Defn: Conditional PMFs/PDFs

Given RVs X and Y the conditional PMF/PDF of $X \mid Y = y$ is

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Defn: Conditional Expectation

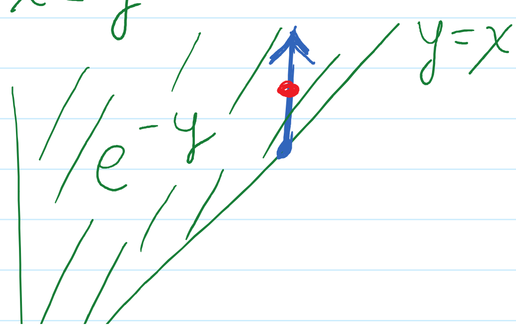
If $g: \mathbb{R} \to \mathbb{R}$ then the conditional expectation of $g(X)$ given $Y = y$ is

$$\mathbb{E}\left[g(X) \mid Y = y\right] = \begin{cases} \sum_x g(x) f(x|y) & \text{(discrete)} \\ \int_R g(x) f(x|y)\, dx & \text{(cts)} \end{cases}$$

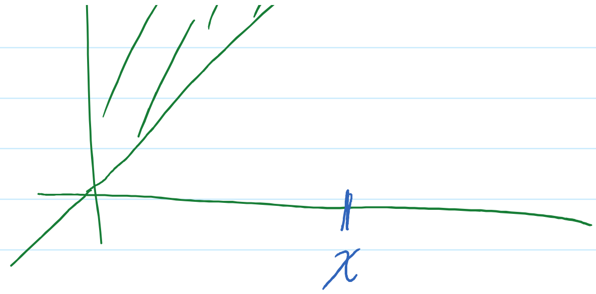
Ex. $f(x,y) = e^{-y}$ for $0 < x < y$

Last time:

$f(y|x) = e^{-(y-x)}$ for $y > x$



for $y > x$



$$E[Y \mid X = x]$$
$$= \int_{\mathbb{R}} y f(y \mid x)\, dy$$
$$= \int_x^\infty y e^{-(y-x)}\, dy = \cdots = 1 + x$$

Defn : Conditional Variance

$$\mathrm{Var}(Y \mid X = x) = \mathbb{E}\left[\left(Y - \mathbb{E}[Y \mid X = x] \right)^2 \,\middle|\, X = x \right]$$

Short-cut formula:

$$\mathrm{Var}(Y \mid X = x) = \mathbb{E}[Y^2 \mid X = x] - \mathbb{E}[Y \mid X = x]^2$$

Ex, continue from above

$$\mathbb{E}[Y^2 \mid X = x] = \int_{\mathbb{R}} y^2 f(y \mid x)\, dy$$
$$= \int_x^\infty y^2 e^{-(y-x)}\, dy = x^2 + 2x + 2$$

$$Var(Y|X=x) = E[Y^2|X=x] - E[Y|X=x]^2$$

$$= (x^2+2x+2) - (1+x)^2$$

$$= \cancel{x^2} + \cancel{2x} + 2 - \cancel{x^2} - \cancel{2x} - 1$$

$$= 1$$

<u>Independence:</u>

<u>For events:</u> If $A, B \subset S$ then

$$A \perp\!\!\!\perp B \iff P(AB) = P(A)P(B)$$

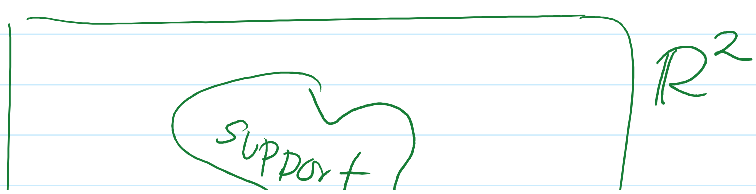
<u>For RVs:</u>

$$X \perp\!\!\!\perp Y \iff P(X \in A, Y \in B) = P(X \in A)P(Y \in B) \quad \forall A, B \subset R$$

