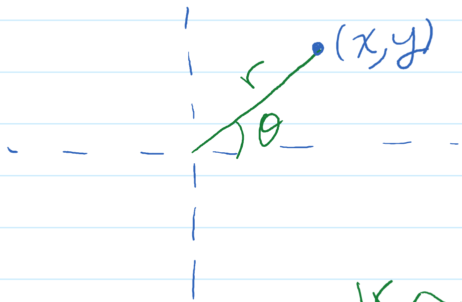
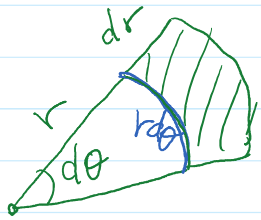
Lecture 11 - Expectation

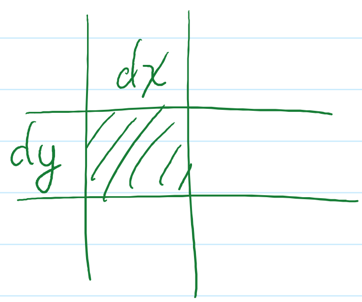
Tuesday, October 12, 2021 2:00 PM

Polar coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$$





$$dxdy = r\,dr\,d\theta$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(x^2+y^2)\right) dxdy = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi}\underbrace{\int_{r=0}^{r=\infty} \exp\left(-\frac{1}{2}r^2\right) r\,dr}\,d\theta$$

u-substitution

$\underline{u = \frac{1}{2}r^2}$ then $du = r\,dr$

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \exp(-u)\,du\,d\theta$$

$$\frac{1}{e^{\infty}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(-e^{-u}\Big|_0^{\infty}\right)d\theta \qquad = e^{-\infty} - (-e^{0})$$

$$= \frac{1}{2\pi}\int_0 \underbrace{\left(-e^{-u}\Big|_0\right)}_{0-(-1)\;=\;1} d\theta$$

$$-e - (-e)$$
$$= -0 + 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta = \frac{1}{2\pi}2\pi = 1 = I^2$$

so $I = 1$.

Expected Value

If X is a RV then the mean or expected value of X

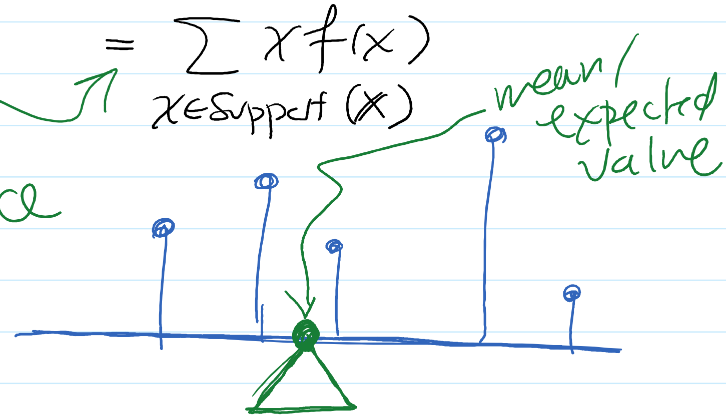
denoted $\mathbb{E}[X]$

is defined as

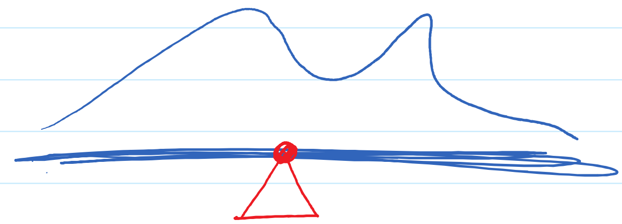
(1) discrete $\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x f(x)$ ← PMF

$$= \sum_{x \in \text{Support}(X)} x f(x)$$

weighted average or sum of the values in the support weighted by their prob. of occurence



