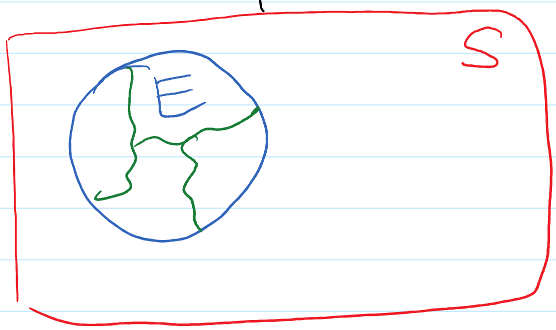
Probability as an $\boxed{\text{Area}}$:



$$\boxed{P(E) = \frac{\text{Area } E}{\text{Area } S}}$$

Countable Unions:

Countable Additivity Axiom

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i) \quad \text{as long as } (E_i) \text{ are disjoint.}$$

$\boxed{\text{Finite Additivity?}}$

$$(E_i)_{i=1}^{n} \rightsquigarrow (B_i)_{i=1}^{\infty}$$

$B_i = E_i$ for $i = 1, \ldots, n$

$B_i = \emptyset$ for $i > n$

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right)$$

if E_i disjoint then B_i are disjoint

$$= \sum_{i=1}^{\infty} P(B_i)$$

$$= \sum_{i=1}^{n} \mathbb{P}(B_i) + \underbrace{\sum_{i>n} \mathbb{P}(\cancel{B_i})}_{}$$

$\emptyset$

$$\boxed{P(\emptyset) = 0}$$

$$= \sum_{i=1}^{n} \mathbb{P}(E_i).$$

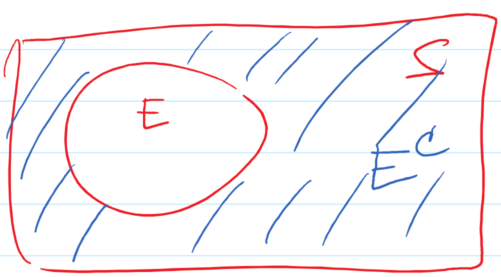
Basic Theorems

Ex, $E =$ "its raining" ; $\mathbb{P}(E) = 1/3$

$$\mathbb{P}(\underbrace{\text{"not raining"}}_{E^c}) = 2/3 = 1 - 1/3$$

Theorem: $\boxed{\mathbb{P}(E^c) = 1 - \mathbb{P}(E) \quad \forall E \subset S}$

pf. $S = E \cup E^c$, $EE^c = \emptyset$ so E, E^c disjoint



(1) Axiom 2: $\mathbb{P}(S) = 1$

$$\mathbb{P}(E \cup E^c) = 1$$

(2) Axiom 3: $\mathbb{P}(E \cup E^c)$

$$= \mathbb{P}(E) + \mathbb{P}(E^c)$$

Combine!

$$1 = \mathbb{P}(S) = \mathbb{P}(E) + \mathbb{P}(E^c)$$

re-arrange: $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$.

Lemma: Finite Measure $\boxed{\mathbb{P}(E) \leq 1 \quad \forall E \subset S}$

Lemma: Finite Measure | $\underline{\mathbb{P}(E) \le 1 \quad \forall E \subset S}$

$$\mathbb{P}(E^c) = 1 - \mathbb{P}(E) \quad \text{(by prev. theorem)}$$

$$\mathbb{P}(E^c) \ge 0 \quad \text{(by Axiom 1)}$$

So $1 - \mathbb{P}(E) \ge 0$

hence $\mathbb{P}(E) \le 1$.

