Lecture 17 - Bivariate Expectation and Conditional Distributions

Defn: Bivariate Expectation

If (X, Y) is Biv RV and $g: \mathbb{R}^2 \to \mathbb{R}$

then

joint pmf

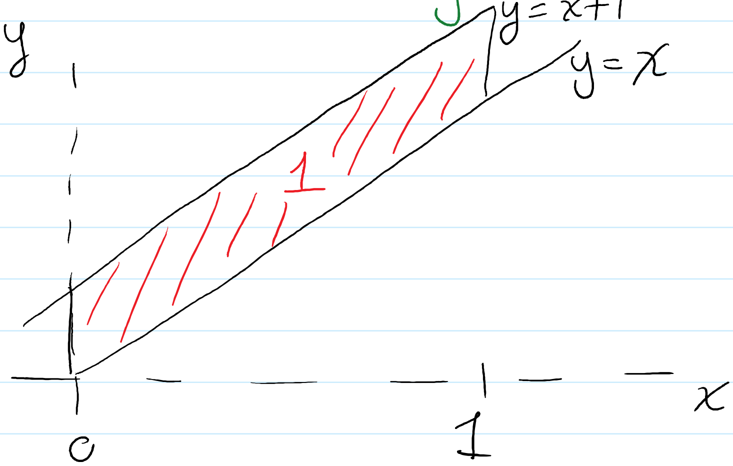
$$E[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) & \text{discrete} \\ \iint g(x,y) f(x,y)\,dx\,dy & \text{cts} \end{cases}$$

analogy: univariate $E[g(X)] = \int g(x) f(x)\,dx$

Ex. (X, Y) biv continuous w/ joint pdf

$g(x,y) = xy$ $\quad f(x,y) = 1$ for $0 < x < 1$

$x < y < x+1$



$E[XY]$

$= \int g(x,y) f(x,y)\,dx\,dy$

$= \int_0^1 \int_x^{x+1} xy\,(1)\,dy\,dx$

$= \int_0^1 \int_x^{x+1} xy\,dy\,dx = \text{for you} = 7/12.$

Theorem: Biv Expectation is Linear

If $g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$

then for any $a, b \in \mathbb{R}$

$$\mathbb{E}\left[a g_1(X,Y) + b g_2(X,Y)\right]$$

$$= a\mathbb{E}[g_1(X,Y)] + b\mathbb{E}[g_2(X,Y)].$$

Pf. follows from linearity of summation (discrete) or integration (cts).

Defn: Covariance

We define the covariance between two RVs X and Y as

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$$

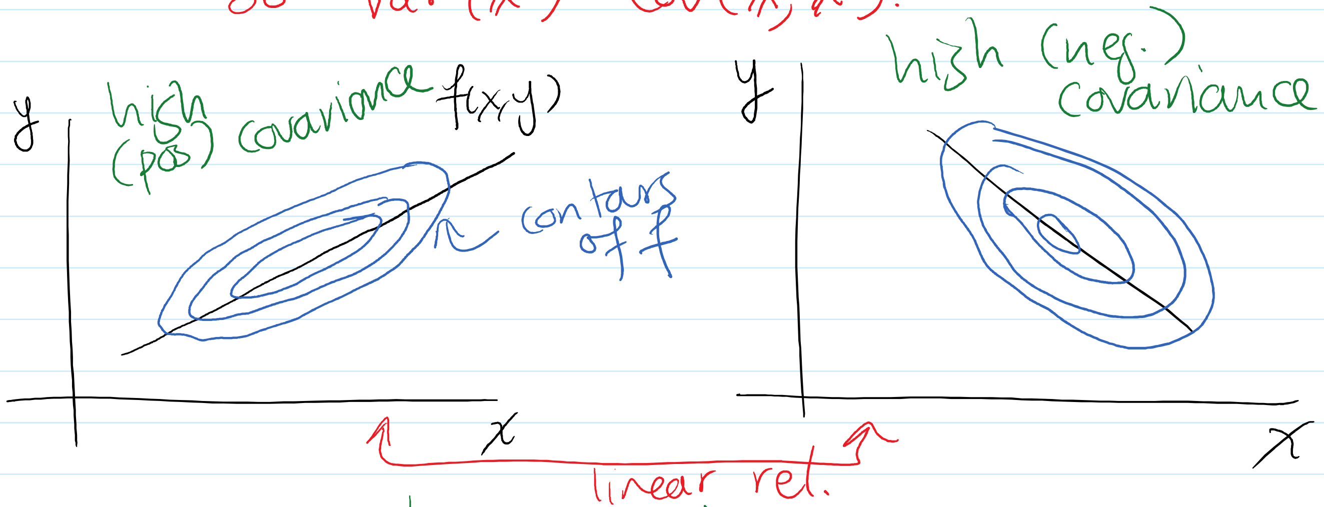
$\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$

