Theorem: $a, b \in \mathbb{R}$ then

$$\mathrm{Cov}(aX+b, Y) = a\,\mathrm{Cov}(X,Y)$$

Recall: $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$

Pf. $\mathrm{Cov}(aX+b, Y) =$

$$= \mathbb{E}\big[\big((aX+b) - E[aX+b]\big)(Y - EY)\big]$$

$$= E\big[(aX \cancel{+b} - aE[X] \cancel{-b})(Y - EY)\big]$$

$$= \mathbb{E}\big[a(X - EX)(Y - EY)\big]$$

$$= a\,\mathrm{Cov}(X,Y)$$

Corollary: ① $\mathrm{Cov}(X, cY+d) = c\,\mathrm{Cov}(X,Y)$

② $\mathrm{Cov}(aX+b, cY+d) = ac\,\mathrm{Cov}(X,Y)$

Theorem:

$$\mathrm{Cor}(aX+b, cY+d) = \mathrm{Sign}(a)\,\mathrm{sign}(c)\,\mathrm{cor}(X,Y)$$

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Ex. $\text{Cor}(-5X+1, Y) = -\text{Cor}(X, Y)$

Corollary: if $a, c > 0$ then

$$\text{Cor}(aX+b, cY+d) = \text{Cor}(X, Y)$$

Pf. $a, c \neq 0$

Fact: if $x \neq 0$ then $\text{sign}(x) = \dfrac{x}{|x|}$

$\text{Cor}(aX+b, cY+d)$

$$= \frac{\text{Cov}(aX+b, cY+d)}{\sqrt{\text{Var}(aX+b)\,\text{Var}(cY+d)}}$$

$$= \frac{ac\,\text{Cov}(X, Y)}{\sqrt{a^2\,\text{Var}(X)\,c^2\,\text{Var}(Y)}}$$

$$= \frac{a}{}\;\frac{c}{}\;\frac{\text{Cov}(X, Y)}{\phantom{\sqrt{\text{Var}(X)\text{Var}(Y)}}}$$

$$= \frac{a}{|a|} \frac{c}{|c|} \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$= \mathrm{sign}(a)\,\mathrm{sign}(c)\ \mathrm{Cor}(X, Y).$$

Theorem: $\boxed{-1 \le \mathrm{Cor}(X, Y) \le 1}$

Pf. $\boxed{\tilde{X} = \frac{X - \mathbb{E}X}{\sqrt{\mathrm{Var}(X)}}}$; $\boxed{\tilde{Y} = \frac{Y - \mathbb{E}Y}{\sqrt{\mathrm{Var}(Y)}}}$

Claim: $\mathbb{E}\tilde{X} = 0$ and $\mathrm{Var}(\tilde{X}) = 1$

$$\tilde{X} = \underbrace{\frac{1}{\sqrt{\mathrm{Var}(X)}}}_{a} X - \underbrace{\frac{\mathbb{E}[X]}{\sqrt{\mathrm{Var}(X)}}}_{b} = aX + b$$

$$\mathbb{E}[\tilde{X}] = a\,\mathbb{E}[X] + b = \frac{1}{\sqrt{\mathrm{Var}(X)}}\mathbb{E}[X] - \frac{\mathbb{E}[X]}{\sqrt{\mathrm{Var}(X)}}$$

$$= 0$$

$$\mathrm{Var}(\tilde{X}) = a^2\,\mathrm{Var}(X) = \left(\frac{1}{\sqrt{\mathrm{Var}(X)}}\right)^2 \mathrm{Var}(X) = 1$$

Note:

$$\boxed{Cor(X,Y)} = Cor(\tilde{X},\tilde{Y}) \quad \text{[by prev. thm]}$$

$$= \frac{Cov(\tilde{X},\tilde{Y})}{\sqrt{Var(\tilde{X})\,Var(\tilde{Y})}} \quad \text{[defn]}$$

both vars = 1

$$= \boxed{Cov(\tilde{X},\tilde{Y})}$$

Note: $\underline{Var(\tilde{X} \pm \tilde{Y})} = \underbrace{Var(\tilde{X})}_{1} + \underbrace{Var(\tilde{Y})}_{1} \pm 2\,Cov(\tilde{X},\tilde{Y})$

