Lecture 15 - Even More Common Distributions

Thursday, October 28, 2021 2:00 PM

Gamma function:

1. If a is an integer then

$$\Gamma(a) = (a-1)!$$
$$\Gamma(a+1) = a!$$

Ex. $\Gamma(1) = 0! = 1$

$\Gamma(2) = 1! = 1$

$\Gamma(3) = 2! = 2$

$\vdots$

2. For an integer x, $x! = x(x-1)!$

For an integer a

$$\Gamma(a) = (a-1)! = \overbrace{(a-1)(a-2)!}$$
$$= (a-1)\Gamma(a-1)$$

$$\boxed{\text{So } \Gamma(a) = (a-1)\Gamma(a-1) \quad \forall a > 0}$$

or

$$\Gamma(a+1) = a\Gamma(a)$$

Important facts for Γ

1. $\Gamma(a+1) = a!$ for integers a

① $\Gamma(a+1) = a!$ for integers a

② $\Gamma(a+1) = a\Gamma(a) \quad \forall a>0$

Gamma Distribution generalize exponential

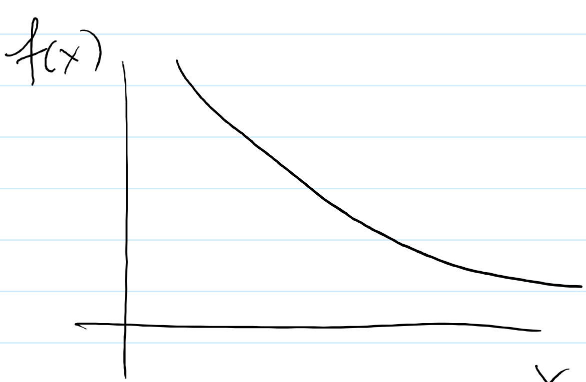
$$X \sim \text{Gamma}(\underbrace{a}_{\text{shape}}, \underbrace{\lambda}_{\text{rate}})$$

PDF:

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)} \quad \text{for } x>0$$

notice: if $a=1$ then $f(x) = \lambda e^{-\lambda x}$ for $x>0$

so $X \sim \text{Exp}(\lambda)$.



change a

Expectation

Expectation

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^\infty x\,\frac{\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)}\,dx$$

$$\frac{\Gamma(a+1)}{\Gamma(a)\lambda}\underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x}\,\lambda^{a-1}\,x^{a+1-1}\,\lambda\,dx}{\Gamma(a+1)}}_{\text{integrates to 1}}$$

$$c\underbrace{\int \text{Gamma}(a+1,\lambda)}_{1}$$

$$\frac{\lambda e^{-\lambda x}(\lambda x)^{(a+1)-1}}{\Gamma(a+1)}$$

$$= \frac{\Gamma(a+1)}{\Gamma(a)}\,\frac{1}{\lambda}$$

$$= \frac{a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}}\,\frac{1}{\lambda} = \boxed{\frac{a}{\lambda} = E[X]}$$

when $a=1$
then
$E[X] = 1/\lambda$ ✓

moments:

$$E[X^r] = \int_0^\infty \frac{x^r\,\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)}\,dx$$

$$\frac{\Gamma(a+r)}{\Gamma(a)\lambda^r}\underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x}\,\lambda^{a-1}\,x^{a+r-1}\,\lambda^r\,dx}{\Gamma(a+r)}}_{\text{integrates to 1}}$$

$$c\underbrace{\int \text{Gamma}(a+r,\lambda)}_{1}$$

$$\lambda e^{-\lambda x}(\lambda x)^{a+r-1}$$

$$\boxed{= \frac{\Gamma(a+r)}{\Gamma(a)} \frac{1}{\lambda^r} = \mathbb{E}[X^r]} \qquad \frac{\lambda e^{-\lambda x}(\lambda x)^{a+r-1}}{\Gamma(a+r)}$$

$r=2$

$$\mathbb{E}[X^2] = \frac{\Gamma(a+2)}{\Gamma(a)} \frac{1}{\lambda^2} = \frac{(a+1)\Gamma(a+1)}{\Gamma(a)} \frac{1}{\lambda^2}$$

$$= \frac{(a+1)a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}} \frac{1}{\lambda^2}$$

$$= \frac{(a+1)a}{\lambda^2}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{(a+1)a}{\lambda^2} - \left(\frac{a}{\lambda}\right)^2$$

$$\boxed{= a/\lambda^2}$$

if $a=1 \Rightarrow \mathrm{Va}(X) = 1/\lambda^2$

$= \mathrm{Var}$ of $\mathrm{Exp}(\lambda)$

Geometric Distribution (lec 9)

Canonical experiment:

If I flip coins (independently), each has prob of $p \in [0,1]$ of a H.

Do this until I get my first H.

