# Lecture 14 - More Common Distributions

## Poisson Distribution

- discrete r.v.
- support non-neg. integers : $\{0, 1, 2, 3, \ldots, \}$

canonical experiment:

count the number of "events" in some time period

Ex. - radioactive decay
- ecological models : fish capture
- microbio : RNA count

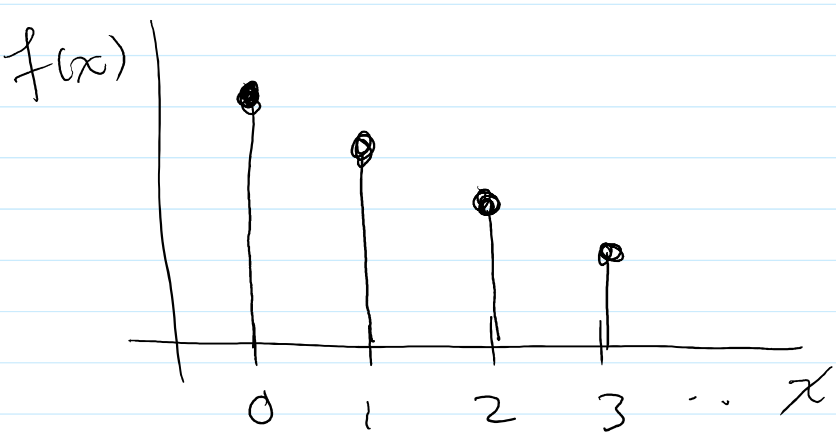
$X \sim \text{Pois}(\lambda)$

# of events in some time period

rate of events occuring

PMF

$$\boxed{f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots}$$




$\frac{x}{} = \frac{x}{} - 1$

$0 \quad 1 \quad 2 \quad 3 \quad \cdots \quad \infty$

$$\frac{x}{x!} = \frac{x}{x(x-1)!} = \frac{1}{(x-1)!}$$

## Expected Value:

$$E[X] = \sum_{x=0}^{\infty} x f(x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda} \lambda \overbrace{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!}}^{e^{\lambda}}$$

$$= e^{-\lambda} \lambda e^{\lambda}$$

$$= \lambda$$

Fact:

$$e^a = \sum_{x=0}^{\infty} \frac{a^x}{x!}$$

$$\boxed{E[X] = \lambda}$$

avg./expected # events in one time period.

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$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) f(x) = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\sum_{x=2}^{\infty} \frac{x(x-1)}{x!} e^{-\lambda} \lambda^x$$

$$\frac{\cancel{x(x-1)}}{\cancel{x(x-1)}(x-2)!} = \frac{1}{(x-2)!}$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$\lambda^2 e^{-\lambda} \sum \lambda^x \qquad \lambda^2 e^{-\lambda} e^{\lambda} \quad \lambda^2$$

$x=2$ ...

$$= \lambda^2 e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!}}_{e^{\lambda}} = \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$$

$$\mathbb{E}[X(X-1)] = \lambda^2$$

$$\mathbb{E}[X^2 - X^2] = \mathbb{E}[X^2] - \overbrace{\mathbb{E}(X]}^{\lambda} = \lambda^2$$

So... $\mathbb{E}[X^2] = \lambda^2 + \lambda$

<u>Variance:</u>

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 \boxed{= \lambda}$$

<u>MGF:</u> $M(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} f(x) = \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\lambda} \lambda^x}{x!}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\overbrace{\lambda e^t}^{a})^x}{x!} = e^{-\lambda} e^{\lambda e^t}$$

