Defn: Sample Space

The sample space S is the set of possible outcomes.

Ex. Flip a coin

$$S = \{H, T\}.$$

Ex. Roll a 6-sided die:

$$S = \{1, 2, \dots, 6\}$$

Ex. Roll two dice

$$S = \{(1,1), (1,2), (2,1), \dots\}$$

Ex. Waiting for bus to arrive, wait time

$$S = [0, \infty)$$

Ex. Number of customers arriving at my restaurant

$$S = \{0, 1, 2, 3, \dots\} = \mathbb{N}_0$$

Types of sample spaces:

1. finite $|S| < \infty$ (e.g. $\{H, T\}$)

2. infinite $|S| = \infty$
   - (i) countable (e.g. $\mathbb{N}_0$)
   - (ii) uncountable (e.g. $[0, \infty)$)

Defn: Outcome

We call elements of the sample space outcomes:

$$\underset{\text{outcome}}{s} \in \underset{\text{sample space}}{S}$$

Ex. $S = \{1, \dots, 6\}$

then $1 \in S$ so 1 is a outcome.

Defn: Event

An event E is a subset of S:

$$E \subset S.$$

Ex. $S = \{1, \dots, 6\}$ then

$$E = \{1, 2\} \subset S$$

is the event that I roll either a 1 or 2.

Ex. $S = \{(i,j) \mid 1 \le i \le 6,\ 1 \le j \le 6\}$

then

$$E = \{(2,1), (3,2)\} \subset S$$

$$F = \{(1,2), (2,3)\}$$

We say an event "happens" or "occurs" if the observed outcome of our experiment is in E

Ex. $S \subset S$ so S is an event.

the event that something happens.

Ex. $\emptyset \subset S$ so $\emptyset$ is an event.

the event nothing happens???

Axiomatic Probability

Given: a sample space S

Want: for any event $E \subset S$ want to assign some measure of the probability of E occuring.

probability function

Mathematically:

For each $E \subset S$ we assign $\mathbb{P}(E)$

prob. of E

What are the rules for building $\mathbb{P}$?

1. mathematically consistent
2. encode some intuitions about probability

Defn: Probability Function

Given a sample space S a prob. fn $\mathbb{P}$ is a function

$$P : 2^S \to \mathbb{R}$$

that satisfies the Kolmogorov Axioms :

① non-negativity

$$P(E) \geq 0 \quad \forall E \subset S$$

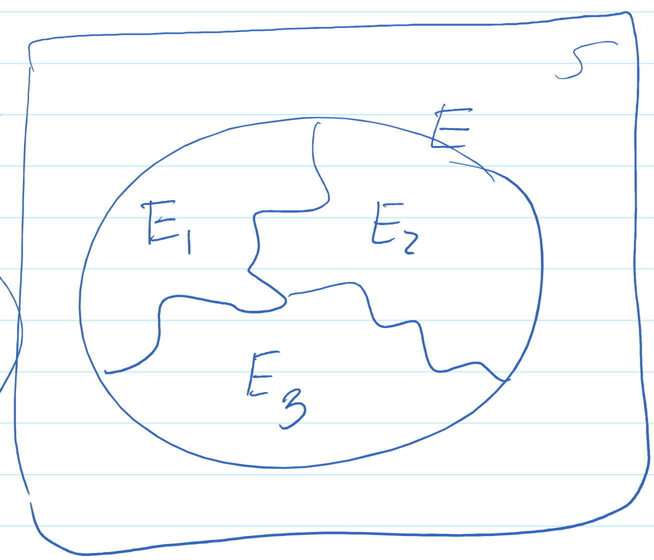
② unit-measure

$$P(S) = 1$$

③ Countable Additivity

If $(E_i)_{i=1}^{\infty}$ is a partition of E.

$\left(E = \bigcup_i E_i \; ; \; E_i E_j = \emptyset\right)$



then

$$P(E) = \sum_{i=1}^{\infty} P(E_i).$$

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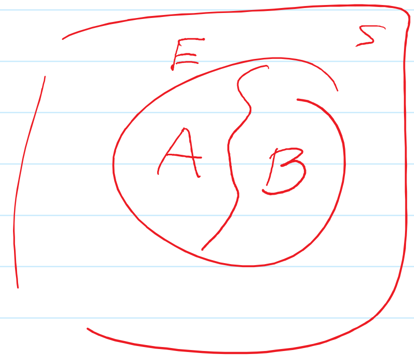
① Axiom 3 is basically a distributive law for disjoint

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

② It also holds for finite partitions.

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$



in particular:

$E = A \cup B$ and $AB = \emptyset$

then $P(E) = P(A) + P(B)$.

<u>Ex.</u> Flip a coin

$$S = \{H, T\}$$

what is a possible valid P on S?

$$P(\{H\}) = 1/2 \qquad P(\overbrace{\{H, T\}}^{S}) = 1$$

$$P(\{T\}) = 1/2 \qquad P(\emptyset) = 0$$

Is this a valid P?

