Ex. Flip a coin 3 times.

X = # heads among 3 flips of coin.

summarize into a number

$s \in S$	$X(s)$
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

function!

Defn: Random Variable

A random variable (r.v.) X is a function

$$X : S \longrightarrow \mathbb{R}$$

also called a radom variate

→ a $\boxed{\text{real valued}}$ random variable

→ a univariate radom variable

($\mathbb{R}$ not $\mathbb{R}^n$)

later!

Ex. ① toss two dice, X = sum of the dice

(2) toss a coin 25 times

X = largest chain of consecutive H's.

(3) observe rainfall and let X = yield crops

We'd really like to make statements like

$\mathbb{P}(\text{"}X = 1\text{"})$ ⟵ abusive of notation

"prob. that X is 1"

<u>recall</u>:

$$\mathbb{P} : \mathcal{P}(S) \to \mathbb{R}$$

really mean X = # heads in 3 coin flips

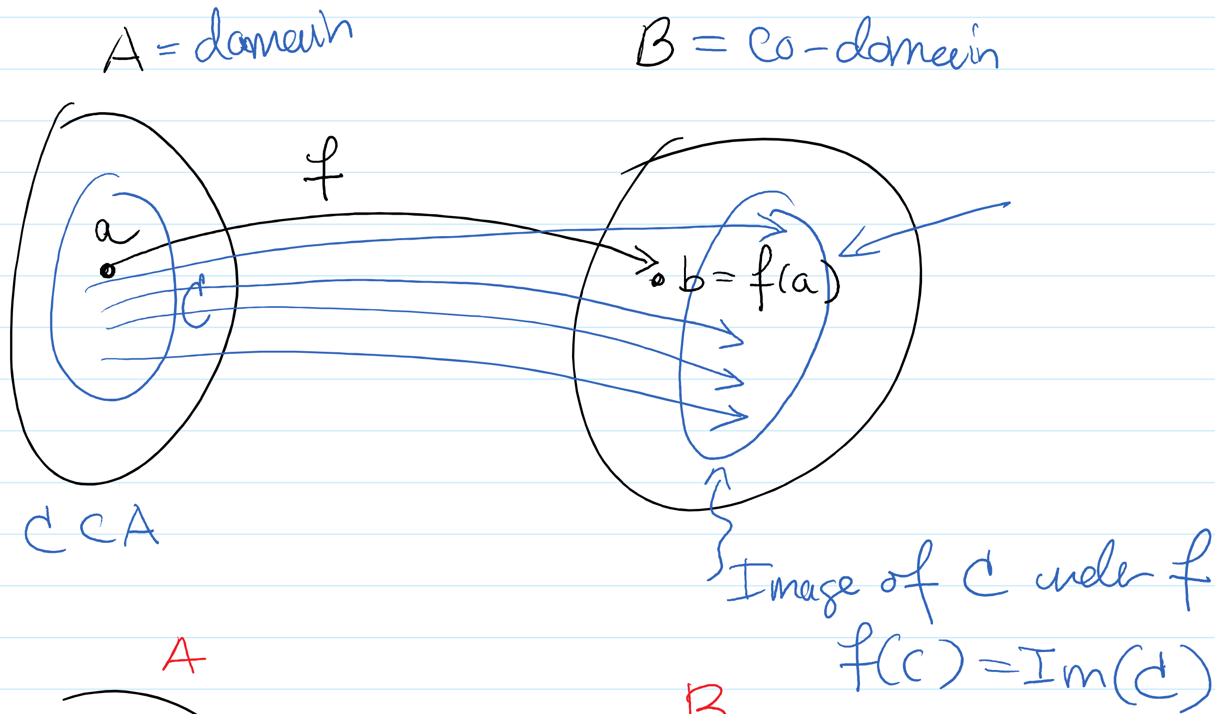
$$\mathbb{P}(X=1) = \mathbb{P}(\overbrace{\{HTT, THT, TTH\}}^{\text{an event}}) = 3/8$$

