Laymen's defn:

→ things don't affect each other

→ events are indep. if the occurence (or not) of one doesn't affect the prob. of other.

Defn: Independence (of Events)

If $A, B \in S$ we say "A is independent of B" denoted $A \perp\!\!\!\perp B$, if

$$\mathbb{P}(AB) = \mathbb{P}(A)\,\mathbb{P}(B).$$

→ independence is a distributive law

→ intuition for product notation for intersection.

Theorem: If $A \perp\!\!\!\perp B$ then

$$\mathbb{P}(A \mid B) = \mathbb{P}(A).$$

Pf.

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\,\cancel{\mathbb{P}(B)}}{\cancel{\mathbb{P}(B)}} = \mathbb{P}(A).$$

Ex. Consider rolling two dice (independently)

$$\mathbb{P}(\text{at least one } 6)$$

$$= 1 - \mathbb{P}(\text{"at least one 6"}^c)$$

$$= 1 - \mathbb{P}(\text{no } 6\text{s})$$

A_1 = no 6 on first roll

A_2 = " second "

$$= 1 - \mathbb{P}(A_1 A_2)$$

by independence

$$= 1 - \mathbb{P}(A_1)\mathbb{P}(A_2)$$

$$= 1 - (5/6)(5/6) = 11/36$$

Counting perspective:

sampling from $\{1, \ldots, 6\}$ $(n = 6)$

two times $(r = 2)$ w/ replacement

Ordered: $|S| = 6^2 = 36$

E = "at least one 6s"

$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5)\}$

$|E| = 11$

$P(E) = 11/36$

Unordered: $|S| = \binom{n+r-1}{r} = \binom{6+2-1}{2} = \binom{7}{2}$

$= 21$

and $E = \{\{1,6\}, \{2,6\}, \{3,6\}, \{4,6\}, \{5,6\}, \{6,6\}\}$

So

$P(E) = 6/21$

Ex. why does ordered counting encode independence?

Roll two dice (independently)

$E = \{$ 1 or 2 on first roll,

3, 4, 5 on second}

Solve w/ counting (in ordered way)

$$|S| = n^r = 6^2 = 36 = 6 \cdot 6$$

$$E = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$$

$$= \{1,2\} \times \{3,4,5\}$$

cartesian product

$$|E| = |\{1,2\} \times \{3,4,5\}|$$

$$= 2 \cdot 3$$

$$P(E) = \frac{|E|}{|S|} = \frac{2 \cdot 3}{6 \cdot 6} = \left(\frac{2}{6}\right)\left(\frac{3}{6}\right)$$

prob. first roll is 1 or 2

second roll is 3, 4, 5

Theorem: Independence and Complements

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If $A \perp\!\!\!\perp B$ then

① $A \perp\!\!\!\perp B^c$

② $A^c \perp\!\!\!\perp B$

③ $A^c \perp\!\!\!\perp B^c$

pf. Case 1:

$$\mathbb{P}(AB^c) = \mathbb{P}(A) - \mathbb{P}(AB)$$
$$= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)$$
$$= \mathbb{P}(A)(1 - \mathbb{P}(B))$$
$$= \mathbb{P}(A)\mathbb{P}(B^c)$$

Defn: Mutual Independence

Generalize independence to multiple events.

If $(A_i)_{i=1}^n$ is a seq of events, we say they are (mutually) independent if for any (all) subsequence $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$

$$\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2})\cdots\mathbb{P}(A_{i_k})$$
$$= \prod_{j=1}^{k} \mathbb{P}(A_{i_j})$$

Q: Do I really have to check *all* possible subsequences?

Can I just check

$$\mathbb{P}(A_1 A_2 \cdots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_n) \ ?$$

No.

Ex, Roll two dice.

$A =$ "doubles" $= \{(1,1),(2,2),\ldots,(6,6)\}$.

$|A| = 6$

$B =$ "Sum between 7 and 10"

$= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),$
$(2,6),(3,5),(4,4),(5,3),(6,2),$
$(3,6),(4,5),(5,4),(6,3),$
$(4,6),(5,5),(6,4)\}$

$|B| = 18$

$C =$ "sum is 2, 7, 8"

C = "sum is 2, 7, 8"

$= \{(1,1), \quad \}$

$|C| = 12$

Mutually Independent?

$$P(ABC) = \underbrace{P(A)}_{4/36 = 1/6}\underbrace{P(B)}_{18/36 = 1/2}\underbrace{P(C)}_{12/36 = 1/3}. \quad \checkmark$$

$$\underbrace{\{(4,4)\}}_{1/36} = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{3}$$

"sum 7 or 8"

$$\underbrace{P(BC)}_{11/36} \neq \underbrace{P(B)}_{1/2}\underbrace{P(C)}_{1/3}$$

doesn't work.

A, B, C not mutually independent.

<u>Defn: Pairwise Independence</u>

If (A_i) is a seq of events, we call them pairwise independent,

$$\mathbb{P}(A_i A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j) \quad \forall\, i \neq j.$$

Can $A \perp\!\!\!\perp A$? Yes, in certain cases.

$$\mathbb{P}(A) = \mathbb{P}(AA) = \mathbb{P}(A)\mathbb{P}(A) = \mathbb{P}(A)^2$$

so $\mathbb{P}(A) = \mathbb{P}(A)^2$ recall, $\mathbb{P}(A) \in [0, 1]$

hence either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

Ex. Pairwise Independence $\neq$ Mutual Independence

$S = \{abc, acb, bac, bca, cab, cba,$
$aaa, bbb, ccc\}$

$|S| = 9$, all equally likely.

$A_i = \{i^{th}$ place has an $a\}$

$A_1 = \{$first is an $a\} = \{abc, acb, aaa\}$

$A_2 = \{\text{second} \quad \text{''} \quad \} = \{bac, cab, aaa\}$

$A_3 = \{\text{third} \quad \text{''} \quad \} = \{bca, cba, aaa\}$

Pairwise independent? ✓

$$P(A_i A_j) = \underbrace{P(A_i)}_{1/3}\,\underbrace{P(A_j)}_{1/3}$$

$$1/9 = 1/3 \quad 1/3$$

Mutually Independent? ✗

$$\underbrace{P(A_1 A_2 A_3)}_{1/9} \neq \underbrace{\underbrace{P(A_1)}_{1/3}\,\underbrace{P(A_2)}_{1/3}\,\underbrace{P(A_3)}_{1/3}}_{1/27}$$

Ex. Serial System

$$\to \boxed{1} \to \boxed{2} \to \boxed{3} \to \cdots \to \boxed{n} \to$$

→ system only works if all components work

→ all components fail independently

W_i = i^{th} component works

W_i^c = i^{th} component fails; $P(W_i^c) = p_i$

P(system works)

$$= \mathbb{P}\left(\bigcap_{i=1}^{n} W_i\right)$$

← defn of mutual independence

$$= \prod_{i=1}^{n} \mathbb{P}(W_i)$$

$$= \prod_{i=1}^{n} (1 - P(W_i^c))$$

$$\boxed{= \prod_{i=1}^{n} (1 - p_i).}$$

Random Variables

Ex. Flip a coin 3 times.

X = # heads among 3 flips

$s \in S$	$X(s)$

$s \in S$	$X(s)$
H H H	3
H H T	2
H T H	2
H T T	1
T H H	2
T H T	1
T T H	1
T T T	0

a function