ct. pf. last time.

$$\boxed{③ \quad E = \bigcup_{i=1}^{\infty} E_i \text{ where } E_i \text{ are disjoint then } P(E) = \sum_{i=1}^{\infty} P(E_i).}$$

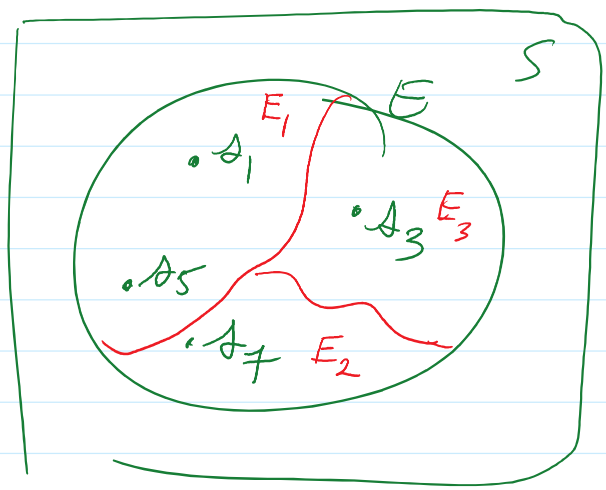
$$P(E) = P\Big(\bigcup_{i=1}^{\infty} E_i\Big)$$

$$= \sum_{j:\, s_j \in \bigcup_{i=1}^{\infty} E_i} p_j$$

$$= \sum_{i=1}^{\infty} \underbrace{\sum_{j:\, s_j \in E_i} p_j}_{P(E_i)}$$

$$= \sum_{i=1}^{\infty} P(E_i)$$

$$P(F) = \sum_{j:\, s_j \in F} p_j$$



$$P(E) = p_1 + p_3 + p_5 + p_7$$
$$= \underline{(p_1 + p_5)} + \underline{(p_7)} + \underline{(p_3)}$$

Basic Theorems

$$\boxed{\text{Theorem: } P(\emptyset) = 0.}$$

pf. Al[illegible] axiom ③

Pf. Abuse axiom 3

$$S = S \cup \emptyset \cup \emptyset \cup \emptyset \cdots$$
$$= S \cup \bigcup_{i=1}^{\infty} \emptyset \leftarrow \text{partitions } S$$

So $\mathbb{P}(S) = \mathbb{P}(S) + \sum_{i=1}^{\infty} P(\emptyset)$

thus $\sum_{i=1}^{\infty} P(\emptyset) = 0$

this only works if $P(\emptyset) = 0$.

Pf 2 $\mathbb{P}(\emptyset \cup \emptyset \cup \cdots) = \sum_{i=1}^{\infty} \mathbb{P}(\emptyset) < \infty$

only possible if $\mathbb{P}(\emptyset) = 0$.

Countable additivity: $(E_i)_{i=1}^{\infty}$ partition E then

$$\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Finite Additivity: $(E_i)_{i=1}^{N}$ partition E then

$$\mathbb{P}(E) = \sum_{i=1}^{N} \mathbb{P}(E_i)$$

Pf. of finite additivity

$$\mathbb{P}\left(\bigcup_{i=1}^{N} E_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{N} E_i \cup \emptyset \cup \emptyset \cup \cdots\right) = \sum_{i=1}^{N} \mathbb{P}(E_i) + \overbrace{\sum_{i=N+1}^{\infty} \mathbb{P}(\emptyset)}^{0}$$

$$= \sum_{i=1}^{N} \mathbb{P}(E_i)$$

Corollary: $A \cap B = \emptyset$ and $C = A \cup B$

then

$$\mathbb{P}(C) = \mathbb{P}(A) + \mathbb{P}(B)$$

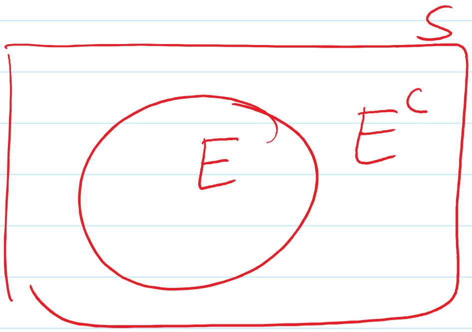
Ex. $E =$ "its raining"

