Lecture 13 - More Common Distributions

Poisson Distribution

discrete RV

- support : $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

Canonical Experiment:

count the number of events in some time period

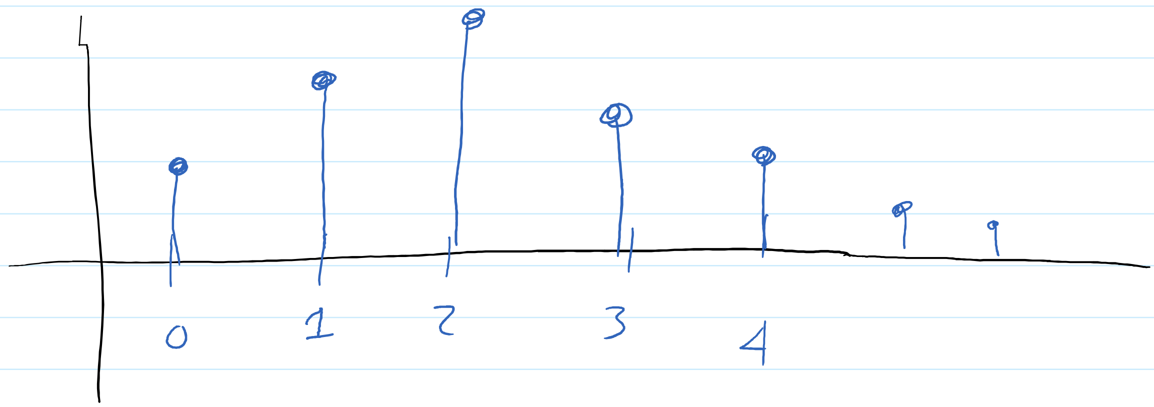
Ex. - model fish capture in a river
- count # of RNA in a cell
- radioactive decay

$X \sim Pois(\lambda)$ rate of occurence $\lambda > 0$

# of occurences

PMF:

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad \text{for } x = 0, 1, 2, 3, \dots$$



Expected Value

Expected Value

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x f(x) = \sum_{x=\cancel{0}1}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad \frac{x}{x!} = \frac{1}{(x-1)!}$$

$$\boxed{e^{x} = \sum_{i=0}^{\infty} \frac{x^{i}}{i!}}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+1}}{x!} = \lambda e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \boxed{\lambda = \mathbb{E}[X]}$$

Consider:

$$\mathbb{E}[X(X-1)] = \sum_{x=\cancel{0}2}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad \frac{x(x-1)}{x!} = \frac{x(x-1)}{x(x-1)(x-2)!} = \frac{1}{(x-2)!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-2)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+2}}{x!}$$

$$= \lambda^{2} e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}} = \lambda^{2} e^{-\lambda} e^{\lambda} = \lambda^{2}$$

$e^{\wedge}$

$$\mathbb{E}[X(X-1)] = \mathbb{E}[X^2 - X] = \mathbb{E}[X^2] - \mathbb{E}[X] = \lambda^2$$

$$\mathbb{E}[X^2] = \lambda^2 + \mathbb{E}[X] = \lambda^2 + \lambda$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda}$$

MGF:

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\lambda} \lambda^x}{x!}$$

$(\lambda e^t)^x$

$$\boxed{\sum_{i=0}^{\infty} \frac{y^i}{i!} = e^y}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{(\lambda e^t)}$$

$$\boxed{= \exp(\lambda(e^t - 1)) = M(t)}$$

Gamma Distribution:

→ generalization of exponential dist.

→ cts dist w/ support $(0, \infty)$

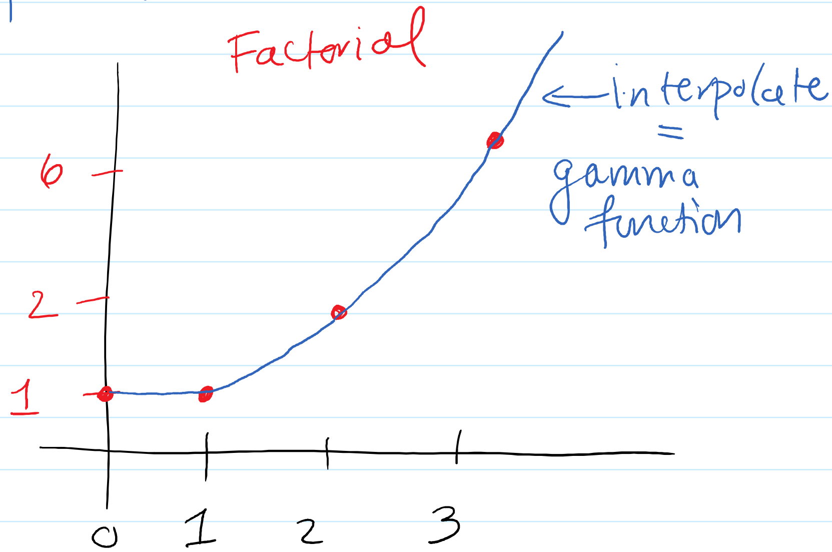
Let's talk about gamma function:

Extends factorial to pos. numbers.

