Lecture 6 - Conditional Probability

Tuesday, September 21, 2021 9:30 AM

Ex. Survey W&M student about political affiliation.

Political Afil

		A	B	
gender	men	501	238	739
	women	782	123	905
		1283	361	1644

Q1: What is the prob a randomly selected student is a woman?

$$\mathbb{P}(\text{woman}) = 905/1644 \approx 55\%$$

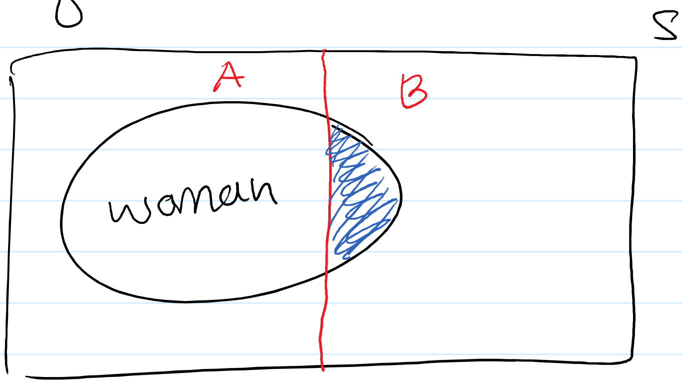
Q2: Given a student is a member of part B, what is the prob. they are a woman?

conditioning on information

$$\mathbb{P}(\text{woman GIVEN party B}) = \frac{123}{361}.$$

$$\approx 34\%$$

Venn Diagram



Q1: $$P(\text{women}) = \frac{\text{area of woman}}{\text{area of } \boxed{S}}$$

Q2: $$P(\text{women GIVEN } B) = \frac{\text{area of } D}{\text{area of } \boxed{B}}$$

$$= \frac{\text{area } \text{woman} \cap \boxed{B}}{\text{area } \boxed{B}}$$

Defn: Conditional Probability

If $A, B \subset S$ and $P(B) > 0$. Then

$$P(A \mid B) = \frac{P(AB)}{P(B)}.$$

$P(B)$.

read:
A given B

Facts:

(1) $\boxed{P(B|B) = 1}$

Pf. $P(B|B) = \frac{P(BB)}{P(B)} = \frac{P(B)}{P(B)} = 1$.

(2) $\boxed{\text{If } AB = \emptyset \text{ then } P(A|B) = 0.}$

Ex. Roll two dice.

Q: What is the prob. the first roll is a 2 given the sum of the two is ≤ 5.

A

B

$S = \{(i,j) \text{ for } 1 \leq i,j \leq 6\}$

So $|S| = 36 = 6^2$

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$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{|AB|/|S|}{|B|/|S|} = \frac{|AB|}{|B|} = \frac{3}{10}.$$

O = A
X = B

second \ first	1	2	3	4	5	6
1	X	⊗	X	X		
2	X	⊗	X			
3	X	⊗				
4	X	O				
5		O				
6		O				

Facts: Basically $P(\cdot|B)$ behaves like any other prob function.

E.g. $P(A^c|B) = 1 - P(A|B)$

E.g. $P(A_1 \cup A_2 | B) = P(A_1|B) + P(A_2|B) - P(A_1A_2|B)$.

Theorem: Compound Probability

$$P(AB) = P(A|B)P(B) = P(B|A)P(A).$$

Pf. If $P(B) > 0$

then $P(A|B) = \dfrac{P(AB)}{P(B)}$

... re-arrange ...

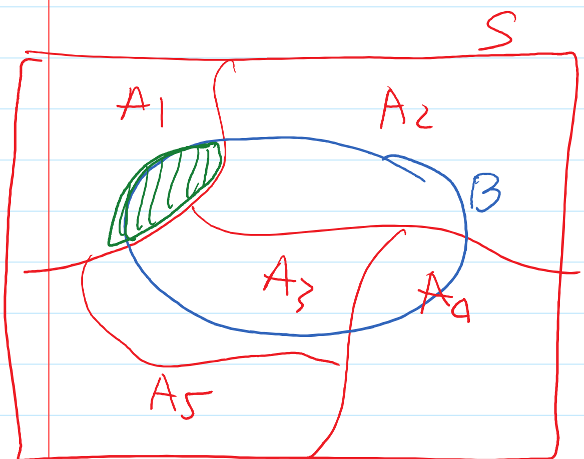
Recall: (A_i) partition S then

$$P(B) = \sum_i \boxed{P(BA_i)}.$$

Lec 3

Theorem: Law of Total Probability

If (A_i) partitions S then for any $B \subset S$



$$P(B) = \underbrace{\sum_i \underbrace{P(B|A_i)}_{\text{area of } BA_i \text{ rel. to area of } A_i} \underbrace{P(A_i)}_{\text{area of } A_i \text{ rel. to area of } S}}_{\text{area of } BA_i \text{ rel to } S}$$

