Lecture 13 - MGFs and Common Distributions

Thursday, October 21, 2021 1:59 PM

$$M(t) = \mathbb{E}[e^{tX}].$$

Theorem: If X is a RV w/ MGF $M(t)$ then

$$\left.\frac{d^r M}{dt^r}\right|_{t=0} = M^{(r)}(0) = \mathbb{E}[X^r] = \mu_r$$

pf. recall: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

so $e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \cdots$

needs to converge in some neighborhood of zero

$$M(t) = \mathbb{E}[e^{tX}] = \underline{1 + t\mathbb{E}[X] + \frac{t^2}{2!}\mathbb{E}[X^2] + \frac{t^3}{3!}\mathbb{E}[X^3] + \cdots}$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \mathbb{E}[X] + \frac{2\cancel{t}^{\,0}}{2!}\mathbb{E}[X^2] + \frac{3\cancel{t^2}^{\,0}}{3!}\mathbb{E}[X^3] + \cdots$$

$$= \mathbb{E}[X] + 0 + 0 + 0 + 0$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \frac{2}{2!}\mathbb{E}[X^2] + \frac{3\cdot 2\cdot \cancel{t}^{\,0}}{3!}\mathbb{E}[X^3] + \cdots \to 0 \;\cdots$$

$$= \mathbb{E}[X^2]$$

Ex. $X \sim Bin(n, p)$

Binomial theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$(a+b)^2 = a^2 + 2ab + b^2 = \sum_{i=0}^{2} \binom{2}{i} a^i b^{2-i}$$

$$= \binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2$$

$$M(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} f(x)$$

$$e^{ab} = (e^a)^b$$

$$= \sum_{x=0}^{n} (e^{t})^{x} \binom{n}{x} p^x (1-p)^{n-x}$$

apply binomial theorem

$$= \sum_{x=0}^{n} \binom{n}{x} (\overbrace{pe^t}^{a})^x (\overbrace{1-p}^{b})^{n-x} = (a+b)^n$$

$$\boxed{M(t) = (pe^t + 1-p)^n}$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \left. n(\underbrace{p\cancel{e^t}^{1} + 1-p}_{1})^{n-1} p\cancel{e^t}^{1} \right|_{t=0} = np = E[X]$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = n(n-1)(\cancel{pe^t + 1-p}^{1})^{n-2} p\cancel{e^t}^{1} p\cancel{e^t}^{1}$$

$$+ n(\cancel{pe^t + 1-p}^{1})^{n-1} p\cancel{e^t}^{1}$$

$$= n(n-1)p^2 + np = E[X^2]$$

Theorem: For constents a and b let

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$$Y = aX + b$$

then

$$M_Y(t) = e^{tb} M_X(at)$$

MGF of Y MGF of X

$$e^{a+b} = e^a e^b$$

pf. $M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(aX+b)}]$

$$= \mathbb{E}[e^{(at)X} e^{tb}]$$

$$= e^{tb}\,\mathbb{E}[e^{(at)X}]$$

$$= e^{tb} M_X(at)$$

Theorem: If X and Y are RVs and

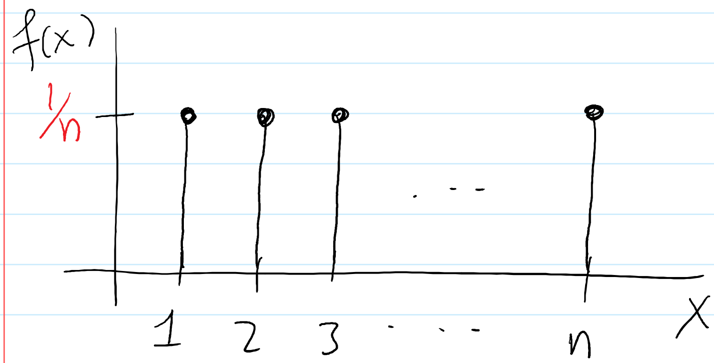
$$M_X(t) = M_Y(t)$$

for t in some neighborhood of zero

then $X \overset{d}{=} Y$.

