**Defn: Identically Distributed**

We say that two RVs $X$ and $Y$ are equal in distribution if $\forall A \subset \mathbb{R}$

$$\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$$

we denote this as

$$X \overset{d}{=} Y$$

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Ex. This doesn't mean $X = Y$ (as functions)

Flip 3 coins.

$X = \#$ heads

$Y = \#$ tails

$$X(HTT) = 1 \quad \text{but} \quad Y(HTT) = 2$$

different functions

However $X \overset{d}{=} Y$.

$$P(X=0) = 1/8 = P(Y=0)$$
$$P(X=1) = 3/8 = P(Y=1)$$

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Theorem:

$$X \overset{d}{=} Y \quad \text{iff} \quad F_X = F_Y.$$

CDF of $X$ CDF of $Y$.

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Ex. Toss a coin (independently) until a H appears.

$$S = \{H, TH, TTH, TTTH, \ldots\}$$

note $|S| = \infty$

Let $p$ be the prob. of getting a H on any flip.

Let $X$ = # flips until I get a H

| $s \in S$ | $X(s)$ |
|---|---|
| H | 1 |
| TH | 2 |
| TTH | 3 |
| $\vdots$ | $\vdots$ |

| $s \in S$ | $X(s)$ |
|---|---|
| H | 1 |
| TH | 2 |
| TTH | 3 |
| $\vdots$ | $\vdots$ |

Q: what is the CDF? $F(x) = \mathbb{P}(X \le x)$

We'll look at $P(X = x)$

Let $H_i$ = $i^{th}$ toss is a H, $T_i = H_i^c$

$$\text{"}X = i\text{"} = \underbrace{\underbrace{T_1 \cdots T_{i-1}} \; \underbrace{H_i}}_{\text{independent}}$$

$$P(X = i) = P(T_1 \cdots T_{i-1} H_i)$$

$$= P(T_1) \cdots P(T_{i-1}) P(H_i)$$

$$= (1-p) \cdots (1-p) p$$

$$= (1-p)^{i-1} p$$

Lets consider "$X \le x$" = "$X=1$" $\cup$ "$X=2$" $\cup \cdots \cup$ "$X=x$"

disjoint

So

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}(X=1) + \mathbb{P}(X=2) + \cdots + \mathbb{P}(X=x)$$

$$= \sum_{i=1}^{x} \mathbb{P}(X=i)$$

$$= \sum_{i=1}^{x} (1-p)^{i-1} p$$

$$= p \sum_{i=1}^{x} (1-p)^{i-1}$$

$$= p \sum_{i=0}^{x-1} (1-p)^{i}$$

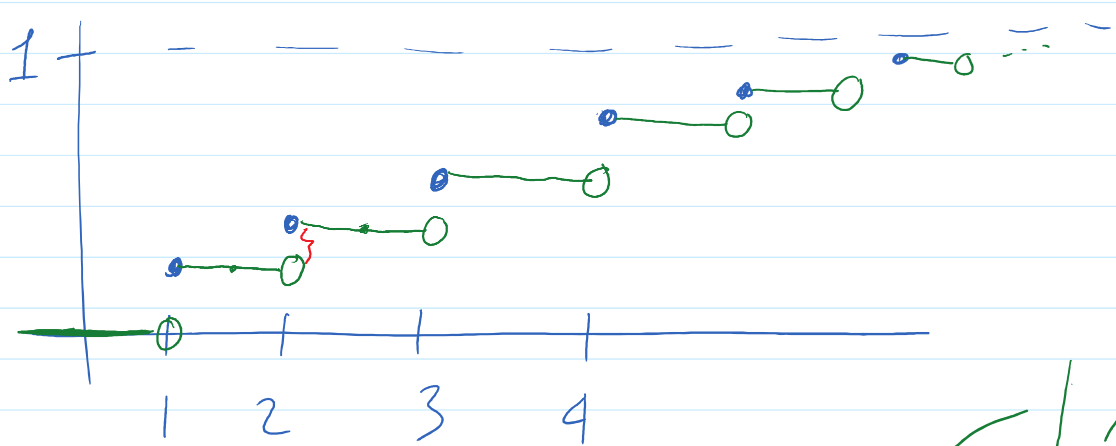
$$= \cancel{p} \, \frac{1-(1-p)^x}{\cancel{1-(1-p)}}$$

Geometric Sum:

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$$

$r = 1-p$

$$\boxed{F(x) = 1-(1-p)^x}$$




Really,

$$\begin{cases} 0, & x < 1 \\ \ldots \end{cases}$$

$\lfloor x \rfloor \overset{=}{} $ round down

$$F(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor}, & x \geq 1 \end{cases}$$

