Lecture 9 - CDFs and PMFs

Thursday, September 30, 2021 9:28 AM

Additional Office Hours: Friday 10-11 am
Monday 12-1 pm

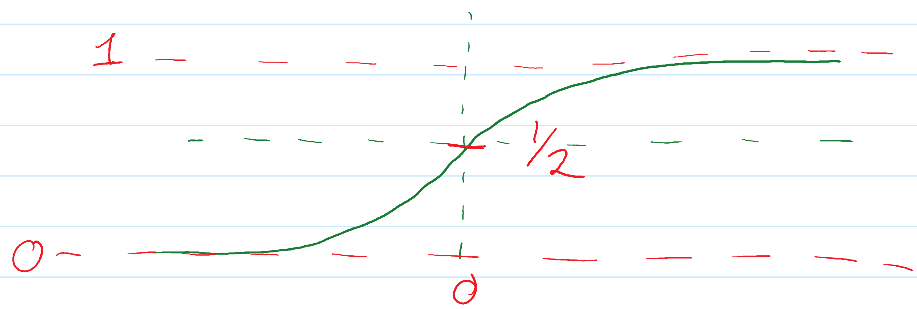
Theorem: F is the CDF of some RV iff

① $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$

② F is non-decreasing

③ F is right-continuous

Ex. $F(x) = \dfrac{1}{1+e^{-x}} \quad \forall x \in \mathbb{R}$



Is this a valid CDF?

Check our 3 conditions:

① $\lim_{x \to \infty} F(x) = \dfrac{1}{1+e^{-\infty}} = \dfrac{1}{1+0} = 1$

$\lim_{x \to -\infty} F(x) = \dfrac{1}{1+e^{-(-\infty)}} = \dfrac{1}{\infty} = 0$

(2) $\frac{d}{dx}F(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$

so increasing function

(3) Right cts?

It is a continuas function.

So, yes, this is a valid CDF.

Defn: Identically Distributed RVs (equal in distribution)

Two RVs X and Y are equal in distribution if $\forall A \subset \mathbb{R}$

$$\boxed{\mathbb{P}(X \in A) = \mathbb{P}(Y \in A).}$$

We denote this as $X \overset{d}{=} Y$.

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This is different than $X = Y$ as functions.

Ex. 3 coin flips

$X = \#$ heads and $Y = \#$ tails

notice: $X(HTT) = 1$ but $Y(HTT) = 2$

however:

$$P(X=0) = 1/8 = P(Y=0)$$
$$P(X=1) = 3/8 = P(Y=1)$$
$$\vdots$$

So $X \overset{d}{=} Y$ but $X \neq Y$ (as functions).

Theorem: $X \overset{d}{=} Y$ iff $F_X = F_Y$ (as fns).

CDF of X CDF Y

Ex. Toss coins independently until a H appears.

$$S = \{H, TH, TTH, TTTH, \dots\}$$

so $|S| = \infty$.

Let p as the prob. of a H on any flip.
Let $X = \#$ of flips to get H.

$s \in S$	$X(s)$
H	1
TH	2
TTH	3

TH	2
TTH	3
TTTH	4
⋮	⋮

Q: What is the CDF of X?

$$F(x) = \boxed{\mathbb{P}(X \le x)}$$

To determine F lets consider

$$\boxed{\mathbb{P}(X = x)} \text{ for } x \in \mathbb{R}$$

it takes x flips to get H

Let $H_i = H$ on i^{th} flip and $T_i = H_i^c$.

