# Lecture 18 - Bivariate RVs

Thursday, November 11, 2021 9:28 AM

Joint PMF: $f(x,y) = P(X=x, Y=y)$

Marginal PMFs: $f_X(x) = \sum_y f(x,y)$

$f_Y(y) = \sum_x f(x,y)$ (*)

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Ex. Flip 3 coins,

$$X = \begin{cases} 0 & \text{last flip is T} \\ 1 & \text{last flip is H} \end{cases}$$

$Y$ = # heads among 3 flips

| $f(x,y)$ | $Y=0$ | $Y=1$ | $Y=2$ | $Y=3$ | |
|---|---|---|---|---|---|
| $X=0$ | $f(0,0)=1/8$ | $f(0,1)=3/8$ | $1/8$ | $0$ | $f_X(0)=1/2$ |
| $X=1$ | $0$ | $1/8$ | $2/8$ | $1/8$ | $f_X(1)=1/2$ |
| | $f_Y(0)=1/8$ | $f_Y(1)=3/8$ | $f_Y(2)=3/8$ | $f_Y(3)=1/8$ | |

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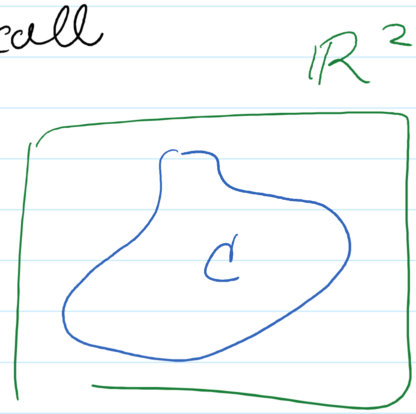
Defn: Joint PDF

## Defn: Joint PDF

If $X$ and $Y$ are continuous we call

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

the joint PDF if $\forall C \subset \mathbb{R}^2$




$$\mathbb{P}((X,Y) \in C) = \int_C f(x,y)\,dx\,dy$$

Univariate case:

$$P(X \in A) = \int_A f(x)\,dx$$

Facts: (1) $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$

$$\left(\text{uni: } F(x) = \int_{-\infty}^{x} f(t)\,dt\right)$$

(2) $f(x,y) = \dfrac{\partial^2 F}{\partial x \partial y}$

$$\left(\text{uni: } f(x) = \frac{dF}{dx}\right)$$

③ $f(x,y) \geq 0$ and $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$

(Univ: $f(x) \geq 0$ and $\int_{\mathbb{R}} f(x)\,dx = 1$)

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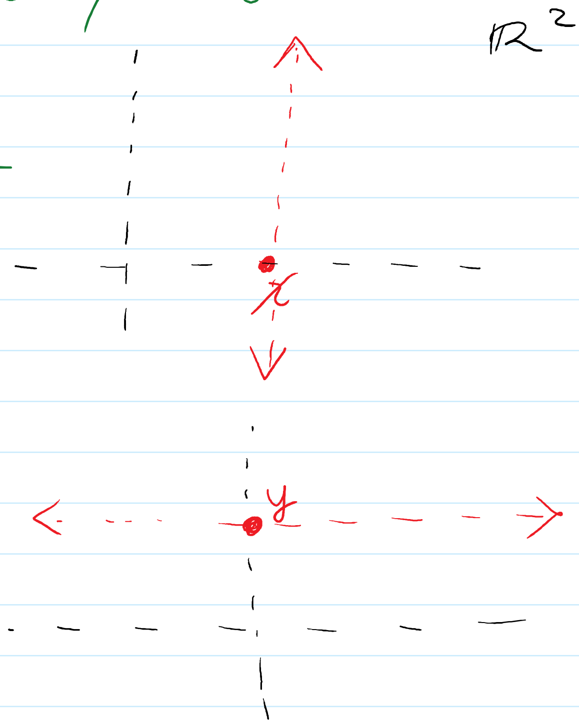
Theorem: Rel. btwn Joint/Marginals

