Ex. E = "its raining"

$P(E) = 1/3$

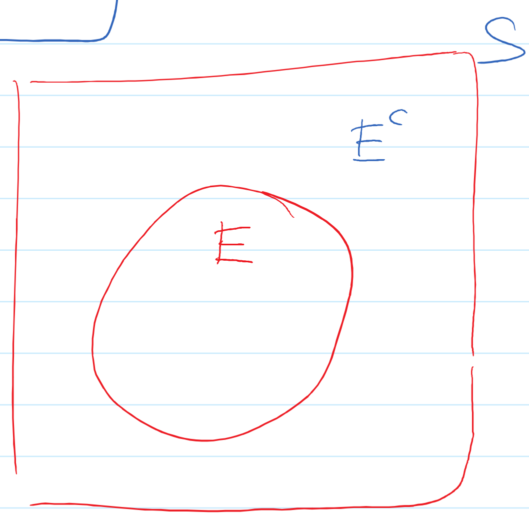
$P(\text{not raining}) = 1 - 1/3$

$P(E^c) = 1 - 2/3$

$$\boxed{\text{Theorem: } \quad P(E^c) = 1 - P(E)}$$

pf. $S = E \cup E^c$

a partition



So

$1 = P(S) = P(E) + P(E^c)$

$1 = P(E) + P(E^c)$

rearrange

$P(E^c) = 1 - P(E)$.

Theorem: $0 \le P(E) \le 1$

pf. $P(E) \ge 0$ by Axiom 1

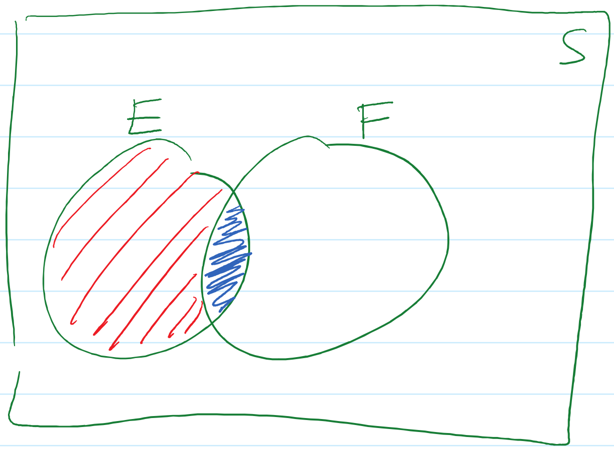
$$P(E^c) \geq 0$$

and so $1 - P(E) \geq 0$

rearrange to get $P(E) \leq 1$.

Theorem: If $E, F \subset S$ then

$$P(E \setminus F) = P(EF^c) = P(E) - P(EF)$$



pf. $E = EF \cup EF^c$

partition of E

$$P(E) = P(EF) + P(EF^c)$$

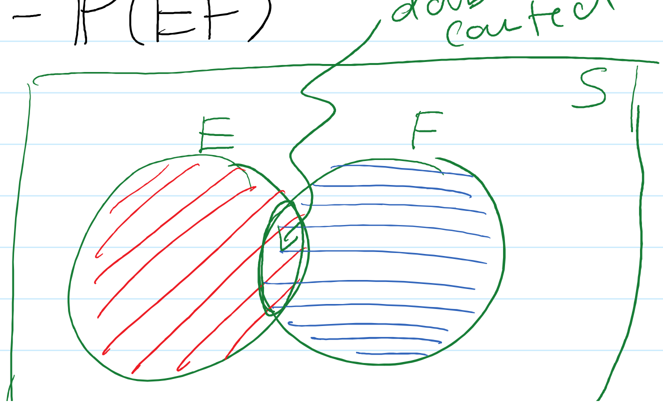
rearrange

$$P(EF^c) = P(E) - P(EF).$$

Theorem: Let $E, F \subset S$

(maybe not disjoint)

$$P(E \cup F) = P(E) + P(F) - P(EF)$$



pf.

