Lecture 10 - PDFs

Thursday, October 7, 2021 9:15 AM

Discrete RVs

$f(x) = P(X = x)$ PMF

$F(x) = \mathbb{P}(X \le x)$ CDF

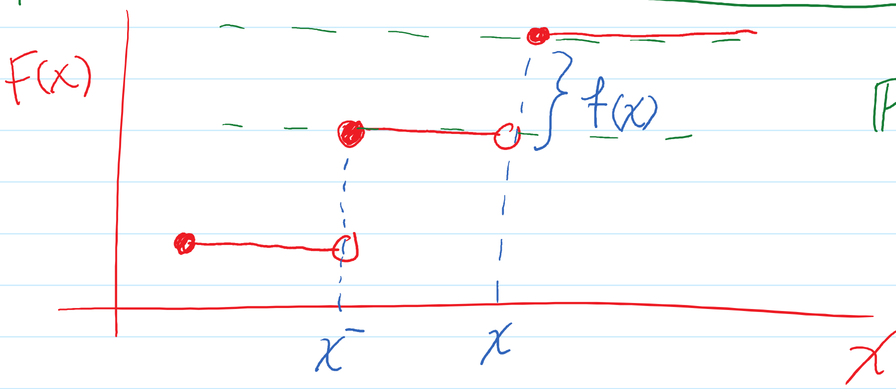
$$\boxed{F(x) = \sum_{i \le x} f(i)}$$ from PMF to CPF

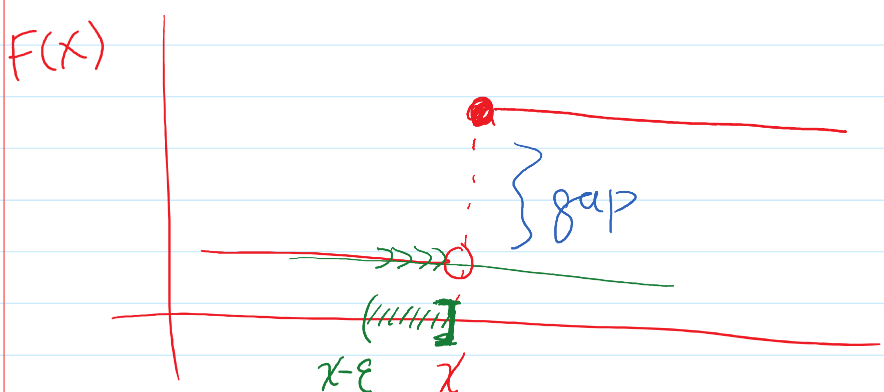
$$\boxed{\mathbb{P}(a < X \le b) = F(b) - F(a)}$$ from CPF to PMF

$\mathbb{P}(X = x)$

$= \mathbb{P}(x^- < X \le x)$

$= F(x) - F(x^-)$



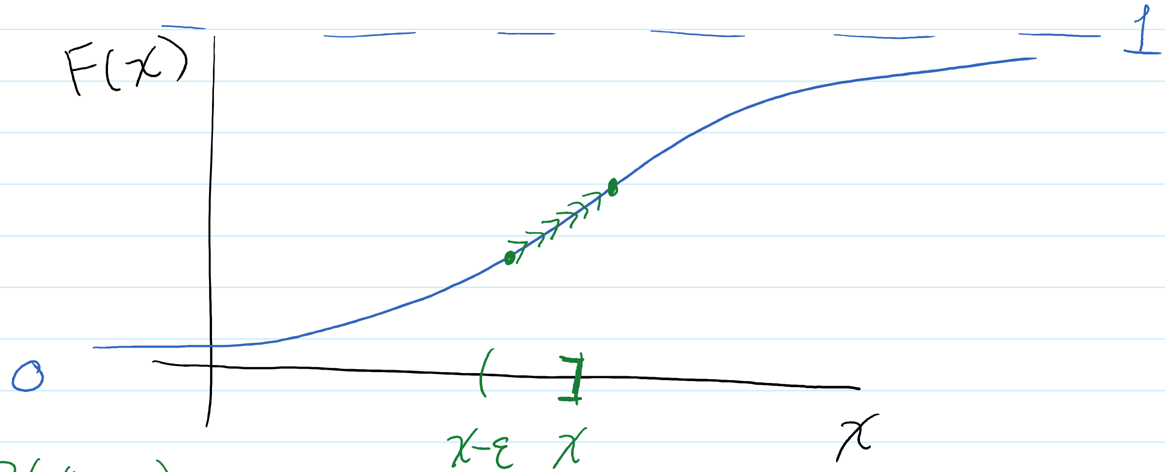


$$\mathbb{P}(X = x) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(x - \varepsilon < X \le x)$$

