# Lecture 2 - Axiomatic Probability

Tuesday, September 7, 2021 1:57 PM

## Defn: Sample Space

The sample space $S$ is the set of possible outcomes.

Ex. Flip a coin.

$$S = \{H, T\}$$

Ex. Roll a six-sided die.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Ex. Roll two dice.

$$S = \{(1,1), (1,2), (1,3), \ldots, (2,1), \ldots, (6,6)\}$$

Ex. Waiting for my bus to arrive, experiment is how long bus takes.

$$S = [0, \infty) \subset \mathbb{R}$$

Ex. Number of customers arriving in my shop today:

$$S = \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$$

types of sample spaces:

(1) finite $(|S| < \infty)$

(2) infinite $(|S| \geq \infty)$

→ (i) countable (e.g. $\mathbb{N}_0$)

→ (ii) uncountable (e.g. $[0, \infty)$)

---

**Defn: Outcome**

We call elements of $S$ "outcomes"

$$\underset{\text{an outcome}}{s} \in \underset{\text{sample space}}{S}$$

Ex. $S = \{1, \ldots, 6\}$

then $1 \in S$ so $1$ is a possible outcome.

---

**Defn: Event**

An event $E$ is a subset of $S$.

---

Ex. $S = \{1, \dots, 6\}$ then

$$E = \{1, 2\} \subset S$$

and so $E$ is the event that I roll a 1 or 2.

Ex. $S = \{(i, j)$ for $1 \le i \le 6$ and $1 \le j \le 6\}$

$$E = \{(2,1), (3,2)\}$$

different

$$F = \{(1,2), (2,3)\}$$

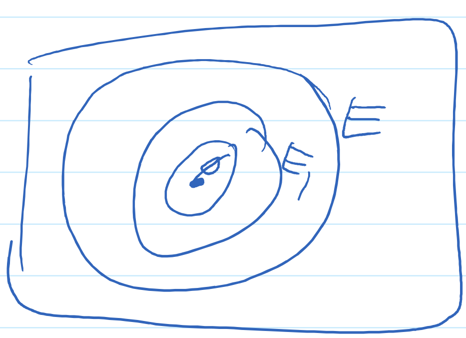
We say an event E "happens" if the observed outcome is in E

Ex. $\emptyset \subset S$ so $\emptyset$ is an event.

nothing happens???

Ex. $S \subset S$ so $S$ is an event.

something happening



# Axiomatic Probability

Given: an experiment and assoc. sample space $S$.

Want: for any event $E \subset S$ want to assign some measure of the likelihood of $E$ occuring.

probability function

Mathematically:

For each $E \subset S$ assign a prob. $\mathbb{P}(E)$

What are the rules for building $\mathbb{P}$?

want

1. mathematically consistent
2. encode some intuition about probability

---

Defn: Probability Function

Given a sample space $S$ a prob. function $\mathbb{P}$ is a function

$$\mathbb{P}: 2^S \longrightarrow \mathbb{R}$$

$$P: 2^{S} \longrightarrow \mathbb{R}$$

domain co-domain

that satisfies the Kolmogorov Axioms

① non-negativity

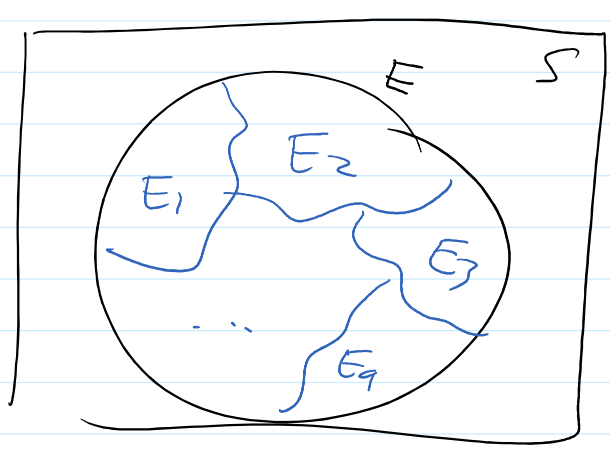
$$P(E) \geq 0 \quad \forall E \subset S$$

② unit-measure

$$P(S) = 1$$

③ If $(E_i)_{i=1}^{\infty}$ is a partition of $E$

$\left( \bigcup_i E_i = E \text{ and } E_i E_j = \emptyset \quad \forall i \neq j \right)$




then

$$P(E) = \sum_{i=1}^{\infty} P(E_i).$$

→ distributive law for partitions

$$\mathbb{P}(E) = \mathbb{P}\left(\bigcup_i E_i\right) = \sum_i \mathbb{P}(E_i)$$

→ One-directional implication

(3) ⇒ same thing for finite partitions.