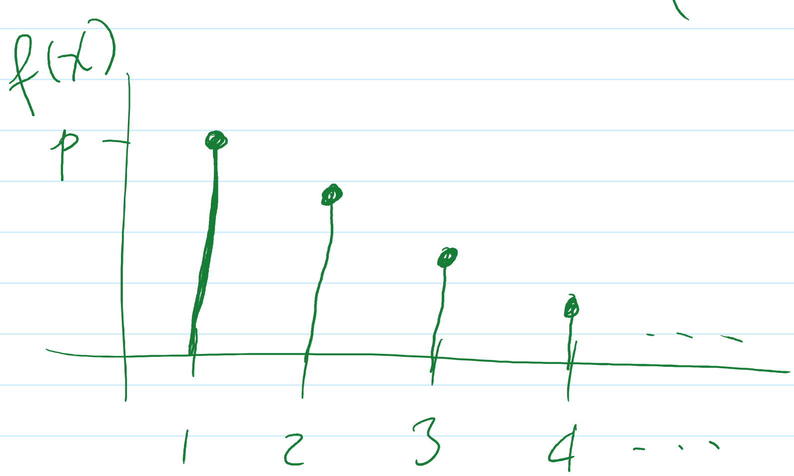
## Defn: Discrete / Continuous RVs

A discrete RV has a CDF that is a step function.

A continuous RV has a continuous CDF.

Ex.

$$f(x) = \mathbb{P}(X = x) = \begin{cases} (1-p)^{x-1} p, & x = 1, 2, 3, \ldots \\ 0 & \text{else} \end{cases}$$




## Defn: Probability Mass Function (PMF)

For a discrete RV $X$, the PMF of $X$ is defined as a function $f: \mathbb{R} \to \mathbb{R}$ where $x \in \mathbb{R}$

$$f(x) = P(X = x)$$

Also called the distribution of $X$.

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Theorem: For discrete RVs

$$F(x) = \sum_{i \leq x} f(i)$$

pf "$X \leq x$" $= \bigcup_{i \leq x}$ "$X = i$"

disjoint

so

$$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}\left(\bigcup_{i \leq x} \text{"}X = i\text{"}\right)$$

$$= \sum_{i \leq x} \mathbb{P}(X = i)$$

$$= \sum_{i \leq x} f(i)$$

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Ex. We say $X$ has a discrete uniform distribution over $1, \ldots, n$ if

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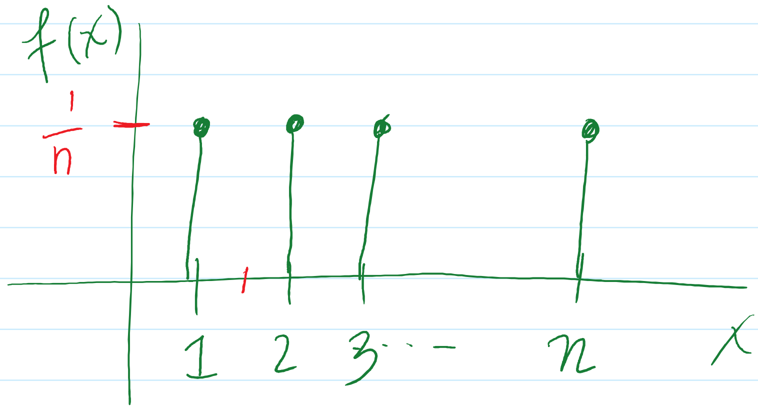


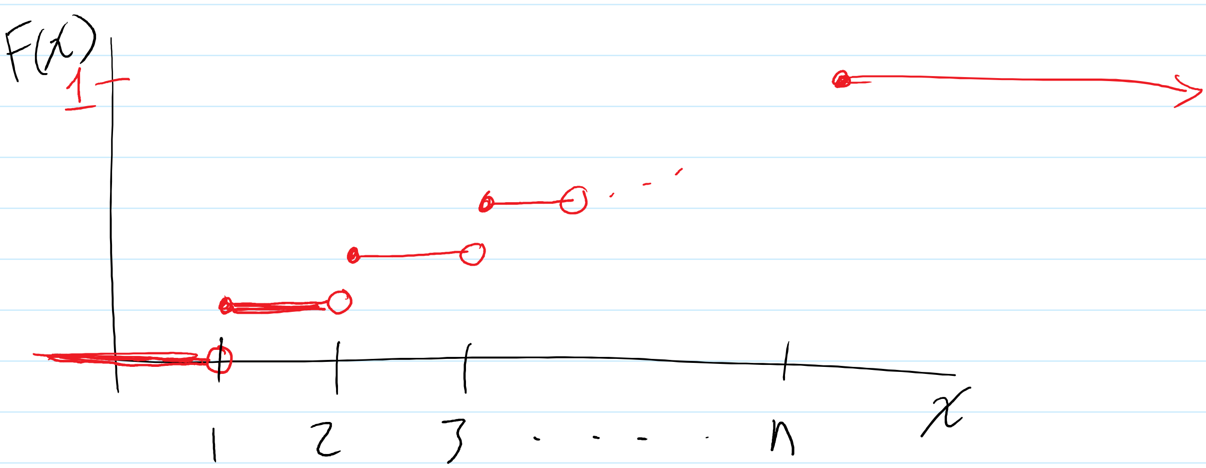

**Notation:**

$X \sim U(\{1, \ldots, n\})$

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x = 1, 2, \ldots, n \\ 0 & \text{else} \end{cases}$$