$$= \boxed{2 \pm 2\,Cor(X,Y) \geq 0}$$

So $1 \pm Cor(X,Y) \geq 0$

$1 + Cor(X,Y) \geq 0$ $\quad$ $1 - Cor(X,Y) \geq 0$

$\downarrow$ $\quad$ $\downarrow$

$\boxed{Cor(X,Y) \geq -1}$ $\quad$ $\boxed{Cor(X,Y) \leq 1}$

Theorem: Short-Cut Formula for Cov

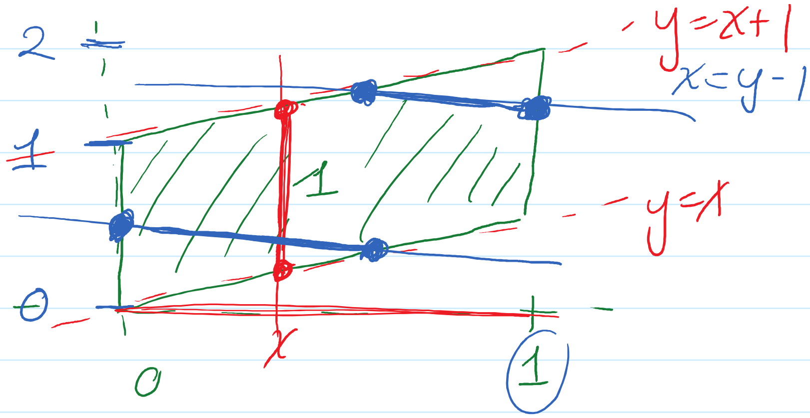
$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

<u>Recall:</u> $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

<u>Ex.</u> $f(x,y) = 1$ for $0 < x < 1$
$x < y < x+1$

<u>Last time:</u>

$\mathbb{E}[XY] = 7/12$



$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y$

need

<u>Marginal of X:</u>

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_x^{x+1} 1\,dy = (x+1) - x = 1$$

for $0 < x < 1$

$X \sim U(0,1)$ so $\boxed{E[X] = 1/2, \ Var(X) = 1/12}$

Marginal of Y:

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \begin{cases} \int_0^y 1\,dx & 0<y<1 \\ \int_{y-1}^1 1\,dx & 1<y<2 \end{cases}$$

$$f_Y(y) = \begin{cases} y & 0<y<1 \\ 2-y & 1<y<2 \end{cases}$$

$$EY = \int_0^2 f_Y(y)\,dy = \cdots \boxed{= 1}$$

$$Var(Y) = \boxed{1/6}$$

$$Cov(X,Y) = E[XY] - EX\,EY = \left(7/12\right) - \left(1/2\right)(1)$$

$$= 1/12$$

$$Cor(X,Y) = \frac{Cov(X,Y)}{\quad} \qquad 1/12$$

$$\operatorname{Cor}(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} = \frac{1/12}{\sqrt{\frac{1}{12}\cdot 1/6}}$$

<u>Conditional Probability:</u>

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$$

If X and Y are discrete

$$A = \{X = x\} \quad \text{and} \quad B = \{Y = y\}$$

$$\overset{f_{X|Y=y}(x)}{\overset{\parallel}{\mathbb{P}(X=x \mid Y=y)}} = \mathbb{P}(A \mid B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$$

$$= \frac{f(x,y)}{f_Y(y)}$$

<u>Defn: Conditional PMF</u>

If X, Y are discrete then <u>the conditional PMF of X given $Y=y$</u> is defined as

$$f(x|y) = f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

The quantity "$X|Y=y$" is just some univariate RV, w/ PMF $f(x|y)$.

Ex. Joint PMF $\to f(x,y)$

Y \ X	0	1	2
30	0	3/18	4/18
20	2/18	4/18	0
10	2/18	3/18	0

$f_X(0) = 4/18$

Let's get PMF of $Y|X=0$.

$$f(y|0) = \frac{f(0,y)}{f_X(0)} = \frac{f(0,y)}{4/18} = \begin{cases} \frac{2/18}{4/18} & y=10 \\ \frac{2/18}{4/18} & y=20 \\ \frac{0}{4/18} & y=30 \end{cases}$$

$$= \begin{cases} 1/2 & y=10 \\ 1/2 & y=20 \end{cases}$$

What about cts RVs?

