Lecture 23 - Multivariate RVs

Thursday, December 2, 2021 9:30 AM

If $X_1, \dots, X_n$ are RVs then

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

is called a <u>multivariate Random variable</u>

or a <u>random vector</u>.

<u>Defn</u>: <u>PMF/PDF</u>

If the X_i's are discrete then <u>joint PMF</u> is

$$f(\underset{\sim}{x}) = \mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

$\underset{\sim}{x} \in \mathbb{R}^n$ where $\underset{\sim}{x} = (x_1, \dots, x_n)$

<u>Cts case</u>: if X_i's are continuous we define the joint PDF as the function $f: \mathbb{R}^n \to \mathbb{R}$ so that $\forall A \subset \mathbb{R}^n$ then

$$\mathbb{P}(\underset{\sim}{X} \in A) = \int_A f(\underset{\sim}{x})\, d\underset{\sim}{x}$$

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$$= \underbrace{\int\int_A \cdots \int}_{n\text{-times}} f(x_1, x_2, x_3, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

Expectation

If $g : \mathbb{R}^n \to \mathbb{R}$

$$\mathbb{E}[g(\underset{\sim}{X})] = \begin{cases} \displaystyle\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n) & \text{(discrete)} \\ \displaystyle\int \cdots \int_{\mathbb{R}^n} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n & \text{(cts)} \end{cases}$$

Defn: Marginal Dists

The marginal dist. of X_i is

$$f_{X_i}(x_i) = \begin{cases} \displaystyle\sum_{x_1}\sum_{x_2}\cdots\sum_{x_{i-1}}\sum_{x_{i+1}}\cdots\sum_{x_n} f(x_1, \ldots, x_n) & \text{(discrete)} \\ \displaystyle\int \cdots \int f(x_1, \ldots, x_n)\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n & (\ldots) \end{cases}$$

$$\int \cdots \int f(x_1, \dots, x_n)\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n \ (\text{cts})$$

If I want marginal dist of some subsequence
$X_{i_1}, \dots, X_{i_m}$

we just sum or integrate joint PMF/PDF
over all vars. but my subseq.

If I have two seqs of RVs
$X_1, \dots, X_m$ and $Y_1, \dots, Y_n$

the conditional dist. of the Xs given Ys
is

$$f(x_1, \dots, x_m \mid y_1, \dots, y_n) = \frac{f(x_1, \dots, x_m, y_1, \dots, y_n)}{f(y_1, \dots, y_n)}$$

Ex. Let $X_1, \dots, X_4$ have a joint PDF
given by

$$f(x_1, \ldots, x_4) = \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4)$$

for $0 < x_i < 1$.

① $P(X_1 < 1/2, X_2 < 3/4, X_4 > 1/2)$

$$= \underset{A}{\iiiint} f(x_1, \ldots, x_4)\, dx_1\, dx_2\, dx_3\, dx_4 \qquad 0 < X_1 < 1/2$$

$$= \int_{1/2}^{1}\int_{0}^{1}\int_{0}^{3/4}\int_{0}^{1/2} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_1\, dx_2\, dx_3\, dx_4$$

$$= \cdots = 3/256$$

② Joint of X_1 and X_2?

$$f(x_1, x_2) = \iint f(x_1, \ldots, x_4)\, dx_3\, dx_4$$

$$= \int_0^1 \int_0^1 \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_3\, dx_4$$

$$= \cdots$$

$$= \frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)$$

③ $E[\ldots] = \int^1\int^1\int^1\int^1$

(3) $\mathbb{E}[X_1 X_2] = \int_0^1\int_0^1\int_0^1\int_0^1 x_1 x_2 \frac{3}{4}(x_1^2+x_2^2+x_3^2+x_4^2)\, dx_3\, dx_4\, dx_1\, dx_2$

$$= \int_0^1\int_0^1 x_1 x_2\left(\frac{1}{2}+\frac{3}{4}(x_1^2+x_2^2)\right) dx_1\, dx_2$$

$$= \cdots = 5/16$$

(4) Conditional Dist

$$f(x_3,x_4 \mid x_1,x_2) = \frac{f(x_1,x_2,x_3,x_4)}{f(x_1,x_2)}$$

$$\boxed{= \frac{\frac{3}{4}(x_1^2+x_2^2+x_3^2+x_4^2)}{\frac{1}{2}+\frac{3}{4}(x_1^2+x_2^2)}}$$

