Defn: Conditional PMF/PDF

Given RVs X and Y the conditional PMF/PDF of $X \mid Y=y$ is

$$f_{X|Y=y}(x) = f(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

"$X \mid Y=y$" think of as a univariate RV.

Defn: Conditional Expectation

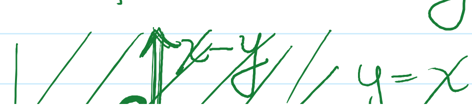
If $g: \mathbb{R} \to \mathbb{R}$ then the conditional expectation of $g(X)$ given $Y=y$ is

$$\mathbb{E}[g(X) \mid Y=y] = \begin{cases} \sum_x g(x) f(x|y) & \text{(discrete)} \\ \int g(x) f(x|y)\, dx & \text{(cts)} \end{cases}$$

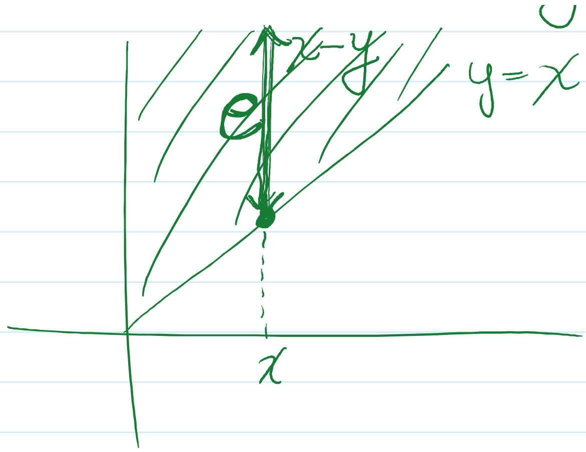
Analogy: $\mathbb{E}[g(X)] = \int g(x) f(x)\, dx$

Ex. Last time

$$f(y|x) = e^{x-y} \quad \text{for } 0 < x < y$$



Q: $E[Y|X=x]$



$$= \int y f(y|x)\,dy$$

$$= \underbrace{\int_x^\infty y e^{x-y}\,dy}$$

integration by parts
$u = y \quad dv = e^{-y}$

$$\vdots$$

$$= 1 + x$$

Defn: Conditional Variance

$$Var(Y|X=x) = E\left[(Y - E[Y|X=x])^2 \,\middle|\, X = x \right]$$

$$\left(Var(Y) = E[(Y - E[Y])^2] \right)$$

Short-cut formula:

$$Var(Y|X=x) = E[Y^2|X=x] - E[Y|X=x]^2.$$

Ex. Continue from above.

$E[Y|X=x] = 1 + x$

$$\mathbb{E}[Y^2 | X = x] = \int y^2 f(y|x)\,dy = \int_x^\infty y^2 e^{x-y}\,dy$$

$\vdots$ integration by parts

$$= x^2 + 2x + 2$$

hence

$$\mathrm{Var}(Y | X = x) = x^2 + 2x + 2 - (1+x)^2 = \boxed{1}.$$

Independence for RVs.

For event $A \perp B$ if $P(AB) = \mathbb{P}(A)\mathbb{P}(B)$.

For RVs. If $\forall\, A, B \subset \mathbb{R}$ and

$$\mathbb{P}(X \in A,\ Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

then we say X is independent of Y denoted $X \perp\!\!\!\perp Y$.

Another way: $X^{-1}(A) \perp Y^{-1}(B)$ $\forall\, A, B \subset \mathbb{R}$.