$$= \lim_{\varepsilon \downarrow 0} F(x) - F(x - \varepsilon)$$

$= \underline{\text{gap size at } x}$

What about for Continuous RVs?



$\mathbb{P}(X=x) =$
$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(x-\varepsilon < X \le x)$$

$$= \lim_{\varepsilon \downarrow 0} F(x) - F(x-\varepsilon)$$

$$= F(x) - \lim_{\varepsilon \downarrow 0} F(x-\varepsilon)$$ F is continuous

$$= F(x) - F(x)$$

$$= 0$$

punchline: - can't define a PMF as a jump size
- $\mathbb{P}(X=x) = 0 \quad \forall \; x \in \mathbb{R}$

Can we do something similar for cts RVs?

Want something like: $F(x) = \sum f(i)$

Want something like: $\boxed{F(x) = \sum_{i \le x} f(i)}$

<u>Defn: Probability Density Function</u> (PDF)

Analog of PMF for cts RVs.

The PDF is a function $f : \mathbb{R} \to \mathbb{R}$ defined so that $\forall x \in \mathbb{R}$

$$\boxed{F(x) = \int_{-\infty}^{x} f(t)\,dt}$$

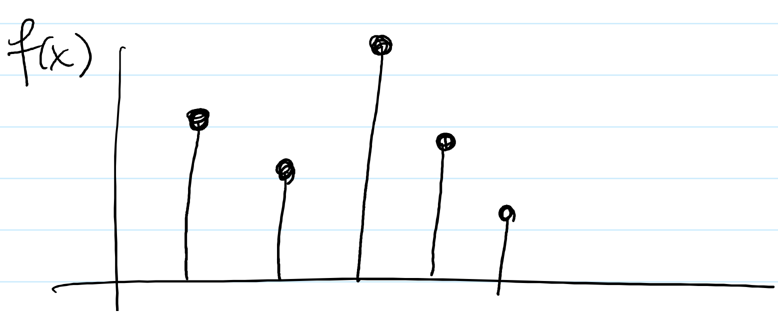
<u>note</u>: by Fundamental Theorem of Calculus

$$\frac{dF}{dx} = \frac{d}{dx} \int_{-\infty}^{x} f(t)\,dt = f(x).$$

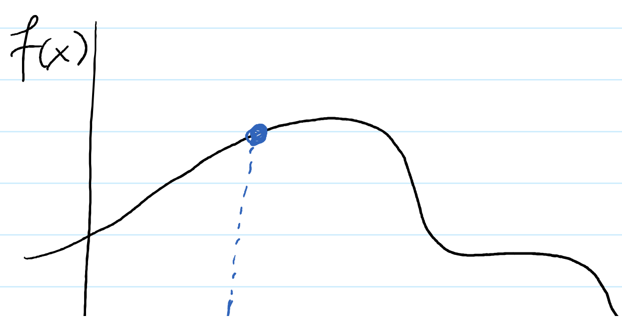
<u>punchline</u>: $f(x) = \dfrac{dF}{dx}$

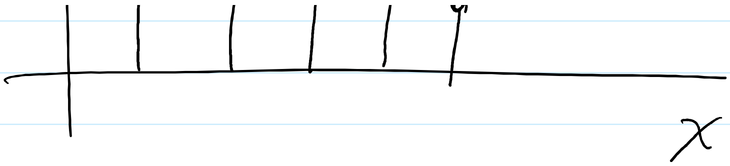
PDF = derivative of CDF.

