Lecture 13 - More Common Distributions

Poisson Distribution

- discrete RV
- support is $\{0, 1, 2, \dots\}$

Canonical Experiment

Count the number of "events" that occur in some time period

Ex. - capture fish in a river
- count # mRNA molecules in a cell
- radioactive decay

$X \sim \text{Pois}(\lambda)$

λ: rate of occurrence per time interval

X: # of events in time interval

PMF: $f(x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$ for $x = 0, 1, 2, 3, \dots$

Expected Value

$E[X] = \sum^{\infty} \textcircled{x}\, \dfrac{e^{-\lambda}\lambda^{x}}{}$

$\longrightarrow \dfrac{x}{x!} = \dfrac{\cancel{x}}{\cancel{x}(x-1)!} = \dfrac{1}{(x-1)!}$

$$E[X] = \sum_{x=\cancel{0}\ \underline{1}}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Taylor Series:

$$e^{y} = \sum_{i=0}^{\infty} y^{i}/i!$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+1}}{x!} = \lambda e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \boxed{\lambda = E[X]}$$

$$E[X(X-1)] = \sum^{\infty}$$

$$\frac{x(x-1)}{x!} = \frac{\cancel{x}\cancel{(x-1)}}{\cancel{x}\cancel{(x-1)}(x-2)!}$$

$$= \frac{1}{(x-2)!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-2)!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x+2}}{x!} = e^{-\lambda}\lambda^{2} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}}$$

$$= e^{-\lambda} e^{\lambda} \lambda^{2} = \lambda^{2}$$

$$\underbrace{E[X(X-1)]}_{\lambda^{2}} = E[X^{2}-X] = E[X^{2}] - \underbrace{E[X]}_{\lambda}$$

So $\boxed{E[X^{2}] = \lambda^{2} + \lambda}$

So $\boxed{\mathbb{E}[X^2] = \lambda^2 + \lambda}$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

<u>MGF:</u> $M(t) = \mathbb{E}[e^{tX}]$ $\quad (\lambda e^t)^x$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}}_{e^{(\lambda e^t)}}$$

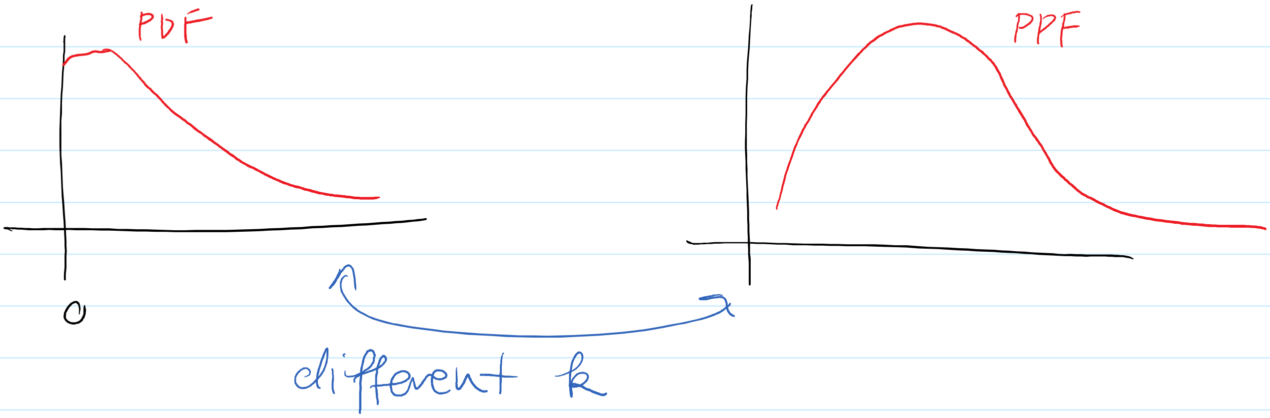
$$= e^{-\lambda} e^{\lambda e^t} = \boxed{\exp(\lambda(e^t - 1)) = M(t)}$$