✓ 1. $P(E) \geq 0$

✓ 2. $P(S) = 1$

3. $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$ for disjoint E_i

One example: $E = S$

$E_1 = \{H\}$, $E_2 = \{T\}$

then E_1 and E_2 partition E

$$1 = P(S) = P(E) = P(E_1) + P(E_2) = \frac{1}{2} + \frac{1}{2}$$

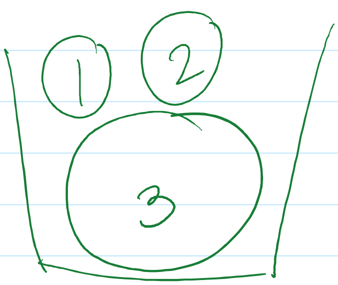
Ex. $S = \{H, T\}$

$P(S) = 1$ and $P(\{T\}) = \alpha$

$P(\emptyset) = 0$ $\quad$ $P(\{H\}) = 1 - \alpha$

$0 \le \alpha \le 1$

Ex.



$$S = \{1, 2, 3\}$$

$$p_1 = \frac{1}{4} \quad p_2 = \frac{1}{4} \quad p_3 = \frac{1}{2}$$

(pos. and sum to 1)

$$P(\{2,3\}) = p_2 + p_3 = 3/4$$

$$P(\{1,3\}) = p_1 + p_3 = 3/4$$

Theorem: Discrete Sample Space Theorem

Theorem: Discrete Sample Space Theorem

If $S = \{s_1, \dots, s_n\}$ so that $|S| = n$

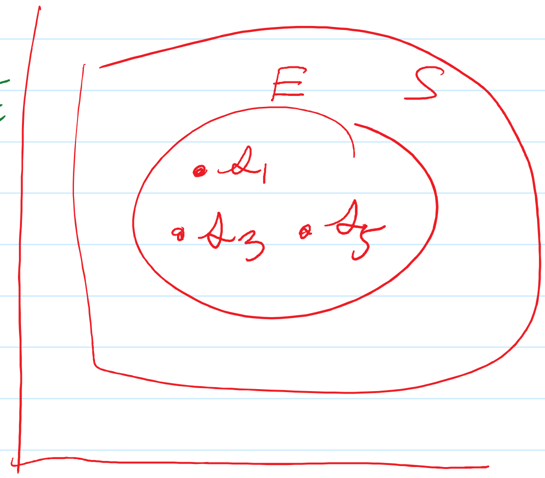
and we choose $p_1, p_2, \dots, p_n$ so that

$$\text{(1) } p_i \geq 0 \quad \text{and} \quad \text{(2) } \sum_{i=1}^{n} p_i = 1$$

and we define a $\mathbb{P}$ so that for $E \subset S$

$$\mathbb{P}(E) = \text{sum } p_i \text{ with corresponding } s_i \in E$$

$$= \sum_{i:\, s_i \in E} p_i$$



$P(E) = p_1 + p_3 + p_5$

Then $\mathbb{P}$ is a valid prob. fn.

Pf. Need to check sats. Komogorov Axioms

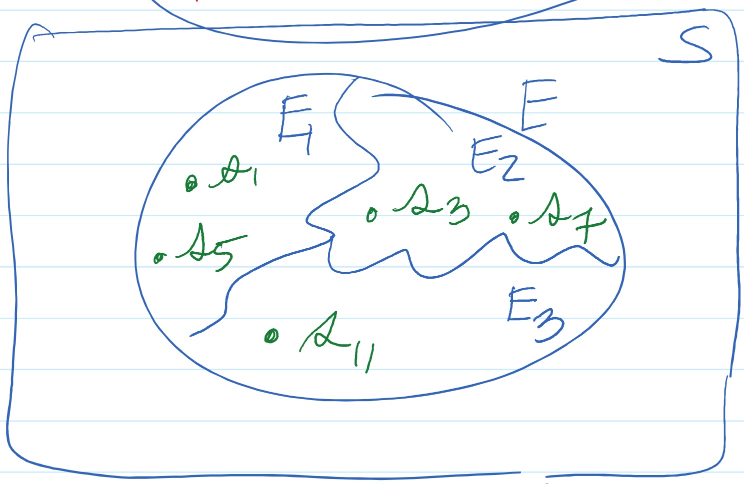
(1) $P(E) \geq 0 \quad \forall E \subset S$

$$P(E) = \sum_{\text{some } i\text{s}} p_i \geq 0 \qquad p_i \geq 0$$

(2) $P(S) = 1$

$$\mathbb{P}(S) = \sum_{i:s_i \in S} p_i = \sum_{i=1}^{n} p_i = 1$$

(3) E_i partition E then $\sum_{i=1}^{\infty} \mathbb{P}(E_i) = P(E)$



$$\mathbb{P}(E) = (p_1 + p_5) + (p_3 + p_7) + (p_{11})$$
$$= \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3).$$

$\boxed{\text{Theorem: } \mathbb{P}(\emptyset) = 0.}$

Pf.

$S = S \cup \emptyset \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots$

then by Axiom 3

$$\underbrace{P(S)}_{1} = P(S) + P(\emptyset) + P(\emptyset) + P(\emptyset) + \cdots$$
$$= \underbrace{P(S)}_{1} + \sum_{i=1}^{\infty} P(\emptyset)$$

I can add any number of $\emptyset$ to a partition and it remains a partition

So $\sum_{i=1}^{\infty} P(\emptyset) = 0$. $\Rightarrow$ This only works if $\mathbb{P}(\emptyset) = 0$

So $\sum_{i=1} \mathbb{P}(\emptyset) = 0. \rightarrow$

$$\mathbb{P}(\emptyset) = 0.$$

Third Axiom: $\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$ if E_i disjoint

Finite Additivity: $\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i)$ if E_i disjoint

Pf $E = A \cup B$, $AB = \emptyset$.

Notice $E = A \cup B \cup \emptyset \cup \emptyset \cup \emptyset \cup \ldots$

add countable number of $\emptyset$

Apply Axiom 3

$$\boxed{\mathbb{P}(E) = \mathbb{P}(A) + \mathbb{P}(B)} + 0 + 0 + 0 + 0 + \ldots$$

For ≥ 2 sets use induction.
