1)$$

$$+ n(\underbrace{p(1)+1-p}_{1})^{n-1}p(1)$$

$$= n(n-1)p^2 + np$$

$$= \underline{n^2p^2 - np^2 + np} = E[X^2] = \mu_2$$

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**Theorem:** For constants $a$ and $b$ let

$$Y = aX + b$$

then $\boxed{M_Y(t) = e^{bt}M_X(at)}$

MGF of $Y$ MGF of $X$

MGF of $Y$ MGF of $X$

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Pf. $\underline{M_Y(t)} = E[e^{tY}] = E[e^{t(aX+b)}]$

$= E[e^{(at)X} e^{tb}]$

$= e^{tb} E[e^{(at)X}] \longrightarrow M_X(t) = E[e^{tX}]$

$= \underline{e^{tb} M_X(at)}$

---

Theorem: If $X$ and $Y$ are RVs. and

$$M_X(t) = M_Y(t)$$

for all $t$ in some neighborhood of $0$, then

$$X \stackrel{d}{=} Y.$$

equal in dist.

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