Q: What is the CDF of $X$?

$$F(x) = \sum_{i \le x} f(i) = \sum_{i=1}^{x} f(i) = \sum_{i=1}^{x} \frac{1}{n} = \frac{x}{n}$$




$$F(x) = \begin{cases} 0, & x < 1 \\ \ldots \end{cases}$$

$$F(x) = \begin{cases} 0, & x < 1 \\ \lfloor x \rfloor / n & 1 \le x < n \\ 1, & x \geq n \end{cases}$$

Saw: $F(x) = \sum_{i \le x} f(i)$

$\mathbb{P}(X \le x)$

Generalized: $A \subset \mathbb{R}$,

$$\mathbb{P}(X \in A) = \sum_{i \in A} f(i)$$

Ex. Continue prev. $n = 7$

$$P(2 \le X \le 5) = P(X \in \{2,3,4,5\})$$

$$= \sum_{i=2,3,4,5} f(i) \quad 1/7$$

$$= 4/7$$

Ex. Roll a die 60 times. (independently)

$X = \#$ of 6s I roll

Q: What is $f(x)$?

$$f(0) = P(X=0) = \underbrace{\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)}_{60} = \left(\tfrac{5}{6}\right)^{60}$$

$$f(1) = P(X=1) = \binom{60}{1}\left(\tfrac{1}{6}\right)\underbrace{\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)}_{59}$$

$$= \binom{60}{1}\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)^{59}$$

$$f(2) = \binom{60}{2}\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)$$

$$= \binom{60}{2}\left(\tfrac{1}{6}\right)^{2}\left(\tfrac{5}{6}\right)^{58}$$

pattern

$$\boxed{f(x) = P(X=x) = \binom{60}{x}\left(\tfrac{1}{6}\right)^{x}\left(\tfrac{5}{6}\right)^{60-x}}$$

We call this a Binomial RV.

Generally, if I do a sequence of $n$ yes/no experiments (independently), and I have

a prob. $p$ of a "Yes" on each experiment,

and $X = \#$ Yes

then $X$ has binomial distribution where

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

above: $n = 60$, $p = 1/6$

Notation: $X \sim Bin(n, p)$

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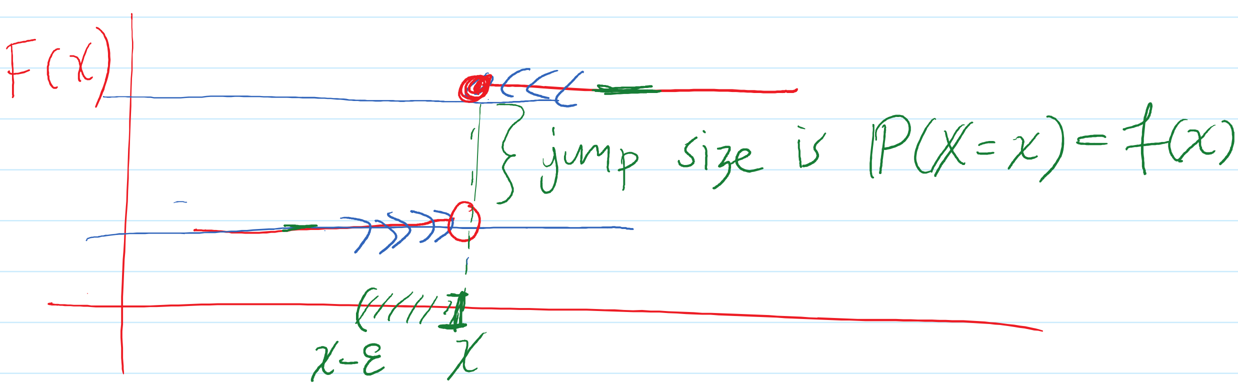
PMF: $f(x) = P(X = x)$

CDF: $F(x) = P(X \le x)$

$\Rightarrow F(x) = \sum_{i \le x} f(i)$




Consider $(x - \varepsilon, x]$

$\lim P(x - \varepsilon < X \le x)$

$\varepsilon \to 0$

$$= \lim_{\varepsilon \to 0} F(x) - F(x-\varepsilon) = \text{jump size.}$$

Consider lim as $\varepsilon \to 0$ $= \mathbb{P}(X = x)$.