Product Spaces

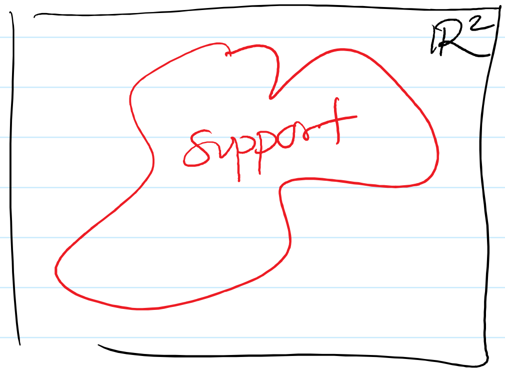
Support $(X, Y) = \{$all pts when $f(x,y) > 0\} \subset \mathbb{R}^2$

Consider X, Y w/ pdf/pmf

$f(x,y) = \ldots$ for $\underline{x \in A}$ and $\underline{y \in B}$

doesn't depend on y

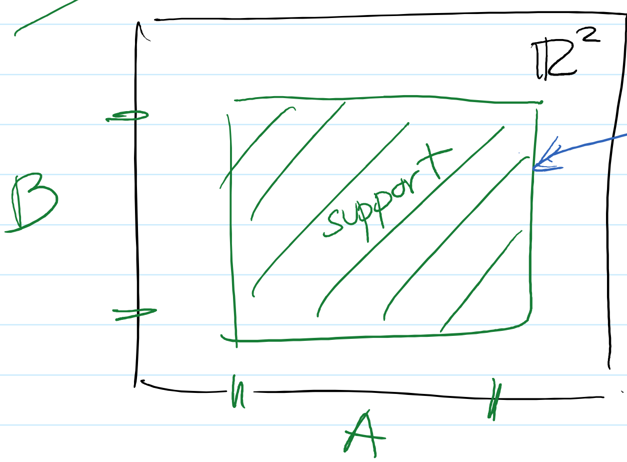
doesn't depend on x



e.g. $f(x,y) = 6xy^2$ for $x \in \underbrace{[0,1]}_{A}$, $y \in \underbrace{[5,7]}_{B}$

this means that $\text{Support}(X,Y) = A \times B$

We call this a product space



Square or retangle

Theorem: Factorization Theorem

If X and Y have a support that is a product space (square/retangle) then

$$X \perp\!\!\!\perp Y \iff ① \; F(x,y) = F_X(x) F_Y(y)$$

or

$$② \; f(x,y) = f_X(x) f_Y(y)$$

$$② \quad \underline{f(x,y) = f_X(x)\,f_Y(y)}$$

Ex. (discrete)

$f(x,y)$

$Y \backslash X$	10	20
3	$1/5$	$3/10$
2	$1/5$	$1/10$
1	$1/10$	$1/10$

Q: Is $X \perp\!\!\!\perp Y$?

① clearly the support is a retangle

$$A \times B$$

$A = \{10, 20\}$ $B = \{1, 2, 3\}$

②

Get marginals

x	10	20
$f(x)$	$1/2$	$1/2$

y	1	2	3
$f(y)$	$1/5$	$3/10$	$1/2$

$$f(10,3) = 1/5 \neq f(10)f(3) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

So $X \not\!\perp\!\!\!\perp Y$.

Corollary: If support is a product space & only a fn of X

Corollary: If support is a product space &

$$X \perp\!\!\!\perp Y \iff f(x,y) = h(x)\,g(y)$$

only a fn of y

Don't need to verify that h is f_X and g is f_Y.

Ex. Let $f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)}$, $x>0$ and $y>0$

product space

$$\underline{f(x,y)} = \underbrace{\left(\frac{1}{384} x^2 e^{-x/2}\right)}_{h(x)} \underbrace{\left(e^{-y}\right)}_{g(y)} \quad (0,\infty)\times(0,\infty)$$

So $X \perp\!\!\!\perp Y$.

Fact: If $X \perp\!\!\!\perp Y$ then

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\cancel{f(x)}\,f(y)}{\cancel{f(x)}} = f(y)$$

