Defn: Conditional PMF/PDFs

Given X and Y the conditional pmf/pdf of $X|Y=y$ is

$$f_{X|Y=y}(x) = f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Univariate RV - like X

Defn: Conditional Expectation

If $g:\mathbb{R}\to\mathbb{R}$ then the conditional expectation of $g(X)$ given $Y=y$ is

$$\mathbb{E}\left[g(X)\,\middle|\,Y=y\right] = \begin{cases} \sum_x g(x) f(x|y) & (\text{discrete}) \\ \int g(x) f(x|y)\,dx & (\text{cts}) \end{cases}$$

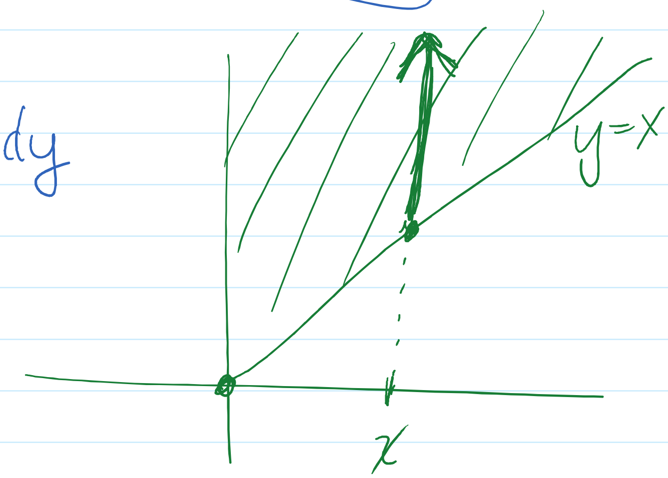
Ex. Last time

$-y$

$f(x,y) = e \qquad 0 < x < y$

Showed:
$$f(y|x) = e^{x-y} \quad \boxed{0<x<y}$$

$$E[Y|X=x] = \int_x^\infty y\, f(y|x)\, dy$$



$$= \int_x^\infty y\, e^{x-y}\, dy$$

$$= \dots$$

$$= 1 + x$$

Defn: Conditional Variance

$$Var(Y|X=x) = E\left[(Y - E[Y|X=x])^2 \,\middle|\, X=x\right]$$

short-cut formula:

$$Var(Y|X=x) = E[Y^2|X=x] - E[Y|X=x]^2.$$

Ex. Continue from above

Shown: $E[Y|X=x] = 1+x$

Need: $E[Y^2|X=x]$

$$= \int_{\mathbb{R}} y^2 f(y|x)\,dy$$

$$= \int_x^\infty y^2 e^{x-y}\,dy = \cdots = x^2+2x+2$$

So $\mathrm{Var}(Y|X=x) = x^2+2x+2-(1+x)^2 = 1$

Independence:

For events: for $A, B \subset S$

$A \perp\!\!\!\perp B$ if $P(AB) = P(A)P(B)$.

For RVs we say $X \perp\!\!\!\perp Y$ if:

$$P(X\in A, Y\in B) = P(X\in A)P(Y\in B)$$

$\forall A, B \subset \mathbb{R}$.

Product Spaces

$\text{Support}(X,Y) = \{(x,y) \mid f(x,y) > 0\} \subset \mathbb{R}^2$

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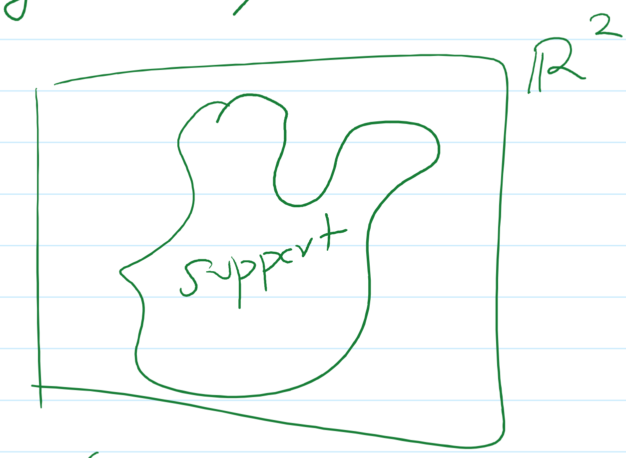