<u>discrete case</u> (PMF)

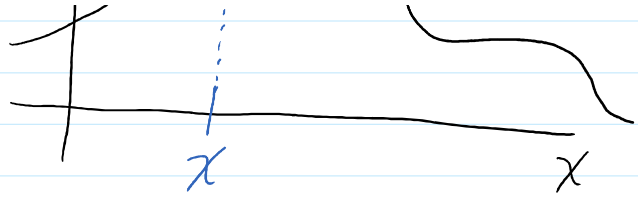


Continuous (PDF)





$f(x) = \mathbb{P}(X = x)$

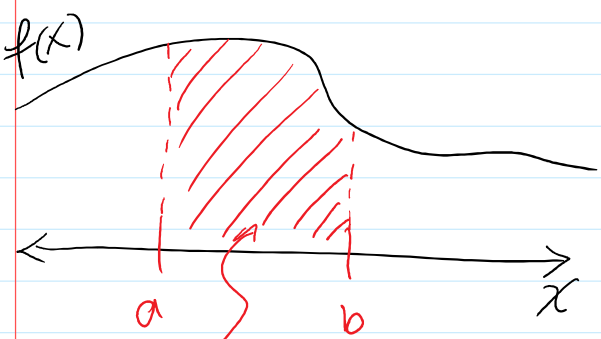


$f(x) \neq \mathbb{P}(X = x) = 0$

Properties for cts RVs

$$\mathbb{P}(a < X \leq b) = F(b) - F(a)$$

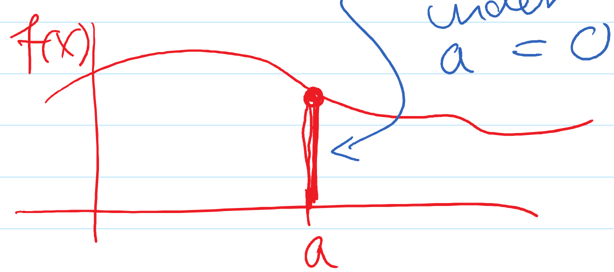
$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$



$$= \int_{a}^{b} f(t)\,dt$$

$\mathbb{P}(a < X \leq b)$ = area btw a and b under PDF

We said: $\mathbb{P}(X = a) = 0 = \mathbb{P}(X = b)$



$\mathbb{P}(a < X \leq b)$

$= \mathbb{P}(a \leq X \leq b)$

$= \mathbb{P}(a < X < b)$

$= \mathbb{P}(a \le X < b)$

More generally:

For discrete: $\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$

For cts: $\mathbb{P}(X \in A) = \int_A f(x)\,dx$

Ex. $\mathbb{P}(X \in (2,3]) = \int_2^3 f(t)\,dt$

$\mathbb{P}(X \in \{2\}) = \mathbb{P}(X = 2) = \int_2^2 f(t)\,dt = 0$

Ex. $F(x) = \dfrac{1}{1+e^{-x}}$

what is the associated PDF?

$$f(x) = \frac{dF}{dx} = \dots = \frac{e^{-x}}{(1+e^{-x})^2}$$

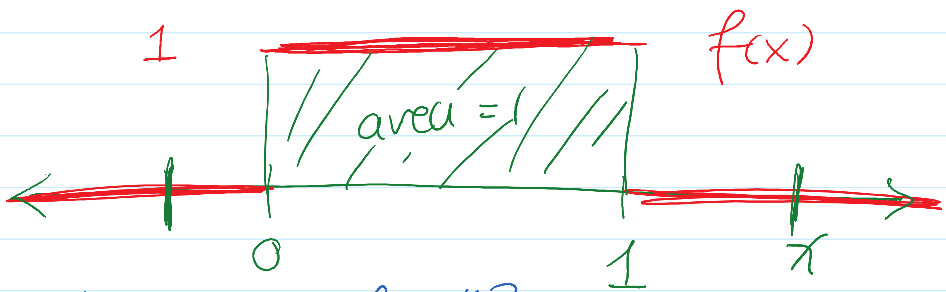
Ex. Continuas Uniform Distribution (on $[0,1]$)

