Lecture 5 - Unordered Counting

Thursday, September 16, 2021 1:54 PM

Sampling w/o replacement and unordered

Ex. ① ② ③ draw $r=2$ from $n=3$

If order matters:

$(1,2)$ $(1,3)$ $(2,3)$

$(2,1)$ $(3,1)$ $(3,2)$

$$\frac{n!}{(n-r)!} = \frac{3!}{(3-2)!} = 3! = 6$$

If order doesn't matter

$\{1,2\}$, $\{1,3\}$, $\{2,3\}$ 3 possibilities

General fact: for each unordered sample of size r I can permute that to make $r!$ ordered samples

$\{1,2,5\}$ → $(1,2,5)$, $(2,1,5)$, ...

$$\left(\#\text{ ordered samples}\right) = r!\left(\#\text{ unordered samples}\right)$$

$$\begin{pmatrix}\text{\# ordered} \\ \text{samples}\end{pmatrix} = r! \begin{pmatrix}\text{\# unordered} \\ \text{samples}\end{pmatrix}$$

So

$$\begin{pmatrix}\text{\# unordered} \\ \text{samples}\end{pmatrix} = \frac{(\text{\# ordered})}{r!}$$

$$= \frac{n!}{(n-r)!} \Big/ r!$$

$$= \frac{n!}{r!(n-r)!}$$

Theorem: Unordered w/o Replacement

If sample r things from n w/o ordering or replacement I can do this in

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{ways.}$$

Binomial coefficient

Ex. I have 10 $=n$ proffesors, how may Co-equal commitees of size 4 $= r$ can I create?

unordered

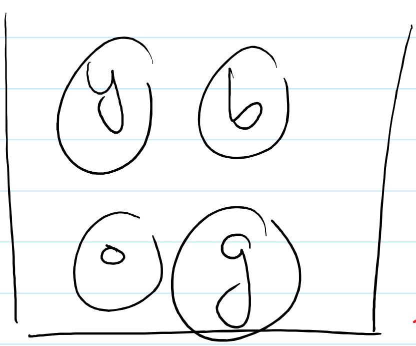
Ans: $\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!\,6!}$

$$= \frac{10\cdot \overset{3}{\cancel{9}}\cdot \cancel{8}\cdot 7\cdot \cancel{6\cdots 1}}{\cancel{4\cdot 3\cdot 2}\cdot 1\ \cancel{6\cdot 5\cdot 4\cdots 1}} = 210$$

Ex. How many 5-card poker hands can I create?

Theorem says: $\binom{52}{5} \approx 2.5$ mil

Ex. I have a jar w/ 4 marbles of colors yellow, blue, orange, green.



I choose w/o replacement 3 marbles (all choices equally likely) what is the prob I have a

 the prob I have a yellow and blue in my selection?

E = yellow and blue in my selection

$$P(E) = \frac{|E|}{|S|}$$

$S = \{$ all 3-samples from 4 w/o repl. or ordering $\}$

$\{y,b,o\}, \{y,b,g\}$
$\{b,o,g\}, \{o,g,y\}$

$$|S| = \binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$$

$$E = \{\{y,b,o\}, \{y,b,g\}\}$$

So $P(E) = \dfrac{|E|}{|S|} = 2/4 = 1/2$.

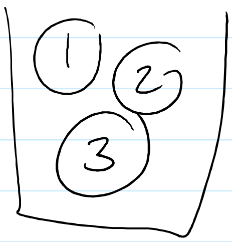
Last case: sampling w/ replacement and w/o ordering.

temptation!

for w/o repl.

$$\cancel{(\#\text{ ordered}) = r!\,(\#\text{ unordered})}$$

Consider: $n=3$, $r=2$



Ordered: formula: $n^r = 3^2 = 9$

$(1,1)$, $(2,1)$, $(3,1)$, $(2,3)$, $(2,2)$
$(1,2)$, $(1,3)$, $(3,2)$, $(3,3)$

Unordered:

$\{1,1\}$, $\{1,2\}$, $\{1,3\}$, $\{2,2\}$, $\{3,3\}$, $\{2,3\}$] 6

Game of Partitioning

Ex. How many ways can I partition $r=2$ objects using $n-1=4$ walls

Ex 1

| O | | O |

0 1 0 1 0

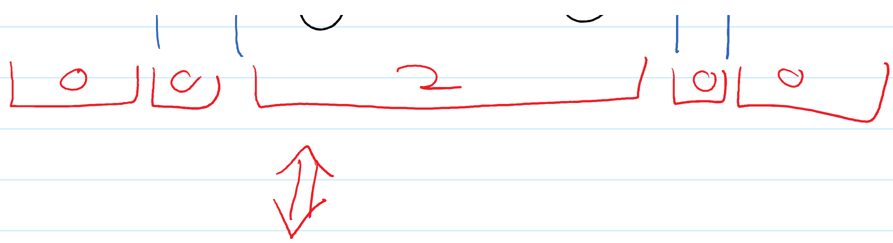
$\{2,4\}$

sample w/ repl. w/o ordering of $r=2$ from $n=5$

Ex 2

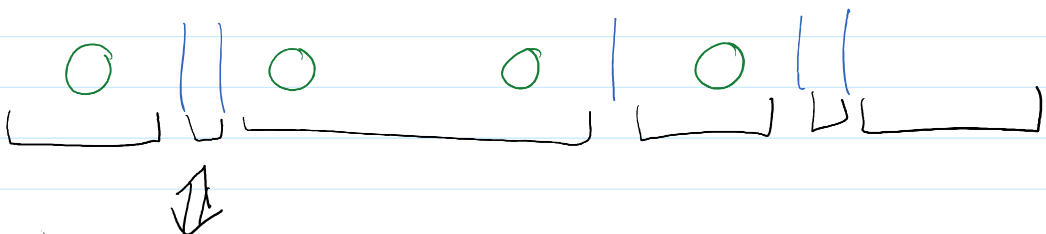
| | O O | |

0 0 2 0 0



$\{3,3\}$

Ex. $r=4$, $n-1=5$ walls

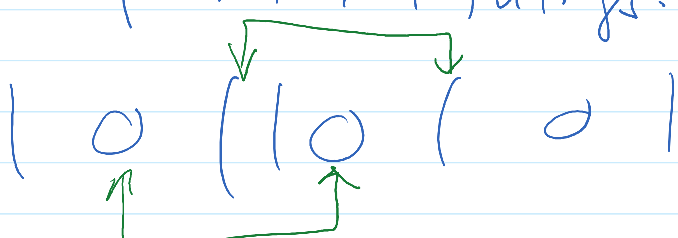


$\{1,3,3,4\}$

So there is a 1-1 corresp. btwn # samples and # arrangements of objects/walls

How many arrangements possible?

I have r objects and $n-1$ walls for a total of $r+n-1$ things.



each arrangement is a permutation of these

$r+n-1$ things.

However we can permute the walls among each other (similarly for objects) and basically have same arrangement.

So the total # of distinct arrangements is

$$\frac{(r+n-1)!}{r!\,(n-1)!}.$$

Theorem: ~~Sampling w/o ordering w/ replacement.~~

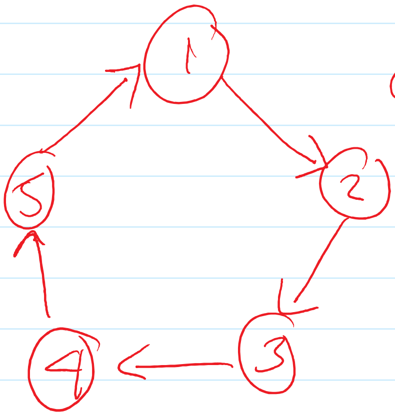
The number of ways to draw a sample of size r from n w/o ordering and w/ replacement is

$$\frac{(r+n-1)!}{(n-1)!\,r!} = \binom{r+n-1}{r} = \binom{r+n-1}{n-1}.$$

Ex, 10 passengers on a bus route w/ 5 hotels on the route.

The bus driver counts how many people

The bus driver counts how many people get off at each stop.



hotel	# people
1	0
2	3
3	1
4	2
5	4

Q: how many possible records are there?

$\Longleftrightarrow \{2,2,2,3,4,4,5,5,5,5\}$

Theorem says: $r=10$, $n=5$

$$\binom{r+n-1}{r} = \binom{10+5-1}{10} = \binom{14}{10} \doteq 1001$$

Ex. Jar w/ 4 marbles: yellow, blue, orange, green.

Draw a sample of size $r=3$ w/ replacement.

(all such samples are equally likely)

Q: What is the prob my sample has a yellow and blue?

$$S = \{\text{all samples}\} \Rightarrow |S| = \binom{r+n-1}{r} = \binom{3+4-1}{3}$$

$$= \binom{6}{3} = 20$$

$$E = \{\{y, b, g\}, \{y, b, o\}, \{y, b, b\}, \{y, b, y\}\}$$

So $|E| = 4$

and $\mathbb{P}(E) = 4/20 = 1/5$.

Four sampling possibilities

	w/o repl.	w/ repl.
ordered	$\frac{n!}{(n-r)!}$	n^r
unordered	$\frac{n!}{r!(n-r)!}$	$\frac{(r+n-1)!}{r!\,(n-1)!}$

The point of counting:

I have S w/ equally likely outcomes

then

$$\mathbb{P}(E) = \frac{|E|}{|S|}$$

need to count

The most imporant fact is that we assume all outcomes are equally likely.

Q: ordering? w/ replacement?

need to respect this fact

Ex. Flip a coin twice.

What is the prob. of getting a H and T,

Option 1: Unordered Sample Space

$$S = \{HH, TT, HT\} \text{ so } |S| = 3$$

and $E = \{HT\}$

and so $P(E) = 1/3$.

Option 2:

$$\underbrace{\underbrace{H}_{1/2}\underbrace{T}_{1/2}}_{1/4} \text{ or } \underbrace{\underbrace{T}_{1/2}\underbrace{H}_{1/2}}_{1/4}$$

$$1/4 + 1/4 = 1/2$$

equivalent to ordered sample space:

$$S = \{HH, TT, HT, TH\}$$
then $|S| = 4$

and $E = \{HT, TH\}$ so $|E| = 2$

and $P(E) = 2/4 = 1/2$.

General rule:

If I assume my sampling comes about from a seq of independent actions then counting S and E as ordered typically gives the right answer.

When sampling w/ replacement we need to be careful about ordering.

We don't have to be as careful when sampling w/o replacement

$$P(E) = \frac{\# E r!}{\# S r!}$$