Pf. Compound prob. theorem says

$$P(BA_i) = P(B|A_i)P(A_i)$$

Partitioning theorem from Lec 3 says

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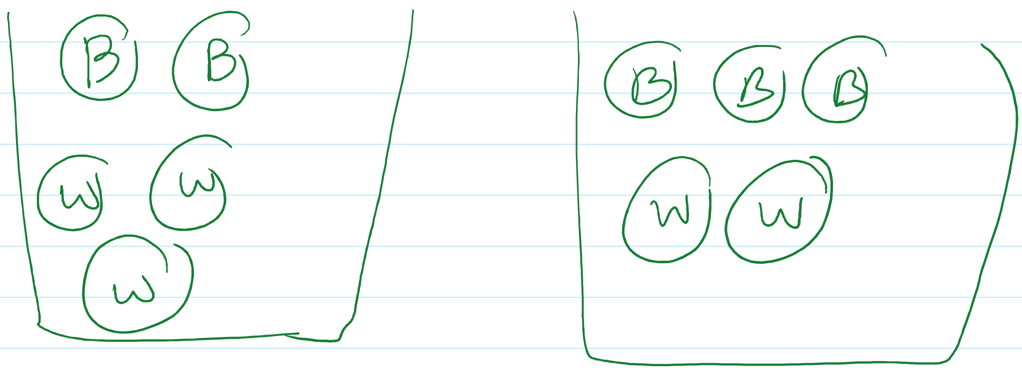
$$P(B) = \sum_i P(BA_i) = \sum_i P(B|A_i)P(A_i).$$

Ex. Basket 1 Basket 2



Game: 1 Randomly select a ball from basket 1 and place in basket 2

2 Randomly select a ball from basket 2

Q: what is the prob. of choosing a B on step 2?

Let W = choose white on step 1

W^c = " black "

Let B = choose black on step 2

B^c " white "

Want is $P(B)$.

Notice that W and W^c partition S.

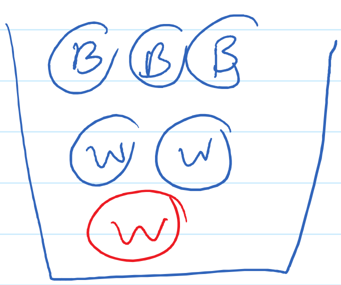
The Law of Tot. Prob says

$$P(B) = \underbrace{P(B|W)}_{(1/2)}\,\underbrace{P(W)}_{(3/5)} + \underbrace{P(B|W^c)}_{(2/3)}\,\underbrace{P(W^c)}_{(2/5)} = 17/30$$

$P(B|W) = 1/2$

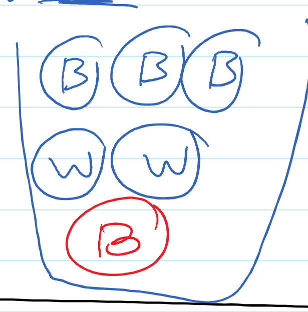
Given W

Basket 2



$P(B|W^c) = 4/6 = 2/3$

Given W^c



Theorem: Bayes' Theorem

Know $P(B|A)$ can I use to calc. $P(A|B)$.

If $A, B \subset S$ and $P(A) > 0$, $P(B) > 0$ then

$$P(A|B) = P(B|A)\,\frac{P(A)}{P(B)}.$$

$$P(B)$$

Pf. $P(A|B) = \dfrac{P(AB)}{P(B)} = \dfrac{P(B|A)\,P(A)}{P(B)}$.

compound prob: $P(BA) = P(B|A)P(A)$

Ex. Continue previous.

Given I chose a black ball on second step, what is the prob. I chose a white on first step?

B

W

Know all these

Want: $P(W|B) = \dfrac{P(B|W)\,P(W)}{P(B)}$

$$= \frac{(1/2)(3/5)}{(17/30)} = \ldots$$

Theorem: Law of Total Prob. + Bayes.

If (A_i) partition S and $B \subset S$, then

$$P(A_i|B) = \frac{P(B|A_i)\,P(A_i)}{\quad}$$

$$\sum_j P(B|A_j)P(A_j)$$

Pf.

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$$ ← Bayes'

P(B) ← expand w/ Law of Total Prob.

$$= \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)}.$$

Ex. COVID has a prevalence rate of 1%

We test for COVID and get + or −.

→ The test accurately reports a + 95% of time (sensitivity)

→ " a − 99% " (specificity)

Q: I get a COVID test, and get a +. What is the prob this is correct?

→ D = have COVID, D^c = I don't

$P(D) = .01$; $P(D^c) = .99$

→ $P(+|D) = .95$ and $P(-|D^c) = .99$

→ <u>want</u>: $\boxed{P(D|+) = \dfrac{P(+|D)P(D)}{P(+|D)P(D) + \underbrace{P(+|D^c)}_{1-P(-|D^c)}\underbrace{P(D^c)}_{1-P(D)}}}.$

$$= \frac{(.95)(.01)}{(.95)(.01) + (1-.99)(1-.01)}$$

$\approx 50\%$.