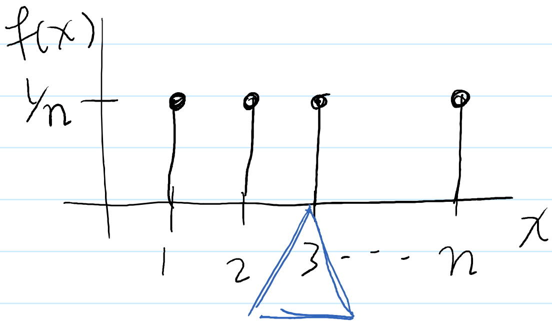
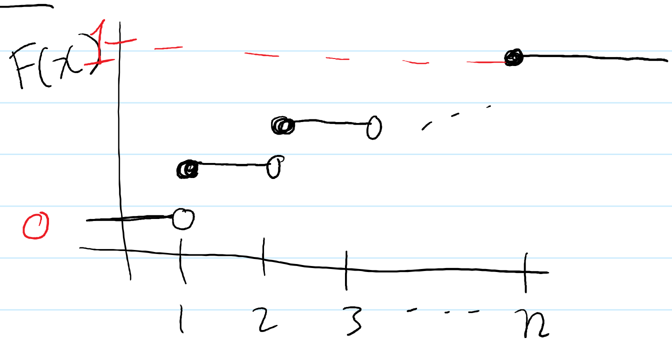
Discrete Uniform

$$X \sim U(\{1, \dots, n\})$$

$$f(x) = \frac{1}{n} \quad \text{for } x = 1, 2, 3, \dots, n$$



CDF:



$$F(x) = \begin{cases} 0, & x < 1 \\ \lfloor x \rfloor / n & 1 \le x < n \\ 1 & x \ge n \end{cases}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad *$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Expectation:

$$E[X] = \sum_{x=1}^{n} x\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{x=1}^{n} x = \frac{1}{n}\,\frac{n(n+1)}{2} = \boxed{\frac{n+1}{2}}$$

$$E[X^2] = \sum_{x=1}^{n} x^2\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{1}{n}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \cdots = \boxed{\frac{n^2-1}{12}}$$

MGF:

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=1}^{n} e^{tx}\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{x=1}^{n}(e^t)^x$$

Geometric Sum:

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r} \quad \text{for } r \neq 1$$

$$r = e^t$$

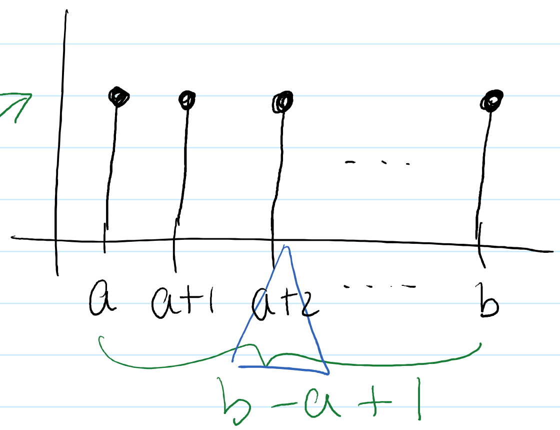
$$\frac{1}{n}\sum_{x=1}^{n} r^x = \frac{1}{n}\sum_{x=0}^{n-1} r^{x+1} = \frac{r}{n}\sum_{x=0}^{n-1} r^x = \frac{r}{n}\,\frac{1-r^n}{1-r}$$

$$= \frac{e^t(1-(e^t)^n)}{n(1-e^t)}$$

$$\boxed{= \frac{e^t - e^{t(n+1)}}{n(1-e^t)}}$$

$$= \frac{e^{t} - e^{\dots}}{n(1-e^{t})}$$

Consider $Y \sim U(\{a, \dots, b\})$



Let $n = b - a + 1$

then $\boxed{Y = X + (a-1)}$

linear transformation

$$f(y) = \frac{1}{b-a+1} \text{ for } y = a, \dots, b$$

$$\mathbb{E}[Y] = \mathbb{E}[X + (a-1)] = \mathbb{E}[X] + (a-1)$$

$$= \frac{n+1}{2} + (a-1)$$

$$= \frac{(b-a+1)+1}{2} + (a-1)$$

$$= \dots = \boxed{\frac{a+b}{2}}$$

$$\mathrm{Var}(Y) = \mathrm{Var}(X + a - 1)$$

$$= \mathrm{Var}(X) = \frac{n^2 - 1}{\ } = \frac{(b-a+1)^2 - 1}{\ }$$

$$= \operatorname{Var}(X) = \frac{n^2-1}{12} = \boxed{\frac{(b-a+1)^2-1}{12}}$$

$Y = X + a - 1$

$$M_Y(t) = e^{(a-1)t} M_X(t)$$

$$= e^{(a-1)t}\,\frac{e^t - e^{t(n+1)}}{n(1-e^t)}$$

$$= \boxed{\frac{e^{at} - e^{t(b+1)}}{(b-a+1)(1-e^t)}}$$

Continuous Uniform

$X \sim U(a,b)$

PDF:

$f(x) = \frac{1}{b-a}$ for $a < x < b$

$a < x < b$

CDF:

$F(x) = \int^{x} \quad = \int^{x} 1 \, dt \quad = t \Big|^{x}$

CDF:

