Lecture 9 - CDFs and PMFs

Thursday, September 30, 2021 1:56 PM

Additional Office Hours: Friday 10-11 am
Monday 12-1 pm

Theorem: F is the CDF of some RV iff

① $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to -\infty} F(x) = 0$

② F is non-decreasing

③ F is right-continuous

Ex. Let

$$F(x) = \frac{1}{1+e^{-x}} \quad \forall x \in \mathbb{R}$$

Q: Is this the CDF of some RV?

Check 3 conditions:

① $\lim_{x\to\infty} F(x) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$

$\lim_{x\to -\infty} F(x) = \frac{1}{1+e^{-(-\infty)}} = \frac{1}{\infty} = 0$

$$\lim_{x \to -\infty} F(x) = \frac{1}{1+e^{-(-\infty)}} = \frac{1}{\infty} = 0$$

(2) Non-decreasing?

$$\frac{dF}{dx} = \frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$$

(increasing)

(3) Right-continuous?

Yes its a cts fn.

So F is the CDF of some RV.

Defn: Identically Distributed RVs

We say two RVs X and Y are equal in distribution if $A \subset \mathbb{R}$

$$\mathbb{P}(X \in A) = \mathbb{P}(Y \in A).$$

We denote this as

$$X \overset{d}{=} Y$$

This doesn't mean $X = Y$.

Ex. 3 coin flips.

$X = \#$ heads and $Y = \#$ tails.

$X(HTT) = 1$ and $Y(HTT) = 2$

but $X \overset{d}{=} Y$.

$$\mathbb{P}(X=0) = 1/8 = \mathbb{P}(Y=0)$$

$$\mathbb{P}(X=1) = 3/8 = \mathbb{P}(Y=1)$$

$$\vdots$$

Theorem:

$$X \overset{d}{=} Y \quad \text{iff} \quad \underbrace{F_X}_{\text{CDF of } X} = \underbrace{F_Y}_{\text{CDF of } Y}.$$

Ex. Toss coins (independently) until a H appears.

$$S = \{H, TH, TTH, TTTH, \dots\}$$

note $|S| = \infty$

Let p be prob of getting a H on any flip.

Let X = # flips until I get a H

$s \in S$	$X(s)$
H	1
TH	2
TTH	3
TTTH	4
⋮	⋮

Q: What is CDF of X? $F(x) = P(X \le x)$

we'll look at $P(X = x)$.

Let H_i = i^{th} toss is a H and $T_i = H_i^c$

then

$$W_i = \text{"}X = i\text{"} = \underbrace{T_1 T_2 T_3 \cdots T_{i-1} H_i}_{\text{independent}}$$

then

$$\boxed{\mathbb{P}(X = i)} = \mathbb{P}(W_i) = \mathbb{P}(T_1 \cdots T_{i-1} H_i)$$

$$= \mathbb{P}(T_1) \cdots \mathbb{P}(T_{i-1}) \mathbb{P}(H_i)$$

$$= (1-p) \cdots (1-p)\ p$$

$$\boxed{= (1-p)^{i-1} p}$$

$$\text{"}X \le x\text{"} = W_1 \cup W_2 \cup W_3 \cup \cdots \cup W_x$$

disjoint union

$$F(x) = P(X \le x) = \sum_{i=1}^{x} P(W_i) = \sum_{i=1}^{x} (1-p)^{i-1} p$$

$$= p \sum_{i=0}^{x-1} \underbrace{(1-p)}_{r}{}^{i}$$

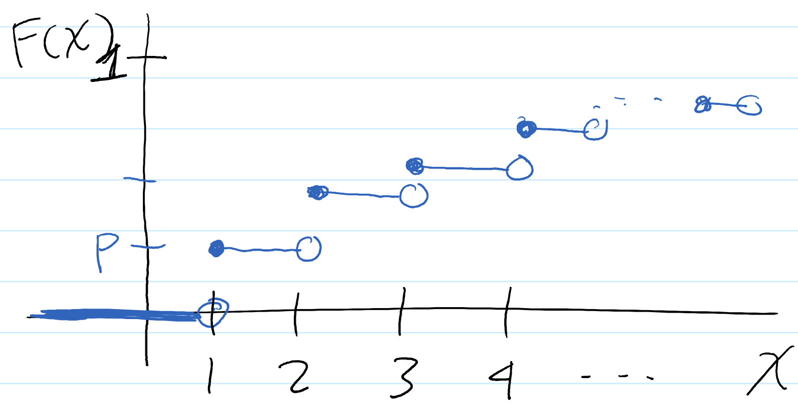
$$= \cancel{p}\,\frac{1-(1-p)^x}{1-\cancel{(1-p)}}$$

