Lecture 2 - Axiomatic Probability

Tuesday, September 7, 2021 9:29 AM

Defn: Sample Space

The sample space S is the set of possible outcomes.

Ex. Flip a coin.

$$S = \{H, T\}$$

Ex. Roll a six-sided die

$$S = \{1, 2, 3, 4, 5, 6\}$$

Ex. Roll two dice

$$S = \{(1,1), (1,2), (1,3), \ldots (2,1), \ldots, (6,6)\}$$

Ex. Waiting time for a bus to arrive

$$S = [0, \infty) \subset \mathbb{R}$$

Ex. Number of customers arriving at my restaurant

$$S = \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$$

Two major types of sample spaces:

1. finite $(|S| < \infty)$
2. infinite $(|S| \geq \infty)$
   - $\hookrightarrow$ countable (e.g. $\mathbb{N}$)
   - $\hookrightarrow$ uncountable (e.g. $[0, \infty)$)

Defn: Outcome

We call the elements of S "outcomes"

$$\underset{\text{outcome}}{x} \in \underset{\text{sample space}}{S}$$

Ex. $S = \{1, 2, \ldots, 6\}$

then $1 \in S$ so 1 is an outcome.

Defn: Event

An event E is a subset of S,

$E \subset S.$

Ex. Roll a die. $S = \{1, \dots, 6\}$

then

$$E = \{1, 2\} \subset S$$

is the event of rolling a 1 or 2.

We say an event "happens" if the observed outcome of the experiment is an element of E.

Ex. $S \subset S$ hence S is an event.

is the event that something happens

Ex. $\emptyset \subset S$ so $\emptyset$ is an event.

the event that nothing happens???

Axiomatic Probability

Given an experiment (and a sample space S)

Want: For any event $E \subset S$ want to assign to E some measure of how likely E is to occur.

→ a probability function

Mathematically:

For each $E \subset S$ assign a probability $\mathbb{P}(E)$.

What are the rules for $\mathbb{P}$?

1. mathematically consistent
2. encode some of our intuitions about probability

Defn: Probability Function $\mathbb{P}$

Given a sample space S a prob. fn. $\mathbb{P}$ is a function

$$\mathbb{P} : 2^S \longrightarrow \mathbb{R}$$

domain   co-domain

that satisfies the Kolmogorov Axioms

① non-negativity

$$\mathbb{P}(E) \geq 0 \quad \forall E \subset S$$

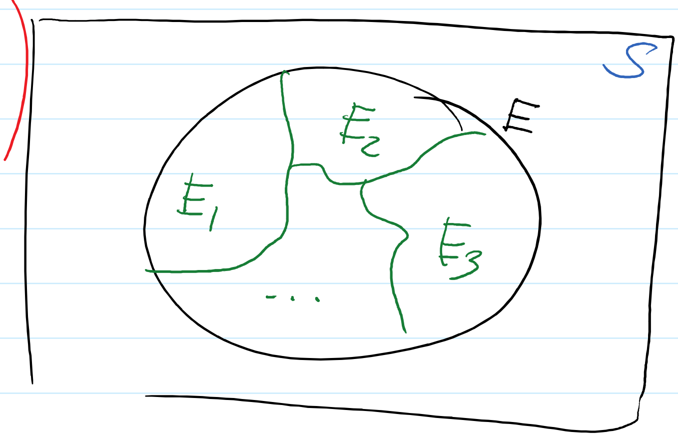
② unit-measure

$$\mathbb{P}(S) = 1$$

③ countable additivity

If $(E_i)_{i=1}^{\infty}$ is a partition of E

$\left(\bigcup_i E_i = E \text{ and } E_i E_j = \emptyset \text{ for } i \neq j \right)$



then

$$\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

One-way implication:

$(E_i)_{i=1}^{N}$ partion E then $\mathbb{P}(E) = \sum_{i=1}^{N} \mathbb{P}(E_i)$

Ex. Flip a coin.

$$S = \{H, T\}$$

What is a valid prob. function for S?