① $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$

② $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$

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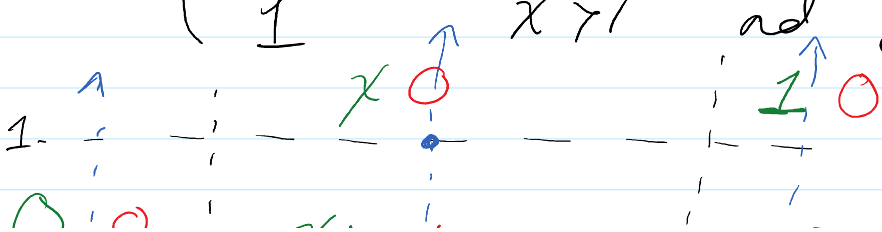
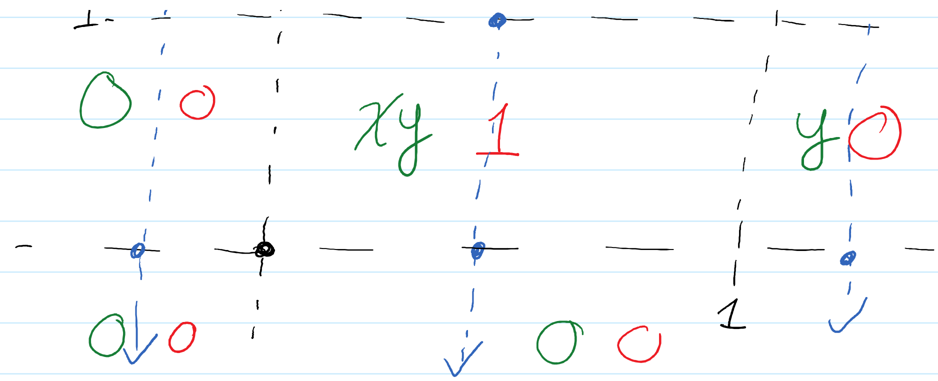
Ex.

$$F(x,y) = \begin{cases} 0 & x \leq 0 \text{ or } y \leq 0 \\ xy & (x,y) \in [0,1]^2 \\ x & y > 1 \text{ and } 0 < x < 1 \\ y & x > 1 \text{ and } 0 < y < 1 \\ 1 & x > 1 \text{ and } y > 1 \end{cases}$$






① What is the joint PDF?

Recall: $f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$

So $f(x,y) = 1$ for $0<x<1$ and $0<y<1$

② What are the marginal dist of $X$ and $Y$?

(i) $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_0^1 1\,dy = 1$

for $0<x<1$

If $x>1$ or $x<0$ then $f_X(x) = 0$

So $X \sim U(0,1)$

Similarly for $Y \sim U(0,1)$.

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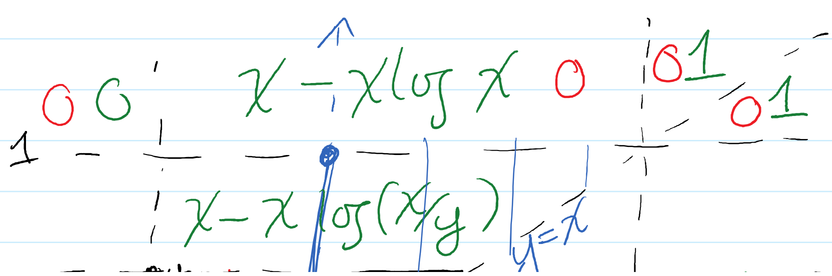
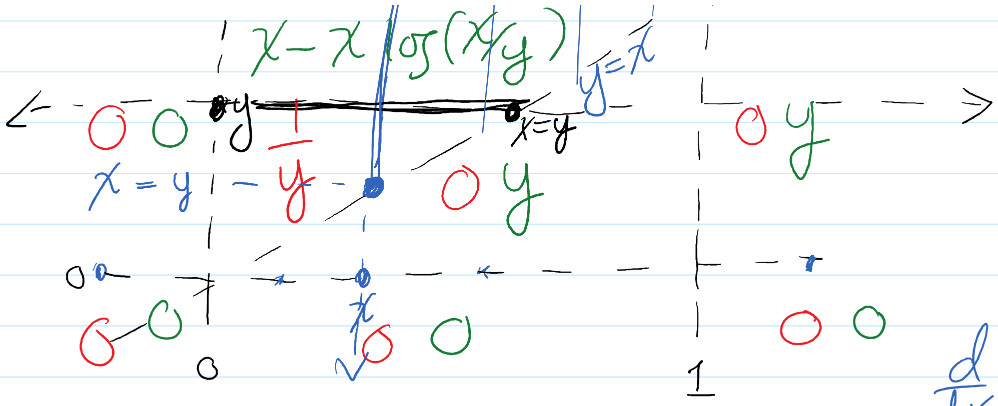
Ex.




