Lecture 18 - Bivariate RVs

Thursday, November 11, 2021 1:58 PM

Joint PMF: $f(x,y) = P(X=x, Y=y)$

<u>Marginal PMFs</u>: (i) $f_X(x) = \sum_y f(x,y)$

(ii) $f_Y(y) = \sum_x f(x,y)$

<u>Ex.</u> Flip 3 coins.

$$X = \begin{cases} 0 & \text{if last flip is } T \\ 1 & \quad \text{"} \qquad H \end{cases}$$

Y = # heads

$f(x,y)$

X \ Y	0	1	2	3	
0	$f(0,0) = \frac{1}{8}$	$f(0,1) = \frac{2}{8}$	$1/8$	0	$f_X(0) = 1/2$
1	0	$1/8$	$2/8$	$1/8$	$f_X(1) = 1/2$
	$f_Y(0) = \frac{1}{8}$	$f_Y(1) = \frac{3}{8}$	$f_Y(2) = \frac{3}{8}$	$f_Y(3) = \frac{1}{8}$	

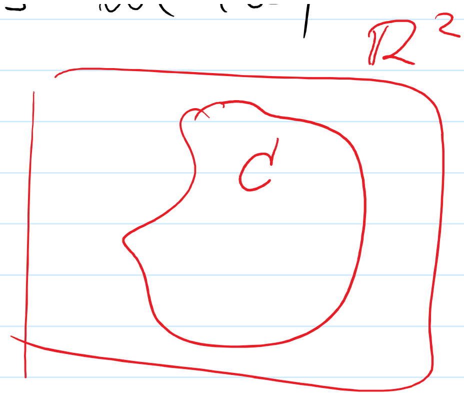
$f_X(0) = \sum_y f(0,y)$

<u>Defn: Joint PDF</u>

If X and Y are cts RVs we call $\mathbb{R}^2$

$$f : \mathbb{R}^2 \to \mathbb{R}$$

the joint PDF if $\forall C \subset \mathbb{R}^2$



$$\mathbb{P}((X,Y) \in C) = \int_C f(x,y)\,dx\,dy$$

$$\left(\text{uni: } \mathbb{P}(X \in A) = \int_A f(x)\,dx\right)$$

Facts:

① $$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

$$\left(\text{uni: } F(x) = \int_{-\infty}^{x} f(t)\,dt\right)$$

② $$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$\left(\text{uni: } f(x) = \frac{dF}{dx}\right)$$

③ $f(x,y) \geq 0$ and $\iint_{\mathbb{R}^2} f(x,y) = 1 \iff f$ PDF of some RVs (X,Y)

$$\left(\text{Uni: } f(x) \geq 0 \text{ and } \int f(x)\,dx = 1\right)$$

(Uni: $f(x) \geq 0$ and $\int_{\mathbb{R}} f(x)\,dx = 1$)

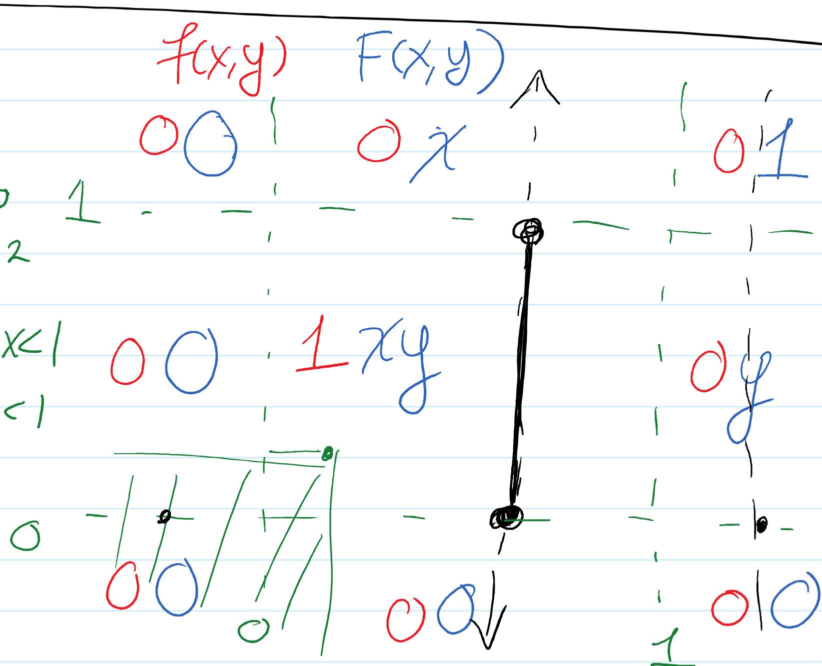
Theorem: Rel. btwn Joint/Marg PDFs

(i) $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$

(ii) $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$

Ex.

