Lecture 16 - Transformations

Tuesday, November 2, 2021 2:02 PM

This lecture:

① Know something about X

② What do I know about $Y = g(X)$?

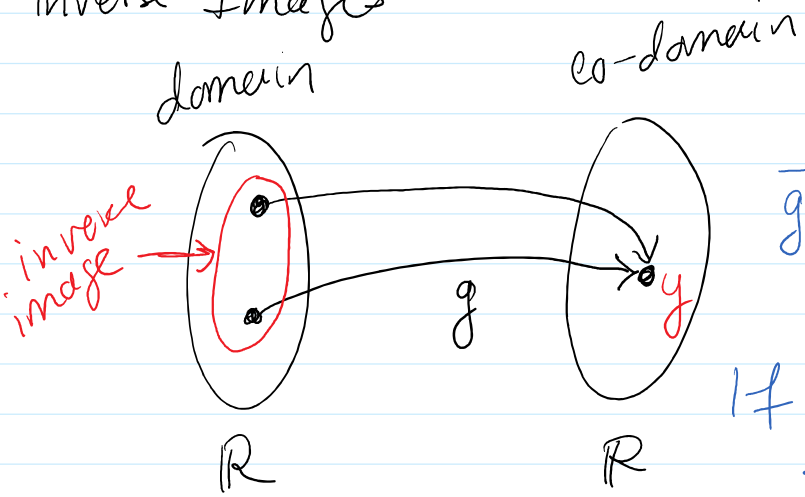
Discrete RVs (PMFs)

Let X be discrete and know f_X ← PMF of X

Let $Y = g(X)$.

What is f_Y? ← PMF of Y.

Inverse Images



g^{-1} = inverse image

$g^{-1}(\{y\}) = g^{-1}(y)$ ← slightly abusive

If g is invertible: inverse image = true inverse

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notice:

$$f_Y(y) = \mathbb{P}(Y=y) = \mathbb{P}(g(X)=y)$$

if g is invertible

$$= \mathbb{P}(X = g^{-1}(y))$$

$$= f_X(g^{-1}(y))$$

What if g not invertible

$$f_Y(y) = \mathbb{P}(Y=y) = \mathbb{P}(g(X)=y)$$

$$= \mathbb{P}(X \in \underbrace{g^{-1}(y)}_{A})$$

$$= \sum_{x \in g^{-1}(y)} f_X(x)$$

$x : g(x) = y$

$$\mathbb{P}(X \in A) = \sum_{x \in A} f_X(x)$$

Theorem: If X discrete and $Y = g(X)$

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x)$$

Ex. Let $X \sim Bin(n,p)$

"$\hookrightarrow$ # of H in n indep coin flips each w/ prob. p of H.

Let $Y = n - X \leftarrow$ # tails

$y = g(x) = n - x \iff x = n - y$

so $g^{-1}(y) = n - y$

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x) = \sum_{x = n-y} f_X(x)$$

$$= f_X(n-y)$$

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x = 0, 1, \ldots, n$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{\cancel{n-(n-y)} \to y} \quad \text{for } y = 0, 1, \ldots, n$$

$$q = 1 - p \qquad \binom{n}{y} = \frac{n!}{y!(n-y)!} = \binom{n}{n-y}$$

$$f_Y(y) = \underbrace{\binom{n}{y} q^y (1-q)^{n-y}}_{\text{Bin}(n,q)} \quad \text{for } y = 0, \ldots, n$$

So $\boxed{Y \sim \text{Bin}(n, 1-p)}$

So $\boxed{Y \sim Bin(n, 1-p)}$

Lets look at continuous RVs (CDFs).