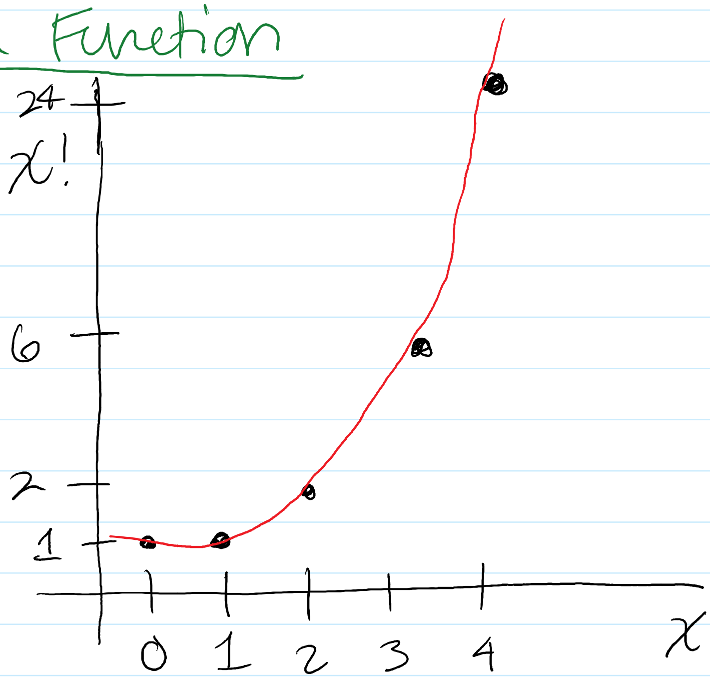
$$\boxed{= \exp(\lambda(e^t - 1))}$$

$\exp(a) = e^a$

# Gamma Distribution : generalize exponential

## Lets talk about Gama Function

Gamma function:
extend $x!$ to
pos. real numbers




$\Gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

$a \in \mathbb{R}^+$

(1) If $a$ is an integer :

$$\Gamma(a) = (a-1)! \iff \Gamma(a+1) = a!$$

Ex,
$\Gamma(1) = 0! = 1$
$\Gamma(2) = 1! = 1$
$\Gamma(3) = 2! = 2$
$\Gamma(4) = 3! = 6$ . . .

(2) Notice: $x! = x(x-1)!$ for $x$ integers

For $\Gamma$, $\Gamma(a+1) = a\Gamma(a)$

$a! = a(a-1)!$ for $a$ integer

or $\Gamma(a) = (a-1)\Gamma(a-1)$

## Important Facts for $\Gamma$:

1. $\Gamma(x+1) = x!$ for $x$ integer
2. $\Gamma(x+1) = x\Gamma(x)$

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**Gamma dist:** generalizes exponential

$$X \sim \text{Gamma}(\underset{\text{shape}}{a}, \underset{\text{rate}}{\lambda})$$

**PDF:**

$$\boxed{f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)} \quad \text{for } x > 0}$$

notice: $a=1$ then $f(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{0}}{\Gamma(1)} = \lambda e^{-\lambda x}$



Expected Value:

## Expected Value:

$$E[X] = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{x\,\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)}\,dx$$

trick:

$$c \underbrace{\int \text{PPF of Gama}}_{1}$$

Gama$(a+1, \lambda)$

$$= \frac{1}{\lambda}\int_0^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{a}}{\Gamma(a)}\,dx$$

$$= \frac{\Gamma(a+1)}{\lambda\Gamma(a)} \underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{(a+1)-1}}{\Gamma(a+1)}\,dx}_{\text{PPF of Gama}(a+1,\lambda)\text{ integrates to } 1}$$

$$\frac{\lambda e^{-\lambda x}(\lambda x)^{(a+1)-1}}{\Gamma(a+1)}$$

$$= \frac{\Gamma(a+1)}{\lambda\Gamma(a)} \leftarrow \Gamma(a+1) = a\Gamma(a)$$

$$= \frac{a\cancel{\Gamma(a)}}{\lambda\cancel{\Gamma(a)}} \boxed{= \frac{a}{\lambda}} = E[X]$$

## Moments:

$$E[X^r] = \int_0^\infty \frac{x^r\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)}\,dx$$

