# Lecture 20 - Conditional Distributions

Thursday, November 18, 2021 2:00 PM

## Defn: Conditional PMF/PDFs

Given RVs $X$ and $Y$ the conditional PMF/PDF of "$X \mid Y = y$" is

$$\underbrace{f_{X \mid Y = y}(x)}_{\text{a univariate RV}} = f(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

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## Defn: Conditional Expectation

If $g : \mathbb{R} \to \mathbb{R}$ then the conditional expectation of $g(X)$ given $Y = y$ is

$$E[g(X) \mid Y = y] = \begin{cases} \sum_x g(x) f(x \mid y) & \text{(discrete)} \\ \int_{\mathbb{R}} g(x) f(x \mid y)\, dx & \text{(cts)} \end{cases}$$

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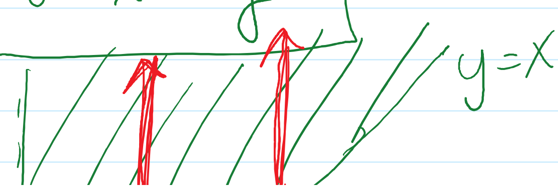
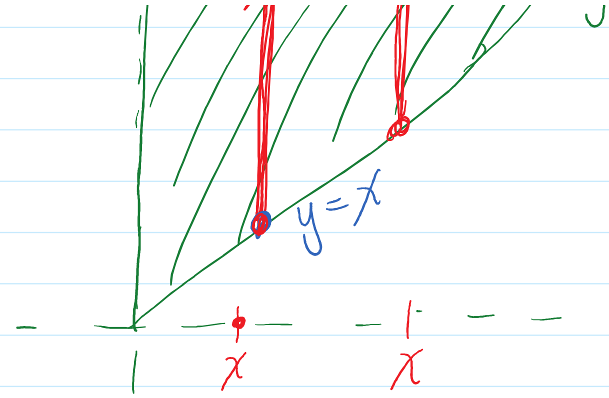
Ex. $f(x,y) = e^{-y}$ for $\boxed{0 < x < y}$

$f(y \mid x) = e^{x-y}$



$$f(y|x) = e^{x-y}$$




$$\mathbb{E}[Y|X=x] = \int_x^{\infty} y f(y|x)\,dy$$

$$= \int_x^{\infty} y e^{x-y}\,dy = \cdots = 1+x$$

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Defn: Conditional Variance

$$\mathrm{Var}(Y|X=x) = \mathbb{E}\left[\left(Y - \mathbb{E}[Y|X=x]\right)^2 \,\middle|\, X=x\right]$$

Short-cut formula:

$$\mathrm{Var}(Y|X=x) = \mathbb{E}[Y^2|X=x] - \mathbb{E}[Y|X=x]^2$$

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Ex. (continue from above)

$$\mathbb{E}[Y^2|X=x] = \int y^2 f(y|x)\,dy$$

$$= \int_x^{\infty} y^2 e^{x-y}\,dy = \cdots = x^2+2x+2$$

$$\text{Var}(Y \mid X = x) = (x^2 + 2x + 2) - (1 + x)^2$$
$$= 1$$

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## Independence

**For Events:** If $A, B \subset S$

$$A \perp\!\!\!\perp B \iff \mathbb{P}(AB) = \mathbb{P}(A)\,\mathbb{P}(B)$$

**For RVs:**

$$X \perp\!\!\!\perp Y \iff \mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$$

$\rightarrow$ $\quad \forall A, B \subset \mathbb{R}.$

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## Product Spaces

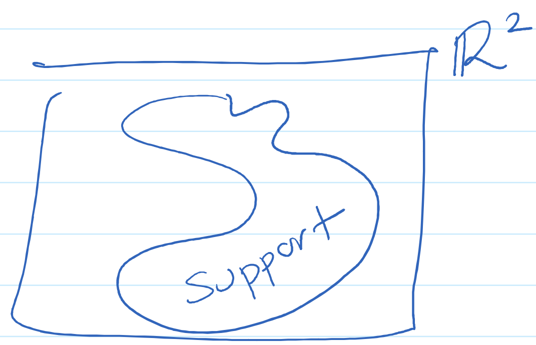
$$\text{Support}(X, Y) = \{(x, y) \text{ where } f(x,y) > 0\}$$