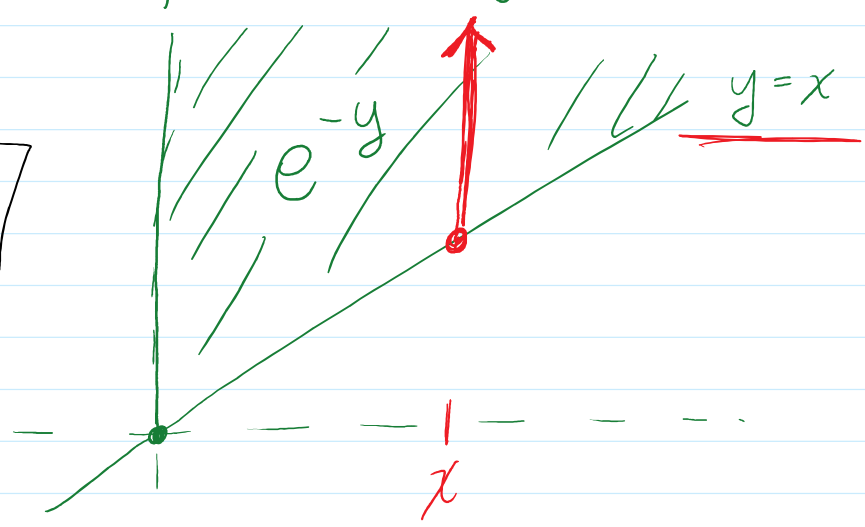
Defn: Conditional PDF

If X and Y are cts then the conditional PDF of X given $Y=y$ is

$$f(x|y) = f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

Ex. $f(x,y) = e^{-y}$ for $0<x<y$

What is the PDF of $\boxed{Y|X=x}$



$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_{y=x}^{\infty} e^{-y}\,dy = -e^{-y}\Big|_x^{\infty}$$

$$= (-0) - (-e^{-x})$$

$\text{Exp}(\lambda = 1)$ $\quad e^{-x}$

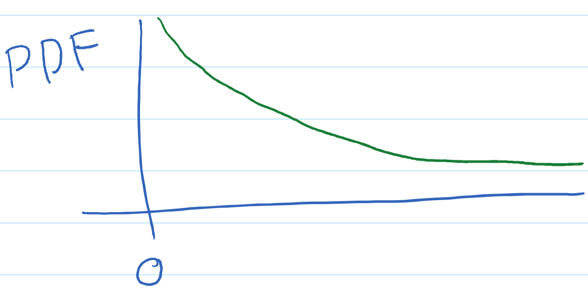
$$\text{Exp}(\lambda = 1) \longleftarrow \boxed{= e^{-x} \text{ for } x > 0}$$

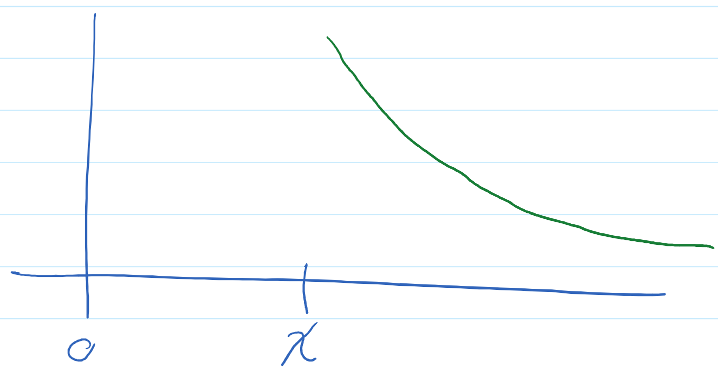
$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} \quad \text{for } 0 < x < y$$

$$\boxed{= e^{-(y-x)} \quad \text{for } y > x}$$

Called a Shifted Exp. dist.

Exp(1)





Defn: Conditional Expectation

If $g: \mathbb{R} \to \mathbb{R}$ then the conditional expectation of $g(X)$ given $Y = y$ is

$$\begin{cases} \sum g(x) f(x|y) & \text{(discrete)} \end{cases}$$

$$\mathbb{E}[g(X) \mid Y = y] = \begin{cases} \sum_x g(x) f(x|y) & \text{(discrete)} \\ \int_{\mathbb{R}} g(x) f(x|y)\, dx & \text{(cts)} \end{cases}$$