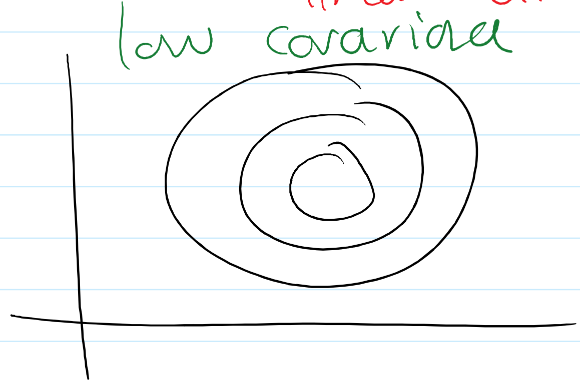
$$= \mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$g(x,y) = (x-\mu_X)(y-\mu_Y)$

Facts: recall that $Var(X) = E[(X - E[X])^2]$

so $Var(X) = Cov(X, X)$.





Defn: Correlation

Re-scaled covariance so that it is between ± 1.

$+1$ = perfect linear rel.

-1 = perfect (neg.) lin. rel.

0 = not lin. related.

We define the correlation btw X and Y as

$$\boxed{\operatorname{cor}(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}}$$

alt. define $sd(X) = \sqrt{\operatorname{Var}(X)}$
$sd(Y) = \sqrt{\operatorname{Var}(Y)}$

then

$$\boxed{\operatorname{cor}(X, Y) = \frac{\operatorname{Cov}(X, Y)}{sd(X)\, sd(Y)}.}$$

<u>Theorem:</u> If $a, b \in \mathbb{R}$ then

$$\boxed{\operatorname{Var}(aX + bY) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y) + 2ab \operatorname{Cov}(X, Y).}$$

<u>pf.</u> $Z = aX + bY$ want $\operatorname{Var}(Z)$

$$= E\left[(Z - E[Z])^2\right]$$

(1)

$$Z - E[Z] = aX + bY - E[aX + bY]$$

$$= aX + bY - a\underbrace{E[X]}_{\mu_X} - b\underbrace{E[Y]}_{\mu_Y}$$

$$= a(X - \mu_X) + b(Y - \mu_Y)$$

(2) $(Z - \mathbb{E}[Z])^2 = a^2(X-\mu_X)^2 + b^2(Y-\mu_Y)^2 + 2ab(X-\mu_X)(Y-\mu_Y)$

(3) $\mathbb{E}[\quad] = a^2 \overbrace{\mathbb{E}[(X-\mu_X)^2]}^{Var(X)} + b^2 \overbrace{\mathbb{E}[(Y-\mu_Y)^2]}^{Var(Y)} + 2ab\,\underbrace{\mathbb{E}[(X-\mu_X)(Y-\mu_Y)]}_{Cov(X,Y)}$.

