Lecture 15 - Even More Common Distributions

Thursday, October 28, 2021 9:32 AM

$\Gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$

1. If a is an integer then

$$\Gamma(a) = (a-1)! \iff \Gamma(a+1) = a!$$

Ex,

$\Gamma(1) = 0! = 1$

$\Gamma(2) = 1! = 1$

$\Gamma(3) = 2! = 2$

$\vdots$

2. notice: for integer x, $x! = x(x-1)!$

For Γ we have

$$\Gamma(a+1) = a\Gamma(a) \quad \forall a > 0$$

if a is an integer

$$a! = a(a-1)!$$

equiv. $\Gamma(a) = (a-1)\Gamma(a-1)$

Important facts about Γ

1. $\Gamma(a+1) = a!$ for integer a
2. $\Gamma(a+1) = a\Gamma(a)$

Gamma Distribution: generalize exponential

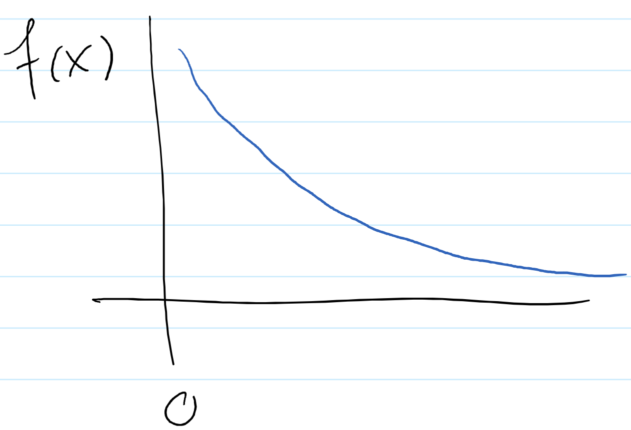
$$X \sim \text{Gamma}(a, \lambda)$$

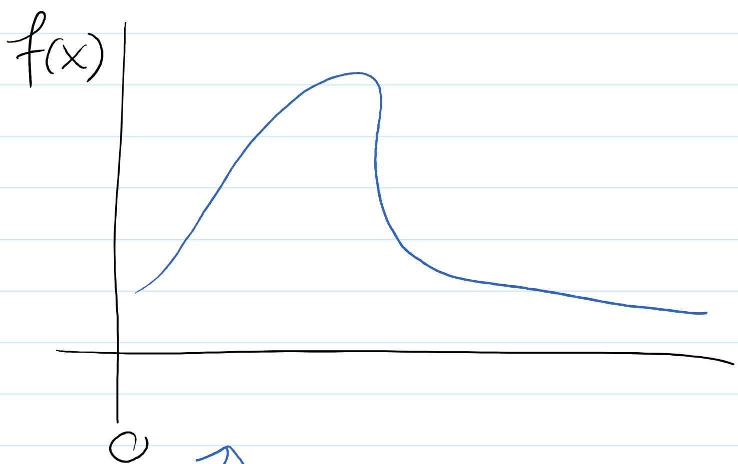
shape (a), rate (λ)

PDF:

$$\boxed{f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)} \quad \text{for } x > 0}$$

notice if $a = 1$ then $X \sim \text{Exp}(\lambda)$.





different shapes a

Expected:

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^\infty x \frac{\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)}\,dx$$

$$= \underbrace{\frac{1}{\lambda}\frac{\Gamma(a+1)}{\Gamma(a)}}\int_0^\infty \underbrace{\frac{\lambda e^{-\lambda x}(\lambda x)^a}{\Gamma(a+1)}}_{1} dx$$

$$\int \underbrace{\text{PDF Gamma}(a+1, \lambda)}_{1}$$

$$\frac{\lambda e^{-\lambda x}(\lambda x)^{(a+1)-1}}{\Gamma(a+1)}$$

$$= \frac{1}{\lambda}\frac{\Gamma(a+1)}{\Gamma(a)}$$

$$= \frac{1}{\lambda}\frac{a\Gamma(a)}{\Gamma(a)} = \boxed{\frac{a}{\lambda}}$$

notice if $a = 1$ then $EX = 1/\lambda$

Moments:

$$E[X^r] = \int_0^\infty x^r \frac{\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)} dx$$

$$= \frac{\Gamma(a+r)}{\Gamma(a)}\frac{1}{\lambda^r}\int_0^\infty \underbrace{\frac{\lambda e^{-\lambda x} x^{a+r-1} \lambda^{a-1} \lambda^r}{\Gamma(a+r)}}_{1} dx$$

$$\int \underbrace{\text{Gamma}(a+r, \lambda)}_{1}$$

$$\frac{\lambda e^{-\lambda x}(\lambda x)^{(a+r)-1}}{\Gamma(a+r)}$$

$$= \boxed{\frac{\Gamma(a+r)}{\Gamma(a)}\frac{1}{\lambda^r}}$$

$r = 1$ $\quad E[X] = \frac{a}{\ldots}$

$\underline{r=1}$ $\quad E[X^1] = \frac{a}{\lambda}$

$\underline{r=2}$ $\quad E[X^2] = \frac{\Gamma(a+2)}{\Gamma(a)} \frac{1}{\lambda^2} = \frac{(a+1)\Gamma(a+1)}{\Gamma(a)} \frac{1}{\lambda^2}$

$$= \frac{(a+1)\,a\,\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}} \frac{1}{\lambda^2}$$

$$= \frac{(a+1)a}{\lambda^2}$$

$$\mathrm{Var}(X) = \frac{(a+1)a}{\lambda^2} - \left(\frac{a}{\lambda}\right)^2 = \boxed{\frac{a}{\lambda^2}}$$

If $a=1$ the $\mathrm{Var}(X) = 1/\lambda^2$

Geometric Distribution (lecture 9)

Canonical experiment:

If I flip coins independently each w/ a prob $p \in [0,1]$ of a H.

If X = # flips to get first H

$$X \sim \text{Geometric}(p)$$

outcome	X
H	1
TH	2
TTH	3
⋮	⋮

PMF: $f(x) = (1-p)^{x-1} p$ for $x = 1, 2, 3, \ldots$

CDF: $F(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor} & \text{for } x \geq 1 \\ 0 & \text{else} \end{cases}$

lec 9

Geometric series

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

for $|r| < 1$

Expectation:

$$E[X] = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$$x(1-p)^{x-1} = -\frac{d}{dp}(1-p)^x$$

$$= -p \sum_{x=1}^{\infty} \frac{d}{dp}(1-p)^x$$

$$= -p \frac{d}{dp} \sum_{x=1}^{\infty} (1-p)^x$$

$$= -p \frac{d}{dp} \sum_{x=0}^{\infty} (1-p)^{x+1}$$

$$= -p \frac{d}{dp}\left[(1-p)\underbrace{\sum_{x=0}^{\infty} (1-p)^{x}}_{\text{geometric series w/ } r=1-p}\right]$$

$$= -p \frac{d}{dp}\left[(1-p)\frac{1}{1-(1-p)}\right]$$

$$= -p \frac{d}{dp}\left[\frac{1-p}{p}\right] = -p\left(-\frac{1}{p^2}\right) = \boxed{\frac{1}{p} = E[X]}$$

MGF:

$$M(t) = E\left[e^{tX}\right] = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p$$

$$= \frac{p}{1-p}\sum_{x=1}^{\infty} e^{tx}(1-p)^{x}$$

$$= \frac{p}{1-p}\sum_{x=1}^{\infty} \big(\underbrace{(1-p)e^{t}}_{r}\big)^{x}$$

$$= \frac{p}{1-p}\sum_{x=0}^{\infty} \left((1-p)e^{t}\right)^{x+1}$$

$$= \frac{p}{\cancel{1-p}}\cancel{(1-p)}e^{t}\underbrace{\sum_{x=0}^{\infty}\left((1-p)e^{t}\right)^{x}}_{\text{Geometric series}}$$