<u>Product Spaces</u>

$$\text{Support}(X, Y) = \{(x,y) \mid f(x,y) > 0\}$$

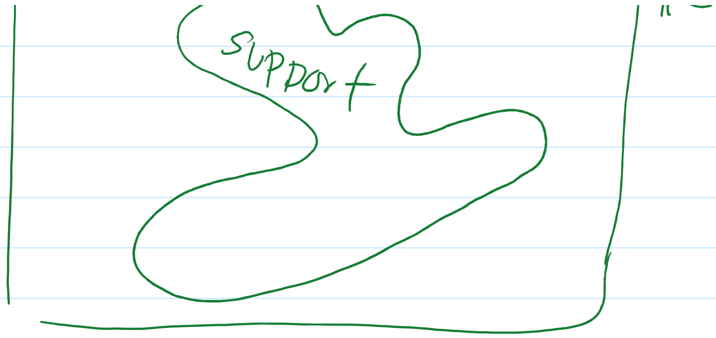
$\boxed{\text{If } f(x,y) = \ldots}$



If $f(x,y) = \text{~~~~}$ for $x \in A$ and $y \in B$

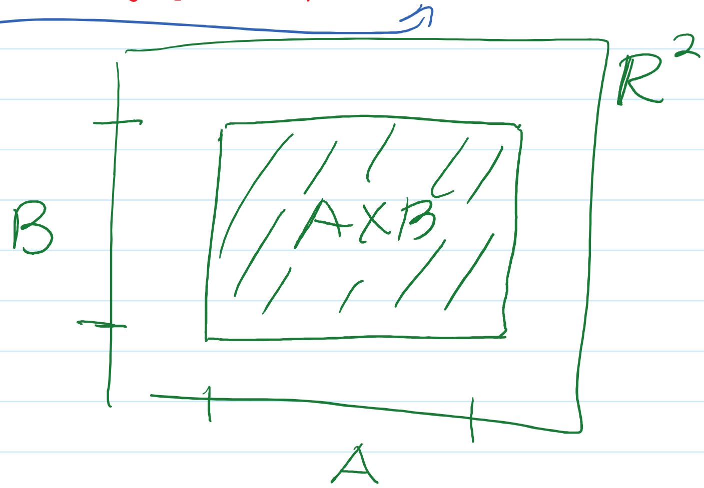
doesn't depend on y

doesn't depend on x

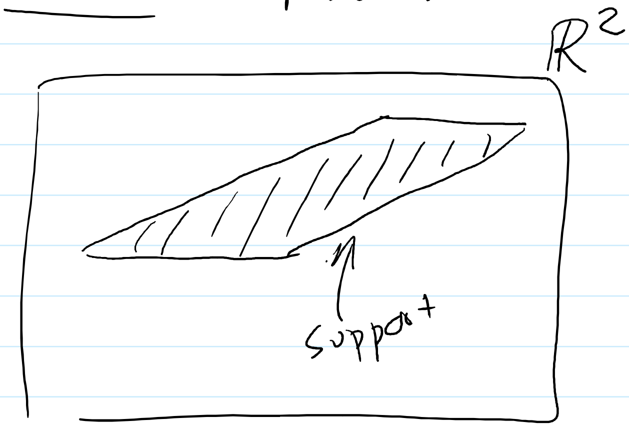


then the support is the set $A \times B$.

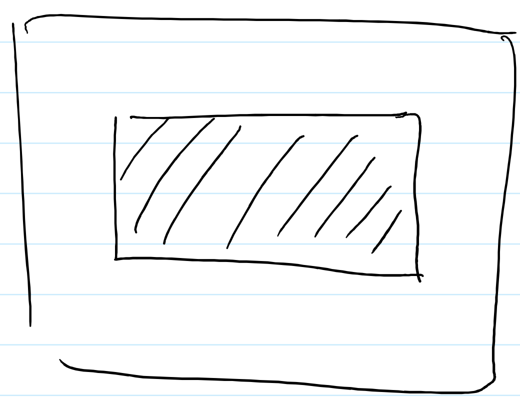
call a product space



Recall: Rhombus



product space



Theorem: Factorization Theorem

X and Y are independent

iff

① Support of X and Y is a product space

AND

② either $F(x,y) = F_X(x) F_Y(y)$

or $f(x,y) = f_X(x) f_Y(y)$

Ex.

Y \ X	10	20	f_Y
f_X	$1/2$	$1/2$	
3	$1/5$	$3/10$	$1/2$
2	$1/5$	$1/10$	$3/10$
1	$1/10$	$1/10$	$2/10$

$f(x,y)$

Q: $X \perp\!\!\!\perp Y$?

✓ ① Product space? $A = \{10, 20\}$, $B = \{1, 2, 3\}$

Support is $A \times B$

② $f(x,y) = f_X(x) f_Y(y) \quad \forall x, y$

Ex. $f(10,3) = \frac{1}{5} \overset{?}{\neq} f_X(10) f_Y(3) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

So not independent.

Corollary: If support of X and Y is a product space

$$X \perp\!\!\!\perp Y \iff f(x,y) = h(x)\,g(y)$$

fn only of x fn only of y

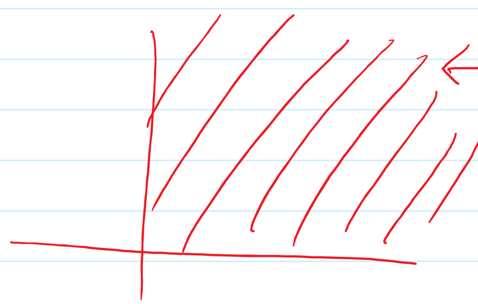
Don't even need to get f_X and f_Y.