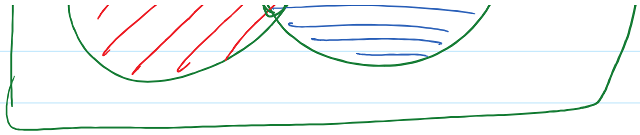
disjoint

$E \cup F = E \cup FE^c$

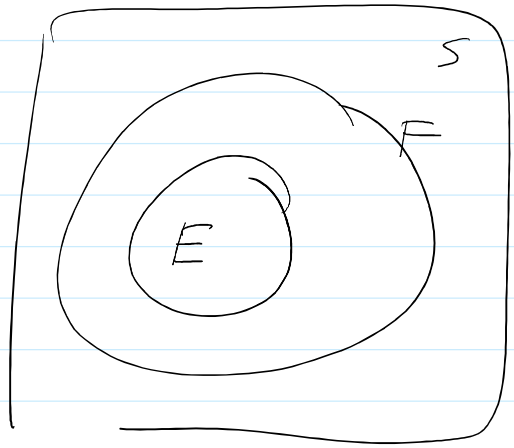
So

$P(E \cup F) = P(E) + P(FE^c)$

So



$$P(E \cup F) = P(E) + P(FE^c)$$
$$= P(E) + P(F) - P(FE)$$



Theorem: If $E \subset F$ then

$$P(E) \leq P(F).$$

Pf. By Axiom 1,

$$P(FE^c) \geq 0$$

so $P(F) - P(EF) \geq 0$

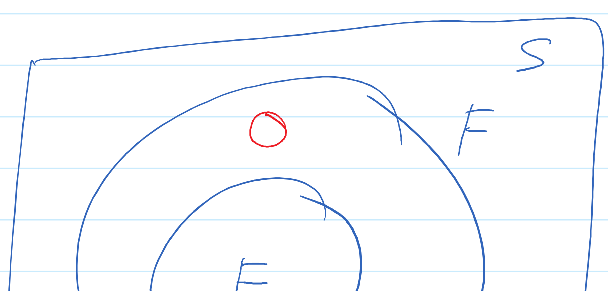
hence $P(EF) \leq P(F)$ $\quad$ $E \subset F$ so $EF = E$

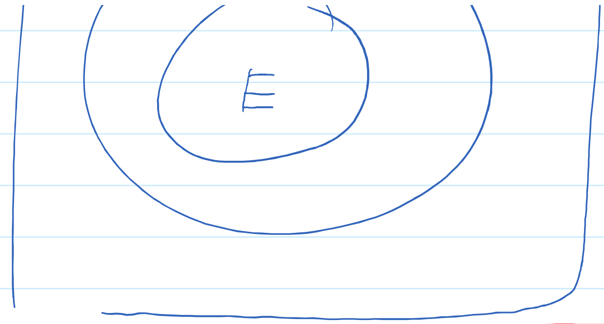
so $P(E) \leq P(F)$.

Consider $E \subset F$ but $E \neq F$.

$$P(E) \cancel{<} P(F) \text{ ?}$$

Generally can't say.





<u>Said</u>:

$$P(E \cup F) = P(E) + P(F) - \overbrace{P(EF)}^{\geq 0}$$

$$\leq P(E) + P(F)$$

<u>Generalize this: Boole's Inequality</u>

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i).$$

<u>Pf.</u> Replace E_i w/ B_i where

(1) $\bigcup_i B_i = \bigcup_i E_i$.

(2) B_i are disjoint.

<u>defn</u>:

$B_1 = E_1$

$B_2 = E_2 E_1^c$

$B_3 = E_3 E_2^c E_1^c$

Convince yourself that this satisfies (1) & (2).

Notice: $B_i \subset E_i$

$B_3 = E_3 E_2^c E_1^c$

$B_4 = E_4 E_3^c E_2^c E_1^c$

Notice: $B_i \subset E_i$

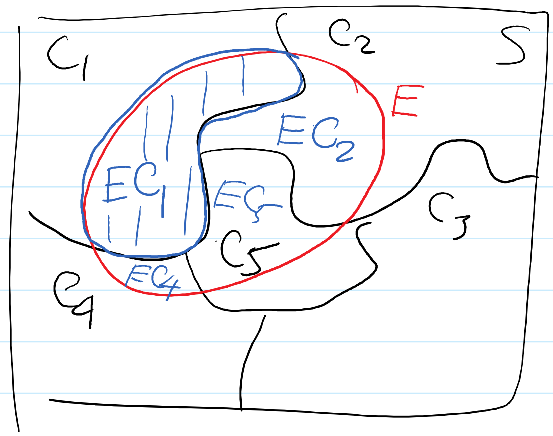
$P(B_i) \le P(E_i)$

then

$$P\left(\bigcup_i E_i\right) = P\left(\bigcup_i B_i\right) = \sum_i P(B_i) \le \sum_{i=1}^{\infty} P(E_i)$$

Theorem: If (C_i) are a partition of S and $E \subset S$

$$P(E) = \sum_i P(EC_i).$$



Pf. ① (EC_i) partitions E

② by Additivity

$$P(E) = \sum_i P(EC_i).$$

Equally Likely Outcomes in a Finite Sample Space

I have a sample space

$S = \{s_1, \dots, s_n\}$ so that $|S| = n$

assume that

$$\frac{1}{n} = \underline{P(\{s_i\}) = P(\{s_j\})} \quad \forall i,j$$

Reasons

$$1 = \mathbb{P}(S) = \sum_{i=1}^{n} \underline{\mathbb{P}(\{s_i\})}$$

the only way this works is if $\mathbb{P}(\{s_i\}) = \frac{1}{n}$

More generally:

$$\mathbb{P}(E) = \frac{\#\text{ elements in } E}{\#\text{ elements in } S} = \frac{|E|}{|S|}$$

Ex. Roll a six-sided die.

$$S = \{1, \dots, 6\}$$

and all rolls are equally likely

and

$$E = \{2, 6\}$$

then

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{2}{6} = 1/3.$$

Counting

Ex. An experiment has 3 factors

1. 2 temp. settings
2. 2 pressure settings
3. 4 humidity settings.