$$\mathbb{P}(E) = 1/3$$

$$\mathbb{P}(\underbrace{\text{"not raining"}}_{E^c}) = \mathbb{P}(E^c) = 2/3$$

$$\boxed{\text{Theorem: } \mathbb{P}(E^c) = 1 - \mathbb{P}(E).}$$

pf. $S = E \cup E^c$ so $\underbrace{P(S)}_{1} = P(E) + P(E^c)$



So $1 = P(E) + P(E^c)$

or

$P(E^c) = 1 - P(E)$.

Theorem: Finite Measure

$$\boxed{0 \le P(E) \le 1}$$

pf. Axiom 1 says $P(E) \ge 0$

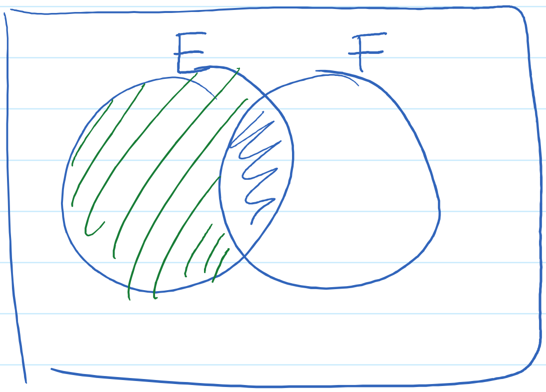
also $P(E^c) \ge 0$

then $1 - P(E) \ge 0$

prev. result.

So $P(E) \le 1$.

Theorem: $P(E \setminus F) = P(E) - P(EF)$



Pf $E = EF \cup EF^c$

EF^c: $E \setminus F$

$EF \cup EF^c$: a partition of E

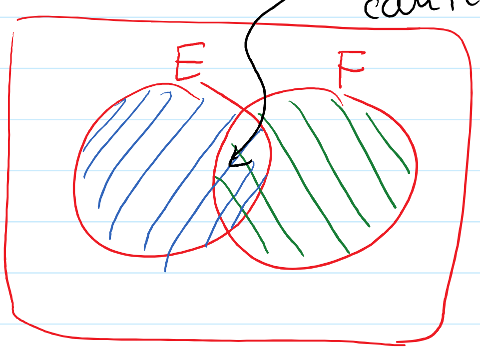
So

$$P(E) = P(EF) + P(EF^c)$$

rearrange

$$P(E \setminus F) = P(EF^c) = P(E) - P(EF)$$

Theorem: $$P(E \cup F) = P(E) + P(F) - P(EF)$$



(*) Notice E and F don't have to be disjoint.

Pf: $E \cup F = E \cup FE^c$

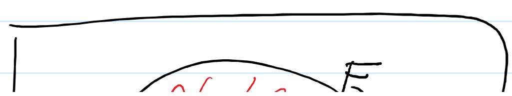
disjoint

$$P(E \cup F) = P(E \cup FE^c)$$

$$= P(E) + P(FE^c) \quad \text{b/c disjoint}$$

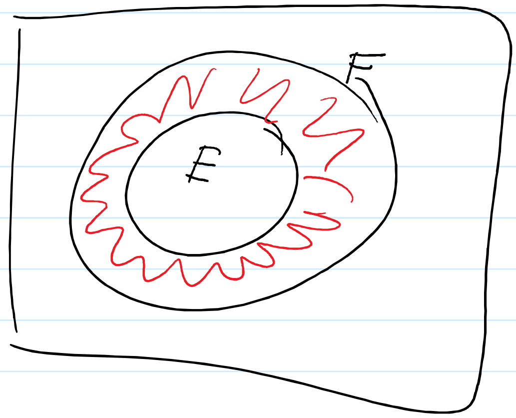
$$= P(E) + P(F) - P(FE) \quad \text{prev. theorem.}$$