PPF of

$$\int_0 \quad \Gamma(a)$$

$$= \frac{\Gamma(a+r)}{\Gamma(a)\lambda^r}\int_0^\infty \underbrace{\frac{\lambda e^{-\lambda x}(\lambda x)^{(a+r)-1}}{\Gamma(a+r)}}_{\text{PDF integrates to } 1}$$

PDF of Gamma$(a+r, \lambda)$

$$\boxed{\mathbb{E}[X^r] = \frac{\Gamma(a+r)}{\Gamma(a)\lambda^r}}$$

$\boxed{r=1}$

$$\mathbb{E}[X] = \frac{a}{\lambda}$$

(notice if $a=1 \Rightarrow \frac{1}{\lambda}$) $\mathbb{E}[\text{exponential r.v.}]$

$\boxed{r=2}$

$$\mathbb{E}[X^2] = \frac{\Gamma(a+2)}{\Gamma(a)\lambda^2} = \frac{(a+1)\Gamma(a+1)}{\Gamma(a)\lambda^2}$$

$$= \frac{(a+1)\,a\,\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}\lambda^2}$$

$$\boxed{= \frac{a^2+a}{\lambda^2}}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= \frac{a^2+a}{\lambda^2} - \left(\frac{a}{\lambda}\right)^2$$
$$\boxed{= \frac{a}{\lambda^2}}$$

(when $a = 1 \Rightarrow \frac{1}{\lambda^2}$
i.e. variance of exponential)

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## Geometric Distribution

Canonical experiment

If I flip coins (independently), each has a prob. $p \in [0,1]$ of H,

let $W$ = waiting time until I get a first H.

| outcome | W |
|---|---|
| H | 1 |
| TH | 2 |
| TTH | 3 |
| ⋮ | ⋮ |

Then $W \sim \text{Geometric}(p)$

Support: $\{1, 2, 3, 4, \ldots\}$

PMF: $\boxed{f(x) = (1-p)^{x-1} p}$ for $x = 1, 2, 3, \ldots$

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make $x$ flips to get H

CDF: $F(x) = P(W \le x) = \sum_{i=1}^{x} f(i) = \sum_{i=1}^{x} p(1-p)^{i-1}$

$$= p \sum_{i=0}^{x-1} (1-p)^i$$

Calc II: $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$

$$= \cancel{p}\,\frac{1-(1-p)^x}{\cancel{1-(1-p)}}$$

$$\boxed{= 1-(1-p)^x} = F(x)$$

Expected Value:

$$E[W] = \sum_{i=1}^{\infty} i p (1-p)^{i-1} = p \sum_{i=1}^{\infty} i(1-p)^{i-1}$$

$$\boxed{i(1-p)^{i-1} = \frac{d}{dp}(1-p)^i}$$

$$= -p \sum_{i=1}^{\infty} \frac{d}{dp}(1-p)^i$$

$r = 1-p$

$\infty \quad \infty \quad i$

$$= -p \frac{d}{dp} \sum_{i=1}^{\infty} \underbrace{(1-p)}_{r=1-p}{}^{i} \qquad \sum_{i=1}^{\infty} r^i$$

$$= -p \frac{d}{dp}\left[(1-p) \sum_{i=0}^{\infty} (1-p)^i\right] \qquad \sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

$$= -p \frac{d}{dp}\left[(1-p) \frac{1}{1-(1-p)}\right] \quad {}^{1/p}$$

$$= -p \frac{d}{dp}\left[(1-p)/p\right]$$

$$= -p\left(-1/p^2\right) = \boxed{1/p} = E[W]$$

MGF: $M(t) = E[e^{tW}] = \sum_{i=1}^{\infty} e^{ti} p(1-p)^{i-1}$

$$= pe^t \sum_{i=1}^{\infty} e^{t(i-1)} (1-p)^{i-1}$$

$$= pe^t \sum_{i=0}^{\infty} e^{ti} (1-p)^i$$

$$= pe^t \sum_{i=0}^{\infty} \left(e^t(1-p)\right)^i$$

$$= pe^t \sum_{i=0} (e^t(1-p))$$

$$= \text{geometric series} \ldots \boxed{= pe^t \frac{1}{1-e^t(1-p)}} = M(t)$$

$$E[X^2] = \left.\frac{d^2M}{dt^2}\right|_{t=0} = \cdots = \frac{2-p}{p^2}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \boxed{\frac{1-p}{p^2}}$$

