Lecture 8 - Random Variables

Tuesday, September 28, 2021 1:52 PM

Ex. Flip a coin 3 times.

X = # heads among my 3 flips

$s \in S$	$X(s)$
H H H	3
H H T	2
H T H	2
H T T	1
T H H	2
T H T	1
T T H	1
T T T	0

← a functions

Defn: Random Variable

A random variable (RV) X is a function

$$X : S \to \mathbb{R}$$

also called a random variate

or real-valued random variable

or a univariate random variable

($\mathbb{R}$ not $\mathbb{R}^n$) later

Ex. (1) toss two dice

X = sum of dice

(2) toss a coin 25 times,

X = length of the longest chain of consecutive Hs

(3) observe rainfall amount,

X = crop yield

We'd like to say, e.g.,

$P(X=1)$ abusive of notation

read: prob. X is 1.

recall: $P : 2^S \to \mathbb{R}$

what we really mean

if X = # heads in 3 flips

$$P(X=1) = P(\{HTT, THT, TTH\}) = 3/8$$

"$X=1$" short-hand for $\underbrace{\{s \in S \mid X(s) = 1\}}_{\text{inverse image of } \{1\} \text{ under } X.} \subseteq S$

Review:

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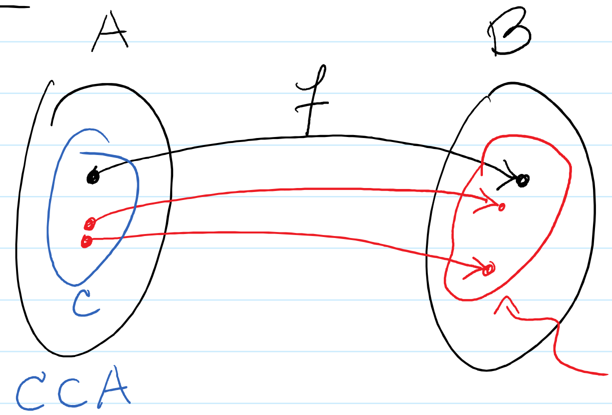
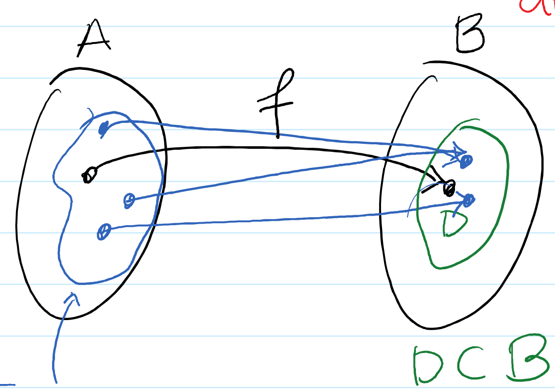


Image of C under f

denoted $\mathrm{Im}(C)$ or $f(C)$.
$\subseteq$
B



Inverse Image of D under f

denote: $f^{-1}(D) \subseteq A$

Image: $f(C) = \{f(x) \mid x \in C\}$

Inverse Image: $f^{-1}(D) = \{a \in A \mid f(a) \in D\}$

"$X = 1$" shorthand for $\underbrace{\{s \in S \mid X(s) = 1\}}_{X^{-1}(\{1\})}$

Notation: If X is a RV we write

$$P(X \in A) \quad \text{where } A \subset \mathbb{R}$$

means

$$\mathbb{P}(\underbrace{X^{-1}(A)}_{X^{-1}(A) \subseteq S})$$

Ex. $X = \#$ heads in 3 tosses of a coin

① $P(X=1) = \mathbb{P}(X \in \underbrace{\{1\}}_{A})$

$= P(X^{-1}(\{1\}))$

$= \mathbb{P}(\{HTT, THT, TTH\}) = 3/8$

② $\mathbb{P}(X = 1 \text{ or } 2) = \mathbb{P}(X \in \{1,2\})$

$= \mathbb{P}(X^{-1}(\{1,2\}))$

$= \mathbb{P}(\{HTT, THT, TTH,$
$HHT, HTH, THH\}) = 6/8$

Defn: Support of a RV

If X is a RV its support is the set of possible values of X,

$$\text{Support}(X) = X(S)$$

Image of S under X.

