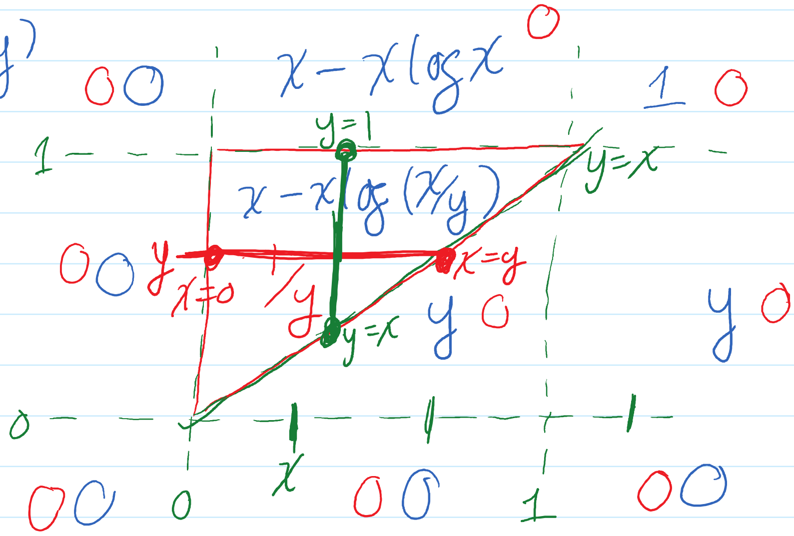
Ex. $F(x,y)$



What is the joint PDF?

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

For $0 < x < y < 1$, $F(x,y) = x - x\log(x/y)$

$$f(x,y) = \frac{\partial F}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(x - x\log(x/y)\right)$$

$$= \frac{\partial}{\partial x}\left[x/y\right]$$

$$= 1/y$$

$$\frac{\partial}{\partial y}\left(x - x\log(x/y)\right) = -x\,\frac{-x/y^2}{x/y} = x/y$$

All together: $\boxed{f(x,y) = \frac{1}{y} \text{ for } 0 < x < y < 1}$

What is the marginal PDF of X?

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_x^1 \frac{1}{y}\,dy = \log(y)\Big|_x^1$$

$$= \log(1) - \log(x)$$

$$= -\log(x)$$

$$\boxed{f_X(x) = -\log(x) \quad \text{for} \quad 0 < x < 1}$$

What is the marginal of Y?

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \int_0^y \frac{1}{y}\,dx = \frac{1}{y}\int_0^y dx$$

$$= \frac{1}{y}\, x\Big|_0^y = \frac{1}{y}\left[y - 0\right]$$

$$= 1$$

for $0 < y < 1$

So $Y \sim U(0,1)$

Ex. Let $f(x,y) = 6xy^2$ for $0 < x < 1$, $0 < y < 1$

$\to x + y \geq 1 \iff y \geq 1 - x$

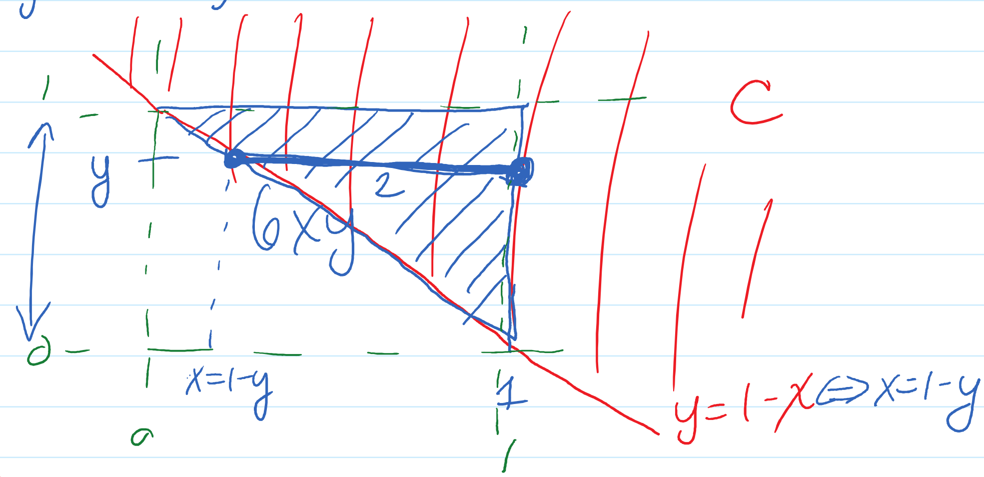
$P(X+Y \geq 1)$ → $x+y \geq 1 \iff y \geq 1-x$