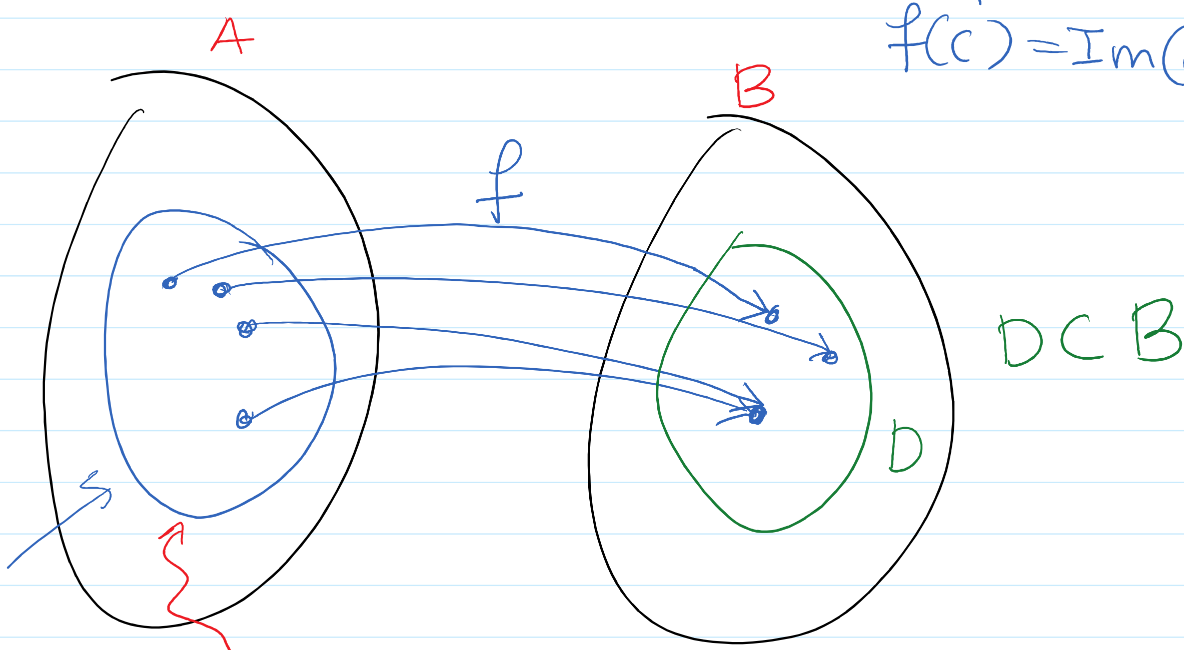
$$\text{"}X = 1\text{"} \quad \text{short-hand} \quad \{\underline{s \in S} \mid X(\underline{s}) = \underline{1}\} \subseteq S$$

this is called the <u>inverse-image</u> of 1 under X.

<u>Review: Inverse Images</u>

A = domain $\qquad B$ = co-domain





Inverse Image of D under f

denoted: $f^{-1}(D)$

① $f(C) = \{x \in B \mid \exists c \in C \text{ when } f(c) = x\}$

Image $= \{f(x) \mid x \in C\}$

all things mapped to

② $f^{-1}(D) = \{a \in A \mid f(a) \in C\}$ by C

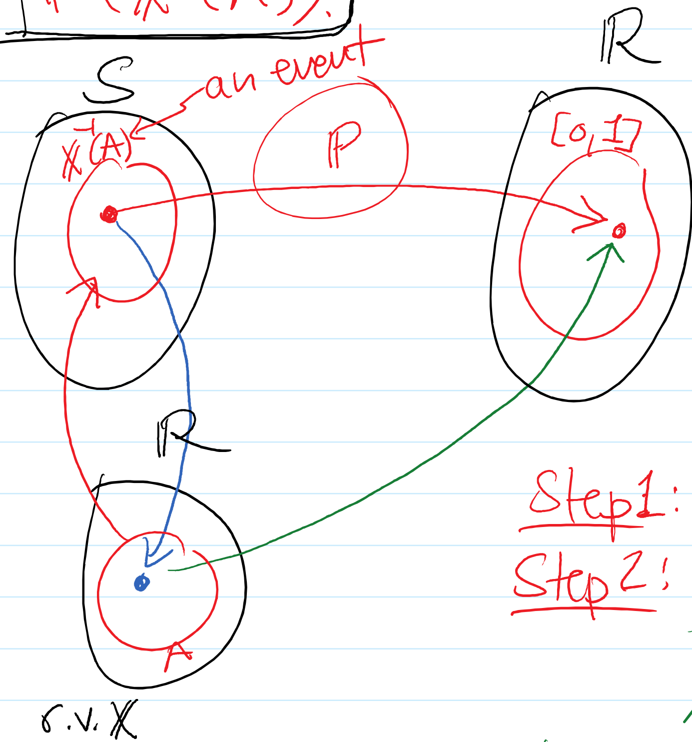
Inverse Image — all things in A that map to C

