Ex. Flip a coin 3 times.

X = # heads among my 3 flips.

$s \in S$	$X(s)$
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

← a function

Defn: Random Variable

A random variable (RV) X is a function

$$X : S \to \mathbb{R}$$

also called a random variate

or a real-valued random variable

or a univariate random variable

($\mathbb{R}$ not $\mathbb{R}^n$)

($\mathbb{R}$ not $\mathbb{R}$)

Ex. (1) toss two dice,

X = sum of dice

(2) toss a coin 25 times

X = length of longest chain of consecutive Hs

(3) observe rainfall in region

X = crop yield

We'd like to say,

$$\underbrace{P(X=1)}_{\text{abuse of notation}}$$

recall: $\mathbb{P}: 2^S \to \mathbb{R}$

what we really mean

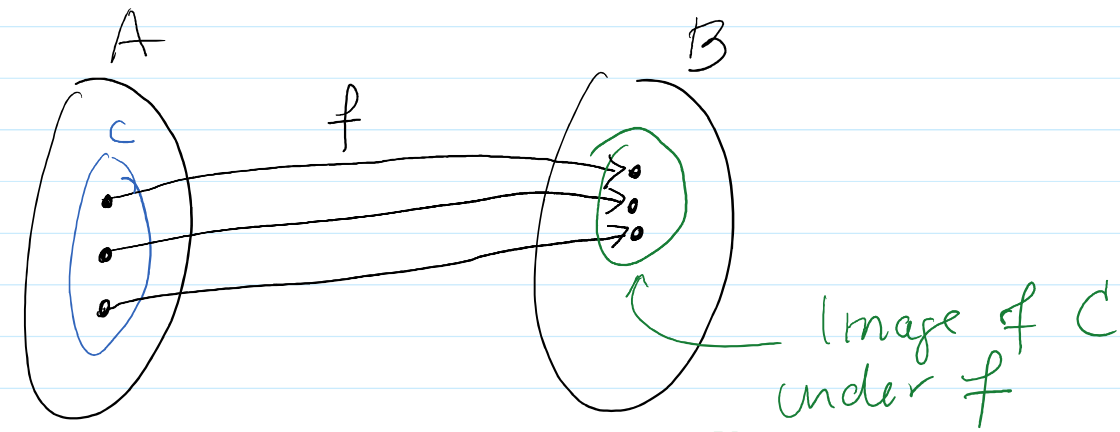
X = # heads in 3 flips

$$P(X=1) = P(\{HTT, THT, TTH\}) = 3/8$$

"$X=1$" short-hand for $\{a \in S \mid X(a) = 1\}$

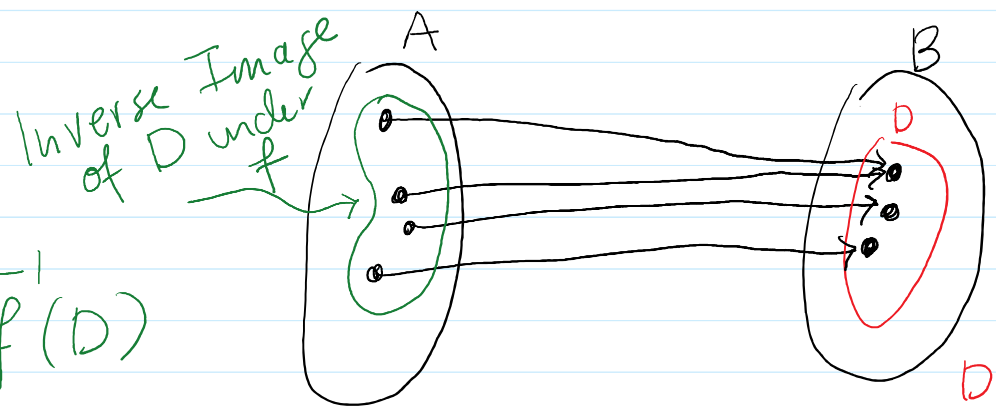
"$X=1$" short-hand for $\underbrace{\{\omega \in S \mid X(\omega)=1\}}_{\text{inverse image of } \{1\} \text{ under } X}$

Review:



$C \subset A$

$f(C)$.



