Lecture 12 - Variance and Moments

Thursday, October 14, 2021 2:01 PM

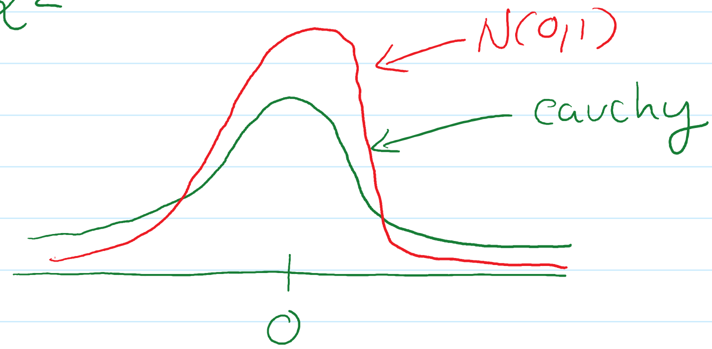
$$\mathbb{E}[X] = \begin{cases} \sum_x x f(x) & \text{(discrete)} \\ \int_{\mathbb{R}} x f(x)\,dx & \text{(cts)} \end{cases}$$

Q: Does expectation always exist? No.

Ex. Cauchy Distribution

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$

$f(x) \propto e^{-x^2}$



$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\pi}\frac{1}{1+x^2}\,dx$$

looks like

$$\int_{-\infty}^{\infty} \frac{x}{x^2}\,dx = \int_{-\infty}^{\infty} \frac{1}{x}\,dx \quad \text{doesn't converge}$$

analogy

$$\sum_i \frac{1}{i^2} < \infty \quad \text{but} \quad \sum_i 1/i = \infty$$

$$\int^{\infty} \frac{1}{x^2}\,dx < \infty \quad \text{but} \quad \int^{\infty} 1/x\,dx = \infty$$

$$\int_0 \frac{1}{x^2}dx < \infty \quad \text{but} \quad \int_0 1/x \, dx = \infty$$

Theorem: Properties of Expectation

① Expectation is linear:

$$E[aX+b] = aE[X]+b$$

pf. (cts)

$$E[aX+b] = \int (ax+b)f(x)\,dx = \int [axf(x) + bf(x)]\,dx$$

$$= \int axf(x)\,dx + \int bf(x)\,dx$$

$$= a\underbrace{\int xf(x)\,dx}_{E[X]} + b\underbrace{\int f(x)\,dx}_{1}$$

$$= aE[X] + b$$

② If $X \geq 0$ then $E[X] \geq 0$.

support $\subseteq [0,\infty)$

pf. (cts)

$$E[X] = \int_0^\infty x f(x)\,dx \geq 0$$

$x \geq 0$ $\quad$ $f(x) \geq 0$

(3) If g_1 and g_2 are functions

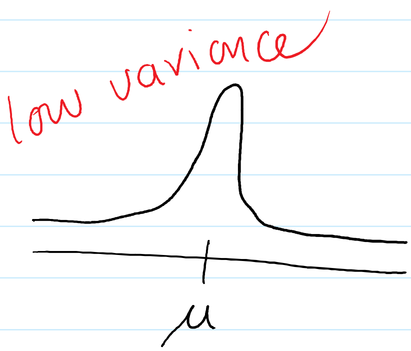
(i) $\mathbb{E}[g_1(X) + g_2(X)] = \mathbb{E}[g_1(X)] + \mathbb{E}[g_2(X)]$

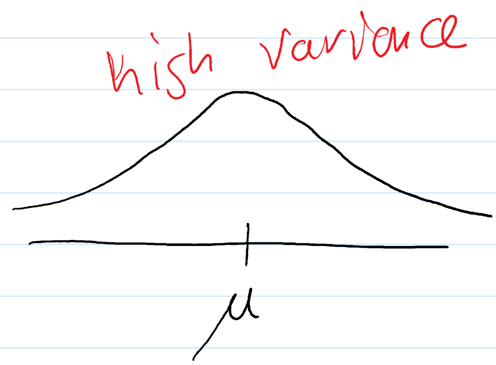
(ii) If $g_1(x) \le g_2(x)$ then

$$\mathbb{E}[g_1(X)] \le \mathbb{E}[g_2(X)]$$

(4) If $a \le X \le b$ then $a \le \mathbb{E}[X] \le b$.

