# Lecture 8 - Random Variables

Tuesday, September 28, 2021 9:25 AM

Ex. Flip a coin 3 times.

$X$ = # heads among 3 flips.

| $s \in S$ | $X(s)$ |
|---|---|
| H H H | 3 |
| H H T | 2 |
| H T H | 2 |
| H T T | 1 |
| T H H | 2 |
| T H T | 1 |
| T T H | 1 |
| T T T | 0 |

← a function

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Defn: Random Variable

A random variable (RV) $X$ is a function

$$X : S \to \mathbb{R}$$

also called a random variate

or a real-valued random variable

($\mathbb{R}$ not $\mathbb{R}^n$) ← later

or a univariate random variable

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Ex. (1) toss two dice, $X$ = sum of dice

(2) toss a coin 25 times,

$X$ = length of the longest chain of consecutive Hs

(3) observe rainfall,

$X$ = yield of crops

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We'd like to make statements like

$$\mathbb{P}(X=1)$$

abuse of notation

"prob. that $X=1$"

recall: $\mathbb{P}: 2^S \to \mathbb{R}$

really mean e.g. if $X$ = # heads in 3 coin flips

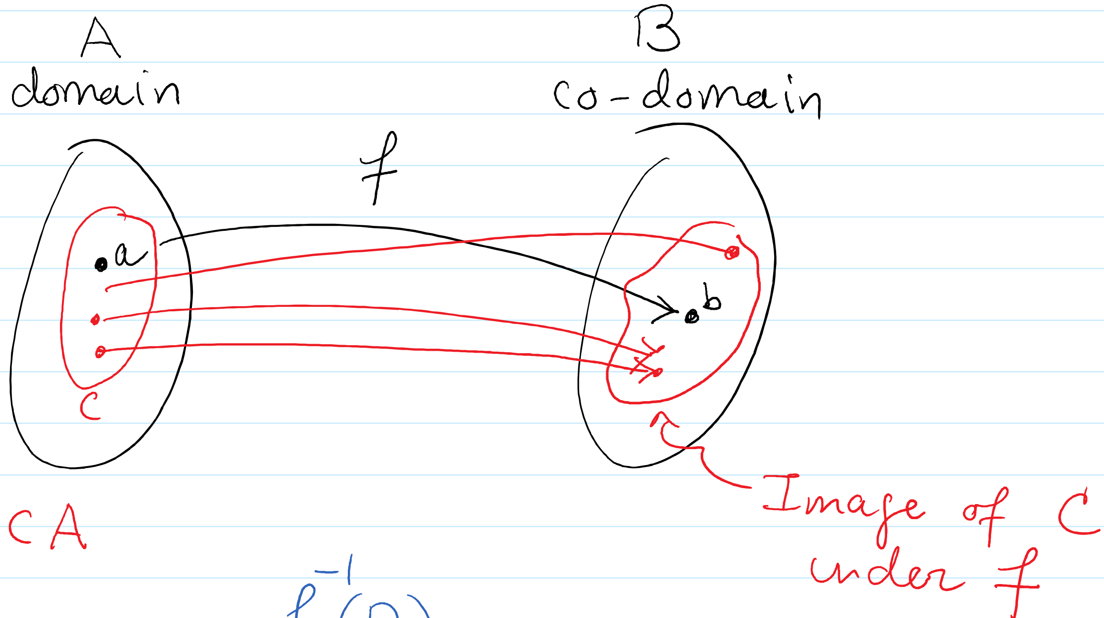
$$\mathbb{P}(X=1) = \mathbb{P}(\{HTT, THT, TTH\}) = 3/8$$

define "$X=1$" as short-hand for

"$X=1$" $= \{ \omega \in S \mid X(\omega) = 1 \}$

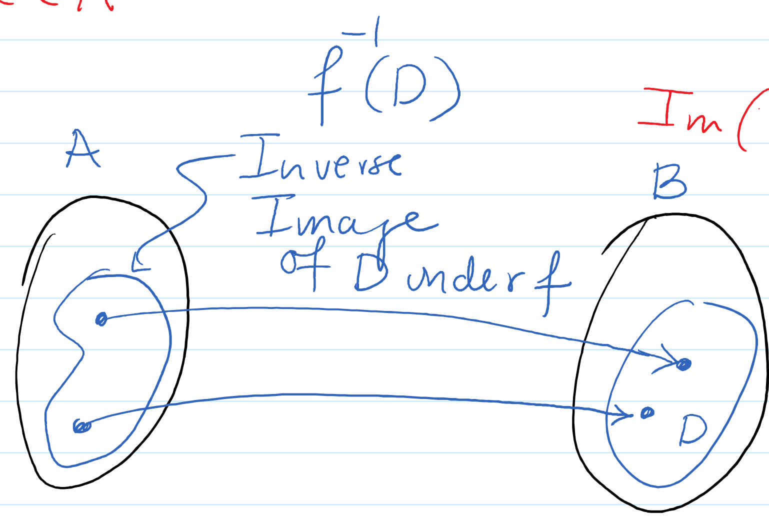
$$\text{"}X = 1\text{"} = \underbrace{\{ s \in S \mid X(s) = 1 \}}_{\text{inverse-image of } \{1\} \text{ under } X}$$

