Lecture 6 - Conditional Probability

Tuesday, September 21, 2021 1:53 PM

Ex. Survey W&M students and ask about political affiliation.

gender	A	B	total
men	501	238	739
women	782	123	905
		361	1644

Q1: If I randomly select a student, what is the prob they are a woman?

$P(\text{woman}) = 905/1644 \approx 55\%$

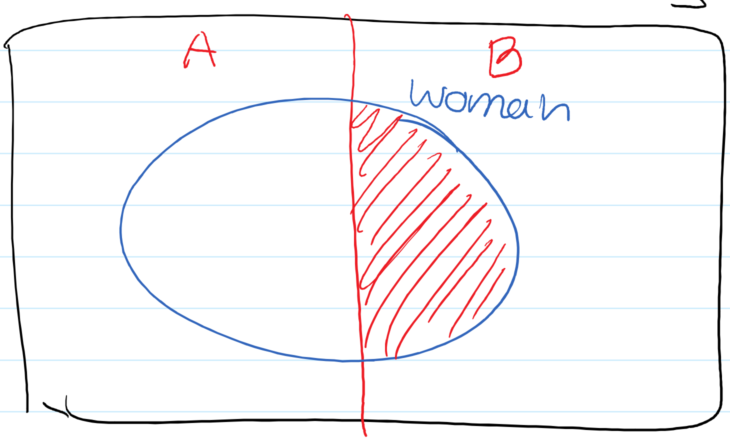
Q2: Given that a student is a member of party B, what is the prob. they are a woman?

conditioning

$P(\text{woman GIVEN in party B}) = \frac{123}{361}$

$\approx 34\%$.

Venn Diagram



Q1: $$P(\text{woman}) = \frac{\text{area}(\text{woman})}{\text{area}\,\boxed{S}}$$

Q2: $P(\text{woman GIVEN in } B)$

$$= \frac{\text{area of}}{\text{area of } \boxed{B}}$$

$$= \frac{\text{area of } (\text{woman}) \cap \boxed{B}}{\text{area of } \boxed{B}}$$

Defn: Conditional Probability

If $A, B \subset S$ and $P(B) > 0$ then the conditional prob. of A given B is

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

read

Facts: Assume $P(B) > 0$.

① $\boxed{P(B|B) = 1}$

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pf: $P(B|B) = \dfrac{P(BB)}{P(B)} = \dfrac{P(B)}{P(B)} = 1.$

② $\boxed{\text{If } AB = \emptyset \text{ then } P(A|B) = 0}$

pf. $P(A|B) = \dfrac{P(AB)}{P(B)} = \dfrac{P(\emptyset)}{P(B)} = \dfrac{0}{P(B)} = 0.$

Ex. Roll two dice.

Q: What is the prob. the first roll is a 2 given the sum is ≤ 5.

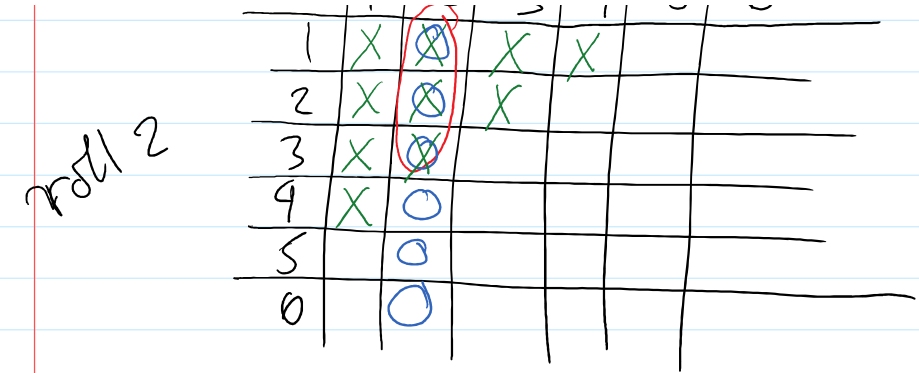
$\underbrace{\text{first roll is a 2}}_{A}$ $\quad$ $\underbrace{\text{the sum is} \le 5}_{B}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{|AB|/|S|}{|B|/|S|} = \frac{|AB|}{|B|} = \frac{3}{10}$$

roll 1

	1	2	3	4	5	6
1	X	ⓧ	X	X		
2	X	ⓧ	X			

A denote w/ O



A denote w/ O

B denote w/ X

Theorem: Compound Probability

$$\mathbb{P}(AB) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A).$$

Pf. $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(AB)}{\mathbb{P}(B)}$

and rearrange ...

Recall: if (A_i) partition S then

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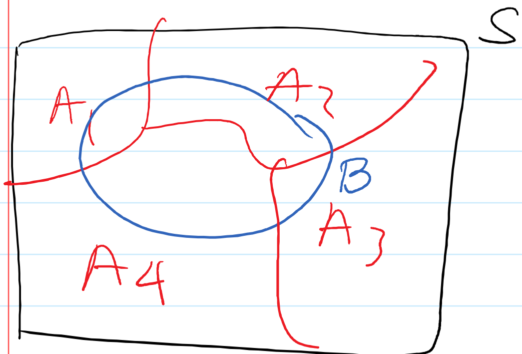
$\boxed{\text{lect } 4}$

