Ex Consider flipping a coin 3 times

$$X = \begin{cases} 0 & \text{if last flip is a T} \\ 1 & \text{if last flip is a H} \end{cases}$$

Y = # heads among 3 flips

$Z = (X, Y)$

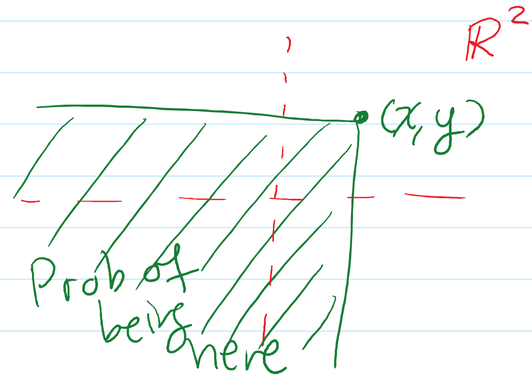
$s \in S$	$Z(s)$
H H H	(1, 3)
H H T	(0, 2)
H T H	(1, 2)
H T T	(0, 1)
T H H	(1, 2)
T H T	(0, 1)
T T H	(1, 1)
T T T	(0, 0)

Defn: Bivariate / Joint CDF

The joint CDF is a function

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

so that for $(x,y) \in \mathbb{R}^2$



$$F(x,y) = \mathbb{P}(X \le x, Y \le y)$$

where

$$F(x,y) = \mathbb{P}(X \le x, Y \le y)$$

Univariate: $F(x) = \mathbb{P}(X \le x)$

Properties of Joint CDF

① $F(x,y) \geq 0$

② $\lim_{x,y \to \infty} F(x,y) = 1$

$\boxed{\text{Uni: } \lim_{x \to \infty} F(x) = 1}$

③ $\lim_{x \to -\infty} F(x,y) = 0$

$\lim_{y \to -\infty} F(x,y) = 0$

$\boxed{\text{Uni: } \lim_{x \to -\infty} F(x) = 0}$

④ F is non-decreasing and right-continuous in argument

Defn: Marginal RVs / Distributions

If (X, Y) is a biv. RV then X and Y are called the marginal RVs and their properties are also called marginal their properties are also called marginal

e.g. their PMFs/PDFs are called the marginal PMFs/PPFs ...

Theorem: Relation between Joint/Marginal CDFs

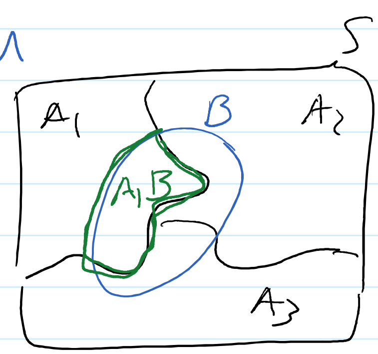
① $F_X(x) = \lim_{y\to\infty} F(x,y)$

marginal CDF of X — joint CDF

② $F_Y(y) = \lim_{x\to\infty} F(x,y)$

Recall: A_i that partition S then

$$P(B) = \sum_i P(B \cap A_i)$$



---

pf:

$$F_X(x) = P(X \le x) = P(X \le x, Y = \text{anything})$$

$$= P(X \le x, Y < \infty)$$

$$= \lim_{y\to\infty} P(X \le x, Y \le y)$$

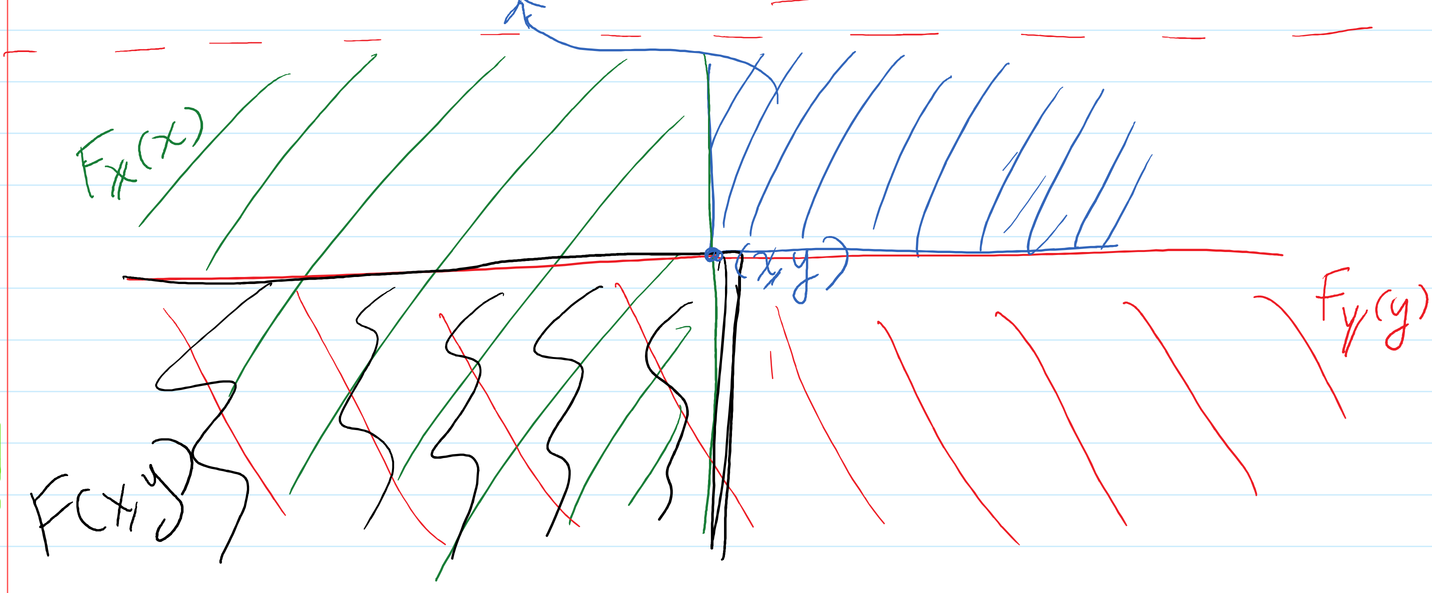
$$= \lim_{y \to \infty} P(X \le x, Y \le y)$$

$$= \lim_{y \to \infty} F(x,y)$$

Univariate $P(X > x) = 1 - F(x)$

Bivariate Case:

$$P(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F(x,y)$$



Defn: Joint PMF

If X and Y are discrete RVs then the joint PMF is defined as

$$f(x,y) = P(X = x, Y = y)$$

$f(x,y) = \mathbb{P}(X = x, Y = y)$

[Univariate Analg: $f(x) = \mathbb{P}(X = x)$]

Theorem: Valid PMF

A function f is a valid PMF iff

(1) $f(x,y) \geq 0 \quad \forall x, y$

(2) $\sum_x \sum_y f(x,y) = 1$

Theorem: Rel. btwn joint/marginal PMF

(1) $f_X(x) = \sum_y f(x,y)$

marginal PMF of X

joint PMF

(2) $f_Y(y) = \sum_x f(x,y)$

Pf. Notice $A_y = \{Y = y\}$ for all possible y,

$\subset S$

These A_y partition S

So let $B = $ "$X = x$"

then

So let B = "X = x"
then

$$f_X(x) = \mathbb{P}(X=x) = \mathbb{P}(B) = \sum_y \mathbb{P}(B \cap A_y)$$

$$= \sum_y \mathbb{P}(\text{"}X=x\text{"} \cap \text{"}Y=y\text{"})$$

$$= \sum_y \mathbb{P}(X=x, Y=y)$$

$$= \sum_y f(x,y)$$

Ex. Revisit prev. ex.

Flip 3 coins,

$$X = \begin{cases} 0 & \text{if last } T \\ 1 & \text{'' } H \end{cases}$$

$$Y = \#\text{ heads}$$

$f(x,y)$

X \ Y	0	1	2	3	
0	$f(0,0) = 1/8$	$f(0,1) = 2/8$	$f(0,2) = 1/8$	$f(0,3) = 0$	$f_X(0) = $ sum of row $= 1/2$
1	0	1/8	2/8	1/8	$f_X(1) = 1/2$
	$f_Y(0)$	$f_Y(1)$	$f_Y(2)$	$f_Y(3)$	