If $f(x,y) = \sim\sim\sim$

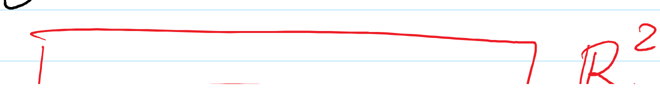
for $x \in A$ and $y \in B$

doesn't depend on $y$

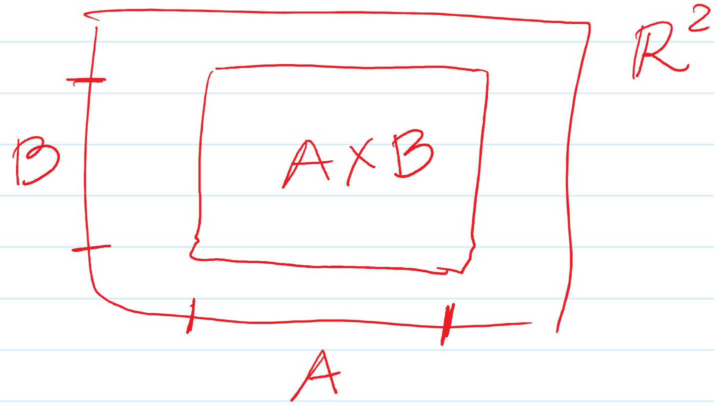
doesn't depend on $x$




Support of $X, Y$ is $A \times B$



Support of $X, Y$ is $A \times B$

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## Theorem: Factorization Theorem

If $X$ and $Y$ have support on a product space then

$X \perp\!\!\!\perp Y \iff$ ① $F(x,y) = F_X(x) F_Y(y)$

OR

② $f(x,y) = f_X(x) f_Y(y)$

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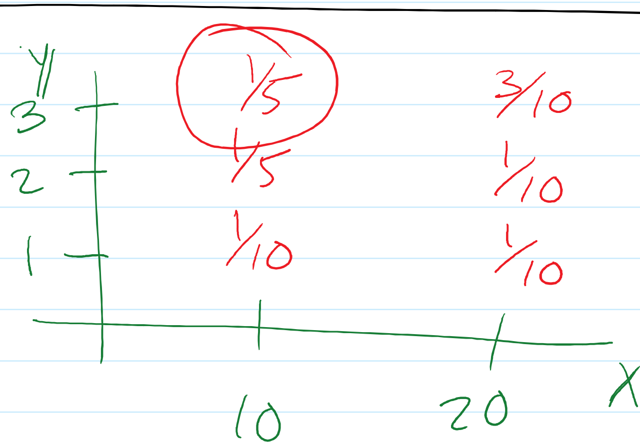
Ex.




$f(x,y)$

| $f(x)$ | $1/2$ | $1/2$ |
|---|---|---|
| $x$ | 10 | 20 |

| $f(y)$ | $1/5$ | $3/10$ | $1/2$ |
|---|---|---|---|
| $y$ | 1 | 2 | 3 |

Q: $X \perp\!\!\!\perp Y$?

① clearly support is $\{10, 20\} \times \{1, 2, 3\}$

② $f(x,y) = f_X(x) f_Y(y)$

(2) $f(x,y) = f_X(x) f_Y(y)$

Ex. $f(10,3) = \frac{1}{5} \overset{?}{\cancel{=}} f_X(10) f_Y(3) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

Not independent.

Corrollary: If support of $X$ and $Y$ is a product space

$$X \perp\!\!\!\perp Y \iff f(x,y) = h(x) g(y)$$

only $x$ only $y$

Don't need $f_X$ and $f_Y$.