$X \sim U(0,1)$

means

means

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$



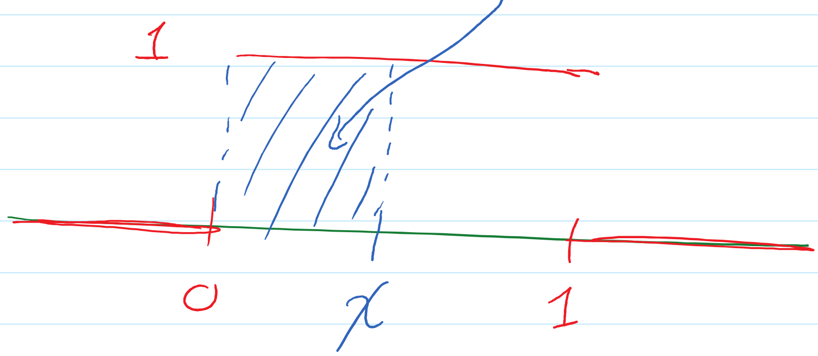
What is the CDF of X?

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

If $x < 0$ then $F(x) = \int_{-\infty}^{x} 0\,dt = 0$

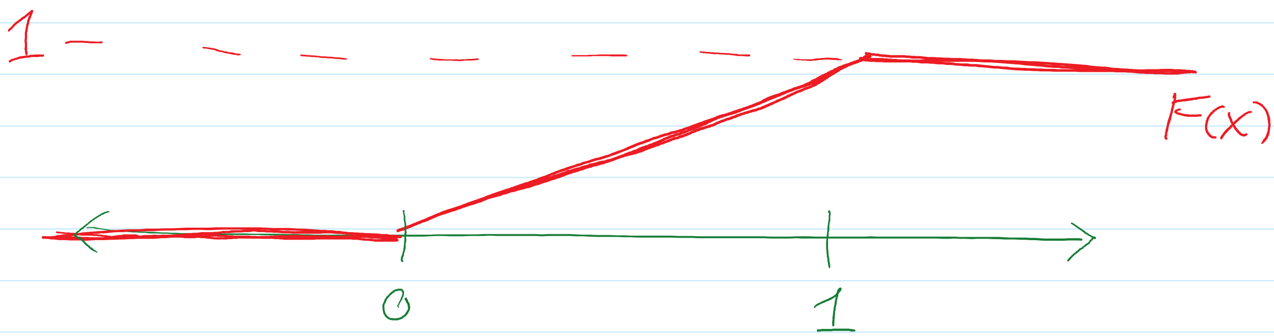
If $x > 1$ then $F(x) = \int_{-\infty}^{x} f(t)\,dt = 1$

If $0 < x < 1$ then $F(x) =$ area $= x$



$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

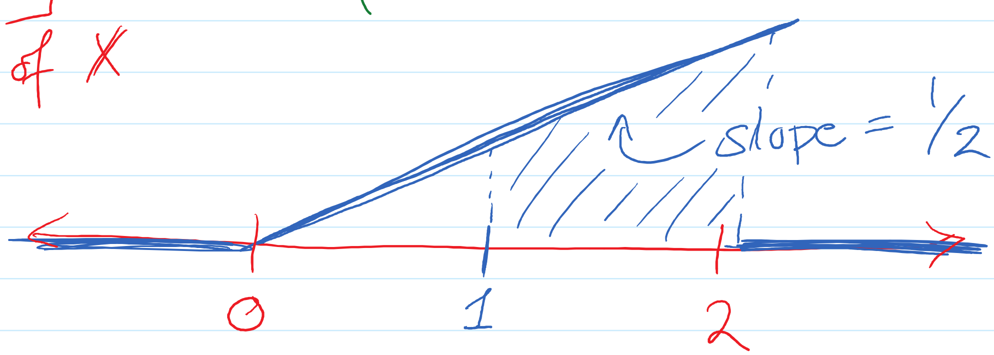
$$\begin{cases} & \\ 1 & x > 1 \end{cases}$$