Mutual Independence

We say $X_1, \ldots, X_n$ are mutually independent

if

$\mathbb{P}(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n)$

$$= \mathbb{P}(X_1 \in A) \cdots \mathbb{P}(X_n \in A_n)$$

$$\forall A_i \subset \mathbb{R}$$

Theorem: $X_1, \ldots, X_n$ are mutually independent iff

ⓐ $f(x_1, \ldots, x_n) = f(x_1) f(x_2) \cdots f(x_n)$

or

ⓑ $F(x_1, \ldots, x_n) = F(x_1) F(x_2) \cdots F(x_n)$

Theorem: If $(X_i)_{i=1}^n$ are independent and

$g_i : \mathbb{R} \to \mathbb{R}$

then

① $(g_i(X_i))_{i=1}^n$ are independent

② $\mathbb{E}[g_1(X_1) g_2(X_n) \cdots g_n(X_n)]$

$$= \mathbb{E}[g_1(X_1)] \mathbb{E}[g_2(X_2)] \cdots \mathbb{E}[g_n(X_n)]$$

Corollary: MGF of Sums of Independent RVs

If X_i are independent and

If X_i are independent and

$$Z = \sum_{i=1}^{n} X_i$$

then

$$M_Z(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$$

$$= \prod_{i=1}^{n} M_{X_i}(t)$$

Follow-on:

$$Z = \sum_{i=1}^{n} (a_i X_i + b_i)$$

$$M_{aX+b}(t) = e^{tb} M_X(at)$$

univariate

then

$$M_Z(t) = e^{t\sum_{i=1}^{n} b_i} \prod_{i=1}^{n} M_{X_i}(a_i t)$$

Ex. $X_i \sim N(\mu_i, \sigma_i^2)$

and independent

$$Z = \sum_{i=1}^{n} (a_i X_i + b_i) \sim N\left(\sum_{i=1}^{n}(a_i \mu_i + b_i), \sum_{i=1}^{n} b_i^2 \sigma_i^2\right)$$

pf. uses prev. theorem.

Multivariate Transformations

I have $\underset{\sim}{X} = (X_1, \ldots, X_n)^T$

and $g : \mathbb{R}^n \to \mathbb{R}^n$

consider $U = g(\underset{\sim}{X})$.

$U_i = g_i(X_1, \ldots, X_n)$

If the X_i are continuous,

and g is invertible,

and g^{-1} is differentiable

then

$$f_U(u) = f_{\underset{\sim}{X}}(g^{-1}(u)) \left| \det J \right|$$

$n \times n$ mtx

$$J_{ij} = \frac{\partial g_i^{-1}}{\partial u_j}$$

Means / Variance for MV-RVs

Uni: $\mathbb{E}[X] \in \mathbb{R}$

$$Var(X) = \mathbb{E}[(X - \mathbb{E}X)^2] \in \mathbb{R}$$

MV case: $\underset{\sim}{X} = (X_1, \ldots, X_n)^T$ – n-dim'l

$$\mu = \mathbb{E}[\underset{\sim}{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix} \in \mathbb{R}^n \quad \text{(mean vector)}$$

covariance matrix $\Sigma = Cov(\underset{\sim}{X}) \in \mathbb{R}^{n \times n}$

where $\Sigma_{ij} = Cov(X_i, X_j)$

notice: $\Sigma_{ii} = Cov(X_i, X_i) = Var(X_i)$

$$\Sigma = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & \cdots & \\ Cov(X_2, X_1) & Var(X_2) & & \\ \vdots & & \ddots & \\ & & & Var(X_n) \end{bmatrix}$$

Uni: $Var(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$

mv: $\text{Cov}(\underset{\sim}{X}) = \mathbb{E}\left[(\underset{\sim}{X} - \mathbb{E}\underset{\sim}{X})(\underset{\sim}{X} - \mathbb{E}\underset{\sim}{X})^T\right]$

Theorems: If $a \in \mathbb{R}^m$ and $B \in \mathbb{R}^{m \times n}$ and $\underset{\sim}{X}$ is a n-dim'l RVector then

$a + B\underset{\sim}{X}$ is a m-dim'l Rand Vector

then

(a) $\mathbb{E}[a + B\underset{\sim}{X}] = a + B\,\mathbb{E}[\underset{\sim}{X}]$

(b) $\text{Cov}(a + B\underset{\sim}{X}) = B\,\text{Cov}(\underset{\sim}{X})\,B^T$

Multivariate Normal

$\underset{\sim}{X} \sim N(\mu, \Sigma)$

$\mu \in \mathbb{R}^n$ $\qquad$ $\Sigma \in \mathbb{R}^{n \times n}$

then

$$f(\underset{\sim}{X}) = (2\pi)^{-n/2} (\det \Sigma)^{-1/2} \exp\left(-\tfrac{1}{2}(\underset{\sim}{X} - \mu)^T \Sigma^{-1} (\underset{\sim}{X} - \mu)\right)$$

Special case: $\mu = 0$ and $\Sigma = I$

this is called the Standard MV normal.

Theorem: if $a \in \mathbb{R}^m$, $B \in \mathbb{R}^{n \times m}$ then

$$a + B\underset{\sim}{X} \sim N(a + B\mu, B\Sigma B^T)$$