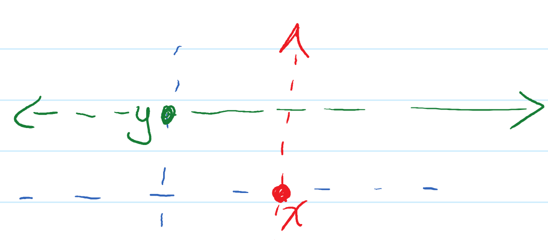
$$F(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ xy, & (x,y) \in [0,1]^2 \\ x, & y>1 \text{ and } 0<x<1 \\ y, & x>1 \text{ and } 0<y<1 \\ 1, & x>1 \text{ and } y>1 \end{cases}$$



What is f?

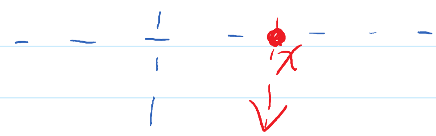
$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$ so $f(x,y) = 1$ for $0<x<1$, $0<y<1$

What are marginals?



$f_X(x) = \int f(x,y)\,dy$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$



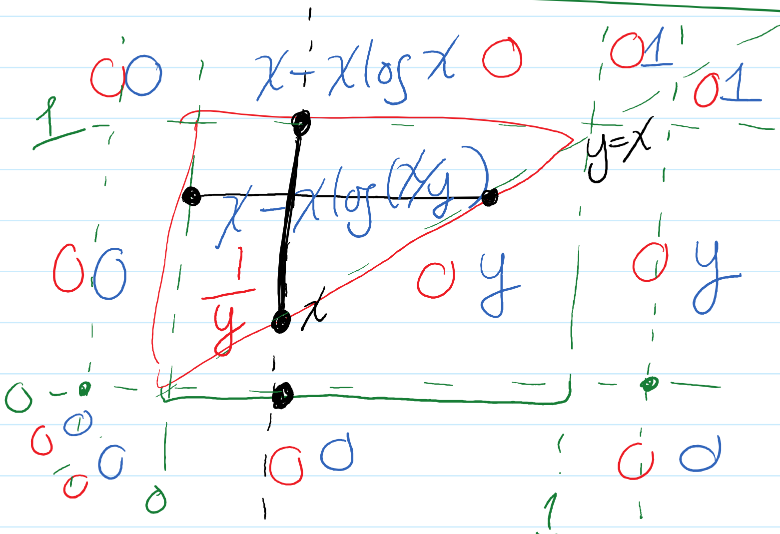
for $\underline{0<x<1}$ $= \int_0^1 1\,dy = \underline{1}$

for $x<0$ or $x>1$ $f_X(x) = 0$

$\Bigg\}$ $X \sim U(0,1)$

Similarly, $Y \sim U(0,1)$.

$\underline{\text{ex,}}$



$F(x,y)$

what is the joint PDF? $f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$

for $0<x<y<1$

$$f(x,y) = \frac{\partial^2}{\partial x \partial y}\left(x - x\log\left(\frac{x}{y}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$-\log\left(\frac{x}{y}\right) = \log\left(\frac{y}{x}\right)$$

$$\frac{\partial}{\partial y}\log\left(\frac{y}{x}\right) = \frac{1/x}{y/x} = \frac{1}{y}$$

$$\boxed{f(x,y) = \frac{1}{y} \quad \text{for } 0<x<y<1}$$

$$\boxed{f(x,y) = \frac{1}{y} \quad \text{for} \quad 0 < x < y < 1}$$

What are the marginals?

(i) $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_x^1 \frac{1}{y}\,dy = \log(y)\Big|_x^1$ $\quad$ For $0<x<1$

$$= \log 1 - \log x$$

$$= -\log x$$

(ii) $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \int_0^y \frac{1}{y}\,dx = \frac{1}{y}\int_0^y dx = 1$ $\quad$ For $0<y<1$

So $Y \sim U(0,1)$.