Ex. Let

$$f(x) = \begin{cases} x/2 & 0 < x < 2 \\ 0 & \text{else} \end{cases}$$

PDF of X

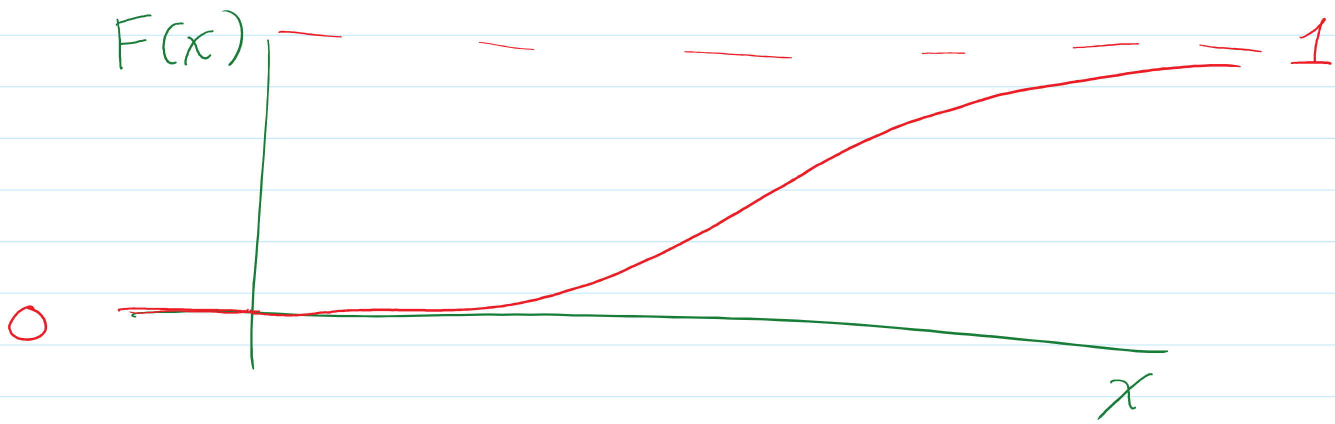


$$\mathbb{P}(X > 1) = \int_1^{\infty} f(t)\,dt = \int_1^2 t/2\,dt = \left.\frac{t^2}{4}\right|_1^2$$

$$= \frac{1}{4}(2^2 - 1^2) = 3/4$$

Ex. Let

$$F(x) = \begin{cases} 1-e^{-x} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases}$$



Q: $P(1 < X < 2)$

Way 1

<u>Theorem</u>: $P(a < X \leq b) = F(b) - F(a)$

$$\begin{aligned} P(1 < X < 2) &= F(2) - F(1) \\ &= (1-e^{-2}) - (1-e^{-1}) \\ &= e^{-1} - e^{-2} \end{aligned}$$

Way 2:

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}(1-e^{-x}) = e^{-x}$$

and so

$$P(1 < X < 2) = \int_1^2 f(x)\,dx = \int_1^2 e^{-x}\,dx$$

$$= -e^{-x}\Big|_1^2 = -e^{-2} - (-e^{-1})$$

$$= e^{-1} - e^{-2}$$

Theorem: Valid PMF/PDFs

There is some RV w/ f as its PMF/PDF iff

① $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

② $\begin{cases} \text{(discrete)} & \sum_{x \in \mathbb{R}} f(x) = 1 \\ \text{(continuous)} & \int_{\mathbb{R}} f(x)\,dx = 1 \end{cases}$

note: For ① (discrete) $P(X \in A) = \sum_A f(x) \geq 0$

(cts) $P(X \in A) = \int_A f(x)\,dx \geq 0$

② $1 = P(S) = P(X \in \mathbb{R})$

$$= \begin{cases} \sum_{\mathbb{R}} f(x) = 1 \\ \int_{\mathbb{R}} f(x)\,dx = 1 \end{cases}$$

