Lecture 13 - MGFs and Common Distributions

Thursday, October 21, 2021 9:31 AM

Moment generating functions:

$$M(t) = E[e^{tX}]$$

<u>Theorem:</u> If M is the MGF of a RV X then

$$\left.\frac{d^r M}{dt^r}\right|_{t=0} = M^{(r)}(0) = E[X^r] = \mu_r .$$

proof.

recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \cdots$$

need this to converge for some neighborhood around zero

$$M(t) = E[e^{tX}] = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3 E[X^3]}{3!} + \cdots$$

$$\frac{dM}{dt} = E[X] + \frac{2t}{2!}E[X^2] + \frac{3t^2 E[X^3]}{3!} + \cdots$$

$$\left.\frac{dM}{dt}\right|_{t=0} = E[X] + 0 + 0 + 0 \cdots$$

$$\frac{d^2M}{dt^2} = E[X^2] + \frac{3\cdot 2\cdot t}{3!}E[X^3] + \cdots$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = E[X^2] + 0 + 0 + \cdots$$

Ex. Binomial Distribution

Binomial theorem:

$$(a+b)^n = \underline{\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}}$$

$$(a+b)^2 = a^2 + 2ab + b^2 = \underline{\binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2}$$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{n} e^{tx} \underbrace{\binom{n}{x} p^x (1-p)^{n-x}}_{f(x)}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (\overset{a}{pe^t})^x (\overset{b}{1-p})^{n-x}$$

= ... binomial theorem ...

$$M(t) = (pe^t + 1 - p)^n$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \left. \underbrace{n(pe^t + 1 - p)^{n-1}} \underbrace{pe^t} \right|_{t=0} = n(p(1) + 1 - p)^{n-1} p(1)$$

$$= np = \mathbb{E}[X]$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = n(n-1)(\overset{1}{pe^t + 1 - p})^{n-2} (\overset{p^2}{pe^t\, pe^t})$$

$$+ \; n(\underset{1}{pe^t + 1 - p})^{n-1} (\underset{p}{pe^t})$$

$$= n(n-1)p^2 + np = \mathbb{E}[X^2].$$

Theorem: Linear transformations of MGFs

For constants a, b let

$$Y = aX + b$$

$$\underbrace{M_Y(t)}_{\text{MGF of } Y} = e^{bt} \underbrace{M_X(at)}_{\text{MGF of } X}$$

$$e^{a+b} = e^a e^b$$

Pf. $M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(aX+b)}]$

$$= \mathbb{E}[e^{atX} e^{bt}]$$

$$= e^{bt} \mathbb{E}[e^{(at)X}]$$

$$= e^{bt} M_X(at)$$

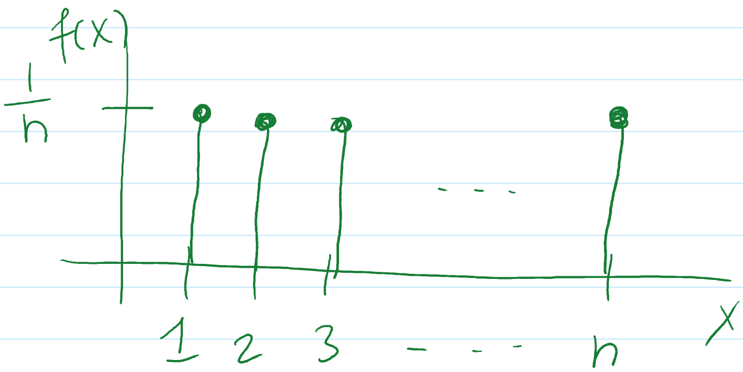
Theorem: If X and Y are RVs w/ MGFs M_X and M_Y and

$$M_X(t) = M_Y(t) \quad \forall t \text{ in some}$$

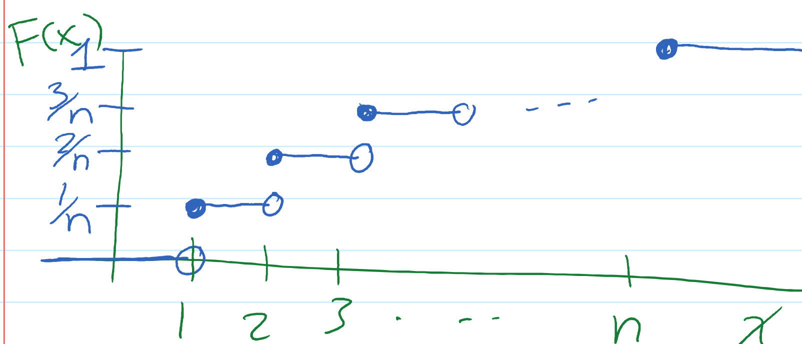
neighborhood of zero

then $X \stackrel{d}{=} Y$.

