Quiz Problem 8
Due Oct. 28st, 11:59 pm EST

Problem.
Let X be a random variable with moment generating function

$$M(t) = \frac{p}{1-(1-p)e^t} \text{ for } t < -log(1-p).$$

What is Var (X)?
□

$\frac{dM}{dt} = \frac{p(1-p)e^t}{(1-(1-p)e^t)^2} \Rightarrow \frac{dM}{dt}\Big|_{t=0} = \frac{p(1-p)}{p^2} = \frac{1-p}{p}$ $\mathbb{E}[X]$

$\frac{d^2M}{dt^2} = \frac{(1-(1-p)e^t)^2 p(1-p)e^t + p(1-p)e^t 2(1-(1-p)e^t)(1-p)e^t}{(1-(1-p)e^t)^4}$

$\frac{d^2M}{dt^2}\Big|_{t=0} = \frac{p^2 p(1-p) + p(1-p)2p(1-p)}{p^4}$

$= \frac{p(1-p)(p^2+2p(1-p))}{p^4} = \frac{p(1-p)(-p^2+2p)}{p^4}$

$= \underbrace{\frac{(1-p)(2-p)}{p^2}}_{\mathbb{E}[X^2]}$

$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{(1-p)(2-p)}{p^2} - \frac{(1-p)^2}{p^2}$

$= \frac{1-p}{p^2}(2-p-1+p) = \boxed{\frac{1-p}{p^2}}$