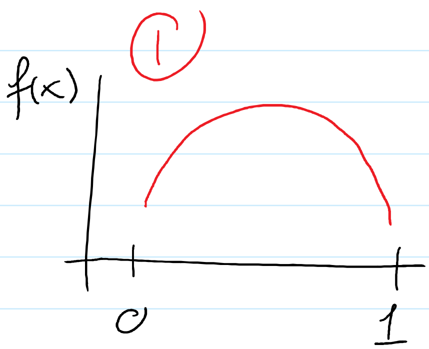
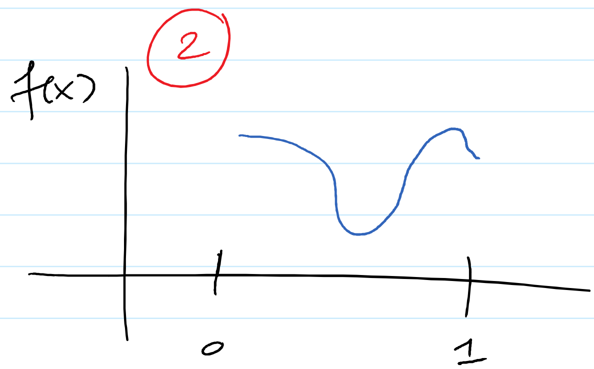
Beta Distribution

- a cts RV w/ support between 0 and 1





Beta Function $a, b \in \mathbb{R}^+$

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

Fact:

$$\boxed{B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}$$

$$\frac{1}{B(a,b)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \approx \frac{(a+b)!}{a!\,b!} = \binom{a+b}{a}$$

Beta Dist:

$$X \sim \text{Beta}(a,b)$$

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } 0 < x < 1$$

$B(a,b)$

Expectation:

$$E[X] = \int_0^1 x \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

looks like

$$\underline{c}\underbrace{\int \frac{\text{PDF Beta}(a+1,b)}{}}_{\text{goes to } 1}$$

$$= \int_0^1 \frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+1,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}\,dx}_{1}$$

$$\frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}$$

$$= \frac{B(a+1,b)}{B(a,b)} = \frac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+b+1)} \Big/ \frac{\Gamma(a)\cancel{\Gamma(b)}}{\Gamma(a+b)}$$

$$\Gamma(a+1) = a\,\Gamma(a)$$

$$= \frac{\Gamma(a+b)\Gamma(a+1)}{\Gamma(a+b+1)\Gamma(a)}$$

$$= \frac{\cancel{\Gamma(a+b)}\,a\,\cancel{\Gamma(a)}}{(a+b)\cancel{\Gamma(a+b)}\cancel{\Gamma(a)}}$$

$$(a+b)\Gamma(\cancel{a+b})\cancel{\Gamma(a)}$$

$$= \boxed{\frac{a}{a+b} = \mathbb{E}[X]}$$

Moments:

$$\mathbb{E}[X^r] = \int_0^1 x^r \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+r,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{(a+r)-1}(1-x)^{b-1}}{B(a+r,b)}\,dx}_{\text{integrates to } 1}$$

$$\boxed{\mathbb{E}[X^r] = \frac{B(a+r,b)}{B(a,b)}}$$

$$\mathbb{E}[X^2] = \frac{B(a+2,b)}{B(a,b)} = \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)} \Big/ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{(a+1)a\,\cancel{\Gamma(a)}\,\cancel{\Gamma(b)}\,\cancel{\Gamma(a+b)}}{(a+b+1)(a+b)\,\cancel{\Gamma(a+b)}\,\cancel{\Gamma(a)}\,\cancel{\Gamma(b)}}$$

$$= \frac{a(a+1)}{(a+b)(a+b+1)}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2$$

$$= \cdots = \boxed{\frac{ab}{(a+b+1)(a+b)^2} = Var(X)}$$

Transformations:

1. If I know something about X
2. What do I know about $Y = g(X)$

Discrete RVs

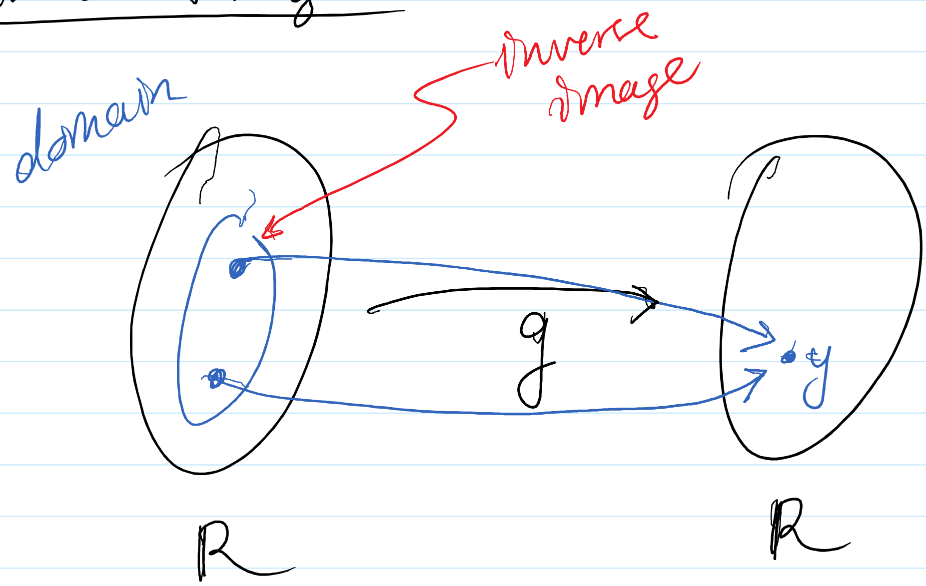
Let X be a discrete RV

Know f_X ← PMF of X

and let $Y = g(X)$

Q: what is f_Y ?

Inverse Image



$$g^{-1}(\{y\}) = \{ x : g(x) = y \}$$

If g is really invertible then inverse image = true inverse

$$f_Y(y) = \mathbb{P}(Y = y) = \mathbb{P}(g(X) = y) \quad (*)$$

want

If g is invertible then

$$\circledast = \mathbb{P}(X = g^{-1}(y))$$

$$= f_X(g^{-1}(y))$$

If g is not invertible then

$$\circledast = \mathbb{P}(X \in g^{-1}(y))$$

$$= \sum_{x \in g^{-1}(y)} f_X(x)$$

$$= \sum_{x : g(x) = y} f_X(x)$$

$\mathbb{P}(X \in A)$

$= \sum_{x \in A} f_X(x)$

$\circledast$ Theorem: If X is discrete and $Y = g(X)$ then

$$f_Y(y) = \sum_{x : g(x) = y} f_X(x)$$

Ex. Let $X \sim \text{Bin}(n, p)$

# of H in n coin flips each w/ a prob. p of getting H

$Y = n - X$ ← # tails

$$\mathbb{Y} = g(\mathbb{X}) \quad \text{where } y = g(x) = n - x$$

$$\Updownarrow \quad x = n - y$$

$$f_{\mathbb{Y}}(y) = \sum_{x : g(x) = y} f_{\mathbb{X}}(x) = \sum_{\boxed{x = n-y}} f_{\mathbb{X}}(x)$$

$$= f_{\mathbb{X}}(n-y)$$

$$f_{\mathbb{X}}(x) = \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{\overset{y}{n-(n-y)}}$$

note: $\binom{n}{n-y} = \binom{n}{y}$; $q = 1-p$

$$= \underbrace{\binom{n}{y} q^{y} (1-q)^{n-y}}$$

PMF of $\text{Bin}(n, q)$

i.e. $\mathbb{Y} \sim \text{Bin}(n, 1-p)$

Let's consider continuous RVs

Theorem: If $\mathbb{X}$ is continuous and $\mathbb{Y} = g(\mathbb{X})$

then

① if g is increasing then

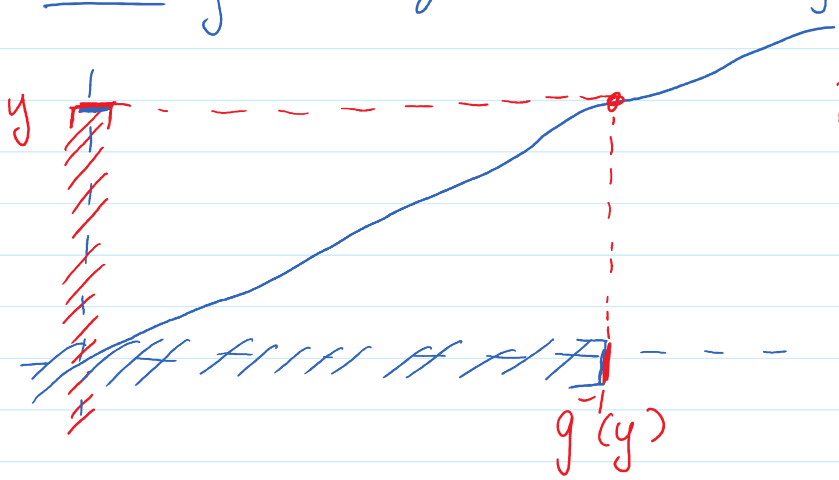
$$F_Y(y) = F_X(g^{-1}(y))$$

inverse exists

② if g is decreasing then

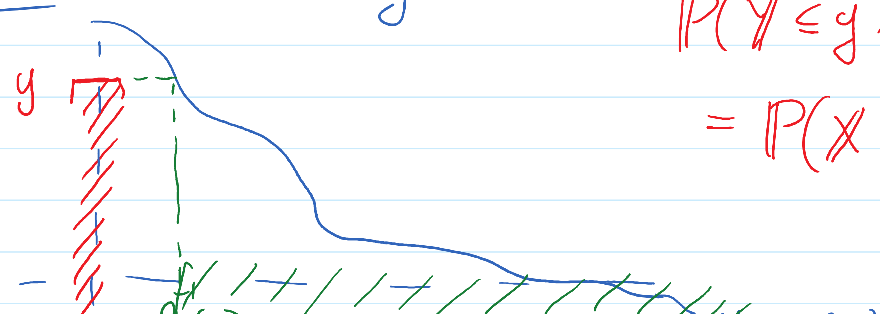
$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