$= \iint_C f(x,y)\,dx\,dy$

$= \int_0^1 \int_{x=1-y}^{x=1} 6xy^2\,dx\,dy$

$= \ldots$

$= 9/10$



Uni: $P(X \in A) = \int_A f_X(x)\,dx$

biv: $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$

Ex. $f(x,y) = e^{-y}$ for $0 < x < y$

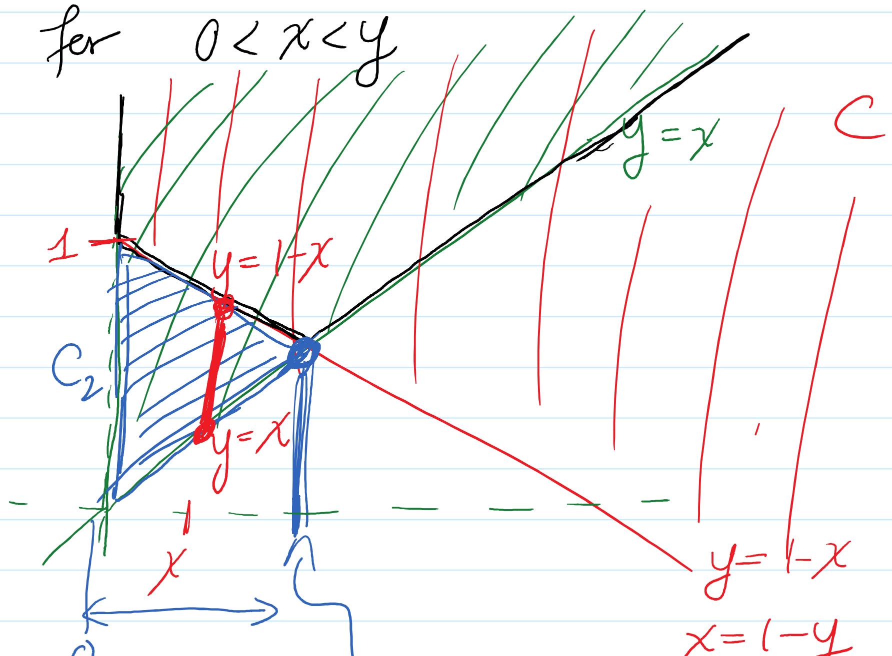
$P(X+Y \geq 1)$

$= \iint_C f(x,y)\,dx\,dy$

$= 1 - P(X+Y < 1)$

$= 1 - \iint_{C_2} f(x,y)\,dx\,dy$

$= 1 - \int_0^{1/2} \int_{y=x}^{y=1-x} e^{-y}\,dy\,dx$



intersection of $y=x$ and $y=1-x$

$x = 1-x$

$\therefore x = 1/2$

$$= 1 - \int_0 \int_{y=x} e^{-y}\, dy\, dx$$

$x = 1 - x$

i.e. $x = 1/2$

$$= \cdots = \boxed{2e^{-1/2} - e^{-1}}$$

Defn: Bivariate Expectation

If (X, Y) is a biv. RV and $g: \mathbb{R}^2 \to \mathbb{R}$

then

$$\mathbb{E}[g(X, Y)] = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) & \text{(discrete)} \\ \iint_{\mathbb{R}^2} g(x,y) f(x,y)\, dx\, dy & \text{(cts)} \end{cases}$$

$$\left[\underline{\text{Uni:}} \quad \mathbb{E}[g(X)] = \int g(x) f(x)\, dx \right]$$

