Defn: Conditional PMF/PDF

Given X and Y then the conditional PMF/PDF of $X \mid Y = y$ is

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Defn: Conditional Expectation

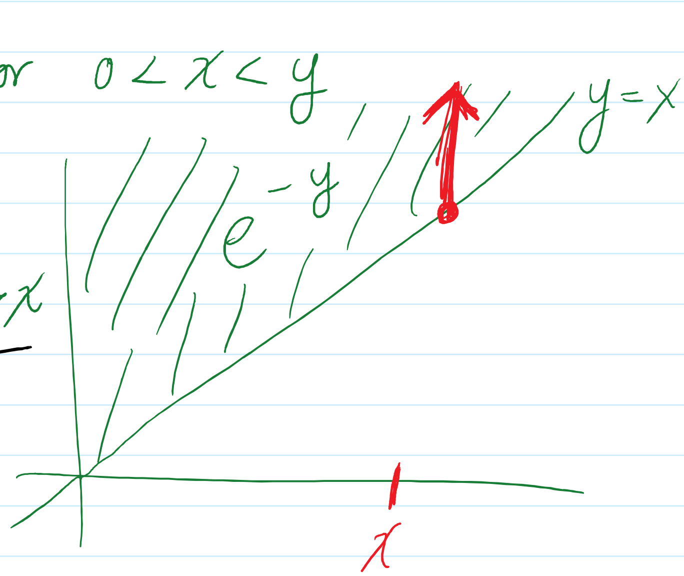
If $g: \mathbb{R} \to \mathbb{R}$ then the conditional expectation of $g(X)$ given $Y = y$ is

$$\mathbb{E}[g(X) \mid Y = y] = \begin{cases} \sum_x g(x) f(x|y) & \text{discrete} \\ \int_{\mathbb{R}} g(x) f(x|y)\,dx & \text{cts} \end{cases}$$

Ex. $f(x,y) = e^{-y}$ for $0 < x < y$

Had shown:

$f(y|x) = e^{-(y-x)}$ for $\underline{y > x}$



$$\mathbb{E}[Y|X=x]$$

$$= \int_{\mathbb{R}} y f(y|x)\,dy$$

$$= \int_x^\infty y e^{-(y-x)}\,dy = \cdots = 1+x$$

Defn: Conditional Variance

$$\mathrm{Var}(Y|X=x) = \mathbb{E}[(Y - \mathbb{E}[Y|X=x])^2 \mid X=x]$$

Short-cut formula:

$$\mathrm{Var}(Y|X=x) = \mathbb{E}[Y^2|X=x] - \mathbb{E}[Y|X=x]^2$$

Ex. $\mathbb{E}[Y^2|X=x] = \int_{\mathbb{R}} y^2 f(y|x)\,dy$

$$= \int_x^\infty y^2 e^{-(y-x)}\,dy = \cdots = x^2+2x+2$$

$$\mathrm{Var}(Y \mid X=x) = \mathbb{E}[Y^2 \mid X=x] - \mathbb{E}[Y \mid X=x]^2$$

$$= (x^2+2x+2) - (1+x)^2$$

$$= \cancel{x^2} + \cancel{2x} + 2 - \cancel{x^2} - \cancel{2x} - 1$$

$$= 2-1$$

$$= 1$$

Independence

For events: If $A, B \subset S$ then

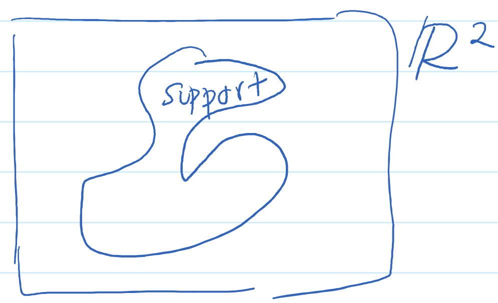
$$A \perp\!\!\!\perp B \Leftrightarrow \mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$$

For RVs

$$X \perp\!\!\!\perp Y \Leftrightarrow \mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$
$$\forall A, B \subset \mathbb{R}$$