$f_Y(0) = 1/8$	$f_Y(1) = 3/8$	$f_Y(2) = 3/8$	$f_Y(3) = 1/8$

Defn: Joint PDF

If X and Y are continuous we call the function $f : \mathbb{R}^2 \to \mathbb{R}$ the joint PDF if $\forall C \subset \mathbb{R}^2$

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x,y)\, dx\, dy$$

Univariate: Analogy $P(X \in A) = \int_A f(x)\, dx$

Facts:

① $F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v)\, du\, dv$

Univ: $F(x) = \int_{-\infty}^{x} f(t)\, dt$

② $f(x,y) = \dfrac{\partial^2 F}{\partial x \partial y}$

Univ: $f(x) = \dfrac{dF}{dx}$

③ f is a valid joint PDF iff

① $f(x,y) \geq 0 \quad \forall x,y$

② $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$

Theorem: Rel. between joint/marginal PDFs

① $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$

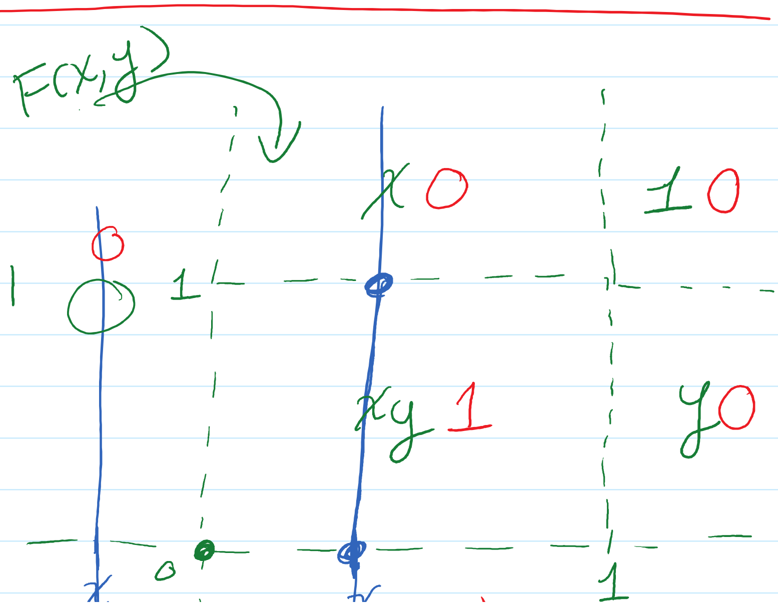
marginal PDF of X

joint PDF

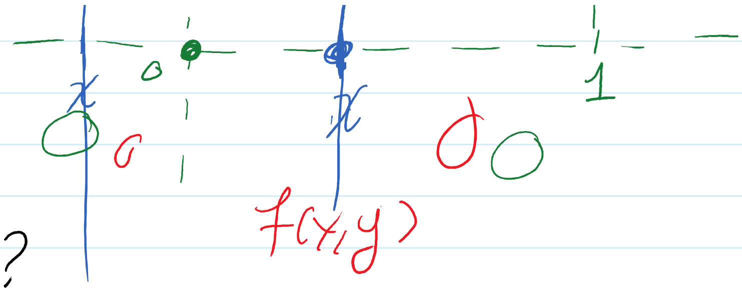
② $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$

Ex.

$$F(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ xy, & 0<x<1 \text{ and } 0<y<1 \\ x, & 0<x<1 \text{ and } y>1 \\ y, & x>1 \text{ and } 0<y<1 \\ 1, & x>1 \text{ and } y>1 \end{cases}$$



$1, \quad x > 1$ and $y > 1$



What's the joint PDF f?

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$f(x,y) = \begin{cases} 1 & 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

What is the marginal dist of X?

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\, dy$$

If $x < 0$ or $x > 1$ then $f(x,y) = 0$

so $f_X(x) = \int 0 \, dy = 0$

if $0 < x < 1$ $\quad f(x,y) = 1$ for $0 < y < 1$

$$f_X(x) = \int_0^1 1 \, dy = \underline{1}$$

All together: $f_X(x) = \begin{cases} 0 & x < 0 \text{ or } x > 1 \\ 1 & 0 < x < 1 \end{cases}$

i.e. $\boxed{X \sim U(0,1)}$

Similarly $Y \sim U(0,1)$