Theorem: $E \subset F$



Theorem: $E \subset F$

$$P(E) \le P(F)$$



Pf. $P(FE^c) \ge 0$ (Axiom 1)

$\downarrow$

$P(F) - P(FE) \ge 0$

$E \subset F$ then $EF = E$

rearrange

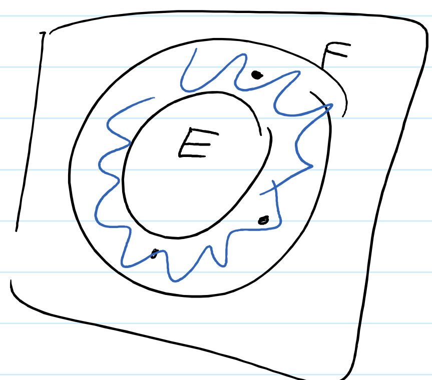
$P(FE) \le P(F)$

So $P(E) \le P(F)$.

Q: If $E \subset F$ but $E \neq F$.

(proper subset)

Q: $\cancel{P(E) < P(F)}$???



Still only $P(E) \le P(F)$

what if $F \setminus E$ has prob. zero.

Previously: $P(E \cup F) = P(E) + P(F) - \underbrace{P(EF)}_{\ge 0}$

it is true that

$$P(E \cup F) \leq P(E) + P(F)$$

Can generalize into <u>Boole's Inequality</u>.

If we have a sequence of sets $(E_i)_{i=1}^{\infty}$ then

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i)$$

<u>pf.</u> Replace E_i w/ B_i where

1. $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} B_i$

2. B_i are disjoint

<u>define</u>:

these satisfy 1 & 2

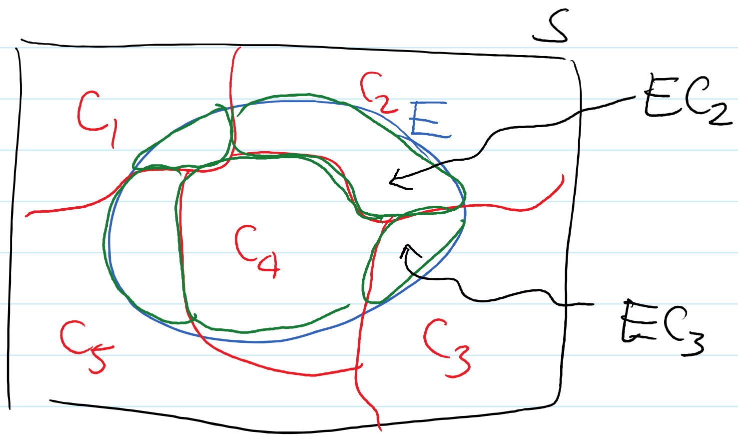
$$B_1 = E_1$$
$$B_2 = E_2 \setminus E_1$$
$$B_3 = E_3 \setminus E_2 \setminus E_1$$
$$B_4 = E_4 \setminus E_3 \setminus E_2 \setminus E_1$$
$$\vdots$$

but notice $B_i \subset E_i$

so $\boxed{P(B_i) \leq P(E_i)}$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) \leq \sum_{i=1}^{\infty} P(E_i).$$

Theorem: Event Partitioning



Notice that $(EC_i)_{i=1}^{\infty}$ partitions E.

Theorem: If $(C_i)_{i=1}^{\infty}$ partions S then for any $E \subset S$

$$P(E) = \sum_{i=1}^{\infty} P(EC_i)$$

Pf. $(EC_i)_{i=1}^{\infty}$ partions E

(1) For $i \neq j$ $\underbrace{EC_i \cap EC_j}_{E\underbrace{C_iC_j}_{\emptyset}E} = \emptyset$

(2) $\bigcup_{i=1}^{\infty} EC_i = E$

So

$$P(E) = P\left(\bigcup_{i=1}^{\infty} EC_i\right) = \sum_{i=1}^{\infty} P(EC_i).$$