$\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$

So that for $k \geq 0$

$$\boxed{\Gamma(k) = \int_0^\infty x^{k-1} e^{-x}\, dx}$$



Properties of Γ:

1. If k is an integer then

$$\Gamma(k) = (k-1)!$$

$$\Gamma(k+1) = k!$$

Ex. $\Gamma(1) = 0! = 1$

$\Gamma(2) = 1! = 1$

$\Gamma(3) = 2! = 2$ ...

Notice! $x! = x(x-1)!$

So for an integer k,

$$\Gamma(k) = (k-1)! = (k-1)(k-2)! = (k-1)\Gamma(k-1)$$

or $\Gamma(k+1) = k\Gamma(k)$

This is also true for all $k \geq 0$.

Important Facts about Γ

1. $\Gamma(k+1) = k\Gamma(k)$ or $\Gamma(k) = (k-1)\Gamma(k-1)$
2. If k is an integer,

   $\Gamma(k) = (k-1)!$ or $\Gamma(k+1) = k!$

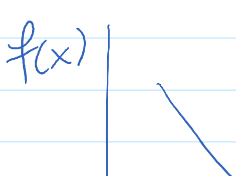
Gamma Dist:

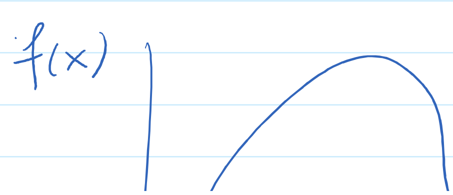
$$X \sim \text{Gamma}(\underset{\text{shape}}{k}, \underset{\text{rate}}{\lambda})$$

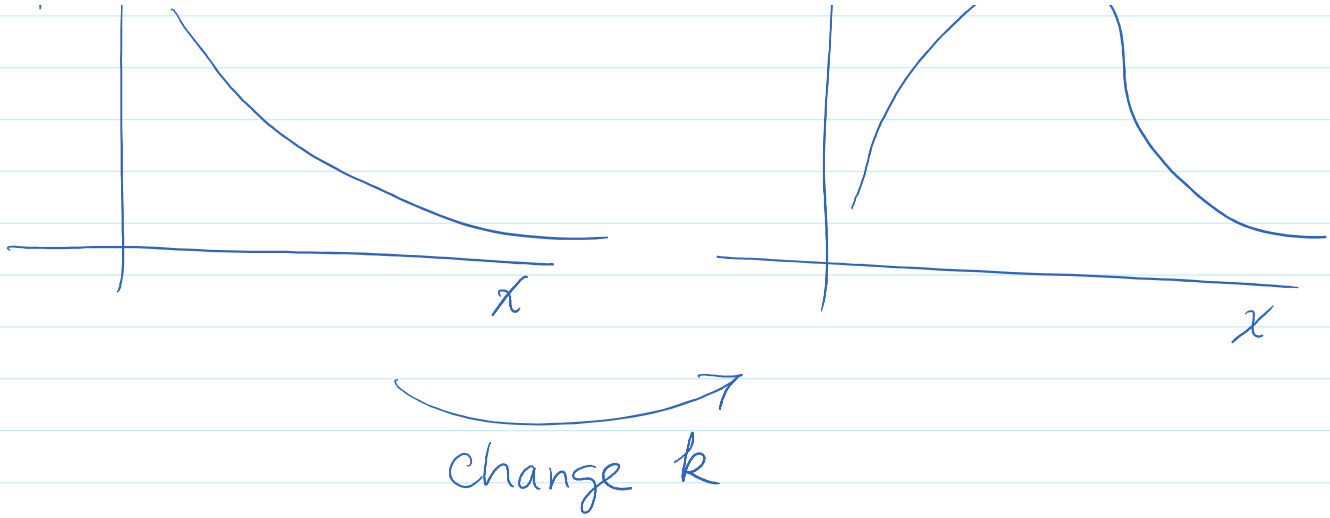
PDF:

$$f(x) = \frac{\overbrace{\lambda e^{-\lambda x}}^{\text{Exp}(\lambda)} (\lambda x)^{k-1}}{\Gamma(k)} \qquad \forall\, x \geq 0$$

