Indicator Function

$$\mathbb{1}(\text{statement}) = \begin{cases} 1 & \text{statement true} \\ 0 & \text{statement false} \end{cases}$$

indicates the truth of statement

Ex. $\mathbb{1}(x \in A) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$

alt. $\mathbb{1}_A(x)$

PDF of a r.v. $X \sim \text{Exp}(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

" $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ "

using indicator function

$$\boxed{f(x) = \lambda e^{-\lambda x} \mathbb{1}(x > 0)} = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$\mathbb{1}(x > 0) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Independence:

$$\boxed{\text{Theorem: } X \perp\!\!\!\perp Y \iff f(x,y) = f(x)f(y)}$$

Ex. $f(x,y) = \underbrace{e^{-x}}_{f(x)}\underbrace{e^{-y}}_{f(y)}$ for $x>0$ and $y>0$

$$f(x,y) = e^{-x}e^{-y}\mathbb{1}(x>0 \text{ and } y>0)$$

Fact! $\mathbb{1}(A \text{ and } B) = \mathbb{1}(A)\mathbb{1}(B)$

$$= e^{-x}e^{-y}\mathbb{1}(x>0)\mathbb{1}(y>0)$$

$$= \underbrace{\left(e^{-x}\mathbb{1}(x>0)\right)}_{f(x)}\underbrace{\left(e^{-y}\mathbb{1}(y>0)\right)}_{f(y)} \qquad X \perp\!\!\!\perp Y$$

Ex. $f(x,y) = c\,e^{-x}e^{-y}$ for $0<x<y<1$

$$\boxed{= c\,e^{-x}e^{-y}\mathbb{1}(0<x<y<1)}$$

No way to factor. So $X \not\perp\!\!\!\perp Y$

Generally:

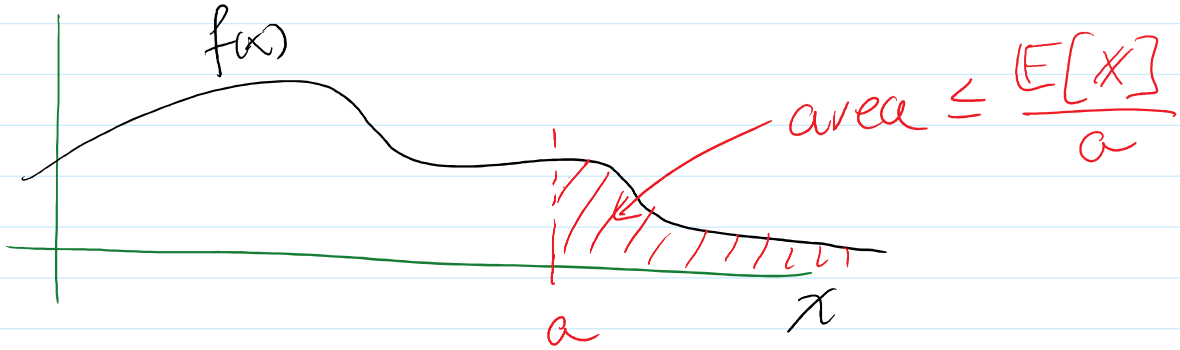
$$f(x) = (\ldots\ldots)\,\mathbb{1}(x \in \text{Support})$$

Inequalities

Theorem: Markov's Inequality

$X \geq 0$ then for $a > 0$

$$P(X > a) \leq \frac{\mathbb{E}[X]}{a}.$$



pf. (cts case)

$$\boxed{\mathbb{E}[X]} = \int_{\mathbb{R}} x f(x)\,dx = \int_0^\infty x f(x)\,dx$$

$$= \underbrace{\int_0^a x f(x)\,dx}_{A} + \underbrace{\int_a^\infty x f(x)\,dx}_{B}$$

