Lecture 22 - Bivariate Transformations

Tuesday, November 30, 2021 9:29 AM

Uni: $g : \mathbb{R} \to \mathbb{R}$ consider $g(X)$.

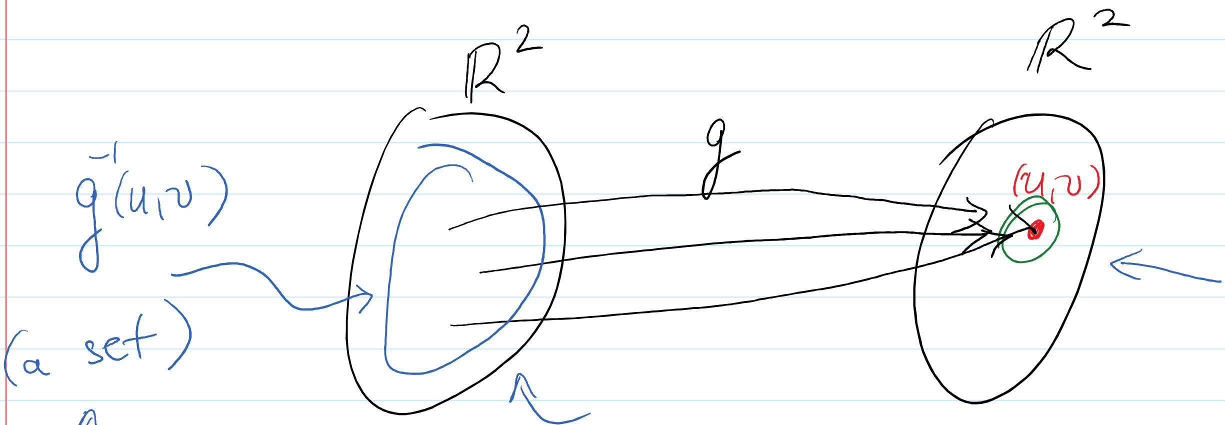
bivariate: $g : \mathbb{R}^2 \to \mathbb{R}^2$ and consider $g(X, Y)$

notation: $(X, Y) \overset{g}{\rightsquigarrow} (U, V)$

ex, $(U, V) = (X^2 Y, -\log Y)$

Discrete $(U, V) = (g_1(X, Y), g_2(X, Y))$

Assume X and Y discrete.



$\{(x,y) \mid g_1(x,y) = u \text{ and } g_2(x,y) = v\}$

Want: PMF of (U, V)

$f_{U,V}(u,v) = \mathbb{P}(U = u, V = v)$

$= \mathbb{P}((U,V) \in \{(u,v)\})$

$$= P((X,Y) \in g^{-1}(u,v))$$

$$= \sum_{(x,y) \in g^{-1}(u,v)} f_{X,Y}(x,y)$$

Ex, $X \perp\!\!\!\perp Y$

and $X \sim \text{Pois}(\theta)$ } discrete
$Y \sim \text{Pois}(\lambda)$

let $\boxed{U = X + Y \quad \text{and} \quad V = Y}$

① $f(x,y) = f(x)f(y) = \dfrac{\theta^x e^{-\theta}}{x!} \dfrac{\lambda^y e^{-\lambda}}{y!}$

② Solve for X and Y

$u = g_1(x,y) = x + y \quad \text{and} \quad v = g_2(x,y) = y$

$u - v = x + y - y = x$

$\boxed{g_2^{-1}(u,v) = v}$

so $\boxed{g_1^{-1}(u,v) = u - v}$

③ Theorem says

$$f_{U,V}(u,v) = f_{X,Y}(\overbrace{g_1^{-1}(u,v)}^{u-v}, \overbrace{g_2^{-1}(u,v)}^{v})$$

$$\boxed{= \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^{v} e^{-\lambda}}{v!}}$$

$U = X+Y$ so lets get marginal.

$u = x+y$
$v = y$
$v < u$

$$f_U(u) = \sum_{v=0}^{u} f(u,v)$$

$$= \sum_{v=0}^{u} \frac{\theta^{u-v} e^{-\theta} \lambda^{v} e^{-\lambda}}{(u-v)!\, v!}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^{u} \frac{u!\, \theta^{u-v} \lambda^{v}}{(u-v)!\, v!}$$

$\longrightarrow \binom{u}{v}$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \underbrace{\sum_{v=0}^{u} \binom{u}{v} \theta^{u-v} \lambda^{v}}$$

Binomial Theorem $= (\theta+\lambda)^u$

$$f(u) = \frac{e^{-(\theta+\lambda)} (\theta+\lambda)^u}{}$$

$u = X+Y$

$u!$

$\leftarrow Pois(\theta+\lambda)$

<u>Theorem</u>: If $X \perp Y$, $X \sim Pois(\theta)$
$Y \sim Pois(\lambda)$

then $X+Y \sim Pois(\theta+\lambda)$

<u>Continuous</u>

$$(U, V) = (g_1(X,Y), g_2(X,Y))$$

If X and Y are continuous and

1. g is invertible
2. g^{-1} is differentiable

then

$$f_{U,V}(u,v) = f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v)) \left| \det J \right|$$

Jacobian mtx of g^{-1}

$\left(\underline{\text{Uni}}: f(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right| \right)$

Jacobian: $h: \mathbb{R}^2 \to \mathbb{R}^2$

$$h(x,y) = (h_1(x,y), h_2(x,y))$$

then

$$J = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix}$$

So in our case:

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix}$$

Recall determinant

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(A) = ad - cb$

Steps: $(u, v) = (g_1(x, y), g_2(x, y))$

1. get g_1^{-1}, g_2^{-1}
2. Find J and get $\det J$
3. plug in formula

Ex,

$$(U,V) = (\underbrace{X+Y}_{g_1}, \underbrace{X-Y}_{g_2})$$

1 get inverses

$$\begin{aligned} u &= x+y \\ v &= x-y \end{aligned} \Rightarrow u+v = 2x \Rightarrow \frac{u+v}{2} = x$$

so $\boxed{g_1^{-1}(u,v) = \frac{u+v}{2}}$

Similarly $\boxed{\frac{u-v}{2} = y = g_2^{-1}(u,v)}$

2 get J and det J

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1/2 \end{bmatrix} \Rightarrow |\det J| = \left|\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right|$$

$$= \left|-\frac{1}{2}\right|$$

$$= 1/2$$

3 Plug in to formula!

$$\boxed{f(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)\frac{1}{2}}$$

$$\boxed{f(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)\frac{1}{2}}$$

Assume $X, Y \overset{iid}{\sim} N(0,1)$

independent and identically distributed

by independence

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

So

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u+v}{2}\right)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-v}{2}\right)^2} \frac{1}{2}$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\pi} \quad e^{-\frac{1}{2}\left(\frac{u^2+v^2}{2}\right)} \quad \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi}}$$

$$\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2 = \frac{1}{4}\left(u^2 \cancel{+2uv} + v^2 + u^2 \cancel{-2uv} + v^2\right)$$

$$= \frac{1}{4}\left(2u^2 + 2v^2\right)$$

$$= \frac{1}{2}\left(u^2 + v^2\right)$$

$N(0,2)$ $N(0,2)$

$$= \frac{1}{2}(u^2+v^2)$$

$$f(u,v) = \underbrace{\overbrace{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{u^2}{2}}}^{N(0,2)}}_{\text{fn of just } u} \; \underbrace{\overbrace{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{v^2}{2}}}^{N(0,2)}}_{\text{fn of just } v}$$

Then $U \perp\!\!\!\perp V$.

$$X \pm Y \sim N(0,2)$$

<u>Ex.</u> $X \sim \text{Beta}(\alpha, \beta)$, $Y \sim \text{Beta}(\alpha+\beta, \gamma)$ and $X \perp\!\!\!\perp Y$.

Consider $U = XY$ and $V = X$.

Notice: $0 \le X, Y \le 1$

so $0 \le U \le V \le 1$

(1) get inverse transf

$u = xy$ and $v = x$

$$\Downarrow$$

$$x = \boxed{g_1^{-1}(u,v) = v}$$

$$\Downarrow$$

$$\frac{u}{x} = y \Rightarrow \boxed{y = g_2^{-1}(u,v) = u/v}$$

② get J, $\det J$

$$J = \begin{bmatrix} 0 & 1 \\ \frac{1}{v} & -u/v^2 \end{bmatrix} \Rightarrow |\det J| = \left| -\frac{1}{v} \right| = \frac{1}{v}$$

③ plug in

$$f_{X,Y}(x,y) \overset{\text{independence}}{=} f_X(x) \underbrace{f_Y(y)}_{\text{Beta}(\alpha+\beta,\, \gamma)}$$

$$= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \; \frac{y^{\alpha+\beta-1}(1-y)^{\gamma-1}}{B(\alpha+\beta,\gamma)}$$

$$f(u,v) = \frac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha,\beta)} \; \frac{(u/v)^{\alpha+\beta-1}(1-u/v)^{\gamma-1}}{B(\alpha+\beta,\gamma)} \; \frac{1}{v}$$

for $0 < u < v < 1$

Ex. $X \sim \text{Gamma}(\alpha, \lambda)$
$Y \sim \text{Gamma}(\beta, \lambda)$ and $X \perp Y$

$$f(x,y) = f(x)\, f(y) = \lambda e^{-\lambda x} (\lambda x)^{\alpha-1} \quad \lambda e^{-\lambda y} (\lambda y)^{\beta-1}$$

$$f(x,y) = f(x)\,f(y) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \; \frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)}$$

$U = X+Y$ and $V = \frac{X}{X+Y}$

(1) get inverse

$$x = g_1^{-1}(u,v) = uv$$

$$y = g_2^{-1}(u,v) = u - uv = u(1-v)$$

(2) $J = \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} \Rightarrow |\det J| = |-uv - u(1-v)|$

$= |-u|$

$= u$

(3) plug-in

$$\boxed{f(u,v) = \frac{\lambda e^{-\lambda uv}(\lambda uv)^{\alpha-1}}{\Gamma(\alpha)} \; \frac{\lambda e^{-\lambda u(1-v)}(\lambda u(1-v))^{\beta-1}}{\Gamma(\beta)}\, u}$$

$= \cdots$ algebra $\cdots$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{u^{\alpha+\beta-1} e^{-\lambda u}}_{\text{fn of only } u} \underbrace{v^{\alpha-1} (1-v)^{\beta-1}}_{\text{fn of only } v}$$

(proportional to PDF of u)   (... prop to PDF of v)

$U \perp\!\!\!\perp V$ and $U \sim \text{Gamma}(\alpha+\beta, \lambda)$

$V \sim \text{Beta}(\alpha, \beta)$

<u>Theorem:</u> Independence and Transformations

If $X \perp\!\!\!\perp Y$ and

$U = g(X) \leftarrow$ only X

$V = h(Y)$

only Y

then $U \perp\!\!\!\perp V$.

<u>Ex.</u> $U = X^2$ and $V = -\log Y$.