Product Spaces

$$\text{Support}(X,Y) = \{(x,y) \mid f(x,y) > 0\}$$

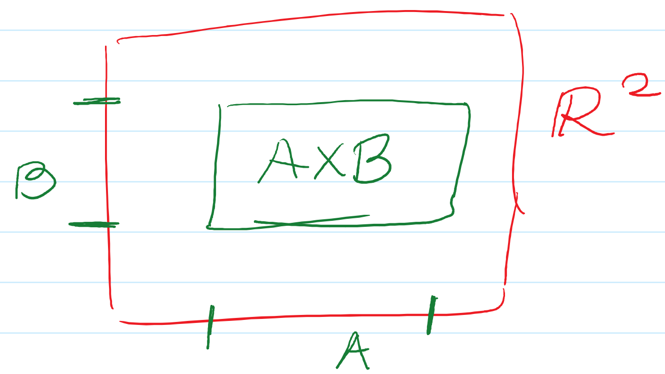


If $f(x,y) =$ ~~~~

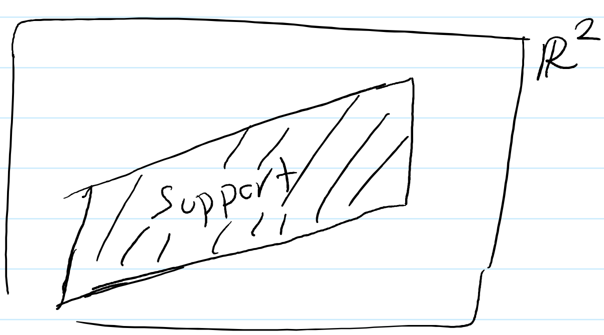
for $x \in A$ and $y \in B$

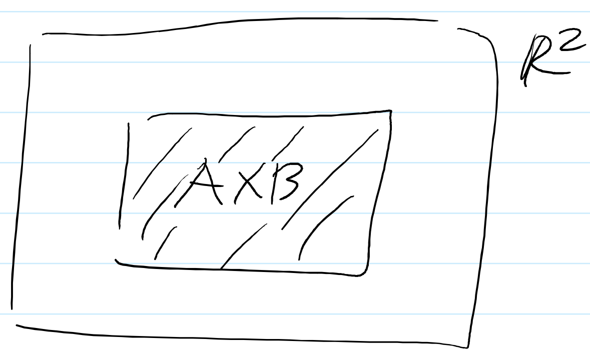
doesn't depend on y doesn't depend on x

then the support is a product space $A \times B$



Two different support examples





Theorem: Factorization Theorem

$X \perp\!\!\!\perp Y$

iff

① support of X and Y is a product space

② either $F(x,y) = F_X(x)\,F_Y(y)$

or $f(x,y) = f_X(x)\,f_Y(y)$

Ex.

Y \ X	10	20	f_Y
3	1/5	3/10	5/10
2	1/5	1/10	3/10
1	1/10	1/10	2/10
f_X	1/2	1/2	

Q: $X \perp\!\!\!\perp Y$?

① Product space? Yes: $A = \{10, 20\}$, $B = \{1, 2, 3\}$

then $A \times B$ is my support

② $f(x,y) = f_X(x)\,f_Y(y)$?

Ex. $f(10,3) = \frac{1}{5} \overset{?}{\cancel{=}} f_X(10)\,f_Y(3) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

$\underline{\text{Ex.}}$ $f(10,3) = \frac{1}{5} \neq f_X(10)\, f_Y(3) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

So $X \not\perp\!\!\!\perp Y$.

$\underline{\text{Corollary:}}$

$X \perp\!\!\!\perp Y$

iff

① support is a product space

② $f(x,y) = g(x)\,h(y)$

some fn only of x

h fn only of y

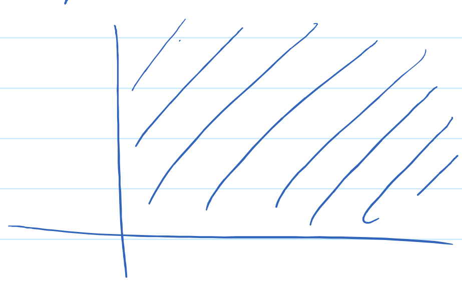
$\underline{\text{Ex.}}$

$$f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)} \qquad x>0,\ y>0$$

$\underline{X \perp\!\!\!\perp Y?}$

① product space? Yes

Support $= (0,\infty) \times (0,\infty)$



② $$f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)}$$

$$= \frac{1}{384} x^2 e^{-y} e^{-(x/2)}$$

$$= \underbrace{\left(\frac{1}{384} x^2 e^{-x/2}\right)}_{g(x)} \underbrace{\left(e^{-y}\right)}_{h(y)}$$