$f^{-1}(D)$

$D \subset B$

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\}$$

$$\text{"}X=1\text{"} = X^{-1}(\{1\}) = \{\omega \in S \mid X(\omega)=1\}$$

Notation: If X is a RV and $A \subset \mathbb{R}$,

we write

we write

$$\mathbb{P}(X \in A)$$

means

$$\mathbb{P}(\underbrace{X^{-1}(A)}_{\subset S})$$

Ex. $\mathbb{P}(X = 1 \text{ or } 2)$

$X = \#$ heads in 3 flips

$$= \mathbb{P}(X \in \{1, 2\})$$

$$= \mathbb{P}(X^{-1}(\{1, 2\}))$$

$$= \mathbb{P}(\{THT, TTH, HTT, HHT, THH, HTH\})$$

$$= 6/8$$

Defn: Support of a RV

If X is a RV its support is the set of possible values of X. (i.e. the image of S under X)

Ex. Prev. Example,

$$\text{Support}(X) = \{0, 1, 2, 3\}$$

Notice: $\mathbb{P}(X=5)=0$.

More generally, $A \subset \mathbb{R}$ and $\text{Support}(X) \cap A = \emptyset$ then $\mathbb{P}(X \in A) = 0$.

Informal Defn Types of RVs

1. discrete RV: support is finite or countable

   Ex. X = sum of two dice

   Ex. X = # number of customers arriving in a shop

2. continuous RVs: support is uncountably infinite

   Ex. time or space

Defn: Cumulative Distribution Function (CDF)

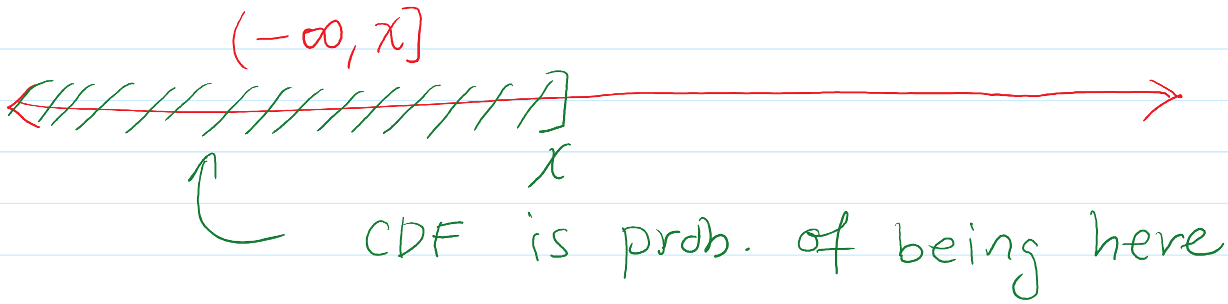
If X is a RV then its CDF is a function $F: \mathbb{R} \to \mathbb{R}$ defined for $x \in \mathbb{R}$

a RV

$$F(x) = \mathbb{P}(X \le x)$$

a number in $\mathbb{R}$

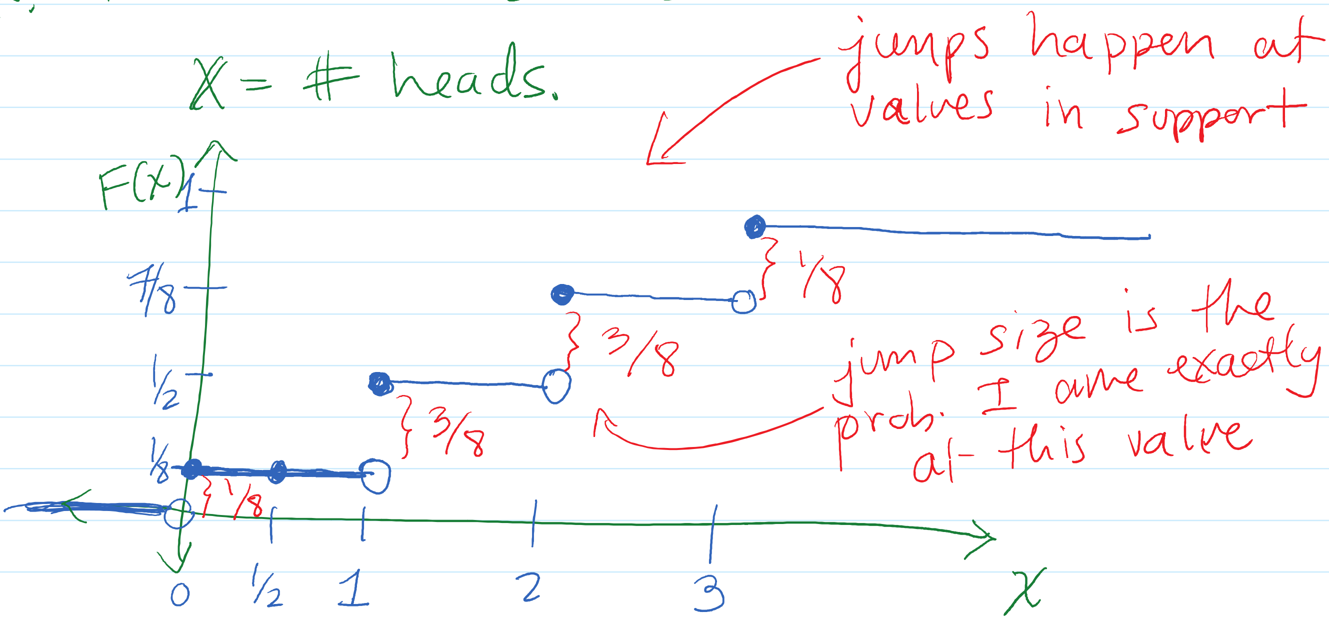
$F(x) = P(X \le x)$ a number in $\mathbb{R}$



