Lecture 17 - More Transformations and Bivariate RVs

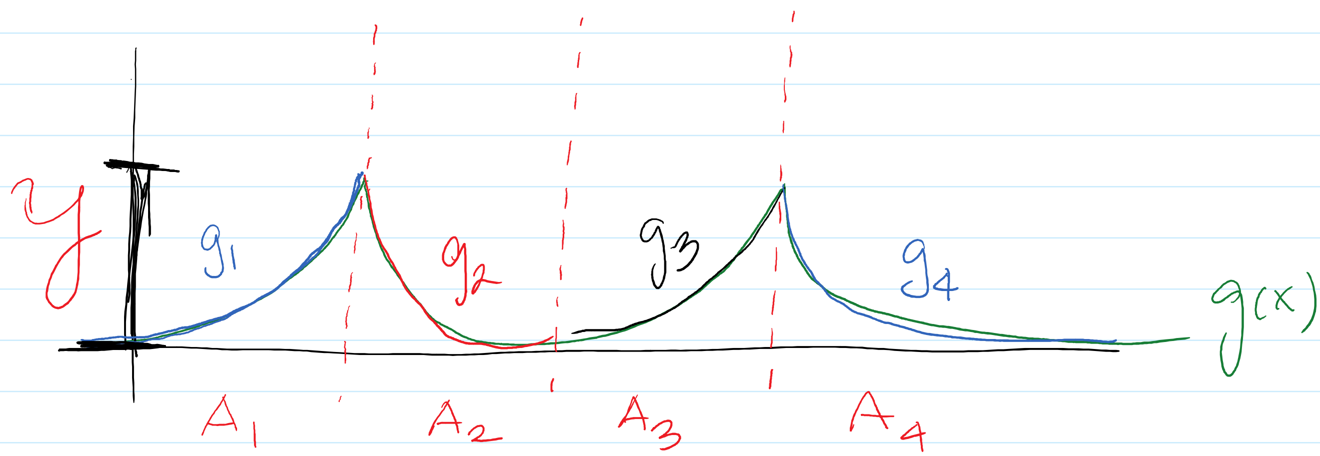
Thursday, November 4, 2021 1:58 PM

If X is cts and $Y = g(X)$ and

① g is invertible

② g^{-1} is differentiable

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$



Theorem:

If X is cts w/ support $\mathcal{X}$ and $(A_i)_{i=1}^{K}$ partitions $\mathcal{X}$ so that $g_i = g$ restricted to A_i

and

① the prev theorem applies for each g_i on A_i

[- g_i invertible (on A_i)

$- g_i^{-1}$ is differentiable

(2) Image of each A_i under g_i is the same $\forall i$ $(\mathcal{Y})$

$$f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{d g_i^{-1}}{dy} \right|$$

for $y \in \mathcal{Y}$

Ex. <u>Chi-Squared Distribution</u>

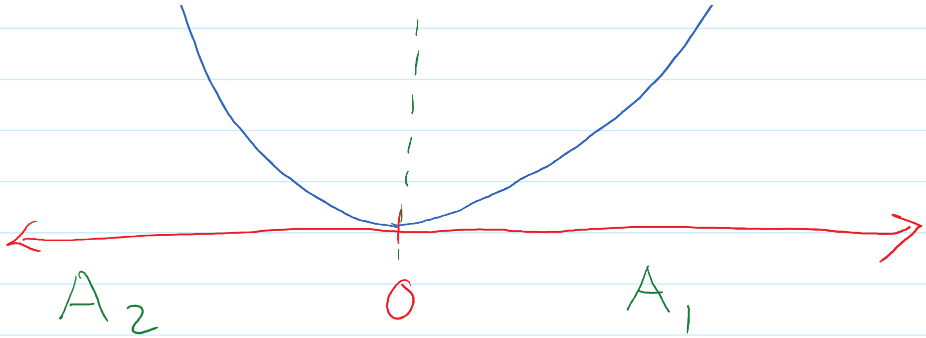
If $X \sim N(0,1)$ and $Y = X^2$

then we say Y has a chi-squared distribution w/ <u>one degree of freedom</u>.

$$Y \sim \chi^2(1)$$

What is the PDF of Y?

