## Theorem: Cov/Cor of Independent RVs

If $X \perp\!\!\!\perp Y$ then $\text{Cov}(X, Y) = \text{Cor}(X, Y) = 0$.

Pf: $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y$

$= (\mathbb{E}X)(\mathbb{E}Y) - (\mathbb{E}X)(\mathbb{E}Y) = 0$

So $\text{Cor}(X, Y) = 0$ b/c Cor is just re-scaled Cov.

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Converse is generally false.

If $\text{Cor}(X, Y) = 0$ they may or may not be independent.

Ex $X \sim N(0, 1)$ and $Y = X^2$

not independent

$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y$

$= \mathbb{E}[X^3] - (\cancel{\mathbb{E}X})^{0}(\mathbb{E}[X^2])$

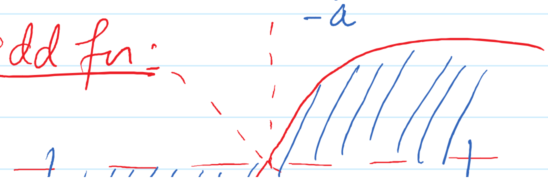
$= \mathbb{E}[X^3]$

$= 0$

$\mathbb{E}[X^3] = \int_{-\infty}^{\infty} \overbrace{x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2}}^{\text{odd}}\, dx = 0$

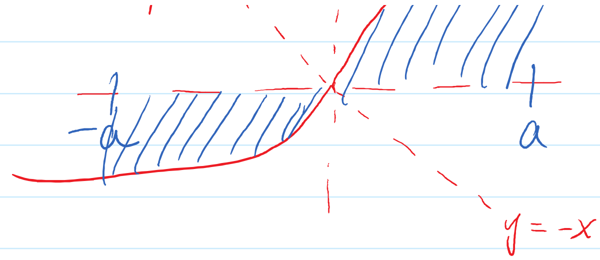
$\int_{-a}^{a} \text{odd fn} = 0$

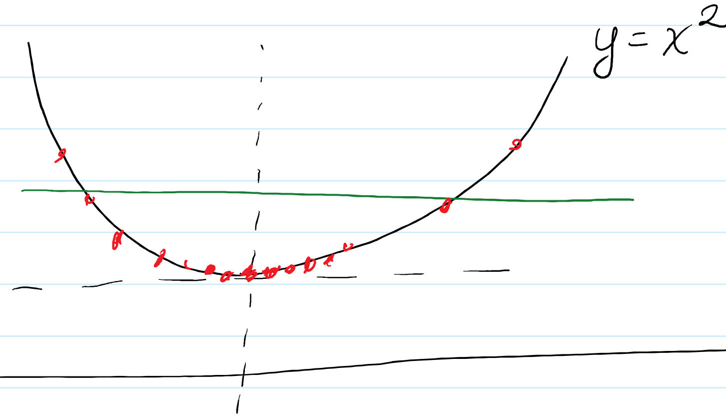
odd fn:



$= 0$

So $\mathrm{Cor}(X, Y) = 0$.




## Bayes' Theorem

Events: $$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$$

RVs: $$\boxed{f(x \mid y) = \frac{f(y \mid x)\, f_X(x)}{f_Y(y)}}$$

## Law of Total Probability

Events: $(C_i)$ partition $S$ then

$$\mathbb{P}(A) = \sum_i P(A \mid C_i) P(C_i)$$

RVs: discrete:

$$f(y) = \sum_x f(y|x) f(x)$$

cts:

$$f(y) = \int_{\mathbb{R}} f(y|x) f(x)\, dx$$

Pf. (cts)

(1) $f(y|x) = \dfrac{f(x,y)}{f(x)} \iff f(x,y) = f(y|x) f(x)$

(2) $f(y) = \int_{\mathbb{R}} f(x,y)\, dx = \int_{\mathbb{R}} f(y|x) f(x)\, dx$

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Ex. $X \sim Exp(\lambda)$

$Y | X = x \sim Pois(x)$

What is the dist of $Y$?

Law of Total Prob.

$Z \sim Pois(\lambda)$

$f(z) = \lambda^z e^{-\lambda}$

$$f(y) = \int_{\mathbb{R}} f(y|x) f(x)\, dx = \int_0^\infty \underbrace{\frac{x^y e^{-x}}{y!}}_{Pois(x)} \underbrace{\lambda e^{-\lambda x}}_{Exp(\lambda)}\, dx$$

$\underbrace{\qquad}_{Pois(x)} \underbrace{\qquad}_{Exp(\lambda)}$

$$= \frac{\lambda}{y!} \int_0^\infty x^{\overset{a-1}{y}} e^{-\overset{b}{(\lambda+1)} x} \, dx$$

PDF Gamma$(a, b)$

$$\underline{x^{a-1} e^{-bx}} \frac{b^a}{\Gamma(a)}$$

$$= \frac{\lambda}{y!} \frac{\Gamma(a)}{b^a} \underbrace{\int_0^\infty x^{a-1} e^{-bx} \frac{b^a}{\Gamma(a)} dx}_{\text{integral of a Gamma PDF} = 1}$$

$$= \frac{\lambda}{\cancel{y!}} \frac{\overset{y!}{\cancel{\Gamma(y+1)}}}{(\lambda+1)^{y+1}}$$

$$\boxed{f(y) = \frac{\lambda}{(\lambda+1)^{y+1}} \quad \text{for } y = 0, 1, 2, 3, \ldots}$$