Recall: $A \perp\!\!\!\perp B$ then $\mathbb{P}(A|B) = \mathbb{P}(A)$

Theorem: $\mathbb{E}$ for independent RVs

If $X \perp\!\!\!\perp Y$, then $g_1 : \mathbb{R} \to \mathbb{R}$, $g_2 : \mathbb{R} \to \mathbb{R}$

then
$$\mathbb{E}[g_1(X)g_2(Y)] = \mathbb{E}[g_1(X)]\,\mathbb{E}[g_2(Y)].$$

Pf.
$$\begin{aligned}
\mathbb{E}[g_1(X)g_2(Y)] &= \iint g_1(x)g_2(y)\,f(x,y)\,dx\,dy \\
&= \iint g_1(x)g_2(y)\,f(x)f(y)\,dx\,dy \\
&= \underbrace{\int g_2(y)f(y)\,dy}_{\mathbb{E}[g_2(Y)]}\ \underbrace{\int g_1(x)f(x)\,dx}_{\mathbb{E}[g_1(X)]}
\end{aligned}$$

Ex. Let $X, Y \sim \mathrm{Exp}(\lambda = 1)$ and $X \perp\!\!\!\perp Y$

then
$$\begin{aligned}
\mathbb{E}[X^2Y] &= \mathbb{E}[X^2]\,\mathbb{E}[Y] \\
&= (2)(1) = 2.
\end{aligned}$$

Lemma: MGFs for Independent RVs.

If $X \perp\!\!\!\perp Y$ then

$$\boxed{\text{If } X \perp\!\!\!\perp Y \text{ then } \quad M_{X+Y}(t) = M_X(t)\,M_Y(t)}$$

$M(t) = \mathbb{E}[e^{tX}]$

Pf. $M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[e^{tX}e^{tY}]$

$$= \mathbb{E}[e^{tX}]\,\mathbb{E}[e^{tY}]$$

$$= M_X(t)\,M_Y(t)$$

Ex. Let $X \sim N(\mu, \sigma^2)$ and $\underline{Y \sim N(\gamma, \tau^2)}$ and $X \perp\!\!\!\perp Y$.

$$M_{X+Y}(t) \overset{\text{our prev. theorem}}{=} M_X(t)\,M_Y(t)$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\exp\left(\gamma t + \frac{\tau^2 t^2}{2}\right)$$

$$= \exp\Big(\underbrace{(\mu+\gamma)t + \frac{(\sigma^2+\tau^2)t^2}{2}}_{}\Big)$$

MGF of a $N(\mu+\gamma, \sigma^2+\tau^2)$

$X+Y$ has the $\underline{\text{MGF of } N(\mu+\gamma, \sigma^2+\tau^2)}$

$X + Y$ has the MGF of $N(\mu+\gamma, \sigma^2+\tau^2)$

hence $\boxed{X + Y \sim N(\mu+\gamma, \sigma^2+\tau^2)}$

Theorem: Cov/Cor for Independent

If $X \perp\!\!\!\perp Y$ then $\mathrm{Cov}(X, Y) = \mathrm{Cor}(X, Y) = 0$.

$\Rightarrow$

Pf. $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$= \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] = 0$

Since $\mathrm{Cor}(X, Y) \propto \mathrm{Cov}(X, Y)$ it is also zero.

Converse is generally false.

If $\mathrm{Cor}(X, Y) = 0$ they may / may not be independent (we don't know)

Ex. $X \sim N(0,1)$ and $Y = X^2$.

then $\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, X^2)$

$= \mathbb{E}[X X^2] - \mathbb{E}[X^2]\mathbb{E}[X]$

aside: $X \sim N(0,1)$ $\quad = \mathbb{E}[X^3] - \mathbb{E}[X^2]\mathbb{E}[X]$

$$= \underbrace{\mathbb{E}[X^3]}_{0} - \mathbb{E}[X^2]\underbrace{\mathbb{E}[X]}_{0}$$

$$= 0$$

aside: $X \sim N(0,1)$

$$\mathbb{E}[X^r] = 0 \quad \forall \text{ odd } r$$

reason:

$$\mathbb{E}[X^r] = \int_{-\infty}^{\infty} \underbrace{\underbrace{x^r}_{\text{odd}}\underbrace{f(x)}_{\text{even}}}_{\text{odd}}\,dx = 0$$

Baye's Theorem for RV

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$\boxed{f(y|x) = \frac{f(x|y)f(y)}{f(x)}}$$

