Lecture 19 - Covariance and Correlation

Tuesday, November 16, 2021 1:58 PM

<u>Defn:</u>

$$\mathrm{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}X)\right]$$

$$g(x,y) = (x-\mu_x)(y-\mu_y)$$

$$\mu_x = \mathbb{E}X, \ \mu_y = \mathbb{E}Y$$

<u>Defn:</u> Correlation

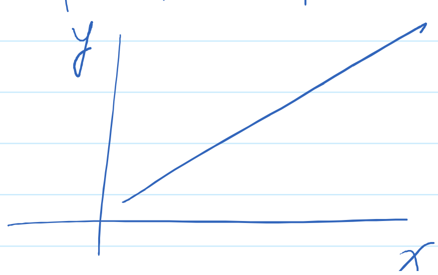
Re-scaled covariance so that it is btwn ± 1.

$$\boxed{\mathrm{Cor}(X, Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}}$$

$$sd(X) = \sqrt{\mathrm{Var}(X)} \ ; \ sd(Y) = \sqrt{\mathrm{Var}(Y)}$$

$$= \frac{\mathrm{Cov}(X,Y)}{sd(X)\,sd(Y)}$$

<u>Note:</u> $\mathrm{Cor} = +1$ perfect pos. lin. rel.



$\text{cor} = -1$ " neg. lin. rel

$\text{cor} \approx 0$ no lin. rel.

Theorem: If $a, b \in \mathbb{R}$,

$$\boxed{Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X, Y)}$$

pf. $Z = aX + bY$

$$Var(Z) = \mathbb{E}\left[(Z - \mathbb{E}[Z])^2\right]$$

$$= \mathbb{E}\left[(aX + bY - \mathbb{E}[aX + bY])^2\right]$$

$$= \mathbb{E}\left[(aX + bY - a\mathbb{E}X - b\mathbb{E}Y)^2\right]$$

$$= \mathbb{E}\Big[\big(\underbrace{a(X - \mathbb{E}X)}_{\alpha} + \underbrace{b(Y - \mathbb{E}Y)}_{\beta}\big)^2\Big]$$

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$= \mathbb{E}\left[a^2(X - \mathbb{E}X)^2 + b^2(Y - \mathbb{E}Y)^2 + 2ab(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$$

$$= a^2 \underbrace{\mathbb{E}\left[(X - \mathbb{E}X)^2\right]}_{Var(X)} + b^2 \overbrace{\mathbb{E}\left[(Y - \mathbb{E}Y)^2\right]}^{Var(Y)} +$$

$$2ab\, \mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$$

$$\overline{Var(X)} \qquad 2ab\,\underbrace{E[(X-EX)(Y-EY)]}_{Cov(X,Y)}$$

$$Var(aX+bY)$$
$$= a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X,Y).$$

Theorem: $a,b \in \mathbb{R}$,

$$Cov(aX+b, Y) = a\,Cov(X,Y)$$

recall: $Var(aX+b) = a^2 Var(X)$

pf.

$$Cov(aX+b, Y) = E[(aX+b - E[aX+b])(Y - E[Y])]$$
$$= E[(\underbrace{aX\cancel{+b} - aEX \cancel{-b}}_{a(X-EX)})(Y-EY)]$$
$$= a\,E[(X-EX)(Y-EY)]$$
$$= a\,Cov(X,Y).$$

Note: $Cov(X, aY+b) = a\,Cov(X,Y)$

$$Cov(aX+b, cY+d) = ac\,Cov(X,Y)$$

Theorem: If $a,b,c,d \in \mathbb{R}$

Theorem: If $a, b, c, d \in \mathbb{R}$

$$\boxed{\operatorname{cor}(aX+b,\ cY+d) = \operatorname{Sign}(a)\operatorname{Sign}(c)\operatorname{cor}(X,Y)}$$

$$\operatorname{Sign}(x) = \begin{cases} +1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$x \neq 0$ then $\operatorname{Sign}(x) = \dfrac{x}{|x|}$