$x=0$

geometric series

$$\boxed{= \frac{pe^t}{1-(1-p)e^t} = M(t)}$$

$$E[X^2] = \frac{d^2M}{dt^2}\Big|_{t=0} = \cdots = \frac{2-p}{p^2}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$$

$$\boxed{= \frac{1-p}{p^2}}$$

Beta Distribution continuous RV

Beta function: $a, b \in \mathbb{R}^+$

$$B(a,b) = \int_0^1 \underbrace{x^{a-1}(1-x)^{b-1}} dx$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

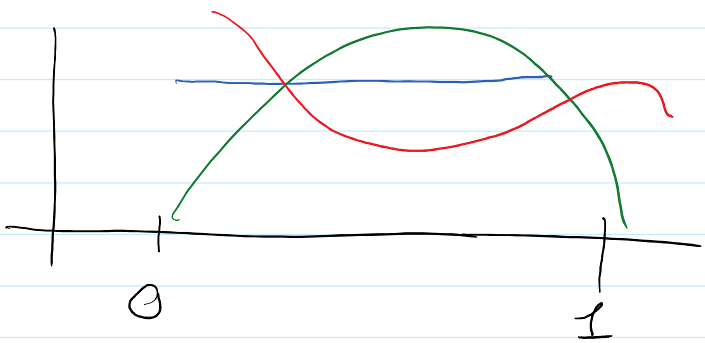
$\Gamma(a+b) \quad (a+b)!$

$$\binom{a+b}{a} = \frac{(a+b)!}{a!\,b!}$$

$X \sim \text{Beta}(a, b)$

PDF:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } 0<x<1$$



$$E[X] = \int_0^1 x f(x)\,dx = \int_0^1 \frac{x\, x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+1,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}\,dx}_{1}$$

$$\underbrace{c\int \text{Beta}(a+1, b)}_{1}$$

$$\frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}$$

$= B(a+1, b)$

$$\frac{x \qquad (1-x)}{B(a+1,b)}$$

$$= \frac{B(a+1,b)}{B(a,b)}$$

$$= \frac{\dfrac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+1+b)}}{\dfrac{\Gamma(a)\cancel{\Gamma(b)}}{\Gamma(a+b)}} = \frac{\Gamma(a+1)\,\Gamma(a+b)}{\Gamma(a)\,\Gamma(a+b+1)}$$

$$= \frac{a\cancel{\Gamma(a)}\;\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}(a+b)\;\cancel{\Gamma(a+b)}}$$

$$E[X] \boxed{= \frac{a}{a+b}}$$

$$E[X^r] = \int_0^1 x^r \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$\frac{B(a+r,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{(a+r)-1}(1-x)^{b-1}\,dx}{B(a+r,b)}}_{1}$$

$$c\underbrace{\int \text{Beta}(a+r,b)}_{1}$$

$$\frac{x^{(a+r)-1}(1-x)^{b-1}}{B(a+r,b)}$$

$$= \frac{B(a+r,b)}{\quad} = E[X^r]$$

$$= \boxed{\frac{B(a+r,b)}{B(a,b)} = \mathbb{E}[X^r]}$$

$r=2$

$$\mathbb{E}[X^2] = \frac{B(a+2,b)}{B(a,b)} = \frac{\frac{\Gamma(a+2)\cancel{\Gamma(b)}}{\Gamma(a+2+b)}}{\frac{\Gamma(a)\cancel{\Gamma(b)}}{\Gamma(a+b)}}$$

$$= \frac{\Gamma(a+2)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+2)}$$

$$= \frac{(a+1)a\cancel{\Gamma(a)}\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}(a+b+1)(a+b)\cancel{\Gamma(a+b)}}$$

$$= \boxed{\frac{a(a+1)}{(a+b)(a+b+1)} = \mathbb{E}[X^2]}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \ldots \text{algebra} \ldots$$

$$\boxed{= \frac{ab}{(a+b+1)(a+b)^2}}$$

$$\boxed{= \frac{ab}{(a+b+1)(a+b)^2}}$$