Discrete Uniform $X \sim U(\{1, \ldots, n\})$



$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x = 1, \ldots, n \\ 0 & \text{else} \end{cases}$$



$$F(x) = \begin{cases} 0, & x < 1 \\ 1/n & 1 \le x < 2 \\ 2/n & 2 \le x < 3 \\ \vdots & \\ 1 & x \ge n \end{cases}$$

$$\mathbb{E}[X] = \sum_{x=1}^{n} x f(x) = \sum_{x=1}^{n} x \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x$$

$$= \frac{1}{n} \frac{n(n+1)}{2}$$

$$= \frac{n+1}{2}$$

Aside

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\mathbb{E}[X^2] = \ldots = \sum_{x=1}^{n} x^2 \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{}$$

$$= \frac{n+1}{2}$$

$$\sum_{i=1} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$E[X^2] = \cdots = \sum_{x=1}^{n} x^2 \, \tfrac{1}{n} = \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{1}{n}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{2^2}$$

$$= \ldots \text{ algebra } = \boxed{\frac{n^2-1}{12}}$$

aside: $(e^a)^b = e^{ab}$

$$M(t) = E[e^{tX}] = \sum_{x=1}^{n} e^{tx} \underbrace{\tfrac{1}{n}}_{f(x)} = \frac{1}{n}\sum_{x=1}^{n} (e^t)^x$$

recall: partial sum formula for geometric series

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r} \quad \text{for } |r| < 1$$

$$\frac{1}{n}\sum^{n-1}(e^t)^{x+1} = \frac{e^t}{n}\sum^{n-1}(\overset{r}{e^t})^x = \frac{e^t}{n}\,\frac{1-(e^t)^n}{\ldots}$$

$$\frac{1}{n}\sum_{x=0}^{n-1}(e^t)^{x+1} = \frac{e^t}{n}\sum_{x=0}^{n-1}(\overset{r}{e^t})^x = \frac{e^t}{n}\,\frac{1-(e^t)^n}{1-e^t}$$

$$\boxed{= \frac{e^t - e^{t(n+1)}}{n(1-e^t)}}$$

as long as $|e^t| < 1$

or $\log e^t < \log(1)$

for $\boxed{t < 0}$

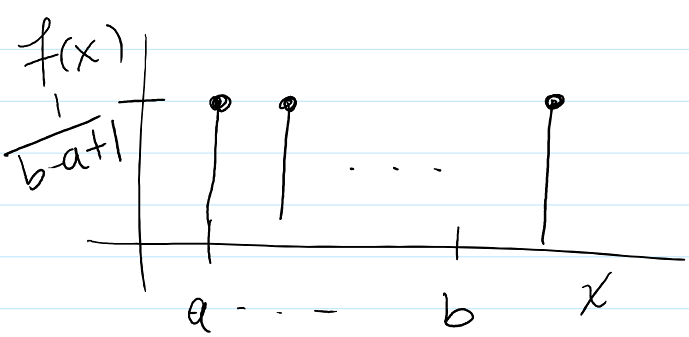
Consider $\underline{X \sim U(\{a, \dots, b\})}$

$\uparrow$ a, b integers

If $Y \sim U(\{1, \dots, n\})$ let $n = b - a + 1$

then

$$X = \underline{(a-1)} + Y$$



Need: $\sum_{x=a}^{b} f(x) = 1$

then $\sum_{x=a}^{b} c = 1$

$c \sum_{x=a}^{b} 1 = 1$

$c(b-a+1) = 1$

so $c = \frac{1}{}$

$b-a+1$

$$\mathbb{E}[X] = \mathbb{E}[(a-1)+Y]$$
$$= (a-1) + \mathbb{E}[Y]$$
$$= (a-1) + \frac{n+1}{2}$$
$$= (a-1) + \frac{(b-a+1)+1}{2} = \ldots \text{algebra} = \boxed{\frac{a+b}{2}}$$

