MV RV:

If $X_1, X_2, \dots, X_n$ are RVs then

RV $\rightarrow$ $$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = (X_1, X_2, \dots, X_n)^T$$

is called a <u>multivariate random variable</u>

or a <u>random vector</u>.

Defn: PMF/PDF

If X_is are <u>discrete</u> then the <u>joint pmf</u>

$x \in \mathbb{R}^n$

$$f(\underset{\sim}{x}) = f(x_1, x_2, x_3, \dots, x_n)$$
$$= \mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

<u>Note:</u> $f: \mathbb{R}^n \rightarrow \mathbb{R}$.

<u>Cts case:</u> if X_is are continuous. Then the <u>joint pdf</u> is defined as a function

the joint pdf is defined as a function

$$f : \mathbb{R}^n \to \mathbb{R}$$

so that $A \subset \mathbb{R}^n$ then

$$\mathbb{P}(\underset{\sim}{X} \in A) = \mathbb{P}((X_1, X_2, X_3, \dots, X_n)^T \in A)$$

$$= \int_A f(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$= \underbrace{\int\int\int \cdots \int}_{n-\text{times}} f(x_1, x_2, \dots, x_n)\, dx_1\, dx_2 \dots dx_n$$

Expectation:

If $g : \mathbb{R}^n \to \mathbb{R}$ then

$$\mathbb{E}[g(\underset{\sim}{X})] = \begin{cases} \sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} g(x_1,\dots,x_n) f(x_1,\dots,x_n) & \text{(discrete)} \\ \int \cdots \int g(x_1,\dots,x_n) f(x_1,\dots,x_n)\, dx_1 \cdots dx_n & \text{(cts)} \end{cases}$$

Theorem: Marginal Distributions

The marginal dist. of X_i is

$$f_{X_i}(x_i) = \begin{cases} \sum_{x_1}\sum_{x_2}\cdots\sum_{x_{i-1}}\sum_{x_{i+1}}\cdots\sum_{x_n} f(x_1,\dots,x_n) & \text{(discrete)} \\ \int\cdots\int f(x_1,\dots,x_n)\,dx_1\cdots dx_{i-1}\,dx_{i+1}\cdots dx_n & \text{(cts)} \end{cases}$$

Conditional Distributions

If I have two sequences of RVs

$$X_{i_1}, X_{i_2}, \dots, X_{i_m}$$

and

$$X_{j_1}, X_{j_2}, \dots, X_{j_k}$$

then the conditional pmf/pdf of $X_{i_1},\dots,X_{i_m}$ given $X_{j_1},\dots,X_{j_k}$ is

$$f(x_{i_1},\dots,x_{i_m} \mid x_{j_1},\dots,x_{j_k}) = \frac{f(x_{i_1},\dots,x_{i_m},x_{j_1},\dots,x_{j_k})}{f(x_{j_1},\dots,x_{j_k})}$$

Ex. Let X_1, X_2, X_3, X_4 have a joint PDF of

$$f(x_1, x_2, x_3, x_4) = \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$$

for $0 < x_i < 1$.

Check: $\int \cdots \int f(x_1, \ldots, x_4)\, dx_1 \cdots dx_4 = 1$

(a) $P(\overbrace{X_1 < 1/2, X_2 < 3/4, X_4 > 1/2}^{A})$

$$= \underbrace{\int_{1/2}^{1}\int_{0}^{1}\int_{0}^{3/4}\int_{0}^{1/2}}_{A} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_1\, dx_2\, dx_3\, dx_4$$

$$= \text{Calc } 3 = 3/256.$$

(b) Joint pdf of X_1 and X_2?

$$f(x_1, x_2) = \int_0^1\int_0^1 \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_3\, dx_4$$

$$= \text{Calc } 3 = \boxed{\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)}$$

(c) $$E[X_1 X_2] = \int_0^1\int_0^1\int_0^1\int_0^1 x_1 x_2 \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_1\, dx_2\, dx_3\, dx_4$$

$$= \int_0^1\int_0^1 x_1 x_2 \left(\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)\right) dx_1\, dx_2$$

$$= \int_0^1 \int_0^1 x_1 x_2 \left(\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2) \right) dx_1 dx_2$$

$$= \text{Calc } 3 = 5/16$$

d) Conditional Dists

What is the conditional dist. of X_3 and X_4 given X_1 and X_2.

