Lecture 12 - Variance and Moments

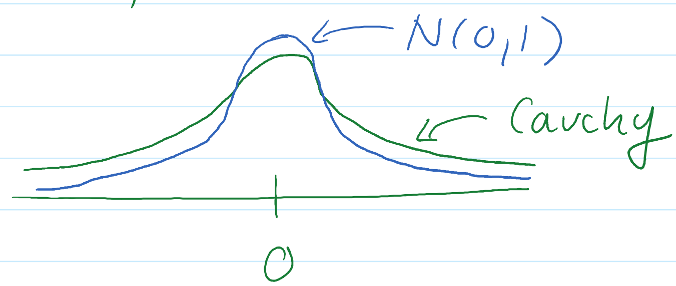
Thursday, October 14, 2021 9:28 AM

Expected value: $\mathbb{E}[X] = \begin{cases} \int x f(x)\,dx & \text{cts} \\ \sum x f(x) & \text{discrete} \end{cases}$

Q: Does expected value always exist? No.

Ex. Cauchy Distribution

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad \text{for all } x \in \mathbb{R}$$



$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\,dx = \infty$$

looks like $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2}\,dx = \int_{-\infty}^{\infty} \frac{1}{x}\,dx = \infty$

Facts from calc II:

$$\sum_k 1/k^2 < \infty \quad \text{but} \quad \sum_k 1/k = \infty$$

$$\int_0^{\infty} 1/x^2\,dx < \infty \quad \text{but} \quad \int_0^{\infty} 1/x\,dx = \infty$$

Theorem: Properties of Expectation

(1) Expectation is linear

$$E[aX+b] = a\,\mathbb{E}[X] + b$$

pf. (cts)

$$\mathbb{E}[aX+b] = \int_{\mathbb{R}} (ax+b) f(x)\,dx = \int [a x f(x) + b f(x)]\,dx$$

$$= \int a x f(x)\,dx + \int b f(x)\,dx$$

$$= a \underbrace{\int x f(x)\,dx}_{\mathbb{E}[X]} + b \underbrace{\int f(x)\,dx}_{1}$$

$$= a\,\mathbb{E}[X] + b$$

(2) If $X \geq 0$ then $E[X] \geq 0$.

support $\subseteq [0, \infty)$

pf. (cts)

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\infty} \underbrace{x}_{\geq 0} \underbrace{f(x)}_{\geq 0}\,dx \geq 0$$

(3) If g_1 and g_2 are functions then

(i) $\mathbb{E}[g_1(X) + g_2(X)] = \mathbb{E}[g_1(X)] + \mathbb{E}[g_2(X)]$

(ii) if $g_1(x) \le g_2(x)$ then

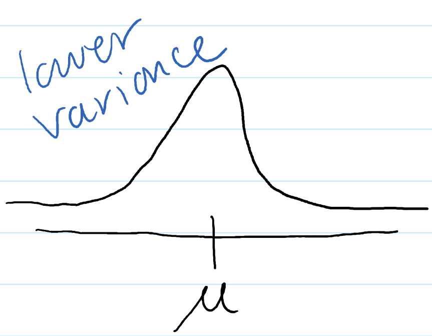
$$\mathbb{E}[g_1(X)] \le \mathbb{E}[g_2(X)]$$

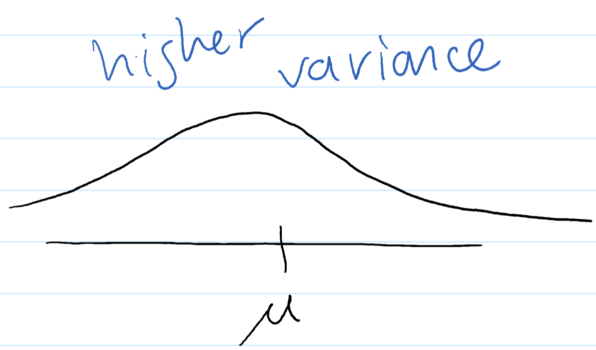
Pf. Use (1), (2), Law of Unconscious Stat.

(4) If $a \le X \le b$ then $a \le \mathbb{E}[X] \le b$.