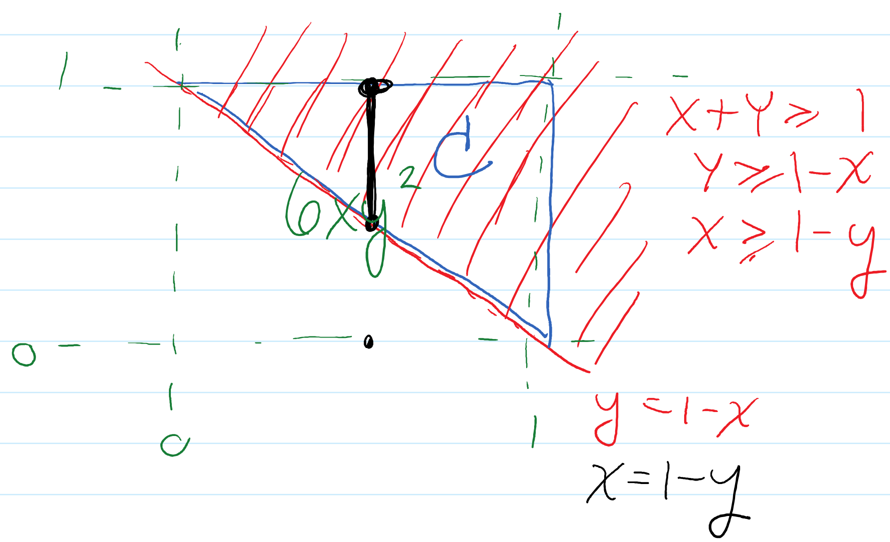
Ex. Let

$$f(x,y) = 6xy^2 \quad \text{for} \quad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$$

$P(X+Y \geq 1)$

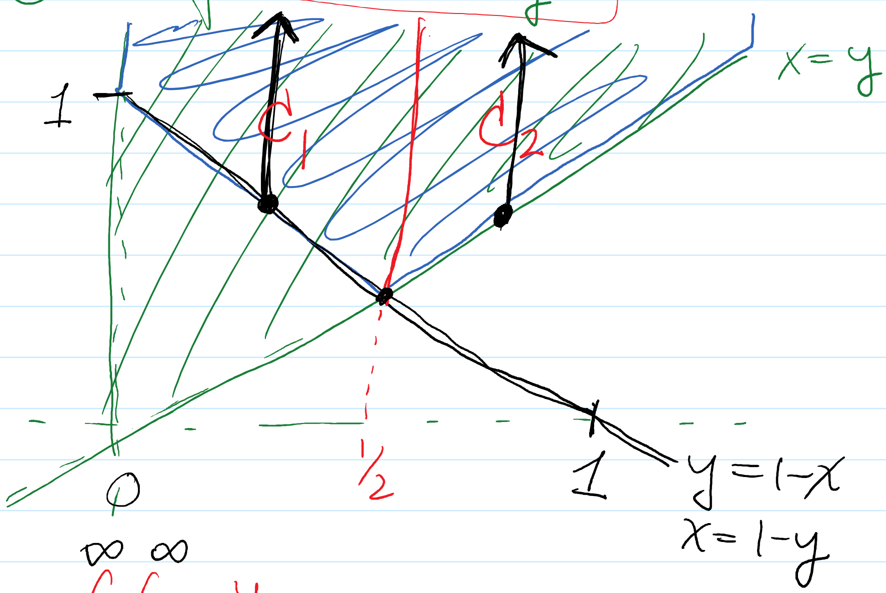
$$= \int_C f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_{1-y}^1 6xy^2\,dx\,dy$$



$$= \cdots = 9/10$$

<u>Ex.</u> $f(x,y) = e^{-y}$ for $0 < x < y$



$P(X+Y \geq 1)$

$$= \int_{C_1} f + \int_{C_2} f$$

$$= \int_0^{1/2}\int_{1-x}^{\infty} e^{-y}\,dy\,dx + \int_{1/2}^{\infty}\int_{x}^{\infty} e^{-y}\,dy\,dx$$

$$= \cdots = \boxed{2e^{-1/2} - e^{-1}}$$

<u>Defn: Bivariate Expectation</u>

If (X,Y) is Biv RV and $g : \mathbb{R}^2 \to \mathbb{R}$

$$\mathbb{E}[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) & \text{discrete} \\ \iint_{\mathbb{R}^2} g(x,y) f(x,y)\,dx\,dy & \text{continuous} \end{cases}$$

(<u>Uni:</u> $\mathbb{E}[g(X)] = \int g(x) f(x)\,dx$)

