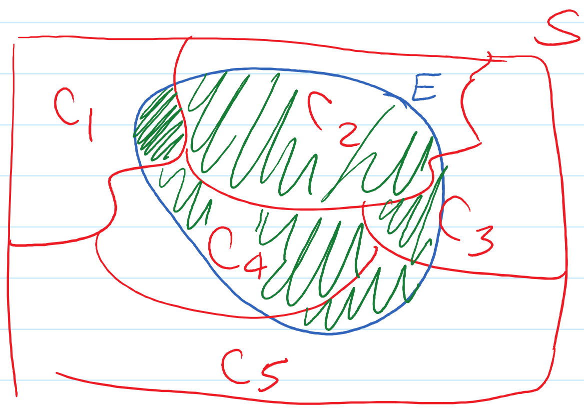
Theorem: If (C_i) are a partition of S then.



$$P(E) = \sum_i P(EC_i).$$

Pf. (1) (EC_i) partitions E

need to show

$$- \bigcup_i EC_i = E$$

$$= EC_i \cap EC_j = \emptyset \quad \forall i \neq j$$

(2) By Additivity

$$P(E) = P\left(\bigcup_i EC_i\right) = \sum_i P(EC_i)$$

Equally Likely Outcomes

I have a sample space S

$$S = \{s_1, s_2, \ldots, s_n\} \text{ so that } |S| = n.$$

assume that

$$\frac{1}{n} = \mathbb{P}(\{s_i\}) = \mathbb{P}(\{s_j\}) \quad \forall\, i,j$$

Rationale: $\{s_i\}$ for $i=1,\dots,n$ partition S

so

$$1 = \mathbb{P}(S) = \sum_{i=1}^{n} \underbrace{\mathbb{P}(\{s_i\})}_{\text{same } \forall i}$$

the only way this works is if

$$\mathbb{P}(\{s_i\}) = 1/n.$$

More generally: If $E \subset S$ then if all outcomes are equally likely

$$\mathbb{P}(E) = \frac{\#\text{ elements in } E}{\#\text{ elements in } S} = \frac{|E|}{|S|}$$

Ex. Roll a six-sided die.

$$S = \{1,\dots,6\}$$

If all rolls are equally likely then

$$E = \{2,6\}$$

then $\quad |E|/ \quad = 2/$

$P(E) = |E|/|S| = 1/6$

Ex, An experiment consist of 3 factors:

1. 2 temperature settings
2. 2 pressure settings
3. 4 humidity settings

Q: How many possible experiments? $16 = 2 \cdot 2 \cdot 4$

Theorem: Fundamental Theorem of Counting (FTC)

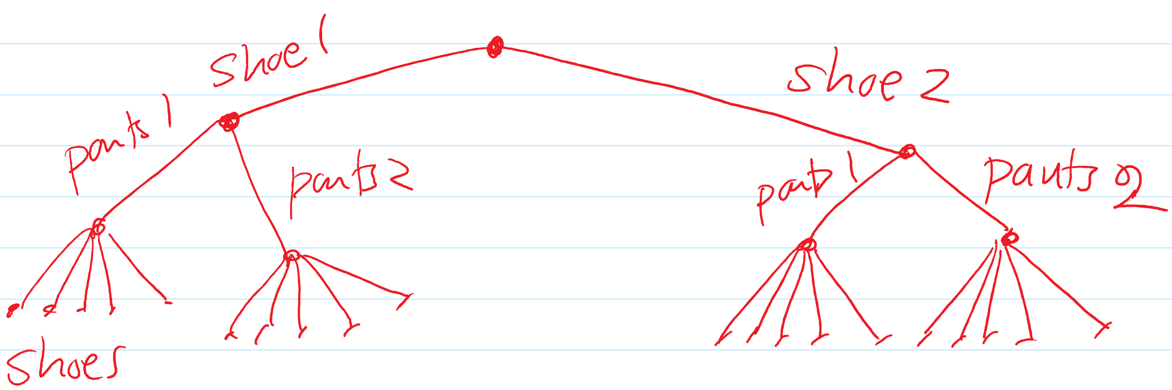
If I have a task consisting of k different subtasks where the i^{th} task can be done in n_i ways. The total number of ways I can complete the overall task is

$$N = n_1 \cdot n_2 \cdot n_3 \cdot \cdots n_k$$
$$= \prod_{i=1}^{n} n_i$$

Ex. A man has 5 shirts, 2 pairs of

Ex. A man has 5 shirts, 2 pairs of pants and 2 pair of shoes.
How many outfits does he have?

A: By FTC $5 \cdot 2 \cdot 2 = 20$.



Ex. I have a deck of 52 cards.

I shuffle the cards so that each ordering is equally likely.

13 denominations: A, 2, 3, ..., 10, J, Q, K

4 suits: C, D, H, S

Q: What is the prob. after the shuffle that the cards are in order?