Ex.

$$f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)} \quad \text{for } x>0, y>0$$

(1) Support is a product space:

$(0,\infty) \times (0,\infty)$

(2) $$f(x,y) = \underbrace{\left(\frac{1}{384} x^2 e^{-x/2}\right)}_{h(x)} \underbrace{\left(e^{-y}\right)}_{g(y)}$$

So $X \perp\!\!\!\perp Y$

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Fact:

For events: $A \perp\!\!\!\perp B$ then $P(A|B) = P(A)$

For RVs: $X \perp\!\!\!\perp Y$ then

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)$$

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Theorem: expectation for independent

If $X \perp\!\!\!\perp Y$ and $g_1 : \mathbb{R} \to \mathbb{R}$, $g_2 : \mathbb{R} \to \mathbb{R}$

then

$$\mathbb{E}[g_1(X)g_2(Y)] = \mathbb{E}[g_1(X)]\,\mathbb{E}[g_2(Y)]$$

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Pf: (cts)

$$E[g_1(X)g_2(Y)] = \iint_{\mathbb{R}^2} g_1(x)g_2(y)f(x,y)\,dx\,dy$$

$$= \iint g_1(x)g_2(y)f_X(x)f_Y(y)\,dx\,dy$$

$$= \int g_1(x) f(x)\,dx \int g_2(y) f(y)\,dy$$

$$= \mathbb{E}[g_1(X)]\, E[g_2(Y)]$$

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Let $X, Y \sim \text{Exp}(\lambda = 1)$ and $X \perp\!\!\!\perp Y$

then

$$\mathbb{E}[X^2 Y] = \mathbb{E}[X^2]\,\mathbb{E}[Y]$$

$$= (2)(1) = 2$$

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**Theorem: MGF of Independent**

If $X \perp\!\!\!\perp Y$ then

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

Pf.

$$M_{X+Y}(t) = \mathbb{E}\left[e^{t(X+Y)}\right] = \mathbb{E}\left[e^{tX} e^{tY}\right]$$

$$= \mathbb{E}[e^{tX}]\,\mathbb{E}[e^{tY}]$$

$$= M_X(t)\, M_Y(t)$$

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Ex. Let $X \sim N(\mu, \sigma^2)$, $Y \sim N(\gamma, \tau^2)$

and $X \perp\!\!\!\perp Y$

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$
$$= e^{\mu t + \sigma^2 t^2/2}\, e^{\gamma t + \tau^2 t^2/2}$$
$$= e^{\underbrace{(\mu+\gamma)}\,t + \underbrace{(\sigma^2+\tau^2)}\,t^2/2}$$

MGF of $N(\mu+\gamma, \sigma^2+\tau^2)$

So $\boxed{X+Y \sim N(\mu+\gamma, \sigma^2+\tau^2)}$

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**Theorem:** Cor/Cov of Independent

If $X \perp\!\!\!\perp Y$ then $\mathrm{Cov}(X,Y) = \mathrm{Cor}(X,Y) = 0$.

Pf. $\mathrm{Cov}(X,Y) = E[XY] - EX\,EY$
$$= EX\,EY - EX\,EY = 0$$

Cor = re-scaled cov so it is also zero.

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Converse is generally false.

If $\mathrm{Cor}(X,Y) = 0$ they may or may not be independent.

Ex. $X \sim N(0,1)$ and $Y = X^2$

$X$ and $Y$ not independent.

but

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$
$$= \underbrace{E[X^3]}_{0} - \underbrace{E[X]}_{0} E[X^2] = 0$$

Bayes' Theorem:

Events: $\mathbb{P}(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

$$\boxed{\text{RVs: } f(x|y) = \frac{f(y|x) f_X(x)}{f_Y(y)}}$$

Law of Total Probability

Events: $(C_i)$ partition $S$ then

$$P(A) = \sum_i P(A|C_i)P(C_i)$$

$$\boxed{\begin{array}{ll}\underline{\text{RVs}}: & (\text{discrete})\ f(y) = \sum_x f(y|x)f(x) \\ & (\text{cts})\ f(y) = \int_{\mathbb{R}} f(y|x)f(x)\,dx\end{array}}$$

<u>Pf</u> (cts case)

① $f(y|x) = \dfrac{f(x,y)}{f_X(x)} \iff \underbrace{f(y|x)f(x) = f(x,y)}$

② $f(y) = \int_{\mathbb{R}} f(x,y)\,dx$

$= \int_{\mathbb{R}} f(y|x)f(x)\,dx$