Lecture 7 - Independence

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Laymen's definition:

→ things don't affect each other
→ events are independent if the occurence (or not) of one doesn't affect the prob. of the other

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<u>Defn: Independence</u> (of Events)

If $A, B \subset S$ we say "A is independent of B" denoted $A \perp\!\!\!\perp B$, if

$$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B).$$

→ kind of a distributive law
→ hints at notation for intersection

<u>Theorem:</u> (intuition for independence)

If $A \perp\!\!\!\perp B$ then

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

<u>Pf:</u>

$$\mathbb{P}(A|B) \overset{\text{defn of cond. prob.}}{=} \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} \overset{\text{use independence}}{=} \frac{\mathbb{P}(A)\cancel{\mathbb{P}(B)}}{\cancel{\mathbb{P}(B)}} = \mathbb{P}(A).$$

Ex. Consider rolling two dice (independently)

$P(\text{at least one } 6)$

$= 1 - P(\underbrace{\text{"at least one 6"}^c}_{\text{no } 6})$ $A_1 A_2$

$= 1 - P(A_1 A_2)$

A_1 = no 6 on 1^{st} roll

$= 1 - P(A_1)P(A_2)$

A_2 = no 6 on 2^{nd} roll

$= 1 - (5/6)(5/6)$

$= 1 - 25/36 = \boxed{11/36}$.

Counting Perspective:

Sampling from $\{1, .., 6\}$ $(n = 6)$ two times $(r = 2)$ w/ replacement.

Ordered Way: $|S| = 6^2 = 36$

E = at least one 6

$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$

$|E| = 11$

So $P(E) = 11/36$

Unordered: $|S| = \binom{n+r-1}{r} = \binom{7}{2} = 21$

$E = \{\{1,6\}, \{2,6\}, \{3,6\}, \{4,6\}, \{5,6\}, \{6,6\}\}$

$|E| = 6$

So $P(E) = 6/21$

Ex. Roll two dice.

$E = \{$1 or 2 on first roll and 3, 4 or 5 on second roll$\}$

Solve w/ ordered counting:

$E = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

$= \{1,2\} \times \{3,4,5\}$

$|E| = 6 = |\{1,2\}| \cdot |\{3,4,5\}| = 2 \cdot 3$

$|S| = 36 = 6 \cdot 6$

both ordered counting and independence have a product

$$|S| = 36 = 6\cdot 6$$

... and independence with a product structure

$$\text{So } \mathbb{P}(E) = \frac{2\cdot 3}{6\cdot 6} = \underbrace{\left(\frac{2}{6}\right)}_{\text{prob. I get 1 or 2 on 1st}}\underbrace{\left(\frac{3}{6}\right)}_{\text{prob. I get 3, 4, or 5 on second}}$$

Theorem: Complementary Independence

Complements don't harm independence.

If $A \perp\!\!\!\perp B$ then

1. $A \perp\!\!\!\perp B^c$
2. $A^c \perp\!\!\!\perp B$
3. $A^c \perp\!\!\!\perp B^c$.

Pf. Case 1:

$$\begin{aligned}\mathbb{P}(AB^c) &= \mathbb{P}(A) - \mathbb{P}(AB)\\ &= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)\\ &= \mathbb{P}(A)(1-\mathbb{P}(B))\\ &= \mathbb{P}(A)\mathbb{P}(B^c)\end{aligned}$$

Defn: Mutual Independence

Generalize independence to multiple events.

If $(A_i)_{i=1}^n$ of events, we say they are mutually independent if

for any subsequence $A_{i_1}, A_{i_2}, A_{i_3}, \dots, A_{i_k}$

of length k

$$\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2})\cdots\mathbb{P}(A_{i_k})$$

$$= \prod_{j=1}^{k} \mathbb{P}(A_{i_j}).$$

Q: Do we really need to check all subsequences?

Can I just check that

$$\mathbb{P}(A_1 A_2 A_3 \cdots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)\cdots\mathbb{P}(A_n)?$$

Ex. Roll two dice.

