Theorem: Law of Total Variance

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y])$$

Ex. $P \sim \text{Beta}(\alpha, \beta)$

$X | P = p \sim \text{Bin}(n, p)$

$\text{Var}(X)$?

1. $\mathbb{E}[X|P=p] = np$

   $\text{Var}(X|P=p) = np(1-p)$

2. $\mathbb{E}[X|P] = nP$

   $\text{Var}(X|P) = nP(1-P)$

3. $\text{Var}(X) = \mathbb{E}[\text{Var}(X|P)] + \text{Var}(\mathbb{E}[X|P])$

$$= \mathbb{E}[nP(1-P)] + \text{Var}(nP)$$

$$= n\mathbb{E}[P(1-P)] + n^2\text{Var}(P)$$

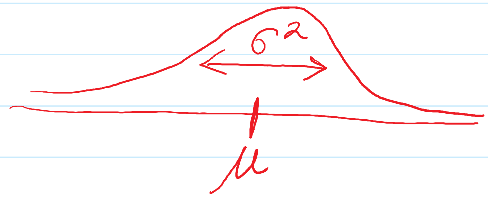
$$= n(\mathbb{E}[P] - \mathbb{E}[P^2]) + n^2\text{Var}(P)$$

$$= \ldots$$

$$= n \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)} + n^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