<u>Defn : Variance</u> $\mu = \mathbb{E}[X] = \text{mean}$

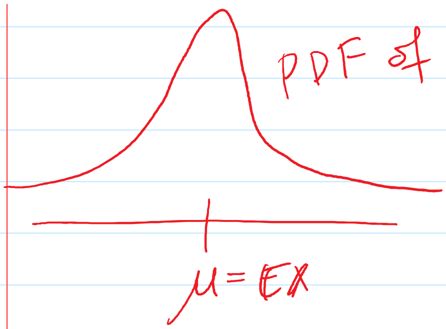


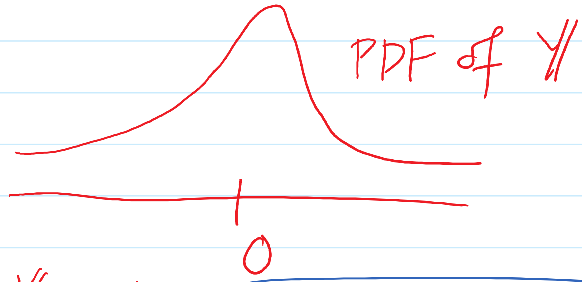


variance $\approx$ how spread are values around the mean.

$$Var(X) = \mathbb{E}[(X - \mu)^2]$$

$$= \mathbb{E}[(X - \mathbb{E}[X])^2]$$





$Y = X - \mu$

then $\mathrm{Var}(X) = \mathbb{E}[Y^2]$

$$\mathbb{E}[Y] = \mathbb{E}[X - \mu] = \mathbb{E}[X] - \mu = \mu - \mu = 0$$

Ex. $X \sim \mathrm{Exp}(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

$$\mu = \mathbb{E}[X] = 1/\lambda \quad \text{and} \quad \mathbb{E}[X^2] = 2/\lambda^2$$

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] = \cancel{\int_{\mathbb{R}} (x-\mu)^2 f(x)\,dx}$$

ok, ... but takes a lot of effort

$$= \mathbb{E}[X^2 - 2X\mu + \mu^2]$$

$$= \mathbb{E}[X^2] - 2\mu\,\mathbb{E}[X] + \mu^2$$

$$= 2/\lambda^2 - 2\left(\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}\right) + \left(1/\lambda\right)^2$$

$$= 2/\lambda^2 - 2/\lambda^2 + 1/\lambda^2$$

$$\boxed{Var(X) = 1/\lambda^2}$$

Theorem: Short-Cut Formula for Variance

$$Var(X) = E[X^2] - E[X]^2$$

expected sq − sq of expectation

Pf.

$$Var(X) = E[(X-\mu)^2]$$
$$= E[X^2 - 2\mu X + \mu^2]$$
$$= E[X^2] - 2\mu E[X] + \mu^2$$
$$= E[X^2] - 2E[X]E[X] + E[X]^2$$
$$= E[X^2] - E[X]^2$$

Ex. $X \sim Exp(\lambda)$

$$Var(X) = E[X^2] - E[X]^2$$
$$= 2/\lambda^2 - (1/\lambda)^2 = 1/\lambda^2$$

Theorem:

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X).$$

① multiply X by a $\rightsquigarrow$ Var is mult. by a^2

② ignore additive shifts

Pf. $\mathrm{Var}(aX+b) = \mathbb{E}[(aX+b)^2] - (\mathbb{E}[aX+b])^2$

$$= \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\mathbb{E}(X)+b)^2$$

$$= a^2E[X^2] + \cancel{2abE[X]} + \cancel{b^2} - (a^2E[X]^2 + \cancel{2abE[X]} + \cancel{b^2})$$

$$= a^2(\mathbb{E}[X^2] - E[X]^2)$$

$$= a^2\,\mathrm{Var}(X).$$

Ex. $X \sim \mathrm{Bin}(n,p)$ $\quad f(x) = \binom{n}{x} p^x (1-p)^{n-x}$

recall: $E[X] = np$

$$E[X^2] = \sum_{x=\cancel{0}1}^{n} x^2 f(x) = \sum_{x=1}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