$X = \#$ of flips to get a H.

outcome	X
H	1
TH	2
TTH	3
⋮	⋮

then $X \sim \text{Geometric}(p)$

PMF: $f(x) = (1-p)^{x-1} p$ for $x = 1, 2, 3, \ldots$

CDF: $F(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor} & x \geq 1 \\ 0 & x < 1 \end{cases}$

recall: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $|r| < 1$

geometric series

Expected value:

$$E[X] = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$$\downarrow \quad x(1-p)^{x-1} = -\frac{d}{dp}(1-p)^{x}$$

$$= -p \sum_{x=1}^{\infty} \frac{d}{dp}(1-p)^{x}$$

$$= -p \frac{d}{dp} \sum_{x=1}^{\infty} (1-p)^{x}$$

$$= -p \frac{d}{dp}\left[(1-p)\sum_{x=1}^{\infty}(1-p)^{x-1}\right]$$

$$= -p \frac{d}{dp}\left[(1-p)\underbrace{\sum_{x=0}^{\infty}(1-p)^{x}}_{\text{geometric series w/ } r=1-p}\right]$$

$$= -p \frac{d}{dp}\left[(1-p)\frac{1}{1-(1-p)}\right]$$

$$= -p \frac{d}{dp}\left[\frac{1-p}{p}\right]$$

$$= -p\left(-1/p^{2}\right) = \boxed{\frac{1}{p} = \mathbb{E}[X]}$$

MGF:

$$M(t) = \mathbb{E}[e^{tX}] = \sum^{\infty} e^{tx}(1-p)^{x-1} p$$

$$M(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p$$

$$= pe^t \sum_{x=1}^{\infty} e^{t(x-1)}(1-p)^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} (\underbrace{e^t(1-p)}_{r})^{x-1}$$

$$= pe^t \sum_{x=0}^{\infty} (e^t(1-p))^x \qquad \frac{1}{1-r}$$

$$M(t) \boxed{= pe^t/(1-e^t(1-p))}$$

$$\left.\frac{d^2M}{dt}\right|_{t=0} = \frac{2-p}{p^2} = E[X^2]$$

So $$\mathrm{Var}(X) = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \boxed{\frac{1-p}{p^2}}$$

Beta Distribution – a continuous RV

Beta Function: $a, b \in \mathbb{R}^+$

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

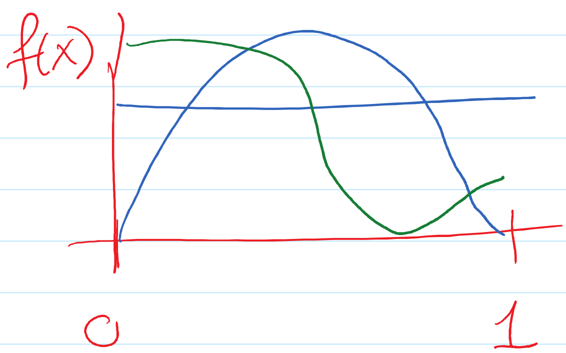
For integers a, b

$$\frac{1}{B(a,b)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \approx \frac{(a+b)!}{a!\,b!} = \binom{a+b}{a}$$

Beta distribution:

$$X \sim \text{Beta}(a, b)$$

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } 0 < x < 1$$



$$\mathbb{E}[X] = \int_0^1 x \, \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \, dx$$

$$= \frac{B(a+1,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)} \, dx}_{\text{integrate to } 1}$$

PDF trick

$$c \int \text{Beta}(a+1, b)$$

$$\frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}$$

$$= \frac{B(a+1,b)}{B(a,b)}$$

$$= \frac{\dfrac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+1+b)}}{\dfrac{\Gamma(a)\cancel{\Gamma(b)}}{\Gamma(a+b)}} = \frac{\Gamma(a+1)\Gamma(a+b)}{\Gamma(a)\,\Gamma(a+b+1)}$$

$$= \frac{a\cancel{\Gamma(a)}\,\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}(a+b)\cancel{\Gamma(a+b)}}$$

$$\boxed{E[X] = \frac{a}{a+b}}$$

Moments:

$$E[X^r] = \int_0^1 \frac{x^r x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$\frac{B(a+r,b)}{B(a,b)} \int_0^1 \frac{x^{(a+r)-1}(1-x)^{b-1}}{\underbrace{B(a+r,b)}_{\text{integrates to 1}}}\,dx$$

make look like PDF of Beta$(a+r, b)$

$$\Rightarrow \boxed{E[X^r] = \frac{B(a+r,b)}{B(a,b)}}$$

$$\Rightarrow \boxed{\mathbb{E}[X'] = \frac{\quad}{B(a,b)}}$$

$$E[X^2] = \frac{B(a+2,\,b)}{B(a,b)} = \frac{\dfrac{\Gamma(a+2)\cancel{\Gamma(b)}}{\Gamma(a+b+2)}}{\dfrac{\Gamma(a)\,\cancel{\Gamma(b)}}{\Gamma(a+b)}}$$

$$= \frac{(a+1)a\cancel{\Gamma(a)}\,\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}(a+b+1)(a+b)\cancel{\Gamma(a+b)}}$$

$$= \frac{(a+1)a}{(a+b+1)(a+b)}$$

$$Var(X) = EX^2 - (EX)^2$$

$$= \frac{(a+1)\;a}{(a+b+1)(a+b)} - \left(\frac{a}{a+b}\right)^2$$

$$= \;\ldots\; \text{algebra}$$

$$\boxed{= \frac{ab}{(a+b+1)(a+b)^2}}$$