<u>Theorem</u>: If X is continuous and $Y = g(X)$ then

① if <u>g is increasing</u> then

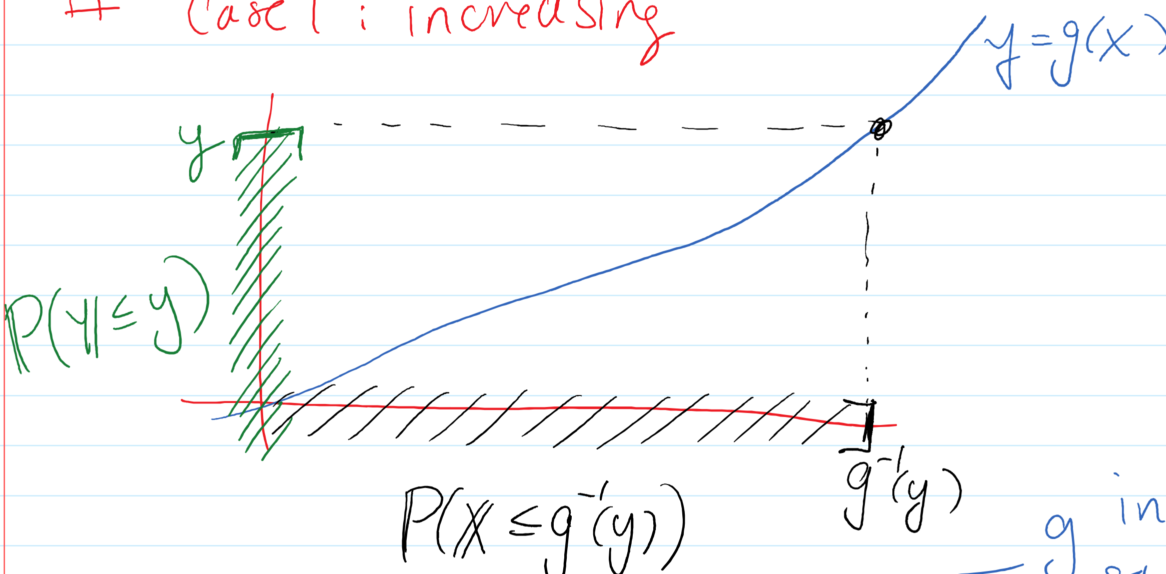
invertible

$$F_Y(y) = F_X(g^{-1}(y))$$

② if <u>g is decreasing</u> then

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

Pf: Case 1: increasing

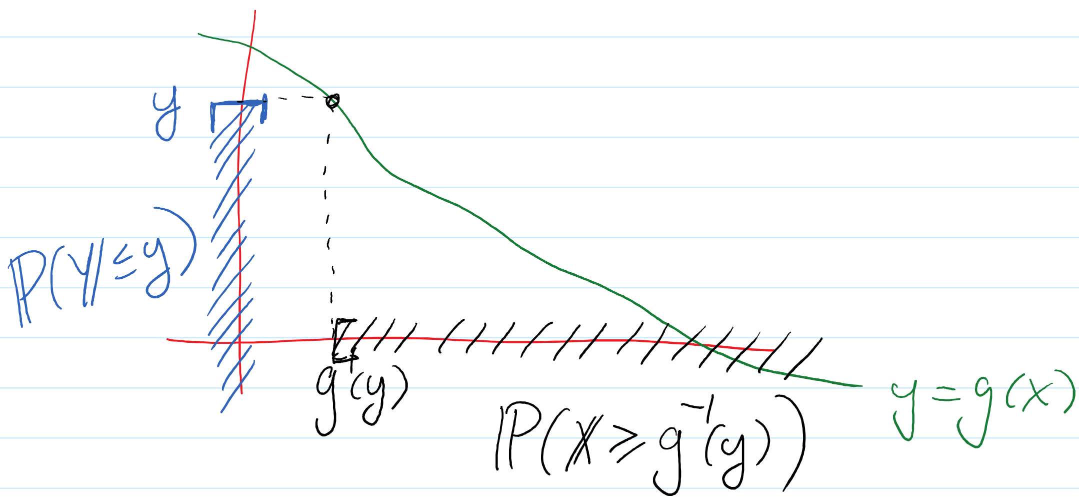


g increasing so is g^{-1}

$F_Y(y) = P(Y \le y) = P(g(X) \le y)$

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y)$$
$$= \mathbb{P}(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

