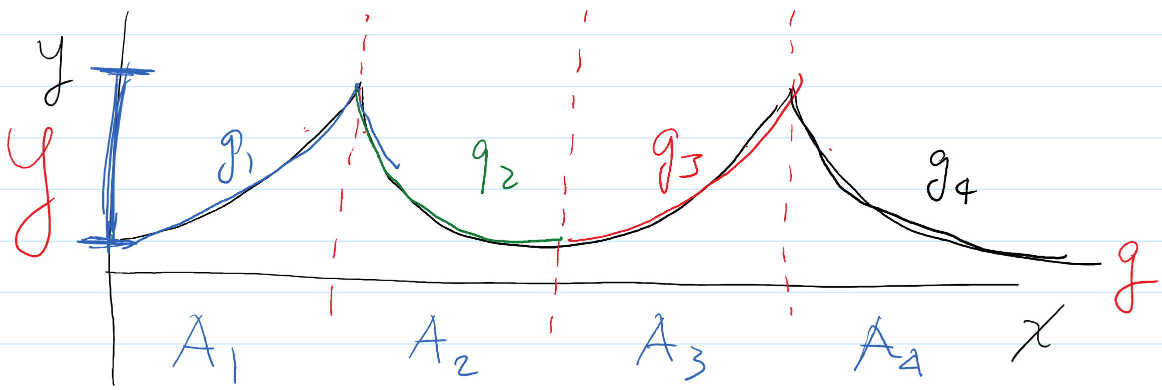
Lecture 17 - More Transformations and Bivariate RVs

Thursday, November 4, 2021 9:29 AM

If X continuous and $Y = g(X)$ and
① g invertible
② g^{-1} differentiable

then
$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$



If X is continuous w/ support $\mathcal{X}$
and $(A_i)_{i=1}^{K}$ partition $\mathcal{X}$
and $Y = g(X)$ so that
g_i is g restricted to A_i
and
① our prev. theorem applies to each g_i on A_i
[g_i invertible, g_i^{-1} differentiable]
② the image of A_i under g_i is

Same for all $i=1,\dots,K$

(call this y)

then $$\boxed{f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}}{dy} \right|}$$

Ex. <u>Chi-squared Distribution</u>

If $X \sim N(0,1)$ and $Y = X^2$

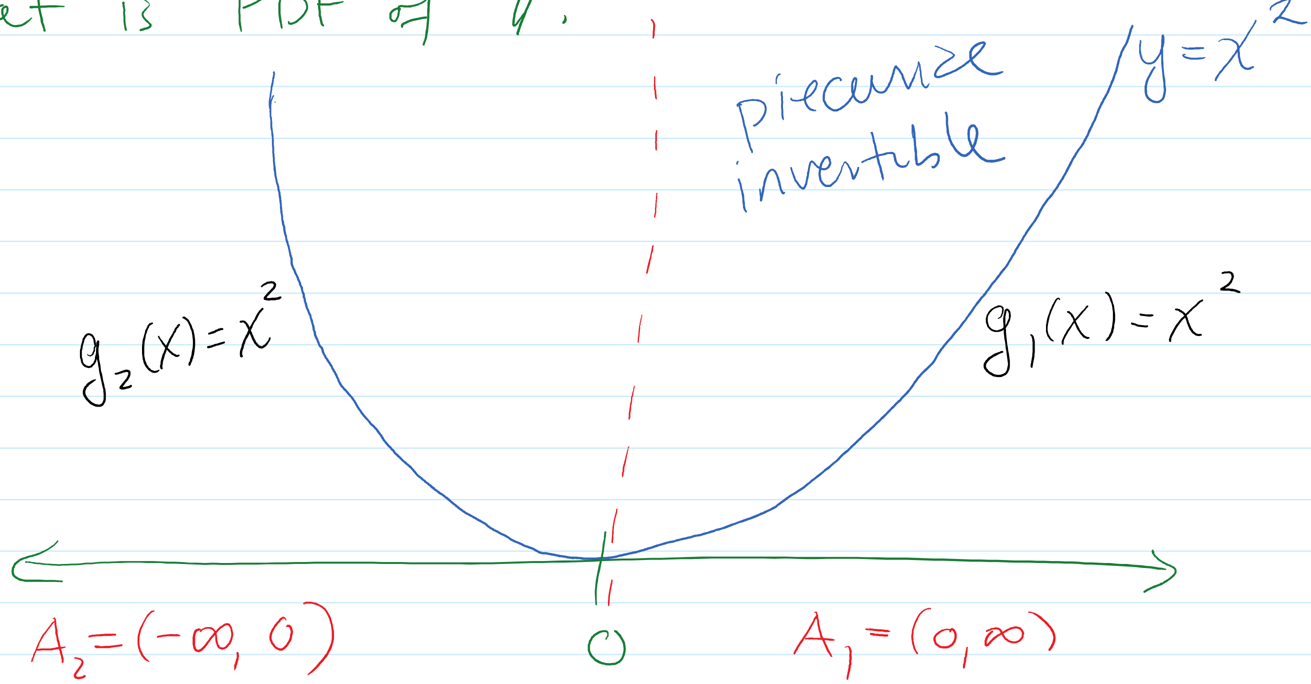
then we say Y has a chi-sq. dist.

