The point of counting is:

If I have a S w/ equally likely outcomes;

then

$$P(E) = \frac{|E|}{|S|}$$

need to count

Q: ordering? w/ replacement?

The answer to this that all outcomes must be equally likely.

Ex. Flip a coin twice.

What is the prob. of getting a H and T.

Option 1: Unordered sample space

$$S = \{HH, TT, HT\}$$

and $E = \{HT\}$

So $P(E) = \frac{|E|}{\quad} = \frac{1}{\quad}$.

$|S| \quad 3$

<u>Option 2:</u>

$$\underbrace{H}_{\frac{1}{2}}\underbrace{T}_{\frac{1}{2}} \text{ or } \underbrace{T}_{\frac{1}{2}}\underbrace{H}_{\frac{1}{2}}$$

$$\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

basically we are counting <u>w/ ordering</u>

$$S = \{HH, TT, HT, TH\}$$

$$E = \{HT, TH\} \text{ and so } \mathbb{P}(E) = \frac{|E|}{|S|} = \frac{2}{4} = \frac{1}{2}.$$

<u>General rule:</u>

If I build S through a seq. of independent actions then <u>typically</u> counting in an ordered way is correct.

Really only a big deal when sampling <u>w/ replacement.</u>

Sample $\boxed{\text{w/o replacement}}$

$$P(E) = \frac{|E| r!}{\ldots}$$

1|5|r!

Ex. Survey W&M students and ask about political afil.

gender	A	B	
men	501	238	739
women	782	123	905
		361	1644

Q1: If I randomly select a student, what is the prob. they are a woman?

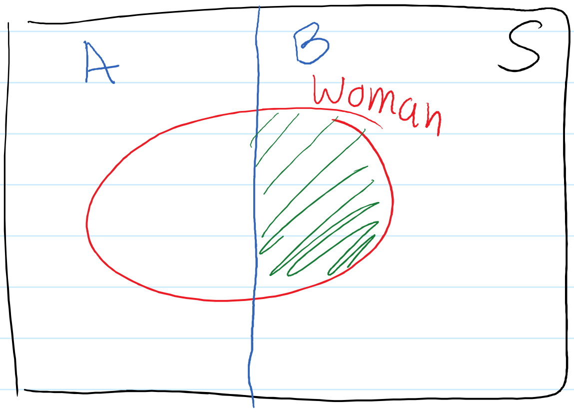
$$P(\text{women}) = 905/1644$$

Q2: Given the student is in party B, what is the prob. they are a woman?

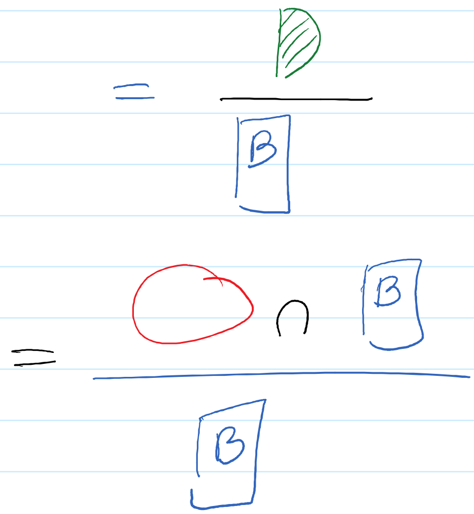
$$P(\text{Women } \underline{\text{given}} \text{ B}) = 123/361$$

Venn Diagram

Q1: $P(\text{woman}) = \dfrac{\bigcirc}{\square}$



Q2: P(woman GIVEN B)



Defn: Conditional Probability

If $A, B \subset S$ and $P(B) > 0$ then the conditional prob. of A given B

$$P(A|B) = \frac{P(AB)}{P(B)}$$

"given"

Facts:

① $\boxed{P(B|B) = 1}$

Pf. $P(B|B) = \frac{P(BB)}{\quad} = \frac{P(B)}{\quad} = 1$

Pf. $P(B|B) = \dfrac{P(BB)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(2) $\boxed{\text{If } AB = \emptyset \text{ then } P(A|B) = 0}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0$$

Ex. Roll two dice.