$$P(B) = \sum_i P(BA_i).$$

Theorem: Law of Total Probability

If (A_i) partition S and $P(A_i) > 0$ then for any $B \subset S$

$$P(B) = \sum_i P(B|A_i)P(A_i).$$



Pf. Prev. partitioning theorem says

$$P(B) = \sum_i P(BA_i)$$

apply compound prob rule

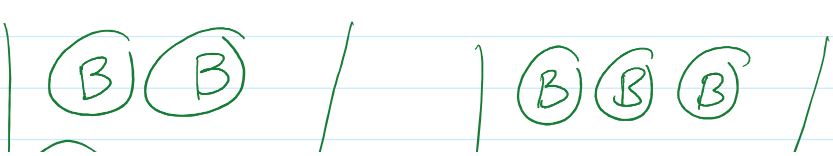
$$= \sum_i P(B|A_i)P(A_i).$$

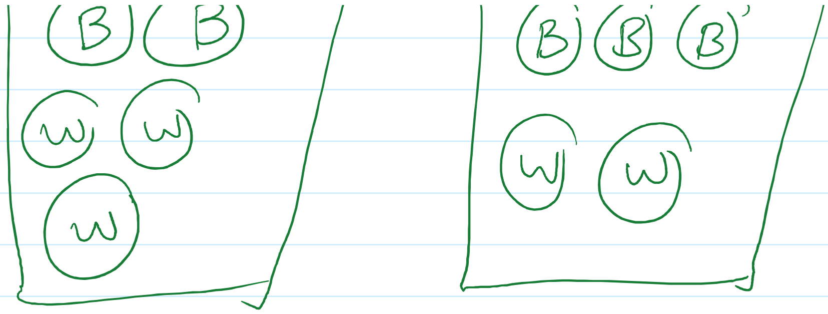
Note: the event A and A^c partition S.

In this case the Law says

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c).$$

Ex. Basket 1 Basket 2





Game: ① randomly select marble from basket 1 and place in basket 2

② randomly select marble from basket 2

Q: what is the prob I select a black marble on step 2?

let W = choose white on step 1

W^c " black "

B = choose black on step 2

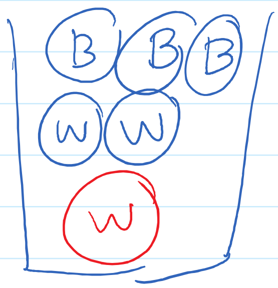
B^c = " white "

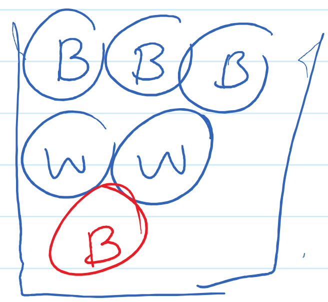
Want: $P(B)$. Use Law of Tot. prob. by partitioning/conditioning on W and W^c

$$P(B) = \underbrace{P(B|W)}_{(1/2)}\underbrace{P(W)}_{(3/5)} + \underbrace{P(B|W^c)}_{(2/3)}\underbrace{P(W^c)}_{(2/5)}$$

$(\ \ )(\ \ ) + (\ /3)(\ /5)$ $= 17/30$

$P(B|W) = 1/2$ | $P(B|W^c)$





Theorem: Bayes' Theorem

Way to calc. $P(A|B)$ from $P(B|A)$?

If $A, B \subset S$ and $P(A) > 0$, $P(B) > 0$. Then

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}$$

Pf. $P(A|B) = \dfrac{P(AB)}{P(B)} = \dfrac{P(B|A)P(A)}{P(B)}$.

$P(AB) = P(BA) = P(B|A)P(A)$ ← compound prob.

Ex. Continue previous.

Q: Given I choose a black marble on the second step, what is the prob I chose a white on first.

Bayes' rule:

$$P(W|B) = \frac{P(B|W)\,P(W)}{P(B)}$$

$$= \frac{(1/2)(3/5)}{(17/30)}$$

Theorem: Law of Total Prob + Bayes'

If (A_i) are partition of S and $P(A_i) > 0$, then

$$P(A_i|B) = \frac{P(B|A_i)\,P(A_i)}{\sum_j P(B|A_j)\,P(A_j)}$$

Pf. By Bayes'

$$P(A_i|B) = \frac{P(B|A_i)\,P(A_i)}{P(B)}$$

expand w/ Law of Tot Prob

$$= \frac{P(B|A_i)\,P(A_i)}{}$$

$\sum_j P(B|A_j)P(A_j)$

Note: A and A^c partition S so for these two events

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}}$$

Ex. Covid has a prevalence rate of 1%.

$P(D) = .01$; $P(D^c) = .99$

We test for COVID and get either a + or −.

→ The test accurately reports a + 95% of time (sensitivity) $P(+|D) = .95$

→ the test accurately reports a − 99% of time. (specificity) $P(-|D^c) = .99$

Q! If I get a + test, what is the prob. I have COVID?

Let D = I have COVID, D^c = I don't

$+$ = pos. test, $-$ = neg. test.

$P(+|D)P(D)$

$$\mathbb{P}(D \mid +) = \frac{\qquad}{\mathbb{P}(+ \mid D)\mathbb{P}(D) + \mathbb{P}(+ \mid D^c)\mathbb{P}(D^c)}$$

$$\mathbb{P}(+ \mid D^c) = 1 - \mathbb{P}(- \mid D^c)$$

$$= \frac{(.95)(.01)}{(.95)(.01) + (1-.99)(.99)}$$

$$\cong .49$$