$A \geq 0$

$A + B \geq B$

$x \geq 0$ $f(x) \geq 0$

whole integral ≥ 0

$$\geq \int_a^\infty x f(x)\,dx \quad B$$

when $x \in [a, \infty)$, $x \geq a$

$$\geq \int_a^\infty a f(x)\,dx = a \int_a^\infty f(x)\,dx = \boxed{a\,P(X \geq a)}$$

$$\int_a a f(x)\,dx = a \int_a f(x)\,dx = \boxed{a\,\mathbb{P}(X \geq a)}$$

Says: $\mathbb{E}[X] \geq a\,\mathbb{P}(X \geq a)$

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

Theorem: Chebyshev's Inequality

If $\mu = \mathbb{E}[X]$ and $\sigma^2 = Var(X)$

then
$$\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2.$$

Pf. $Y = \dfrac{(X-\mu)^2}{\sigma^2}$ and $a = k^2$

notice: $Y \geq 0$ so by Markov's Ineq.

$$\mathbb{P}(Y \geq k^2) \leq \frac{\mathbb{E}[Y]}{k^2}$$

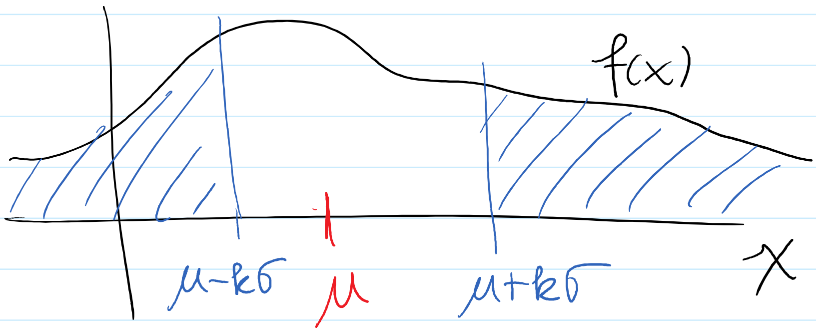
$$\mathbb{E}[Y] = \mathbb{E}\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2}\overbrace{\mathbb{E}\left[(X-\mathbb{E}[X])^2\right]}^{Var(X)=\sigma^2} = 1$$

$$\mathbb{P}(Y \geq k^2) \leq 1/k^2$$

or
$$\mathbb{P}\left(\frac{(X-\mu)^2}{\sigma^2} \geq k^2\right) \leq 1/k^2$$

$$\mathbb{P}\left(\frac{|X-\mu|^2}{\sigma^2} \geq k^2\right) = 1/k^2$$

or $$\boxed{\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2}$$



prob. $\geq k$ stds away from μ is $\leq 1/k^2$

Various Forms

① $\boxed{\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2}$

② $\boxed{\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \leq k\right) \geq 1 - 1/k^2}$

③ $\varepsilon = k\sigma$ then $\boxed{\mathbb{P}(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}}$

$k = \varepsilon/\sigma$

$1/k^2 = \sigma^2/\varepsilon^2$

④ $\boxed{\mathbb{P}(|X-\mu| \leq \varepsilon) \geq 1 - \sigma^2/\varepsilon^2}$

Ex. Let X = # nails in a manufactured box by same factory.

$\mu = E[X] = 1000$

$\sigma^2 = Var(X) = 25$

What is the prob. that

$$994 \le X \le 1006$$

"$X = 1000 \pm 6$"

$\sigma^2 = 25$ so $\boxed{\sigma = 5}$

$$\boxed{P\left(\frac{|X-\mu|}{\sigma} \le k\right) \ge 1 - \frac{1}{k^2}}$$

$$\boxed{P(994 \le X \le 1006)}$$

$$= P(|X - 1000| \le 6)$$

$$= P\Big(\frac{|X-1000|}{5} \le \underbrace{1.2}_{k}\Big) \ge 1 - 1/k^2$$

by Chebyshev

$$1 - \frac{1}{(1.2)^2} \approx .3055$$

Convergence of Random Variables

Case II: considered

$x_n \longrightarrow x$ where x_n, $x \in \mathbb{R}$

Case II: Considered

$$x_n \to x \quad \text{where } x_n, x \in \mathbb{R}$$

Now: Consider $X_n \to X$ where

X_n, X are r.v.s

Recall: a r.v. is a function

$$X_n : S \to \mathbb{R}$$

so is $X : S \to \mathbb{R}$

Soln: Consider r.v.s as functions and talk about convergence of fns.

