Theorem: $Cov(aX+b, Y) = a\, Cov(X,Y)$

Recall: $Var(aX+b) = a^2 Var(X)$

Pf. $Cov(aX+b, Y) = \mathbb{E}\left[\left((aX+b) - E[aX+b]\right)(Y - EY)\right]$

$= \mathbb{E}\left[(aX+\cancel{b} - a\mathbb{E}X - \cancel{b})(Y - EY)\right]$

$= \mathbb{E}\left[a(X - \mathbb{E}X)(Y - EY)\right]$

$= a\,\mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$

$= a\, Cov(X, Y)$

Corollaries: ① $Cov(X, cY+d) = c\, Cov(X,Y)$

② $Cov(aX+b, cY+d) = ac\, Cov(X,Y)$.

Theorem:

$cor(aX+b, cY+d) = Sign(a)Sign(c)cor(X,Y)$

$$Sign(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Ex. $\mathrm{Cor}(-5X, Y) = -\mathrm{Cor}(X, Y)$

pf. $a, c \neq 0$

if $x \neq 0$ then $\mathrm{sign}(x) = \frac{x}{|x|}$

$$\mathrm{Cor}(aX+b, cY+d) = \frac{\mathrm{Cov}(aX+b, cY+d)}{\sqrt{\mathrm{Var}(aX+b)\,\mathrm{Var}(cY+d)}}$$

$$= \frac{ac\,\mathrm{cov}(X, Y)}{\sqrt{a^2\,\mathrm{Var}(X)\,c^2\,\mathrm{Var}(Y)}}$$

$$= \frac{a\quad c}{|a|\,|c|} \boxed{\frac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}}$$

$$= \mathrm{sign}(a)\,\mathrm{sign}(c)\,\mathrm{Cor}(X, Y).$$

Claim: $\boxed{-1 \leq \mathrm{Cor}(X, Y) \leq 1}$

pf. $$\tilde{X} = \frac{X - EX}{\sqrt{\mathrm{Var}(X)}} = \underbrace{\frac{1}{\sqrt{\mathrm{Var}(X)}}}_{a>0} X - \underbrace{\frac{E[X]}{\sqrt{\mathrm{Var}(X)}}}_{} = aX + b$$

$$\underbrace{}_{a>0} \qquad \underbrace{}_{b}$$

$$E[\tilde{X}] = \frac{EX - EX}{\sqrt{Var(X)}} = 0$$

$$Var(\tilde{X}) = \left(\frac{1}{\sqrt{Var(X)}}\right)^2 Var(X) = 1$$

$$\tilde{Y} = \frac{Y - EY}{\sqrt{Var(Y)}}$$

Notice: $Cor(\tilde{X}, \tilde{Y}) = Cor(X, Y)$ (*)

Also: $$Cor(\tilde{X}, \tilde{Y}) = \frac{Cov(\tilde{X}, \tilde{Y})}{\sqrt{\underbrace{Var(\tilde{X})}_{1}\underbrace{Var(\tilde{Y})}_{1}}} = Cov(\tilde{X}, \tilde{Y})$$

Chain: $Cor(X, Y) = Cor(\tilde{X}, \tilde{Y}) = Cov(\tilde{X}, \tilde{Y})$

Consider

$$Var(\tilde{X} \pm \tilde{Y}) = \underbrace{Var(\tilde{X})}_{1} + \underbrace{Var(\tilde{Y})}_{1} \pm 2\underbrace{Cov(\tilde{X}, \tilde{Y})}_{Cor(X,Y)}$$

$$= 2 \pm 2\, Cor(X, Y) \geq 0$$

So $1 \pm Cor(X, Y) \geq 0$

So $1 \pm \mathrm{Cor}(X,Y) \geq 0$

$1 + \mathrm{Cor}(X,Y) \geq 0$ $\Rightarrow$ $\boxed{-1 \leq \mathrm{Cor}(X,Y)}$

$1 - \mathrm{Cor}(X,Y) \geq 0$ $\Rightarrow$ $\boxed{\mathrm{Cor}(X,Y) \leq 1}$

Theorem: Short-Cut formula for Covariance

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

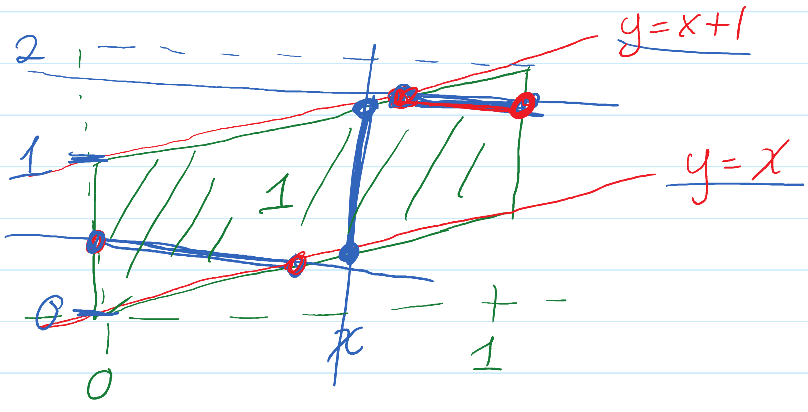
Analogy: $\mathrm{Var}(X) = E[X^2] - (EX)^2$

Ex. $f(x,y) = 1$ for $0 < x < 1$, $x < y < x+1$

We had calculated

$$\boxed{\mathbb{E}[XY] = 7/12}$$

What is Cov/Cor?