Gamma Distribution

→ cts distribution w/ support on $(0, \infty)$

→ generalization of $\text{Exp}(\lambda)$

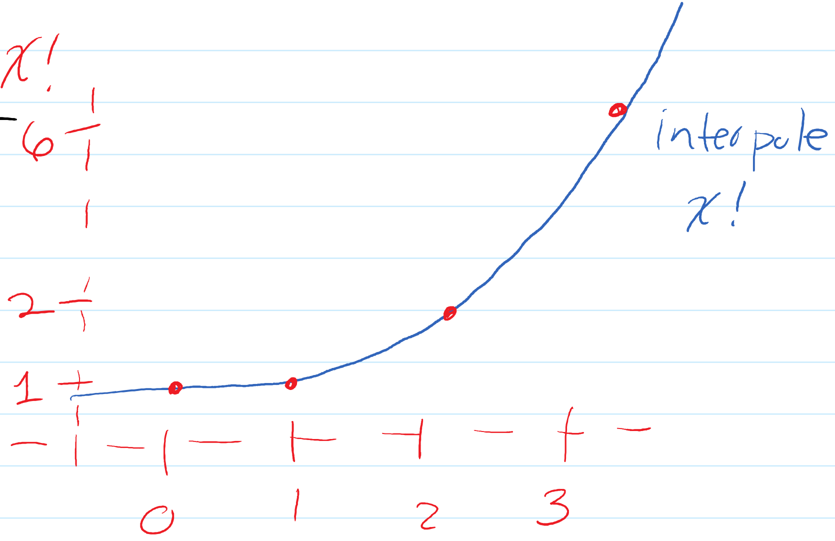
$$X \sim \text{Gamma}(\underset{\text{shape}}{k}, \underset{\text{rate}}{\lambda})$$



Gamma Function

$\Gamma : R^+ \to R^+$

extends factorial (basically)



For $k \geq 0$ then

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} \, dx$$

0

Properties:

① If k is an integer then

$$\boxed{\Gamma(k) = (k-1)! \quad \text{or} \quad \Gamma(k+1) = k!}$$

Notice: if k is an integer

$$\Gamma(k) = (k-1)! = (k-1)(k-2)!$$
$$= (k-1)\Gamma(k-1)$$

or $\Gamma(k+1) = k\Gamma(k)$

② This is generally true:

$k \geq 0$ then

$$\boxed{\Gamma(k) = (k-1)\Gamma(k-1) \quad \text{or} \quad \Gamma(k+1) = k\Gamma(k).}$$

Let $X \sim \text{Gamma}(k, \lambda)$ then

PDF: $f(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}$ for $x > 0$

Notice! if $k=1$ then this is $\text{Exp}(\lambda)$.

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Expectation:

$$E[X] = \int_0^\infty x \, \frac{\lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma(k)} \, dx$$

$$c \underbrace{\int \text{Gamma PPF}}_{1}$$

$$= \int_0^\infty \frac{\lambda e^{-\lambda x} \lambda^{k-1} x^{k}}{\Gamma(k)} \, dx$$

looks like $\text{Gamma}(k+1, \lambda)$

$$\frac{\lambda e^{-\lambda x} (\lambda x)^{k}}{\Gamma(k+1)}$$

$$= \frac{\Gamma(k+1)}{\Gamma(k)\lambda} \int_0^\infty \underbrace{\frac{\lambda e^{-\lambda x} \lambda^{k-1} \lambda x^{k}}{\Gamma(k+1)}}_{\text{integrate to } 1} \, dx$$

PPF

$$= \frac{\Gamma(k+1)}{\Gamma(k)} \frac{1}{\lambda}$$

recall: $\Gamma(k+1) = k\Gamma(k)$

$$= \frac{k \cancel{\Gamma(k)}}{\cancel{\Gamma(k)}} \frac{1}{\lambda} = \boxed{\frac{k}{\lambda} = E[X]}$$

