Lecture 11 - Expectation

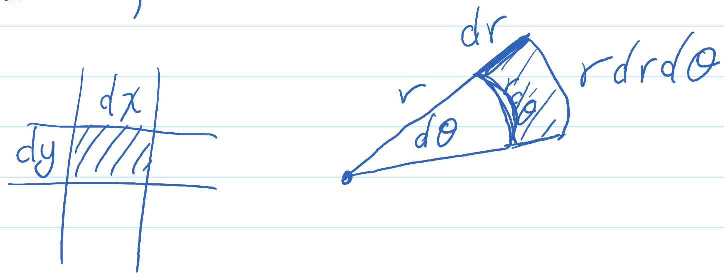
Tuesday, October 12, 2021 9:29 AM

Continue Normal example

Recall polar coordinates:

$$x, y \rightsquigarrow r, \theta$$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2=r^2 \end{array}\right\} \quad dx\,dy = r\,dr\,d\theta$$



$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{2}(x^2+y^2)\right)dx\,dy = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} \exp\left(-\tfrac{1}{2}r^2\right) r\,dr\,d\theta$$

u-substitution

let $u = \frac{1}{2}r^2$, $du = r\,dr$

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \exp(-u)\,du\,d\theta$$

$$\frac{1}{e^{\infty}} \to 0$$

$$-e^{-\infty} - (-e^{-0})$$
$$= 0 + 1$$
$$= 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{-e^{-u}\Big|_0^{\infty}}_{1} d\theta$$

2π

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta = \frac{1}{2\pi}\,2\pi = 1 = I^2$$

So $I = 1$.

Is $\int_{-\infty}^{\infty} e^{-x^2}\,dx < \infty$?

$$\propto \int_0^{\infty} e^{-x^2}\,dx = \underbrace{\int_0^{1} e^{-x^2}\,dx}_{<\infty} + \underbrace{\int_1^{\infty} \overbrace{e^{-x^2}}^{< e^{-x}}\,dx}_{< \int_1^{\infty} e^{-x}\,dx}$$

Expected Value

If X is a RV then the mean or expected value of X is denoted

$$\mathbb{E}[X]$$

is defined as

① discrete: $\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x f(x)$

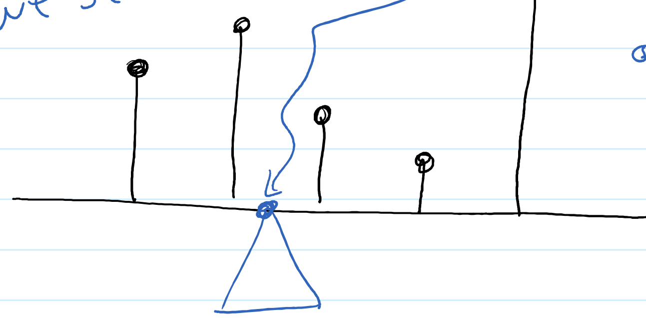
a weighted average or sum of all values X can ... $= \sum_{x \in \text{Support}(X)} x f(x)$

$\mathbb{E}[X]$ = balancing

or sum $\quad x \in$ support(X)

all values X can attain — weighted by the prob. we see them

$E[X]$ = balancing point of distribution



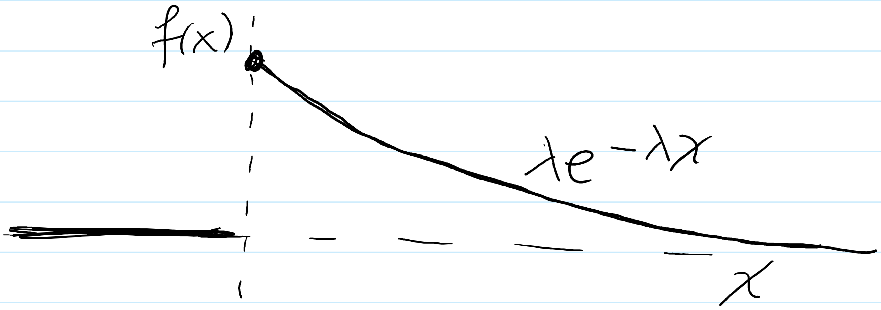
(2) Continuous: $E[X] = \int_{\mathbb{R}} x f(x)\,dx$

Ex. $\quad X \sim \mathrm{Exp}(\lambda) \quad \lambda > 0$

Exponential Distribution

means

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$



Q: what is $E[X]$?

