Lecture 3 - Basic Theorems

Thursday, September 9, 2021 2:04 PM

③ $(E_i)_{i=1}^{\infty}$ that partition E then

$$\boxed{P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)}$$

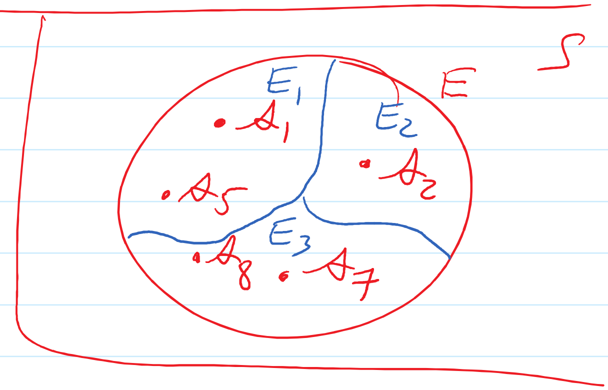
$P(F) = \sum_{j : s_j \in F} p_j$

$$P(E) = P\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$= \sum_{j : s_j \in \bigcup_{i=1}^{\infty} E_i} p_j$$

$$= \sum_{i=1}^{\infty} \underbrace{\sum_{j : s_j \in E_i} p_j}_{P(E_i)}$$

$$= \sum_{i=1}^{\infty} P(E_i)$$



$$P(E) = p_1 + p_2 + p_5 + p_7 + p_8$$

$$= (p_1 + p_5) + (p_2) + (p_7 + p_8)$$

Theorem: $\boxed{P(\emptyset) = 0.}$

Pf: $S = \underbrace{S \cup \emptyset \cup \emptyset \cup \cdots}_{\text{partition of } S}$

$$P(S) = P(S \cup \emptyset \cup \emptyset \cup \cdots)$$

$$= \mathbb{P}(S) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \cdots$$

So $\sum_{i=1}^{\infty} \mathbb{P}(\emptyset) = 0$

So it must be that $\mathbb{P}(\emptyset) = 0$.

<u>Third Axiom:</u> $(E_i)_{i=1}^{\infty}$ partition E then

(Countable additivity) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$

<u>Finite Additivity:</u> $(E_i)_{i=1}^{N}$ partition E then

$$\mathbb{P}\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} \mathbb{P}(E_i).$$

<u>Pf</u> $\mathbb{P}\left(\bigcup_{i=1}^{N} E_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{N} E_i \cup \emptyset \cup \emptyset \cup \emptyset \cdots\right)$

$$= \sum_{i=1}^{N} \mathbb{P}(E_i) + \underbrace{\cancel{\mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \cdots}}_{0}$$

$$= \sum_{i=1}^{N} \mathbb{P}(E_i)$$

<u>Corrollary:</u> If $AB = \emptyset$

then $P(A \cup B) = P(A) + P(B)$.

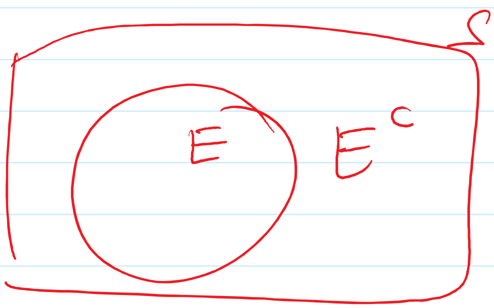
Ex. If E = "its raining"

and $P(E) = 1/3$

$P(\text{Its not raining}) = P(E^c) = 2/3 = 1 - 1/3$

$$\boxed{\text{Theorem: } P(E^c) = 1 - P(E)}$$

Pf. $S = E \cup E^c$ partition



So $P(S) = P(E \cup E^c)$

$\overset{\shortparallel}{1} = P(E) + P(E^c)$

rearrange

$P(E^c) = 1 - P(E)$

$$\boxed{\text{Theorem: } 0 \le P(E) \le 1}$$

By Axiom 1, $P(E) \geq 0$

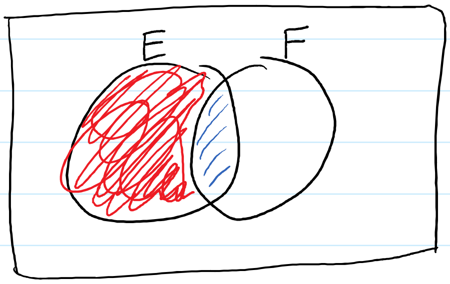
Note that (Axiom 1) says $P(E^c) \geq 0$

but $0 \le P(E^c) = 1 - P(E)$

rearrange $P(E) \le 1$.