Consider $E[X^r]$

$$E[X^r] = \int_0^\infty x^r \frac{\lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma(k)} \, dx$$

looks like a $\boxed{\text{Gamma}(k+r, \lambda)}$ PPF

$$\lambda e^{-\lambda x} (\lambda x)^{k+r-1}$$

$\Gamma(k)$

$$\frac{\lambda e^{-\overset{\cup}{\lambda x}}(\lambda x)^{k+r-1}}{\Gamma(k+r)}$$

$$= \frac{\Gamma(k+r)}{\Gamma(k)}\frac{1}{\lambda^r}\underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x}\lambda^{k-1}\lambda^{r}x^{k+r-1}}{\Gamma(k+r)}\,dx}_{\text{integerates to } \underline{1}}$$

$$\boxed{E[X^r] = \frac{\Gamma(k+r)}{\Gamma(k)}\frac{1}{\lambda^r}}$$

$$E[X^2] = \frac{\Gamma(k+2)}{\Gamma(k)}\frac{1}{\lambda^2} = \frac{(k+1)\Gamma(k+1)}{\Gamma(k)}\frac{1}{\lambda^2}$$

$$= \frac{(k+1)k\cancel{\Gamma(k)}}{\cancel{\Gamma(k)}}\frac{1}{\lambda^2} = \frac{k(k+1)}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{k(k+1)}{\lambda^2} - \left(\frac{k}{\lambda}\right)^2 = \dots = \boxed{k/\lambda^2}$$

Geometric Distribution

Canonical Experiment

If I flip coins (independently), each w/ a prob

If I flip coins (independently), each w/ a prob p of H, until I get my first H.

X = # flips to get my first H

Outcome	X
H	1
TH	2
TTH	3
⋮	⋮

$X \sim \text{Geometric}(p)$

PMF: $f(x) = (1-p)^{x-1} p$ for $x = 1, 2, 3, \ldots$

CDF: $F(x) = 1 - (1-p)^{\lfloor x \rfloor}$ for $x \geq 1$

Recall: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $|r| < 1$

↑ Geometric Series

Expectation:

$$\frac{d}{dx} r^x = x r^{x-1}$$

$$E[X] = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$$= p \underbrace{\sum_{x=1}^{\infty} x(1-p)^{x-1}}$$

→ looks like $-\frac{d}{dp}(1-p)^x$

$$= p \underbrace{\sum_{x=1} x(1-p)} \quad -\frac{}{dp}(1-p)$$

$$= -p \sum_{x=1}^{\infty} \frac{d}{dp}(1-p)^{x}$$

$$= -p \frac{d}{dp} \underbrace{\sum_{x=1}^{\infty} (1-p)^{x}} \quad \sum_{x=0}^{\infty} (1-p)^{x+1}$$

$$= (1-p) \underbrace{\sum_{x=0}^{\infty} (\overset{r}{1-p})^{x}}_{\frac{1}{1-(1-p)} = \frac{1}{p}}$$

$$= -p \frac{d}{dp}\left[\frac{1-p}{p}\right]$$

$$= -p\left(-1/p^2\right) = \boxed{\frac{1}{p} = \mathbb{E}[X]}$$

MGF:

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p$$

$$= \sum_{x=0}^{\infty} e^{t(x+1)}(1-p)^{x}p$$

$$= pe^{t} \sum_{x=0}^{\infty} (1-p)^{x} e^{tx} \quad ((1-p)e^{t})^{x}$$

$$= pe^{t} \sum_{x=0}^{\infty} \underbrace{((1-p)e^{t})^{x}} \leftarrow \text{Geometric}$$

$$= \frac{pe^{t}}{} = M(t) \quad \text{for } (1-p)e^{t} < 1$$

$$e^{t} < \frac{1}{}$$

$$= \frac{pe^{t}}{1-(1-p)e^{t}} = M(t)$$

for

$$e^{t} < \frac{1}{1-p}$$

$$\boxed{t < -\log(1-p)}$$

$$E[X^2] = \frac{d^2M}{dt^2}\Big|_{t=0} = \frac{2-p}{p^2}$$

So $Var(X) = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$

$$= \cdots = \boxed{\frac{1-p}{p^2}}$$