Ex. $\underline{Y \sim Pois(\lambda)}$

$X \mid Y = y \sim Bin(y, p)$ $\quad$ $\boxed{0 \le X \le Y}$ $\quad$ $0 < p < 1$

$$\binom{y}{x} \frac{1}{y!} = \frac{\cancel{y!}}{x!(y-x)!} \frac{1}{\cancel{y!}} = \frac{1}{x!(y-x)!}$$

What is the dist of $X$?

$$f(x) = \sum f(x|y) f(y) = \sum^\infty \binom{y}{x} p^x (1-p)^{y-x} \lambda^y e^{-\lambda}$$

$$f(x) = \sum_y \underset{\sim\sim\sim}{f(x|y)} f(y) = \sum_{y=x}^{\infty} \underbrace{\binom{y}{x} p^x (1-p)^{y-x}}_{Bin(y,p)} \underbrace{\frac{\lambda^y e^{-\lambda}}{y!}}_{Pois(\lambda)}$$

$$= \frac{p^x e^{-\lambda}}{x!} \lambda^x \sum_{y=x}^{\infty} \frac{1}{(y-x)!} (1-p)^{y-x} \lambda^{y-x}$$

$$= \frac{p^x e^{-\lambda}}{x!} \lambda^x \sum_{y=x}^{\infty} \frac{1}{(y-x)!} \left[(1-p)\lambda\right]^{y-x}$$

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!}$$

$$= \frac{p^x e^{-\lambda} \lambda^x}{x!} \underbrace{\sum_{y=0}^{\infty} \frac{1}{y!} [\underbrace{(1-p)\lambda}_{z}]^y}_{e^{(1-p)\lambda}}$$

$$= \frac{p^x \cancel{e^{-\lambda}} \lambda^x e^{(1-p)\lambda}}{x!}$$

$$Z \sim Pois(\lambda)$$

$$\frac{\lambda^3 e^{-\lambda}}{3!}$$

$$\boxed{f(x) = \frac{(p\lambda)^x e^{-(p\lambda)}}{x!}}$$

$$\boxed{X \sim Pois(p\lambda)}$$

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**Theorem:** Iterated Expectation

If $X$ and $Y$ are RVs then

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a RV

$$\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}[X|Y]\right]$$

$$\mathbb{E}[X|Y=y] = \int x f(x|y)\,dx = g(y)$$

For each $y \in \mathbb{R}$ this defines some fn

$$g(y) = \mathbb{E}[X|Y=y]$$

a real valued fn

e.g. $g(y) = y^2$ or $g(y) = 1+y$

We can plug $Y$ into $g$ to get $g(Y)$

e.g. $g(Y) = Y^2$ or $g(Y) = 1+Y$

Might want to write

$$g(Y) = \mathbb{E}[X\cancel{|Y=Y}]$$

awkward

$$= \mathbb{E}[X|Y]$$

notation

a RV

$$\boxed{g(y) = \mathbb{E}[X|Y=y]}$$

a number

Summary: $\mathbb{E}[X \mid Y=y] =$ a number

$\mathbb{E}[X \mid Y] =$ a RV

ex. $\mathbb{E}[X \mid Y=y] = y^2$

then $\mathbb{E}[X \mid Y] = Y^2$

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Ex. Prev. Ex. $Y \sim \text{Pois}(\lambda)$

$X \mid Y=y \sim \text{Bin}(y, p)$

what is $\mathbb{E}[X]$?

Iterated Expectation: $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$

1. $\mathbb{E}[X \mid Y=y] = yp$

2. $\mathbb{E}[X \mid Y] = Yp$

3. $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[Yp] = p\mathbb{E}Y$

$\boxed{= p\lambda}$

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Ex. $P \sim \text{Beta}(\alpha, \beta)$

$X \mid P=p \sim \text{Bin}(n, p)$

$\hookleftarrow$ Known

$E[X]$?

① $E[X \mid P=p] = np$

② $E[X \mid P] = nP$

③ $E[E[X \mid P]] = E[nP] = nE[P]$

$$\boxed{= n\frac{\alpha}{\alpha+\beta} = EX}$$

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pf. (cts case)

→ ① $\boxed{f(x) = \int f(x,y)\,dy}$

② $f(x|y) = \dfrac{f(x,y)}{f(y)} \iff \boxed{f(x,y) = f(x|y)f(y)}$

③ $\boxed{g(y) = E[X \mid Y=y] = \int x f(x|y)\,dx}$

$$EX = \int x f(x)\,dx \overset{①}{=} \int x \int f(x,y)\,dy\,dx$$

$$\overset{②}{=} \int x \int f(x|y)f(y)\,dy\,dx$$

rearrange

$$= \int \underbrace{\int x f(x|y)\,dx}_{g(y)}\, f(y)\,dy$$

$$= \int g(y) f(y)\, dy$$

$$= \mathbb{E}[g(Y)]$$

notation for $g(Y)$

$$= \mathbb{E}[\mathbb{E}[X \mid Y]]$$