Lecture 21 - Iterated Expectation and BivNormal

Tuesday, November 23, 2021 9:30 AM

Bayes' Theorem: $f(y|x) = \dfrac{f(x|y)\,f(y)}{f(x)}$

<u>Law of Tot. Prob:</u>

$$\begin{cases} f(y) = \sum_x f(y|x)\,f(x) & \text{(discrete)} \\ f(y) = \int f(y|x)\,f(x)\,dx & \text{(cts)} \end{cases}$$

<u>Ex.</u> $X \sim Exp(\lambda)$

$Y | X = x \sim Pois(x)$

what is dist of $\boxed{Y?}$

$$f(y) = \int f(y|x) f(x)\, dx \longleftarrow$$

$$= \int_0^\infty \underbrace{\frac{x^y e^{-x}}{y!}}_{Pois(x)} \underbrace{\lambda e^{-\lambda x}}_{Exp(\lambda)} dx$$

$$= \frac{1}{y!} \int_0^\infty \lambda e^{-(\lambda+1)x} x^y \, dx$$

PDF trick notice its basically Gamma$(y+1, \lambda+1)$

basically Gamma $(y+1, \lambda+1)$

$$= \frac{\lambda \Gamma(y+1)}{y!\,(\lambda+1)^{y+1}} \int_0^\infty \frac{(\lambda+1)^{y+1} x^{y} e^{-(\lambda+1)x}}{\Gamma(y+1)}\,dx$$

$$= \frac{\lambda \cancel{\Gamma(y+1)}}{\cancel{y!}\,(\lambda+1)^{y+1}} = \boxed{\frac{\lambda}{(\lambda+1)^{y+1}} \quad \text{for } y = 0, 1, 2, \ldots}$$

Ex.

$Y \sim \text{Pois}(\lambda)$

$X \mid Y = y \sim \text{Bin}(y, p)$ $\quad p \in [0, 1]$

$y \geq x$

what is the dist of X?

$$f(x) = \sum_y f(x \mid y) f(y)$$

$$= \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{\lambda^y e^{-\lambda}}{y!}$$

$$\binom{y}{x} \frac{1}{y!} = \frac{\cancel{y!}}{x!\,(y-x)!} \frac{1}{\cancel{y!}} = \frac{1}{x!\,(y-x)!}$$

$x!\,y!$ $\quad x!(y-x)!\,\cancel{y!}$ $\quad x!(y-x)!$

$$= \frac{p^x e^{-\lambda} \lambda^x}{x!} \sum_{y=x}^{\infty} \frac{(1-p)^{y-x}}{(y-x)!} \lambda^{y-x}$$

$$= \frac{p^x e^{-\lambda} \lambda^x}{x!} \boxed{\sum_{y=0}^{\infty} \frac{(1-p)^{y}}{y!} \lambda^{y}}$$

$$\sum_{y=0}^{\infty} \frac{((1-p)\lambda)^y}{y!} = e^{(1-p)\lambda}$$

$$= \frac{(\lambda p)^x \cancel{e^{-\lambda}}}{x!} e^{(\cancel{1}-p)\lambda}$$

$$f(x) \boxed{= \frac{(\lambda p)^x e^{-\lambda p}}{x!}} \longleftarrow \text{Pois}(\lambda p)$$

So $\boxed{X \sim \text{Pois}(\lambda p)}$

Theorem: Iterated Expectation

If X and Y are RVs then

$$E[X] = E_Y[\underbrace{E[X|Y]}_{\text{a fn } Y}].$$

$E[X|Y=y] = \int x f(x|y)\,dx \in \mathbb{R}$

For each $y \in \mathbb{R}$ I get a potentially diffent $\mathbb{E}[X \mid Y=y]$

This defines a function

$$g(y) = \underline{\mathbb{E}[X \mid Y=y]}$$

what happens I plus Y into g?

$$g(Y) = \cancel{\mathbb{E}[X \mid Y=Y]}$$ weird notation

$$= \mathbb{E}[X \mid Y]$$

① $\mathbb{E}[X \mid Y=y]$ is a number

② $\mathbb{E}[X \mid Y]$ is a RV (a fn of Y)

Pf. Facts: (cts)

① $f(x) = \int f(x,y)\, dy$

② $f(x \mid y) = \dfrac{f(x,y)}{f(y)} \iff f(x,y) = f(x \mid y) f(y)$

③ $\mathbb{E}[X \mid Y=y] = \int x f(x \mid y)\, dx$

③ $\mathbb{E}[X \mid Y=y] = \int x f(x|y)\,dx$

$$\mathbb{E}[X] = \int x \underbrace{f(x)}\,dx \overset{①}{=} \int x \underbrace{\int f(x,y)\,dy}\,dx$$

$$\overset{②}{=} \int x \int \underbrace{f(x|y)\,f(y)}\,dy\,dx$$

$$= \int \underbrace{\int x f(x|y)\,dx}_{\mathbb{E}[X|Y=y]=g(y)}\, f(y)\,dy \qquad [\text{rearrange}]$$