w/ one degree of freedom.

$$Y \sim \chi^2(1)$$

$y = g(x) = x^2$

what is PDF of Y?



$$
\begin{cases}
A_1 = (0,\infty), \quad g_1(x) = x^2, \quad g^{-1}(y) = \sqrt{y}; \quad \dfrac{dg_1^{-1}}{dy} = \dfrac{1}{2\sqrt{y}} \\[2ex]
A_2 = (-\infty, 0), \quad g_2(x) = x^2; \quad g^{-1}(y) = -\sqrt{y}; \quad \dfrac{dg_2^{-1}}{dy} = -\dfrac{1}{2\sqrt{y}}
\end{cases}
$$

$$
f_Y(y) = f_X(g_1^{-1}(y)) \left| \frac{dg_1^{-1}}{dy} \right| + f_X(g_2^{-1}(y)) \left| \frac{dg_2^{-1}}{dy} \right|
$$

$$
f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2)
$$

$$
= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}}
$$

$$
= \frac{1}{\sqrt{2\pi}} \exp(-\sqrt{y}^2) \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} \exp(-(-\sqrt{y})^2) \frac{1}{2\sqrt{y}}
$$

$$
= \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} \left(e^{-y} + e^{-y} \right)
$$

$$
\boxed{= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y}}
$$

PDF of $\chi^2(1)$

Probability Integral Transformation

If X is continuous w/ CDF F_X then

$$F_X(X) \sim U(0,1)$$

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pf Assume that F_X is strictly increasing

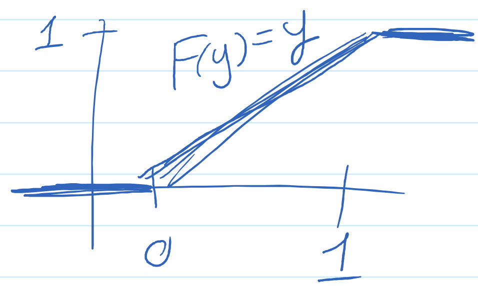
Let $Y = g(X)$ when $g = F_X$

Our CDF theorem says

$$F_Y(y) = F_X(g^{-1}(y))$$

$$= F_X(F_X^{-1}(y)) = y$$

CDF of $U(0,1)$



So F_Y is the CDF of a $U(0,1)$

hence $Y \sim U(0,1)$.

$$g(X) \sim U(0,1) \iff g = F_X$$

Generate Random Numbers on Computer

(1) $Y \sim U(0,1)$

② Let $X = F_X^{-1}(Y)$

then X has CDF F_X.

