Ex. $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

$$\mu = EX = 1/\lambda \quad ; \quad \mathbb{E}[X^2] = 2/\lambda^2$$

$$Var(X) = \mathbb{E}[(X - EX)^2]$$

$$= E[(X-\mu)^2] = \int_{\mathbb{R}} (x-\mu)^2 f(x)\,dx$$

correct, but probably time intensive

$$= \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

$\mathbb{E}[aX+b] = a\mathbb{E}X + b$

$$= \mathbb{E}[X^2] - 2\mu EX + \mu^2$$

$$= \left(2/\lambda^2\right) - 2\left(1/\lambda\right)\left(1/\lambda\right) + \left(1/\lambda\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{2}{\lambda^2} + \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2} = Var(X)}$$

Theorem: Short-Cut Formula For Variance

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

= expected sq − sq. expectation

pf. $\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2]$

$= \mathbb{E}[X^2 - 2\mu X + \mu^2]$

$= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2$ $\quad \mu = \mathbb{E}X$

$= \mathbb{E}[X^2] - 2\,\mathbb{E}[X]^2 + \mathbb{E}[X]^2$

$\mathbb{E}[X^2] - \mathbb{E}[X]^2$

Ex. Revisit prev.

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$

$= 2/\lambda^2 - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Theorem:

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

① multiply X by a $\rightsquigarrow$ Var is multiplied by a^2

② ignore additive shifts

pf. $\mathrm{Var}(aX+b) = \mathbb{E}[(aX+b)^2] - (\mathbb{E}[aX+b])^2$

pf. $\operatorname{Var}(aX+b) = \mathbb{E}[(aX+b)^2] - (\mathbb{E}[aX+b])^2$

$$= \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\mathbb{E}X + b)^2$$

$$= a^2\mathbb{E}[X^2] + \cancel{2ab\mathbb{E}X} + \cancel{b^2} - (a^2(\mathbb{E}X)^2 + \cancel{2ab\mathbb{E}X} + \cancel{b^2})$$

$$= a^2(\mathbb{E}[X^2] - (\mathbb{E}X)^2)$$

$$= a^2 \operatorname{Var}(X)$$

Ex. $X \sim \operatorname{Bin}(n,p)$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\mathbb{E}[X] = np$$

turns out that $\underbrace{\mathbb{E}[X^2] = np(np-p+1)}$

So $\operatorname{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$

$$= np(np-p+1) - (np)^2$$

$$= np(1-p)$$

Standard deviation: $sd(X) = \sqrt{\operatorname{Var}(X)}$

so $Sd(X) = \sqrt{np(1-p)}$

Defn: Moments of a RV

If r is a pos. integer we define the r^{th} moment of a RV X as

$$\mu_r = \mathbb{E}[X^r]$$

Ex. $\mu_1 = \mathbb{E}X$, $\mu_2 = \mathbb{E}[X^2]$, $\mu_3 = \mathbb{E}[X^3]$, ...

Defn: Moment Generating Function (MGF)

If X is a RV the MGF of X is a function

$$M : \mathbb{R} \to \mathbb{R}$$

defined for $t \in \mathbb{R}$

$$M(t) = \mathbb{E}[e^{tX}].$$

For discrete:

$$M(t) = \mathbb{E}e^{tX} = \sum_x e^{tx} f(x)$$

for continuous:

$$M(t) = Ee^{tX} = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

<u>Ex.</u> $X \sim Exp(\lambda)$

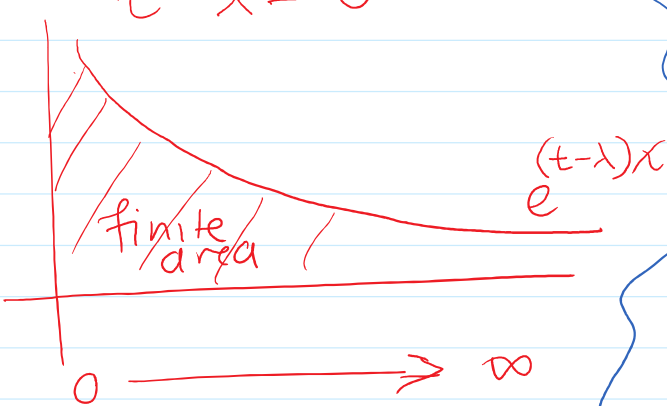
$$\boxed{f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0}$$