Notation: $F(x) = P(X \le x)$
$= P(X \in (-\infty, x])$

Ex. Toss a coin 3 times.

$X = \#$ heads.



$F(0) = P(X \le 0) = P(X = 0) = 1/8$

$F(1/2) = P(X \le 1/2) = P(X = 0) = 1/8$

$F(.9) = P(X \le .9) = P(X = 0) = 1/8$

$F(1) = P(X \le 1) = P(X = 0 \text{ or } 1) = 4/8 = 1/2$

$F(1) = \mathbb{P}(X \le 1) = \mathbb{P}(X = 0 \text{ or } 1) = 4/8 = 1/2$

$F(1.5) = \mathbb{P}(X \le 1.5) = \mathbb{P}(X = 0 \text{ or } 1) = 1/2$

$F(2) = \mathbb{P}(X \le 2) = 7/8$

$F(3) = \mathbb{P}(X \le 3) = 1$

$F(4) = \mathbb{P}(X \le 4) = 1$

Facts!

① $\boxed{0 \le F(x) \le 1}$

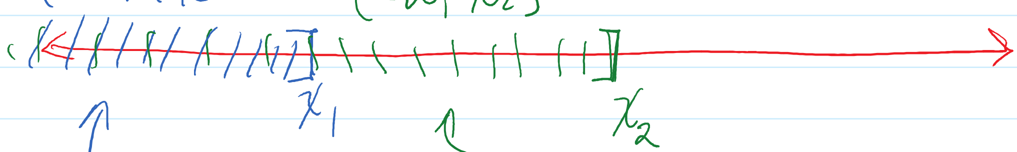
pf. $F(x) = \mathbb{P}(\sim\sim) \in [0, 1]$

② $\boxed{\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1}$

③ $\boxed{F \text{ is non-decreasing.}}$

If $x_1 < x_2$ then $F(x_1) \le F(x_2)$.

pf. $(-\infty, x_1] \subset (-\infty, x_2]$



$F(x_1) = \mathbb{P}(X \le x_1)$

$F(x_2) = \mathbb{P}(X \le x_2)$

$= \mathbb{P}(X \le x_1)$ $\qquad$ $\mathbb{P}(X \le x_2)$

$= \mathbb{P}(X \in (-\infty, x_1])$ $\qquad$ $= \mathbb{P}(X \in (-\infty, x_2])$

$$= \mathbb{P}(\underbrace{X^{-1}(\underbrace{(-\infty, x_1]}))}) \quad \le \quad = \mathbb{P}(\underbrace{X^{-1}(\underbrace{(-\infty, x_2]}))})$$

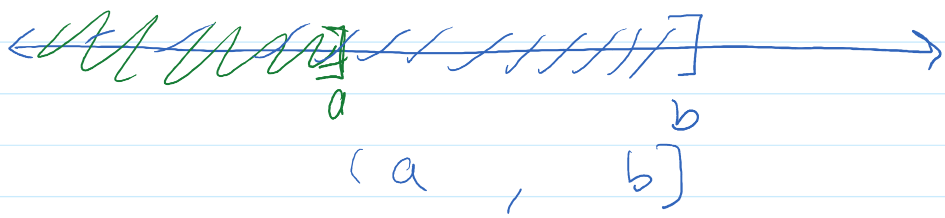
inverse images preserve subsets:

$A \subset B$,

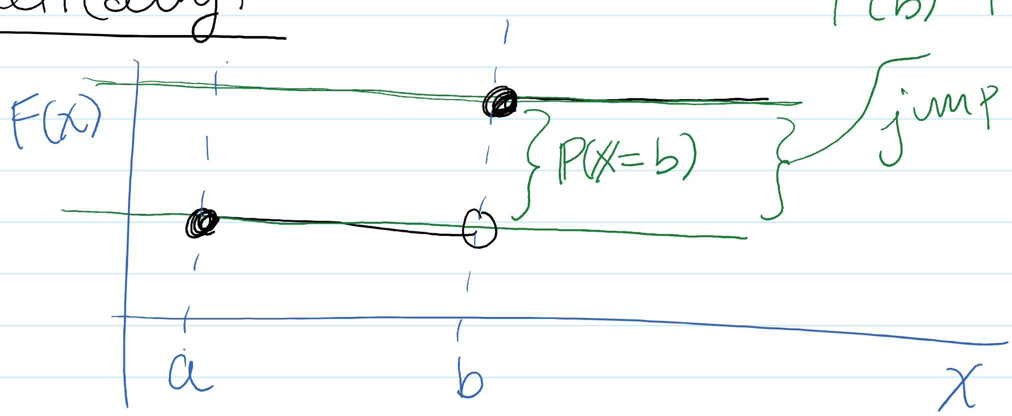
$f^{-1}(A) \subset f^{-1}(B)$

(4) $\boxed{\mathbb{P}(a < X \le b) = F(b) - F(a)}$

$(a, b] = (-\infty, b] \setminus (-\infty, a]$



practically:



$F(b) - F(a)$

jump size

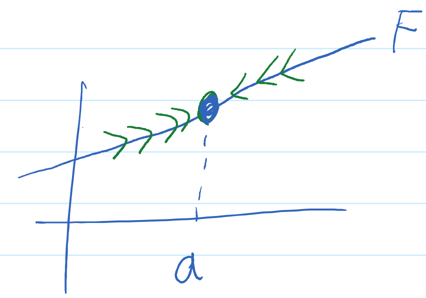
$\therefore\ \mathbb{P}(a < X \le b) = \mathbb{P}(X = b)$

$\checkmark$ $\mathbb{P}(a < X \le b) = \mathbb{P}(X = b)$

⑤ $\boxed{F \text{ is right-continuous.}}$

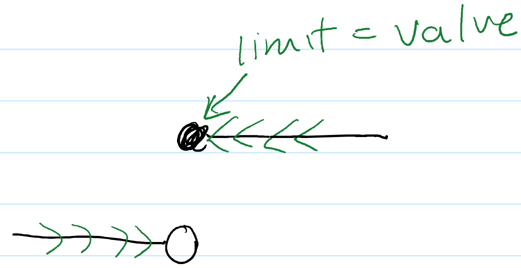
Recall: cts function

$$\lim_{x \to a} F(x) = F(a)$$



Right cts:

$$\lim_{x \to a^+} F(x) = F(a)$$



Note: a cts function is right cts.

Theorem: F is the CDF of some RV iff

(1) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to} F(x) = 1$

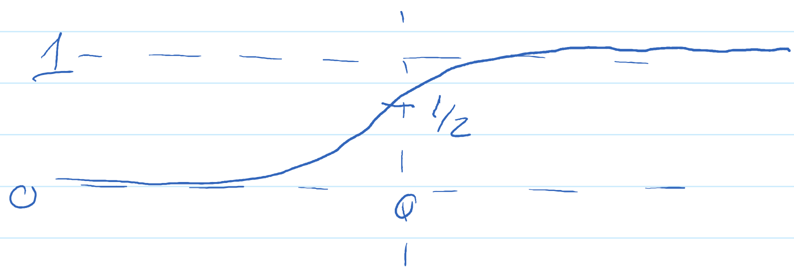
(2) F is non-decreasing

(3) F is right continuous.

Ex. Let

$$F(x) = \frac{1}{1+e^{-x}}$$

Q: Is this a valid CDF?



Check 3 conditions

✓ (1) $\lim_{x \to -\infty} F(x) = \frac{1}{1+e^{-(-\infty)}} = \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0$

$\lim_{x \to \infty} F(x) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$

✓ (2) non-decreasing.

$$\frac{dF}{dx} = \frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$$

✓ (3) Right cts?

This is cts, so its right cts.