$$Var(X) = Var((a-1)+Y)$$
$$= Var(Y) = \frac{n^2-1}{12} = \boxed{\frac{(b-a+1)^2-1}{12}}$$

$$X = (a-1)+Y$$

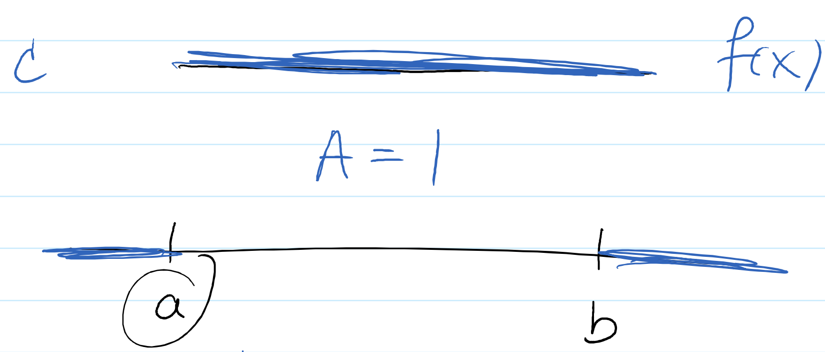
$$M_X(t) = e^{(a-1)t} M_Y(t) = e^{(a-1)t} \frac{e^t - e^{t(n+1)}}{n(1-e^t)}$$
$$= \frac{e^{at} - e^{t(n+a)}}{n(1-e^t)}$$
$$= \frac{e^{at} - e^{t(b\cancel{-a}+1\cancel{+a})}}{(b-a+1)(1-e^t)}$$
$$= \boxed{\frac{e^{at} - e^{t(b+1)}}{(b-a+1)(1-e^t)}}$$

$$(b-a+1)(1-e^t)$$

Continuas Uniform

$$X \sim U(a,b)$$

means: density of X is same over (a,b)



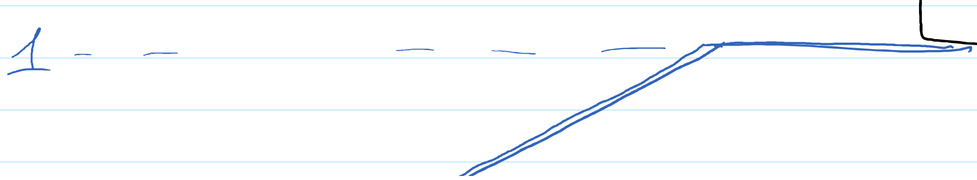
Need: $\displaystyle\int_a^b c\,dx = 1 \iff c = \frac{1}{b-a}$

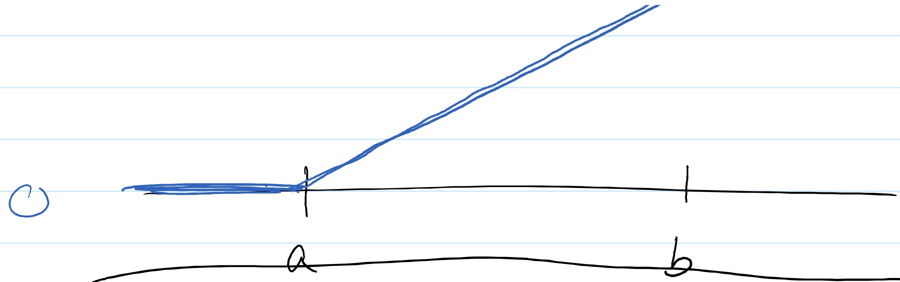
$$\boxed{\text{PDF: } f(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{else} \end{cases}}$$

$a<x<b$

CDF: $\displaystyle F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_a^x \frac{1}{b-a}\,dt = \frac{1}{b-a}\left[t\right]_a^x$

$$= \boxed{\frac{x-a}{b-a}}$$





$$\boxed{F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a < x < b \\ 1 & x > b \end{cases}}$$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a} \frac{x^2}{2}\Big|_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$= \frac{a+b}{2}$$

$$\mathbb{E}[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a} \frac{x^3}{3}\Big|_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2+ba+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ba+a^2}{3}$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{b^2+ba+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \cdots = \boxed{\frac{(b-a)^2}{12}}$$

<u>MGF:</u> $\mathbb{E}[e^{tX}] = \cdots$