$F(x,y)$

$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$




$= \dfrac{\partial^2 F}{\partial x \partial y}$

$\dfrac{d}{dx}\log u(x) = \dfrac{u'(x)}{u(x)}$

$$\frac{\partial^2}{\partial x \partial y}\left(x - x\log(x/y)\right) = \frac{\partial}{\partial x}(x/y) = \frac{1}{y}$$

$$\frac{d}{dy}\log(y/x) = \frac{1/x}{y/x}$$

So $\boxed{f(x,y) = \frac{1}{y} \quad \text{for } 0<x<y<1}$

What are the marginals?

(i) $f_X(x) = \displaystyle\int_{\mathbb{R}} f(x,y)\,dy = \int_x^1 \frac{1}{y}\,dy = \log(y)\Big|_x^1$

$= \log(1) - \log(x)$

$= -\log(x)$

for $0<x<1$

(ii) $f_Y(y) = \displaystyle\int_{\mathbb{R}} f(x,y)\,dx = \int_0^y \frac{1}{y}\,dx = \frac{y}{y} = 1$

for $0<y<1$

$Y \sim U(0,1)$.

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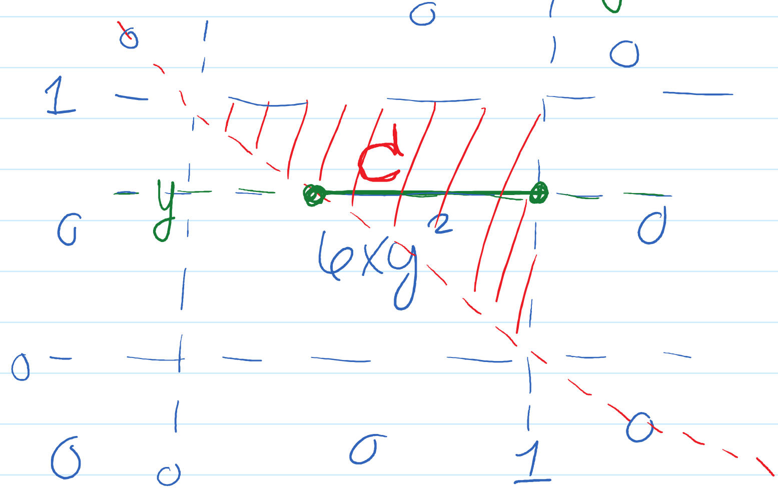
Ex. Let

$$f(x,y) = 6xy^2 \quad \text{for} \quad 0<x<1, \; 0<y<1$$

$$\mathbb{P}(X+Y>1) = \mathbb{P}((X,Y)\in C) = \int_C f(x,y)\,dx\,dy = \int_0^1 \int_{1-y}^1 6xy^2 \,dx\,dy = \dots = 9/10$$




$x+y=1$

$x = 1-y$ or $y = 1-x$

$x+y>1$ then $y>1-x$

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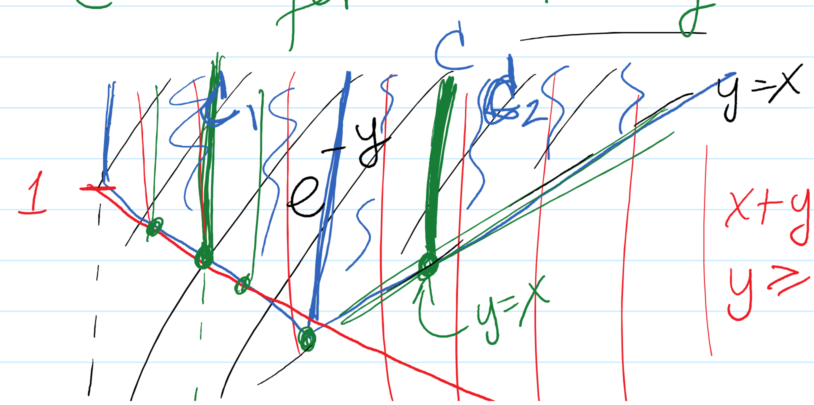
Ex. Let $f(x,y) = e^{-y}$ for $0<x<y$

