Lecture 7 - Independence

Thursday, September 23, 2021 1:59 PM

Laymen's definition:

→ things don't affect each other
→ events are independent if knowing the occurence (or not) ~~of~~ of one event doesn't affect the prob of another.

Defn: Independence (of events)

If $A, B \subset S$ we say "A is independent of B" denoted $A \perp B$, if

$$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B).$$

→ independence is distributive law
→ intuition for product notation for intersection

Theorem:

If $A \perp\!\!\!\perp B$ then

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

Pf.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

defn of cond. prob.

defn of independence

defn of Cond. prob. — defn of independence

Ex. Consider rolling two dice. (independently)

$P(\text{at least one } 6)$

$= 1 - P(\underbrace{\text{"at least one 6"}^c}_{\text{no } 6})$ — $A_1 A_2$

$A_1 = \text{no } 6 \text{ on first roll}$

$A_2 = \text{no } 6 \text{ on second roll}$

$= 1 - P(A_1 A_2)$

$= 1 - P(A_1)P(A_2)$

$= 1 - (5/6)(5/6) = 11/36$

Counting Perspective:

Sampling from $\{1, \ldots, 6\}$ $(n = 6)$ two times $(r = 2)$ w/ replacement

Ordered: $|S| = 6^2 = 36$

E = "at two 6"

$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5)\}$

$|E| = 11$

$$\mathbb{P}(E) = \frac{|E|}{|S|} = 11/36$$

Unordered: $|S| = \binom{n+r-1}{r} = \binom{7}{2} = 21$

$E = \{\{1,6\}, \{2,6\}, \{3,6\}, \{4,6\}, \{5,6\}, \{6,6\}\}$

$|E| = 6$

So $\mathbb{P}(E) = 6/21$.

Ex. Roll two dice. (independently)

E = { 1 or 2 on first roll and 3, 4, 5 on second roll. }

Ordered

$|S| = n^r = 6^2 = 6 \cdot 6$

<u>Note:</u> $E = \{(1,3), (1,4), (1,5),$
$(2,3), (2,4), (2,5)\}$ so $|E| = 6$

$$E = \{1,2\} \times \{3,4,5\}$$

and $|E| = |\{1,2\}| \cdot |\{3,4,5\}|$
$= 2 \cdot 3$

$$P(E) = \frac{|E|}{|S|} = \frac{2 \cdot 3}{6 \cdot 6} = \left(\frac{2}{6}\right)\left(\frac{3}{6}\right)$$

prob. of rolling 1 or 2 on first

prob. of rolling 3, 4 or 5 on second.

<u>Theorem: Complements and Independence</u>

If $A \perp\!\!\!\perp B$ then

1. $A \perp\!\!\!\perp B^c$
2. $A^c \perp\!\!\!\perp B$
3. $A^c \perp\!\!\!\perp B^c$

<u>pf.</u>

$$\begin{aligned} P(AB^c) &= P(A) - P(AB) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B^c) \end{aligned}$$

<u>Defn: Mutual Independence</u>

Defn: Mutual Independence

Generalize independence to multiple events.

If $(A_i)_{i=1}^{n}$ is a seq of events we say they are (mutually) independent if for any subsequence $A_{i_1}, A_{i_2}, A_{i_3}, \dots, A_{i_k}$ of size k

$$\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k})$$
$$= \prod_{j=1}^{k} \mathbb{P}(A_{i_j})$$

Q: Do we really have to check all subsequences?

Can I just check

$$\mathbb{P}(A_1 A_2 A_3 \cdots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2) \cdots \mathbb{P}(A_n)$$

No.

Ex. Roll two dice.