<u>Theorem:</u> $a, b \in \mathbb{R}$, then

$$\boxed{Cov(aX+b, Y) = a\,Cov(X,Y).}$$

recall $Var(aX+b) = a^2 Var(X)$

Pf. $\mathbb{E}[(aX+b-\mathbb{E}[aX+b])(Y-\mathbb{E}[Y])]$

$= \mathbb{E}[(aX+\cancel{b} - a\mathbb{E}[X]-\cancel{b})(Y-\mathbb{E}[Y])]$

$= \mathbb{E}[a(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$

$= a\,Cov(X,Y)$.

additions:

$\to$ by symmetry: $\boxed{Cov(X, aY+b) = a\,Cov(X,Y).}$

→ by symmetry: $\boxed{\operatorname{Cov}(X, aY+b) = a\operatorname{Cov}(X,Y).}$

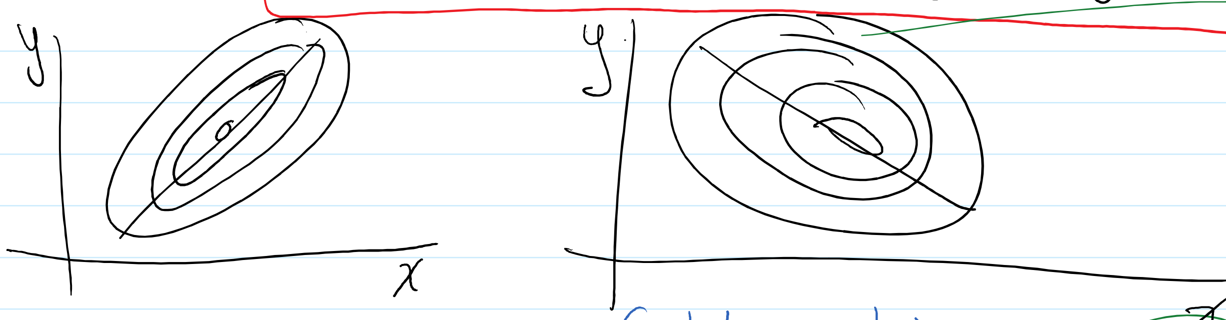
→ apply twice:

$$\rightarrow \boxed{\operatorname{Cov}(aX+b, cY+d) = \underline{ac\operatorname{Cov}(X,Y)}.}$$

<u>Theorem:</u> linear transformations of correlation.

If $a,b,c,d \in \mathbb{R}$ then

$$\boxed{\operatorname{Cor}(aX+b, cY+d) = \operatorname{sign}(a)\operatorname{sign}(c)\underline{\operatorname{Cor}(X,Y)}}$$



here $\operatorname{sign}(b) = \begin{cases} +1 & b>0 \\ -1 & b<0 \\ 0 & b=0 \end{cases} = \dfrac{b}{|b|}$

$\boxed{\underline{\text{alt}}: \quad |\operatorname{Cor}(aX+b, cY+d)| = |\operatorname{Cor}(X,Y)|.}$

<u>Ex.</u> $\operatorname{Cor}(X,Y) = -\operatorname{Cor}(X,-5Y)$

<u>pf.</u> $a, c \neq 0$

$$\operatorname{Cor}(aX+b, cY+d) = \frac{\operatorname{Cov}(aX+b, cY+d)}{\sqrt{\operatorname{Var}(aX+b)\operatorname{Var}(cY+d)}}$$

$$= \frac{ac\operatorname{Cov}(X,Y)}{\sqrt{a^2\operatorname{Var}(X)c^2\operatorname{Var}(Y)}}$$

$$\overline{\sqrt{a^2 Var(X) c^2 Var(Y)}}$$

$$= \frac{a}{|a|}\frac{c}{|c|} Cor(X, Y)$$

$$\operatorname{sign}(a)\operatorname{sign}(c).$$

<u>Claim:</u> $\boxed{-1 \le Cor(X, Y) \le 1}$ ✓

<u>Pf</u> eq. $|Cor(X,Y)| \le 1$.

$$\left.\begin{aligned} \tilde{X} &= \frac{X-\mu_X}{\sigma_X} \leftarrow sd(X) \\ \tilde{Y} &= \frac{Y-\mu_Y}{\sigma_Y} \end{aligned}\right\} \text{linear transf.}$$

<u>prove to yourself:</u>

$E[\tilde{X}] = 0 \quad Var(\tilde{X}) = 1$

Similar for $\tilde{Y}$.

