Lecture 16 - Transformations

Tuesday, November 2, 2021 9:28 AM

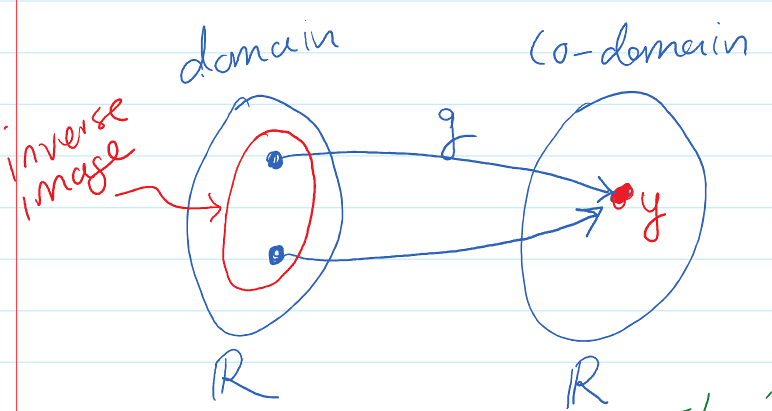
This lecture:

1. I know something about X
2. what can I say about $Y = g(X)$?

transformation of X

Discrete RVs (PMFs)

1. I know f_X — PMF of X
2. How do I get f_Y? — PMF of $Y = g(X)$



Recall:

g^{-1} = inverse image = a set in domain

$g^{-1}(\{y\})$ = all things in domain that map to y

$= g^{-1}(y)$ ← abuse notation

<u>Note:</u> If g is truly invertible then g^{-1} is the true inverse

$$f_Y(y) = \mathbb{P}(Y=y) = \mathbb{P}(g(X)=y)$$

$$= \mathbb{P}(X = g^{-1}(y))$$ if g is invertible

$$= f_X(g^{-1}(y))$$

If g isn't invertible

$$\boxed{\begin{aligned} f_Y(y) = \mathbb{P}(Y=y) &= \mathbb{P}(g(X)=y) \\ &= \mathbb{P}(X \in \underbrace{g^{-1}(y)}_{A}) \\ &= \sum_{x \in g^{-1}(y)} f_X(x) \end{aligned}}$$

even if g not invertible

$\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$

Theorem: If X is discrete and $Y = g(X)$ then

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x)$$

all x where $g(x) = y$

Ex. $X \sim \text{Bin}(n,p) \leftarrow$ # of H in n coin flips let $Y = n - X$

$\uparrow$ # of T ...

So $Y = g(X)$ where $y = g(x) = n - x$

turns out g is invertible

$x = g^{-1}(y) = n - y$

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x) = \sum_{\underline{x = n-y}} f_X(x)$$

$$= \underline{f_X(n-y)}$$

$$\underline{f_X}(x) = \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x = 0, 1, 2, \ldots, n$$

$$= \underbrace{\binom{n}{n-y}} p^{n-y} (1-p)^{\cancel{n-(n-y)} \to y} \text{ for } y = 0, \ldots, n$$

$q = 1 - p$

$$f_Y(y) = \underbrace{\binom{n}{y} q^y (1-q)^{n-y}} \text{ for } y = 0, \ldots, n$$

PMF of $Bin(n, q = 1-p)$

So $\boxed{Y \sim Bin(n, 1-p)}$

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What about continuous X? (CDFs)