Fact: if I have a fn $g(x) \geq 0$

in **cts case:** $\displaystyle\int_{\mathbb{R}} g(x)\,dx = c < \infty$

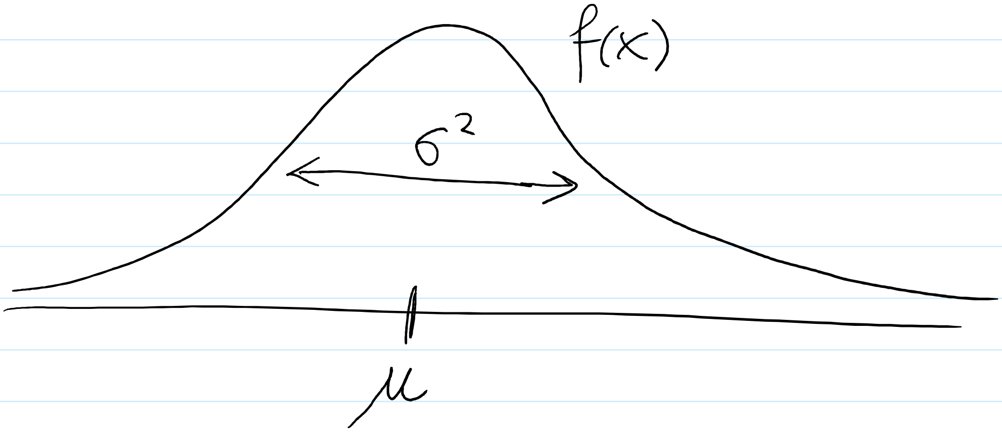
then
$$f(x) = \frac{1}{c} g(x)$$

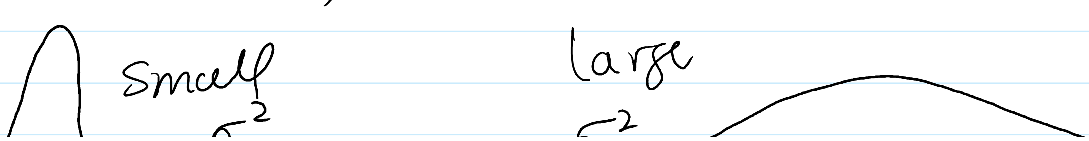
we have $f(x) \geq 0$ and
$$\int_{\mathbb{R}} f(x)\,dx = \frac{1}{c}\int_{\mathbb{R}} g(x)\,dx = \frac{1}{c}\,c = 1.$$

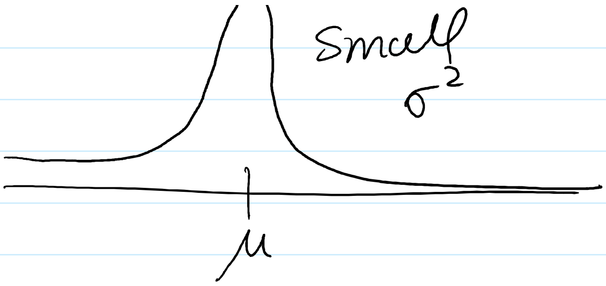
Ex. **Normal Distribution (Gaussian Distribution)**

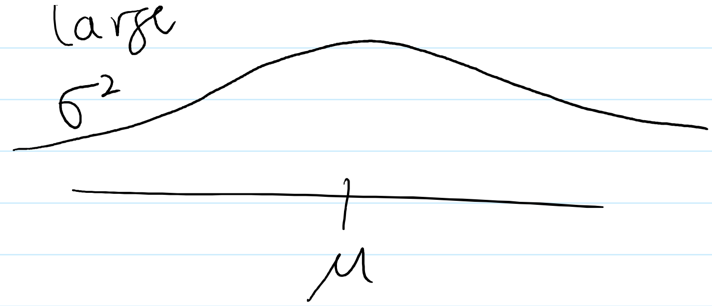
$$X \sim N(\mu, \sigma^2)$$

mean: $\mu \in \mathbb{R}$ Variance: $\sigma^2 > 0$









Special case: standard normal

when $\mu = 0$ and $\sigma^2 = 1$

$X \sim N(0,1)$.

the PDF of X is

$\exp(a) = e^a$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \quad \forall x \in \mathbb{R}$$

Is this a valid PDF?

① $f(x) \geq 0$ ✓

② $\int f(x)\,dx = 1$

$$\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx}_{\text{a number } I} \overset{?}{=} 1$$

Want to show: $I = 1 \iff \boxed{I^2 = 1}$

$\int^{\infty}$ $\int^{\infty}$

$$I^2 = I \cdot I = \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}x^2\right) dx}_{I} \; \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}y^2\right) dy}_{I}$$

$$\overset{\text{Calc III}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(-\tfrac{1}{2}x^2\right) \exp\left(-\tfrac{1}{2}y^2\right) dx\, dy$$

$$= \iint_{\mathbb{R}^2} \frac{1}{2\pi} \exp\left(-\tfrac{1}{2}\left(x^2+y^2\right)\right) dx\, dy$$

Polar Coordinates?