$\boxed{\text{Equally Likely}}$ Outcomes

If I have a sample space

$$S = \{s_1, s_2, \dots, s_n\}$$

and assume

$$\underbrace{\mathbb{P}(\{s_i\}) = \mathbb{P}(\{s_j\})}_{1/n} \quad \forall\, i, j.$$

(Rationale: $\mathbb{P}(S) = 1$ $\quad S = \underbrace{\{s_1\} \cup \{s_2\} \cup \cdots \cup \{s_n\}}_{\text{disjoint (partition } S)}$

$$1 = \mathbb{P}(S) = \sum_{i=1}^{n} \mathbb{P}(\{s_i\})$$ each is equal $p \in [0,1]$

$$1 = \sum_{i=1}^{n} p = np \Rightarrow p = 1/n = 1/|S|$$ note: $n = |S|$)

More generally: if $E \subset S$ then

$$\mathbb{P}(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S} = \frac{|E|}{|S|}$$

Ex. Roll a six-sided (fair) die.

$$S = \{1, 2, 3, \ldots, 6\}$$

<u>if all outcomes are equally likely</u> then

$$E = \{2, 6\}$$

then $$P(E) = \frac{|E|}{|S|} = 2/6 = 1/3.$$

<u>note:</u>

$$\mathbb{P}(E) = \frac{|E|}{|S|}$$

$|E|$ ← need to <u>count</u> # of outcomes in E

$|S|$ ← need to <u>count</u> # in S

<u>Counting</u>

<u>Theorem</u>: <u>Fundamental Theorem of Counting (FTC)</u>

If a task or job consists of k <u>subtasks</u> and the i^{th} <u>subtask</u> can be done in $\boxed{n_i}$ ways, then overall the number of ways to do the task is

$$N = n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot \ldots \cdot n_k$$
$$= \prod_{i=1}^{k} n_i$$

<u>Ex.</u> An experiment consists of 3 factors:

Ex: An experiment consists of 3 factors:

(1) 2 temperature settings
(2) 2 pressure settings
(3) 4 humidity setting

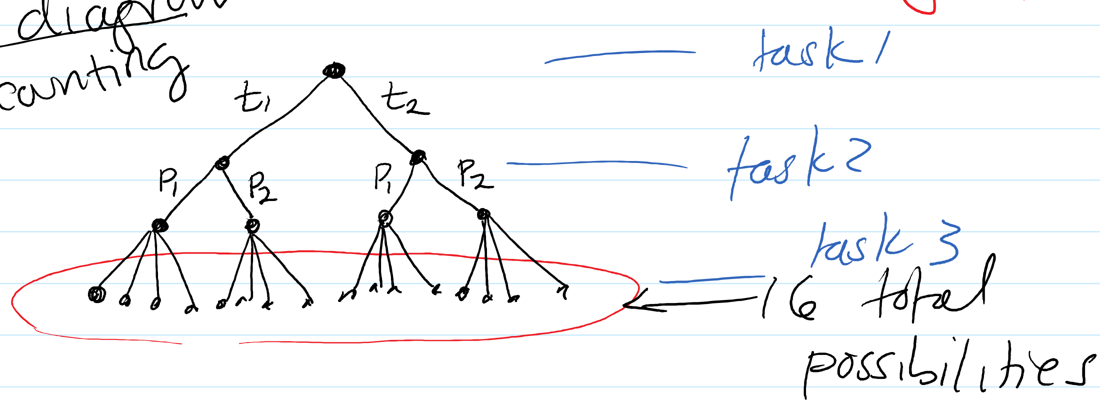
Q: How many ways can I run the experiment?

$k=3$, $n_1=2$, $n_2=2$, $n_3=4$

overall I have

$N = 2 \cdot 2 \cdot 4 = 16$ ways to choose my experiment

Tree diagram for counting



Ex, A man has 5 shirts, 2 pairs of pants, 2 pairs of shoes.

Q: How many outfits does the man have?

$N = 5 \cdot 2 \cdot 2 = 20$ outfits.

Ex, I have a deck of 52 cards.

denominations: A, 2, 3, ..., 10, J, Q K (13)

Suits : C, D, H, S (4)

I shuffle the cards so that each ordering of the cards is equally likely

Q: what is the prob. that I get the cards "in order" (A–K, C, D, H, S)

E = "in order"

S = all possible orderings of 52 cards

then $P(E) = \frac{|E|}{|S|}$ ← $|E| = 1$, ← ??

k = 52 subtasks to create an ordering of the cards

task 1 : choose card 1 → 52

task 2 : " 2 → 51

task 3 : " 3 → 50

⋮

task 52 : " 52 → 1

multiply

$$|S| = 52 \cdot 51 \cdot 50 \cdot 49 \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

hence

$$\mathbb{P}(E) = 1/|S| = \frac{1}{52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1}$$

$$\approx 10^{-68}$$

Defn: Factorial

For any non-negative integer n we define **n-factorial** (denoted $n!$) as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$
$$= \prod_{i=1}^{n} i$$

note: $0! = 1$.

