Ex. (X,Y) // Z is Biv RV

Flip a coin 3 times

$$X = \begin{cases} 0 & \text{if last flip is a T} \\ 1 & \text{if " H} \end{cases}$$

Y = # heads among 3 flips

$s \in S$	$Z(s)$
H H H	(1, 3)
T H H	(1, 2)
H T H	(1, 2)
T T H	(1, 1)
H H T	(0, 2)
T H T	(0, 1)
H T T	(0, 1)
T T T	(0, 0)

Defn: Biv CDF

The bivariate CDF is a function

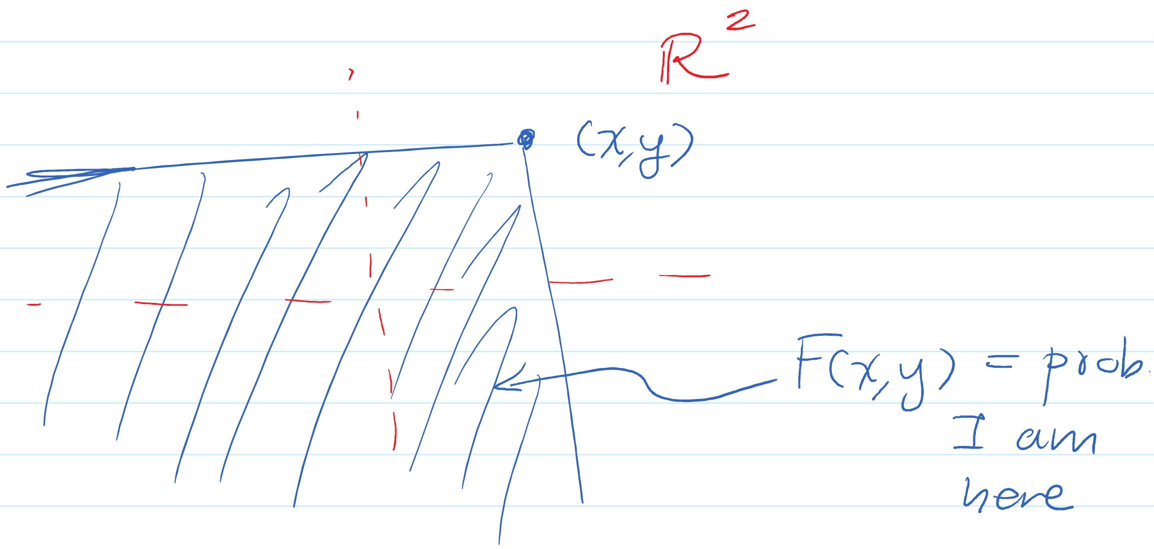
$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

so that for $x, y \in \mathbb{R}$ then

$$F(x,y) = P(X \le x, Y \le y)$$

Univarite Case:

$F(x) = P(X \le x)$



Properties of Biv CDF

① $F(x,y) \geq 0 \quad \forall x,y$

② $\lim\limits_{x,y \to \infty} F(x,y) = 1$ [Uni: $\lim\limits_{x \to \infty} F(x) = 1$]

③ $\lim\limits_{x \to -\infty} F(x,y) = 0$ [Uni: $\lim\limits_{x \to -\infty} F(x) = 0$]

$\lim\limits_{y \to -\infty} F(x,y) = 0$

④ F is non-decreasing and right continuous in each argument x and y

Defn: Marginal Distributions

If (X, Y) is a Biv RV then we say X and Y (individually) are the marginal RVs

and their distributions (and other properties) are called the marginal dists.

Theorem: Rel. between marginal and Biv CDFs

If (X, Y) is a Biv RV w/ Biv. CDF F and marginal CDFs F_X and F_Y, resp.

Then

① $F_Y(y) = \lim_{x \to \infty} F(x, y)$

② $F_X(x) = \lim_{y \to \infty} F(x, y)$

pf.

