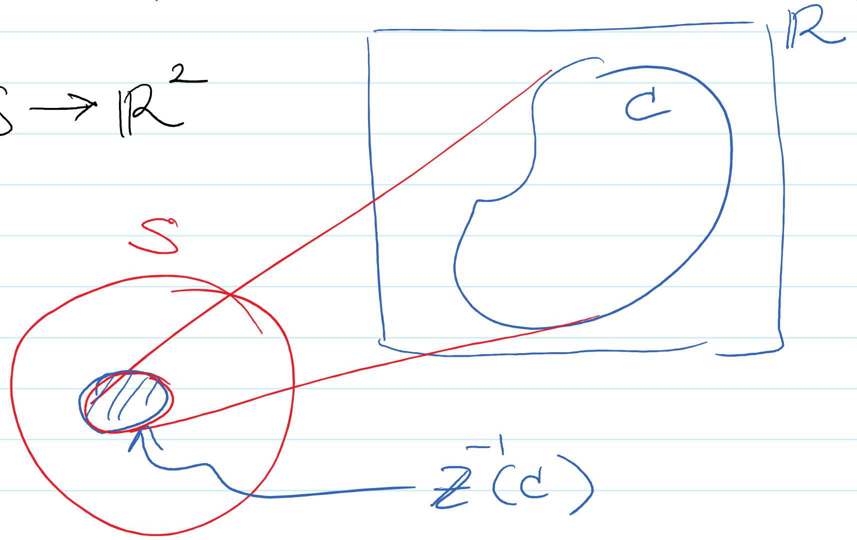
Lecture 16 - Bivariate Random Variables

If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$ then

(X, Y) is a <u>bivariate RV</u>.

<u>Say:</u> $$P(\underbrace{(X,Y)}_{Z} \in \overset{C \subset \mathbb{R}^2}{C}) = P(Z^{-1}(C))$$

$$Z = (X, Y) : S \to \mathbb{R}^2$$

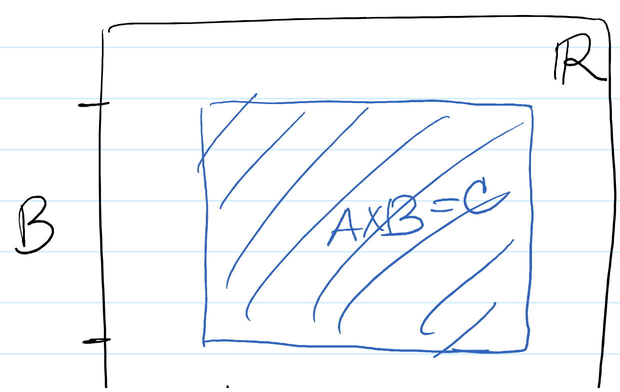


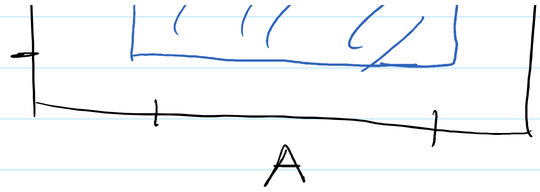
Often $C = A \times B$ where $A, B \subset \mathbb{R}$

So $C \subset \mathbb{R}^2$

Say $$P((X,Y) \in A \times B) = \boxed{\overset{\text{write}}{P(X \in A, Y \in B)}}$$ "and"

$$= P(X^{-1}(A) \cap Y^{-1}(B))$$





Ex.

Flip a coin 3 times.

$$X = \begin{cases} 0 & \text{if last flip is a } T \\ 1 & \text{if} \quad \text{"} \quad H \end{cases}$$

Y = # heads among 3 flips.