$y = x^2$



$$\begin{cases} A_1 = (0, \infty) \,;\; g_1(x) = x^2 \,;\; g_1^{-1}(y) = \sqrt{y} \,;\; \dfrac{dg_1^{-1}}{dy} = \dfrac{1}{2\sqrt{y}} \\ A_2 = (-\infty, 0) \,;\; g_2(x) = x^2 \,;\; g_2^{-1}(y) = -\sqrt{y} \,;\; \dfrac{dg_2^{-1}}{dy} = \dfrac{-1}{2\sqrt{y}} \end{cases}$$

$$f_Y(y) = f_X(g_1^{-1}(y)) \left| \frac{dg_1^{-1}}{dy} \right| + f_X(g_2^{-1}(y)) \left| \frac{dg_2^{-1}}{dy} \right|$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2)$$

$$= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} \exp(-\sqrt{y}^2) \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} \exp(-(-\sqrt{y})^2) \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} \left(e^{-y} + e^{-y} \right)$$

$$\boxed{= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y} \quad \text{for } y > 0}$$

Probability Integral Transformation

If X is cts w/ CDF F_X then

$$Y = F_X(X) \sim U(0,1).$$

Pf. Assume F_X is strictly increasing, so F_X^{-1} exists.

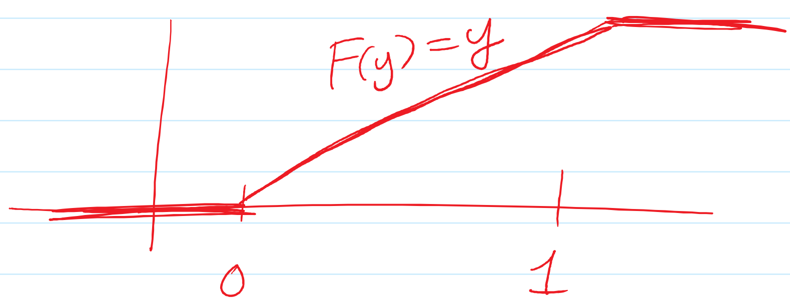
Prev. CDF theorem $Y = g(X) \Rightarrow F_Y(y) = F_X(g^{-1}(y))$

So

$$Y = F_X(X) \Rightarrow F_Y(y) = F_X(F_X^{-1}(y))$$

$$= y \Rightarrow \text{CDF of } Y \text{ is the CDF of a } U(0,1)$$

recall the CDF of $U(0,1)$



$$g(X) \sim U(0,1) \iff g = F_X^{-1}.$$

How do I generate random numbers on

a computer:

want X w/ CDF F_X

<u>idea:</u> ① generate $Y \sim U(0,1)$

② $X = F_X^{-1}(Y)$

has correct dist.

