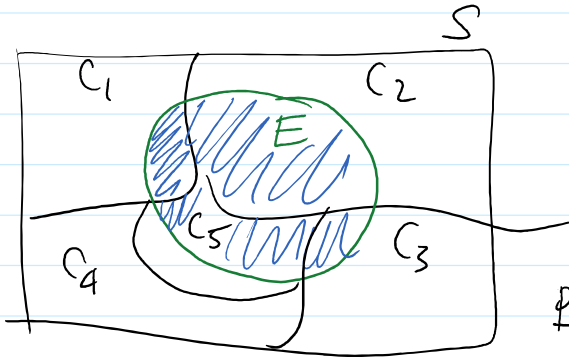
Lecture 4 - Ordered Counting

Tuesday, September 14, 2021 9:32 AM

Theorem: If (C_i) are a partition of S then for any event E



$$P(E) = \sum_i P(EC_i).$$

Pf.

① (EC_i) is a partition of E

Ⓐ $E = \bigcup_i EC_i$

Ⓑ $EC_i \cap EC_j = \emptyset \quad \forall i \neq j$

② By Axiom 3

$$P(E) = P\left(\bigcup_i EC_i\right) = \sum_i P(EC_i).$$

Equally Likely Outcomes

If I have a sample space

$$S = \{s_1, \ldots, s_n\} \text{ so that } |S| = n.$$

Then assume that

$$\underline{\frac{1}{n} = \mathbb{P}(\{s_i\})} = \mathbb{P}(\{s_j\}) \quad \forall\, i,j$$

Rational: $\{s_i\}$ $i = 1, \dots, n$ partition S

So

$$1 = \mathbb{P}(S) = \sum_{i=1}^{n} \underbrace{\mathbb{P}(\{s_i\})}_{\text{equal } \forall i}$$

must be $1/n$

More generally: if $E \subset S$ then

$$\mathbb{P}(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S}$$

$$= |E|/|S| = |E|/n.$$

Ex. $E = \{s_1, s_2\}$ then

$$\mathbb{P}(E) = \mathbb{P}(\{s_1\}) + \mathbb{P}(\{s_2\}) = 1/n + 1/n = \frac{2}{n}$$

Counting

Theorem: Fundamental Theorem of Counting (FTC)

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If I have a task consisting of k subtasks and the i^{th} subtask can be done in n_i then the total number of ways to achieve my task is

$$N = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$
$$= \prod_{i=1}^{k} n_i .$$

Ex. An experiment consist of 3 factors

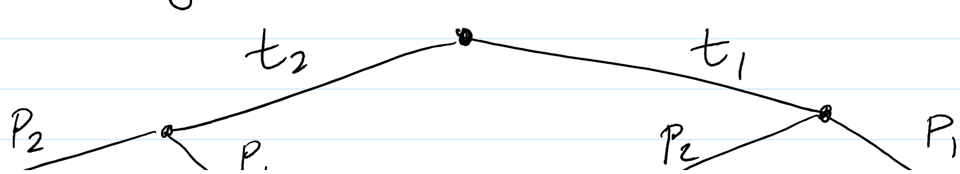
- 2 temp settings
- 2 pressure settings

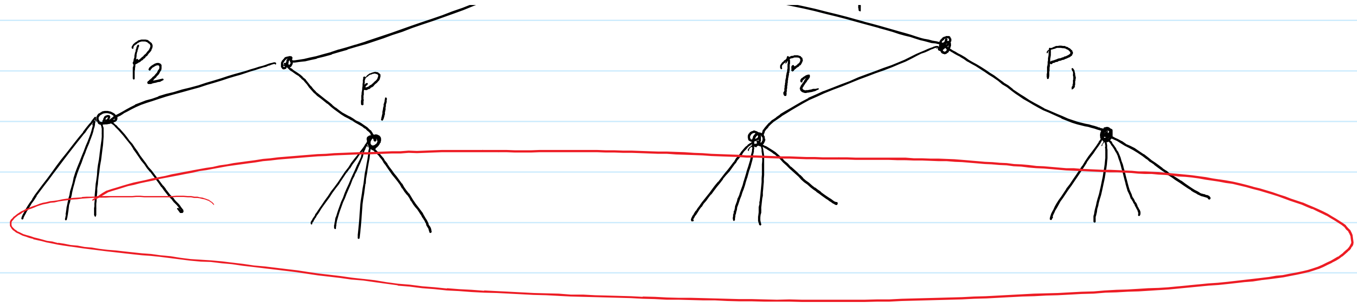
4 humidity settings.

How many ways can I do my experiment?

A: FTC says $2 \cdot 2 \cdot 4 = 16$ ways

Tree diagram





16 possibilities.

Ex. A man has 5 shirts, 2 pairs of pants and 2 pairs of shoes.

Q: How many outfit does he have?

$$5 \cdot 2 \cdot 2 = 20 \text{ outfits.}$$

Ex. I have a deck of 52 cards

13 denominations
A, 2, 3, ..., 10, J, Q, K

4 suits: C, D, H, S

If I shuffle the cards so that each ordering is equally likely, what is the prob I get them back in order after the shuffle?

$\hookrightarrow A - K,\ C,D,H,S$

$E =$ "in order" $\longleftarrow 1$

then $P(E) = |E| / |S|$.

$S =$ all possible orderings.