$$= \int g(y)\,f(y)\,dy$$

$$= \mathbb{E}[g(Y)] \quad \text{notation}$$

$$= \mathbb{E}[\,\mathbb{E}[X|Y]\,]$$

Ex. $Y \sim \text{Pois}(\lambda)$

$X \mid Y=y \sim \text{Bin}(y, p)$

$\mathbb{E}[X]$? $\quad (\mathbb{E}X = \mathbb{E}[\mathbb{E}[X|Y]]$

① get $\mathbb{E}[X \mid Y=y] = yp$

② get $\mathbb{E}[X|Y] = Yp$

③ get $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Yp] = p\mathbb{E}Y$

$\boxed{= p\lambda}$

answer

Ex. $P \sim \text{Beta}(\alpha, \beta)$

$X | P = p \sim \text{Bin}(n, p)$

$\mathbb{E}[X]$? $= \mathbb{E}[\mathbb{E}[X|P]]$

① get $\mathbb{E}[X|P=p] = np$

② get $\mathbb{E}[X|P] = nP$

③ get $\mathbb{E}[\mathbb{E}[X|P]] = \mathbb{E}[nP] = n\mathbb{E}[P]$

$$\boxed{= n\frac{\alpha}{\alpha+\beta}}$$

Theorem: Law of Total Variance

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y])$$

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calc. $\mathrm{Var}(X|Y=y)$

then promote y to Y in formula

Ex. $\boxed{P \sim \text{Beta}(\alpha, \beta)}$

$X \mid P = p \sim \text{Bin}(n, p)$

$\underline{\text{Var}(X)?}$

① $E[X \mid P = p] = np$

$\text{Var}(X \mid P = p) = np(1-p)$

② $E[X \mid P] = nP$

$\text{Var}(X \mid P) = nP(1-P)$

③ $\text{Var}(X) = E\,\text{Var}(X \mid P) + \text{Var}\,E[X \mid P]$

$= E[nP(1-P)] + \text{Var}(nP)$

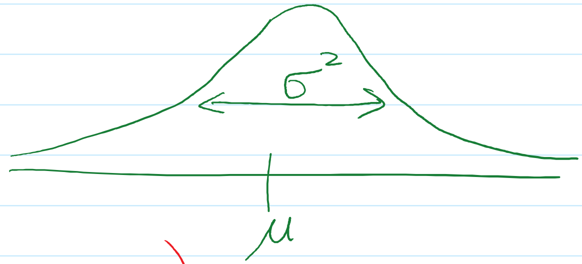
$= n(E[P] - E[P^2]) + n^2\,\text{Var}(P)$

$= \cdots$ plug in

$$= n\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)} + n^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

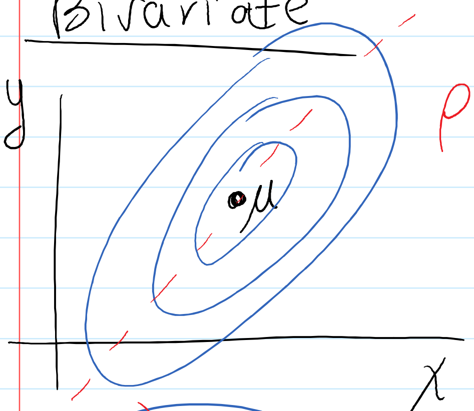
Bivariate Normal

Univariate: $N(\mu, \sigma^2)$

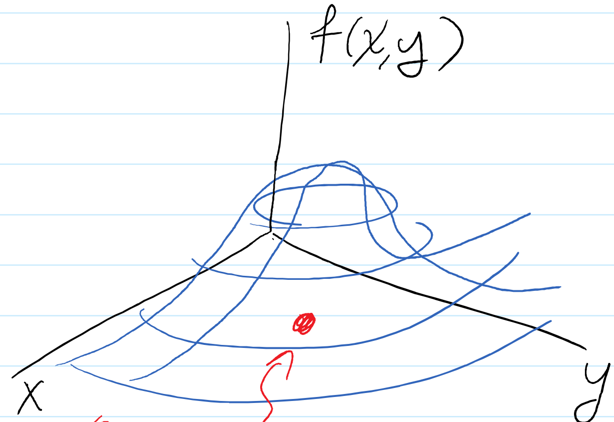


$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Bivariate



$\rho \gg 0$



$\mu = (\mu_x, \mu_y)$

$\sigma_x^2, \sigma_y^2, \rho$ corr btwn X and Y

$\rho \ll 0$

$\rho \approx 0$

PDF: $(X, Y) \sim \mathrm{BivN}(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$

$$f(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2} \frac{1}{1-\rho^2} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho \left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) \right] \right\}$$

mean vector

$\mu = (\mu_X, \mu_Y) \in \mathbb{R}^2$

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \sigma_X \sigma_Y \rho \\ \sigma_X \sigma_Y \rho & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Var}(Y) \end{bmatrix}$$

covariance matrix

$z = (x, y)$

Then

$$\boxed{f(z) = \frac{1}{2\pi} \frac{1}{\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2} (z-\mu)^T \Sigma^{-1} (z-\mu) \right)}$$

Uni: $f(x) = \frac{1}{2\pi} \frac{1}{\sqrt{\sigma^2}} \exp\left(-\frac{1}{2} (x-\mu)(\sigma^2)^{-1}(x-\mu) \right)$

Facts: ① $X \sim N(\mu_X, \sigma_X^2)$

$Y \sim N(\mu_Y, \sigma_Y^2)$

② $\mathrm{Cor}(X, Y) = \rho$

③ $aX + bY \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho)$

④ A characteization of BivN

$(X, Y) \sim \mathrm{BivN} \iff \forall a, b \quad aX + bY \sim N(\dots)$

⑤ Prev: If $X \perp\!\!\!\perp Y$ then $\mathrm{Cor}(X, Y) = 0$

Partial Converse: if $(X, Y) \sim \mathrm{BivN}$

and $\mathrm{Cor}(X, Y) = 0$

then $X \perp\!\!\!\perp Y$.