Then

independent

$$\text{"}X = x\text{"} = T_1 T_2 T_3 \cdots T_{x-1} H_x$$

so

$$\mathbb{P}(X = x) = \mathbb{P}(T_1 \cdots T_{x-1} H_x)$$

$$= \mathbb{P}(T_1) \cdots \mathbb{P}(T_{x-1}) \mathbb{P}(H_x)$$

$$= \underbrace{(1-p) \cdots (1-p)}_{x-1 \text{ of these}} p$$

$$= (1-p)^{x-1} p$$

Let $W_i = \text{"}X = i\text{"}$ so that $\mathbb{P}(W_i) = (1-p)^{i-1} p$

notice that $W_i W_j = \emptyset \quad i \neq j$

and

$$\text{"}X \le x\text{"} = W_1 \cup W_2 \cup W_3 \cup \cdots \cup W_x$$

so $$F(x) = \mathbb{P}(X \le x) = \mathbb{P}(W_1 \cup \cdots \cup W_x)$$

$$= \sum_{i=1}^{x} \mathbb{P}(W_i)$$

$$= \sum_{i=1}^{x} (1-p)^{i-1} p$$

$$= p \underbrace{\sum_{i=0}^{x-1} (1-p)^{i}}$$

$r = 1-p$

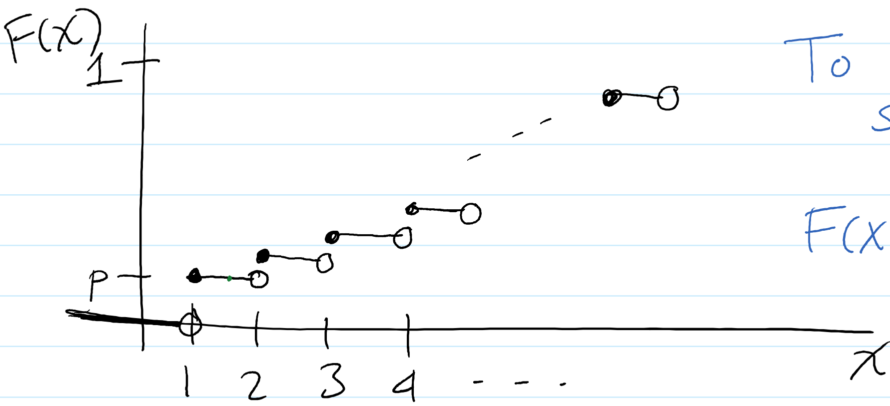
recall:

$$\underbrace{\sum_{i=0}^{n-1} r^i} = \frac{1-r^n}{1-r}$$

$$= \cancel{p}\left(\frac{1-(1-p)^x}{\cancel{1-(1-p)}}\right)$$

$1-(1-p)$
$= p$

$$= 1-(1-p)^x$$



This type of RV is called a geometric RV.

To actually encode step fn behavior

$$F(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor}, & x > 1 \\ 0, & x < 1 \end{cases}$$

$\lfloor x \rfloor$ = round x down to nearest integer

e.g. $\lfloor 3.5 \rfloor = 3$

Defn: Discrete/Continuous RVs

A <u>discrete RV</u> is one whose CDF is a step fn.

A <u>continuous RV</u> is one whose CDF is continuous.

Defn: Probability Mass Function

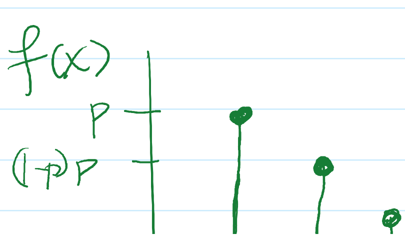
For a discrete RV X the prob. mass function (PMF) is the function $f: \mathbb{R} \to \mathbb{R}$ where

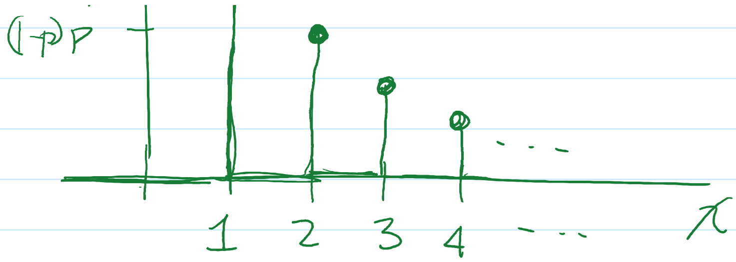
$$f(x) = \mathbb{P}(X = x) \quad \forall x \in \mathbb{R}.$$

Sometimes called the <u>distribution</u> of X.

<u>Ex.</u> For our geometric RV

$$f(x) = \mathbb{P}(X = x) = \begin{cases} (1-p)^{x-1} p, & x = 1, 2, 3, 4, \ldots \\ 0, & \text{else} \end{cases}$$





Theorem: For discrete RVs

$$F(x) = \sum_{i \le x} f(i)$$

$\mathbb{P}(X \le x)$ $\qquad$ $\mathbb{P}(X = i)$

Pf: $\quad$ $"X \le x" = \bigcup_{i \le x} "X = i"$

disjoint union

hence

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}\left(\bigcup_{i \le x} "X = i"\right)$$