To count $|S|$ consider task of choosing the order for the cards consisting of $k = 52$ subtasks

		# ways
task 1	choose card 1	52
task 2	" 2	51
task 3	" 3	50
⋮	⋮	⋮
task 52	" 52	1

So by FTC then

$$|S| = 52 \cdot 51 \cdot 50 \cdot 49 \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

hence

$$\mathbb{P}(E) = \overline{52 \cdot 51 \cdots 3 \cdot 2 \cdot 1}$$

Defn: Factorial

For any non-negative integer n we define n factorial as

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$
$$= \prod_{i=1}^{n} i$$

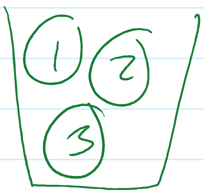
note: $0! = 1$

Ex, Prev. example,

$$\mathbb{P}(E) = 1/52!$$

Sampling w/ or w/o Ordering and Replacement.

Ordering: ordering matters?

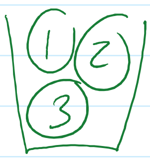


draw 1: 1 3 2

draw 2: 2 3 1

different?

Replacement: can I sample the same item twice?



w/ replacement: possible draw is 1 1 2

~~not~~ possible if I sample w/o replacement

Four scenarios:

	w/o replacement	w/ replacement
ordered (today)	1 $\frac{n!}{(n-r)!}$	2
Unordered (next)	4	3

next time

Defn: permutation

A permutation is an ordering of a collection of objects.

ex. objects $\underbrace{\{A_1, A_2, A_3\}}_{3 \text{ items}}$

permutations:

$$\left.\begin{matrix} A_1A_2A_3 & A_1A_3A_2 & A_2A_1A_3 \\ A_2A_3A_1 & A_3A_1A_2 & A_3A_2A_1 \end{matrix}\right) \begin{matrix} 6 \text{ perms} \\ = \\ 3! \end{matrix}$$

Theorem: Permutation Counting

The number of ways to permute n items is $n!$

Pf. Use FTC. $k = n$ items to choose.

task #	task	# ways
1	choose item 1	n
2	" 2	$n-1$
3	" 3	$n-2$
⋮	⋮	⋮
n	" n	1

then FTC says I can do this in

$$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n! \text{ ways.}$$

Theorem: Sampling W/o Replacement and w/ Ordering

If I have <u>n items</u> and I draw a sample of <u>size r</u> $(r \le n)$ and I do this sampling w/o repl. and w/ ordering, I can draw this sample in

$$(n)_r = \boxed{\frac{n!}{(n-r)!}} \text{ ways}$$

<u>pf.</u> $k = r$ subtasks using FTC

task #	task	# ways
1	draw 1	n
2	draw 2	$n-1$
3	draw 3	$n-2$
$\vdots$	$\vdots$	$\vdots$
r	draw r	$n-r+1$

So by FTC I can do this in

$$\underline{n(n-1)(n-2)\cdots(n-r+1)} \text{ ways.}$$

$$\frac{n!}{(n-r)!} = \frac{n(n-1)\cdots(n-r+2)(n-r+1)\cancel{\cdots 3\cdot 2\cdot 1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cancel{(n-r-2)}\cancel{\cdots 3\cdot 2\cdot 1}}$$

Ex. I have $\boxed{\text{10 students}}$ $n=10$ to put on a committee consisting of:

$\boxed{\text{President, VP, treasurer}}$ $r=3$

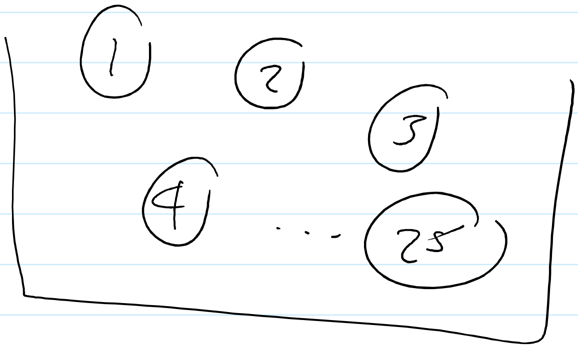
How many ways can I form this committee?

$$\boxed{\frac{10!}{(10-3)!}} = \frac{10!}{7!} = \frac{10\cdot 9\cdot 8\cdot \cancel{7\cdot 6\cdots 1}}{\cancel{7\cdot 6\cdot 5\cdots 1}}$$

$= 10 \cdot 9 \cdot 8 = 720$

Ex. Lotto

I have a box w/ balls numbered 1 to 25



all orderings equally likely

Lotto: draw 4 balls w/o replacement

If I guess the 4 balls in correct order I win.

my choice: 1 3 22 7

Q: What is the prob. I win?

$\{(1,3,22,7)\}$

E = "I win" then

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{1}{|S|}$$

$S = \{\text{all possible draws}\}$ $\qquad \begin{bmatrix} n = 25 \\ r = 4 \end{bmatrix}$

and $|S| = \dfrac{25!}{(25-4!)} = \dfrac{25!}{21!}$

$$= \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21 \cdots 1}}{\cancel{21 \cdot 20 \cdots 1}}$$

$$= 25 \cdot 24 \cdot 23 \cdot 22$$

So $\quad P(E) = \dfrac{1}{25 \cdot 24 \cdot 23 \cdot 22}$

Theorem: Sampling w/ Replacement and w/ Ordering

The number of ways to sample r from n:

1. w/ replacement
2. w/ ordering

is

$$n^r.$$