Q: How many experiments possible?

$$16 = 2 \cdot 2 \cdot 4$$

Fundamental Theorem of Counting (FTC)

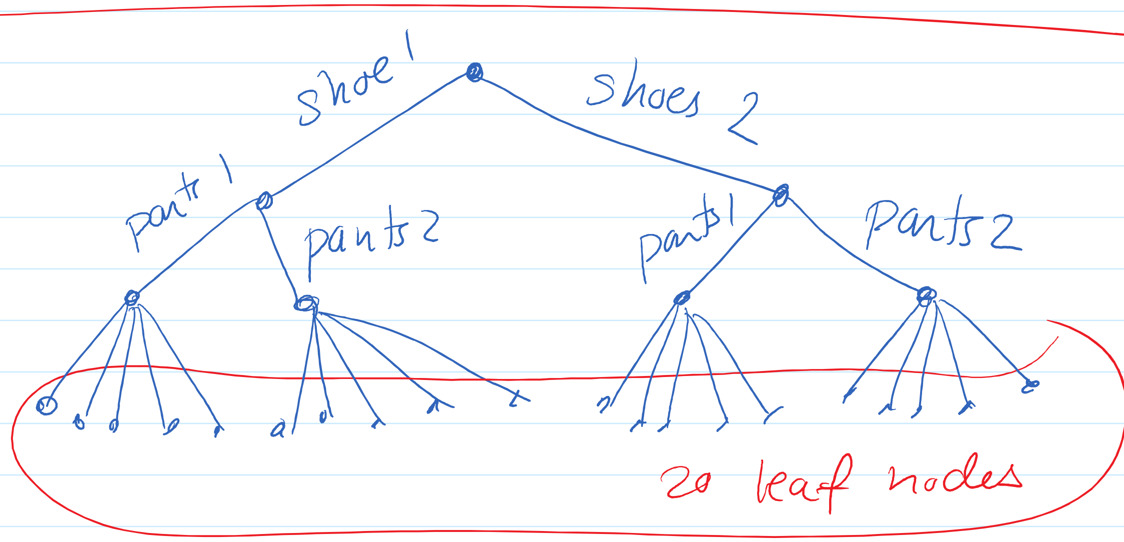
If I have a task that consists of k sub-tasks — where subtask i has n_i ways of being completed. Then the total number of ways to complete the task is

$$N = n_1 n_2 n_3 n_4 \cdots n_k$$

$$= \prod_{i=1}^{k} n_i$$

Ex. A man has 5 shirts, 2 pair pants, 2 pair shoes. How many outfits does he have?

By FTC he has $5 \cdot 2 \cdot 2 = 20$ outfits.



Ex. I have a deck of 52 cards

I shuffle them so each ordering is equally likely.

Q: What is the prob (after shuffle) that the cards are "in order"

↳ A–K, C, D, H, S

E = in order

S = all possible shuffles

$$P(E) = \frac{|E| \leftarrow 1}{|S|}$$

$$P(E) = \frac{}{|S|}$$

Use FTC w/ $k = 52$

task #	task	# ways
1	choose 1^{st} card	52
2	" 2^{nd} "	51
3	" 3^{rd} "	50
$\vdots$	$\vdots$	$\vdots$
52	" 52^{nd} "	1

multiply

$$|S| = 52 \cdot 51 \cdot 50 \cdot \cdots 3 \cdot 2 \cdot 1$$

so $P(E) = 1/(52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1)$

Defn: Factorial

For any non-neg. integer n we define n factorial as

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$
$$= \prod_{i=1}^{n} i$$

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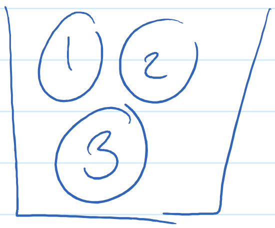
1. more examples:

In prev. example:

$$\mathbb{P}(E) = 1/52!$$

Sampling w/ and w/o Replacement / Ordering

Ordering

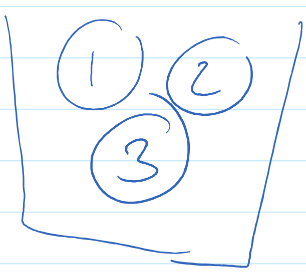


draw 1: 1 3 2

draw 2: 2 1 3

are these different?

Replacement



Can I draw 1 1 2 ?

w/ replacement: Yes

w/o replacement: No

4 options:

	w/o repl.	w/ repl.
ordered	1	2

ordered	1	2
unordered	4	3

Defn: Permutation

A permutation is an ordering of objects.

Ex. 1 2 3

permutations:

$$\left.\begin{array}{cc} 123 & 132 \\ 213 & 231 \\ 312 & 321 \end{array}\right\} \text{6 perms} = 3!$$

Theorem: The number of ways to permute n items is $n!$

pf. FTC w/ $k = n$ tasks.

task #	task	# ways

task #	task	# ways
1	choose 1st	n
2	" 2nd	$n-1$
3	" 3rd	$n-2$
⋮	⋮	
n	" nth	1

multiply

So total num. ways is $n(n-1)\cdots 3\cdot 2\cdot 1$

$= n!$