$Z = (X, Y)$

$s \in S$	$Z(s)$
H H H	(1, 3)
H H T	(0, 2)
H T H	(1, 2)
H T T	(0, 1)
T H H	(1, 2)
T H T	(0, 1)
T T H	(1, 2)
T T T	(0, 0)

Defn: Bivariate CDF

The BiV CDF is a fn

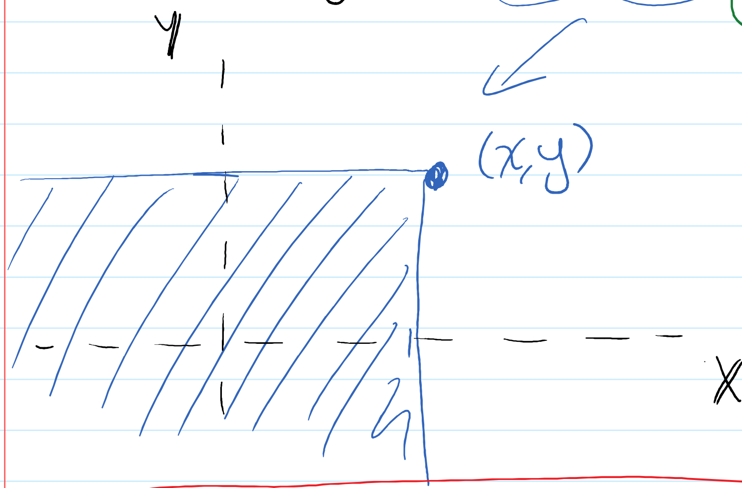
$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

so for $x, y \in \mathbb{R}$,

$$F(x,y) = P(X \le x, Y \le y)$$

"and"

$F(x,y) = \mathbb{P}(X \le x, Y \le y)$ "and"



Theorem: BiV CDF Properties

1. $F(x,y) \geq 0$
2. $\lim\limits_{x,y \to \infty} F(x,y) = 1$ (univariate: $\lim\limits_{x \to \infty} F(x) = 1$)
3. $\lim\limits_{x \to -\infty} F(x,y) = 0$ and $\lim\limits_{x \to -\infty} F(x,y) = 0$ (univariate: $\lim\limits_{x \to -\infty} F(x) = 0$)
4. F is non-decreasing and right-continuous in each argument.

Defn: Marginal Distribution

If (X, Y) is a BiV RV then we say X and Y are the marginal RVs and their distributions are called the marginal distributions.

Theorem: Rel. btw BiV and UniV CDFs

$$\begin{cases} ① \quad F_Y(y) = \lim_{x\to\infty} F(x,y) \\ ② \quad F_X(x) = \lim_{y\to\infty} F(x,y) \end{cases}$$

BiV CDFs.

univariate CDFs

idea: $F_X(x) = P(X \le x) = P(X \le x,\ Y = \text{anything})$

$$= P(X \le x,\ Y \le \infty)$$

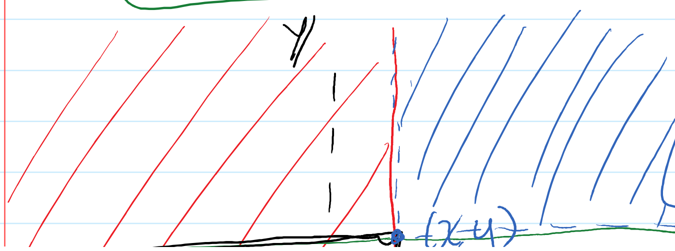
$$= \lim_{y\to\infty} P(X \le x,\ Y \le y)$$

$$= \lim_{y\to\infty} F(x,y).$$

Lemma:

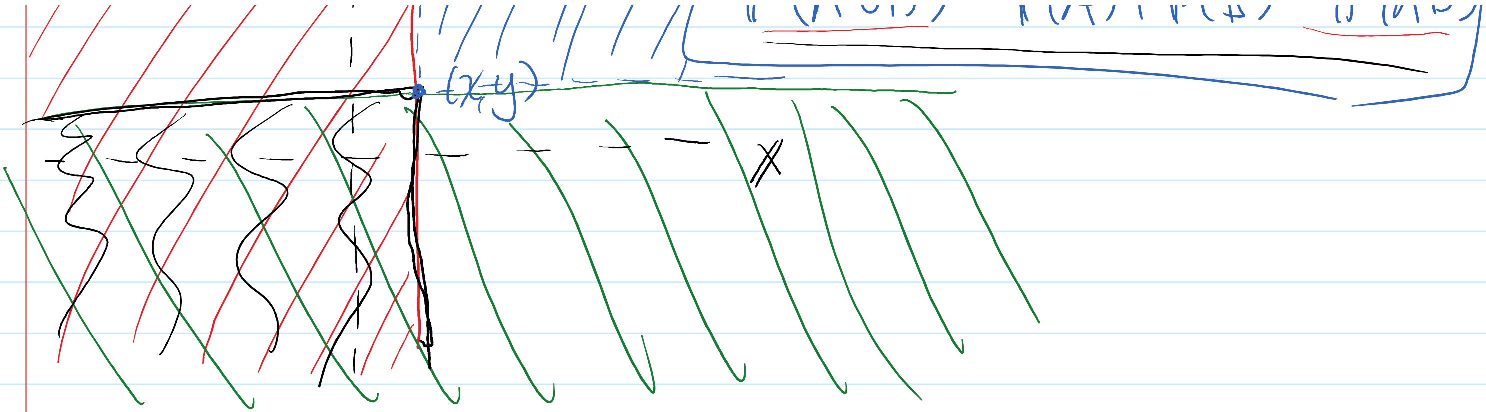
Motivation:
for Univariate $P(X > x) = 1 - F(x)$

$$\boxed{\text{BiV: } P(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F(x,y)}$$



looks like:

$$P(A \cup B) = P(A) + P(B) - P(AB)$$



Extension: $P(x_1 \le X \le x_2,\ y_1 \le Y \le y_2)$

$$= F(x_2,y_2) + F(x_1,y_1) - F(x_1,y_2) - F(x_2,y_1)$$

pf. Similar picture arguement.

Defn: Joint PMF

If X and Y are both discrete then the joint PMF (of X and Y) is

$$\boxed{f(x,y) = P(X=x,\ Y=y).}$$

Theorem: f is a valid joint PMF iff

① $\sum_{x,y} f(x,y) = 1$ ② $f(x,y) \ge 0$.

Theorem: Rel. btwn Joint + Marginal PMFs

① $f_X(x) = \sum f(x,y)$ ... over all

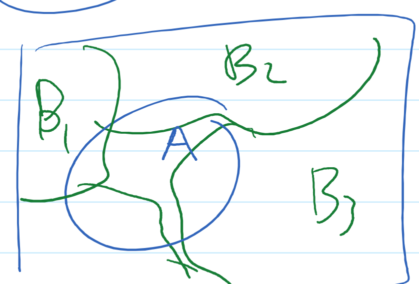
① $f_X(x) = \sum_y f(x,y)$ sum over all possible values

② $f_Y(y) = \sum_x f(x,y)$

pf. relies on the fact that $\{Y=y\}$ over all y partitions S

early lectures on partitioning

$$f_X(x) = \mathbb{P}(X=x) = \sum_y \mathbb{P}(\text{“}X=x\text{”} \cap \text{“}Y=y\text{”})$$



$$= \sum_y \mathbb{P}(X=x,\ Y=y)$$

$$= \sum_y f(x,y)$$

Ex. Flip 3 coins.

$$X = \begin{cases} 0 & \text{last flip is T} \\ 1 & \text{〃 H} \end{cases}$$

Y = # H in 3 flips

check:
① $f(x,y) \geq 0$ ✓
② $\sum_{x,y} f(x,y) = 1$ ✓

$f(x,y)$

X \ Y	0	1	2	3	
0	$f(0,0) = 1/8$	2/8	1/8	0	$1/2 = f_X(0)$
1	$f(1,0) = 0$	1/8	2/8	1/8	$1/2 = f_X(1)$
	1/8	3/8	3/8	1/8	