Ex.

$$f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)} \quad \text{for } x>0,\ y>0$$

$X \perp\!\!\!\perp Y$?

(1) Product Space? Yes. Support $= (0,\infty) \times (0,\infty)$



← rectangle

(2) $f(x,y) = h(x)\,g(y)$

$$\frac{1}{384} x^2 e^{-y-(x/2)}$$

$$= \frac{1}{384} x^2 e^{-y} e^{-(x/2)}$$

$$= \left(\frac{1}{384} x^2 e^{-x/2}\right)\left(e^{-y}\right)$$

First ver. $f(x,y) = f_X(x) f_Y(y)$

$$f_X(x) = \int_0^\infty \frac{1}{384} x^2 e^{-y-(x/2)}\, dy$$

$$\underbrace{\qquad}_{h(x)} \quad \underbrace{\qquad}_{g(y)} \quad \checkmark \quad f_Y(y) = \ldots$$

So $X \perp\!\!\!\perp Y$. $\quad g(y) \propto f_Y(y)$ and $h(x) \propto f_X(x)$

Fact:

For events: $A \perp\!\!\!\perp B$ then $P(A|B) = P(A)$

$$\boxed{\text{For } \underline{\text{RVs}}: X \perp\!\!\!\perp Y \text{ then } f(x|y) = f_X(x)}$$

pf. $$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)$$

Theorem: Expectation of Independent

If $X \perp\!\!\!\perp Y$ and $g_1 : \mathbb{R} \to \mathbb{R}$, $g_2 : \mathbb{R} \to \mathbb{R}$

then

$$\mathbb{E}[g_1(X)g_2(Y)] = \mathbb{E}[g_1(X)]\mathbb{E}[g_2(Y)]$$

Pf. (cts case)

$$\mathbb{E}[g_1(X)g_2(Y)]$$

$$= \iint_{\mathbb{R}^2} g_1(x)g_2(y)f(x,y)\,dx\,dy$$

independence

$$\iint_{\mathbb{R}^2}$$

independence

$$= \iint_{\mathbb{R}} g_1(x)\, g_2(y)\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{\mathbb{R}^2} \boxed{\int g_1(x)\, f_X(x)\, dx}\; g_2(y)\, f_Y(y)\, dy$$

$$= \left[\int_{\mathbb{R}} g_1(x)\, f_X(x)\, dx\right] \left[\int_{\mathbb{R}} g_2(y)\, f_Y(y)\, dy\right]$$

$$= \mathbb{E}[g_1(X)]\, \mathbb{E}[g_2(Y)]$$

Ex. $X, Y \overset{iid}{\sim} \text{Exp}(\lambda = 1)$

independent identically distributed

1. $X \perp\!\!\!\perp Y$
2. $X \sim \text{Exp}(1)$, $Y \sim \text{Exp}(1)$

$$\mathbb{E}[X^2 Y] \underset{X \perp Y}{=} \mathbb{E}[X^2]\, \mathbb{E}[Y] = (\text{Var}(X) + \mathbb{E}[X]^2)\, \mathbb{E}[Y]$$
$$= (1+1)(1) = 2$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

re arrange

$$\mathbb{E}[X^2] = Var(X) + \mathbb{E}[X]^2$$

Theorem: MGF of Independent

If $X \perp\!\!\!\perp Y$ then

MGFs of X, Y resp.

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

MGF of sum

pf. Recall: $\boxed{\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}] \\ M_Y(t) &= \mathbb{E}[e^{tY}] \end{aligned}}$

$e^{a+b} = e^a e^b$

algebra

$$M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[e^{tX} e^{tY}]$$

independence

$$= \mathbb{E}[e^{tX}]\mathbb{E}[e^{tY}]$$

$$= M_X(t) M_Y(t)$$

Ex. Let $X \sim N(\mu, \sigma^2)$, $Y \sim N(\gamma, \tau^2)$ $\Big\} X \perp\!\!\!\perp Y$

What is dist of $X+Y$?

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

$e^{a+b} = e^a e^b$

$$\left(\ldots \frac{\sigma^2 t^2}{\ldots} \right) \left(\gamma t + \frac{\tau^2 t^2}{\ldots} \right)$$

$$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$

$$= \left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)\left(e^{\gamma t + \frac{\tau^2 t^2}{2}}\right)$$

$$= e^{\mu t} e^{\gamma t} e^{\frac{\sigma^2 t^2}{2}} e^{\frac{\tau^2 t^2}{2}}$$

$$\boxed{= e^{(\mu+\gamma)t + (\sigma^2+\tau^2)t^2/2}}$$

Recognize this is the MGF of

$$N(\mu+\gamma,\ \sigma^2+\tau^2)$$

So $\boxed{X+Y \sim N(\mu+\gamma,\ \sigma^2+\tau^2)}$