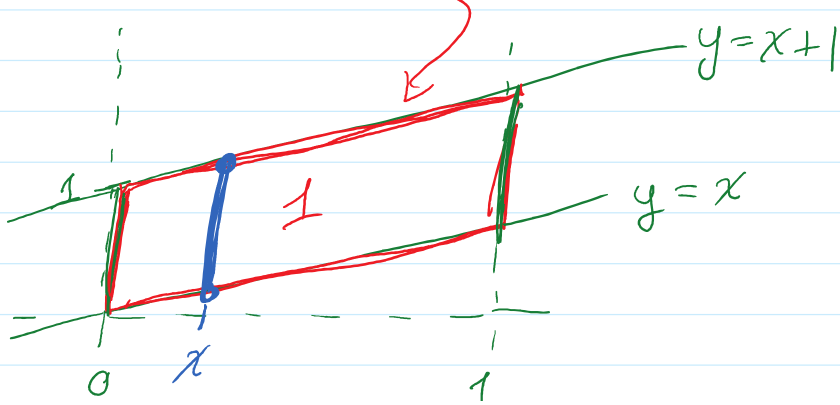
Ex. let $f(x,y) = 1$ for $0 < x < 1$

$x < y < x + 1$

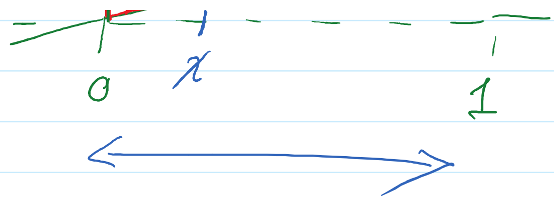
$\mathbb{E}[XY]$ $\quad g(x,y) = xy$

$$= \iint_{\mathbb{R}^2} g(x,y) f(x,y)\, dx\, dy$$

$$= \int^{1} \int^{y=x+1} xy\,(1)\, dy\, dx$$



$$= \int_{0} \int_{y=x} xy\,(1)\,dy\,dx$$



$$= \cdots = 7/12$$

Theorem: Bivariate Expectation is Linear

If $g_1 : \mathbb{R}^2 \to \mathbb{R}$, $g_2 : \mathbb{R}^2 \to \mathbb{R}$ and $a, b \in \mathbb{R}$

then

$$\mathbb{E}[a g_1(X,Y) + b\, g_2(X,Y)] = a\,\mathbb{E}[g_1(X,Y)] + b\,\mathbb{E}[g_2(X,Y)]$$

Define: Covariance

We define the covariance between X and Y as

measuring lin-rel between X and Y

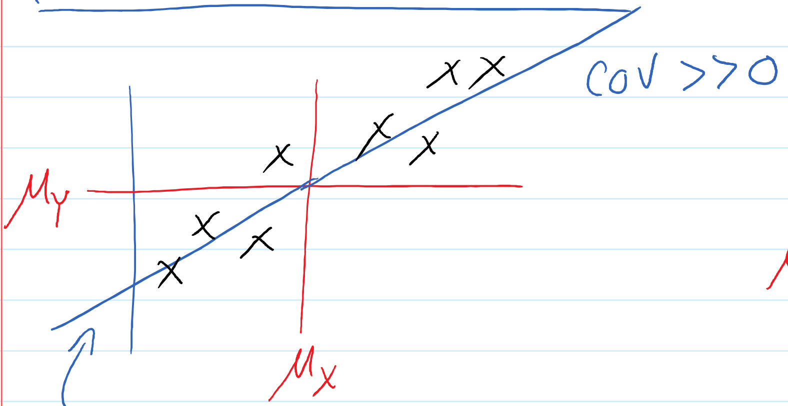
$$\mathrm{Cov}(X,Y) = E[(X - \underbrace{EX}_{\mu_X})(Y - \underbrace{EY}_{\mu_Y})]$$