Q: what is the prob. the first is a 2 (A)

GIVEN the sum of the two ≤ 5. (B)

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$= \frac{|AB|/|S|}{|B|/|S|} = \frac{|AB|}{|B|} = \frac{3}{10}.$$

AB

roll 1

roll 2

	1	2	3	4	5	6
1	O	X	O	O		
2	O	X	O			
3		X				

roll 2

2	O	X	O			
3	O	X				
4	O	X				
5		X				
6		X				

Theorem: Compound Probability

Let $P(A), P(B) > 0$,

$$P(AB) = P(A|B)P(B) = P(B|A)P(A).$$

Pf.

$$P(A|B) = \frac{P(AB)}{P(B)}$$

rearange: $P(A|B)P(B) = P(AB)$.

Recall: partitioning theorem

(A_i) partition S then

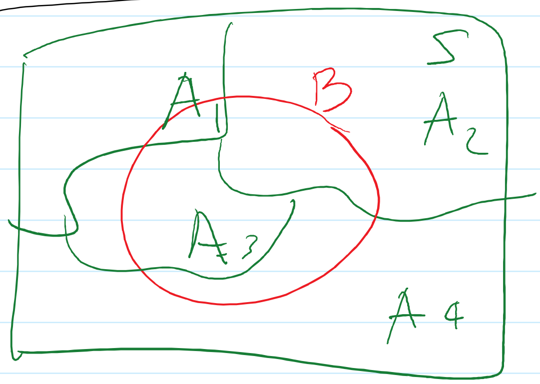
$$P(B) = \sum_i P(BA_i).$$

Theorem: Law of Total Probability

If (A_i) partition S and $P(A_i) > 0$ then

$B \subset S$

$$\mathbb{P}(B) = \sum_i P(B|A_i)\,\mathbb{P}(A_i).$$



Pf. partitioning theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(BA_i)$$

$$P(B|A_i)\,\mathbb{P}(A_i)$$

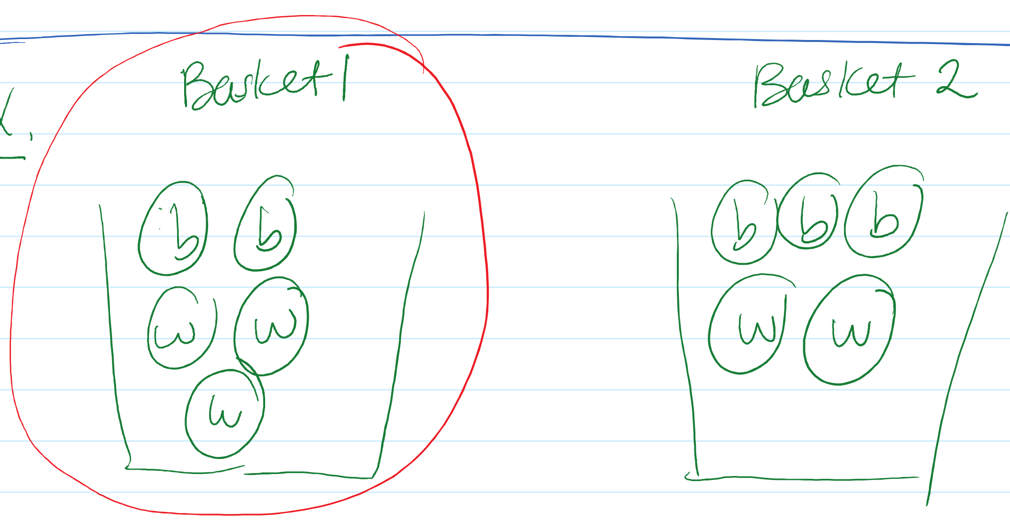
compound prob. theorem.

Special Case: A, A^c always partition S

this theorem says

$$\mathbb{P}(B) = \mathbb{P}(B|A)\,\mathbb{P}(A) + \mathbb{P}(B|A^c)\,\mathbb{P}(A^c).$$

Ex.