Discrete Uniform

$X \sim U(\{1, \dots, n\})$

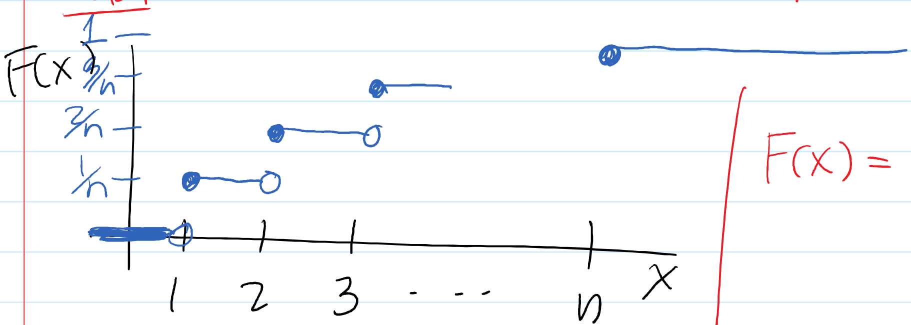


PMF:

$$\boxed{f(x) = \frac{1}{n} \text{ for } x = 1, 2, \dots, n}$$

$$\sum_{x=1}^{n} f(x) = 1 \Leftrightarrow f(x) = \frac{1}{n}$$

CDF:



$$F(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq n \\ 1/n & 1 \leq x < 2 \\ 2/n & 2 \leq x < 3 \\ 3/n & 3 \leq x < 4 \\ \vdots & \end{cases}$$

Expectation:

$$E[X] = \sum_{x=1}^{n} x f(x) = \sum_{x=1}^{n} x \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x$$

$$= \frac{1}{n} \frac{n(n+1)}{2}$$

$$= \boxed{\frac{n+1}{2}}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$E[X^2] = \sum_{x=1}^{n} x^2 \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

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$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}[X]^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \ldots \text{algebra}$$

$$\boxed{\frac{n^2-1}{12}}$$

Moment Generating Function

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=1}^{n} e^{tx}/n = \frac{1}{n}\underbrace{\sum_{x=1}^{n}(e^t)^x}$$

recall: geometric sum

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r} \quad \text{for } |r|<1$$

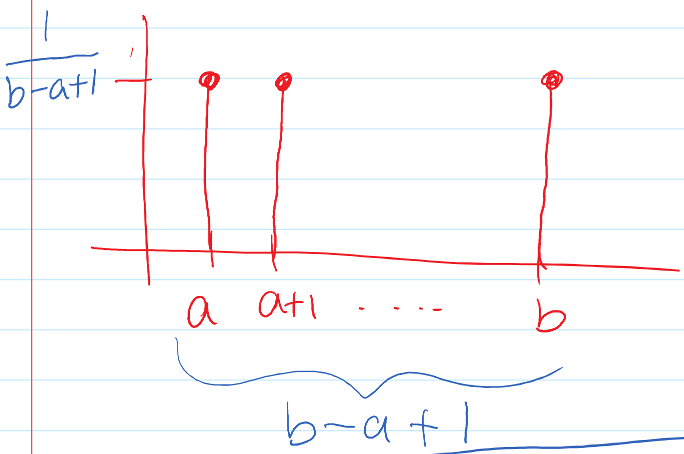
$$= \frac{1}{n}\sum_{x=0}^{n-1}(e^t)^{x+1} = \frac{e^t}{n}\sum_{x=0}^{n-1}(\overbrace{e^t}^{r})^x$$

$$= \frac{e^t}{n}\,\frac{1-(e^t)^n}{1-e^t}$$

$$= \boxed{\frac{e^t - e^{t(n+1)}}{n(1-e^t)}} \quad \text{for } |e^t|<1 \text{ or } t<0$$