Shown: $\boxed{0 \le \mathbb{P}(E) \le 1}$

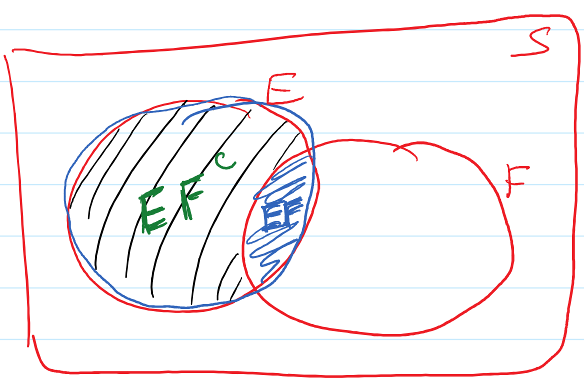
Theorem: $\boxed{\mathbb{P}(\emptyset) = 0.}$

pf. $\mathbb{P}(S) = 1$

$$S^c = \emptyset$$

$$S \setminus S$$

So $\mathbb{P}(\emptyset) = \mathbb{P}(S^c) = 1 - \mathbb{P}(S) = 1 - 1 = 0.$

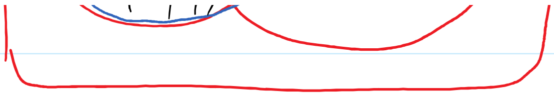


Theorem:

$$\mathbb{P}(EF^c)$$

$$= \mathbb{P}(E \setminus F)$$

$$= \mathbb{P}(E) - \mathbb{P}(EF)$$



$$\boxed{= P(E) - P(EF)}$$

Pf. $E = \underbrace{EF \cup EF^c}_{\text{disjoint}}$

Axiom 3: $P(E) = P(EF \cup EF^c)$

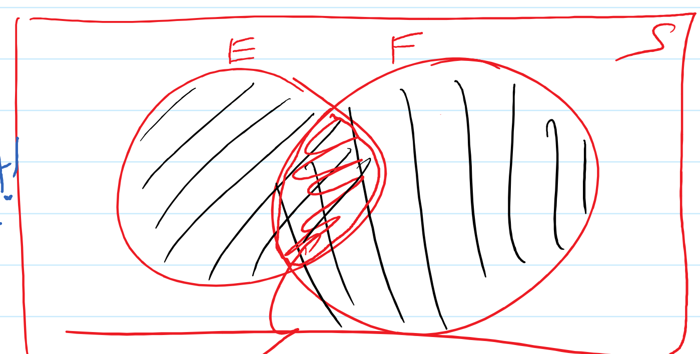
$$= P(EF) + P(EF^c)$$

re-arrange: $\boxed{P(EF^c) = P(E) - P(EF)}$

Theorem:

$$\boxed{P(E \cup F) = P(E) + P(F) - P(EF)}$$

Note: E and F don't need to be disjoint!



double counted

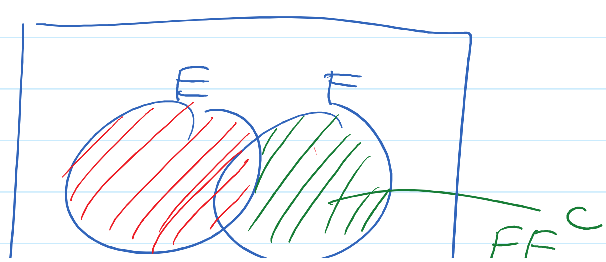
EF

Note: If $EF = \emptyset$ $\quad P(E \cup F) = P(E) + P(F) - \underbrace{P(EF)}_{0}$

Pf.

$E \cup F = \underbrace{E \cup FE^c}_{\text{disjoint}}$

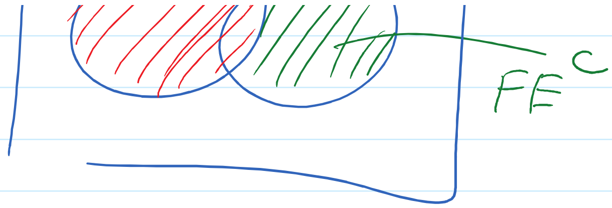
$P(E \cup F) \overset{\text{axiom 3}}{=} P(E) + P(FE^c)$