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \underbrace{x}_{u}\,\underbrace{\lambda e^{-\lambda x}\,dx}_{dv}$$

$-\infty \qquad 0 \quad u \qquad dv$

Integration by parts: $\int u\,dv = uv - \int v\,du$

$u = x \qquad dv = \lambda e^{-\lambda x}dx$

$du = dx \qquad v = -e^{-\lambda x}$

$$= uv - \int v\,du$$

$$= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}dx$$

$$= -xe^{-\lambda x}\Big|_0^\infty - \frac{1}{\lambda}e^{-\lambda x}\Big|_0^\infty$$

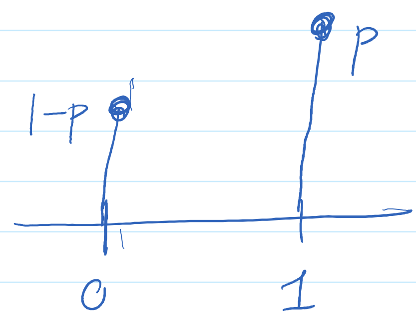
$$= (0 - 0) - \frac{1}{\lambda}(0 - 1) = 1/\lambda = E[X]$$

Ex. $X \sim \text{Bernoulli}(p)$ ← discrete

$0 \le p \le 1$

means

$$f(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \end{cases}$$



Flip a coin w/ prob p of a heads

let $0 =$ tails, $1 =$ heads

$$E[X] = \sum_x x f(x) = (0)(1-p) + (1)(p) = p$$

Ex. $X \sim Bin(n, p)$

$0 \le p \le 1$

number of trials $=$ integer > 0

$X = \#$ heads in n coin flips each w/ prob. p of H and independent.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for integers } x = 0, \dots, n$$

$$\boxed{\text{Justify to yourself: } \sum_{x=0}^{n} f(x) = 1}$$

Binomial Theorem: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

$x = p \qquad 1-p$

$$E[X] = \sum_{x=0}^{n} x f(x) = \sum_{x=\cancel{0}\,1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$n \qquad\qquad n-x$

$$x=0 \qquad\qquad x=\cancel{0}\,1$$

$$= \sum_{x=\cancel{0}1}^{n} n\binom{n-1}{x-1} p^{x}(1-p)^{n-x}$$

$$x\binom{n}{x} = \cancel{x}\frac{n!}{\cancel{x!}_{(x-1)!}(n-x)!}$$

$$= \frac{n!}{(x-1)!(n-x)!}$$

$$= \frac{n(n-1)!}{(x-1)!((n-1)-(x-1))!}$$

$$= \underline{n\binom{n-1}{x-1}}$$

$$y = x-1 \iff x = y+1$$

$$= \sum_{y=0}^{n-1} n\binom{n-1}{y} p^{y+1}(1-p)^{(n-1)-y}$$

$$= np \boxed{\underbrace{\sum_{y=0}^{n-1}\binom{n-1}{y} p^{y}(1-p)^{(n-1)-y}}_{\text{PMF}}}^{\,1}$$

$Y \sim Bin(n-1, p)$

So we have sum of PMF over support $=1$

$$= np$$

Ex. $n = 10$ w/ prob $p = \frac{1}{2}$ of H,

$$E[X] = np = 10\left(\tfrac{1}{2}\right) = 5.$$

General trick: PMF/PDF trick

Often recognize a term in a calculation that is the sum/integral of PMF/PDF over support, and turn this into a 1.

Functions of RVs

Note: any function of a RV is also a RV.

Ex. If X = # heads in some coin flips

then $\left.\begin{array}{l} X^2 = \text{sq. of \# H} \\ \log X = \text{log. of \# H.} \end{array}\right\} g(X)$

$$
\begin{tikzcd}
S \arrow[r, "X"] \arrow[rr, bend right, "g \circ X"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R}
\end{tikzcd}
$$

Theorem: The Law of the Unconscious Statistician

If $g: \mathbb{R} \to \mathbb{R}$ and X is a RV then

$$
\mathbb{E}[\underline{g(X)}] = \begin{cases} \sum_x g(x) f(x) & \text{(discrete)} \\ \int_{\mathbb{R}} g(x) f(x)\, dx & \text{(cts)} \end{cases}
$$

Ex. $X \sim \mathrm{Exp}(\lambda)$, $\lambda > 0$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

Recall: $\mathbb{E}[X] = 1/\lambda$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 \lambda e^{-\lambda x}\,dx$$

Integration by parts: $u = x^2 \qquad dv = \lambda e^{-\lambda x}\,dx$

$du = 2x\,dx \qquad v = -e^{-\lambda x}$

$$= uv - \int v\,du = -x^2 e^{-\lambda x}\Big|_0^{\infty} + 2\int_0^{\infty} e^{-\lambda x} x\,dx$$

$$= 0 + \frac{2}{\lambda}\underbrace{\int_0^{\infty} \overbrace{\lambda e^{-\lambda x}}^{f(x)} \overbrace{x}^{x}\,dx}_{\mathbb{E}[X]}$$

$$= 0 + \frac{2}{\lambda}\left(\frac{1}{\lambda}\right)$$

$$= 2/\lambda^2 \neq \underbrace{\mathbb{E}[X]}_{1/\lambda}{}^2$$