Ex. Prev. example:

$$\mathbb{P}(E) = \frac{1}{52!}$$

Sampling w/ or w/o Ordering and Replacement

Ordering: order matters (or not)

do I care about the order of my sample?

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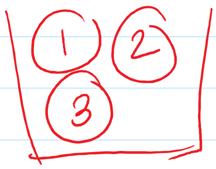


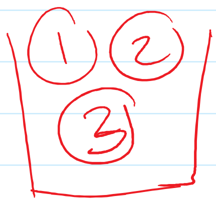
draw 1: 1 3 2

draw 2: 2 1 3

different?

Replacement:

Can I draw the same item more than once?



w/ replacement: possible to draw 1 1 2

w/o replacement: not possible to draw 1 1 2

This yields 4 scenarios

	w/o replacement	w/ replacement
ordered	1 $\frac{n!}{(n-r)!}$	2
un-ordered	4	3

Permutation:

A permutation is an ordering of a collection of objects.

Ex. objects $\{\underbrace{A_1, A_2, A_3}_{3 \text{ items}}\}$

permutations

$A_1A_2A_3$, $A_1A_3A_2$, $A_2A_1A_3$

$A_2A_3A_1$, $A_3A_1A_2$, $A_3A_2A_1$

} 6 possible permutations $= 3!$

Theorem: Permutation Counting

The number of ways to permute n items is $n!$

pf. Use FTC.

$k = n$ subtasks

task #	task	# ways
1	choose item 1	n
2	" 2	$n-1$
3	" 3	$n-2$
⋮	⋮	
n	" n	1

multiply

this gives a total # ways

$$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

Theorem: Ordered Sampling w/o Replacement

If I have n items and I draw a sample of size r (note: $r \le n$)

and I draw my sample w/o replacement and I care about the order of my draws

I can do this (# of possible samples) is

$$\boxed{(n)_r = \frac{n!}{(n-r)!}}$$

pf. $k = r$ subtasks

task #	task	# ways
1	draw 1st	n
2	draw 2nd	$n-1$
3	draw 3rd	$n-2$
⋮	⋮	⋮
r	draw rth	$n-r+1$

multiply

total # ways $= \boxed{n(n-1)(n-2)\cdots(n-r+1)}$

$$\boxed{\frac{n!}{(n-r)!}} = \frac{\boxed{n(n-1)\cdots(n-r+2)(n-r+1)}\cancel{(n-r)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cancel{(n-r-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

Ex. 10 students want to form a committee consisting of 3 members; ... must be ...

Ex. 10 students want to form a committee consisting of 3 members: ← must be distinct

President, VP, treasurer

How many possible different committees can I form?

$n = 10$, $r = 3$, no repl., care about ordering

Sq: 1^{st} = Pres, 2^{nd} = VP, 3^{rd} = treasurer

$$\# \text{ ways} = \frac{n!}{(n-r)!} = \frac{10!}{(10-3)!} = \frac{10!}{7!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}}$$

$$= 10 \cdot 9 \cdot 8 = 720.$$

Ex. Lotto.

I have a box w/ pingpong balls numbered 1, ..., 25

1 2 3 25

Lotto: draws 4 balls w/o replacement, if you guess the 4 in correct order you win.

my choice: 1 3 22 7 ← all possibilities are equally likely

Q: What is the prob. I win?

$E = \{ 1, 3, 22, 7 \}$

$S = \{\text{all possible drawings}\}$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{|S|}$$

$$\boxed{|S|} = \frac{n!}{(n-r)!} = \frac{25!}{(25-4)!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21!}}{\cancel{21!}}$$

$n = 25, \; r = 4$

$$= 25 \cdot 24 \cdot 23 \cdot 22$$

$$\boxed{P(E) = \frac{1}{25 \cdot 24 \cdot 23 \cdot 22}}$$

Theorem: Sampling w/ Replacement and w/ Ordering

The number of ways to draw a sample of size r from n items

(1) w/ replacement

(2) w/o Ordering

is

$$\boxed{n^r}.$$

note: don't need $r \le n$

pf. Use FTC. $k = r$

Pf. Use FTC. $k = r$

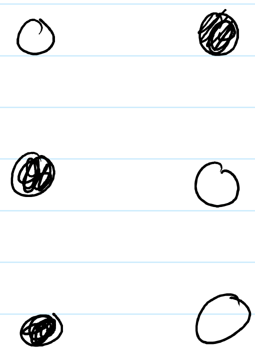
task #	task	# ways
1	select 1st	n
2	select 2nd	n
3	select 3rd	n
$\vdots$	$\vdots$	$\vdots$
r	select rth	n

multiply

total # ways $= \underbrace{n \cdot n \cdot n \cdots n}_{r\text{-times}} = n^r$.

Ex. Braille alphabet

6 locations for bumps in a braille letter



Q: How many braille letters can I make?

idea: draw 6 samples (bumps) from two possibilities "raised" or "not-raised"

Sample w/ repl. w/o ordering.

$n = 2, \ r = 6$

total # ways to do this: $2^6 = 64$.