① $|Cor(X,Y)| = |Cor(\tilde{X}, \tilde{Y})|$

② $Var(\tilde{X} \pm \tilde{Y}) = \underbrace{Var(\tilde{X})}_{1} + \underbrace{Var(\tilde{Y})}_{1} \pm 2\underbrace{Cov(\tilde{X}, \tilde{Y})}_{\rho}$

$$\rho = \underline{Cov(\tilde{X}, \tilde{Y})} = \frac{Cov(\tilde{X}, \tilde{Y})}{\sqrt{1 \cdot 1}} = \boxed{\underline{Cor(\tilde{X}, \tilde{Y})} = \rho}$$

③ $0 \le Var(\tilde{X} \pm \tilde{Y}) = 2 \pm 2\rho$

here $1 \pm \rho \geq 0$

hence $1+\rho \geq 0$ so $\rho \geq -1$

and $1-\rho \geq 0$ so $\rho \leq 1$

thus $\boxed{-1 \leq \rho \leq 1}$

Theorem: Short-cut formula for Covariance

$$\boxed{Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}$$

recall for univariate:

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Ex. from above $(*)$

what is Cor btwn X and Y.

Shown: $\mathbb{E}[XY] = 7/12$.

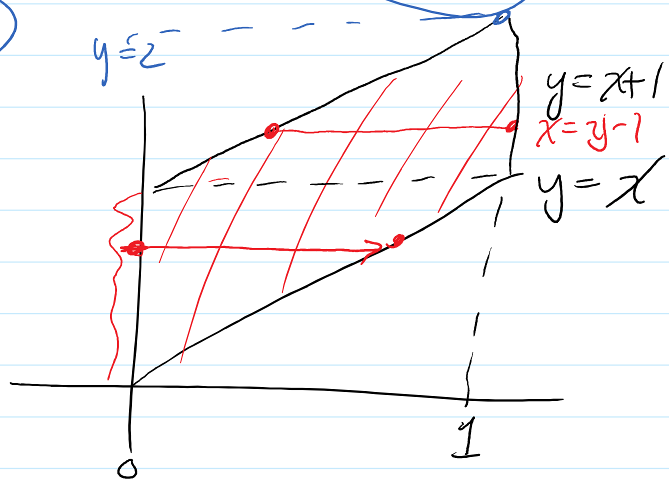
$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\overset{7/12}{\mathbb{E}[XY]} - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{Var(X)Var(Y)}}$$

Get marginals of X, Y

get marginals of X, Y.

X: $f(x) = \int f(x,y)\,dy = \int_x^{x+1} 1\,dy = x+1-x = 1$ for $0<x<1$

So $X \sim U(0,1)$



Y: $f(y) = \int f(x,y)\,dx$

$$= \begin{cases} \int_0^y 1\,dx & 0<y<1 \\ \int_{y-1}^1 1\,dx & 1<y<2 \end{cases}$$

$$= \begin{cases} y & 0<y<1 \\ 2-y & 1<y<2 \end{cases}$$

Show (on your own)

$E[X] = 1/2$; $Var(X) = 1/12$

$E[Y] = 1$; $Var(Y) = 1/6$

hence

$$\boxed{Cor(X,Y) = \frac{7/12 - (1/2)(1)}{\sqrt{(1/12)(1/6)}}}$$

$\sqrt{(1/12)(1/6)}$

Conditional Distributions

Recall: $A, B \subset S$ then

$$P(A|B) = \frac{P(AB)}{P(B)}$$

If X and Y are discrete, then

let $A = \{X = x\}$ and $B = \{Y = y\}$

then,

$$\boxed{P(X=x \mid Y=y) = \frac{\overset{\text{joint}}{P(X=x, Y=y)}}{\underset{\text{marginal}}{P(Y=y)}}}$$

Defn: Conditional PMF (discrete)