part 2: g decreasing

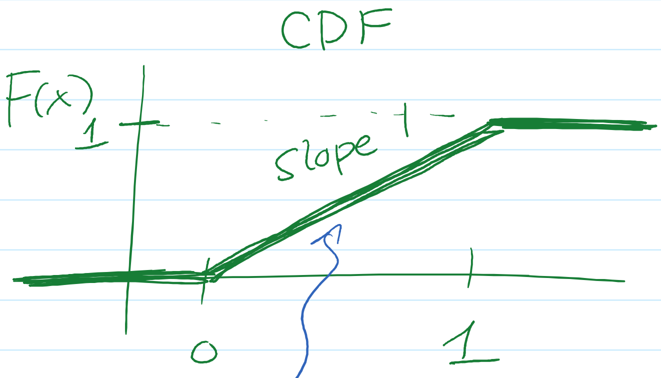


g decreasing so is g^{-1}

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y)$$
$$= \mathbb{P}(X \ge g^{-1}(y))$$
$$= 1 - \mathbb{P}(X \le g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$

Ex. $X \sim U(0,1)$

PDF CDF



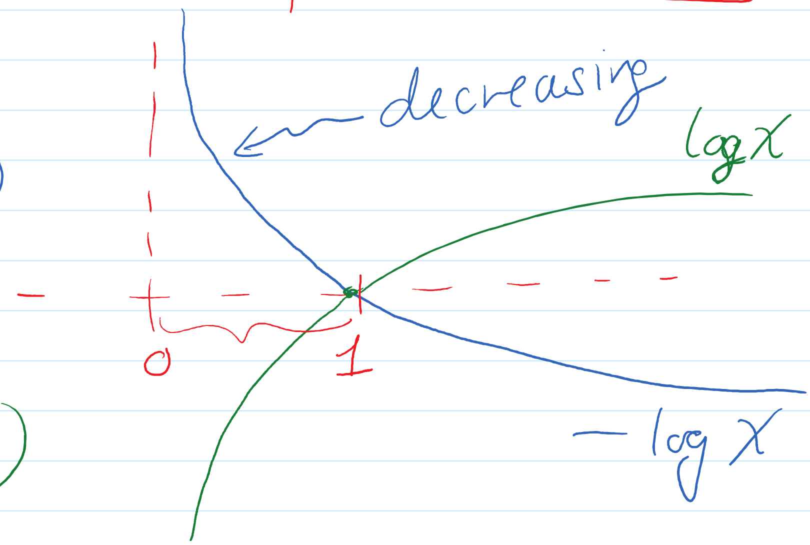
$F_X(x) = x$ for $0 < x < 1$

let $Y = -\log X$

$y = g(x) = -\log(x)$

$\Leftrightarrow -y = \log x$

$\Leftrightarrow e^{-y} = x = g^{-1}(y)$



$F_Y(y) = 1 - F_X(g^{-1}(y))$
(from above)

$= 1 - F_X(\underline{e^{-y}})$

$\boxed{= 1 - e^{-y}}$

CDF of Exp(1)

$y > 0$

$\Leftrightarrow -y < 0$

$e^{-y} < e^0 = 1$

$\Bigg\} \; 0 < e^{-y} < 1$

CDF for Exp(λ)

$1 - e^{-\lambda x}$

So $\boxed{Y \sim \text{Exp}(1)}$

What about PDFs?

