Lecture 24 - CLT

Tuesday, December 7, 2021 1:57 PM

Indicator fuctions

$$\mathbb{1}(\text{statement}) = \begin{cases} 0 & \text{statement false} \\ 1 & \text{statement true} \end{cases}$$

Ex. $$\mathbb{1}(x \in A) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

PDF of a RV

$X \sim \text{Exp}(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$f(x) = \lambda e^{-\lambda x}$ for $x > 0$

$$\boxed{f(x) = \lambda e^{-\lambda x} \mathbb{1}(x > 0)}$$

Independence theorem:

Independence theorem:

$$X \perp Y \iff f(x,y) = g(x)h(y)$$

Fact: $\mathbb{1}(A \text{ and } B) = \mathbb{1}(A)\mathbb{1}(B)$

Ex, $f(x,y) = c e^{-x} e^{-y}$ for $x>0$ and $y>0$

$$= ce^{-x}e^{-y}\mathbb{1}(x>0 \text{ and } y>0)$$

$$= ce^{-x}e^{-y}\mathbb{1}(x>0)\mathbb{1}(y>0)$$

$$= \underbrace{ce^{-x}\mathbb{1}(x>0)}_{g(x)}\underbrace{e^{-y}\mathbb{1}(y>0)}_{h(y)}$$

$\mathbb{1}(x \in A)$

Can also plug in a RV ...

$$E\,\mathbb{1}(X \in A) = P(X \in A).$$

Defn: Random Sample

If $X_1, X_2, X_3, \ldots, X_n \overset{iid}{\sim} f$ then we call these X_is a random sample of size n from f.

Fact:

$$f(\underset{\sim}{X}) = f(x_1, x_2, \ldots, x_n)$$

$$= f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n) \quad [\text{independence}]$$

$$= f(x_1) f(x_2) \cdots f(x_n)$$

$$= \prod_{i=1}^{n} f(x_i)$$

Defn: Statistic

If $(X_i)_{i=1}^{n}$ are a random sample and

$$T : \mathbb{R}^n \to \mathbb{R}^d$$

(typically $d \ll n$) then

$T(\underset{\sim}{X})$ a statistic.

① arithmetic mean

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + X_3 + \cdots + X_n)$$

② Sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

③ Order statistics

$X_{(n)}$ = max of $X_1, \dots, X_n$

$X_{(1)}$ = min of X_i s

$X_{(r)}$ = r^{th} smallest among X_is

<u>Defn: Sampling dist</u>

The sampling dist of $T(\underset{\sim}{X})$ is just its distribution.

<u>Theorem : CLT</u>

of size n

intro stats: draw a sample from pop. w/ pop. mean μ and pop. var. σ^2 then

$$\bar{X} \approx N(\mu, \sigma^2/n)$$

(for large n)

big boy:

converges in dist.

CDF/MGF left $\rightarrow$ CDF/MGF right

$$Y = \sqrt{n}\left(\frac{\bar{X}-\mu}{\sigma}\right) \xrightarrow{d} N(0,1)$$

$$\bar{X} \sim N(\mu, \sigma^2/n) \implies \bar{X}-\mu \sim N(0, \sigma^2/n)$$

$$\implies \underbrace{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}_{\sqrt{n}\left(\frac{\bar{X}-\mu}{\sigma}\right)} \sim N(0,1)$$

Ex. CLT

$X_i \overset{iid}{\sim} \text{Bernoulli}(p)$

$\mathbb{E}X_i = p$ and $Var(X_i) = p(1-p)$

CLT: $\sqrt{\;}\,(\bar{X} - p$

CLT: $$\sqrt{n}\left(\frac{\bar{X}-p}{\sqrt{p(1-p)}}\right) \xrightarrow{d} N(0,1)$$

practically: $$\bar{X} \approx N\left(p, \frac{p(1-p)}{n}\right)$$

intro: CI: $\bar{X} \pm 2\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}$

CLT: If X_i are a radom sample and $EX_i = \mu$ and $Var(X_i) = \sigma^2 < \infty$.

Then

$$Y = \sqrt{n}\left(\frac{\bar{X}-\mu}{\sigma}\right) \xrightarrow{d} N(0,1).$$

Theorem: Taylor's Theorem

If $g: \mathbb{R} \to \mathbb{R}$ is k-times diffentiable then the k^{th} order Taylor polynomial is

$$T_k(x) = \sum_{r=0}^{k} \frac{g^{(r)}(0)}{r!} x^r$$

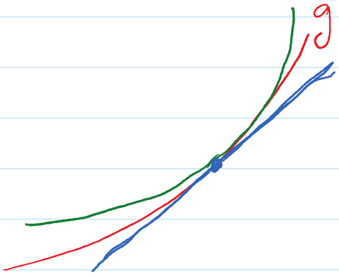
then

$$R = g(x) - T_k(x) \to 0$$

then

$$R = g(x) - T_k(x) \longrightarrow 0$$

as $x \to 0$



---

pf. Show that MGF of Y converges to MGF of a $N(0,1)$ $\quad e^{t^2/2}$

$$Y = \sqrt{n}\left(\frac{\bar{X}-\mu}{\sigma}\right) \qquad ; \quad Y_i = \frac{X_i - \mu}{\sigma}$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i \qquad\qquad EY_i = 0$$

$$Var(Y_i) = 1$$

$$M_{\Sigma Y_i}(t) = M_{Y_1}(t) M_{Y_2}(t) \cdots$$

$$M_{aX}(t) = M_X(at)$$

$$\boxed{M_Y(t) = M\left(t/\sqrt{n}\right)^n}$$

M = mgf of Y_i

$$\underline{M(t)} = \frac{M^{(0)}(0)}{0!}t^0 + \frac{M^{(1)}(0)}{1!}t + \frac{M^{(2)}(0)t^2}{2!} + \dots$$

$$= 1 + \underset{0}{\cancel{E[Y_i]}}\,t + \frac{\underset{1}{\cancel{E[Y_i^2]}}\,t^2}{2}$$

$$\underline{= 1 + t^2/2}$$

$$M_y(t) = M(t/\sqrt{n})^n$$

$$= \left(1 + \frac{\boxed{t^2/2}}{n}\right)^n$$

$$\to e^{t^2/2}$$

$$\left[\left(1 + \frac{c}{n}\right)^n \to e^c\right.$$