Theorem: If $E, F \subset S$ then

$$P(E \setminus F) = P(E) - P(EF)$$



Pf. $E = \underbrace{EF \cup EF^c}_{\text{partition of } E}$

so $P(E) = P(EF) + P(EF^c)$

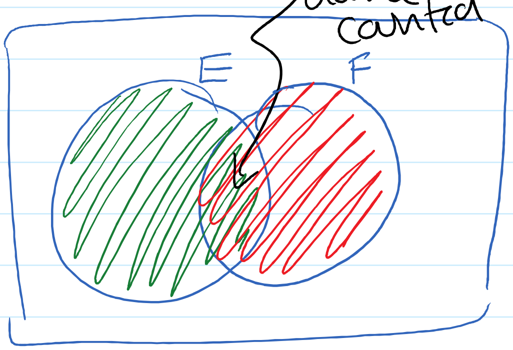
rearrange

$$P(E \setminus F) = P(EF^c) = P(E) - P(EF)$$

Theorem: $P(E \cup F) = P(E) + P(F) - P(EF)$

double counted

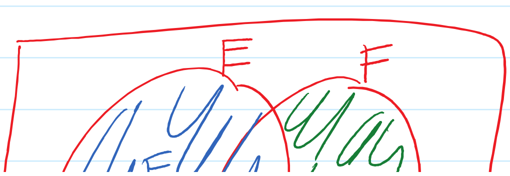
* don't assume disjoint



Notice that if $EF = \emptyset$

$P(E \cup F) = P(E) + P(F) - P(\emptyset)$ 0

Pf. $E \cup F = E \cup FE^c$ disjoint



so

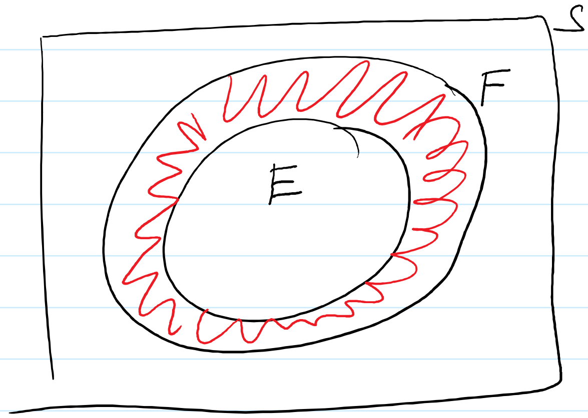
$P(E \cup F) = P(E \cup FE^c)$

$$P(E \cup F) = P(E \cup FE^c)$$
$$= P(E) + P(FE^c)$$
$$= P(E) + P(F) - P(FE)$$

by prev. theorem

Theorem: If $E \subset F$

$$P(E) \leq P(F)$$



Pf: Axiom 1 says
$P(FE^c) \geq 0$

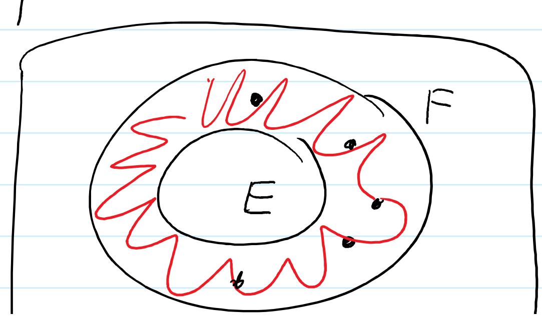
then

$P(F) - P(FE) \geq 0$

So $\underbrace{P(FE)}_{P(E)} \leq P(F)$

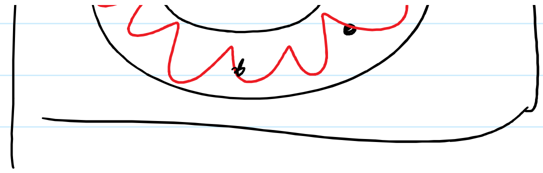
Consider $E \subset F$ but $E \neq F$

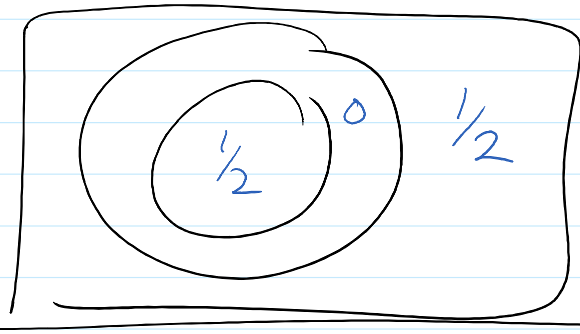
$$\cancel{P(E) < P(F)}\ ?$$



and we know that $P(FE^c) > 0$

Could be that $P(FE^c)=0$





Said that $P(E \cup F) = P(E) + P(F) - P(EF)$

So $P(E \cup F) \leq P(E) + P(F)$.

Generally: (Boole's Ineq)

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i).$$

don't require disjoint.

Pf. Replace the E_i w/ B_i where

① $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} E_i$

② B_i disjoint.

claim is this is true

define: $B_1 = E_1$

Note: $B_i \subset E_i$

$B_1 = E_1$

$B_2 = E_2 \setminus E_1$

$B_3 = E_3 \setminus (E_2 \setminus E_1)$

$B_4 = E_4 \setminus (E_3 \setminus (E_2 \setminus E_1))$

$\vdots$

is this true

Note! $B_i \subset E_i$

& $\mathbb{P}(B_i) \leq \mathbb{P}(E_i)$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(B_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Axiom 3