1/8	3/8	3/8	1/8
$\Vert$	$\Vert$	$\Vert$	$\Vert$
$f_Y(0)$	$f_Y(1)$	$f_Y(2)$	$f_Y(3)$

$$f_X(x) = \sum_y f(x,y)$$

E.g. $f(0) = \sum_{y=0}^{3} f(0,y)$

$$f_Y(2) = \sum_{x=0,1} f(x,2) = f(0,2) + f(1,2)$$

Defn: Joint PDF

If X and Y are continuous we call

$$f : \mathbb{R}^2 \to \mathbb{R}$$

the joint PDF if $\forall C \subset \mathbb{R}^2$

$$P((X,Y) \in C) = \int_C f(x,y)\, dx\, dy$$

Recall in univariate:

$$\mathbb{P}(X \in A) = \int_A f(x)\, dx.$$

Facts:

① $\boxed{F(x,y) = \mathbb{P}(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, du\, dv}$

(Univariate: $F(x) = \int_{-\infty}^{x} f(t)\, dt$)

② $\boxed{f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)}$

(univariate: $f(x) = \frac{d}{dx} F(x)$)

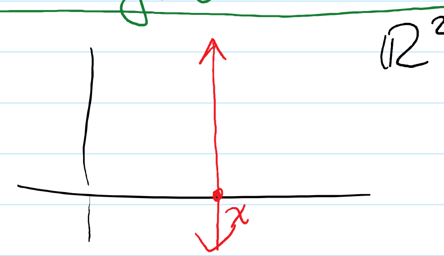
③ f is a valid joint PDF <u>iff</u>

(i) $f(x,y) \geq 0$ and (ii) $\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$.

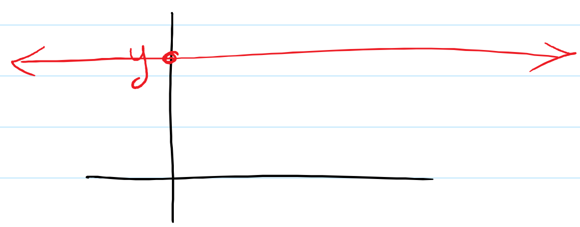
(<u>univariate:</u> $f(x) \geq 0$ and $\int_{\mathbb{R}} f(x)\,dx = 1$).

<u>Theorem: Rel. btw Joint + Marginal PDFs</u>

① $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$



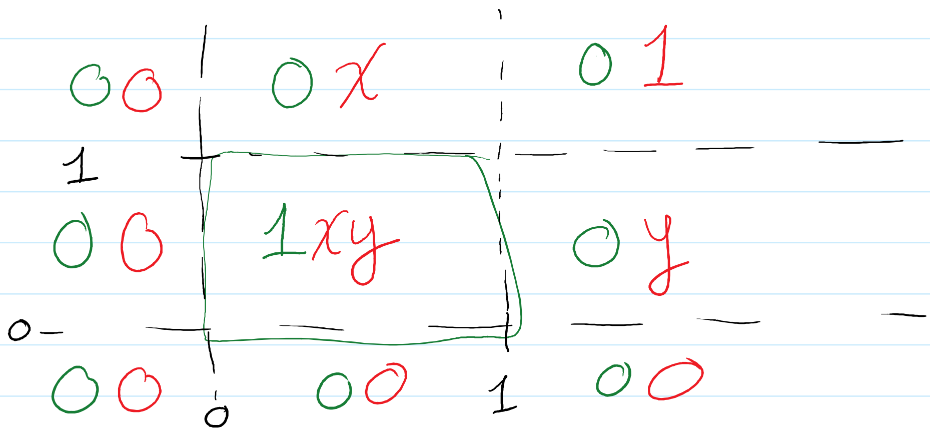
② $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$



<u>Ex.</u>

$$F(x,y) = \begin{cases} 0 & x \leq 0 \text{ or } y \leq 0 \\ xy & (x,y) \in [0,1]^2 \\ x & y > 1 \text{ and } x \in [0,1] \\ y & x > 1 \text{ and } y \in [0,1] \\ 1 & x > 1 \text{ and } y > 1 \end{cases}$$

$x>1$ and $y>1$



① What is the joint PDF? Recall: $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$

hence

$$f(x,y) = 1 \quad \text{for } (x,y) \in [0,1]^2$$

(basically bivarate uniform over square)

② Marginals?