Defn: Convergence of Functions (Pointwise)

If $\{f_n\}_{n=1}^{\infty}$ is a seq. of fns

$$f_n : \mathbb{R} \to \mathbb{R}$$

we say that f_n converge pointwise to f where $f : \mathbb{R} \to \mathbb{R}$

if for all $x \in \mathbb{R}$

convergence as a seq. of numbers.

$$f_n(x) \to f(x)$$

Say $x=1$

$f_1(1), f_2(1), f_3(1), f_4(1), \ldots$ a seq. of #s

better converge to $f(1)$

Say $x=2$ $f_1(2), f_2(2), f_3(2), \ldots \longrightarrow f(2)$

$\vdots$ etc.

Defn: Sure Convergence of r.v.s.

We say $\{X_n\}_{n=1}^{\infty}$ converges to X if they converge pointwise as function.

$$X_n \xrightarrow{\text{ptwsise}} X$$

or

$$X_n \xrightarrow{s} X$$

i.e. $X_n(s) \longrightarrow X(s) \quad \forall s \in S.$

Defn: Almost Sure Convergence (a.s.)

We say $\{X_n\}$ converge almost surley to X $\to$ if X_n converge surely to X on some set A and $\mathbb{P}(A) = 1$

$$X_n(s) \longrightarrow X_n(s) \quad \forall s \in A \subset S$$

$\rightarrow$ if the set where $X_n \not\xrightarrow{\text{ptwise}} X$ is $E \subset S$ and $\mathbb{P}(E) = 0$.

Notation: $X_n \xrightarrow{a.s.} X$

$$\mathbb{P}(\{s : X_n(s) \to X(s)\}) = 1$$

or $$\mathbb{P}(\{s : X_n(s) \not\to X(s)\}) = 0$$

Ex. Let $S = [0,1]$ w/ uniform density

Consider

$$X_n(s) = s + s^n \quad \forall s \in S$$

$$X(s) = s \quad (\text{i.e. } X \sim U(0,1))$$

$\boxed{\text{Does } X_n \xrightarrow{a.s.} X \text{ ?}}$

as $n \to \infty$, $s^n \to 0$

For $s \in [0,1)$

$$X_n(s) = s + s^n \to s = X(s)$$

$$X_n \xrightarrow{\text{ptwise}} X \text{ on } [0,1)$$

When $s = 1$

$$X_n(s) = 1 + 1 = 2 \not\to X(s) = 1$$

Since $\mathbb{P}([0,1)) = 1$, equiv $\mathbb{P}(\{1\}) = 0$

Since $\mathbb{P}([0,1)) = 1$, equiv $\mathbb{P}(\{1\}) = 0$

then we don't care about $\{1\}$

So

$$X_n \xrightarrow{a.s.} X.$$

✓

Defn: Convergence in Probability

We say X_n converges in probability to X

denoted

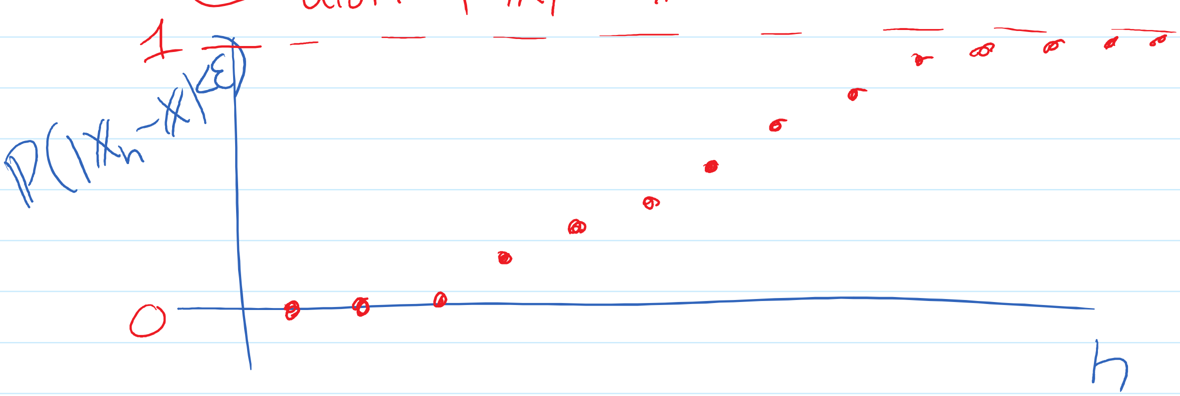
$$X_n \xrightarrow{P} X$$

If $\forall \varepsilon > 0$,

$$\left.\begin{aligned} &\lim_{n\to\infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1 \\ \text{or} \quad &\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0 \end{aligned}\right\}$$

$\mathbb{P}(|X_1 - X| < \varepsilon)$, $\mathbb{P}(|X_2 - X| < \varepsilon)$, $\mathbb{P}(|X_3 - X| < \varepsilon)$, ...

dist. of X_1 to X



Convergence in prob.