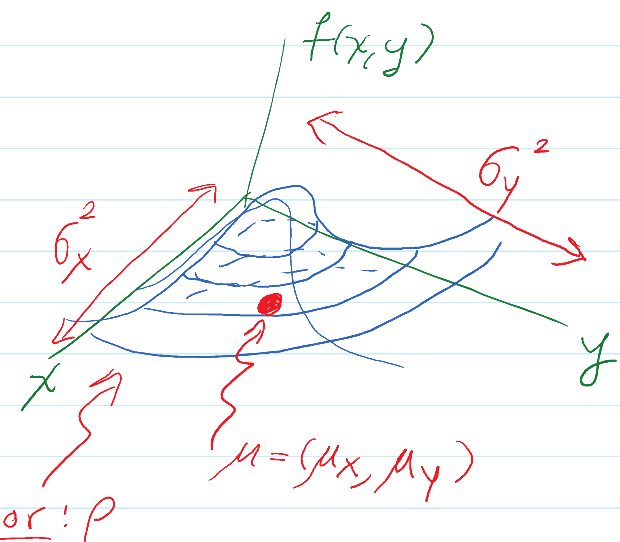
Bivariate Normal Distribution

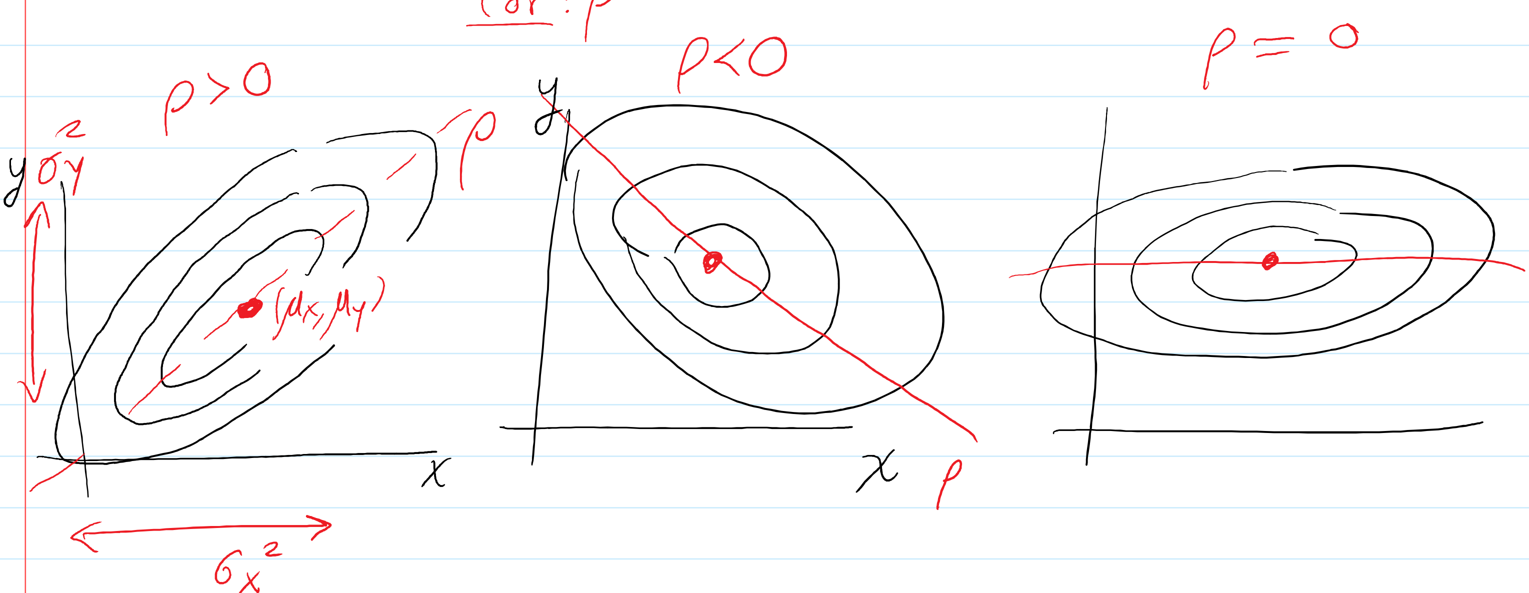
Uni: $N(\mu, \sigma^2)$



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad \forall x \in \mathbb{R}$$

Bivariate:





PDF: $(X, Y) \sim BivN(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\sqrt{1-\rho^2}}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right\}$$

Alt: $\mu = (\mu_X, \mu_Y)$ — mean vector

covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \sigma_X\sigma_Y\rho \\ \sigma_X\sigma_Y\rho & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}$$

$z = (x, y)$

$$f(z) = \frac{1}{2\pi} \frac{1}{\sqrt{\det \Sigma}} \exp\left\{-\frac{1}{2}(z-\mu)^T \Sigma^{-1} (z-\mu)\right\}$$

Uni!

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma^2}} \exp\left\{-\frac{1}{2}(x-\mu)(\sigma^2)^{-1}(x-\mu)\right\}$$

Facts:

1) $X \sim N(\mu_X, \sigma_X^2)$

   $Y \sim N(\mu_Y, \sigma_Y^2)$

2) $Cor(X, Y) = \rho$

③ $aX + bY \sim N(a\mu_X + b\mu_Y,\ a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_X\sigma_Y\rho)$

④ $(X, Y) \sim \text{BivN} \iff \forall a, b \quad aX + bY \sim N$

⑤ Previously: If $X \perp Y$ then $\text{Cor}(X, Y) = 0$

If $(X, Y) \sim \text{BivN}$ and $\rho = 0$ then $X \perp Y$.

Uni: $g : \mathbb{R} \to \mathbb{R}$ what is the dist of $g(X)$?

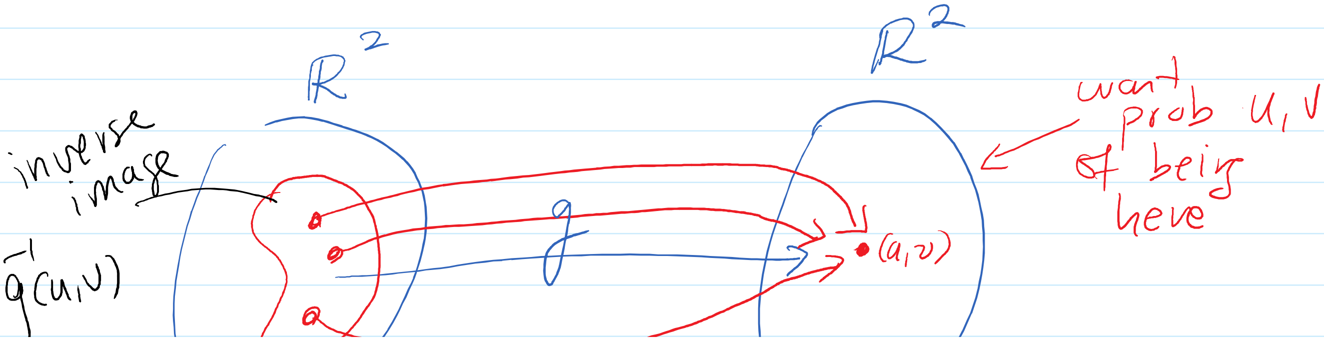
Biv: $g : \mathbb{R}^2 \to \mathbb{R}^2$ what is the dist. of $g(X, Y)$?