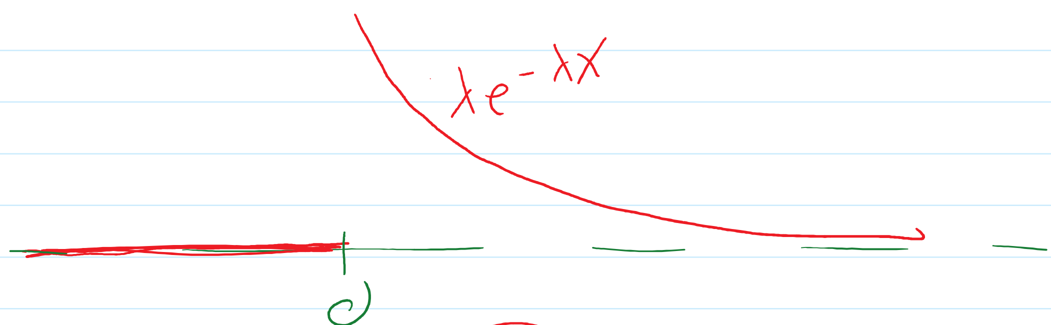
② continuas $\quad E[X] = \int_{\mathbb{R}} x f(x)\,dx$

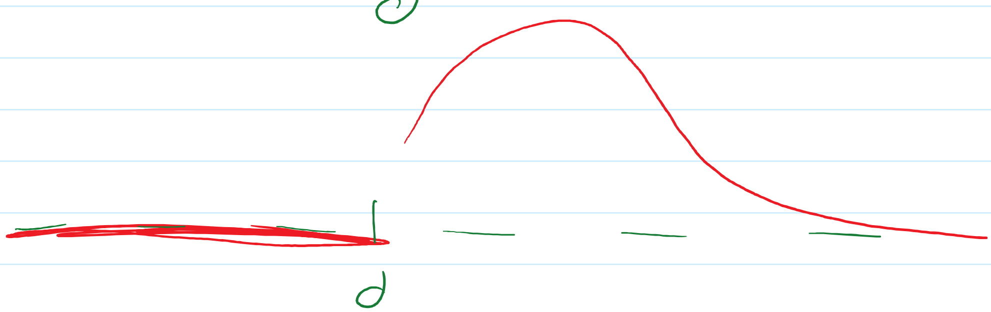


Ex. let $X \sim \text{Exp}(\lambda)$ $\quad \lambda > 0$

means $\quad$ Exponential distribution

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$





Q: what is $E[X]$?

$$E[X] = \int x f(x)\,dx = \int^{\infty} x \lambda e^{-\lambda x}\,dx$$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0 \underbrace{x}_{u}\, \underbrace{\lambda e^{-\lambda x}\,dx}_{dv}$$

integration by parts: $u = x \quad dv = \lambda e^{-\lambda x}\,dx$

$du = dx \quad v = -e^{-\lambda x}$

$$\int u\,dv = uv - \int v\,du$$

$$-xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$

$$= 0 + \left(-\frac{1}{\lambda}e^{-\lambda x}\right)_0^\infty$$

$$= 0 - \left(-\frac{1}{\lambda}\right)$$

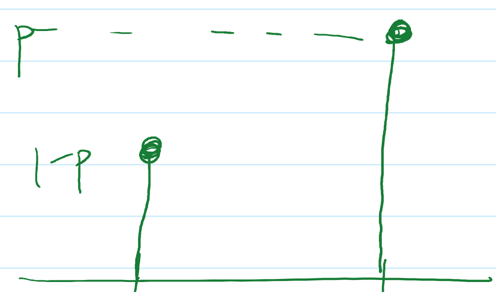
$$\boxed{= 1/\lambda = \mathbb{E}[X]}$$

Ex. $X \sim \text{Bernoulli}(p)$ $\quad 0 \le p \le 1$

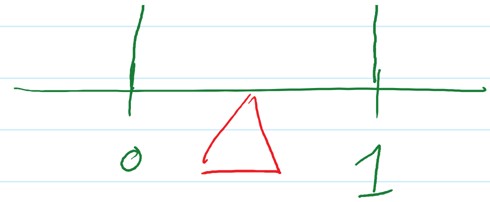
Flip a coin w/ prob. p of H

$X = \#$ heads.