Ex. Prev. ex,

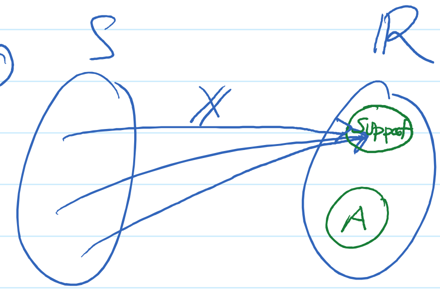
$$\text{Support}(X) = \{0, 1, 2, 3\}$$

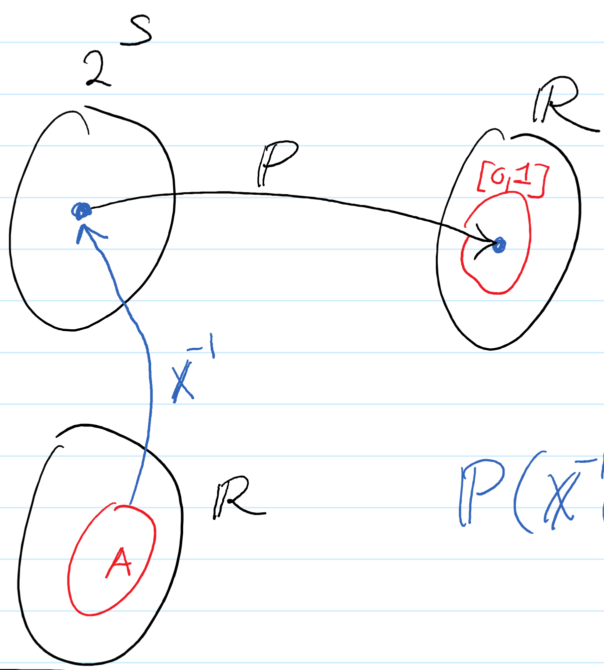
notice $\mathbb{P}(X = 5) = 0$.

If $A \subset \mathbb{R}$ and $\text{Support}(X) \cap A = \emptyset$

then $\mathbb{P}(X \in A) = 0$.

Pf. $\mathbb{P}(X \in A) = \mathbb{P}(\underbrace{X^{-1}(A)}_{\emptyset}) = 0$





$\mathbb{P}(X^{-1}(A))$

Heuristic/Informal Types of RVs

(1) <u>discrete</u> : support is finite or countable

<u>Ex.</u> X = Sum of two dice

<u>Ex.</u> X = # of customers visiting a shop (support is $\mathbb{N}$)

(2) <u>continuous</u> : support is uncountably infinite

<u>Ex.</u> X = waiting time for a bus

Support $= [0, \infty)$

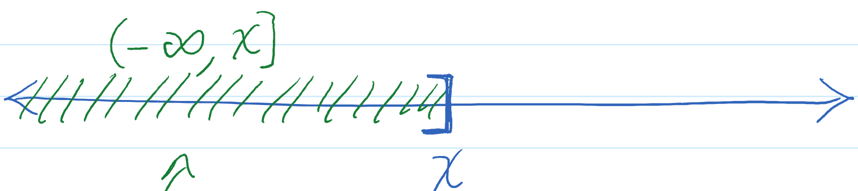
<u>Defn: Cumulative Distribution Function</u> (CDF)

If X is a RV then its CDF is a function $F : \mathbb{R} \to \mathbb{R}$ defined for $x \in \mathbb{R}$

$$F(x) = \mathbb{P}(X \le x)$$

a RV

a number



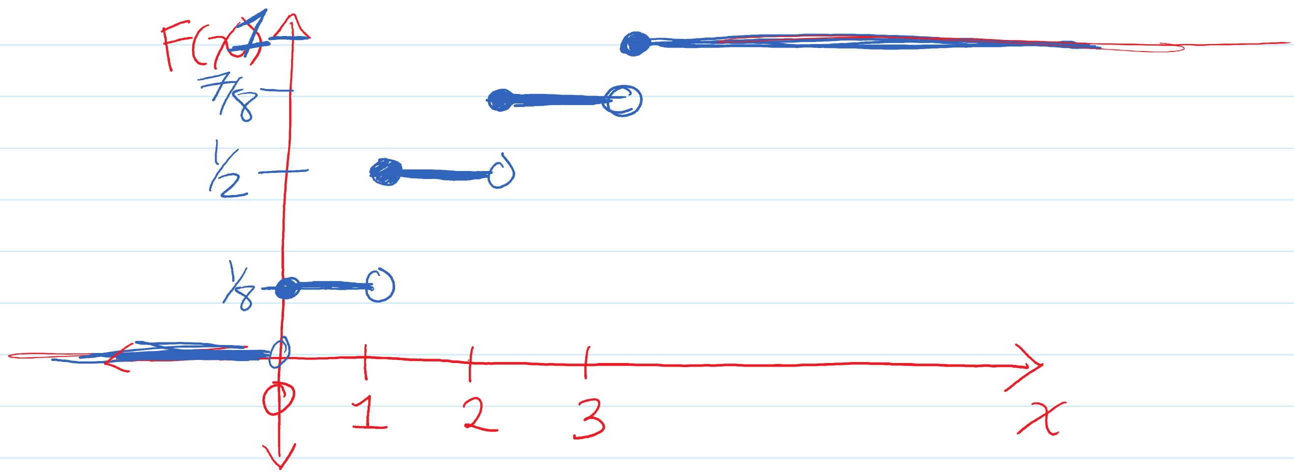
$F(x)$ is the prob of being here

Notation:

$$F(x) = \mathbb{P}(X \le x)$$
$$= \mathbb{P}(X \in (-\infty, x])$$
$$= \mathbb{P}(X^{-1}((-\infty, x]))$$

Ex. Toss a coin 3 times.

$X = \#$ heads.



$$F(0) = \mathbb{P}(X \le 0) = \mathbb{P}(X = 0) = 1/8$$
$$F(1/2) = \mathbb{P}(X \le 1/2) = \mathbb{P}(X = 0) = 1/8$$
$$F(.9) = \mathbb{P}(X \le .9) = \cdots = 1/8$$
$$F(1) = \mathbb{P}(X \le 1) = 4/8 = 1/2$$
$$F(1.5) = \mathbb{P}(X \le 1.5) = \mathbb{P}(X \le 1) = 1/2$$
$$F(2) = \mathbb{P}(X \le 2) = 7/8$$
$$F(3) = \mathbb{P}(X \le 3) = 1$$
$$F(4) = \mathbb{P}(X \le 4) = 1$$

$F(100{,}000) = \mathbb{P}(X \le 100{,}000) = 1$

$F(-1) = \mathbb{P}(X \le -1) = 0$

$\vdots$

Facts:

① $0 \le F(x) \le 1$

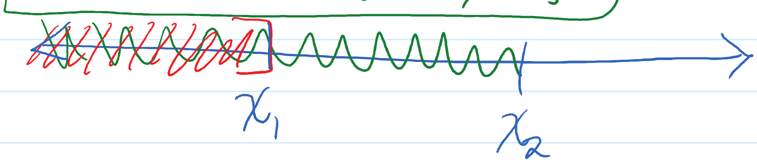
Pf. $F(x) = \mathbb{P}(\cdots) \in [0,1]$

② $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to -\infty} F(x) = 0$

③ F is non-decreasing

if $x_1 < x_2$ then $F(x_1) \le F(x_2)$.

Pf. $(-\infty, x_1] \subset (-\infty, x_2]$



Fact: Inverse image preserve subsets

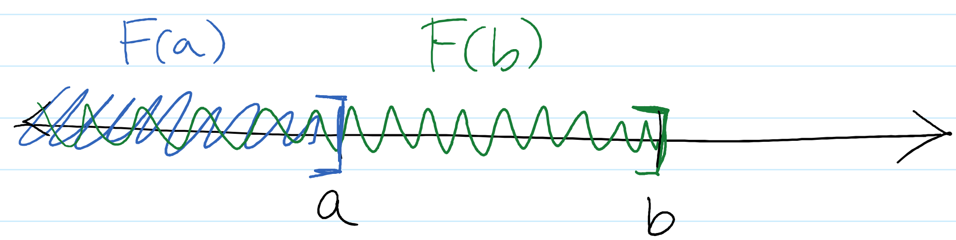
$F(x_1)$ $\qquad$ $F(x_2)$

$\mathbb{P}(X \le x_1)$ $\qquad$ $\mathbb{P}(X \le x_2)$

$$\mathbb{P}(X^{-1}((-\infty,x_1])) \leq \mathbb{P}(X^{-1}((-\infty,x_2]))$$

<u>Fact:</u> $E \subset F$ then $\mathbb{P}(E) \leq \mathbb{P}(F)$

④ $\boxed{\mathbb{P}(a < X \leq b) = F(b) - F(a)}$



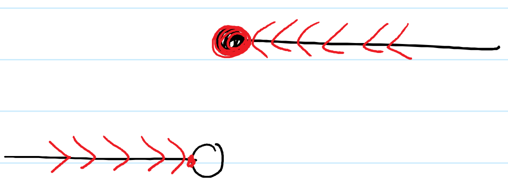
<u>note:</u> $(a,b] = (-\infty,b] \setminus (-\infty,a]$

⑤ $\boxed{F \text{ is right-continuous}}$

$$\lim_{x \to a^+} F(x) = F(a)$$

<u>recall:</u> defn cts fn g

$$\lim_{x \to a} g(x) = g(a)$$



<u>Note:</u> cts fns are right-cts