$E_1, \dots, E_n$ partion $E$ then

$$\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E_i)$$

$$\boxed{\text{Extend finite partita to infinite} \quad E_1 \cup \cdots \cup E_n \cup \emptyset \cup \emptyset}$$

---

Ex, Flip a coin.

$$S = \{H, T\}$$

What is a prob. function on $S$?

$$\mathbb{P}(\{H\}) = 1/2 \qquad \mathbb{P}(S) = 1$$

$$\mathbb{P}(\{T\}) = 1/2 \qquad \mathbb{P}(\emptyset) = 0$$

Valid $\mathbb{P}$?

✓ (1) $\mathbb{P}(E) \geq 0 \quad \forall E \subset S$

✓ (2) $\mathbb{P}(S) = 1$

(3) If $(E_i)$ partition $E$ then $\mathbb{P}(E) = \sum_i \mathbb{P}(E_i)$

<u>One example:</u>

$E = S$ and $E_1 = \{H\}$, $E_2 = \{T\}$

$$\underset{\underset{1}{\parallel}}{\mathbb{P}(S)} = \mathbb{P}(E) = \underset{1/2}{\mathbb{P}(E_1)} + \underset{1/2}{\mathbb{P}(E_2)}$$

---

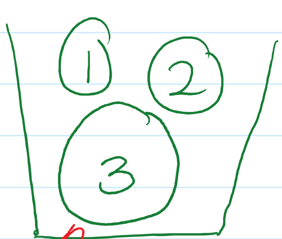
<u>Ex.</u> $S = \{H, T\}$ (another possible $\mathbb{P}$)

$\mathbb{P}(S) = 1$ $\qquad \mathbb{P}(\{H\}) = \alpha$

$\mathbb{P}(\emptyset) = 0$ $\qquad \mathbb{P}(\{T\}) = 1 - \alpha$ where $\alpha \in [0,1]$

---

<u>Ex</u>




$S = \{1, 2, 3\}$

$p_1 = 1/4$ $\quad p_2 = 1/4$ $\quad p_3 = 1/2$

choosing (3) 2x as likely

$\mathbb{P}(\{1,2\}) = p_1 + p_2 = 1/2$

$$\mathbb{P}(\{1,3\}) = p_1 + p_3 = 3/4$$

## Theorem: Finite Sample Space Theorem

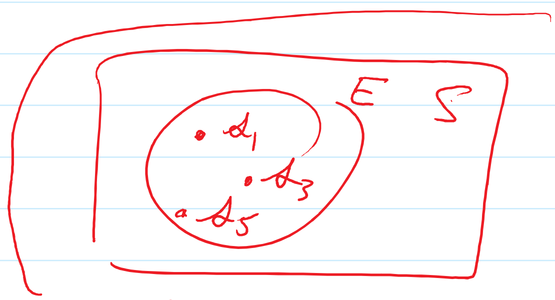
If $S = \{s_1, \ldots, s_n\}$ so that $|S| = n < \infty$

and we choose $p_1, \ldots, p_n$ so that

$$① \; p_i \geq 0 \quad \text{and} \quad ② \; \sum_{i=1}^{n} p_i = 1$$

then the following is a valid prob. fn.

$$\mathbb{P}(E) = \sum_{i : s_i \in E} p_i$$




$$P(E) = p_1 + p_3 + p_5$$

---

✓ ① $\mathbb{P}(E) \geq 0$

$$P(E) = \sum p_i \geq 0 \qquad p_i \geq 0$$

✓ ② $\mathbb{P}(S) = 1$

$$\mathbb{P}(S) = \sum_{i : s_i \in S} p_i = \sum_{i=1}^{n} p_i = 1.$$