$= P(E) + P(F) - P(EF)$

$= P(E) + P(F) - P(FE)$

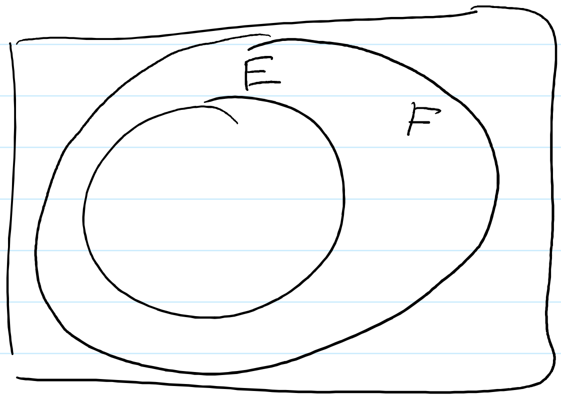
prev. result



Theorem:

If $E \subset F \subset S$

$$P(E) \le P(F).$$



Pf. $P(FE^c) \ge 0$ (Axiom 1)

$P(F) - P(EF) \ge 0$

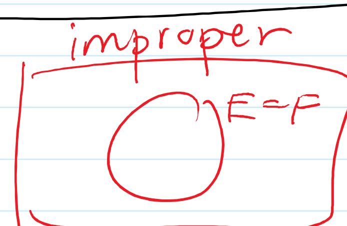
rearrange: $P(EF) \le P(F)$

note: $E \subset F$ hence $EF = E$

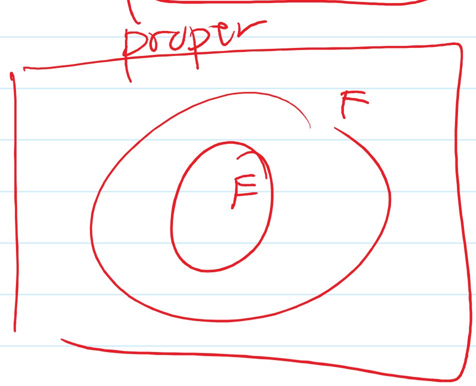
$$P(E) \le P(F).$$

Consider! $E \subset F$ but $E \neq F$

(proper subset)

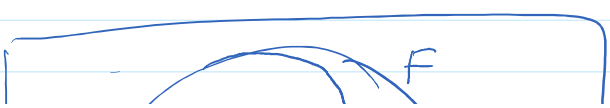


Q: $\cancel{P(E) < P(F)\,?}$

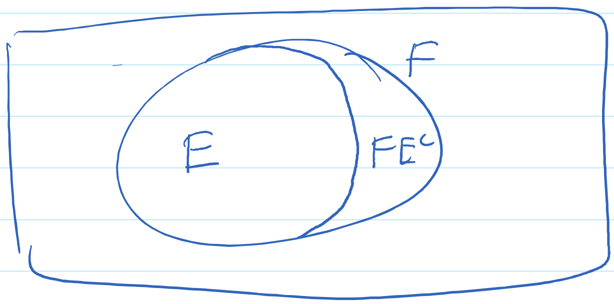


No.

Consider:



what if $P(FE^c) = 0$?



what if $P(FE^c) = 0$?
then even if E proper subset of F,

$$0 = P(FE^c) = P(F) - P(EF)$$
$$= P(F) - P(E)$$
So $P(E) = P(F)$.

Previously: $\boxed{P(E \cup F) = P(E) + P(F) - \underbrace{P(EF)}_{\geq 0}}$

$$P(E \cup F) \leq P(E) + P(F)$$

Can we generalize?

Boole's Inequality

If $(E_i)_{i=1}^{\infty}$ then
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i).$$

---

pf. Calls for Axiom 3.