<u>Ex.</u> Want $X \sim Exp(1)$

$$F_X(x) = 1 - e^{-x} = y$$

$$\Rightarrow \quad y - 1 = -e^{-x}$$

$$\Rightarrow \quad 1 - y = e^{-x}$$

$$\Rightarrow \quad \log(1-y) = -x$$

$$\Rightarrow \quad -\log(1-y) = x$$

$$F_X^{-1}(y) = -\log(1-y)$$

<u>Bivariate RVs</u>

If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$

then

$Z = (X, Y)$ is called a bivariate RV

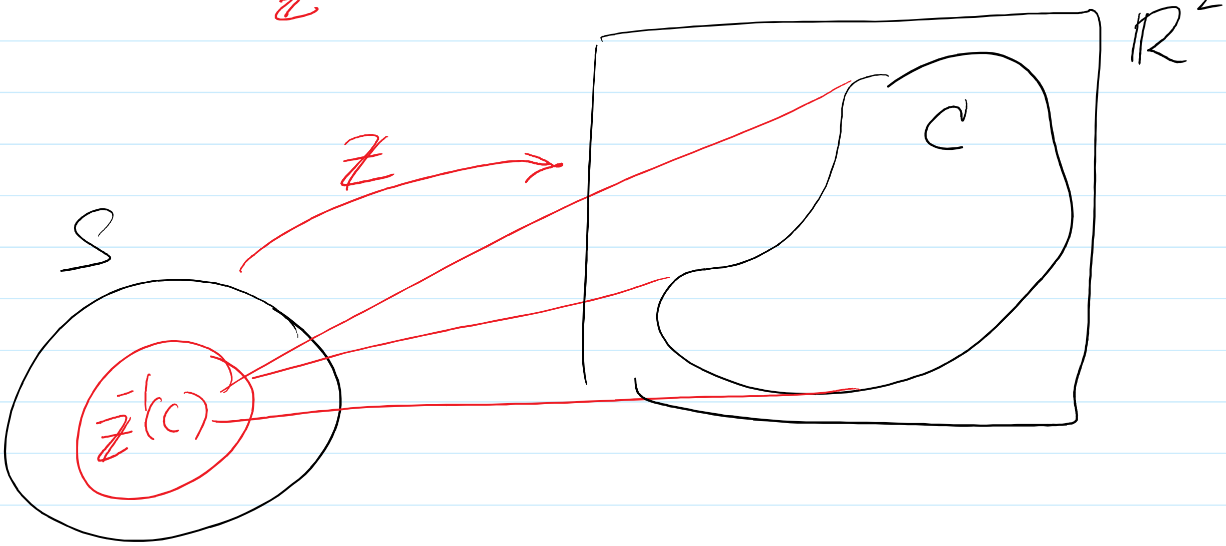
So

$$Z : S \to \mathbb{R}^2$$

$C \subset \mathbb{R}^2$

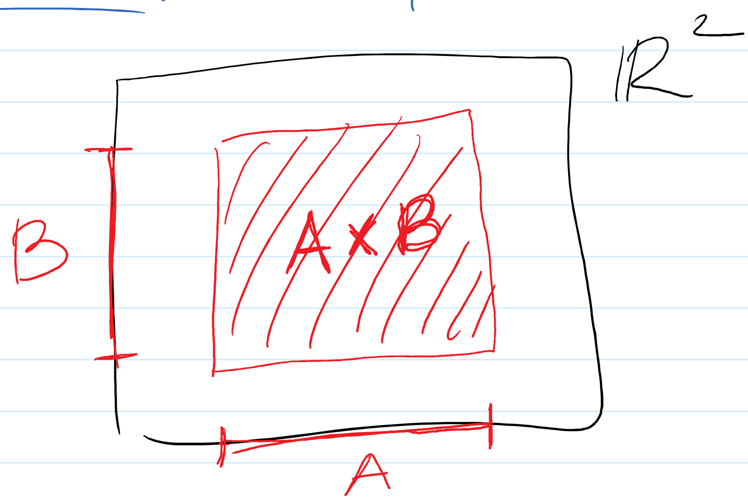
Say: $\mathbb{P}(\underbrace{(X, Y)}_{Z} \in C) = \mathbb{P}(Z^{-1}(C))$

$\mathbb{R}^2$



often $C = A \times B$ where $A, B \subset \mathbb{R}$

Could say



$\mathbb{P}((X,Y) \in C)$

often lazy and write

$\mathbb{P}(X \in A, Y \in B)$

"and"

Ex. Flip a coin 3 times.

$$X = \begin{cases} 0 & \text{if last flip is } T \\ 1 & \text{'' } \quad H \end{cases}$$