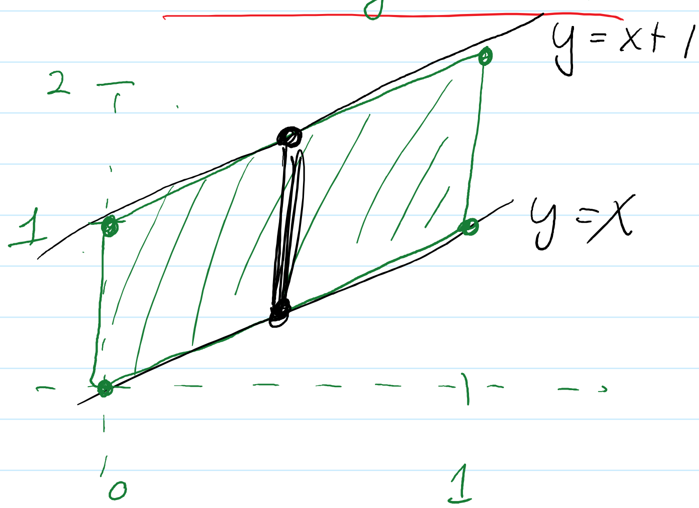
(Uni: $E[g(X)] = \int_{\mathbb{R}} g(x) f(x)\, dx$)

Ex. $f(x,y) = 1$ for $0 < x < 1$, $x < y < x+1$

$g(x,y) = xy$

$E[XY]$

$= \iint g(x,y) f(x,y)\, dx\, dy$



$= \int_0^1 \int_x^{x+1} xy\,(1)\, dy\, dx$

$= \cdots = 7/12$

Theorem: Bivariate Expectation is linear

If $g_1 : \mathbb{R}^2 \to \mathbb{R}$, $g_2 : \mathbb{R}^2 \to \mathbb{R}$ and $a, b \in \mathbb{R}$

then

$$E[a g_1(X,Y) + b g_2(X,Y)]$$

$$= a E[g_1(X,Y)] + b E[g_2(X,Y)]$$

Defn: Covariance

We define the covariance btwn X and Y as

$$\mathrm{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$$

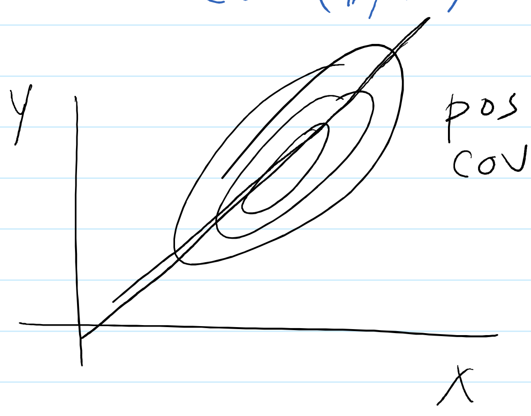
$\mu_X = \mathbb{E}X$ and $\mu_Y = \mathbb{E}Y$

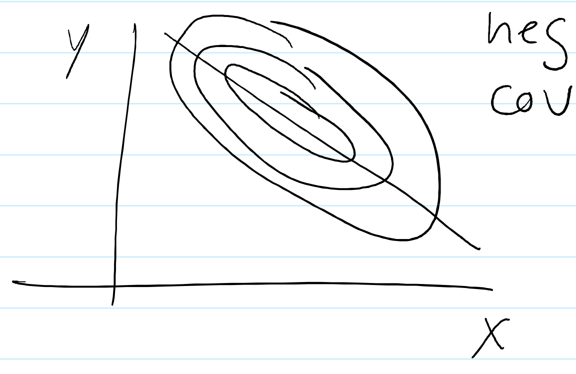
$$= \mathbb{E}\left[(X - \mu_X)(Y - \mu_Y)\right]$$

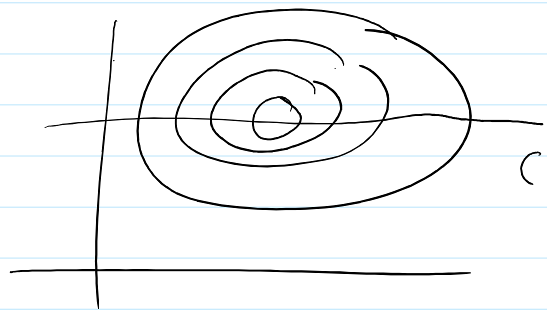
$g(x,y) = (x - \mu_x)(y - \mu_y)$

Recall: $\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}X)^2\right]$

$\mathrm{Cov}(X, X) = \mathrm{Var}(X)$







$\mathrm{cov} \approx 0$,