$$F_X(x) = P(X \le x)$$

$$= P(X \le x, \; Y = \text{anything})$$

$$= P(X \le x, \; Y \le \infty)$$

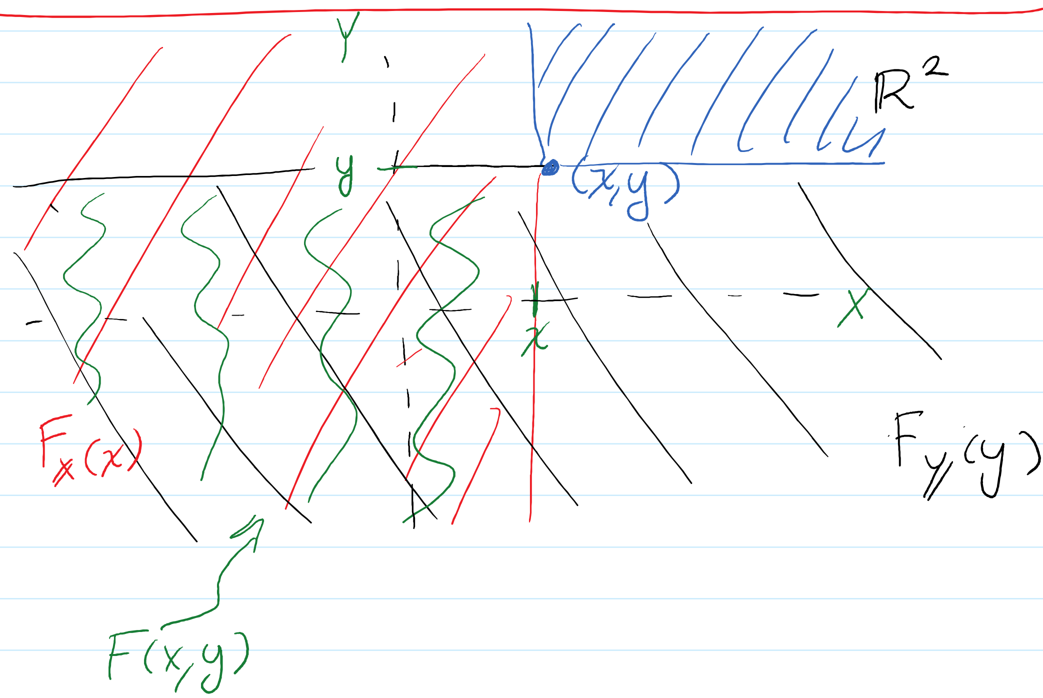
$$= \lim_{y \to \infty} \mathbb{P}(X \le x, Y \le y)$$

$$= \lim_{y \to \infty} F(x,y)$$

$$\boxed{\underline{\text{Uni}}: \quad \mathbb{P}(X > x) = 1 - F(x)}$$

For Biv RVs:

$$\boxed{\begin{aligned}&\mathbb{P}(X > x, Y > y) \\ &\quad = 1 - F_X(x) - F_Y(y) + F(x,y)\end{aligned}}$$



<u>Defn: Bivariate / Joint PMF</u>

If X and Y are discrete then the joint PMF is

$$f(x,y) = \mathbb{P}(X=x, Y=y)$$

Properties of Joint PMF

f is a valid joint PMF if

1. $f(x,y) \geq 0 \quad \forall x,y$
2. $\sum_x \sum_y f(x,y) = 1$

Theorem: Rel. btwn joint/marginal PMFs

joint PMF

1. $f_X(x) = \sum_y f(x,y)$

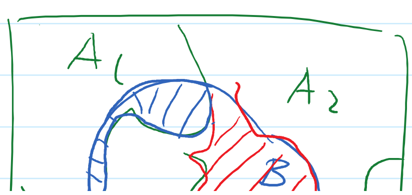
marginal PMF of X

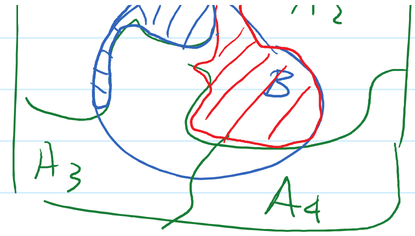
2. $f_Y(y) = \sum_x f(x,y)$

Pf.

Theorem earlier: If A_i partitions S then

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B \cap A_i)$$





Consider the partition $\{Y=y\}\ \forall y$

e.g. $\{Y=1\}, \{Y=2\}, \{Y=3\}, \ldots$

So

$$f_X(x) = \mathbb{P}(X=x) = \sum_y \mathbb{P}(\text{"}X=x\text{"} \cap \text{"}Y=y\text{"})$$

$$= \sum_y \mathbb{P}(X=x, Y=y)$$

$$= \sum_y f(x,y)$$

Ex. Flip 3 coins

$$X = \begin{cases} 0 & \text{last is T} \\ 1 & \text{last is H} \end{cases}$$

Y = # heads

check: $f(x,y) \geq 0$

$\sum_x \sum_y f(x,y) = 1$

$X \backslash Y$	0	1	2	3	
0	$f(0,0) = 1/8$	$f(0,1) = 2/8$	$f(0,2) = 1/8$	$f(0,3) = 0$	$f_X(0) = $ sum of row $= 4/8 = 1/2$
1	0	$1/8$	$2/8$	$1/8$	$f_X(1) = 4/8 = 1/2$

