Lecture 14 - More Common Distributions

Tuesday, October 26, 2021 9:35 AM

MGF of $X \sim U(a,b)$

$$M(t) = \mathbb{E}[e^{tX}] = \int e^{tx} f(x)\,dx = \int_a^b e^{tx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \frac{1}{t} e^{tx} \Big|_{x=a}^{x=b}$$

$$M(t) = \boxed{\frac{e^{tb} - e^{ta}}{t(b-a)}}$$

Bernoulli Distribution

$X \sim \text{Bern}(p)$

Flip a coin, $X = 1$ if H, $X = 0$ if T.

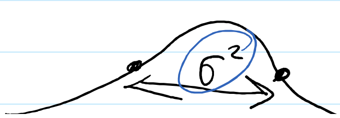
$X \sim \text{Bin}(n=1, p)$

Normal / Gaussian Distribution

$X \sim N(\mu, \sigma^2)$

$\mu \in \mathbb{R}$ $\quad$ $\sigma^2 > 0$

PDF:



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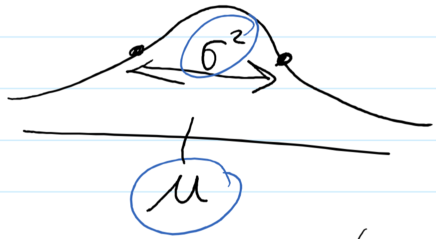


$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad \text{for } x \in \mathbb{R}$$

special case: $\mu = 0$; $\sigma^2 = 1$

Standard normal, $X \sim N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

CDF: no closed form $F(x) = \int_{-\infty}^{x} f(t)\,dt$

Claims: $\mathbb{E}X = \mu$ and $\operatorname{Var} X = \sigma^2$

MGF:

$$M(t) = \mathbb{E}\left[e^{tX}\right] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\underbrace{-\frac{1}{2\sigma^2}(x-\mu)^2}_{\text{expand}}\right) dx$$

$e^a e^b = e^{a+b}$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x^2 - 2\mu x + \mu^2 - 2\sigma^2 t x\right)\right)$$

complete the square

$(a+b)^2 = a^2 + 2ab + b^2$

$x^2 - 2\mu x + \mu^2 - 2\sigma^2 t x$

$= x^2 - 2x(\sigma^2 t + \mu) + \mu^2$

$= \underbrace{(x-(\mu+\sigma^2 t))^2 - (\mu+\sigma^2 t)^2} + \mu^2$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left[(x-(\mu+\sigma^2 t))^2 + \underbrace{\mu^2 - (\mu+\sigma^2 t)^2}_{\text{doesn't depend on } x}\right]\right)$$

$$= \underbrace{\exp\left(-(\mu+\sigma^2 t)^2 + \mu^2\right) / -2\sigma^2}$$

$$\cdot \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2\right)}_{\text{PDF of } N(\mu+\sigma^2 t,\ \sigma^2) \text{ integrates to } 1} dx$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left[\mu^2 - (\mu+\sigma^2 t)^2\right]\right)$$

; algebra

$$\boxed{= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) = M(t)}$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \left.(\mu + \cancel{\sigma^2 t}^{\,0})\exp\left(\cancel{\mu t + \frac{\sigma^2 t^2}{2}}^{\,0}\right)\right|_{t=0}$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \left.(\mu + \cancel{\sigma^2 t})\exp\left(\cancel{\mu t} + \frac{\sigma^2 t^2}{2}\right)\right|_{t=0}$$

$$= \mu \cdot 1 \quad \boxed{= \mu = E[X]}$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \sigma^2 \exp\left(\mu t + \cancel{\frac{\sigma^2 t^2}{2}}\right) + (\mu + \cancel{\sigma^2 t})^2 \exp\left(\cancel{\mu t} + \frac{\sigma^2 t^2}{2}\right)$$

$$= \sigma^2 + \mu^2 = E[X^2]$$

so $\mathrm{Var}(X) = E[X^2] - E[X]^2$

$$= (\sigma^2 + \mu^2) - \mu^2 \boxed{= \sigma^2 = \mathrm{Var}(X)}$$

Theorem: Linear Functions of Normal RVs

Let $X \sim N(\mu, \sigma^2)$ and

$$Y = aX + b$$

then $Y \sim N(\underline{a\mu + b}, \underline{a^2\sigma^2})$.

intuition: $E[Y] = aEX + b = a\mu + b$

$\mathrm{Var}(Y) = a^2 \mathrm{Var}(X) = a^2\sigma^2$

Pf. Need to show ...

