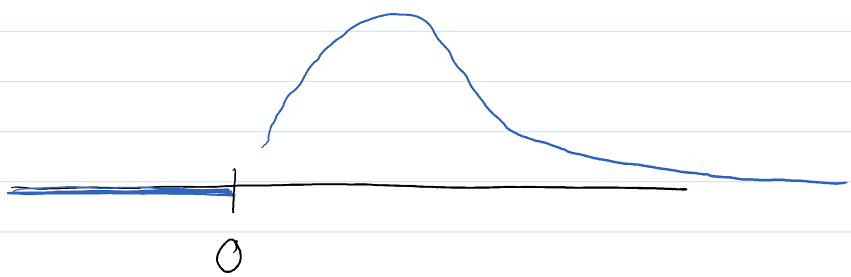
Ex. let $X \sim \mathrm{Exp}(\lambda)$ $\lambda > 0$

X has an exponential dist.

this means

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$



Q: $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x} dx}_{dv} \quad (*)$$

integration by parts:

$$\int u\,dv = uv - \int v\,du$$

$u = x$ $v = -e^{-\lambda x}$

$du = dx$ $dv = \lambda e^{-\lambda x} dx$

$$= \int u\,dv = uv - \int v\,du = \underbrace{x(-e^{-\lambda x})\Big|_0^{\infty}}_{} - \int_0^{\infty} (-e^{-\lambda x})\,dx$$

$$(\ 0\ ) - (\ 0\ )$$

$$= \int_0^\infty e^{-\lambda x}\,dx = \left(-\frac{1}{\lambda}\right)e^{-\lambda x}\Big|_0^\infty$$

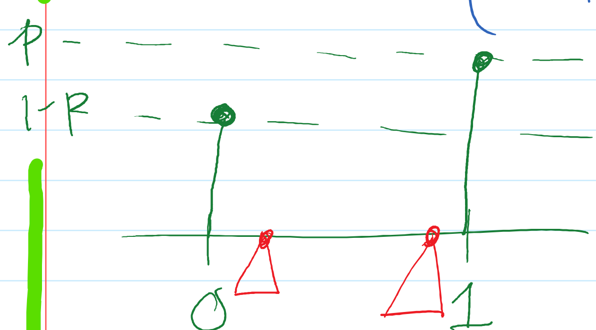
$$= 0 - \left(-\frac{1}{\lambda}\right) = \boxed{\frac{1}{\lambda} = EX}$$

Ex. $X \sim \text{Bern}(p)$ $\quad 0 \le p \le 1$

Bernoulli dist

X = any binary experiment w/ a prob. p of 1

$$f(x) = \begin{cases} p, & X = 0 \\ 1-p, & X = 1 \end{cases}$$



$$E[X] = \sum_x x f(x) = \sum_{x=0,1} x f(x) = (0)f(0) + (1)f(1)$$

$$= (0)(1-p) + (1)(p)$$

$$= p$$

$= p$

Ex. $X \sim Bin(n, p)$ — $p \in [0,1]$, $n \in \mathbb{N}$

Binomial dist

X = # heads among n independent coin flips - where each has a prob. p of H

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$

Binomial Theorem:

$$\boxed{(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}}$$

$$\sum_{x=0}^{n} f(x) = 1 \; ?$$

use binomial theorem w/ $x = p$, $y = 1-p$

$$(x+y)^2 = 1x^2 + 2x^1y^1 + 1y^2$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$E[X] = \sum_{x=\cancel{0}\,1}^{n} x f(x) = \sum_{x=1}^{n} x\binom{n}{x} p^x (1-p)^{n-x}$$

$$x\binom{n}{x} = x\frac{n!}{x!(n-x)!}$$

$$= \sum_{x=1}^{n} n\binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$y = x-1 \iff x = y+1$$

$$x\binom{n}{x} = x\frac{n!}{x!(n-x)!}$$

$$= \frac{\cancel{x}\, n(n-1)!}{\cancel{x}(x-1)!((n-1)-(x-1))!}$$

$$= n\frac{(n-1)!}{(x-1)!((n-1)-(x-1))!}$$

$$= n\binom{n-1}{x-1}$$

$x=1$

$y = x-1 \Leftrightarrow x = y+1$

$$= \sum_{y=0}^{n-1} n\binom{n-1}{y}p^{y+1}(1-p)^{(n-1)-y}$$

$$= np\underbrace{\sum_{y=0}^{n-1}\binom{n-1}{y}p^{y}(1-p)^{(n-1)-y}}_{\text{PMF of Bin}(n-1,p)}$$

Summing PMF over whole support

$$= \boxed{np = \mathbb{E}[X]}$$

General trick! PMF/PDF trick,

Often I can recognize in a calculation some term,

$$\sum_{x} f(x) \quad \text{or} \quad \int_{\mathbb{R}} f(x)\,dx$$

and replace these w/ 1.

Functions of RVs

Note: a function of a RV is also a RV.

Note a function of a RV is also a RV.

e.g. If I have a RV X then

X^2 or $\log X$ or $\sqrt{X}$ is a RV.

Theorem: Law of the Unconscious Statistician

If $g: \mathbb{R} \to \mathbb{R}$ and X is a RV then

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x) f(x) & \text{(discrete)} \\ \int_{\mathbb{R}} g(x) f(x)\, dx & \text{(cts)} \end{cases}$$

Ex. Let $X \sim \text{Exp}(\lambda)$

$f(x) = \lambda e^{-\lambda x}$ for $x > 0$

$\mathbb{E}[X] = 1/\lambda$.