So $X \perp\!\!\!\perp Y$.

w/o this theorem, need

$$f_X(x) = \int_0^\infty \frac{1}{384} x^2 e^{-(x/2)} e^{-y} \, dy$$

$$f_Y(y) = \int_0^\infty \frac{1}{384} x^2 e^{-x/2} e^{-y} \, dx$$

<u>Fact:</u> $A, B \subset S$ and $A \perp B$ then

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)$$

<u>For RVs:</u> $X \perp\!\!\!\perp Y$ then

$$f(x|y) = f_X(x).$$

<u>Pf.</u> $$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)$$

<u>Theorem:</u> Expectation of Product of Independent

If $X \perp\!\!\!\perp Y$ and $g_1 : \mathbb{R} \to \mathbb{R}$, $g_2 : \mathbb{R} \to \mathbb{R}$ then

$$\mathbb{E}[g_1(X)g_2(Y)] = \mathbb{E}[g_1(X)]\mathbb{E}[g_2(Y)].$$

<u>Pf.</u> (cts)

$$\mathbb{E}[g_1(X)g_2(Y)] = \iint_{A \times B} g_1(x)g_2(y)f(x,y)\,dx\,dy$$

$\updownarrow$ independence

$$= \iint g_1(x)g_2(y)f_X(x)f_Y(y)\,dx\,dy$$

$$= \iint_{A \times B} g_1(x) g_2(y) f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_B \boxed{\int_A g_1(x) f_X(x)\, dx}\, g_2(y) f_Y(y)\, dy$$

$$= \underbrace{\int_A g_1(x) f_X(x)\, dx}_{\mathbb{E}[g_1(X)]} \underbrace{\int_B g_2(y) f_Y(y)\, dy}_{\mathbb{E}[g_2(Y)]}$$

<u>Ex.</u> $X, Y \overset{iid}{\sim} Exp(1)$

independent identically distributed

<u>means:</u> $X \perp\!\!\!\perp Y$, $X \sim Exp(1)$
$Y \sim Exp(1)$

$$\mathbb{E}[X^2 Y] = \underbrace{\mathbb{E}[X^2]}_{(1+1)} \underbrace{\mathbb{E}[Y]}_{(1)} = (2)(1) = 2$$

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

rearrange

$$\mathbb{E}[X^2] = Var(X) + (\mathbb{E}X)^2$$

<u>Theorem: MGF of Sum of Independent</u>

Theorem: MGF of Sum of Independent

If $X \perp Y$ then

$$M_{X+Y}(t) = M_X(t) M_Y(t).$$

Pf:
$$M_X(t) = \mathbb{E}[e^{tX}]$$
$$M_Y(t) = \mathbb{E}[e^{tY}]$$

$$M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}]$$
$$= \mathbb{E}[e^{tX}e^{tY}]$$
$$= \underbrace{\mathbb{E}[e^{tX}]}_{M_X(t)}\underbrace{\mathbb{E}[e^{tY}]}_{M_Y(t)}$$

Ex. $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\gamma, \tau^2)$

assume $X \perp\!\!\!\perp Y$.

What is the dist. of $X + Y$?

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} e^{\gamma t + \frac{\tau^2 t^2}{2}}$$

$$= e^{\mu t + \gamma t + \frac{\sigma^2 t^2}{2} + \frac{\tau^2 t^2}{2}}$$

$$= e^{(\mu+\gamma)t + (\sigma^2+\tau^2)t^2/2}$$

MGF of a $N(\mu+\gamma, \sigma^2+\tau^2)$.

i.e. $\boxed{X + Y \sim N(\mu+\gamma, \sigma^2+\tau^2)}$