A = "doubles"

$|A| = 6$ $= \{(1,1), (2,2), (3,3), \ldots, (6,6)\}$

B = "sum is between 7 and 10"

$= \{(2,5), (1,6), (3,4), (4,3), (5,2), (6,1),$
$(2,6), (3,5), (4,4), (5,3), (6,2),$
$(3,6), (4,5), (5,4), (6,3),$
$(6,4), (5,5), (4,6)\}$

$|B| = 18$

$(6,4), (5,5), (4,6)\}$

$C =$ "sum is 2, 7 or 8"

$= \{(1,1), \_ \_ \_ \}$

$|C| = 12$

Q: Mutually Independent?

$$P(ABC) = P(A)P(B)P(C)$$

$$\overbrace{\tfrac{1}{36}}^{\{(4,4)\}} = \underbrace{\left(\tfrac{6}{36}\right)}_{1/6}\underbrace{\left(\tfrac{18}{36}\right)}_{1/2}\underbrace{\left(\tfrac{12}{36}\right)}_{1/3}$$

✓

Consider: $BC =$ "sum is 7 or 8"

$|BC| = 11$

So

$$\underbrace{P(BC)}_{11/36} = P(B)P(C)$$

$$\tfrac{11}{36} \neq \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)$$

mutual independence fails.

Defn: Pairwise Independence

(A_i) are pairwise independent if

$$P(A_i A_j) = P(A_i)P(A_j) \quad \forall\ i \neq j.$$

Can $A \perp\!\!\!\perp A$?

$$P(A) = P(AA) = P(A)\,P(A) = P(A)^2$$

$P(A)$ is in $[0, 1]$.

possible if $P(A) = 0$ or 1.

Ex. Pairwise = Mutual? [Hint: no]

$$S = \{abc, bca, acb, cba, bac, cab, aaa, bbb, ccc\}$$

Assume all 9 outcomes are equally likely.

$A_i = \{ i^{th}$ character is an "a" $\}$

$A_1 = \{abc, acb, aaa\}$

$A_2 = \{bac, cab, aaa\}$

$A_3 = \{bca, cba, aaa\}$

$A_2 = \{bac, cab, aaa\}$

$A_3 = \{bca, cba, aaa\}$

Pairwise independent? $A_iA_j = \{aaa\}$

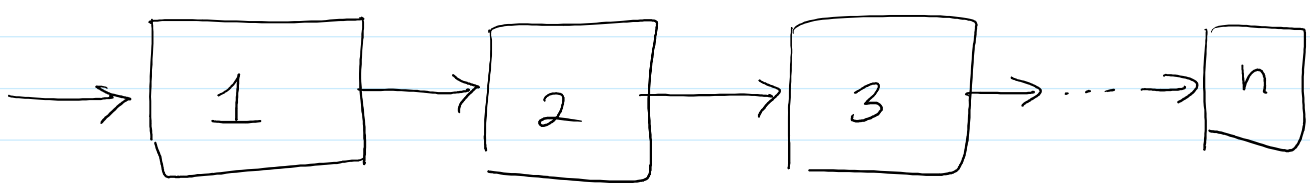
$$\underbrace{P(A_iA_j)}_{1/9} = \underbrace{P(A_i)}_{3/9}\underbrace{P(A_j)}_{3/9} \quad \checkmark$$

Mutually Independent? $\{aaa\}$

$$\underbrace{P(A_1A_2A_3)}_{1/9} \cancel{=} \underbrace{P(A_1)}_{1/3}\underbrace{P(A_2)}_{1/3}\underbrace{P(A_3)}_{1/3} \ ?$$

not mutually independent.

Ex. Failure in a serial systems



fail w/ prob. p_1 p_2 p_3 ... p_n

→ System works only if all steps work.

→ assume failure is independent among steps.

What is the prob. my system works?

Let W_i = i^{th} component works

W_i^c = i^{th} step fails

$$\mathbb{P}(\text{system works})$$

$$= \mathbb{P}\left(\bigcap_{i=1}^{n} W_i\right)$$

$$= \prod_{i=1}^{n} \mathbb{P}(W_i)$$

$$\mathbb{P}(W_i^c) = p_i$$

$$= \prod_{i=1}^{n} \left(1 - \mathbb{P}(W_i^c)\right)$$

$$= \prod_{i=1}^{n} (1-p_i) = (1-p_1)(1-p_2)(1-p_3)\cdots(1-p_n).$$