## Review:




$C \subset A$

$Im(C)$ or $f(C)$




$D \subset B$

## Image:

$$f(C) = \{ f(x) \mid x \in C \} \subset B$$

Inverse Image:

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\} \subset A$$

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Notation: If $X$ is a RV we write

$$\mathbb{P}(\underline{X \in A}) \quad \text{where} \quad A \subset \mathbb{R}$$

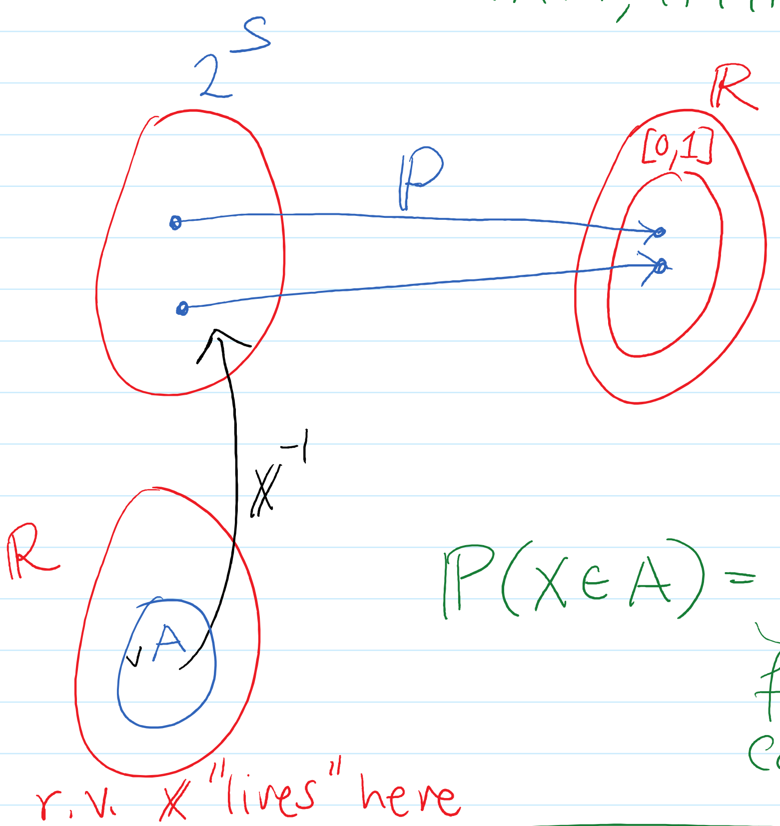
means $\mathbb{P}(\underline{X^{-1}(A)})$.

Ex. $X = \#$ heads in 3 coin tosses

$$\mathbb{P}(X=1) = \mathbb{P}(X \in \{1\})$$
$$= \mathbb{P}(X^{-1}(\{1\}))$$
$$= \mathbb{P}(\{s \in S \mid X(s) = 1\})$$
$$= \mathbb{P}(\{HTT, THT, TTH\}) = 3/8$$

$$\mathbb{P}(X = 1 \text{ or } 2) = \mathbb{P}(X \in \{1,2\})$$
$$= \mathbb{P}(X^{-1}(\{1,2\}))$$
$$= \mathbb{P}(\{s \in S \mid X(s) \in \{1,2\}\})$$
$$= \mathbb{P}(\{HTT, THT, TTH, HHT, HTH, THH\}) = 6/8$$

HHT, HTH, THH}) = 4/8




$$\mathbb{P}(X \in A) = \underbrace{\mathbb{P}(X^{-1}(A))}_{\text{function composition}}$$

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**Defn: Support of a RV**

If $X$ is a RV then the support of $X$ is the set of possible values of $X$

$$X(S)$$

$\uparrow$ Image of $S$ under $X$

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Ex, $X = \#$ Heads in 3 coin flips then $\text{Support}(X) = \{0, 1, 2, 3\}$.

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Notice: $\mathbb{P}(X = 5) = 0$.