Ex. $\operatorname{Cor}(-5X, Y) = -\operatorname{Cor}(X, Y)$

Pf. $a, c \neq 0$

$$\operatorname{Cor}(aX+b, cY+d) = \frac{\operatorname{Cov}(aX+b, cY+d)}{\sqrt{\operatorname{Var}(aX+b)}\sqrt{\operatorname{Var}(cY+d)}}$$

$$= \frac{ac\operatorname{cov}(X,Y)}{\sqrt{a^2\operatorname{Var}(X)}\sqrt{c^2\operatorname{Var}(Y)}}$$

$$= \underbrace{\frac{a}{|a|}}_{\operatorname{Sign}(a)}\ \underbrace{\frac{c}{|c|}}_{\operatorname{Sign}(c)}\ \underbrace{\frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{Var}(X)}\sqrt{\operatorname{Var}(Y)}}}_{\operatorname{Cor}(X,Y)}$$

Claim:

$$-1 \le Cor(X, Y) \le 1$$

pf.

$$\tilde{X} = \frac{X - E[X]}{\sqrt{Var(X)}}$$

$$\tilde{Y} = \frac{Y - E[Y]}{\sqrt{Var(Y)}}$$

claim: $E\tilde{X} = 0$
$Var(\tilde{X}) = 1$

$$Cor(X, Y) = Cor(\tilde{X}, \tilde{Y})$$

(1) $$0 \le Var(\tilde{X} \pm \tilde{Y}) = \underbrace{Var(\tilde{X})}_{1} + \underbrace{Var(\tilde{Y})}_{1} \pm 2Cov(\tilde{X}, \tilde{Y})$$

(2) $$cor(\tilde{X}, \tilde{Y}) = \frac{Cov(\tilde{X}, \tilde{Y})}{1 \cdot 1} = cov(\tilde{X}, \tilde{Y})$$

$$0 \le 2 \pm 2 Cor(\tilde{X}, \tilde{Y})$$

$$\Rightarrow \quad 1 \pm Cor(X, Y) \ge 0$$

$$\Rightarrow \quad 1 - Cor(X, Y) \ge 0 \quad \text{and} \quad 1 + Cor(X, Y) \ge 0$$

$$\Rightarrow \quad Cor(X, Y) \le 1 \quad \text{and} \quad Cor(X, Y) \ge -1$$

Truth: $Y = aX + b \Rightarrow \operatorname{cor}(X, Y) = \pm 1$

Note: $\operatorname{Cov}(X, X) = \operatorname{Var}(X).$

For var: $\operatorname{Var}(X) = E[X^2] - (EX)^2.$

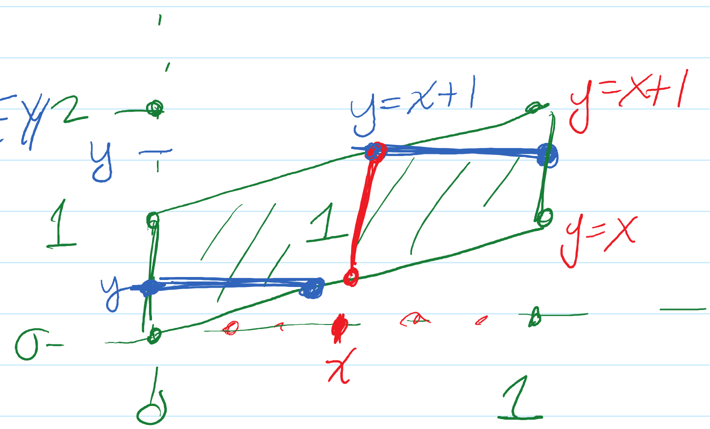
Theorem: Short-cut for Covariance

$$\operatorname{Cov}(X, Y) = E[XY] - E[X]\,E[Y].$$

Ex. $f(x, y) = 1$ for $0 < x < 1$, $x < y < x+1$

$E[XY] = 7/12$

$\operatorname{Cov}(X, Y) = E[XY] - EX\,EY^2$



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_x^{x+1} 1\,dx = x + 1 - x$$

$$\boxed{= 1 \text{ for } 0 < x < 1}$$

So $X \sim U(0,1) \longrightarrow EX = 1/2$

$\text{Var}X = 1/12$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \begin{cases} \int_0^y 1\,dx & 0<y<1 \\ \int_{y-1}^1 1\,dx & 1<y<2 \end{cases} = \begin{cases} y, & 0<y<1 \\ 2-y, & 1<y<2 \end{cases}$$