<u>Ex.</u> want $X \sim \text{Exp}(1)$

want: $F_X(x) = 1 - e^{-x}$

$y = 1 - e^{-x}$

$\Rightarrow 1 - y = e^{-x}$

$\Rightarrow -\log(1-y) = x \longrightarrow F_X^{-1}(y) = -\log(1-y)$

Bivariate RVs

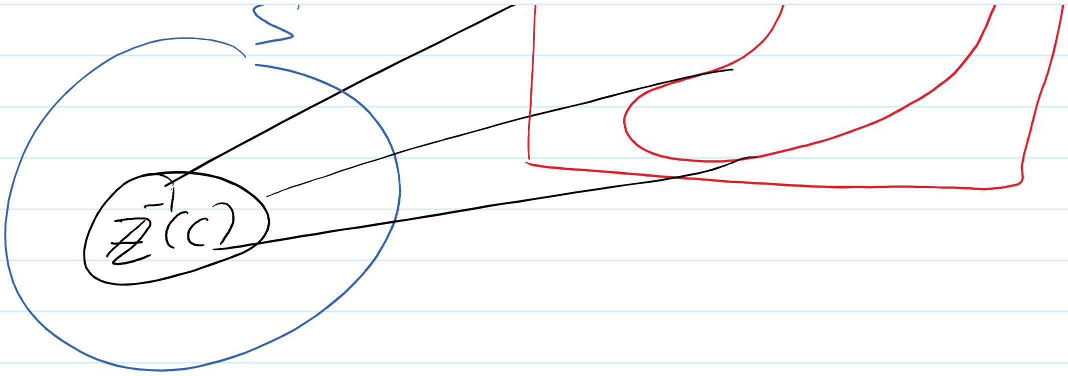
If $X : S \to \mathbb{R}$, and $Y : S \to \mathbb{R}$

then (X, Y) is a <u>bivariate RV</u>

<u>Say:</u> $P(\underbrace{(X,Y)}_{Z} \in C) = P(Z^{-1}(C))$

$C \subset \mathbb{R}^2$

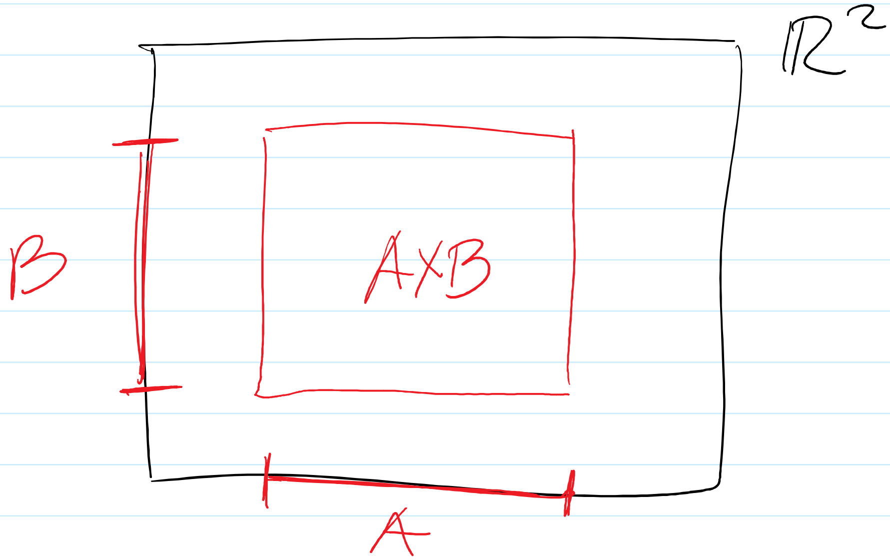
$Z = (X, Y) : S \to \mathbb{R}^2$



Often $C = A \times B$ when $A, B \subset \mathbb{R}$

$P((X,Y) \in A \times B)$

$= P(X \in A, Y \in B)$ ← notation



Ex. Flip a coin 3 times

$$X = \begin{cases} 0 & \text{if last Flip is } T \\ 1 & \text{"} \qquad\qquad H \end{cases}$$