Note: $k = 1$ then we have $\text{Exp}(\lambda)$







Expectation:

$$\mathbb{E}[X] = \int_0^\infty x \, \frac{\lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma(k)} \, dx$$

$$= \frac{1}{\lambda} \int_0^\infty \frac{\lambda e^{-\lambda x} \lambda^{k-1} \lambda x^{k}}{\Gamma(k)} \, dx$$

looks like PDF of Gamma$(k+1, \lambda)$

$$\frac{\lambda e^{-\lambda x} (\lambda x)^{k}}{\Gamma(k+1)}$$

$$= \underbrace{\frac{1}{\lambda} \frac{1}{\Gamma(k)} \Gamma(k+1)}\int_0^\infty \underbrace{\frac{\lambda e^{-\lambda x} \lambda^{k} x^{k}}{\Gamma(k+1)}}_{\text{PDF of Gamma}(k+1,\lambda)} \, dx$$

so integrates to 1

$$= \frac{1}{\lambda} \frac{\Gamma(k+1)}{\Gamma(k)}$$

$$= \frac{1}{\lambda} \frac{k\Gamma(k)}{\Gamma(k)} = \boxed{\frac{k}{\lambda} = E[X]}$$

What about $E[X^r]$

$$E[X^r] = \int_0^\infty \frac{x^r \lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}\,dx$$

$$= \frac{\Gamma(k+r)}{\Gamma(k)} \frac{1}{\lambda^r} \underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x} \lambda^{k-1} \lambda^r x^{k+r-1}}{\Gamma(k+r)}\,dx}_{\text{Gamma}(k+r,\ \lambda)}$$

$$\frac{\lambda e^{-\lambda x}(\lambda x)^{k+r-1}}{\Gamma(k+r)}$$

... integrates to 1

$$\boxed{= \frac{\Gamma(k+r)}{\Gamma(k)} \frac{1}{\lambda^r} = E[X^r]}$$

Ex, $r = 2$

$$E[X^2] = \frac{\Gamma(k+2)}{\Gamma(k)} \frac{1}{\lambda^2} = \frac{(k+1)\Gamma(k+1)}{\Gamma(k)} \frac{1}{\lambda^2}$$

$$= \frac{(k+1)k\cancel{\Gamma(k)}}{\cancel{\Gamma(k)}} \frac{1}{\lambda^2}$$

$$= \frac{(k+1)k}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{k(k+1)}{\lambda^2} - \left(\frac{k}{\lambda}\right)^2$$

$$= \dots = \boxed{k/\lambda^2}$$

Geometric Distribution (discrete)

Canonical Experiment

If I flip coins (independently), each w/ a prob. p of a H, and I do this until I get my first H.

$$X = \#\text{ of flips to get H}$$

outcome	X
H	1
TH	2
TTH	3
⋮	⋮

$X \sim \text{Geometric}(p)$

<u>PMF:</u> $f(x) = (1-p)^{x-1}p$ for $x = 0, 1, 2, 3, \ldots$

<u>CDF:</u>

$$F(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor}, & x \geq 1 \\ 0 & \text{else} \end{cases}$$

<u>Recall:</u> geometric series

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \quad \text{for } |r| < 1$$

Expected Value

$$E[X] = \sum_{x=1}^{\infty} \underbrace{x(1-p)^{x-1}} p$$

$$x(1-p)^{x-1} = -\frac{d}{dp}(1-p)^x$$

$$= -p \sum_{x=1}^{\infty} \frac{d}{dp}(1-p)^x$$

$$\sum^{\infty} (1-p)^{x+1} = (1-p) \sum^{\infty} \overbrace{(1-p)}^{r}{}^{x}$$

$$= -p \sum_{x=1} \frac{d}{dp} (1-p)$$

$$= -p \frac{d}{dp} \sum_{x=1}^{\infty} (1-p)^x$$

$$\sum_{x=0}^{\infty} (1-p)^{x+1} = (1-p) \underbrace{\sum_{x=0}^{\infty} (1-p)^x}_{\text{geometric} \dots}$$

$$= \frac{1}{1-(1-p)} = \frac{1}{p}$$

$$= -p \frac{d}{dp} \left[(1-p) \frac{1}{p} \right]$$

$$= -p \left(-\frac{1}{p^2} \right)$$

$$= \boxed{1/p = E[X]}$$

MGF:

$$M(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} (1-p)^{x-1} p$$

$$= \frac{p}{1-p} \sum_{x=1}^{\infty} \underbrace{((1-p)e^t)}_{r}{}^x$$

$$= \frac{p}{1-p} \sum_{x=0}^{\infty} ((1-p)e^t)^{x+1}$$

if $(1-p)e^t < 1$

$$= \frac{\cancel{(1-p)}e^t}{\cancel{1-p}} p \underbrace{\sum_{x=0}^{\infty} (\underbrace{(1-p)e^t}_{r})^x}_{\frac{1}{1-(1-p)e^t}}$$

t

$$\boxed{= \frac{pe^t}{1-(1-p)e^t} = M(t)}$$

So... $\left.\frac{d^2M}{dt}\right|_{t=0} = \cdots = \frac{2-p}{p^2} = \mathbb{E}[X^2]$

hence $Var(X) = \mathbb{E}[X^2] - \mathbb{E}(X]^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$

$$= \cdots$$

$$= \boxed{\frac{1-p}{p^2} = Var(X)}$$