Pf. Need to show

$$M_Y(t) = \exp\left((a\mu+b)t + \frac{(a^2\sigma^2)t^2}{2}\right)$$

have this theorem:

$$M_Y(t) = e^{bt} M_X(at)$$

$$= e^{bt} \exp\left(\mu(at) + \frac{\sigma^2(at)^2}{2}\right)$$

$$= \exp\left(bt + \mu at + \frac{\sigma^2 a^2 t^2}{2}\right)$$

MGF of $N(\mu, \sigma^2)$ is

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

Poisson Distribution

- discrete RV
- support of non-neg integers: $\{0, 1, 2, 3, \ldots\}$

Canonical experiment:

the count of the number of "events" in some fixed time period

Ex. - radioactive decay (in a day)
- number of fish I capture in a trap (in a day)

count the muber of mRNA in a cell

$X \sim Pois(\lambda)$

rate of events occuring

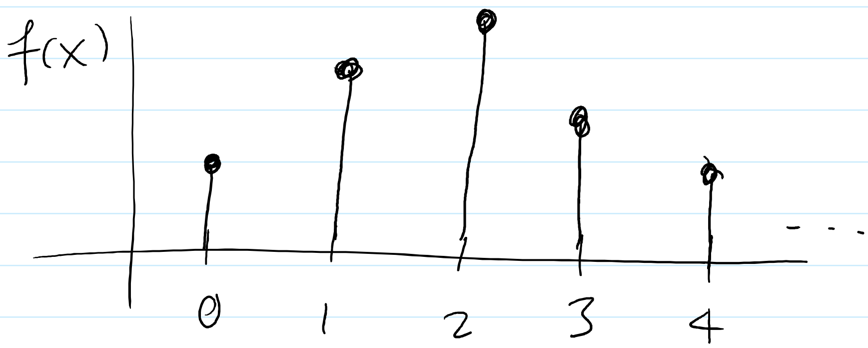
of ... in a ... ined

λ

\# of events in a time period

PMF:

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad \text{for } x = 0, 1, 2, 3, \dots$$



Expected Value:

$$\frac{x}{x!} = \frac{\cancel{x}}{\cancel{x}(x-1)!} = \frac{1}{(x-1)!}$$

$$\mathbb{E}[X] = \sum_{x=\cancel{0}1}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-1)!}$$

$$\boxed{e^{x} = \sum_{i=0}^{\infty} \frac{x^{i}}{i!}}$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\lambda \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}}$$

$$= e^{-\lambda}\lambda e^{\lambda} = \boxed{\lambda = \mathbb{E}[X]}$$

$$\mathbb{E}[X(X-1)] = \sum_{x=\cancel{0}2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-2)!}$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda}\lambda^2 \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}}$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} = \lambda^2 = \mathbb{E}[X(X-1)]$$

$$= \mathbb{E}[X^2 - X]$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]$$

So

$$\mathbb{E}[X^2] = \underbrace{\mathbb{E}[X^2] - \mathbb{E}[X]}_{\lambda^2} + \underbrace{\mathbb{E}[X]}_{\lambda}$$

$$= \lambda^2 + \lambda$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

MGF:

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} (\lambda e^t)^x / x!}_{e^{\lambda e^t}}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

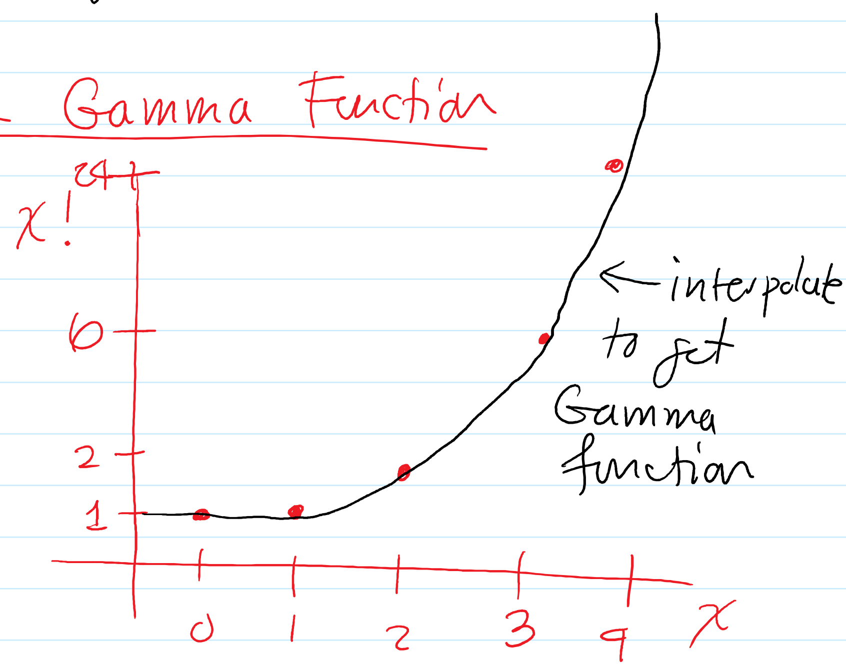
$$= e^{-\lambda} e^{\lambda t}$$

$$\boxed{= \exp(\lambda(e^{t}-1))}$$

Gamma Distribution: generalize exponential dist

Lets talk about the Gamma Function

the Gamma function is extension of $x!$ to pos real numbers.



$$\Gamma : \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$$

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \, dx \quad \text{for } a > 0$$