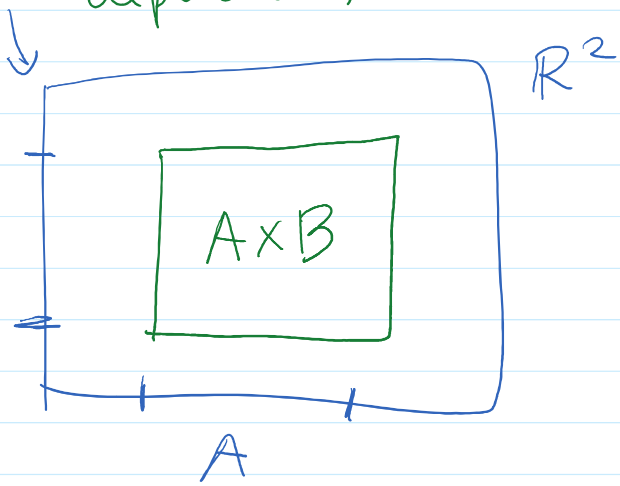
Consider

$f(x,y) = $ ~~~~~~

for $x \in A$ and $y \in B$

doesn't depend on y

doesn't depend on x



Support of (X, Y) is $A \times B$.

Theorem: Factorization Theorem

If X and Y have a support that is a product space $(A \times B)$ then

$$X \perp\!\!\!\perp Y \iff \text{① } F(x,y) = F_X(x) F_Y(y)$$

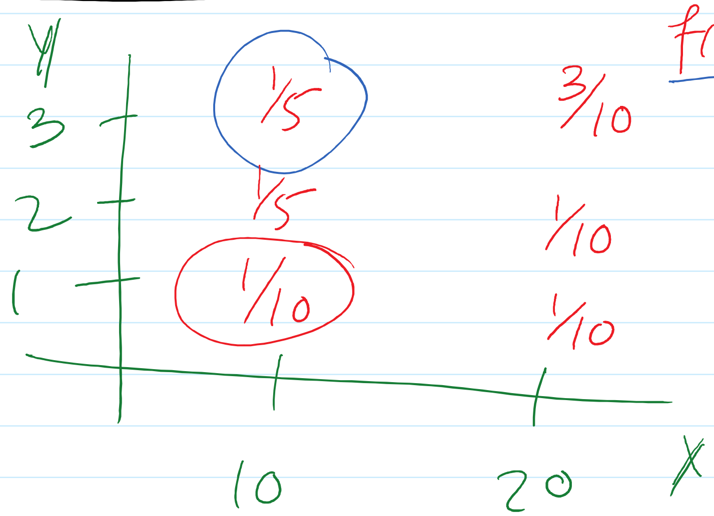
or

$$\text{② } f(x,y) = f_X(x) f_Y(y)$$

Ex $\quad Y \qquad$ $f(x,y)$

x	10	20

Ex.



$f(x,y)$

x	10	20
$f(x)$	$1/2$	$1/2$

y	1	2	3
$f(y)$	$1/5$	$3/10$	$1/2$

Q: Is $X \perp\!\!\!\perp Y$?

Support $= \{10, 20\} \times \{1, 2, 3\}$

Need to check $f(x,y) = f(x) f(y)$

$$f(10,1) = \frac{1}{10} \overset{?}{=} f_X(10) f_Y(1) = \left(\frac{1}{2}\right)\left(\frac{1}{5}\right) = \frac{1}{10}$$

$$f(10,3) = \frac{1}{5} \cancel{\overset{?}{=}} f_X(10) f_Y(3) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

Not independent.

Corollary: If support of X, Y is a product space, then

$$X \perp\!\!\!\perp Y \iff f(x,y) = h(x)\,g(y)$$

where h involves only x, g only y.

don't even need f_X or f_Y

Ex. $$f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)}$$

for $\underbrace{x>0 \text{ and } y>0}_{(0,\infty)\times(0,\infty)}$.