$$f(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \end{cases}$$



$$1-p, \quad x=0$$



$$\mathbb{E}[X] = \sum x f(x) = (0)f(0) + (1)f(1)$$
$$= (0)(1-p) + (1)p$$
$$= p.$$

Ex. Binomial

$$X \sim \mathrm{Bin}(n, p)$$

$0 \le p \le 1$

integer > 0

X = # heads among n independent coin flips w/ a prob. p of H on each

$$f(x) = \underbrace{\binom{n}{x}} p^x (1-p)^{n-x} \quad \text{for } x = 0, \ldots, n$$

Binomial Theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{x} x^i y^{n-i}$$

$$\boxed{\text{Justify: } \sum_{x=0}^{n} f(x) = 1}$$

$x = p$ and $y = 1-p$

$$E[X] = \sum_{x=\cancel{0}1}^{n} x f(x) = \sum_{x=1}^{n} \underbrace{x\binom{n}{x}} p^x (1-p)^{n-x}$$

$$x\binom{n}{x} = \cancel{x}\frac{n!}{\cancel{x!}_{(x-1)!}(n-x)!}$$

$$= \frac{n(n-1)!}{(x-1)!((n-1)-(x-1))!} \quad \binom{n-1}{x-1}$$

$$= n\binom{n-1}{x-1}$$

$$= \sum_{x=1}^{n} n\binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$y = x-1 \iff x = y+1$$

$$= n\sum_{y=0}^{n-1}\binom{n-1}{y} p^{y+1}(1-p)^{n-y-1}$$

$$= np\sum_{y=0}^{n-1}\underbrace{\binom{n-1}{y}p^y(1-p)^{(n-1)-y}}$$

PMF of a $Bin(n-1, p)$

Sum of PMF of $Bin(n-1,p)$ over its support $= \underline{1}$

$$\boxed{E[X] = np}$$

<u>Ex,</u> $n = 10$ w/ $p = 1/2$

$$E[X] = 5$$

<u>General trick:</u> PMF/PDF trick

Often I can recognize in a calculation

some term is either

$$\sum f(x) \quad \text{or} \quad \int f(x)\,dx$$

over the support — I can replace this w/ 1.

Functions of RVs.

Note: a function of a RV is also a RV.

Ex. If I have a RV X then X^2 or $\log X$ is a RV.

Theorem: Law of the Unconscious Statistician

If $g: \mathbb{R} \to \mathbb{R}$ and X is a RV then

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x) f(x) & \text{(discrete)} \\ \int_{\mathbb{R}} g(x) f(x)\,dx & \text{(cts)} \end{cases}$$

<u>Ex.</u> $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

$$\mathbb{E}[X] = 1/\lambda$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^\infty \underbrace{x^2}_{u} \underbrace{\lambda e^{-\lambda x} dx}_{dv}$$

$$u = x^2 \qquad dv = \lambda e^{-\lambda x} dx$$

$$du = 2x\,dx \qquad v = -e^{-\lambda x}$$

$$uv - \int v\,du = \underbrace{-x^2 e^{-\lambda x}\Big|_0^\infty}_{0} + 2\int_0^\infty e^{-\lambda x} x\,dx$$

$$= \frac{2}{\lambda} \underbrace{\int_0^\infty \lambda e^{-\lambda x} x\,dx}_{\mathbb{E}[X] = 1/\lambda}$$

$$\boxed{\mathbb{E}[X^2] = 2/\lambda^2} \neq (\mathbb{E}[X])^2 = 1/\lambda^2$$