trick 1: $x\binom{n}{x} = n\binom{n-1}{x-1}$

$$= \sum^{n} x\, n \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

trick 2: $y = x-1$

$$= \sum_{x=1}^{n} x\, n \binom{n-1}{x-1} p^{x} (1-p)^{n-x}$$

$\underline{\text{trick 2}}$: $y = x-1$

$x = y+1$

$$= \sum_{y=1}^{n-1} (y+1)\, n \binom{n-1}{y} p^{y+1} (1-p)^{(n-1)-y}$$

$$= np \sum_{y=1}^{n-1} (y+1) \binom{n-1}{y} p^{y} (1-p)^{(n-1)-y}$$

$$= np \left[\sum_{y=1}^{n-1} y \underbrace{\binom{n-1}{y} p^{y} (1-p)^{(n-1)-y}}_{f(y)} + \underbrace{\sum_{y=1}^{n-1} \binom{n-1}{y} p^{y} (1-p)^{(n-1)-y}}_{\text{Sum of PMF of } Bin(n-1,p) \,=\, 1} \right]$$

$\mathbb{E}[Bin(n-1, p)]$

$=$

$(n-1)p$

Sum of PMF of $Bin(n-1, p)$

$= 1$

$$= np\left[(n-1)p + 1\right] = np(np - p + 1) = \mathbb{E}[X^2]$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= np(np - p + 1) - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

<u>Standard Deviation</u>: $sd(X) = \sqrt{Var(X)}$

$$sd(X) = \sqrt{np(1-p)}$$

<u>Defn: Moments of RVs</u>

If r is a pos. integer we define the r^{th} moment of X as

$$\mu_r = \mathbb{E}[X^r].$$

<u>Ex</u>

$$\mu_1 = \mu = \mathbb{E}[X]$$

$$\mu_2 = \mathbb{E}[X^2]$$

$$\mu_3 = \mathbb{E}[X^3] \quad \dots$$

<u>Defn: Moment Generating Function</u> (MGF)

If X is a RV the MGF of X is a function

$$M : \mathbb{R} \longrightarrow \mathbb{R}$$

defined for $t \in \mathbb{R}$ as

$$M(t) = \mathbb{E}[e^{tX}].$$

For discrete:

$$M(t) = \sum_x e^{tx} f(x)$$

continuous:

$$M(t) = \int e^{tx} f(x)\, dx$$

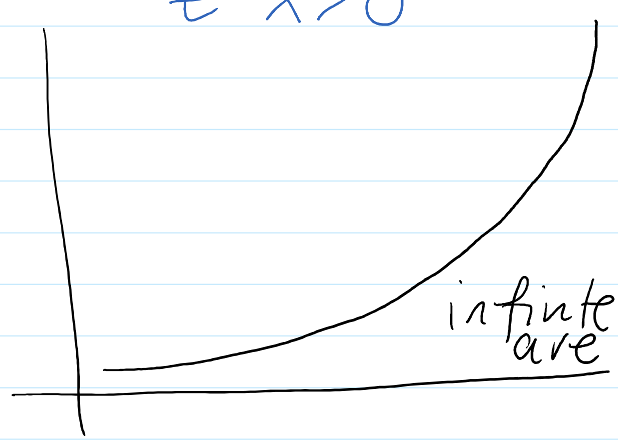
<u>Ex.</u> $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

$$M(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$\boxed{t - \lambda \leq 0}$ $\qquad$ $t - \lambda > 0$



If $t - \lambda \leq 0 \iff \boxed{t \leq \lambda}$

If $t-\lambda \le 0 \iff \boxed{t \le \lambda}$

then

$$M(t) = \lambda \int_0^\infty e^{(t-\lambda)x}\,dx = \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^\infty$$

$$= \frac{\lambda(0-1)}{t-\lambda} = \boxed{\frac{\lambda}{\lambda-t} \quad \text{for } t \le \lambda}$$

$M(t) = \dfrac{\lambda}{\lambda - t}$ for $t \le \lambda$

Recall: $E[X] = 1/\lambda$; $E[X^2] = 2/\lambda^2$

① $\dfrac{dM}{dt} = \dfrac{\lambda}{(\lambda-t)^2} \Rightarrow \left.\dfrac{dM}{dt}\right|_{t=0} = \dfrac{\lambda}{\lambda^2} = 1/\lambda$

② $\dfrac{d^2M}{dt^2} = \dfrac{2\lambda}{(\lambda-t)^3} \Rightarrow \left.\dfrac{d^2M}{dt}\right|_{t=0} = \dfrac{2\lambda}{\lambda^3} = 2/\lambda^2$