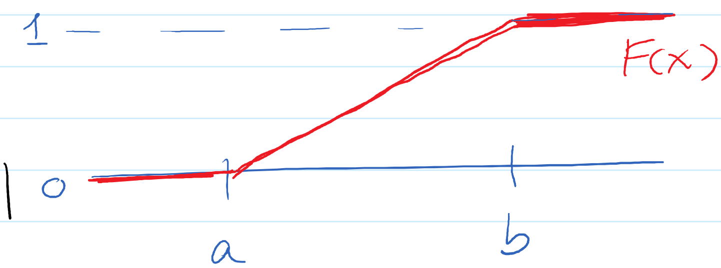
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt = \frac{t}{b-a}\Big|_{a}^{x}$$

$$= \frac{x-a}{b-a}$$

If $x < a$

$F(x) = 0$

If $x \geq b$ then $F(x) = 1$



Expectation

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_{a}^{b} x \frac{1}{b-a}\,dx$$

$$= \frac{x^2}{2(b-a)}\Big|_{a}^{b} = \frac{b^2 - a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

$$\boxed{= \frac{a+b}{2}}$$

$$\mathbb{E}[X^2] = \int_{a}^{b} x^2 \frac{1}{b-a}\,dx = \frac{x^3}{3(b-a)}\Big|_{a}^{b}$$

$$3(b-a)\ \Big|_a$$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{\cancel{(b-a)}(a^2+ab+b^2)}{3\cancel{(b-a)}}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \cdots = \boxed{\frac{(b-a)^2}{12}}$$

<u>MGF:</u>

$$M(t) = \mathbb{E}[e^{tX}] = \int_a^b e^{tx}\frac{1}{b-a}\,dx$$

$$= \frac{e^{tx}}{t(b-a)}\Bigg|_a^b = \boxed{\frac{e^{bt} - e^{at}}{t(b-a)}}$$

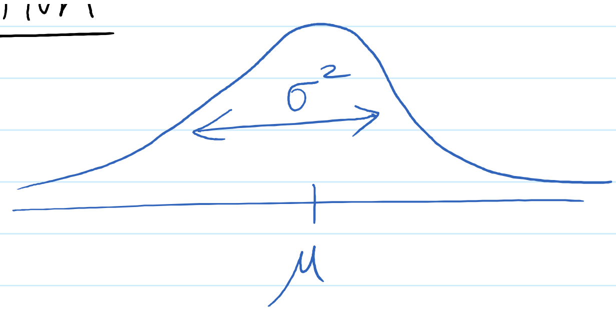
Normal / Gaussian Distribution

$X \sim N(\mu, \sigma^2)$

Normal / Gaussian Distribution

$$X \sim N(\mu, \sigma^2)$$

$\mu \in \mathbb{R}$, $\sigma^2 > 0$



PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad \forall x \in \mathbb{R}$$

CDF: $F(x) = \int_{-\infty}^{x} f(t)\,dt$ no simple closed form for this

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Claim: $E[X] = \mu$ and $Var(X) = \sigma^2$.

MGF:

$$e^a e^b = e^{a+b}$$

$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \quad (*)$$

Combine exponents:

$$tx - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$= tx - \frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)$$

$$= -\frac{1}{2\sigma^2}\left(-2\sigma^2 t x + x^2 - 2\mu x + \mu^2\right)$$

looks like first two terms of $(x-(\mu+\sigma^2 t))^2$

$$= -\frac{1}{2\sigma^2}\left(\underbrace{x^2 - 2x(\mu+\sigma^2 t)} + \mu^2\right)$$

$$= -\frac{1}{2\sigma^2}\left(\underbrace{x^2 - 2x(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2}_{(x-(\mu+\sigma^2 t))^2} - (\mu+\sigma^2 t)^2 + \mu^2\right)$$

$$= -\frac{1}{2\sigma^2}\left([x-(\mu+\sigma^2 t)]^2 - (\mu+\sigma^2 t)^2 + \mu\right)$$

$$e^{a+b} = e^a e^b$$

$$(*) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\quad)\, dx$$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2 t)]^2\right)}_{\text{PDF of } N(\mu+\sigma^2 t,\, \sigma^2)} \underbrace{\exp\left(-\frac{1}{2\sigma^2}\left(-(\mu+\sigma^2 t)^2 + \mu^2\right)\right)}_{\text{doesn't depend on } x} dx$$

integrates to 1

$$= \exp\left(-\frac{1}{2\sigma^2}\left(-(\mu+\sigma^2 t)^2 + \mu^2\right)\right) \cdot 1$$

$$= \cdots = \boxed{\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) = M(t)}$$

$$E[X] = \left.\frac{dM}{dt}\right|_{t=0} = \left.\underbrace{(\mu + \sigma^2 t)}\,\underbrace{\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}\right|_{t=0}$$

$$= \mu$$

$$E[X^2] = \left.\frac{d^2M}{dt^2}\right|_{t=0} = \left.\sigma^2 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) + (\mu + \sigma^2 t)^2 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\right|_{t=0}$$

$$= \sigma^2(1) + (\mu + 0)^2(1)$$

$$= \sigma^2 + \mu^2$$

$$Var(X) = E[X^2] - E[X]^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2.$$

Theorem: Linear Transformations of Normal RVs

If $X \sim N(\mu, \sigma^2)$ and

$$Y = aX + b$$

then $Y \sim N(a\mu+b,\ a^2\sigma^2)$

intuition: $\mathbb{E}[Y] = \mathbb{E}[aX+b] = a\mathbb{E}X + b = a\mu + b$

$\mathrm{Var}(Y) = \mathrm{Var}(aX+b) = a^2\mathrm{Var}(X) = a^2\sigma^2$

Pf. Recall $M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

we also have a theorem that says

$$M_Y(t) = e^{bt} M_X(at)$$

$$= e^{bt} \exp\left(\mu a t + \frac{\sigma^2 a^2 t^2}{2}\right)$$

$$= \exp\left((a\mu+b)t + \frac{(a^2\sigma^2)t^2}{2}\right)$$

this is just the MGF of a $N(a\mu+b,\ a^2\sigma^2)$