$$= \mathbb{E}[(X-\mu_X)(Y-\mu_Y)]$$

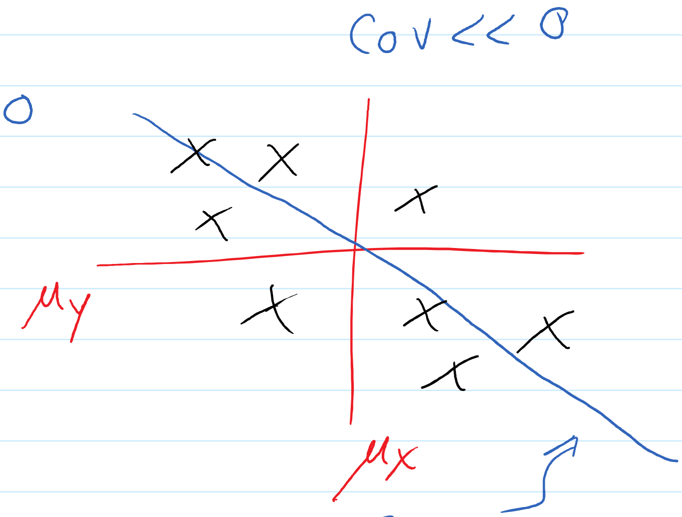
$g(x,y) = (x-\mu_x)(y-\mu_y)$

How to think about:

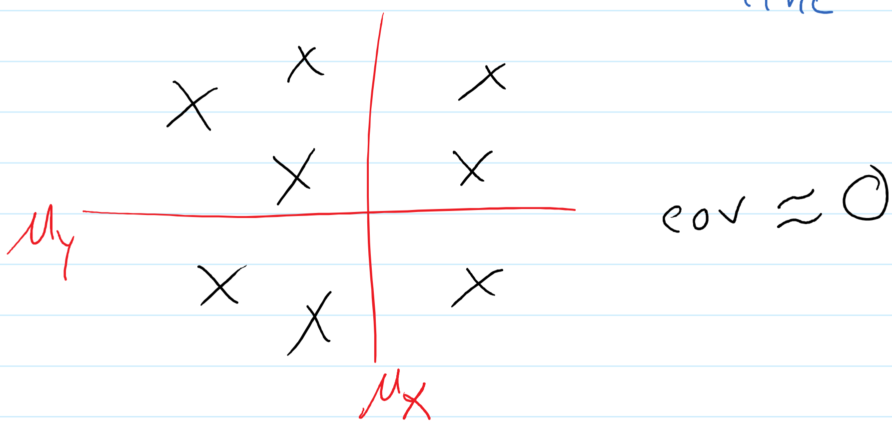
How to think about:



approx samples on line w/ pos slope



Samples $\approx$ line w/ neg. slope



Note: $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$

$$\mathrm{Cov}(X, X) = \mathrm{Var}(X)$$

Notice:

$$\mathrm{Cov}(5X, Y) = \mathbb{E}[(5X - \mathbb{E}[5X])(Y - \mathbb{E}Y)]$$

$$= 5\,\mathrm{cov}(X, Y)$$

Defn: Correlation

Re-scaled covariance so that it is between -1 and 1

$$Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}}$$

$$= \frac{cov(X, Y)}{sd(X)\, sd(Y)}$$

Idea: corr ≈ 1 $\Rightarrow$ strongly pos. lin. rel.

corr ≈ -1 $\Rightarrow$ " neg. "

corr ≈ 0 $\Rightarrow$ no lin. rel.

Theorem: If $a, b \in \mathbb{R}$

$$\boxed{Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X, Y)}$$

pf. $Z = aX + bY$

$$Var(Z) = \mathbb{E}[(Z - \mathbb{E}Z)^2]$$

$$= \mathbb{E}[(aX + bY - \mathbb{E}[aX + bY])^2]$$

$$= \mathbb{E}[(aX + bY - a\mathbb{E}X - b\mathbb{E}Y)^2]$$

$$= \mathbb{E}[(\underbrace{a(X - \mathbb{E}X)}_{\alpha} + \underbrace{b(Y - \mathbb{E}Y)}_{\beta})^2]$$

$$(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$= \mathbb{E}[a^2(X - \mathbb{E}X)^2 + b^2(Y - \mathbb{E}Y)^2 + 2ab(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$$

$$= a^2 \underbrace{\mathbb{E}[(X - \mathbb{E}X)^2]}_{Var X} + b^2 \overbrace{\mathbb{E}[(Y - \mathbb{E}Y)^2]}^{Var(Y)}$$

$$+ 2ab \underbrace{\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]}_{Cov(X, Y)}$$