A-K, C, D, H, S

E = "in order"

S = all possible orderings

$$P(E) = \frac{|E| \leftarrow 1}{|S| \leftarrow ???}$$

Using FTC consider $k = 52$

task #	task	# ways
1	choose card 1	52
2	" 2	51
3	" 3	50
⋮	⋮	
52	" 52	1

multiply

$$|S| = 52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1$$

$$P(S) = 1 / 52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1$$

Defn: Factorial

For any non-negative integer n we define n factorial as

$$n! = (n)(n-1)(n-2)\cdots(3)(2)(1)$$
$$= \prod_{i=1}^{n} i$$

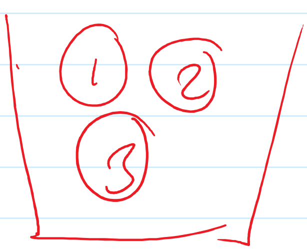
note: $0! = 1$

Ex. Prev. example

$$P(E) = \frac{1}{52!}$$

Sampling w/ and w/o Replacement / Ordering

Ordering:

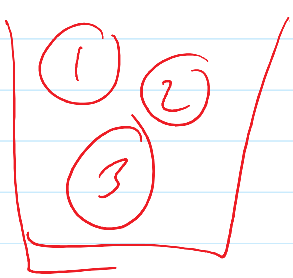


draw 1: 1 3 2

draw 2: 2 1 3

are these different?

Replacement



Can I draw 1 1 2 ?

Yes: w/ replacement

No: w/o replacement.

4 scenarios:

	w/o replacement	w/ replacement
ordered	1 $\frac{n!}{(n-r)!}$	2
Un-ordered	4	3

Permutation:

A permutation is an ordering of a collection of objects

Ex. Object $\underbrace{A_1, A_2, A_3}_{3 \text{ objects}}$ then permutations are

$A_1A_2A_3$, $A_1A_3A_2$, $A_2A_1A_3$

$A_2A_3A_1$, $A_3A_1A_2$, $A_3A_2A_1$

6 permutations = 3!

Theorem: The number of ways to permute n items is $n!$

pf Use FTC w/ $k = n$ subtasks

task #1 task | # ways

task #	task	# ways
1	choose item 1	n
2	" 2	$n-1$
3	" 3	$n-2$
$\vdots$	$\vdots$	$\vdots$
n	" n	1

product is $n!$

Theorem: Ordered Sampling w/o Replacement

If I have n items and I sample r of them ($r \le n$) w/o replacement but w/ ordering. The number of ways I can draw this sample is

$$\boxed{(n)_r = \frac{n!}{(n-r)!}}$$

Pf. Use FTC.

task #	task	# ways
1	choose item 1	n
2	" 2	$n-1$
3	" 3	$n-2$
$\vdots$	$\vdots$	$\vdots$
r	" r	$n-r+1$

$r \quad | \quad r \quad | \quad n-r+1$

$\to$ total # ways is

$$n(n-1)(n-2)\cdots(n-r+1)$$

$$\frac{n!}{(n-r)!} = \frac{\boxed{n(n-1)(n-2)\cdots(n-r+1)}\cancel{(n-r)}\cancel{\cdots 3\cdot 2\cdot 1}}{\cancel{(n-r)(n-r-1)(n-r-2)\cdots 3\cdot 2\cdot 1}}$$

Ex. I form a committee of $\boxed{10}$ students ($n=10$) where the committee has $\boxed{3}$ members ($r=3$): Pres, VP, treasurer.

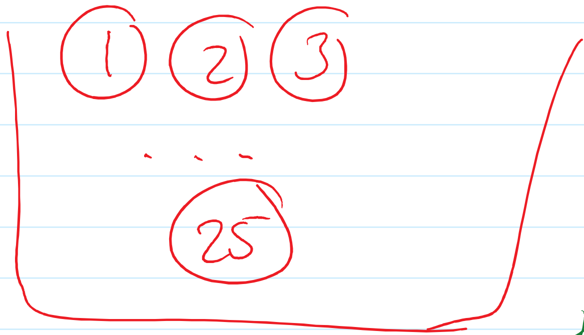
Q: How many ways can I form this committee?

By prev. theorem I can do this in

$$\frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10\cdot 9\cdot 8\cdot \cancel{7}\cdot\cancel{6}\cdots\cancel{1}}{\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdots\cancel{1}}$$

$$= 10\cdot 9\cdot 8 = 720$$

Ex. Lotto. I have 25 balls in a basket



equally likely

Draw 4 w/o replacement. If I can guess the 4 balls in correct order I win.

My choice: 1 3 22 7

Q: prob I win?

$E = \{1\ 3\ 22\ 7\}$; S = all possible draws.

then $P(E) = |E| / |S|$

$|E| = 1$

and $|S| = \dfrac{25!}{(25-4)!}$

$n = 25$
$r = 4$

$$= \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21 \cdot 20 \cdots 1}}{\cancel{21 \cdot 20 \cdot 19 \cdots 1}}$$

$$= 25 \cdot 24 \cdot 23 \cdot 22$$

$$P(E) = 1 / 25 \cdot 24 \cdot 23 \cdot 22$$

Theorem: Sampling w/ replacement w/ ordering

The number of ways to sample r items from n w/ replacement and ordering is

$$n^r$$

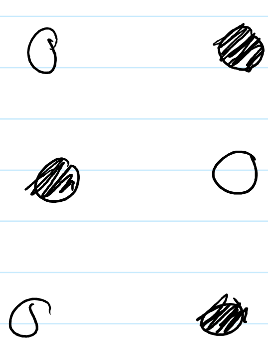
---

pf. Use FTC w/ $k = r$

task #	task	# ways
1	chose item 1	n
2	" 2	n
3	" 3	n
⋮	⋮	⋮
r	r	n

product is $n \cdot n \cdot n \cdots$ $\boxed{= n^r}$

Ex. Braille alpabet. How many different braille configurations?



$k = 2$

○ ●

$\boxed{n=2}$

Idea: sample bumps / not-bump from basket 6 times

$\boxed{r=6}$

So formula says can do in 2^6 ways.