$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx$$

$$\lambda \int_0^\infty e^{(t-\lambda)x}\,dx$$

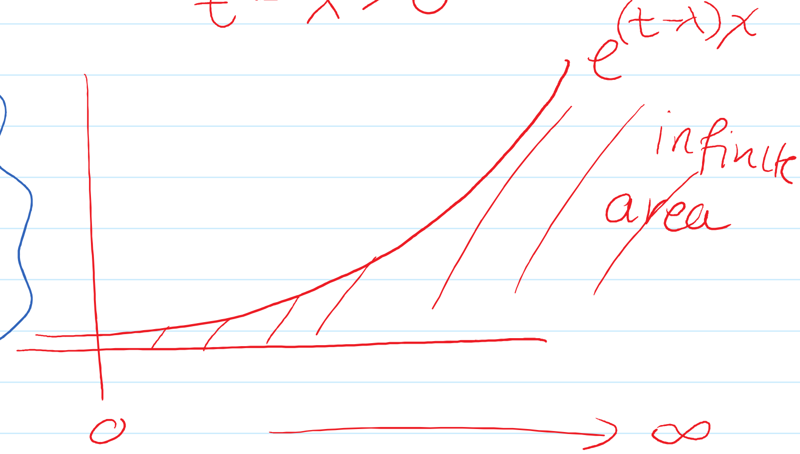
$t \le \lambda$

$t - \lambda \le 0$



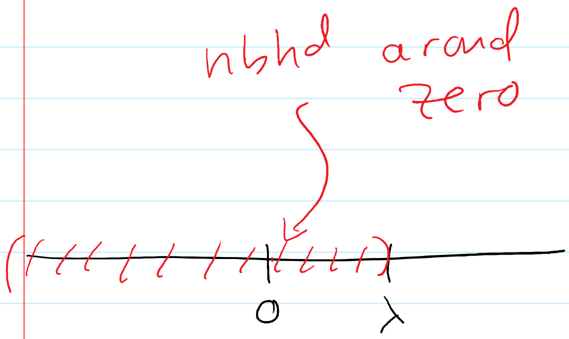
$t > \lambda$

$t - \lambda > 0$



if $t < \lambda$ then

$$M(t) = \lambda \int_0^\infty e^{(t-\lambda)x}\,dx = \lambda \left. \frac{e^{(t-\lambda)x}}{t-\lambda} \right|_0^\infty$$



$$= \frac{\lambda}{t-\lambda}[0-1]$$

$$\boxed{M(t) = \frac{\lambda}{\lambda - t} \quad t < \lambda}$$

__Recall:__ $\mathbb{E}X = \frac{1}{\lambda}$; $\mathbb{E}X^2 = \frac{2}{\lambda^2}$

① $\left.\frac{dM}{dt}\right|_{t=0} = \left.\frac{\lambda}{(\lambda - t)^2}\right|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} = \mathbb{E}X$

② $\left.\frac{d^2M}{dt^2}\right|_{t=0} = \left.\frac{2\lambda}{(\lambda - t)^3}\right|_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2} = \mathbb{E}[X^2]$

__Theorem:__

If X is a RV w/ MGF M then

$$\left.\frac{d^rM}{dt^r}\right|_{t=0} = M^{(r)}(0) = \mathbb{E}[X^r] = \mu_r$$

__pf.__ recall: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots$$

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots$$

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots$$

$$M(t) = E[e^{tX}] = 1 + tEX + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \cdots$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \boxed{EX} + \cancelto{0}{\frac{2t}{2!}} EX^2 + \cancelto{0}{\frac{3t^2}{3!}} E[X^3] + \cdots$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = E[X^2] + \cancelto{0}{\frac{3\cdot 2t}{3!}} E[X^3] + \cdots$$

Ex. $X \sim Bin(n,p)$.

$$E[X^2] = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

Instead use MGF

Recall: binomial theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_x e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (\underbrace{pe^t}_{a = pe^t})^x \underbrace{(1-p)}_{b = 1-p}^{n-x}$$

$$= (a+b)^n = \boxed{(pe^t + 1 - p)^n = M(t)}$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \left. n\underbrace{(pe^t + 1 - p)^{n-1}}\,\underbrace{pe^t}\right|_{t=0} = n(p + 1 - p)^{n-1} p$$

$$= np = \mathbb{E}X$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = n(n-1)(\overset{1}{\cancel{pe^t + 1 - p}})^{n-2} pe^t pe^t$$

$$\left. + \, n(\overset{1}{\cancel{pe^t + 1 - p}})^{n-1} pe^t \right|_{t=0}$$

$$= n(n-1)p^2 + np$$

<u>Theorem:</u>

If $Y = aX + b$ then

$$M_Y(t) = e^{tb} M_X(at)$$

MGF of Y MGF of X

$e^{a+b} = e^a e^b$

Pf.
$$\begin{aligned} M_Y(t) &= E[e^{tY}] = E[e^{t(aX+b)}] \\ &= E[e^{atX} e^{tb}] \\ &= e^{tb} E[e^{(at)X}] \\ &= e^{tb} M_X(at) \end{aligned}$$

<u>Theorem</u>: If X and Y are RVs and

$$M_X(t) = M_Y(t)$$ for t in some neighborhood of zero

then $X \overset{d}{=} Y$.