Can show: $E[Y] = 1$ and $\text{Var}(Y) = 1/6$

$$\text{Cov}(X,Y) = E[XY] - EX\,EY$$

$$= 7/12 - (1/2)(1)$$

$$\text{Cor}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \boxed{\frac{7/12 - (1/2)}{\sqrt{1/12}\sqrt{1/6}}}$$

Conditional Probability

$$P(A|B) = \frac{P(AB)}{}$$

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

If X, Y discrete let

$$A = \{X = x\}, \quad B = \{Y = y\}$$

joint PMF

then

$$P(X = x \mid Y = y) = \frac{P(AB)}{P(B)} = \frac{P(X = x, Y = y)}{P(Y = y)}$$

conditional PMF of X given Y

marginal PMF of Y

$$= \frac{f(x,y)}{f_Y(y)}$$

Defn: Conditional Dists.

If X, Y Biv RVs and discrete then the conditional PMF of X given $Y = y$ is defined as

$$f_{X \mid Y = y}(x) = f(x \mid y) = \boxed{\frac{f(x,y)}{f_Y(y)}}.$$

Ex. Joint PMF

$f_X(0) = 4/18$ $f(1,20)$

$Y \backslash X$	0	1	2
30	$\boxed{0}$	3/18	4/18
20	$\boxed{2/18}$	4/18	0
10	$\boxed{2/18}$	3/18	0

$\leftarrow$ PMF of $Y \mid X = 0$

$$f(y \mid 0) = \frac{f(0,y)}{f_X(0)} = \begin{cases} \frac{f(0,10)}{f_X(0)} = \frac{2/18}{4/18} = \frac{1}{2} & y = 10 \\ \frac{f(0,20)}{f_X(0)} = \frac{2/18}{4/18} = \frac{1}{2} & y = 20 \\ \frac{f(0,30)}{f_X(0)} = \frac{0}{4/18} = 0 & y = 30 \end{cases}$$

What about cts?

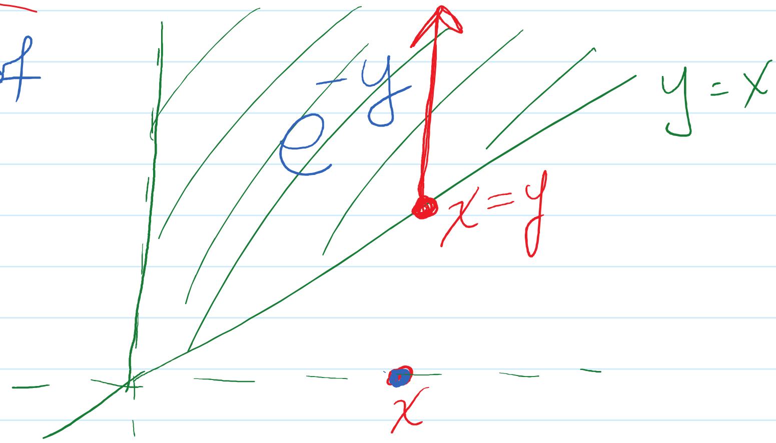
If X, Y cts then then conditional dist of $X \mid Y = y$ is

$$f_{X \mid Y=y}(x) = f(x \mid y) = \frac{f(x,y)}{f_Y(y)}.$$

Ex. $f(x,y) = e^{-y}$ for $0 < x < y$

what is the PDF of

$Y \mid X = x$



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{x}^{\infty} e^{-y}\,dy = e^{-x}$$

$$f(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} \quad \text{for } 0 < x < y$$