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Beta Distribution : continuous r.v.

Beta Function $a, b \in \mathbb{R}^+$

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

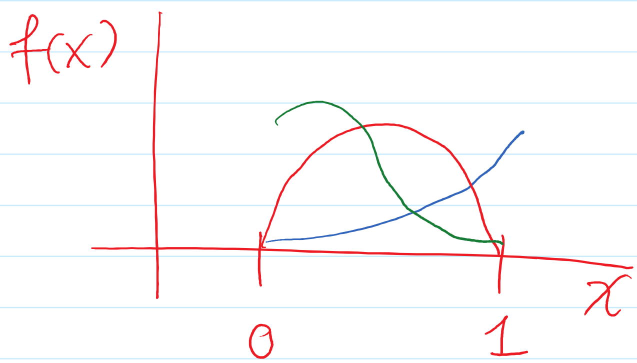
$$\boxed{= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}$$

Beta distribution:

$X \sim \text{Beta}(a,b)$

means

$$\boxed{f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } x \in (0,1)}$$




$$E[X] = \int_0^1 x f(x)\,dx = \int_0^1 x\,\frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$\frac{B(a+1,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}\,dx}_{1}$$

make integral

$$c \underbrace{\int \overset{\text{PPF}}{\text{Beta}(a+1,b)}}_{1}$$

PDF of Beta$(a+1,b)$

$$\boxed{\frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}}$$

$$= \frac{B(a+1,b)}{B(a,b)}$$

$$= \frac{\dfrac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+b+1)}}{\Gamma(a)\Gamma(b)} = \frac{a\cancel{\Gamma(a)}}{(a+b)\cancel{\Gamma(a+b)}}$$

$$\boxed{\frac{a}{\phantom{a+b}}}$$

$$= \frac{\Gamma(a+b+1)}{\frac{\Gamma(a)\cancel{\Gamma(b)}}{\Gamma(a+b)}} = \frac{(a+b)\cancel{\Gamma(a+b)}}{\cancel{\frac{\Gamma(a)}{\Gamma(a+b)}}} \quad \boxed{= \frac{a}{a+b}}$$

$E[X]$

Moments:

$$\mathbb{E}[X^r] = \int_0^1 x^r \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+r,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{(a+r)-1}(1-x)^{b-1}}{B(a+r,b)}\,dx}$$

PDF of Beta(a+r, b)

→ integrates to 1

$$\boxed{\mathbb{E}[X^r] = \frac{B(a+r,b)}{B(a,b)}}$$

$\Gamma(a+2) = (a+1)\Gamma(a+1)$

$= (a+1)\,a\,\cancel{\Gamma(a)}$

$r=2$ $$\frac{B(a+2,b)}{B(a,b)} = \frac{\frac{\Gamma(a+2)\cancel{\Gamma(b)}}{\Gamma(a+b+2)}}{\frac{\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{\cancel{\Gamma(a+b)}}}$$

$\Gamma(a+b+2) = (a+b+1)\Gamma(a+b+1)$

$= (a+b+1)(a+b)\cancel{\Gamma(a+b)}$

$$\mathbb{E}[X^2] = \boxed{= \frac{a(a+1)}{(a+b)(a+b+1)}}$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \ldots \text{ algebra } \boxed{= \frac{ab}{(a+b+1)(a+b)^2}}$$