geometric sum

$$\boxed{\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}}$$

The RV w/ this CDF is called a Geometric RV.

$$\boxed{F(x) = 1-(1-p)^x}$$



$$F(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor} & x \geq 1 \\ 0, & x < 1 \end{cases}$$

$\lfloor x \rfloor$ = round down

$\lfloor 3.14 \rfloor = 3$

Defn: Discrete / Continuous

Defn: Discrete / Continuous

A discrete RV has a CDF that is a step fn.

A continuous RV has a continuous CDF.

Defn: Probability Mass Function

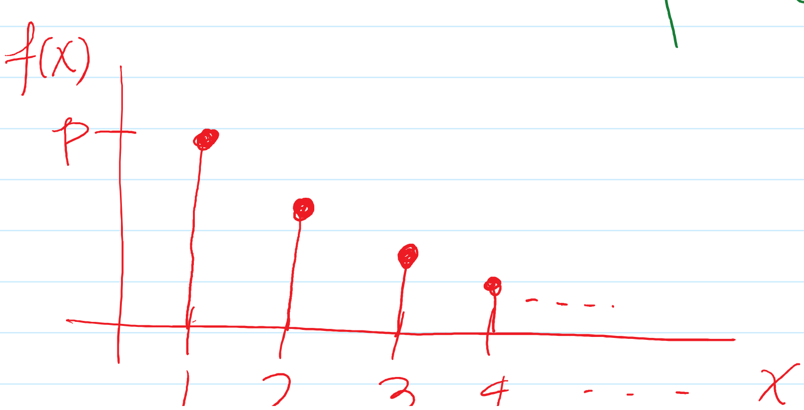
For a discrete RV X the probability Mass function (PMF) is a function $f : \mathbb{R} \to \mathbb{R}$ where

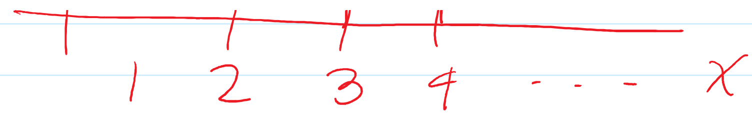
$$f(x) = \mathbb{P}(X = x) \quad \forall x \in \mathbb{R}.$$

Also called the distribution of X.

Ex. Prev. ex.

$$f(x) = \mathbb{P}(X = x) = \begin{cases} (1-p)^{x-1} p, & x = 1, 2, 3, \dots \\ 0 & , \text{else} \end{cases}$$





Theorem: For discrete RVs

$$F(x) = \sum_{i \le x} f(i)$$

Pf.

$$\text{"}X \le x\text{"} = \bigcup_{i \le x} \text{"}X = i\text{"}$$

disjoint union

So

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}\Big(\bigcup_{i \le x} \text{"}X = i\text{"}\Big)$$

$$= \sum_{i \le x} \underbrace{\mathbb{P}(X = i)}_{f(i)}$$

Ex. We say X has a discrete uniform distribution over $1, \dots, n$

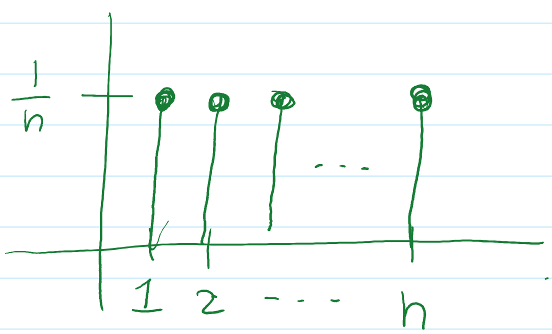
if

$$f(i) = \begin{cases} \frac{1}{n}, & \text{for } i = 1, \dots, n \\ 0, & \text{else} \end{cases}$$

