Theorem: Cov/Cor of Independent RVs

If $X \perp\!\!\!\perp Y$ then $\text{Cov}(X, Y) = \text{Cor}(X, Y) = 0$.

Pf: $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y$

$= (\mathbb{E}X)(\mathbb{E}Y) - (\mathbb{E}X)(\mathbb{E}Y) = 0$

So $\text{Cor}(X, Y) = 0$ b/c Cor is just re-scaled Cov.

Converse is generally false.

If $\text{Cor}(X, Y) = 0$ they may or may not be independent.

Ex $X \sim N(0, 1)$ and $Y = X^2$

not independent

$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y$

$= \mathbb{E}[X^3] - \cancel{(\mathbb{E}X)}^{0}(\mathbb{E}[X^2])$

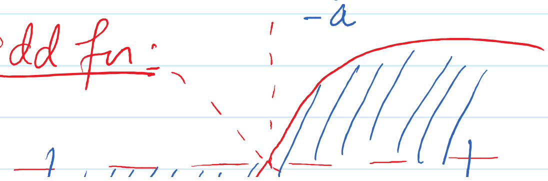
$= \mathbb{E}[X^3]$

$= 0$

$\mathbb{E}[X^3] = \int_{-\infty}^{\infty} \overbrace{x^3 \frac{1}{\sqrt{2\pi}} e^{-x^2}}^{\text{odd}}\, dx = 0$

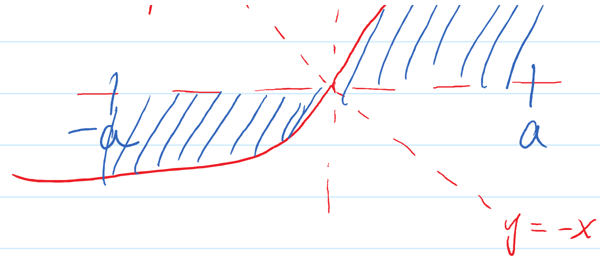
$\int_{-a}^{a} \text{odd fn} = 0$

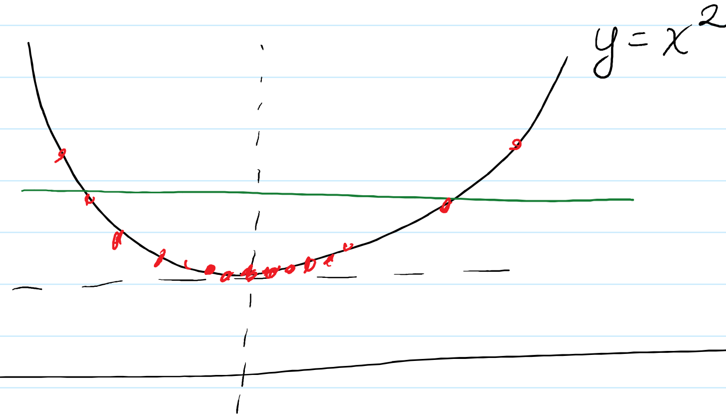
odd fn:



$$= 0$$

So $\mathrm{Cor}(X, Y) = 0$.





Bayes' Theorem

Events: $$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

RVs: $$\boxed{f(x|y) = \frac{f(y|x)\, f_X(x)}{f_Y(y)}}$$

Law of Total Probability

Events: (C_i) partition S then

$$\mathbb{P}(A) = \sum_i P(A \mid C_i) P(C_i)$$

RVs: discrete:

$$f(y) = \sum_x f(y \mid x) f(x)$$

cts:

$$f(y) = \int_{\mathbb{R}} f(y \mid x) f(x) \, dx$$

Pf. (cts)

① $f(y \mid x) = \dfrac{f(x,y)}{f(x)} \iff f(x,y) = f(y \mid x) f(x)$

② $f(y) = \displaystyle\int_{\mathbb{R}} f(x,y) \, dx = \int_{\mathbb{R}} f(y \mid x) f(x) \, dx$

Ex. $X \sim \operatorname{Exp}(\lambda)$

$Y \mid X = x \sim \operatorname{Pois}(x)$

What is the dist of Y? | $Z \sim \operatorname{Pois}(\lambda)$

What is the dist of Y?

Law of Total Prob.

$$z \sim Pois(\lambda)$$
$$f(z) = \lambda^z e^{-\lambda}$$

$$f(y) = \int_{\mathbb{R}} f(y|x) f(x)\, dx = \int_0^\infty \underbrace{\frac{x^y e^{-x}}{y!}}_{Pois(x)} \underbrace{\lambda e^{-\lambda x}}_{Exp(\lambda)}\, dx$$