Theorem: If X is continuous and $Y = g(X)$

and
1. g is invertible
2. g^{-1} is differentiable

then

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$

Pf. Case 1: g increasing $\rightarrow$ g^{-1} increasing so $\frac{dg^{-1}}{dy} > 0$

our prev. result says

$$F_Y(y) = F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = \underline{f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$

Case 2: g decreasing $\rightarrow$ g^{-1} decreasing so $\frac{dg^{-1}}{dy} < 0$

prev. result: $F_Y(y) = 1 - F_X(g^{-1}(y))$

$$f_Y(y) = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right) = - f_X(g^{-1}(y)) \frac{dg^{-1}}{dy}$$

$$= \underline{f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$

$$= f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

$$|-5| = -(-5)$$

Ex. Let $X \sim \text{Gamma}(a, \lambda)$

$$f_X(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{a-1}}{\Gamma(a)} \quad \underline{\text{for } x > 0}$$

Let $Y = 1/X$

so $g(x) = 1/x = y \iff x = g^{-1}(y) = 1/y$

so $\frac{dg^{-1}}{dy} = -1/y^2$

So

$$f_Y(y) = f_X(1/y) \left| -\frac{1}{y^2} \right|$$

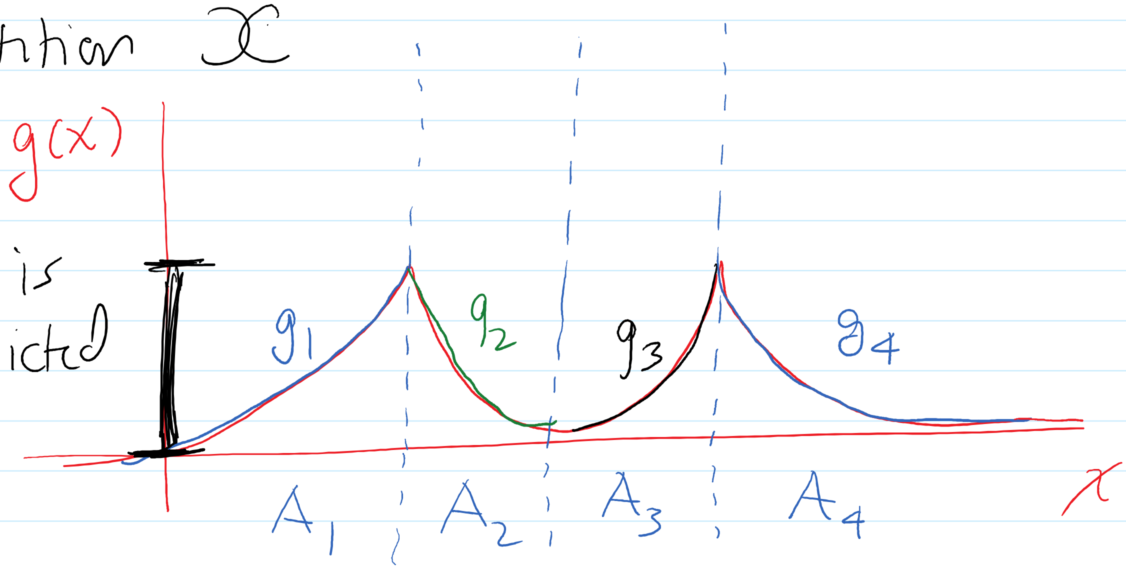
$$\boxed{= \frac{\lambda e^{-\lambda/y} (\lambda/y)^{a-1}}{\Gamma(a)} \frac{1}{y^2} \text{ for } y > 0}$$

called Inverse Gamma distribution

What about non-invertible g?

Theorem: Let X be continuous w/ support $\mathcal{X}$ and for $i = 1, \dots, K$ let $A_1, \dots, A_K$ partition $\mathcal{X}$

Let g_i is g restricted to A_i



If

① my prev. theorem applies on each part of the partition

$\left[\begin{array}{l} g_i \text{ invertible on } A_i \\ g_i^{-1} \text{ is diffentiable} \end{array}\right.$

and

② the image of A_i under g_i is the same $\forall i = 1, \dots, K$

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