notation:

$$X \sim U(\{1, \dots, n\})$$

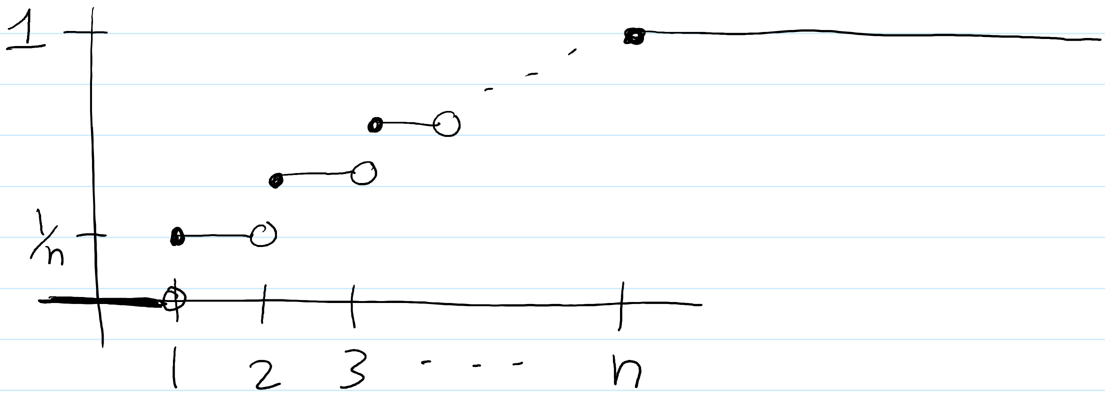
read: distributed as



Q: what is the CDF?

$$F(x) = \sum_{i \le x} f(i) = \sum_{i=1}^{x} \frac{1}{n} = \frac{x}{n}$$

$x \in \{1, \dots, n\}$



$$F(x) = \begin{cases} 0, & x < 1 \\ \lfloor x \rfloor / n & 1 \le x \le n \\ 1, & x \ge n \end{cases}$$

Saw: $F(x) = \sum_{i \le x} f(x)$

$\quad \mathbb{P}(X \le x)$

More generally: $A \subset \mathbb{R}$,

$$\mathbb{P}(X \in A) = \sum_{i \in A} f(i)$$

Ex. $X \sim U(\{1, \dots, 7\})$

$$\mathbb{P}(2 \le X \le 5) = \mathbb{P}(X \in \{2,3,4,5\})$$

$$= \sum_{x=2}^{5} \underbrace{f(x)}_{1/7} = 4/7$$

<u>Ex.</u> Roll a die 60 times (independently)

$X = \#$ of 6s I roll.

What is f?

$$f(0) = \mathbb{P}(X=0) = \underbrace{(5/6)(5/6)(5/6)\cdots(5/6)}_{60 \text{ times}}$$

$$= \left(\frac{5}{6}\right)^{60}$$

$$f(1) = \mathbb{P}(X=1) = \binom{60}{1}\left(\frac{1}{6}\right)\underbrace{\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{5}{6}\right)}_{59 \text{ times}}$$

$$= \binom{60}{1}\left(\frac{1}{6}\right)(5/6)^{59}$$

$$f(2) = \mathbb{P}(X=2) = \binom{60}{2}\left(\frac{1}{6}\right)(1/6)\underbrace{(5/6)\cdots(5/6)}_{58 \text{ times}}$$

$$= \binom{60}{2}(1/6)^2(5/6)^{58}$$

$$\boxed{f(x) = \underline{\binom{60}{x}}\left(\frac{1}{6}\right)^x \left(5/6\right)^{60-x}}$$

We call this a Binomial RV

If I do a sequence of n Yes/No experiments (independently) and each a prob of p of being "Yes"

above:
$n = 60$
$p = 1/6$

and X = # of Yes

We call X a Binomial RV

denoted $X \sim Bin(n, p)$,

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$