Laymen's defn:

→ things don't affect each other

→ events are independent if knowing the occurence (or not) of one event, doesn't change the prob. of the other

Defn: Independence (of Events)

If $A, B \subset S$, we say "A is independent of B", denoted $A \perp\!\!\!\perp B$, if

$$P(AB) = P(A)P(B).$$

→ distributive Law for intersection

→ justifies product notation for intersection

Theorem:

If $A \perp\!\!\!\perp B$ then

$$P(A \mid B) = P(A).$$

Pf:

pf:

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A).$$

Ex. Consider rolling two dice (independently)

$P(\text{at least one } 6)$

$= 1 - P(\text{no } 6\text{s})$

A_1 = no 6 on first roll

A_2 = no 6 on second roll

$= 1 - P(A_1 A_2)$ Assuming $A_1 \perp\!\!\!\perp A_2$

$= 1 - P(A_1)P(A_2)$

$= 1 - (5/6)(5/6)$

$= 11/36$

Counting perspective

Sample twice $(r=2)$ from $\{1, \dots, 6\}$ $(n=6)$ w/ replacement.

Ordered:

Ordered:

$$|S| = n^r = 6^2 = 36$$

E = "at least one 6"

$$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$$

$$|E| = 11$$

$$P(E) = \frac{|E|}{|S|} = 11/36.$$

Unordered:

$$|S| = \binom{n+r-1}{r} = \binom{6+2-1}{2} = \binom{7}{2} = 21$$

$$E = \{\{1,6\}, \{2,6\}, \{3,6\}, \{4,6\}, \{5,6\}, \{6,6\}\}$$

$$|E| = 6$$

$$P(E) = 6/21.$$

Ex. Roll two dice (independently)

$$E = \{1 \text{ or } 2 \text{ on first roll}, \; 3,4,5 \text{ on second}\}$$

Counting in an ordered way:

$$|S| = n^r = 6^2 = 6 \cdot 6$$

$E = \{(1,3),(1,4),(1,5)$
$(2,3),(2,4),(2,5)\}$

note: $E = \{1,2\} \times \{3,4,5\}$

cartesian product

$$|E| = |\{1,2\}| \cdot |\{3,4,5\}|$$
$$= 2 \cdot 3$$

Overall:

$$P(E) = \frac{2 \cdot 3}{6 \cdot 6} = \left(\frac{2}{6}\right)\left(\frac{3}{6}\right)$$

prob. of a 1 or 2

prob. of a 3, 4, 5

Theorem: Complements and Independence

If $A \perp\!\!\!\perp B$ then

1. $A \perp\!\!\!\perp B^c$

Case 1

Pf. $P(AB^c) = P(A) - P(AB)$

$= P(A) - P(A)P(B)$

① $A \perp\!\!\!\perp B^{c}$

② $A^c \perp\!\!\!\perp B$

③ $A^c \perp\!\!\!\perp B^c$

$$= P(A) - P(A)P(B)$$

$$= P(A)(1 - P(B))$$

$$= P(A)P(B^c)$$

Defn: Mutual Independence

Generalize independence to multiple events.

If (A_i) is a seq of events we say they are (mutually) independent if for all subsequences

$$A_{i_1}, A_{i_2}, A_{i_3}, \ldots, A_{i_k}$$

$$P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = P(A_{i_1})P(A_{i_2})\cdots P(A_{i_k})$$

$$= \prod_{j=1}^{k} P(A_{i_j})$$

Q: Do I really need to check all subsequences?

Could I just check:

$$P(A_1 A_2 \cdots A_n) = P(A_1) \cdots P(A_n) \; ?$$

No.

Ex. Roll two dice.

$A =$ "doubles" $= \{(1,1),(2,2),\ldots,(6,6)\}$; $|A| = 6$

$B =$ "sum is between 7 and 10"

$= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),$
$(2,6),(3,5),(4,4),(5,3),(6,2),$
$(3,6),(4,5),(5,4),(6,3),$
$(6,4),(5,5),(4,6)\}$

$|B| = 18$

$C =$ "sum is 2, 7, 8"

$= \{(1,1),$ $\}$

$|C| = 12$

Mutually Independent?

$$\underbrace{P(ABC)}_{1/36} = \underbrace{P(A)}_{\underbrace{6/36}_{1/6}} \underbrace{P(B)}_{\underbrace{18/36}_{1/2}} \underbrace{P(C)}_{\underbrace{12/36}_{1/3}} \; ? \quad \checkmark$$

$\{(4,4)\}$

$$\underbrace{\quad}_{1/6} \quad \underbrace{\quad}_{1/2} \quad \underbrace{\quad}_{1/3}$$

Consider BC, sum is 7 or 8

$$\underbrace{P(BC)}_{11/36} \neq \underbrace{P(B)}_{1/2}\underbrace{P(C)}_{1/3}.$$

Defn: Pairwise Independence

If (A_i) is a seq of events we say they are pairwise independent if

$$P(A_i A_j) = P(A_i)P(A_j) \text{ for } i \neq j.$$

Can $A \perp\!\!\!\perp A$?

$$P(A) = P(AA) = P(A)P(A) = P(A)^2$$

$$P(A) = P(A)^2$$

recall: $P(A) \in [0, 1]$

This works if $P(A) = 0$ or 1.

Pairwise Independence $\neq$ Mutual Indep.

Ex.

$S = \{abc, acb, bac, bca, cab, cba,$
$aaa, bbb, ccc\}.$

$|S| = 9$, all outcomes equally likely

$A_i = i^{th}$ spot is an 'a'

$A_1 = \{abc, acb, aaa\}$ $\quad |A_1| = |A_2| = |A_3| = 3$

$A_2 = \{bac, cab, aaa\}$

$A_3 = \{bca, cba, aaa\}$

Pairwise Independent?

$\{aaa\}$

$$\underbrace{P(A_iA_j)}_{1/9} = \underbrace{P(A_i)}\,P(A_j)$$
$$= (3/9)(3/9) \quad \checkmark$$

Mutual Independence

$\{aaa\}$

$P(A_1A_2A_3) = P(A_1)P(A_2)P(A_3)$

$$P(A_1A_2A_3) = \underbrace{P(A_1)}_{1/3}\underbrace{P(A_2)}_{1/3}\underbrace{P(A_3)}_{1/3}$$

$$1/9 \neq$$

Not mutually independent.

Ex. JWST had ~400 single points of failure.

$$\boxed{T1} \to \boxed{T2} \to \boxed{T3} \to \cdots \to \boxed{T400} \to \text{complete}$$

JWST fails if any fail,
Only works if all tasks succeed.

W_i = i^{th} task works
W_i^c = " fails

Assume that all tasks are independent.

Assume $P(W_i^c) = 1/1000$

$P(\text{JWST works})$

$$= P\left(\bigcap_{i=1}^{400} W_i\right)$$

$$= \mathbb{P}\left(\bigcap_{i=1} W_i\right)$$

$$= \prod_{i=1}^{400} \mathbb{P}(W_i)$$

$$= \prod_{i=1}^{400} \left(1 - \mathbb{P}(W_i^c)\right)$$

$$= \prod_{i=1}^{400} \left(1 - 1/1000\right)$$

$$= \left(1 - 1/1000\right)^{400}$$

Ex. Flip a coin 3 times.

X = # heads among my 3 flips.

$s \in S$	$X(s)$
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

← a function