$$f(x_3, x_4 \mid x_1, x_2) = \frac{f(x_1, x_2, x_3, x_4)}{f(x_1, x_2)}$$

$$\boxed{= \frac{\frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)}{\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)}}$$

Mutual Independence

We say $X_1, \ldots, X_n$ are <u>mutually independent</u> if for any sets $A_1, A_2, A_3, \ldots, A_n \subset \mathbb{R}$

$$P(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n)$$

$$= P(X_1 \in A_1) P(X_2 \in A_2) \cdots P(X_n \in A_n).$$

Theorem: Independence

$X_1, \ldots, X_n$ are mutually independent iff

(a) $f(x_1, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$

pmf/pdf

or

(b) $F(x_1, \ldots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$.

CDF

Theorem: If X_i are independent and

$$g_i : \mathbb{R} \to \mathbb{R}$$

then

(1) $\{g_i(X_i)\}$ are independent

(2) $E[g_1(X_1) g_2(X_2) \cdots g_n(X_n)]$

$$= E[g_1(X_1)] \cdots E[g_n(X_n)].$$

Corollary: MGF of Sums of Independent RVs

Let X_i be mutually independent and

$$Z = \sum_{i=1}^{n} X_i,$$

then

$$M_Z(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$$
$$= \prod_{i=1}^{n} M_{X_i}(t)$$

$$a_1 + a_2 + \cdots a_n = \sum_{i=1}^{n} a_i$$

$$a_1 a_2 a_3 \cdots a_n = \prod_{i=1}^{n} a_i$$

pf.

$$M_Z(t) = E[e^{tZ}]$$
$$= E[e^{t\sum_{i=1}^{n} X_i}] = E[\prod_{i=1}^{n} e^{tX_i}]$$
$$= \prod_{i=1}^{n} E[e^{tX_i}]$$
$$= \prod_{i=1}^{n} M_{X_i}(t)$$

$$e^{a+b} = e^a e^b$$

Follow-on

If a_i and b_i are constants,

and $$Z = \sum_{i=1}^{n} (a_i + b_i X_i)$$

Univariate:

$$M_{a+bX}(t) = e^{at} M_X(bt)$$

n

$i=1$ $= e^{at} M_X(bt)$

then
$$M_Z(t) = e^{t\sum_{i=1}^n a_i} \prod_{i=1}^n M_{X_i}(b_i t)$$

Ex. $X_i \sim N(\mu_i, \sigma_i^2)$ and independent

then
$$Y = \sum_{i=1}^n (a_i + b_i X_i)$$
$$\sim N\left(\sum_{i=1}^n a_i + b_i \mu_i, \sum_{i=1}^n b_i^2 \sigma_i^2\right)$$

prove. this using prev. theorem

Multivariate Transformations

let $\underset{\sim}{X} = (X_1, \ldots, X_n)^T$ and $g: \mathbb{R}^n \to \mathbb{R}^n$

let
$$U = g(\underset{\sim}{X})$$
i.e. $U_i = g_i(X_1, X_2, \ldots, X_n)$ for $i = 1, \ldots, n$.

g_i: ith component fn $\mathbb{R}^n \to \mathbb{R}$

Theorem: In the cts case if g is cts and differentiable, then

$u \in \mathbb{R}^n$

$$\boxed{f_U(u) = f_{\underset{\sim}{X}}(g^{-1}(u))\,|\det J|}$$

$$\underset{n\times n}{J} = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u_1} & \frac{\partial g_1^{-1}}{\partial u_2} & \cdots & \frac{\partial g_1^{-1}}{\partial u_n} \\ \frac{\partial g_2^{-1}}{\partial u_1} & \frac{\partial g_2^{-1}}{\partial u_2} & \cdots & \\ & & & \end{bmatrix}$$

$$J_{ij} = \frac{\partial g_i^{-1}}{\partial u_j}$$

Ex. Let $X_i \overset{iid}{\sim} Exp(\lambda)$

independent indentical distribution

$U = g(\underline{X})$

$u = (u_1, \ldots, u_n)$

$U_1 = X_1 \iff X_1 = U_1 \to \boxed{g_1^{-1}(u) = u_1}$

$U_2 = X_1 + \underline{X_2} \iff X_2 = U_2 - U_1 \to \boxed{g_2^{-1}(u) = u_2 - u_1}$

$U_3 = \underline{X_1 + X_2} + \underline{X_3} \iff X_3 = U_3 - U_2 \to \boxed{g_3^{-1}(u) = u_3 - u_2}$