$X \sim U(\{a, \ldots, b\})$ If $Y \sim U(\{1, \ldots, n\})$

$n = b - a + 1$



$n = b - a + 1$

then

$X = (a-1) + Y$

$\sim U(\{a, \dots, b\})$

$$\boxed{f(x) = \frac{1}{b-a+1} \quad \text{for } x = a, \dots, b}$$

$$\mathbb{E}[X] = \mathbb{E}[(a-1) + Y] = (a-1) + \mathbb{E}[Y]$$

$n = b-a+1$

$$= a - 1 + \frac{n+1}{2}$$

$$= a - 1 + \frac{b-a+1+1}{2}$$

$$= \boxed{\frac{a+b}{2}}$$

$$\mathrm{Var}(X) = \mathrm{Var}((a-1) + Y)$$

$$= \mathrm{Var}(Y)$$

$$= \frac{n^2 - 1}{12} = \boxed{\frac{(b-a+1)^2 - 1}{12}}$$

$$X = (a-1) + Y$$

$$M_X(t) = e^{(a-1)t} M_Y(t)$$

$$M_X(t) = e^{(a-1)t} M_Y(t)$$

$$= e^{(a-1)t} \frac{e^{t} - e^{t(n+1)}}{n(1-e^{t})}$$

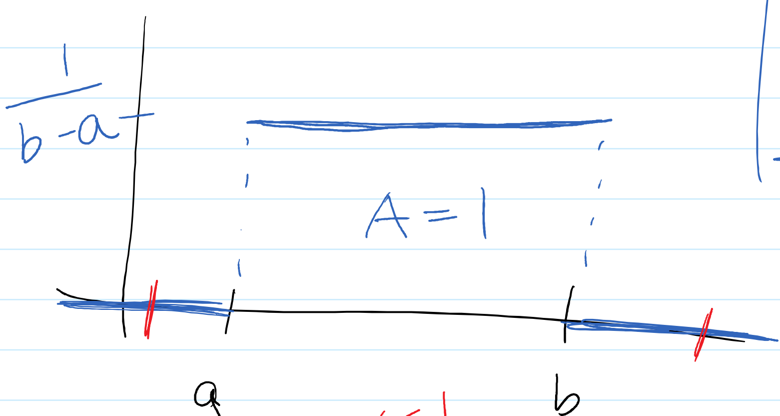
$$= \frac{e^{at} - e^{t(a+n)}}{n(1-e^{t})} \qquad b-a+1$$

$$\boxed{= \frac{e^{at} - e^{t(b+1)}}{(b-a+1)(1-e^{t})}}$$

Continuous Uniform

$$X \sim U(a,b)$$

PDF



$$\boxed{f(x) = \frac{1}{b-a} \text{ for } a<x<b}$$

CDF

$a<X<b$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt = \frac{1}{b-a}\, t \Big|_{a}^{x}$$

$$\boxed{= \frac{x-a}{b-a} \text{ for } a<x<b}$$

$= \dfrac{}{b-a}$ for $a < x < b$

$$\boxed{\begin{array}{ll} \text{If } x < a & F(x) = 0 \\ \phantom{\text{If }} x > b & F(x) = 1 \end{array}}$$

Expectation

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$\boxed{= \frac{a+b}{2}}$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \left.\frac{x^3}{3}\right|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2 + ab + a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2 + ab + a^2}{3}$$

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$Var(X) = \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \ldots \text{ algebra}$$

$$\boxed{= \frac{(b-a)^2}{12}}$$

MGF:

$$M(t) = \mathbb{E}[e^{tX}] = \int_a^b e^{tx} f(x)\, dx$$

$$= \int_a^b \frac{e^{tx}}{b-a}\, dx$$

$$= \frac{1}{b-a} \frac{1}{t} e^{tx} \Big|_a^b$$

$$M(t) \boxed{= \frac{e^{tb} - e^{ta}}{t(b-a)}}$$