Notation: If X is a r.v. we write

(*) $\mathbb{P}(X \in A)$ when $\underline{A \subset \mathbb{R}}$ (*)

think of X as a number

$\underline{\text{means}}$ $\boxed{\mathbb{P}(X^{-1}(A))}$.



$\underline{\text{Step 1}}$: $X^{-1}(A) \subset S$

$\underline{\text{Step 2}}$: $\underline{\mathbb{P}(X^{-1}(A)) \in \mathbb{R}}$

$\underline{\text{we write}}$: $\mathbb{P}(X \in A)$

$\underline{\text{Concrete}}$: $X = \#$ heads in 3 tosses of a coin

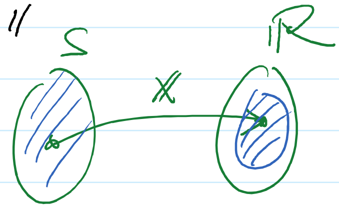
① $\mathbb{P}(X = \underline{1}) = \mathbb{P}(X \in \{1\})$

$$= \mathbb{P}(X^{-1}(\{1\}))$$
$$= \mathbb{P}(\{HTT, THT, TTH\}) = 3/8$$

(2) $\mathbb{P}(X = 1 \text{ or } 2) = \mathbb{P}(X \in \{1,2\})$
$$= \mathbb{P}(X^{-1}(\{1,2\}))$$
$$= \mathbb{P}(\{s \in S \mid X(s) \in \{1,2\}\})$$
$$\# \, ?(\{HTT, THT, TTH, \cdots\}) = 6/8$$

Defn: Support of a RV

If X is a r.v. then the support of X "is the set of possible values"



$$Im(X) = X(S)$$

the image of the sample space under X.

Ex. $X = \#$ heads in 3 flips

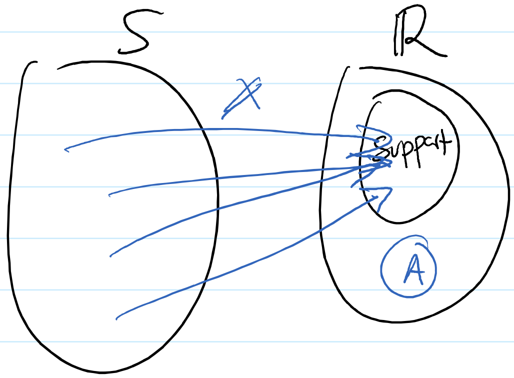
$$\text{Support}(X) = \{0, 1, 2, 3\}.$$

Notice: If $A \cap \text{Support}(X)$ ($A \subset S$) — an event

i.e. an event does overlap w/ support

$$\mathbb{P}(X \in A) = 0.$$

pf $\mathbb{P}(X\in A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\emptyset) = 0.$



Heuristic on Types of Random Variables (informal)

① **discrete** : if support is finite or countable

Ex, roll a die and record outcome
Support $= \{1, \dots, 6\}$

Ex, # of customers arriving at restaurant
Support $= \mathbb{N}$

② **continuous**

Support is uncountably infinite

Ex, waiting time for a bus
Support $= [0, \infty)$

Defn: Cumulative Distribution Function (CDF)

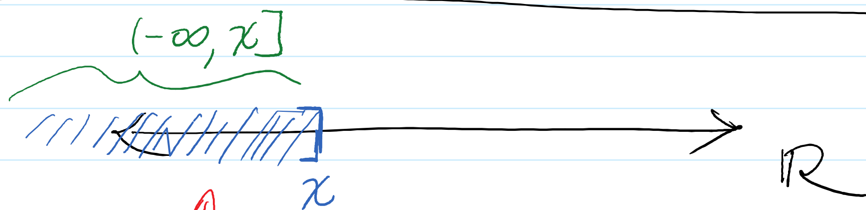
If X is a r.v. then the CDF of X is a function $F: \mathbb{R} \longrightarrow \mathbb{R}$

If X is a r.v. then the CDF of X is a function $F: \mathbb{R} \longrightarrow \mathbb{R}$ defined $\underline{x \in \mathbb{R}}$ as

$$F(x) = \mathbb{P}(X \leq x)$$

non-bold/little-x: a number in $\mathbb{R}$

bold: a r.v.