$P(\{H\}) = .5$ $\quad$ $P(\overbrace{\{H, T\}}^{S}) = 1$

$P(\{T\}) = .5$ $\quad$ $P(\emptyset) = 0$

Check it satisfies axioms:

① $P(E) \geq 0 \ \forall E \subset S$ ✓

② $P(S) = 1$ ✓

③

$S = \{H\} \cup \{T\} \cup \emptyset \cup \emptyset \ldots$

Check for an example:

$E = S$ and $E_1 = \{H\}$, $E_2 = \{T\}$

S partitioned into $E_1 \cup E_2$ and

$$P(E) = \underset{1}{\underset{\parallel}{P(S)}} = \underset{1/2}{P(E_1)} + \underset{1/2}{P(E_2)}$$ ✓

Ex Other ways of forming P

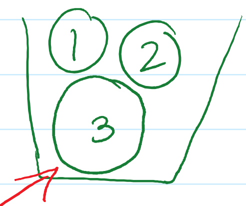
$\mathbb{P}(S) = 1 \qquad \mathbb{P}(\{H\}) = \alpha$

$\mathbb{P}(\emptyset) = 0 \qquad \mathbb{P}(\{T\}) = 1-\alpha$

where $\alpha \in [0,1]$

This will also work.

Ex.



$S = \{1, 2, 3\}$

2x as likely to choose

Practically:

$\mathbb{P}(\{1\}) = 1/4$

$\mathbb{P}(\{2\}) = 1/4$

$\mathbb{P}(\{3\}) = 1/2$

$$\mathbb{P}(\{1,2\}) = \mathbb{P}(\{1\}) + \mathbb{P}(\{2\}) = 1/4 + 1/4 = 1/2$$

Is this valid?

Theorem: Finite Sample Spaces

If $S = \{s_1, \ldots, s_n\}$ so $|S| = n$

and I choose corresp. $p_1, \dots, p_n$

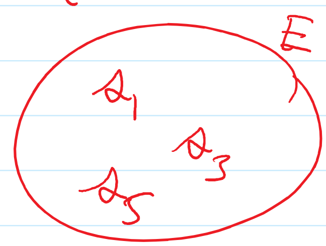
so that

(1) $p_i \geq 0$ and (2) $\sum_{i=1}^{n} p_i = 1$

then a valid prob. function is for $E \subset S$

$$P(E) = \sum_{i : s_i \in E} p_i$$

Sum up p_i corresp. to s_i in E



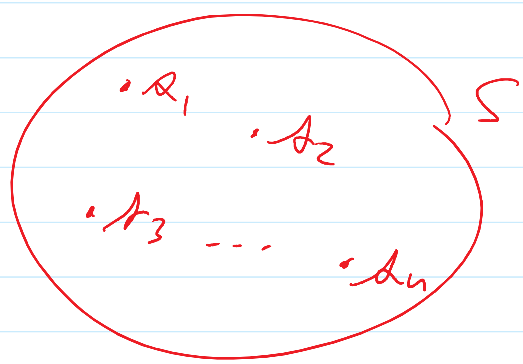
Pf. Need to check Kolmogorov Axioms

$\boxed{(1)\ P(E) \geq 0 \quad \forall E \subset S}$

$$P(E) = \sum p_i \geq 0$$

≥ 0

$\boxed{(2)\ P(S) = 1}$



$$P(S) = \sum p_i = p_1 + p_2 + p_3 + \cdots p_n = 1$$

$$P(S) = \sum_{i: s_i \in S} p_i = p_1 + p_2 + p_3 + \cdots p_n = 1$$

③ If $E \subset S$ and (E_i) partitions E then

$$\mathbb{P}(E) = \sum_i \mathbb{P}(E_i).$$

Recall

$$\mathbb{P}(E) = \mathbb{P}\left(\bigcup_i E_i\right) = \sum_{j: \underbrace{s_j \in \bigcup_i E_i}} p_i$$

$s_j \in \bigcup_i E_i$ then s_j is in exactly 1 E_j