Replace the (E_i) w/ a set of disjoint (B_i) so that

① $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} B_i$

① $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} B_i$

but ② (B_i) are disjoint

define:

$B_1 = E_1$

$B_2 = E_2 \setminus E_1 = E_2 \setminus B_1$

$B_3 = E_3 \setminus E_2 \setminus E_1 = E_3 \setminus B_2$

$B_4 = E_4 \setminus E_3 \setminus E_2 \setminus E_1$

etc.

Convince yourself:

① & ② are true

note: $\mathbb{P}(B_i) \le \mathbb{P}(E_i)$

b/c $B_i = E_i \setminus \sim$

$B_i \subset E_i$

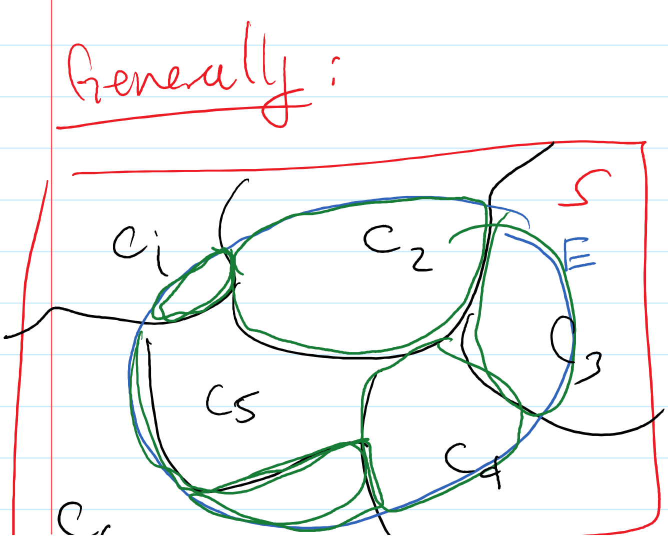
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(B_i) \le \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

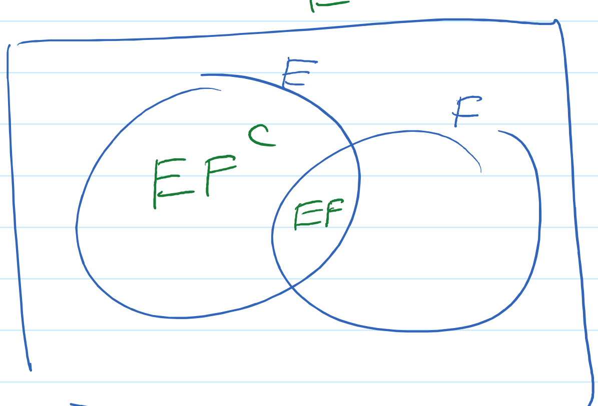
$$\le \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

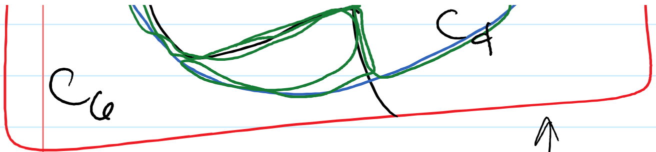
Theorem: Event Partitioning

$E = EF \cup EF^c$

Generally:







Similarly: $E = EC_1 \cup EC_2 \cup EC_3 \cup EC_4 \cup EC_5 \cup EC_6$

also: $P(E) = P(EC_1) + P(EC_2) + \cdots + P(EC_6)$

If $(C_i)_{i=1}^{\infty}$ is a partition of S then for any event $E \subset S$

$$P(E) = \sum_{i=1}^{\infty} P(EC_i)$$

Pf. (1) $(\overbrace{EC_i}^{A_i})(\overbrace{EC_j}^{A_j}) = \emptyset$ (A_i are disjoint)

$A_i = EC_i$; $A_j = EC_j$

$EC_iEC_j = E\underbrace{C_iC_j}_{\emptyset} = \emptyset$

(2) $E = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} EC_i$

Combine these:

axiom 3

$$P(E) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$= \sum^{\infty} P(EC_i)$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(E e_i).$$