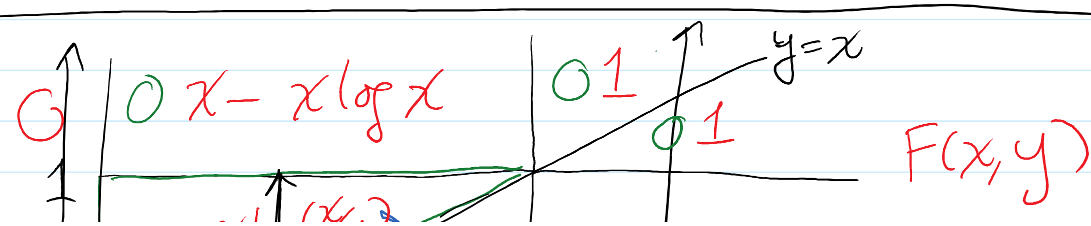
$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

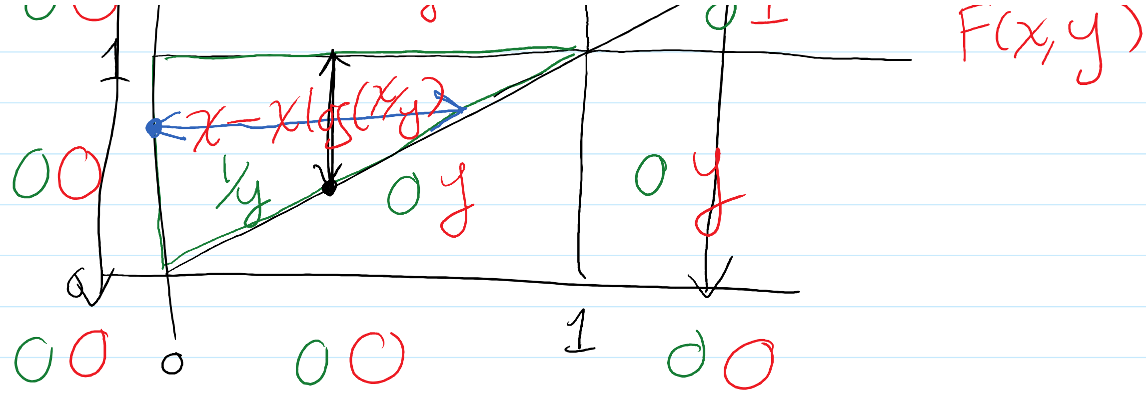
$$= \int_0^1 1\,dy = 1 \quad \text{for } x \in [0,1]$$

So $X \sim U(0,1)$

Similarly, $Y \sim U(0,1)$.

Ex,





Join PDF: $\boxed{f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)}$

So $\boxed{f(x,y) = 1/y \quad \text{for} \quad 0 \le x \le y \le 1}$

$$\frac{\partial^2}{\partial x \partial y}\left(x - x\log(x/y)\right)$$
$$= \frac{\partial}{\partial x}\left(x/y\right)$$
$$= 1/y$$

Marginals?

(1) $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_x^1 \frac{1}{y}\,dy = \log(y)\Big|_x^1$

$$= \log(1) - \log(x)$$
$$= -\log(x)$$

for $x \in [0,1]$

$$= -\log(x) \text{ for } x \in [0,1]$$

② $f_Y(y) = \int_0^y \frac{1}{y}\,dx = \frac{1}{y}\int_0^y dx = \frac{1}{y}\, y = 1$

for $y \in [0,1]$

i.e. $Y \sim U(0,1)$.