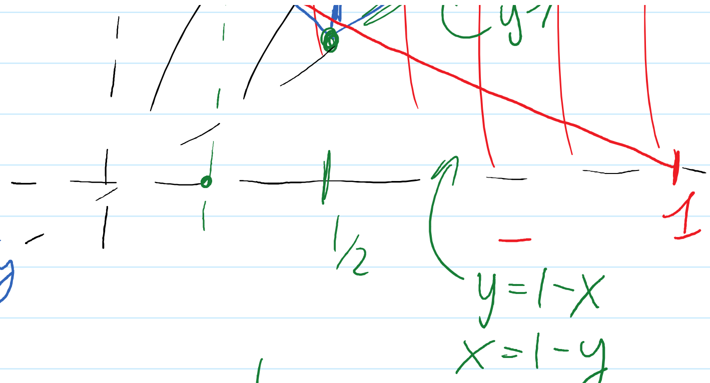
$$\mathbb{P}(X+Y\geq 1) = \int f(x,y)\,dx\,dy$$




$x+y \geq 1$

$y \geq 1-x$

$$= \int_C f(x,y)\,dx\,dy$$




$$= \int_{C_1} f\,dx\,dy + \int_{C_2} f\,dx\,dy$$

$$= \int_0^{1/2}\int_{1-x}^{\infty} e^{-y}\,dy\,dx + \int_{1/2}^{1}\int_{x}^{\infty} e^{-y}\,dy\,dy$$

$$= \cdots = \boxed{2/\sqrt{e} - \frac{1}{e}}$$

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**Defn: Bivariate Expectation**

If $(X,Y)$ is Biv RV and $g : \mathbb{R}^2 \to \mathbb{R}$

then

$$\mathbb{E}[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) \quad \text{PMF} \\ \text{(discrete)} \\ \iint_{\mathbb{R}^2} g(x,y) f(x,y)\,dx\,dy \end{cases}$$

(Univar: $\mathbb{E}[g(X)] = \int g(x) f(x)\,dx$)

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Ex

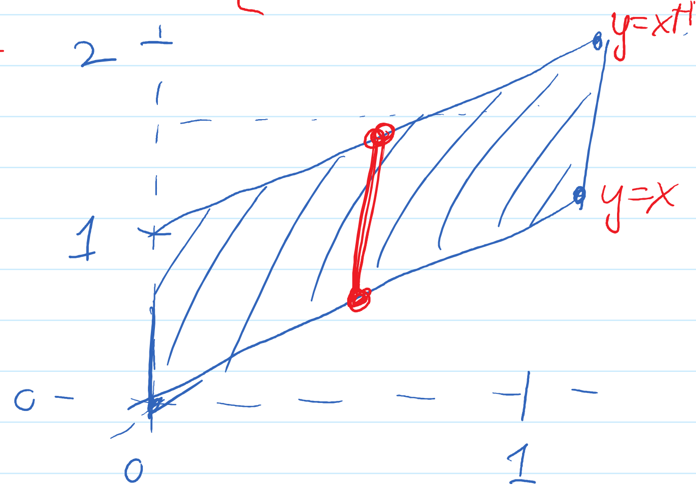
Ex.

Let $f(x,y) = 1$ for $\begin{cases} 0 < x < 1 \\ x < y < x+1 \end{cases}$

$g(x,y) = xy$

$$\mathbb{E}[XY] = \int_0^1 \int_x^{x+1} xy(1)\, dy\, dx$$

$= 7/12$




Theorem: Biv Expectation is Linear

If $g_1 : \mathbb{R}^2 \to \mathbb{R}$, $g_2 : \mathbb{R}^2 \to \mathbb{R}$, and $a, b \in \mathbb{R}$, then

$$\mathbb{E}\left[a g_1(X,Y) + b g_2(X,Y)\right]$$

$$= a\,\mathbb{E}[g_1(X,Y)] + b\,\mathbb{E}[g_2(X,Y)].$$