Pf. apply (2) twice

Defn: Variance $\quad \mu = \mathbb{E}[X] \in \mathbb{R}$ = mean



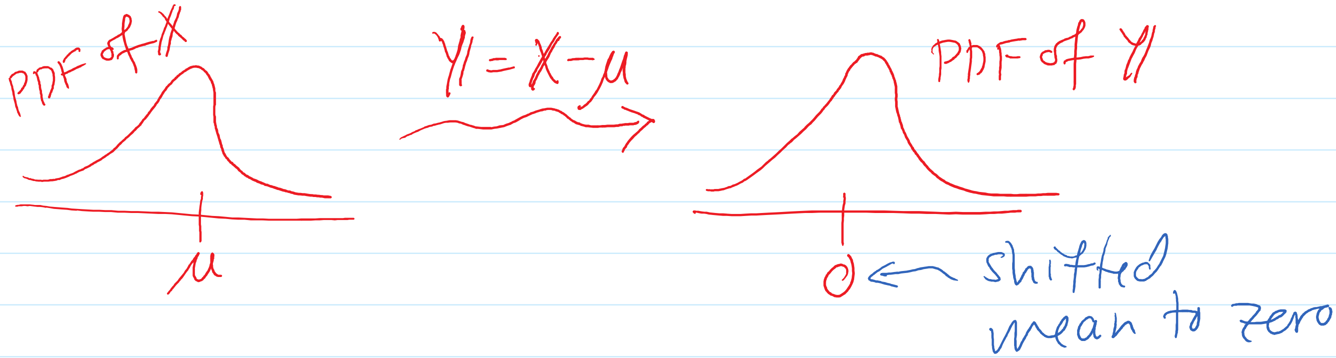


Variance = how spread out values around mean

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2]$$

$$= \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$Y = X - \mu$



$$\mathbb{E}[Y] = \mathbb{E}[X - \mu] = \mathbb{E}[X] - \mu = \mu - \mu = \underline{0}$$

$$\mathrm{Var}(X) = \mathbb{E}[Y^2]$$

Ex. $X \sim \mathrm{Exp}(\lambda)$

Recall: $f(x) = \lambda e^{-\lambda x}$ for $x > 0$

$\mu = \mathbb{E}[X] = 1/\lambda$ and $\mathbb{E}[X^2] = 2/\lambda^2$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[(X-\mu)^2] \\
&= \mathbb{E}[X^2 - 2\mu X + \mu^2] \\
&= \underbrace{\mathbb{E}[X^2]}_{2/\lambda^2} - 2\mu \underbrace{\mathbb{E}[X]}_{1/\lambda} + \mu^2 \\
&= 2/\lambda^2 - 2\,\tfrac{1}{\lambda}\,\tfrac{1}{\lambda} + \left(\tfrac{1}{\lambda}\right)^2
\end{aligned}$$

$$= \cancel{2/\lambda^2} - \cancel{2\,\tfrac{1}{\lambda}\tfrac{1}{\lambda}} + \left(\tfrac{1}{\lambda}\right)^2$$

$$\mathrm{Var}(X) = 1/\lambda^2$$

Notice: only needed to know $\mathbb{E}[X]$, $\mathbb{E}[X^2]$.

Theorem: Short-cut Formula For Variance

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

expected square − sq. of expectation

Pf. $\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$

$$= \mathbb{E}\left[X^2 - 2X\,\mathbb{E}[X] + \mathbb{E}[X]^2\right]$$

$$= \mathbb{E}[X^2] - \underline{2\mathbb{E}[X]\mathbb{E}[X]} + \underline{\mathbb{E}[X]^2}$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Ex. $X \sim \mathrm{Exp}(\lambda)$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}$$

Theorem:

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

① multiply by const ⟿ multiply var by sq

② ignore additive constants

Pf. $\mathrm{Var}(aX+b) = \mathbb{E}[(aX+b)^2] - \mathbb{E}[aX+b]^2$

$$= \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\,\mathbb{E}[X] + b)^2$$

$$= a^2\mathbb{E}[X^2] + \cancel{2ab\,\mathbb{E}[X]} + \cancel{b^2} - (a^2\mathbb{E}[X]^2 + \cancel{2ab\,\mathbb{E}[X]} + \cancel{b^2})$$

$$= a^2\mathbb{E}[X^2] - a^2\mathbb{E}[X]^2$$

$$= a^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2)$$

$$= a^2\,\mathrm{Var}(X)$$

Ex. $X \sim Bin(n, p)$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\mathbb{E}[X] = np$$

trick 1: $x\binom{n}{x} = n\binom{n-1}{x-1}$

trick 2: $y = x - 1$

$x = y + 1$

$$\mathbb{E}[X^2] = \sum_{x=0}^{n} x^2 f(x)$$

$$= \sum_{x=1}^{n} \underline{x^2\binom{n}{x}} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x n \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$= \sum_{y=0}^{n-1} (y+1) \underline{n} \binom{n-1}{y} p^{y+1} (1-p)^{(n-1)-y}$$

$$= np \sum_{y=0}^{n-1} (y+1) \binom{n-1}{y} p^y (1-p)^{(n-1)-y}$$

$$= np \left[\underbrace{\sum_{y=0}^{n-1} y \binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{\mathbb{E}[Bin(n-1,p)] = (n-1)p} + \underbrace{\sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{\text{Sum of } Bin(n-1,p) \text{ PMF} = 1} \right]$$

$$\mathbb{E}[X^2] = np\left((n-1)p + 1\right)$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= np((n-1)p+1) - (np)^2$$

$$np(np - p + 1) - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= -np^2 + np = \boxed{np(1-p)} = Var(X)$$