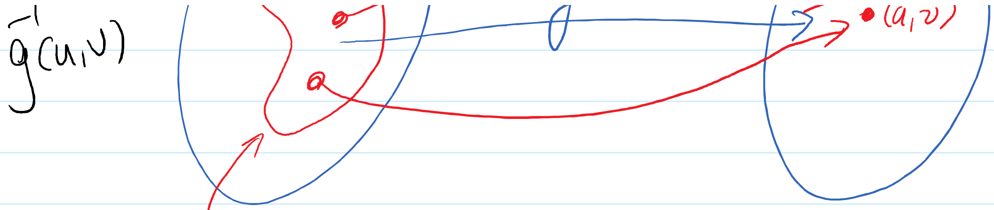
notation: $(X, Y) \xrightarrow{g} (U, V)$

Ex. $(U, V) = (\underbrace{X^2 Y}_{g_1(X,Y)},\ \underbrace{-\log Y}_{g_2(X,Y)}) = g(X, Y)$

Discrete RVs

Assume X and Y are discrete.





instead: calc. prob. of X, Y being here

Want: $\underbrace{P(U=u, V=v)}_{\text{PMF of } U,V}$ from $\underbrace{\mathbb{P}(X= \ldots, Y= \ldots)}_{\text{PMF of } X,Y}$

$$f_{U,V}(u,v) = P(U=u, V=v)$$
$$= P((U,V) \in \{(u,v)\})$$
$$= P(g(X,Y) \in \{(u,v)\})$$
above picture
$$= P((X,Y) \in \underbrace{g^{-1}(u,v)}_{A})$$
$$= \sum_{(x,y) \in g^{-1}(u,v)} f_{X,Y}(x,y)$$

$$\boxed{= \sum_{x,y \,:\, g(x,y) = (u,v)} f_{X,Y}(x,y)}$$

If g invertible then $g^{-1}(u,v)$ is the true inverse and is a single point

$\cup$ and is a single point

$$= \boxed{f_{X,Y}(g^{-1}(u,v))} = \boxed{f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v))} \leftarrow$$

Ex, let $X \perp\!\!\!\perp Y$ and

$$\left.\begin{array}{l} X \sim Pois(\theta) \\ Y \sim Pois(\lambda) \end{array}\right\} \text{discrete}$$

So

independent

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \underbrace{\overbrace{\frac{\theta^x e^{-\theta}}{x!}}^{f(x)} \overbrace{\frac{\lambda^y e^{-\lambda}}{y!}}^{f(y)}}$$

Let $U = X + Y$ and $V = Y$ — $U = V + \text{Something} \geq 0$

$U \geq V$

$$(u,v) = g(x,y) = (x+y, y)$$

$$u = g_1(x,y) = x+y$$

$$v = g_2(x,y) = y$$

Lets get inverse,

notice $u - v = x + y - y = x$

So $\boxed{x = g_1^{-1}(u,v) = u - v}$

$$\boxed{y = v = g_2^{-1}(u,v)}$$

$$\boxed{y = v = g_2^{-1}(u,v)}$$

$$f(u,v) = f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v))$$

$$= f_{X,Y}(u-v, v)$$

$$\boxed{= \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^{v} e^{-\lambda}}{v!} \quad \text{for } u \geq v}$$

Could get marginal of u.

$$f_U(u) = \sum_v f(u,v) = \sum_{v=0}^{u} \frac{\theta^{u-v} e^{-\theta} \lambda^{v} e^{-\lambda}}{(u-v)!\, v!}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^{u} \underbrace{\frac{u!}{(u-v)!\, v!}}_{\binom{u}{v}} \lambda^{v} \theta^{u-v}$$

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^{u} \binom{u}{v} \lambda^{v} \theta^{u-v}$$

$$\boxed{f_U(u) = \frac{e^{-(\theta+\lambda)}}{u!} (\theta+\lambda)^{u}}$$

$X + Y = U \sim \text{Pois}(\theta + \lambda)$

Theorem: $X \perp\!\!\!\perp Y$, $X \sim \text{Pois}(\theta)$
$Y \sim \text{Pois}(\lambda)$

then $X + Y \sim \text{Pois}(\theta + \lambda)$