$X \perp\!\!\!\perp Y$?

$$\frac{1}{384} x^2 e^{-y} e^{-(x/2)} = \Big[\underbrace{\frac{1}{384} x^2 e^{-(x/2)}}_{h(x)}\ \underbrace{e^{-y}}_{g(y)}\Big]$$

So $X \perp\!\!\!\perp Y$.

Recall: $A \perp\!\!\!\perp B$ then $P(A|B) = P(A)$.

Similarly if $X \perp\!\!\!\perp Y$ then

$f(x,y)$ $\cancel{f(x)}f(y)$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\cancel{f(x)}f(y)}{\cancel{f(x)}} = f(y)$$

Theorem: Expectation of Independent

If $X \perp\!\!\!\perp Y$ and $g_1 : \mathbb{R} \to \mathbb{R}$, $g_2 : \mathbb{R} \to \mathbb{R}$,
then

$$\mathbb{E}[g_1(X) g_2(Y)] = \mathbb{E}[g_1(X)]\,\mathbb{E}[g_2(Y)].$$

Pf: (cts)

$$\mathbb{E}[g_1(X) g_2(Y)] = \iint g_1(x) g_2(y) f(x,y)\,dx\,dy$$

$$= \iint g_1(x) g_2(y) f(x) f(y)\,dx\,dy$$

$$= \left[\int g_1(x) f(x)\,dx\right]\left[\int g_2(y) f(y)\,dy\right]$$

$$= \mathbb{E}[g_1(X)]\,\mathbb{E}[g_2(Y)]$$

Ex. If $X, Y \sim \text{Exp}(\lambda = 1)$ and $X \perp\!\!\!\perp Y$

then

$$E[X^2 Y] = E[X^2]E[Y]$$
$$= (2)(1) = 2$$

Theorem: MGFs of Independent

If $X \perp\!\!\!\perp Y$ then

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

pf: $M_{X+Y}(t) = E[e^{t(X+Y)}]$

$$= E[e^{tX} e^{tY}]$$
$$= E[e^{tX}]E[e^{tY}] = M_X(t) M_Y(t)$$

Ex. let $X \sim N(\mu, \sigma^2)$

and $Y \sim N(\gamma, \tau^2)$ and $X \perp\!\!\!\perp Y$

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

$$= \left(e^{\mu t + \sigma^2 t^2/2}\right)\left(e^{\gamma t + \tau^2 t^2/2}\right)$$

$$= e^{\underbrace{(\mu+\gamma)}t + \underbrace{(\sigma^2+\tau^2)}t^2/2}$$

MGF of a $\underbrace{N(\mu+\gamma, \sigma^2+\tau^2)}$

So $\boxed{X+Y \sim N(\mu+\gamma, \sigma^2+\tau^2)}$

Ex. $X \sim N(2,5)$ and $Y \sim N(3,7)$

and $X \perp\!\!\!\perp Y$

then $X+Y \sim N(5,12)$

Theorem: Cor/Cov for Independent.

If $X \perp\!\!\!\perp Y$ then $\text{Cor}(X,Y) = \text{Cov}(X,Y) = 0$.

Pf.

$$\text{Cov}(X,Y) = \mathbb{E}[XY] - (\mathbb{E}X)(\mathbb{E}Y)$$

$$= \mathbb{E}(X)\mathbb{E}(Y) - (\mathbb{E}X)(\mathbb{E}Y) = 0$$

Cor is re-scaled Cov, so it is also

zero.

Converse is generally false.

If $Cov(X,Y)=0$ they may or may not be independent.

Ex. $X \sim N(0,1)$ and $Y = X^2$

then $X \not\perp Y$.

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$= E[X\underbrace{X^2}_{X^3}] - E[X]E[X^2] = 0 - 0$

0

$= 0$

$EX^3 = 0$

Bayes' Theorem

Events: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

$$\boxed{\text{RVs: } f(y|x) = \frac{f(x|y)f(y)}{f(x)}.}$$

Law of Total Prob

Events: C_i partition S

$$P(A) = \sum_i P(A|C_i)P(C_i).$$

RVs:

(discrete) $f(y) = \sum_x f(y|x)f(x)$

(cts) $f(y) = \int_{\mathbb{R}} f(y|x)f(x)\,dx$