↑ from above

more generally if $A \cap \text{Support}(X) = \emptyset$
then $\mathbb{P}(X \in A) = 0$.

pf. $\mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\emptyset) = 0$.

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## Heuristic Types of RVs
(informal)

1) discrete : support is finite or countable

Ex. $X$ = sum of two dice

Ex. $X$ = # of customers arriving in a restaurant

$$\text{Support}(X) = \mathbb{N}$$

2) continuous : support is uncountably infinite

Ex. $X$ = waiting time for a bus to arrive

$$\text{Support}(X) = [0, \infty)$$

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Defn: Cumulative Distribution Function (CDF)

If $X$ is a RV then the CDF of $X$ is a function

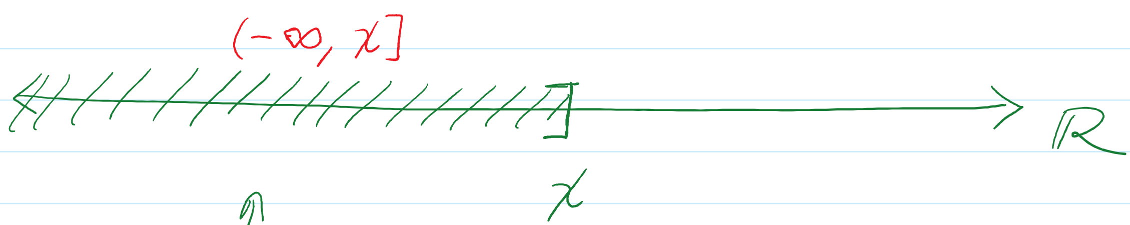
$$F : \mathbb{R} \to \mathbb{R}$$

defined for $x \in \mathbb{R}$ as

$$F(x) = \mathbb{P}(X \le x)$$

random variable

a number




$F(x) =$ prob. $X$ is here

$$\begin{aligned} F(x) &= \mathbb{P}(X \le x) \\ &= \mathbb{P}(X \in (-\infty, x]) \\ &= \mathbb{P}(X^{-1}((-\infty, x])) \end{aligned}$$

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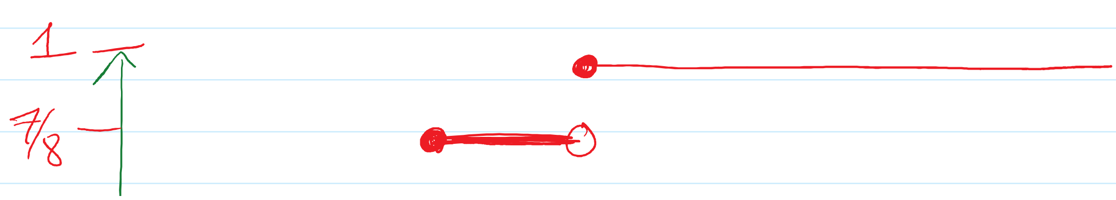
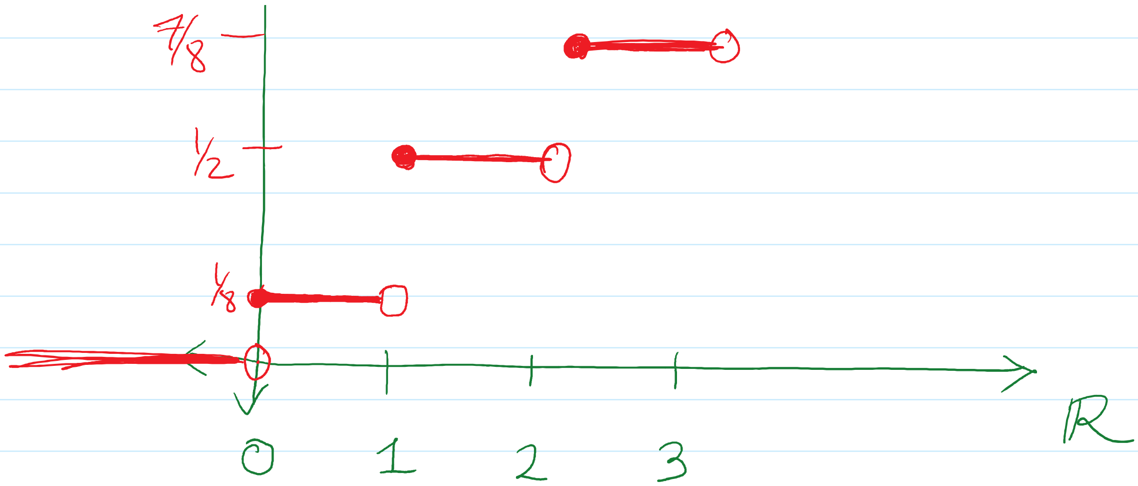
Ex. Flip a coin 3 times.