Pf. case1: g increasing

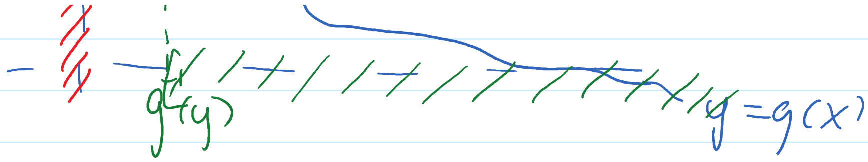


$$F_Y(y) = P(Y \le y)$$
$$= P(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

Case 2: decreasing



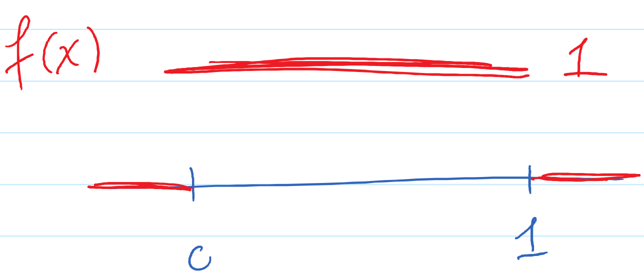
$$P(Y \le y)$$
$$= P(X \ge g^{-1}(y))$$
$$= 1 - P(X \le g^{-1}(y))$$

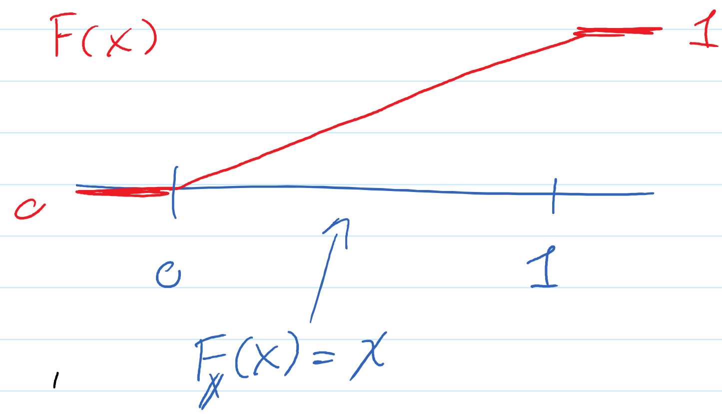


$= 1 - P(X \le g^{-1}(y))$

$= 1 - F_X(g^{-1}(y))$

Ex. $X \sim U(0,1)$

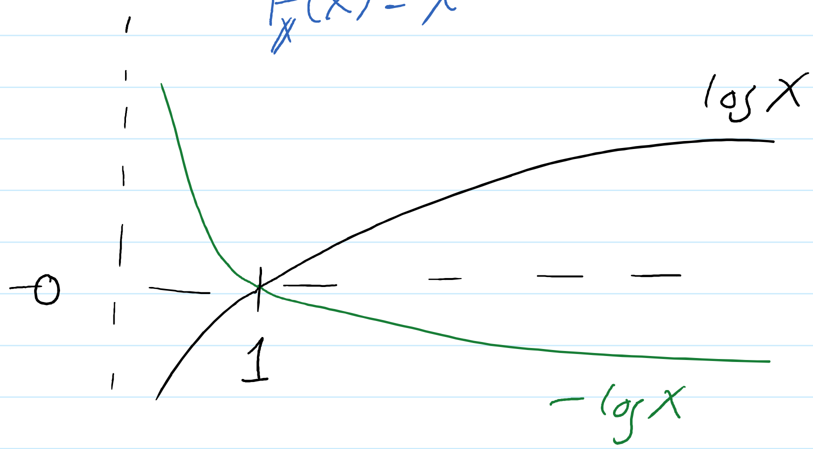




Let $Y = -\log X$

$g(x) = -\log x$

(decreasing)



$y = -\log x = g(x)$

$\Rightarrow -y = \log x$

$\Rightarrow e^{-y} = x = g^{-1}(y)$

$0 < x < 1 \Rightarrow \underbrace{-\log x}_{y} > 0$

$F_Y(y) = 1 - F_X(g^{-1}(y))$

If $y > 0$ then $e^{-y} = \frac{1}{e^{y}} < 1$

$= 1 - F_X(\underline{e^{-y}})$

so $0 < e^{-y} < 1$

$= \underbrace{1 - e^{-y}}_{}$

CDF of a Exp(1)

So $Y \sim Exp(1)$

$Z \sim Exp(1)$

$$\boxed{F(z) = \int_0^z e^{-x}\,dx = 1 - e^{-z}}$$

Theorem: If X is continuous and $Y = g(X)$

and

1. g is invertible
2. g^{-1} is differentiable

then

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$