If (X, Y) is a Biv RV then the conditional PMF of X given $Y=y$ is defined as

joint PMF

$$f_{X|Y=y}(x) = f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_{X|Y=y}(x) \quad f(x|y) \quad \frac{\quad}{f_Y(y)}$$

marginal of Y

Ex. Joint PMF of X and Y

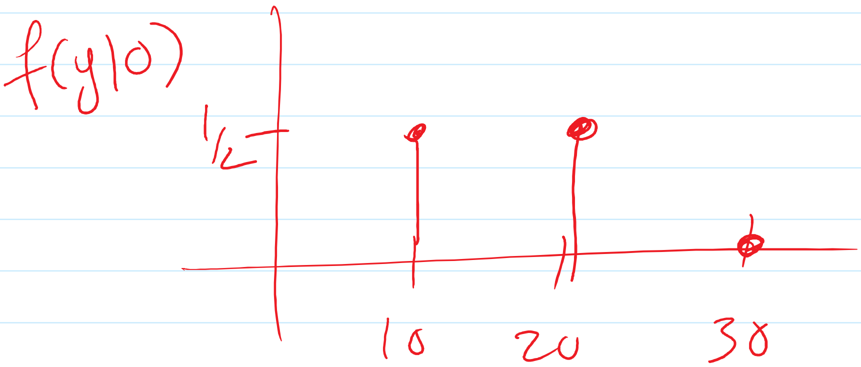
$f(x,y)$

Y \ X	$X=0$	1	2
30	0	3/18	4/18
20	2/18	4/18	0
10	2/18	3/18	0
	$f_X(0) = 4/18$	$f_X(1) = 10/18$	$f_X(2) = 4/18$

$$\boxed{f(y|0)} = f_{Y|X=0}(y) = \frac{f(0,y)}{f_X(0)}$$

$$= \begin{cases} \dfrac{f(0,10)}{f_X(0)} = \dfrac{2/18}{4/18} = 1/2, & y = 10 \\ \dfrac{f(0,20)}{f_X(0)} = \dfrac{2/18}{4/18} = 1/2, & y = 20 \\ \dfrac{f(0,30)}{f_X(0)} = \dfrac{0}{4/18} = 0, & y = 30 \end{cases}$$

$$\boxed{= \begin{cases} 1/2 & y = 10 \\ 1/2 & y = 20 \\ 0 & y = 30 \end{cases}}$$

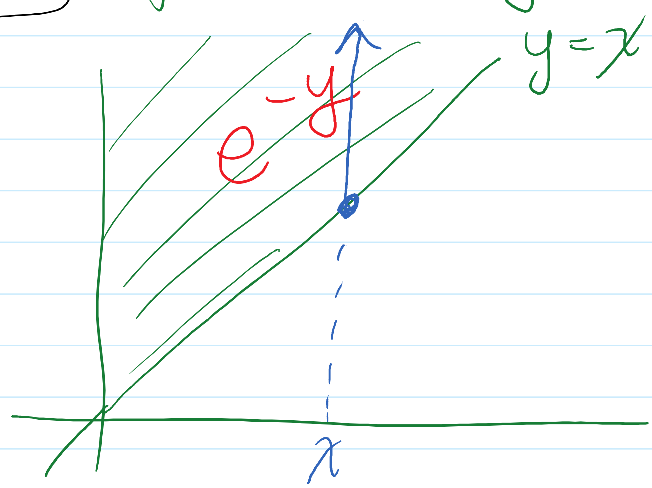


What about the cts case?

Defn: If X, Y are cts then the conditional PDF of X given $Y = y$

$$\boxed{f_{X|Y=y}(x) = f(x|y) = \frac{f(x,y)}{f_Y(y)}.}$$

Ex. $\boxed{f(x,y) = e^{-y}}$ for $0 < x < y$



Lets get $f(y|x)$.
"$Y | X = x$"

Formula:

$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

Marginal of X:

$$f_X(x) = \int f(x,y)\,dy = \int_y^{\infty} e^{-y}\,dy = e^{-x}$$

hence

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} \quad \text{for } 0<x<y$$