Extra OH:

Thurs: 2-3
Mon: 2-3
Tues: 3-4

Defn: Random Sample

If $X_1, X_2, \ldots, X_N$ are mutually independent (← sample size)
all having marginal dist f
then we say these Xs are a random sample (RS) from f.

i.e. $X_n \overset{iid}{\sim} f$

Notation:

$$\underset{\sim}{X} = (X_1, \ldots, X_N)$$ ← a mv-RV or a random vector

$$\underset{\sim}{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

Joint dist of a RS

$$f(\underset{\sim}{x}) = f(x_1, \ldots, x_N)$$

$$= f(x_1) f(x_2) \cdots f(x_N)$$ [by independent]

$$= f(x_1) f(x_2) \cdots f(x_N) \quad \text{[by independent]}$$

$$= \prod_{n=1}^{N} f(x_n)$$

<u>Ex.</u> Let $X_n \overset{iid}{\sim} Exp(\lambda) \leftarrow$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

more explicit

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

<u>indicator</u>

$$\mathbb{1}(\text{statement}) = \begin{cases} 1 & \text{statement true} \\ 0 & \text{statement false} \end{cases}$$

more compact

$$f(x) = \lambda e^{-\lambda x} \mathbb{1}(x > 0)$$

What is the joint of X_ns?

$$f(\underset{\sim}{x}) = \prod_{n=1}^{N} f(x_n) = \prod_{n=1}^{N} \lambda e^{-\lambda x_n} \mathbb{1}(x_n > 0)$$

$$= \lambda^N e^{-\lambda \sum_n x_n} \prod_n \mathbb{1}(x_n > 0)$$

$$e^a e^b = e^{a+b}$$

$$\prod_n e^{a_n} = e^{\sum_n a_n}$$

$$= \lambda^N e^{-\lambda \sum_n x_n} \prod_n \mathbb{1}(x_n > 0) \quad \prod_n e^{\dots} = e^{\dots}$$

$$= \lambda^N e^{-\lambda \sum_n x_n} \mathbb{1}(\text{all } x_n > 0)$$

$\mathbb{1}(A)\mathbb{1}(B) = \mathbb{1}(A \text{ and } B)$

$\prod_n \mathbb{1}(A_n) = \mathbb{1}(\text{all } A_n \text{ true})$

Defn: Statistic

Given a RS $X_n \overset{iid}{\sim} f$

and a function

$$T : \mathbb{R}^N \to \mathbb{R}^d$$

often $d \ll N$

e.g. $d = 1$

then $T(\underset{\sim}{X})$ is called a statistic.

Ex,

1) Arithmetic Mean $(d=1)$

$$T(\underset{\sim}{X}) = \frac{1}{N} \sum_{n=1}^{N} X_n = \overline{X_N}$$

2) Sample Variance

$$S^2_{N-1} = \frac{1}{N-1} \sum_{n=1}^{N} (X_n - \overline{X_N})^2$$

$$S_{N-1} = \frac{\quad}{N-1} \sum_{n=1} (X_n \quad {}_N)$$

③ <u>Minimum</u>: $X_{(1)} = \min\{X_1, \dots, X_N\}$

④ <u>Maximum</u>: $X_{(N)} = \max\{X_1, \dots, X_N\}$

⑤ <u>Range</u>: $X_{(N)} - X_{(1)}$

⑥ <u>Order Statistics</u> $X_{(r)} = r^{th}$ smallest

<u>Defn: Sampling Distribution</u>

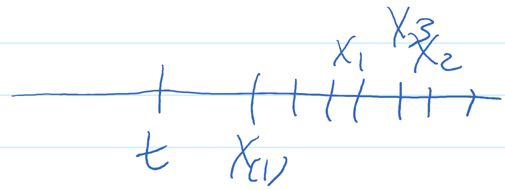
For a stat. T the <u>sampling dist</u> is simply its distribution.

<u>Ex.</u> what is the dist of $X_{(1)}$?

Lets assume $X_n \overset{iid}{\sim} f$ and f cts.

<u>Let F be the CDF of X_n's.</u>

I want the PDF of $X_{(1)}$.



$$P(X_{(1)} > t)$$

$$\mathbb{P}(X_{(1)} \geq t)$$

$$= \mathbb{P}(X_1 \geq t, X_2 \geq t, \ldots, X_N \geq t)$$

independence

$$= \mathbb{P}(X_1 \geq t)\,\mathbb{P}(X_2 \geq t) \cdots \mathbb{P}(X_N \geq t)$$

$$= \mathbb{P}(X_n \geq t)^N$$

$$= \underline{(1 - F(t))^N}$$

$$F_{X_{(1)}}(t) = \mathbb{P}(X_{(1)} \leq t) = 1 - \mathbb{P}(X_{(1)} \geq t)$$

$$= 1 - (1 - F(t))^N$$

$$\boxed{f_{X_{(1)}}(t) = \frac{dF_{X_{(1)}}}{dt} = N(1 - F(t))^{N-1} f(t)}$$

Can play same game for $X_{(N)}$ and look at

$$\mathbb{P}(X_{(N)} \leq t)$$

... and get $\boxed{f_{X_{(N)}}(t) = N F(t)^{N-1} f(t)}$

Ex. iid $-\lambda x$

Ex. Let $X_n \overset{iid}{\sim} \mathrm{Exp}(\lambda)$.

$f(x) = \lambda e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x}$

What is the dist of $X_{(1)}$?

$$f_{X_{(1)}}(t) = N(1-F(t))^{N-1} f(t)$$

$$= N(1-(1-e^{-\lambda t}))^{N-1} \lambda e^{-\lambda t}$$

$$= N(e^{-\lambda t})^{N-1} \lambda e^{-\lambda t}$$

$$f_{X_{(1)}}(t) = (N\lambda) e^{-(N\lambda)t}$$

PDF of $\mathrm{Exp}(N\lambda)$

i.e. $\boxed{X_{(1)} \sim \mathrm{Exp}(N\lambda)}$

So $\mathbb{E} X_{(1)} = \frac{1}{N\lambda}$