Ex. Let (X,Y) have a joint PDF of

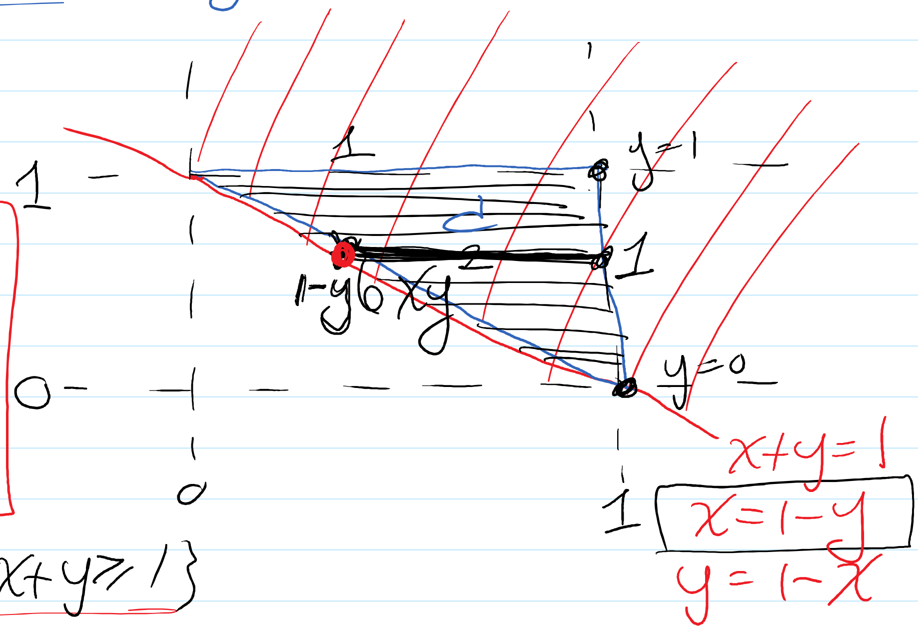
$$f(x,y) = 6xy^2 \text{ for } 0 \le x, y \le 1$$

$P(X+Y \geq 1)$

recall:

$P((X,Y) \in C)$

$= \int_C f(x,y)\,dx\,dy$



here, $C = \{(x,y) : x+y \geq 1\}$

So

1 1

So

$$\underline{\mathbb{P}(X+Y \geq 1)} = \int_{C'} f(x,y)\,dx\,dy = \int_0^1 \int_{1-y}^1 6xy^2\,dx\,dy$$

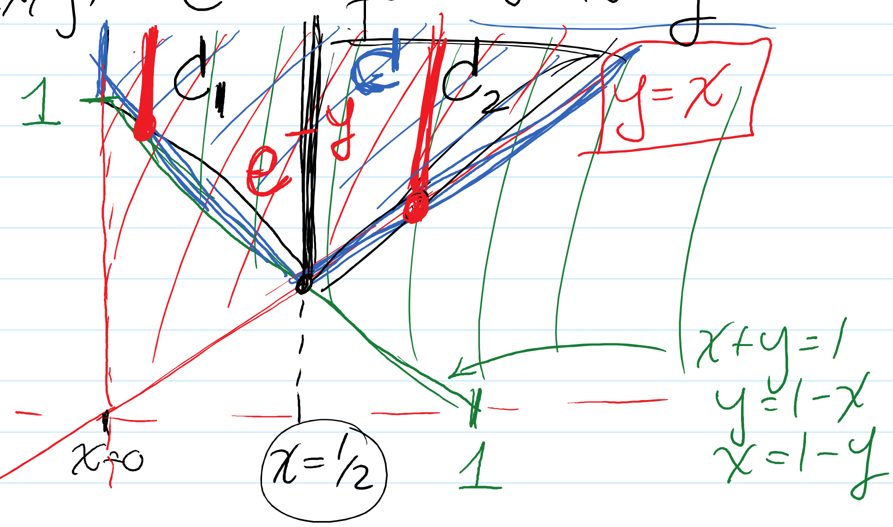
$$= \cdots \text{ Calc III } = \cdots = 9/10.$$

<u>Ex.</u> (X,Y) has a joint PDF of

$$f(x,y) = e^{-y} \text{ for } \underline{0 < x < y}$$

$\mathbb{P}(X+Y \geq 1)$

$$= \int_{d} f(x,y)\,dx\,dy$$



$$= \int_{d_1} f\,dx\,dy + \int_{d_2} f\,dx\,dy$$

$$= \int_0^{1/2}\int_{1-x}^{\infty} e^{-y}\,dy\,dx + \int_{1/2}^{1}\int_{x}^{\infty} e^{-y}\,dy\,dx$$

$$= \text{calc III} = \boxed{2e^{-1/2} - e^{-1}}$$