$Y = \#$ heads among 3 flips

$Z = (X, Y)$

$s \in S$	$Z(s)$
H H H	$(1,3)$
H H T	$(0,2)$

H H H	(1, 3)
H H T	(0, 2)
H T H	(1, 2)
H T T	
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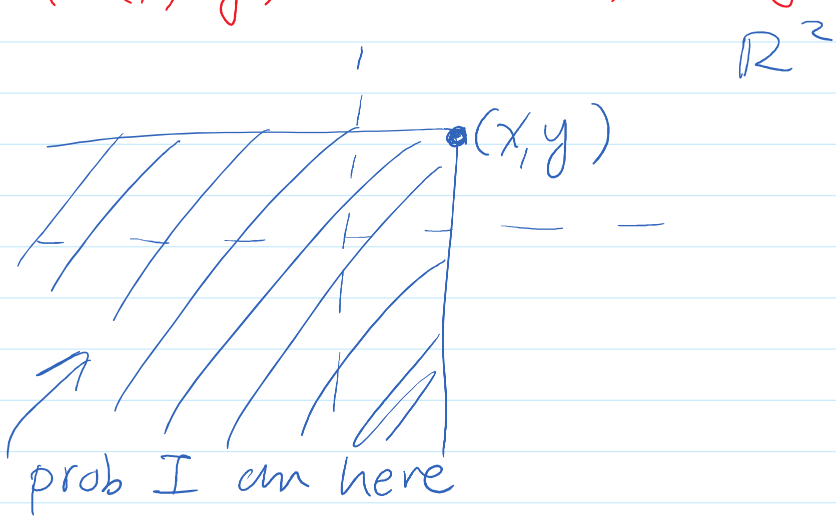
Defn: Biv CDFs

The Biv CDF (joint CDF) is a fn

$$F : \mathbb{R}^2 \to \mathbb{R}$$

so that for $x, y \in \mathbb{R}$ the

$$F(x,y) = \mathbb{P}(X \le x, Y \le y)$$



Theorem: Joint CDF properties

① $F(x,y) \geq 0$

② $\lim_{x,y\to\infty} F(x,y) = 1$ (uni: $\lim_{x\to\infty} F(x) = 1$)

③ $\lim_{x\to-\infty} F(x,y) = 0$ (uni: $\lim_{x\to-\infty} F(x) = 0$)

$\lim_{y\to-\infty} F(x,y) = 0$

④ F is non-decreasing and right-cts in either argument

Defn: If (X, Y) is a Biv RV then X and Y are called the marginal RVs and their properties are prefixed w/ the word marginal.

Theorem: Rel btw Biv CDF and marginal CDFs?

① $F_X(x) = \lim_{y\to\infty} F(x,y)$

② $F_Y(y) = \lim_{x\to\infty} F(x,y)$

idea: $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(X \le x, Y = \text{anything})$