$\mathbb{Y}$ = # heads among 3 flips

then $\mathbb{Z} = (\mathbb{X}, \mathbb{Y})$

$s \in S$	$\mathbb{Z}(s)$
H H H	(1, 3)
H H T	(0, 2)
H T H	(1, 2)
H T T	(0, 1)
T H H	(1, 2)
T H T	(0, 1)
T T H	(1, 2)
T T T	(0, 0)

Defn: Biv CDF

The Biv CDF is a function

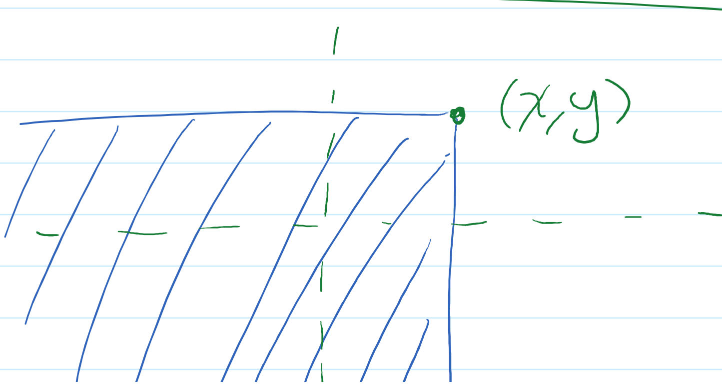
$$F : \mathbb{R}^2 \to \mathbb{R}$$

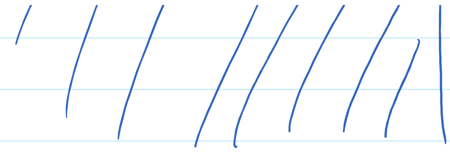
so that for $x, y \in \mathbb{R}$

$$F(x,y) = \mathbb{P}(\mathbb{X} \le x, \mathbb{Y} \le y)$$

Univ:

$F(x) = \mathbb{P}(\mathbb{X} \le x)$





Properties of Biv CDF

1. $F(x,y) \geq 0$
2. $\lim_{x,y \to \infty} F(x,y) = 1$ (uni: $\lim_{x \to \infty} F(x) = 1$)
3. $\lim_{x \to -\infty} F(x,y) = 0$ (uni: $\lim_{x \to -\infty} F(x) = 0$)

   $\lim_{y \to -\infty} F(x,y) = 0$
4. F is non-decreasing and right-cts in each argument

Defn: Marginal Distributions

If (X, Y) is a Biv RV then we say X and Y are the marginal RVs and their dists are called marginal dists.

Theorem: Rel. btwn Biv / Marginal CDFs

① $F_Y(y) = \lim_{x \to \infty} F(x, y)$

② $F_X(x) = \lim_{y \to \infty} F(x, y)$

pf. $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(X \le x, Y = \text{anything})$

$$= \mathbb{P}(X \le x, Y \le \infty)$$

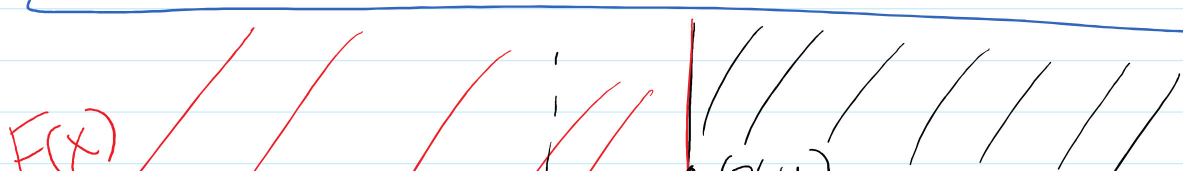
$$= \lim_{y \to \infty} \underbrace{\mathbb{P}(X \le x, Y \le y)}$$