$A =$ "doubles" $= \{(1,1),(2,2),\ldots,(6,6)\}$

$|A| = 6$

$B =$ "sum of two is between 7 and 10"

$= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$

$|B| = 18$ $\quad (2,6),(3,5),(4,4),(5,3),(6,2)$

$(3,6),(4,5),(5,4),(6,3)$

$(4,6),(5,5),(6,4)\}$

$C =$ "sum is 2, 7 or 8"

$= \{(1,1), \ldots \}$

$|C| = 12$

Mutually independent?

$$\underbrace{P(\underbrace{ABC}_{\{(4,4)\}})}_{1/36} = \underbrace{\underbrace{P(A)}_{6/36}}_{1/6}\,\underbrace{\underbrace{P(B)}_{18/36}}_{1/2}\,\underbrace{\underbrace{P(C)}_{12/36}}_{1/3} \quad \checkmark$$

Consider $BC =$ "sum is 7 or 8"

$|BC| = 11$

So

$$\underbrace{P(BC)}_{1/36} \cancel{=} \underbrace{P(B)}_{1/2}\underbrace{P(C)}_{1/3}$$

not mutually independent.

Defn: Pairwise Independence

If (A_i) is a seq of events we say they are pairwise independent if

$$P(A_i A_j) = P(A_i)P(A_j) \quad \forall i \neq j$$

Note: Can $A \perp\!\!\!\perp A$?

$$P(A) = P(AA) = P(A)P(A) = P(A)^2$$

and $P(A) \in [0,1]$

So this works if $P(A) = 0$ or 1.

Ex. Pairwise $\neq$ Mutual Independence.

$$S = \{abc, acb, bac, bca, cab, cba, aaa, bbb, ccc\}$$

$S = \{abc, acb, bac, bca, cab, cba,$
$aaa, bbb, ccc\}$ $\quad |S| = 9$

All outcomes equally likely.

$A_i = \{i^{th}$ place has an $a\}$

$A_1 = \{abc, acb, aaa\}$

$A_2 = \{bac, cab, aaa\}$

$A_3 = \{bca, cba, aaa\}$

Pairwise independent?

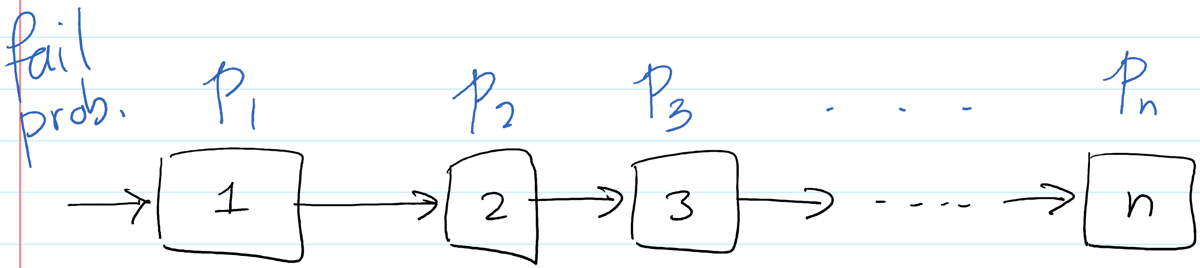
$$\underbrace{P(\underbrace{A_i A_j}_{\{aaa\}})}_{1/9} = \underbrace{P(A_i)}_{\underbrace{3/9}_{1/3}} \underbrace{P(A_j)}_{\underbrace{3/9}_{1/3}} \quad \checkmark$$

Mutual Independence?

$$\underbrace{P(\underbrace{A_1 A_2 A_3}_{\{aaa\}})}_{1/9} \neq \underbrace{P(A_1)}_{1/3} \underbrace{P(A_2)}_{1/3} \underbrace{P(A_3)}_{1/3}$$

not mutually independent.

Ex. Serial System.



→ System only works if all components work.

→ all component fail independently

let W_i = i^{th} component works

W_i^c = i^{th} " fails

P(System works)

$$= \mathbb{P}\left(\bigcap_{i=1}^{n} W_i\right)$$

$$= \prod_{i=1}^{n} \mathbb{P}(W_i)$$

$$\mathbb{P}(W_i^c) = p_i$$

$$= \prod_{i=1}^{n} \left(1 - \mathbb{P}(W_i^c)\right)$$

$$= \prod_{i=1}^{n} (1 - p_i).$$