For large enough n $\mathbb{P}(|X_n - X| \geq \varepsilon) \approx 0$

For large enough n, $P(|X_n - X| \geq \varepsilon) \approx 0$

$$P(|X_n - X| < \varepsilon) \approx 1$$

Convergence almost surely

For large enough n $P(|X_n - X| \geq \varepsilon) = 0$

$$P(|X_n - X| \leq \varepsilon) = 1$$

Theorem: Convergence a.s. implies convergence in prob.

If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{P} X$.

Ex. $S = [0,1]$ w/ uniform density

$X_1(s) = s$ | $|X_1(s) - X(s)| = 0$

$X_2(s) = s + \mathbb{1}(s \in [0, \tfrac{1}{2}])$ | $|X_2(s) - X(s)| = \mathbb{1}(s \in [0, \tfrac{1}{2}))$

$X_3(s) = s + \mathbb{1}(s \in (\tfrac{1}{2}, 1])$ | $|X_3(s) - X(s)| = \mathbb{1}(s \in [\tfrac{1}{2}, 1))$

$X_4(s) = s + \mathbb{1}(s \in [0, \tfrac{1}{3}))$ | $= \mathbb{1}(s \in [0, \tfrac{1}{3}))$

$X_5(s) = s + \mathbb{1}(s \in [\tfrac{1}{3}, \tfrac{2}{3}))$ | $\vdots$

$X_6(s) = s + \mathbb{1}(s \in [\tfrac{2}{3}, 1))$

etc.

$X(s) = s$

$X_n \xrightarrow{P} X$

$X(s) = s$

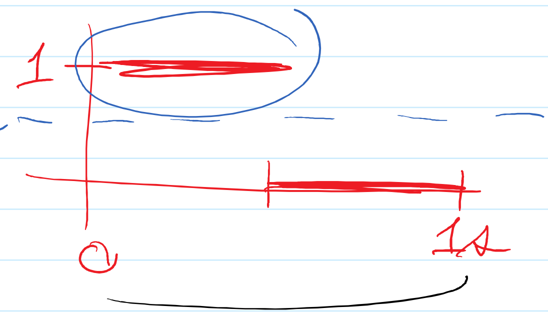
To establish $X_n \xrightarrow{P} X$

$$\boxed{\mathbb{P}(|X_n - X| \geq \varepsilon) \xrightarrow{n} 0}$$

Choose some $0 < \varepsilon < 1$

$$\mathbb{P}(|X_1 - X| \geq \varepsilon) = 0$$

$$\mathbb{P}(|X_2 - X| \geq \varepsilon) = 1/2$$



$$\mathbb{P}(|X_3 - X| \geq \varepsilon) = 1/2$$

$$\mathbb{P}(|X_4 - X| \geq \varepsilon) = 1/3$$

$$\vdots \qquad = 1/3$$

$$= 1/3$$

$$= 1/4$$

$$= 1/4$$

$$\vdots$$

decreasing to zero

So if we believe $\mathbb{P}(|X_n - X| \geq \varepsilon) \xrightarrow{n} 0$

then $X_n \xrightarrow{P} X$.

Q: $X_n \xrightarrow{a.s.} X$? No.

If $A = \{ s \in S \mid X_n(s) \longrightarrow X(s) \}$

then $P(A) = 1$.

For any $s \in [0,1]$ $X_n(s)$ bounces between s and $s+1$

So it won't settle down to $X(s) = s$

So $X_n \overset{a.s.}{\not\to} X$.

$\mathbb{P}(A) = 0$.

Converse: Convergence in prob. doesn't imply convergence a.s.

Defn: Convergence in distribution

We say $X_n \xrightarrow{d} X$ "in distribution" if the CDFs converge pointwise.

i.e.

$$F_{X_n}(x) \longrightarrow F(x) \quad \forall x.$$