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\, dx = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$u = x^2 \qquad dv = \lambda e^{-\lambda x} dx$

$du = 2x\,dx \qquad v = -e^{-\lambda x}$

$du = 2x\,dx \qquad v = -e^{-\lambda x}$

$$= uv - \int v\,du = \underbrace{x^2(-e^{-\lambda x})\Big|_0^{\infty}}_{0-0=0} - \int_0^{\infty} (-e^{-\lambda x})\,2x\,dx$$

$$= 2\frac{1}{\lambda}\underbrace{\int_0^{\infty} x\lambda e^{-\lambda x}\,dx}_{\mathbb{E}[X]}$$

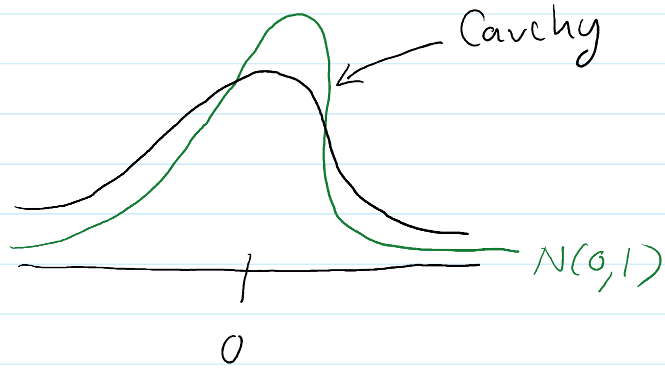
$$= \frac{2}{\lambda}\mathbb{E}X$$

$$= \frac{2}{\lambda}\frac{1}{\lambda} = \boxed{\frac{2}{\lambda^2} = \mathbb{E}[X^2]}$$

$$(\mathbb{E}X)^2 = \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} \neq \frac{2}{\lambda^2} = \mathbb{E}[X^2]$$

Ex. Cauchy Distribution

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$



$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\, dx$$

$$= \frac{1}{\pi} \underbrace{\int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx}$$

$\sim$ asypotically $\frac{x}{1+x^2} \sim \frac{x}{x^2} = \frac{1}{x}$

$$= \infty$$

So X has no expectation.

$\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty \qquad \int \frac{1}{x^2}\, dx < \infty$

$\sum_{i=1}^{\infty} \frac{1}{i} = \infty \qquad \int \frac{1}{x}\, dx = \infty$

Theorem: Properties of Expectation

① Expectation is linear.

$$\mathbb{E}[aX+b] = a\mathbb{E}[X] + b.$$

Pf. (cts)

$$E[aX+b] = \int (ax+b) f(x)\,dx = \int [axf(x) + bf(x)]\,dx$$

$$= \int axf(x)dx + \int bf(x)\,dx$$

$$= a\underbrace{\int xf(x)dx}_{\mathbb{E}X} + b\underbrace{\int f(x)dx}_{1}$$

$$= a\mathbb{E}X + b$$

② If $X \geq 0$ then $\mathbb{E}X \geq 0$

support$(X) \subset (0,\infty)$

pf. (cts)

$$\mathbb{E}X = \underbrace{\int_0^\infty \underbrace{x}_{\geq 0}\, \underbrace{f(x)}_{\geq 0}}_{\geq 0} dx \geq 0$$

③ If g_1 and g_2 are functions

(i) $E[g_1(X)+g_2(X)] = E[g_1(X)] + E[g_2(X)]$

(ii) If $g_1(x) \leq g_2(x)$ then $E[g_1(X)] \leq E[g_2(X)]$.

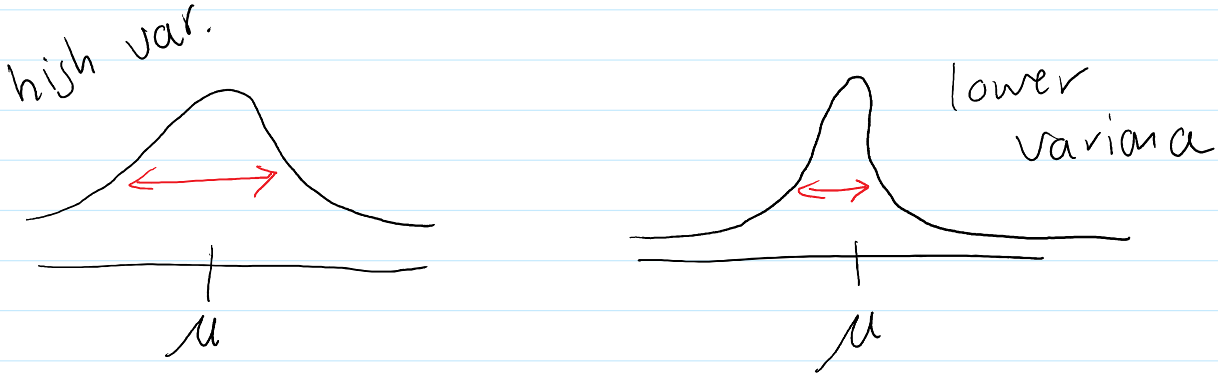
④ If $a \leq X \leq b$ then $a \leq \mathbb{E}X \leq b$.

④ If $a \le X \le b$ then $a \le \mathbb{E}X \le b$.

Defn: Variance

$$\mu = \mathbb{E}X = \text{location of dist}$$

$$\sigma^2 = Var(X) = \text{how spread out dist is}$$



Defn:

$$Var(X) = \mathbb{E}\left[(X-\mu)^2\right]$$

$$= \mathbb{E}\left[(X-\mathbb{E}X)^2\right]$$