PDF Gamma(a, b)

$$\frac{x^{a-1} e^{-bx} b^a}{\Gamma(a)}$$

$$= \frac{\lambda}{y!} \int_0^\infty x^{\overset{a-1}{y}} e^{-\overset{b}{(\lambda+1)} x}\, dx$$

$$= \frac{\lambda}{y!} \frac{\Gamma(a)}{b^a} \underbrace{\int_0^\infty x^{a-1} e^{-bx} \frac{b^a}{\Gamma(a)}\, dx}_{\text{integral of a Gamma PDF} = 1}$$

$$= \frac{\lambda}{\cancel{y!}} \frac{\overset{y!}{\cancel{\Gamma(y+1)}}}{(\lambda+1)^{y+1}}$$

$$\boxed{f(y) = \frac{\lambda}{(\lambda+1)^{y+1}} \quad \text{for } y = 0, 1, 2, 3, \ldots}$$

Ex. $Y \sim Pois(\lambda)$ $\quad 0 \le X \le Y$

$X \mid Y = y \sim Bin(y, p)$ $\quad 0 < p < 1$

$\binom{y}{x}\frac{1}{y!} = \frac{\cancel{y!}}{x!(y-x)!}\frac{1}{\cancel{y!}} = \frac{1}{x!(y-x)!}$

What is the dist of X?

$$f(x) = \sum_y f(x|y)f(y) = \sum_{y=x}^{\infty} \underbrace{\binom{y}{x} p^x (1-p)^{y-x}}_{Bin(y,p)} \underbrace{\frac{\lambda^y e^{-\lambda}}{y!}}_{Pois(\lambda)}$$

$$= \frac{p^x e^{-\lambda}}{x!} \lambda^x \sum_{y=x}^{\infty} \frac{1}{(y-x)!} (1-p)^{y-x} \lambda^{y-x}$$

$$= \frac{p^x e^{-\lambda}}{x!} \lambda^x \sum_{y=x}^{\infty} \frac{1}{(y-x)!} [(1-p)\lambda]^{y-x}$$

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!}$$

$$= \frac{p^x e^{-\lambda} \lambda^x}{x!} \underbrace{\sum_{y=0}^{\infty} \frac{1}{y!} [\underbrace{(1-p)\lambda}_{z}]^y}_{e^{(1-p)\lambda}}$$

$$= \frac{p^x \cancel{e^{-\lambda}} \lambda^x e^{(\cancel{1}-p)\lambda}}{x!}$$

$Z \sim Pois(\lambda)$

$\frac{\lambda^3 e^{-\lambda}}{3!}$

$$\boxed{f(x) = \frac{(p\lambda)^x e^{-(p\lambda)}}{x!}}$$

$x!$

$$\boxed{X \sim Pois(p\lambda)}$$

Theorem: Iterated Expectation

If X and Y are RVs then

a RV

$$E[X] = E[E[X|Y]]$$

$$E[X|Y=y] = \int x f(x|y)\,dx = g(y)$$

For each $y \in \mathbb{R}$ this defines some fn

$$g(y) = E[X|Y=y]$$

a real valued fn

e.g. $g(y) = y^2$ or $g(y) = 1+y$

We can plug Y into g to get $g(Y)$

e.g. $g(Y) = Y^2$ or $g(Y) = 1+Y$

Might want to write

$$g(Y) = E[X | \cancel{Y = Y}]$$

$$\boxed{g(y) = E[X|Y=y]}$$

$$g(Y) = E[\cancel{X \mid Y = Y}]$$

awkward

a number

$$= E[X \mid Y]$$

notation

a RV

Summary: $E[X \mid Y=y]$ = a number

$E[X \mid Y]$ = a RV

ex, $E[X \mid Y=y] = y^2$

then $E[X \mid Y] = Y^2$

Ex. Prev. Ex. $Y \sim \text{Pois}(\lambda)$

$$X \mid Y = y \sim \text{Bin}(y, p)$$

What is $E[X]$?

Iterated Expectation: $E[X] = E[E[X|Y]]$

① $E[X \mid Y = y] = yp$

② $E[X \mid Y] = Yp$

③ $E[E[X|Y]] = E[Yp] = pEY$

$$\boxed{= p\lambda}$$

Ex. $P \sim \text{Beta}(\alpha, \beta)$

$X \mid P = p \sim \text{Bin}(n, p)$

n Known

$E[X]$?

① $E[X \mid P = p] = np$

② $E[X \mid P] = nP$

③ $E[E[X|P]] = E[nP] = nE[P]$

$$\boxed{= n\frac{\alpha}{\alpha + \beta} = EX}$$

pf. (cts case)

→ ① $\boxed{f(x) = \int f(x,y)\,dy}$

② $f(x|y) = \dfrac{f(x,y)}{f(y)} \iff \boxed{f(x,y) = f(x|y)\,f(y)}$

③ $\boxed{g(y) = E[X|Y=y] = \int x f(x|y)\,dx}$

$$EX = \int x f(x)\,dx \overset{①}{=} \int x \int f(x,y)\,dy\,dx$$

$$\overset{②}{=} \int x \int f(x|y) f(y)\,dy\,dx$$

rearrange

$$= \int \underbrace{\int x f(x|y)\,dx}_{g(y)}\, f(y)\,dy$$

$$= \int g(y) f(y)\,dy$$

$$= E[g(Y)]$$

notation for $g(Y)$

$$= E[E[X|Y]]$$