Theorem: If X is continuous and

① if g is increasing and $Y = g(X)$

invertible

then

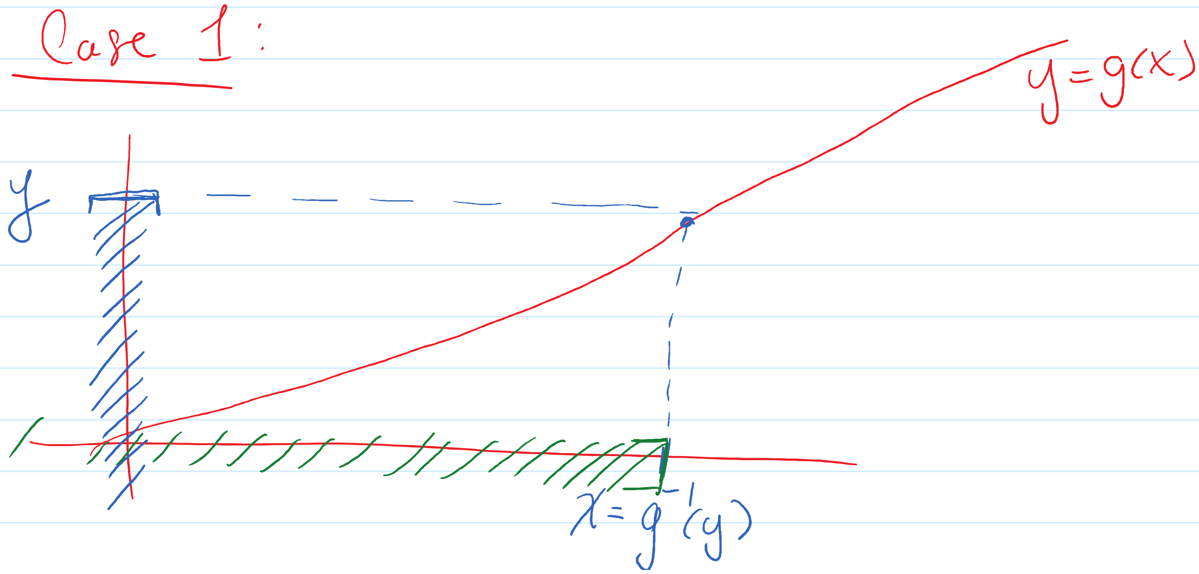
$$\boxed{F_Y(y) = F_X(g^{-1}(y))}$$

② if g is decreasing and $Y = g(X)$

then

$$\boxed{F_Y(y) = 1 - F_X(g^{-1}(y))}$$

Pf: Case 1:



$x = g^{-1}(y)$

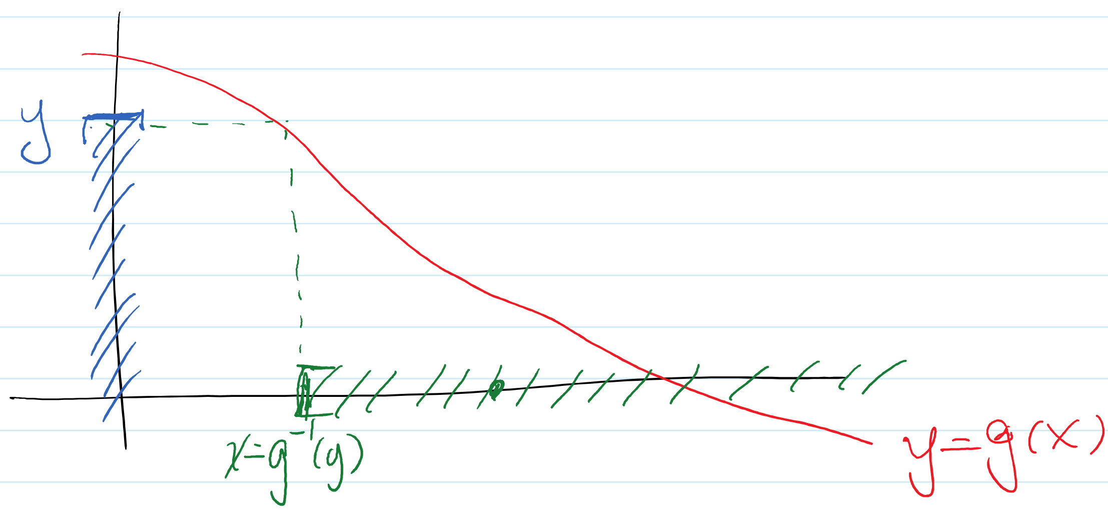
$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y)$$

g increasing so is g^{-1}

$$= \mathbb{P}(X \le g^{-1}(y))$$

$$= F_X(g^{-1}(y))$$

Case 2:



$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y)$$

g dec. then so is g^{-1}

$$= \mathbb{P}(X \ge g^{-1}(y))$$

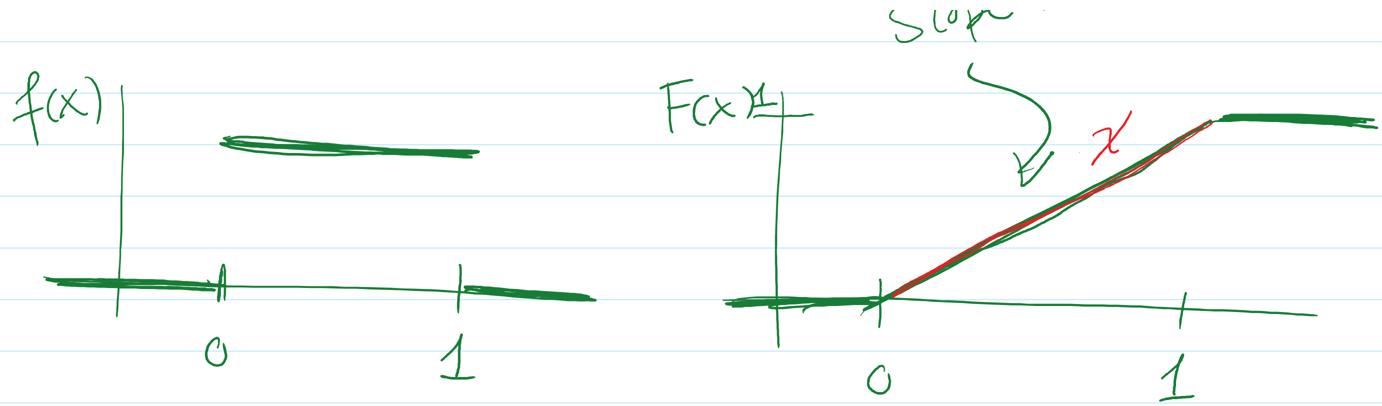
$$= 1 - \mathbb{P}(X \le g^{-1}(y))$$

$$= 1 - F_X(g^{-1}(y))$$

Ex. $X \sim U(0,1)$

Slope 1

$f(x)$ | $F(x)$

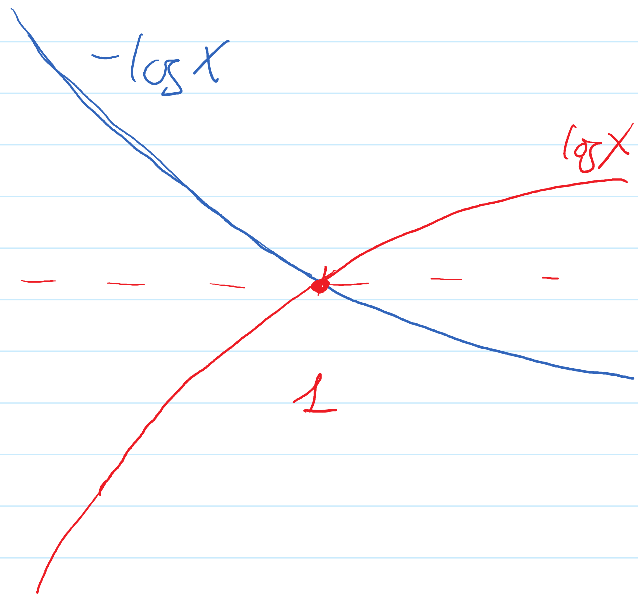


Let $Y = -\log X > 0$

What is F_Y?

$g(x) = -\log x$

$g^{-1}(y) = e^{-y}$



$F_Y(y) = 1 - F_X(g^{-1}(y))$

$y = -\log x$

$-y = \log x$

$e^{-y} = x$

$= 1 - F_X(e^{-y})$

if $y > 0$

$\Rightarrow -y < 0$

$\Rightarrow e^{-y} < 1 = e^{0}$

i.e. $0 < e^{-y} < 1$

btwn 0 and 1, $F_X(x) = x$

$= 1 - e^{-y} = F_Y(y)$

CDF of $\text{Exp}(\lambda = 1)$

CDF of $Exp(\lambda = 1)$

So $\boxed{Y \sim Exp(1)}$

What about PDFs?

Theorem: If X is continuous and $Y = g(X)$

and (1) g is invertible

(2) g^{-1} is differentiable

then

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$

pf. (1) g increasing then so is g^{-1}

and so $\frac{dg^{-1}}{dy} > 0$

prev. theorem said $F_Y(y) = F_X(g^{-1}(y))$

So

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

chain rule

$$= f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

(2) g decreasing then $\frac{dg^{-1}}{dy} < 0$

prev. theorem $F_Y(y) = 1 - F_X(g^{-1}(y))$

So

$$f_Y(y) = \frac{d}{dy} F_Y(y) = -f_X(g^{-1}(y)) \frac{dg^{-1}}{dy} \longleftarrow$$

$$= f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

$$|-5| = -(-5)$$

<u>Ex.</u> $X \sim \text{Gamma}(a, \lambda)$

$$f_X(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{a-1}}{\Gamma(a)} \quad \text{for } x > 0$$

let $Y = 1/X$ equiv. $X = 1/Y$

$$g(x) = 1/x \iff \boxed{g^{-1}(y) = 1/y}$$

$$\frac{dg^{-1}}{dy} = -1/y^2$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

$$= \frac{\lambda e^{-\lambda/y} (\lambda \frac{1}{y})^{a-1}}{\quad} \quad \frac{1}{\quad} \quad \text{for } y > 0$$

$$= \frac{\lambda e^{-\lambda \frac{1}{y}} \left(\lambda \frac{1}{y}\right)^{a-1}}{\Gamma(a)} \frac{1}{y^2} \quad \text{for } y>0$$

$$\boxed{Y \sim \text{Inverse Gamma}(a, \lambda)}$$

<u>what if g isn't invertible?</u>

We are ok as long as g is piecewise invertible.

<u>Theorem:</u>

Let X be continuous w/ support $\mathcal{X}$ and for $i=1,\dots,K$ let $A_1,\dots,A_K$ partition $\mathcal{X}$