Law of Total Probability

Recall: If $\{C_i\}$ partition S then

$$P(A) = \sum_i P(A|C_i)P(C_i)$$

For RVs:

$$\begin{cases} \text{(discrete)} & f(y) = \sum_x f(y|x)f(x) \\ \text{(cts)} & f(y) = \int f(y|x)f(x)\,dx \end{cases}$$

pf. ① $f(y|x) = \dfrac{f(x,y)}{f(x)} \iff \boxed{f(x,y) = f(y|x)f(x)}$

② $f(y) = \int f(x,y)\,dx$

Combine these

$$f(y) = \int f(y|x) f(x)\,dx$$

Ex. Assume $Y | X = x \sim \text{Poisson}(x)$

and $X \sim \text{Exp}(\lambda)$

Q: what is the marginal dist. of Y?

$$f(y) = \int f(y|x)\, f(x)\, dx$$

$$= \int_0^\infty \underbrace{\frac{x^y e^{-x}}{y!}}_{\text{Poison}(x)} \underbrace{\lambda e^{-\lambda x}}_{\text{Exp}(\lambda)} dx = \frac{\lambda}{y!} \int_0^\infty \underbrace{x^{\overset{\alpha-1}{y}} e^{-\overset{\beta}{(\lambda+1)} x}}_{\text{looks like a Gamma PDF}} dx$$

$$= \frac{\lambda}{y!} \frac{\Gamma(y+1)}{(\lambda+1)^{y+1}} \int_0^\infty \underbrace{\frac{(\lambda+1)^{y+1} x^y e^{-(\lambda+1)x}}{\Gamma(y+1)}}_{\text{Gamma PDF:}} dx$$

$$= \frac{\lambda \cancel{\Gamma(y+1)}}{\cancel{y!}(\lambda+1)^{y+1}}$$

Gama PDF:

$$f(z) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z}$$

PDF of Gama$(y+1, \lambda+1)$ so integrates to 1

$$= \boxed{\frac{\lambda}{(\lambda+1)^{y+1}}} \longleftarrow \text{PDF of } Y$$

Ex. $X \mid Y=y \sim \text{Bin}(y, p)$

$Y \sim \text{Poisson}(\lambda)$

discrete

$0 \le x \le y$

Q: what is dist of X?

$$f(x) = \sum_{y=x}^{\infty} f(x|y) f(y) \quad \text{(Law of Total Prob)}.$$

$$= \sum_{y=x}^{\infty} \underbrace{\binom{y}{x} p^{x}(1-p)^{y-x}}_{\text{Bin}(y,p)} \underbrace{\frac{\lambda^{y} e^{-\lambda}}{y!}}_{\text{Poisson}(\lambda)}$$

$$\binom{y}{x}\frac{1}{y!} = \frac{\cancel{y!}}{x!(y-x)!} \frac{1}{\cancel{y!}} = \frac{1}{x!(y-x)!}$$

$$= \sum_{y=x}^{\infty} \frac{1}{x!(y-x)!} p^{x}(1-p)^{y-x} \lambda^{y} e^{-\lambda}$$

$$\sum_{y=x} \frac{}{x!(y-x)!}$$

$$= \frac{1}{x!} p^x e^{-\lambda} \lambda^x \sum_{y=x}^{\infty} \frac{(1-p)^{y-x} \lambda^{y-x}}{(y-x)!}$$

$$= \frac{1}{x!} p^x e^{-\lambda} \lambda^x \sum_{y=0}^{\infty} \frac{((1-p)\lambda)^y}{y!} \quad e^{\lambda(1-p)}$$

fact: $e^a = \sum_{y=0}^{\infty} \frac{a^y}{y!}$

$$= \frac{1}{x!} p^x \cancel{e^{-\lambda}} \lambda^x e^{\cancel{\lambda}(1-p)}$$

$$= \frac{(p\lambda)^x e^{-p\lambda}}{x!} \quad \text{Poisson}(\lambda p)$$

So $\boxed{X \sim \text{Poisson}(\lambda p)}$