Ex. $X \sim \mathrm{Exp}(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

$$\mu = \mathbb{E}X = \frac{1}{\lambda} \quad \text{and} \quad \mathbb{E}[X^2] = 2/\lambda^2$$

$$\mathbb{E}g(X) = \int g(x) f(x)\,dx$$

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] = \int_{\mathbb{R}} (x-\mu)^2 f(x)\,dx$$

correct — but labor intensive

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}[X^2 - 2\mu X + \mu^2] \\ &= \mathbb{E}[X^2] - 2\mu\,\mathbb{E}X + \mu^2 \\ &= \frac{2}{\lambda^2} - 2\left(\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}\right) + \left(\frac{1}{\lambda}\right)^2 \\ &= \frac{2}{\lambda^2} - \frac{2}{\lambda^2} + \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2} = \mathrm{Var}(X)} \end{aligned}$$

(*) $\mathbb{E}[aX+b] = a\mathbb{E}X + b$

Theorem: Short-Cut Formula for Variance

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

= exp. of sq − sq. of exp.

Pf:

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2]$$

$$= \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

$$= \mathbb{E}[X^2] - 2\mu EX + \mu^2$$

$\mu = EX$

$$= \mathbb{E}[X^2] - 2\, EX\, EX + (\mathbb{E}X)^2$$

$$= \mathbb{E}[X^2] - E[X]^2.$$

Revisit example

$$X \sim \mathrm{Exp}(\lambda)$$

$$\mathbb{E}[X] = 1/\lambda \quad \text{and} \quad \mathbb{E}[X^2] = 2/\lambda^2$$

then $\mathrm{Var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = 1/\lambda^2$

Ex. $X \sim \mathrm{Bin}(n, p)$

$$\mathbb{E}[X] = np$$

$$\mathbb{E}[X^2] = np(np - p + 1)$$

So $Var(X) = E[X^2] - (EX)^2$

$$= np(np - p + 1) - (np)^2$$

$$= np(1-p)$$

Standard deviation : $sd(X) = \sqrt{Var(X)}$

$$sd(X) = \sqrt{np(1-p)}$$

Theorem:

$$Var(aX + b) = a^2 Var(X)$$

two rules:

① multiply X by a $\longrightarrow$ Variance gets multiplied by a^2

② ignore additive shifts b

pf.

$$Var(aX+b) = E[(aX+b)^2] - (E[aX+b])^2$$

$$= E[a^2X^2 + 2abX + b^2] - (a\,EX + b)^2$$

$$= \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\,\mathbb{E}X + b)^2$$

$$= a^2\,\mathbb{E}[X^2] + \cancel{2ab\,\mathbb{E}[X]} + \cancel{b^2} - (a^2\,\mathbb{E}[X]^2 + \cancel{2ab\,\mathbb{E}X} + \cancel{b^2})$$

$$= a^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2)$$

$$= a^2\,\mathrm{Var}(X)$$

Defn: Moments of a RV

If r is a pos. integer we define the r^{th} moment of X to be

$$\mu_r = \mathbb{E}[X^r]$$

Ex.

$$\mu_1 = \mathbb{E}[X] = \mu,\ \mu_2 = \mathbb{E}[X^2],\ \mu_3 = \mathbb{E}[X^3],\ \ldots$$

Defn: Moment Generating Function (MGF)

If X is a RV then the MGF of X is a function

$$M : \mathbb{R} \longrightarrow \mathbb{R}$$

defined for $t \in \mathbb{R}$ as

$$M(t) = \mathbb{E}[e^{tX}].$$

$$M(t) = \mathbb{E}[e^{tX}].$$

For discrete

$$M(t) = \mathbb{E}[e^{tX}] = \sum_x e^{tx} f(x)$$

For continuous:

$$M(t) = \mathbb{E}[e^{tX}] = \int e^{tx} f(x)\, dx$$

Ex. $X \sim \mathrm{Exp}(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

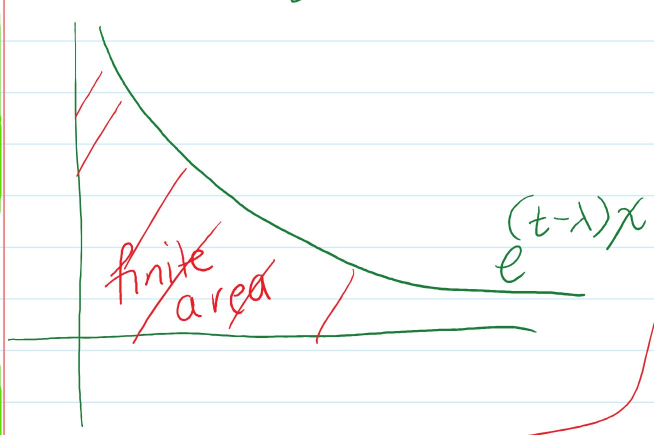
$$e^a e^b = e^{a+b}$$

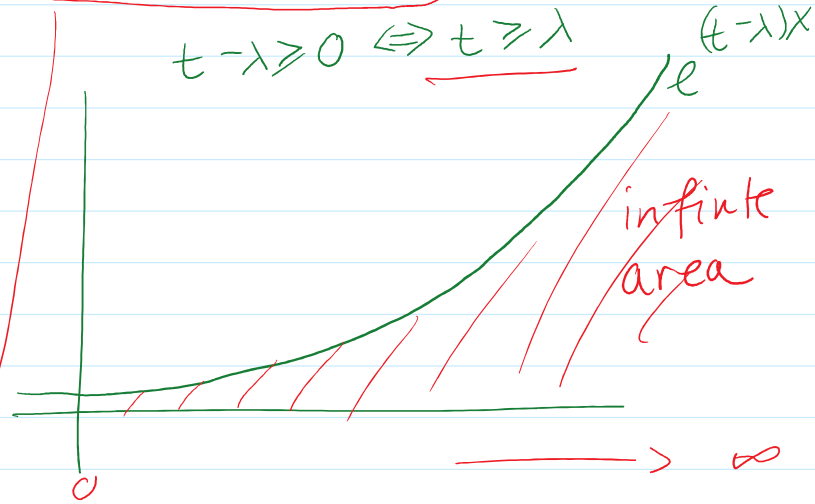
$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x) dx = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

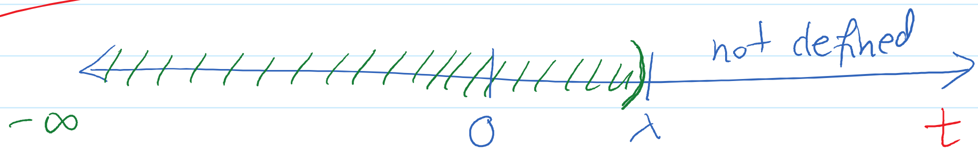
$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$t - \lambda < 0$ $\Leftrightarrow$ $t < \lambda$

$t - \lambda \geq 0 \Leftrightarrow t \geq \lambda$







$t < \lambda$ then

$$M(t) = \lambda \int_0^{\infty} e^{(t-\lambda)x}\, dx = \lambda \left. \frac{e^{(t-\lambda)x}}{t-\lambda} \right|_0^{\infty}$$

$$= \frac{\lambda}{t-\lambda}(0-1) = \boxed{\frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda}$$

Consider

$$\left.\frac{dM}{dt}\right|_{t=0} = \left.\frac{\lambda}{(\lambda-t)^2}\right|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} = E[X]$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \left.\frac{2\lambda}{(\lambda-t)^3}\right|_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2} = E[X^2]$$

$$\vdots$$

Theorem:

$$\frac{d^r M}{dt^r}\Big| \quad M^{(r)} = E[X^r] = \ldots$$

$$\left.\frac{d^rM}{dt^r}\right|_{t=0} = M^{(r)}(0) = \mathbb{E}[X^r] = \mu_r$$

Pf. recall: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots$$

$$M(t) = \mathbb{E}[e^{tX}] = 1 + t\,\mathbb{E}[X] + \frac{t^2}{2!}\mathbb{E}[X^2] + \frac{t^3}{3!}\mathbb{E}[X^3] + \cdots$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \mathbb{E}[X] + \frac{\cancel{2}t}{\cancel{2}!}\mathbb{E}[X^2] + \frac{3t^2}{3!}\mathbb{E}[X^3] + \cdots$$

$$= \mathbb{E}[X]$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \mathbb{E}[X^2] + \frac{3\cdot 2\cdot t}{3!}\mathbb{E}[X^3] + \cdots$$

$$= \mathbb{E}[X^2]$$

Ex. $X \sim Bin(n,p)$

$$\mathbb{E}[X^2] = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

$x=0$

Binomial theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{n} \overbrace{e^{tx}}^{(e^t)^x} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$a = pe^t, \quad b = 1-p$$

$$= (a+b)^n = \boxed{(pe^t+1-p)^n = M(t)}$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \underbrace{n(\overbrace{pe^t+1-p}^{p+(1-p)=1})^{n-1}} \underbrace{p\overbrace{e^t}^{1}} = np = \mathbb{E}X$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = n(n-1)(\overbrace{pe^t+1-p}^{1})^{n-2} \overbrace{pe^t}^{p}\,\overbrace{pe^t}^{p} + n(\overbrace{pe^t+1-p}^{1})^{n-1} \overbrace{pe^t}^{p}$$

$$= n(n-1)p^2 + np = \mathbb{E}[X^2].$$

<u>Theorem:</u> If $Y = aX + b$ then

$$\underset{\text{MGF of } Y}{M_Y(t)} = e^{tb} \underset{\text{MGF of } X}{M_X(at)}$$

Pf: $M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(aX+b)}]$

$$= \mathbb{E}[e^{(at)X} e^{tb}]$$

$$= e^{tb} \underbrace{\mathbb{E}[e^{(at)X}]}_{M_X(at)}$$