Game: ① randomly select ball from basket 1 and put in basket 2

② randomly select ball from basket 2

Q: What is the prob I select a black ball on step 2?

W = choose (w) on step 1

W^c = " (b) "

B = choose (b) on step 2

B^c = " (w) "

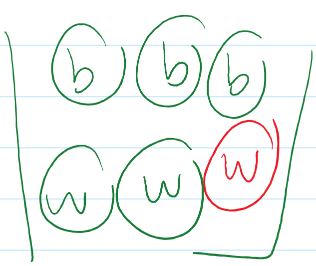
Want: $P(B)$. Solve by partitioning/conditioning on W, W^c.

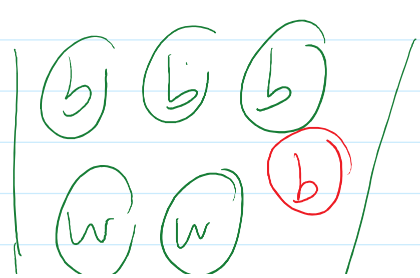
Law of total prob says

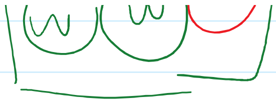
$$P(B) = P(B|W)P(W) + P(B|W^c)P(W^c)$$

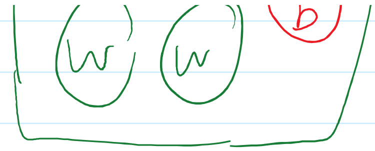
$$\left(\tfrac{1}{2}\right)\left(\tfrac{3}{5}\right) + \left(\tfrac{2}{3}\right)\left(\tfrac{2}{5}\right) = \tfrac{17}{30}$$

$P(B|W) = 3/6 = 1/2$ $\qquad$ $P(B|W^c) = 4/6 = 2/3$









Theorem: Bayes' Theorem

How to calculate $\mathbb{P}(A|B)$ from $\mathbb{P}(B|A)$.

If $A, B \subset S$, $\mathbb{P}(A), \mathbb{P}(B) > 0$ then

$$\mathbb{P}(A|B) = \mathbb{P}(B|A) \frac{\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Pf.

$$\mathbb{P}(A|B) \underset{\text{defn}}{=} \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} \underset{\text{compond prob.}}{=} \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Ex. Continue prev.

Given I choose a black ball on second step, what is the prob. I chose a white on the first.

$$\mathbb{P}(W|B) = \frac{\mathbb{P}(B|W)\,\mathbb{P}(W)}{\mathbb{P}(B)}.$$

$$= \frac{(1/2)(3/5)}{(17/30)}$$

Theorem: Law of Tot. Prob + Bayes'

If (A_i) partition S and $P(A_i) > 0$, $P(B) > 0$ then

$$P(A_i | B) = \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)}$$

Pf.

$$P(A_i|B) \overset{\text{Bayes'}}{=} \frac{P(B|A_i)P(A_i)}{P(B)}$$

expand w/ Law of tot. prob.

$$\boxed{= \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)}.}$$

Note: A, A^c partition S so

$$P(A|B) = \frac{P(B|A)P(A)}{\quad}$$

$$\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c).$$

Ex. COVID has a prevalence rate of 1%.

D = have COVID | $\mathbb{P}(D) = .01$

D^c = no COVID | $\mathbb{P}(D^c) = .99$

We test for COVID and get a + or −.

→ The test accurately reports a + 95%

(sensitivity) $\mathbb{P}(+|D) = .95$

$\mathbb{P}(-|D) = .05$

→ The test acc. reports a − 99%

(specificity) $\mathbb{P}(-|D^c) = .99$

$\mathbb{P}(+|D^c) = .01$

Q: I get a + test.

What is the prob. I have COVID.

$$\mathbb{P}(D|+) = \frac{\mathbb{P}(+|D)\mathbb{P}(D)}{\mathbb{P}(+|D)\mathbb{P}(D) + \mathbb{P}(+|D^c)\mathbb{P}(D^c)}$$

$$= \frac{(.95)(.01)}{(.95)(.01) + (.01)(.99)}$$

$$\approx .49$$