1	0	1/8	2/8	1/8	$f_X(1) = 4/8 = 1/2$
	$f_Y(0) = 1/8$	3/8	3/8	1/8	

$$f_X(x) = \sum_y f(x,y)$$

$$f_Y(y) = \sum_x f(x,y)$$

Defn: Joint PDF

If X and Y are continuous RVs we call the function

$$f : \mathbb{R}^2 \to \mathbb{R}$$

the joint PDF of X and Y if $\forall C \subset \mathbb{R}^2$

$$\mathbb{P}((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$$

$$\boxed{\text{Uni: } \mathbb{P}(X \in A) = \int_A f(x)\,dx}$$

Facts:

① $F(x,y) = \displaystyle\int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,du\,dv$

$-\infty \quad -\infty$

$$\left(\underline{\text{uni:}} \quad F(x) = \int_{-\infty}^{x} f(t)\,dt\right)$$

② $\left(\text{uni:} \quad f(x) = \frac{dF}{dx}\right)$

<u>Biv:</u> $f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$

③ $f(x,y) \geq 0$ and $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$

[iff f is a valid PDF]

<u>Theorem:</u> <u>Rel. btwn Joint/Marginal PDFs</u>

① $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$ joint PDF

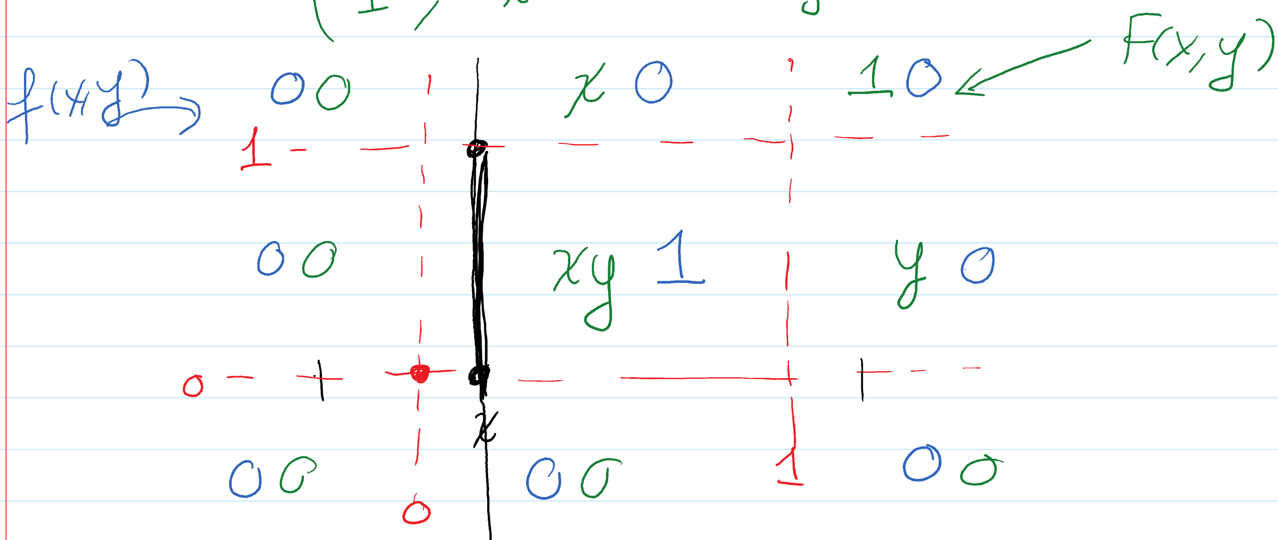
marginal PDF of X

② $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$

Ex. $\{ 0 \quad x<0 \text{ or } y<0$

Ex.

$$F(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ xy, & (x,y) \in [0,1]^2 \\ x, & y>1 \text{ and } 0<x<1 \\ y, & x>1 \text{ and } 0<y<1 \\ 1, & x>1 \text{ and } y>1 \end{cases}$$



What is the joint PDF f?

$$f = \frac{\partial^2 F}{\partial x \partial y}$$

So $f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1]^2 \\ 0 & \text{else} \end{cases}$

What about the marginals?

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\, dy$$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

if $x > 1$ or $x < 0$ then $\int_{\mathbb{R}} f(x,y)\,dy = \int_{\mathbb{R}} 0\,dy = 0$

if $0 < x < 1$ then

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_0^1 1\,dy = 1$$

All together: $f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0 & \text{else} \end{cases}$

$X \sim U(0,1)$

Similarly $\underline{Y \sim U(0,1)}$