$F(x) =$ prob. X is here

Notation: $F(x) = \mathbb{P}(X \leq x)$

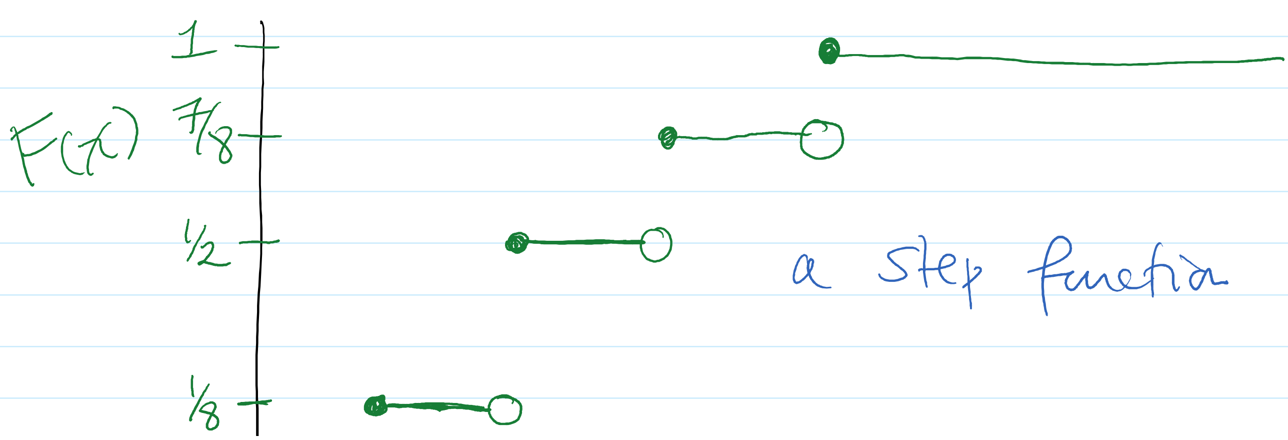
$$= \mathbb{P}(X \in (-\infty, x])$$

$$= \mathbb{P}(X^{-1}((-\infty, x])).$$

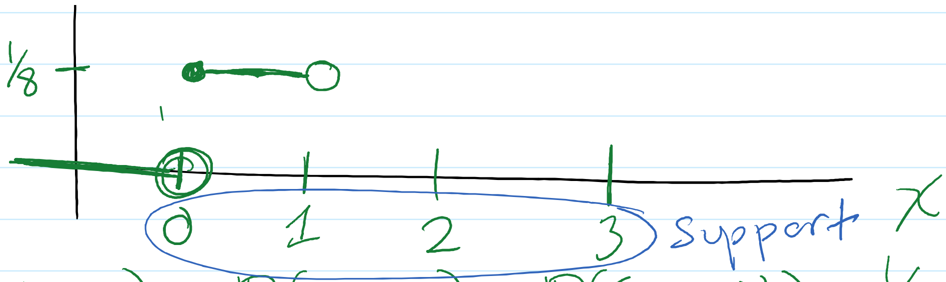
Ex. Toss a coin 3 times.

$X = \#$ heads.

What is F?



a step function



$F(0) = \mathbb{P}(X \le 0) = \mathbb{P}(X=0) = \mathbb{P}(\{TTT\}) = 1/8$

$F(1) = \mathbb{P}(X \le 1) = \cdots = 4/8 = 1/2$

$F(1/2) = \mathbb{P}(X \le 1/2) = \mathbb{P}(X=0) = \cdots = 1/8$

$F(2) = \mathbb{P}(X \le 2) = 7/8$

$F(1.5) = \mathbb{P}(X \le 1.5) = \mathbb{P}(X \le 1) = \cdots = 1/2$

$F(4) = \mathbb{P}(X \le 4) = 1$

$F(-1) = \mathbb{P}(X \le -1) = 0$

<u>Facts</u>: (Theorem)