$$= \mathbb{P}(X \le x, Y < \infty)$$

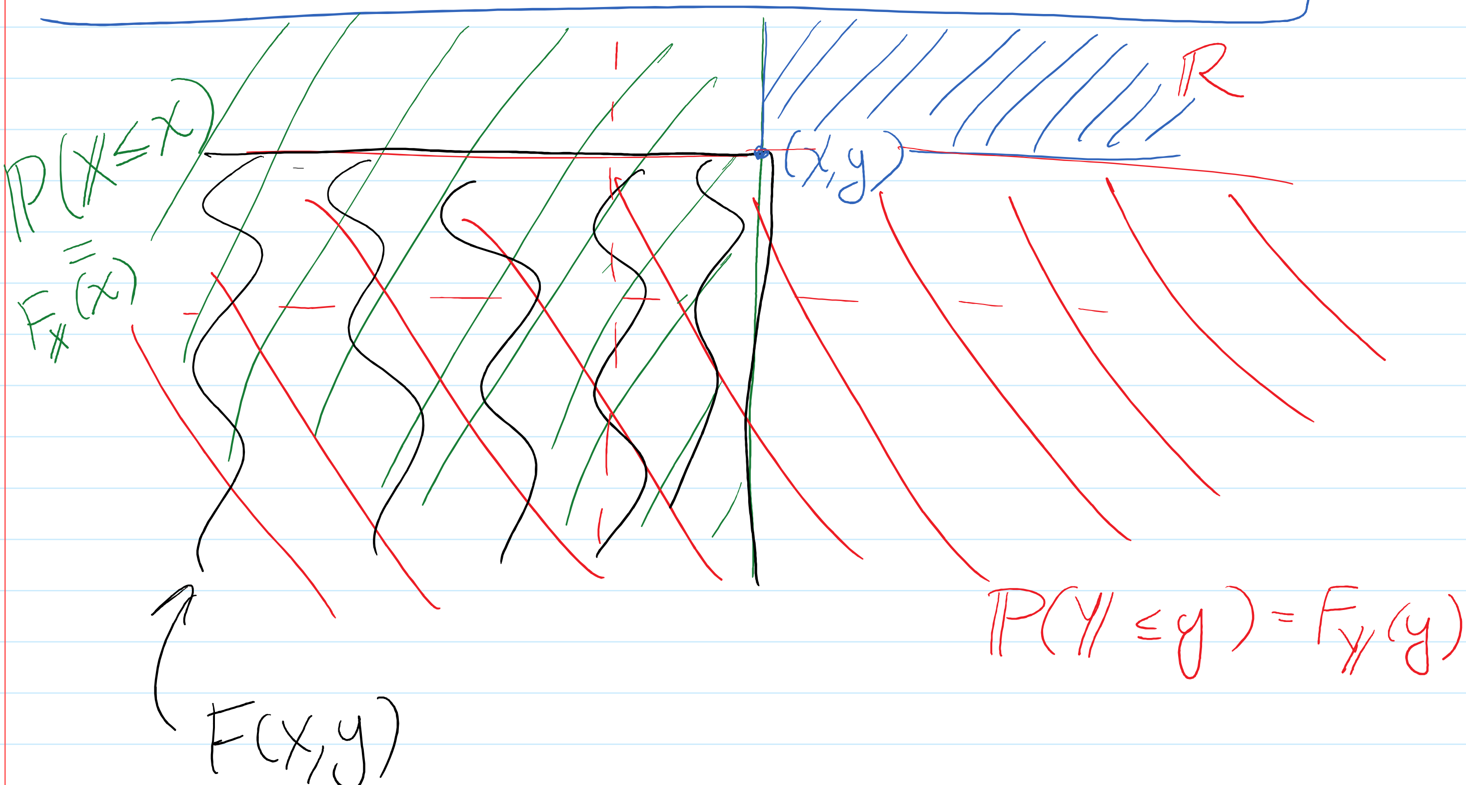
$$= \lim_{y \to \infty} \mathbb{P}(X \le x, Y \le y)$$

$$= \lim_{y \to \infty} F(x, y)$$

Lemma: (uni: $\mathbb{P}(X > x) = 1 - F(x)$)

Biv: $\mathbb{P}(X > x, Y > y)$

$$= 1 - F_X(x) - F_Y(y) + F(x,y)$$



$F(x,y)$

Defn: Joint PMF

If X and Y are discrete then the joint PMF is

$$f(x,y) = P(X=x, Y=y)$$

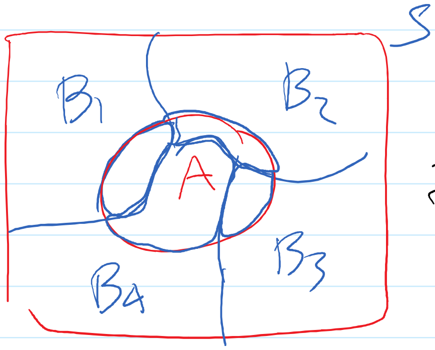
Theorem: f is a valid joint PMF if

1. $f(x,y) \geq 0 \quad \forall x, y$
2. $\sum_x \sum_y f(x,y) = 1$.

Theorem: Rel. btw joint/marginal PMFs

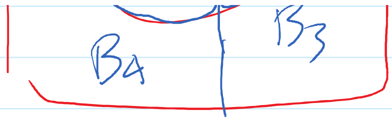
1. $f_X(x) = \sum_y f(x,y)$
2. $f_Y(y) = \sum_x f(x,y)$

pf.



The collection $\{Y=y\}$ $\forall y$ partition S

$$f_X(x) = P(X=x)$$
$$= \sum_y P(\text{"}X=x\text{"} \cap \text{"}Y=y\text{"})$$



$$= \sum_y P(``X=x" \cap ``Y=y")$$

$$= \sum_y P(X=x, Y=y)$$

$$= \sum_y f(x,y) .$$