Marginal of X $\quad 0 < x < 1$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_{y=x}^{y=x+1} 1\,dy = (x+1) - x = 1$$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_{y=x} 1\,dy = (x+1) - x = 1$$

So $f_X(x) = 1$ for $0 < x < 1$

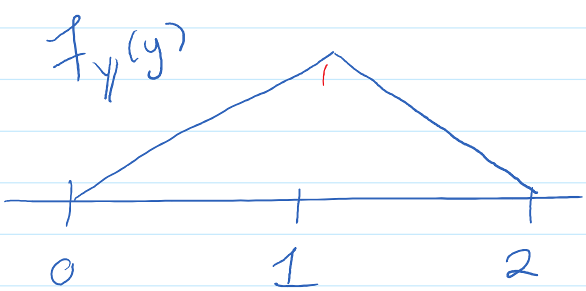
i.e. $X \sim U(0,1)$

So $\boxed{E[X] = 1/2 \quad \text{and} \quad Var(X) = 1/12}$

Marginal of Y

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \begin{cases} \int_0^y 1\,dx & 0 < y < 1 \\ \int_{y-1}^1 1\,dx & 1 < y < 2 \end{cases}$$

$$= \begin{cases} y, & 0 < y < 1 \\ 2-y, & 1 < y < 2 \end{cases}$$



$$\boxed{E[Y] = 1 \quad \text{and} \quad Var(Y) = 1/6}$$

So

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y$$
$$= 7/12 - (1/2)(1) = 1/12$$

$$\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{1/12}{\sqrt{\frac{1}{12}\cdot\frac{1}{6}}}$$

Conditional Probability

$$P(A|B) = \frac{P(AB)}{P(B)}$$

If X and Y are discrete. Consider

$$A = \{X = x\} \text{ and } B = \{Y = y\}$$

$$\underline{\mathbb{P}(X=x \mid Y=y)} = \mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{P(B)} = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$$

$$\boxed{= \frac{f(x,y)}{f_Y(y)}}$$

Defn: Conditional PMF

If X and Y are discrete then the conditional PMF of X given $Y=y$ is

$$f(x|y) = f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

Basically think of $Z =$ "$X|Y=y$" as a univariate RV.

Ex. Joint PMF $\to f(x,y)$

Y \ X	10	20	30
2	0	0	4/18
1	3/18	4/18	3/18
0	2/18	2/18	0

$f_Y(0) = 4/18$

Lets get dist of $X|Y=0$

$$f(x|0) = \frac{f(x,0)}{f_Y(0)} = \begin{cases} \frac{2/18}{4/18} = 1/2 & x = 10 \\ \frac{2/18}{4/18} = 1/2 & x = 20 \\ 0 & x = 30 \end{cases}$$

$X \mid Y = 0$ equally split btwn $(0, 20$

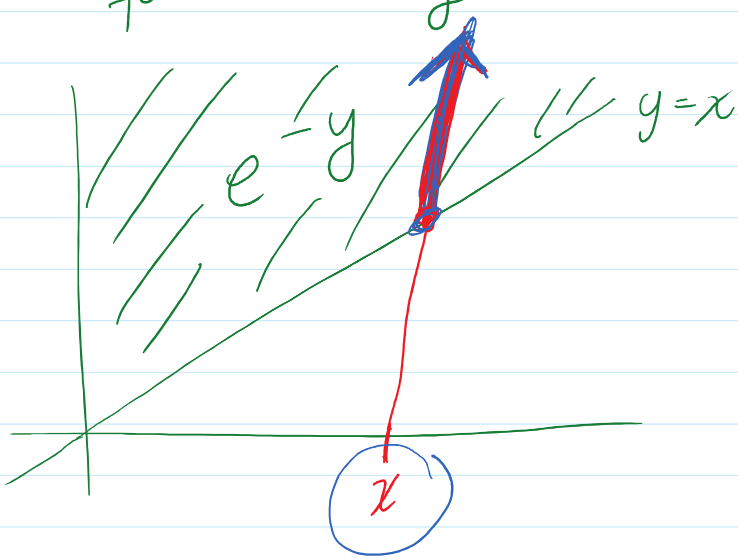
Defn: Conditional PDF

If X and Y are continuous then the conditional PDF of X given $Y = y$ is

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)}.$$

ex. $f(x,y) = e^{-y}$ for $0 < x < y$

What is the PDF of $Y \mid X = x$?



Marginal of X

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_{y=x}^{\infty} e^{-y}\,dy = -e^{-y}\Big|_x^{\infty}$$

$$= 0 - (-e^{-x})$$

$$= 0 - (-e \quad)$$

$$\boxed{\text{for } x > 0 \qquad = e^{-x}}$$

$$X \sim Exp(\lambda = 1)$$

So $$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} \quad \text{for } x < y$$

$$\boxed{= e^{-(y-x)} \quad \text{for } y > x}$$

Called Shifted Exponential dist