Lecture 14 - More Common Distributions

Tuesday, October 26, 2021 2:00 PM

Bernoulli Distribution

$X \sim \text{Bern}(p)$, $p \in [0,1]$.

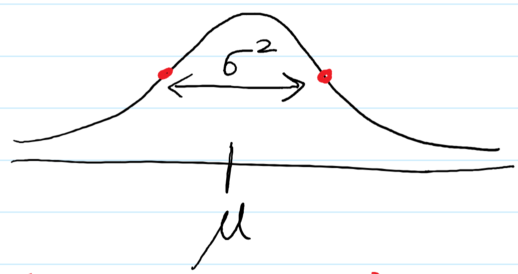
If I flip a coin and let $X=1$ if H, $X=0$ if T.

$X \sim \text{Bin}(n=1, p)$.

Normal / Gaussian Distribution

$X \sim N(\mu, \sigma^2)$

$\mu \in \mathbb{R}$ $\quad \sigma^2 > 0$



PDF: $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad \forall x \in \mathbb{R}$$

special case: $\mu = 0$ and $\sigma^2 = 1$

(standard normal)

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

CDF: no simple form

$$F(x) = \int^{x} f(t)\,dt$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

<u>Claims:</u> $E[X] = \mu$ and $Var(X) = \sigma^2$.

<u>MGF:</u>

$$M(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(\underbrace{-\frac{1}{2\sigma^2}(x-\mu)^2}_{\text{expand}}\Big)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}\big(\underbrace{x^2 - 2\mu x + \mu^2 - 2\sigma^2 t x}_{(a+b)^2 \text{ complete the square}}\big)\Big)\,dx$$

$$x^2 - 2\mu x + \mu^2 - 2\sigma^2 t x$$

$$= \underbrace{x^2 - 2x(\mu + \sigma^2 t)}\, + \mu^2 + \underbrace{(\mu+\sigma^2 t)^2} - (\mu + \sigma^2 t)^2$$

$$= \underbrace{(x - (\mu + \sigma^2 t))^2 + \mu^2 - (\mu + \sigma^2 t)^2}$$

$$= \int \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}\Big[(x-(\mu+\sigma^2 t))^2 + \underbrace{\mu^2 - (\mu+\sigma^2 t)^2}_{\text{don't depend on } x}\Big]\Big)\,dx$$

$$\underbrace{\int \text{PDF of } N(\mu+\sigma^2 t, \sigma^2) = 1}_{} \qquad \text{don't depend on } x$$

$$= \exp\left(\mu^2 - (\mu+\sigma^2 t)^2\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x-(\mu+\sigma^2 t)\right)^2\right) dx}_{1}$$

algebra

$$= \boxed{\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) = M(t)}$$

$$\mathbb{E}[X] = \left.\frac{dM}{dt}\right|_{t=0} = \left.(\mu + \cancel{\sigma^2 t}^{0}) \exp\left(\cancel{\mu t + \frac{\sigma^2 t^2}{2}}^{0}\right)\right|_{t=0}$$

$$= \boxed{\mu = \mathbb{E}[X]}$$

$$\mathbb{E}[X^2] = \left.\frac{d^2 M}{dt^2}\right|_{t=0} = \underbrace{(\sigma^2)\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}_{1} + \underbrace{(\mu + \cancel{\sigma^2 t}^{0})^{2} \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}_{1}$$

$$= \sigma^2 + \mu^2$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sigma^2 + \mu^2 - \mu^2 \boxed{= \sigma^2}$$

Theorem: Linear Functions of Normal RVs

Let $X \sim N(\mu, \sigma^2)$ and

$$Y = aX + b$$

then $$Y \sim N(a\mu + b, a^2\sigma^2).$$

intuition: $$E[Y] = aEX + b = a\mu + b$$
$$Var(Y) = a^2 Var X = a^2\sigma^2$$

Recall the MGF of a $N(\mu, \sigma^2)$ is $\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

just need to show that

$$M_Y(t) = \exp\left((a\mu + b)t + \frac{(a^2\sigma^2)t^2}{2}\right)$$

$$M_Y(t) = e^{tb} M_X(at)$$ ← prev. theorem

$$= e^{tb} \exp\left(\mu(at) + \frac{\sigma^2(at)^2}{2}\right)$$

$$= \exp\left(tb + \mu a t + \frac{\sigma^2 a^2 t^2}{2}\right)$$

algebra

Poisson Distribution

- discrete RV

- Support is non-negative integers
  $\{0, 1, 2, 3, \ldots\}$

Canonical experiment:

count of the number of "events" in some time period

Ex. - radioactive decay
- model fish capture

microbio. count   # mRNA in a cell

$X \sim \text{Pois}(\lambda)$

$\lambda > 0$
rate of occurence

# of occurences

$$\boxed{\underline{\text{PMF}}: \quad f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots}$$

$$\frac{x}{x!} = \frac{x}{x(x-1)!} = \frac{1}{(x-1)!}$$

Expected value:

$x! \quad x(x-1)! \quad (x-1)!$

Expected value:

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x f(x) = \sum_{x=\cancel{0}1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

$$\boxed{e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}}$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\lambda \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \quad e^{\lambda}$$

$$= e^{-\lambda}\lambda e^{\lambda}$$

$$\boxed{\mathbb{E}[X] = \lambda}$$

$$\frac{1}{(x-2)!}$$

$$\mathbb{E}[X(X-1)] = \sum_{x=\cancel{0}2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!}$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda}\lambda^2 \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \quad e^{\lambda}$$

$$\mathbb{E}[X(X-1)] = e^{-\lambda}\lambda^2 e^{\lambda} = \lambda^2$$

$$\overset{=}{\ } \mathbb{E}[X^2] - \mathbb{E}[X]$$

So $\mathbb{E}[X^2] = \mathbb{E}[X^2] - \mathbb{E}[X] + \mathbb{E}[X] = \lambda^2 + \lambda$

So $\mathbb{E}[X^2] = \underbrace{\mathbb{E}[X^2] - \mathbb{E}[X]}_{\lambda^2} + \underbrace{\mathbb{E}[X]}_{\lambda} = \lambda^2 + \lambda$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda}$$

MGF:

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$
$$= e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} (\lambda e^t)^x / x!}_{e^{\lambda e^t}}$$
$$= e^{-\lambda} e^{\lambda e^t}$$
$$\boxed{= \exp(\lambda(e^t - 1))}$$

Gamma Distribution generalize exponential dist

Lets discuss the Gamma function.

Gamma fn extends factorial to $\mathbb{R}^+$

$$\Gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$

so that $a > 0$

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \, dx.$$