$$= \sum_{i \le x} \mathbb{P}(X = i)$$

$$= \sum_{i \le x} f(i) \, .$$

Ex, Say X has a uniform distribution on $1, \dots, n$

denoted

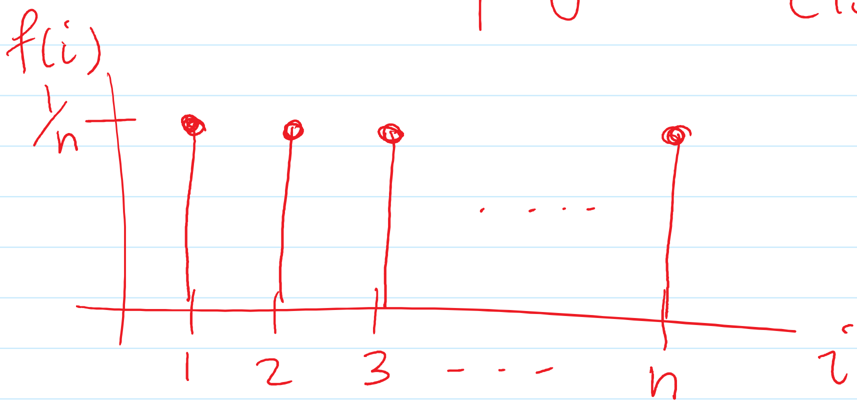
read: distributed as

$$X \sim \underline{U}(\{1, \dots, n\}),$$

$$X \sim \underbrace{U(\{1,\dots,n\})}$$

if

$$f(i) = \begin{cases} \frac{1}{n} & \text{for } i=1,\dots,n \\ 0 & \text{else} \end{cases}$$

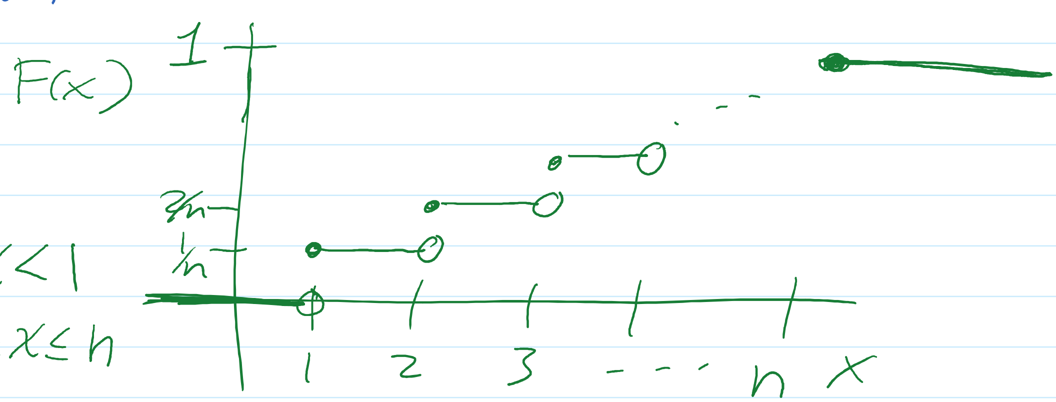


Q: What is the CDF?

$$F(x) = \sum_{i \le x} f(i) = \sum_{i=1}^{x} 1/n = \frac{x}{n} \qquad \underline{x \in \{1,\dots,n\}}$$

(techically:

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{\lfloor x \rfloor}{n} & 1 \le x \le n \\ 1 & x \ge n \end{cases}$$



Said: $F(x) = \sum_{i \le x} f(i)$

$$F(x) = \mathbb{P}(X \le x)$$

More general fact:

$$P(X \in A) = \sum_{i \in A} f(i) \quad (*)$$

Ex. X uniform over $1, \ldots, 7$

$(X \sim U(\{1, \ldots, 7\}))$

$$\mathbb{P}(2 \le X \le 5)$$

$$= \mathbb{P}(X \in \{2,3,4,5\})$$

$$= \sum_{i=2,3,4,5} f(i)$$

$$= \sum_{i=2}^{5} 1/7 = 4/7 .$$

Ex. Roll a die 60 times (independently)

$X = \#$ of 6s I get

lets get the PMF of X.

$f(x) = \mathbb{P}(X = x) =$ prob. I get x 6s among 60 rolls.

$$f(0) = \mathbb{P}(X=0) = \underbrace{(5/6)(5/6)(5/6) \cdots (5/6)}_{} = (5/6)^{60}$$

$$f(1) = \mathbb{P}(X=1) = \binom{60}{1}\left(\frac{1}{6}\right)\underbrace{\left(5/6\right)\left(5/6\right)\cdots\left(5/6\right)}_{59 \text{ times}}$$

60 times

$$= \binom{60}{1}\left(1/6\right)\left(5/6\right)^{59}$$

$$f(2) = \mathbb{P}(X=2) = \binom{60}{2}\left(1/6\right)\left(1/6\right)\underbrace{\left(5/6\right)\cdots\left(5/6\right)}_{58}$$

$$= \binom{60}{2}\left(1/6\right)^{2}\left(5/6\right)^{58}$$

general pattern

$$\boxed{f(x) = \mathbb{P}(X=x) = \binom{60}{x}\left(\frac{1}{6}\right)^{x}\left(5/6\right)^{60-x}}$$

We call this a <u>Binomial</u> RV.

Any experiment where I do n actions independently each w/ a prob. p of occuring and

X = # occurences

then

$X \sim Bin(n, p)$.