$X = \#$ heads.






$F(0) = \mathbb{P}(X \leq 0) = \mathbb{P}(X = 0) = 1/8$

$F(1/2) = \mathbb{P}(X \leq 1/2) = \mathbb{P}(X = 0) = 1/8$

$F(1) = \mathbb{P}(X \leq 1) = 1/2$

$F(1{,}5) = \mathbb{P}(X \leq 1{,}5) = \mathbb{P}(X \leq 1) = 1/2$

$F(2) = \mathbb{P}(X \leq 2) = 7/8$

$F(3) = \mathbb{P}(X \leq 3) = 1$

$F(4) = \mathbb{P}(X \leq 4) = 1$

$F(1000000) = \mathbb{P}(X \leq 1000000) = 1$

$F(-1) = \mathbb{P}(X \leq -1) = 0$

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Facts:

If $F$ is a CDF then

$\boxed{(1)\quad 0 \leq F(x) \leq 1 \quad \forall x \in \mathbb{R}}$

$F(x) = \mathbb{P}(X \leq x) \in [0, 1]$

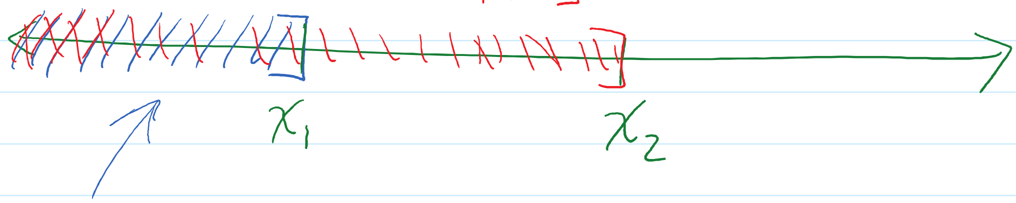
$$\boxed{\textcircled{2}\ \lim_{x\to\infty} F(x) = 1 \quad \text{and} \quad \lim_{x\to -\infty} F(x) = 0}$$

$$\boxed{\textcircled{3}\ F \text{ is non-decreasing}}$$

If $x_1 < x_2$ then $F(x_1) \le F(x_2)$

Pf.

$(-\infty, x_1] \subset (-\infty, x_2]$




$$F(x_1) \le F(x_2)$$

$$\mathbb{P}(X \le x_1) \qquad \mathbb{P}(X \le x_2)$$

$$\mathbb{P}(\underbrace{X^{-1}(\underbrace{(-\infty, x_1]})}) \le \mathbb{P}(\underbrace{X^{-1}(\underbrace{(-\infty, x_2]})})$$

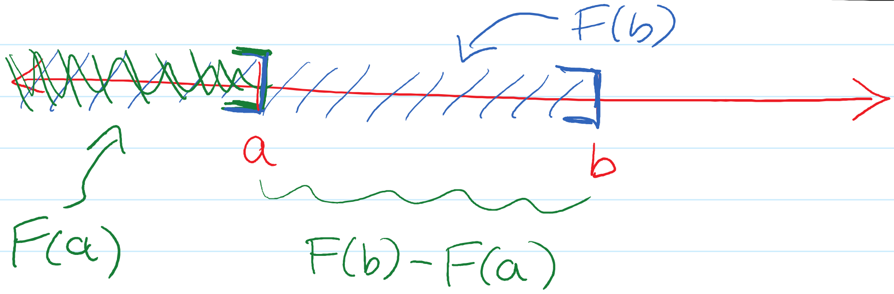
subset

inverse image preserves subsets

$E \subset F$ then $\mathbb{P}(E) \le \mathbb{P}(F)$

$b > a$

$$\boxed{\textcircled{4}\ \mathbb{P}(a < X \le b) = F(b) - F(a)}$$



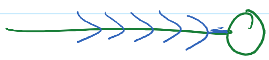
$$\begin{aligned} \mathbb{P}(a < X \le b) &= \mathbb{P}(X \in (a, b]) \\ &= \mathbb{P}(X \in (-\infty, b]) - \mathbb{P}(X \in (-\infty, a]) \\ &= F(b) - F(a) \end{aligned}$$

(5) **F is right-continuous**

$$\lim_{x \to a^+} F(x) = F(a)$$

recall defn cts fn g

$$\lim_{x \to a} g(x) = g(a)$$



Note that cts fns are right continuous.