$\vdots$

$U_n = X_1 + X_2 + \cdots + X_n \qquad \boxed{g_n^{-1}(u) = u_n - u_{n-1}}$

$U_n = X_1 + X_2 + \cdots + X_n$

$$\boxed{g_n^{-1}(u) = U_n - U_{n-1}}$$

$$J_{ij} = \frac{\partial g_i^{-1}}{\partial u_j}$$

$$J = \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ -1 & 1 & 0 & \cdots & & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ & & -1 & 1 & \ddots & 0 \\ 0 & & & -1 & \ddots & \\ & & & & -1 & 1 \end{bmatrix}$$

det of a triangular is product of diags

$$|\det J| = 1$$

$$f_U(u) = f_{\underset{\sim}{X}}(\underline{g^{-1}(u)})$$

① $f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \underline{\lambda e^{-\lambda \sum_{i=1}^{n} x_i}}$

② $f_U(u) = \lambda e^{-\lambda(U_n}$

$\sum_{i=1}^{n} g_i^{-1}(u)$

$= g_1^{-1}(u) + g_2^{-1}(u) + \cdots g_n^{-1}(u)$

$f_U(u) = \lambda e^{-\lambda U_n}$

$$f_U(u) = \lambda e^{-\lambda u_n}$$

$$0 < u_1 < u_2 < u_3 < \cdots < u_n$$

$= g_1(u) + g_2(u) + \cdots + g_n(u)$

$= \cancel{u_1} + \cancel{u_2} - \cancel{u_1} + \cancel{u_3} - \cancel{u_2} + \cancel{u_4} - \cancel{u_3} + \cdots + u_n - \cancel{u_{n-1}}$

Means/Variances For MV RVs.

Univariate: $E[X] \in \mathbb{R}$

$$Var(X) = E[(X - E[X])^2] \in \mathbb{R}$$

Multivariate: $\underset{\sim}{X} = (X_1, \ldots, X_n)^T$ n-dim'l

$$\mu = E[\underset{\sim}{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_3] \end{bmatrix} \in \mathbb{R}^n$$

Covariance matrix

$$Cov(\underset{\sim}{X}) = \Sigma \in \mathbb{R}^{n \times n}$$

when $\Sigma_{ij} = Cov(X_i, X_j)$.

notice $\Sigma_{ii} = Cov(X_i, X_i) = Var(X_i)$

$$\begin{bmatrix} Var(X_1) & \underline{Cov(X_1, X_2)} & \cdots \\ Cov(X_2, X_1) & Var(X_2) & \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & \cdots & \\ Cov(X_2,X_1) & Var(X_2) & & \\ \vdots & & \ddots & \\ & & & Var(X_n) \end{bmatrix}$$

symmetric

Univariate:

$$Var(X) = E[(X-E[X])(X-E[X])]$$

Multivariate:

$$Cov(\underset{\sim}{X}) = E[(\underset{\sim}{X}-E[\underset{\sim}{X}])(\underset{\sim}{X}-E[\underset{\sim}{X}])^T]$$

Theorems:

If $a \in \mathbb{R}^m$ and $B \in \mathbb{R}^{m \times n}$ and $\underset{\sim}{X}$ is a n-dim'l RVector then

ⓐ $E[a + B\underset{\sim}{X}] = a + BE[\underset{\sim}{X}]$ (linearity)

ⓑ $Cov(a + B\underset{\sim}{X}) = B\, Cov(\underset{\sim}{X})\, B^T$ (like $b^2 Var(X)$)

Multivariate Normal

$(X_1, X_2, \ldots, X_n)$

Multivariate Normal

$\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$

$\underset{\sim}{X} \sim N(\mu, \Sigma)$

$\mu \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$

if

$x \in \mathbb{R}^n$

$$\boxed{f(x_1, \ldots, x_n) = f(\underset{\sim}{x}) = (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left(-\tfrac{1}{2}(\underset{\sim}{x}-\mu)\Sigma^{-1}(\underset{\sim}{x}-\mu)^T\right)}$$

Univariate!

$$f(x) = (2\pi)^{-1/2} (\sigma^2)^{-1/2} \exp\left(-\tfrac{1}{2}(x-\mu)(\sigma^2)^{-1}(x-\mu)\right)$$

Special Case:

When $\mu = 0$ and $\Sigma = I$

we call it the Standard MV Normal.