Defn: Standard Deviation $sd(X) = \sqrt{Var(X)}$

for Binomial $sd(X) = \sqrt{np(1-p)}$

Defn: Moments of a RV

If r is a pos. integer then the r^{th} moment of X is defined as

$$\mu_r = E[X^r]$$

Ex.

$$\mu_1 = \mu = E[X]$$

$$\mu_2 = E[X^2]$$

$\mu_3 = \mathbb{E}[X^3]$
$\vdots$

<u>Defn: Moment Generating Function (MGF)</u>

If X is a RV then the MGF of X is a function

$$M : \mathbb{R} \to \mathbb{R}$$

defined for $t \in \mathbb{R}$ as

$$M(t) = \mathbb{E}[e^{tX}]$$

$$\mathbb{E}[g(X)] = \begin{cases} \int g(x) f(x)\, dx \\ \sum g(x) f(x) \end{cases}$$

In our case $g(X) = e^{tX}$

For discrete:

$$M(t) = \mathbb{E}[e^{tX}] = \sum_x e^{tx} f(x)$$

For continuous:

$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\, dx$$

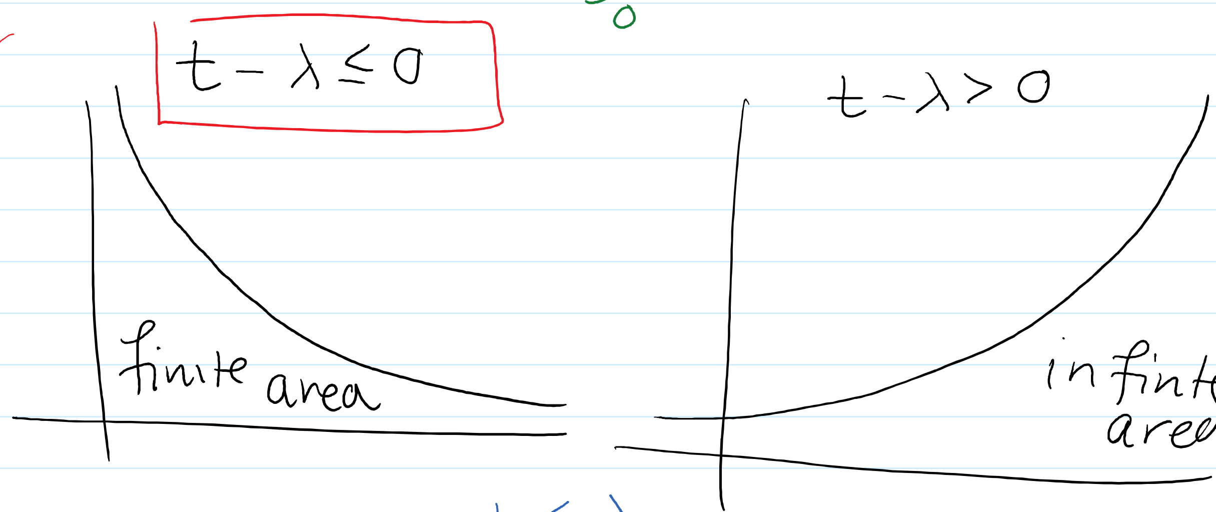
<u>Ex.</u> $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$

$$M(t) = \mathbb{E}[e^{tX}] = \int e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \int_0^\infty \lambda e^{(t-\lambda)x}\,dx$$



If $t - \lambda \le 0 \longrightarrow t \le \lambda$

$$= \lambda \int_0^\infty e^{(t-\lambda)x}\,dx = \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^\infty = \frac{\lambda}{t-\lambda}[0-1]$$

$$\boxed{M(t) = \frac{\lambda}{\lambda - t} \quad \text{for } t \le \lambda}$$

Recall: $\mathbb{E}[X] = 1/\lambda$, $\mathbb{E}[X^2] = 2/\lambda^2$

consider :

Consider:

(1) $$\frac{dM}{dt} = \cdots = \frac{\lambda}{(\lambda - t)^2} \Rightarrow \left.\frac{dM}{dt}\right|_{t=0} = \frac{\lambda}{\lambda^2} = \boxed{\frac{1}{\lambda}}$$

(2) $$\frac{d^2M}{dt^2} = \cdots = \frac{2\lambda}{(\lambda - t)^3} \Rightarrow \left.\frac{d^2M}{dt^2}\right|_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$