If F is a CDF then

(1) $\boxed{0 \le F(x) \le 1 \quad \forall x \in \mathbb{R}}$

<u>Pf</u>. $F(x) = \mathbb{P}(\cdots) \in [0,1]$

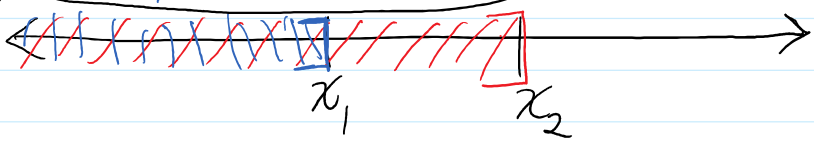
* (2) $\boxed{\lim_{x\to\infty} F(x) = 1 \quad \text{and} \quad \lim_{x\to -\infty} F(x) = 0}$

* (3) F is non-decreasing.

if $x_1 < x_2$ then $F(x_1) < F(x_2)$.

If $x_1 < x_2$ then $F(x_1) \le F(x_2)$.

pf. $(-\infty, x_1] \subset (-\infty, x_2]$



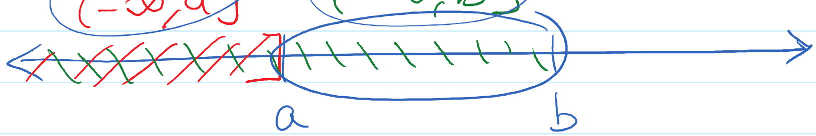
$$\underset{F(x_1)}{=} \mathbb{P}(X \in (-\infty, x_1]) \le \underset{F(x_2)}{=} \mathbb{P}(X \in (-\infty, x_2])$$

$$X^{-1}((-\infty, x_1]) \subset X^{-1}((-\infty, x_2])$$

Fact: if $A \subset B$ then $X^{-1}(A) \subset X^{-1}(B)$

(4) $\boxed{P(a < X \le b) = F(b) - F(a)}$

pf. $(-\infty, a]$ $(-\infty, b]$



$$P(a < X \le b) = P(X \in (a, b])$$

intermediate steps, need justification

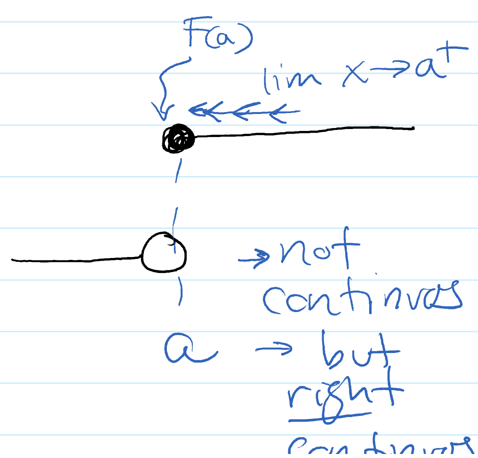
$$= P(X \in (-\infty, b]) - P(X \in (-\infty, a])$$

$$= F(b) - F(a).$$

(5) $\boxed{F \text{ is right-continuous}}$

$$\lim_{x \to a^+} F(x) = F(a)$$

right side



recall: defn cts function f

$$\lim f(x) = f(a)$$

recall: defn cts function f ... right continuous.

$$\lim_{x \to a} f(x) = f(a)$$

note: a continuous function is right cts.

(not nec. vice-versa)

Theorem: Let $F: \mathbb{R} \to \mathbb{R}$. F is the CDF of some random variable if

1. $\lim_{x \to -\infty} F(x) = 0$ $\quad \lim_{x \to \infty} F(x) = 1$
2. F is non-decreasing
3. F is right continuous.

(F continuous $\Rightarrow$ F right cts).

Ex. Let $\boxed{F(x) = \frac{1}{1+e^{-x}}}$ for $x \in \mathbb{R}$.

Q: Is F a "valid" CDF?

there is some r.v. with F as its CDF.

Check 3 conditions:

✓ 1. $\lim_{x \to -\infty} F(x) = \frac{1}{1+e^{-(-\infty)}} = \frac{1}{1+\infty} = 0$

$$\lim_{x\to\infty} F(x) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

② F is non-decreasing?

$$F'(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$$ OK

(F is increasing!)

③ Right cts?

Yes, F is continuos.