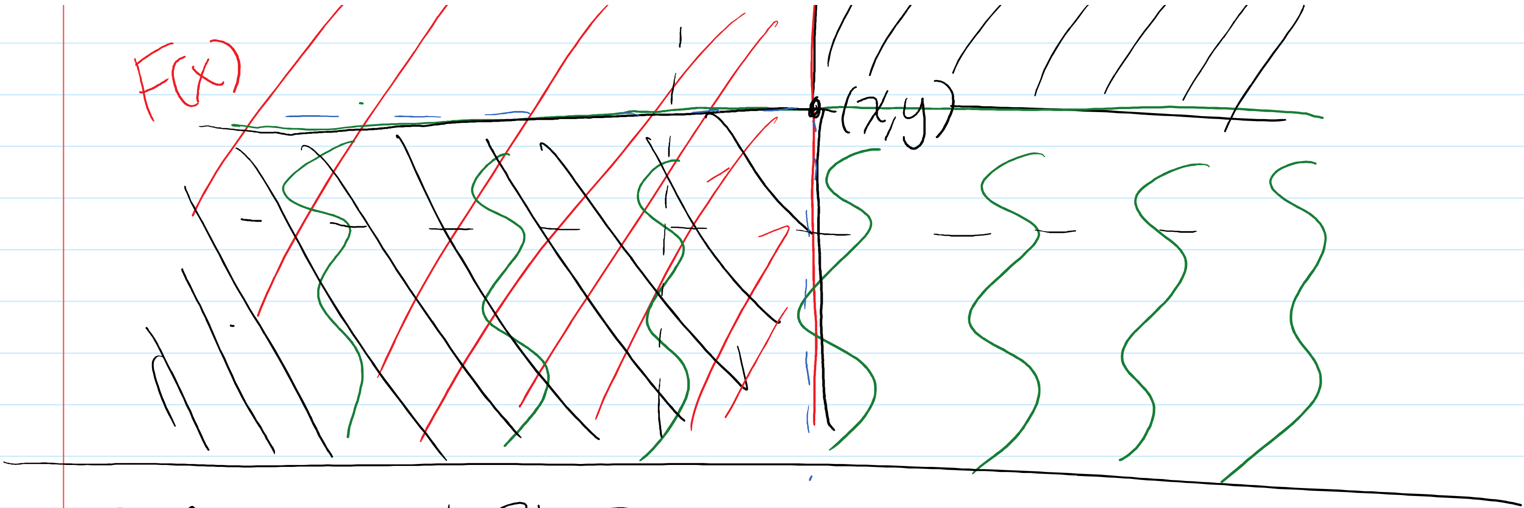
$$= \lim_{y \to \infty} F(x, y)$$

Lemma: $\left(\text{Univ: } \mathbb{P}(X > x) = 1 - F(x) \right)$

For Biv RVs:

$$\boxed{\mathbb{P}(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F(x, y)}$$





Defn: Joint PMF

If X and Y are discrete then the joint PMF is

$$f(x,y) = \mathbb{P}(X=x, Y=y)$$

theorem: f is a valid joint PMF if

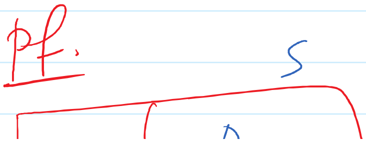
(1) $f(x,y) \geq 0$, (2) $\sum_x \sum_y f(x,y) = 1$

Theorem: Rel. btw joint/marginal PMFs

(1) $f_X(x) = \sum_y f(x,y)$

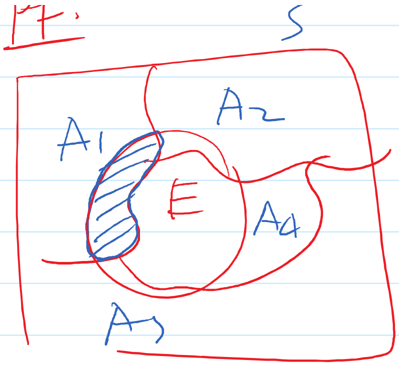
(2) $f_Y(y) = \sum_x f(x,y)$

Pf.



Partitn S w/ collectn of events $\{Y=y\}$ $\forall y$

Ex:



Partition S w/ collection of events $\{Y = y\}$ $\forall y$

$$f_X(x) = \mathbb{P}(X = x)$$

$$= \sum_y \mathbb{P}(``X = x" \cap ``Y = y")$$

$$= \sum_y f(x,y)$$