Defn: Identical Distribution

We say two RVs X and Y are equal in distribution if $\forall A \subset \mathbb{R}$

$$\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$$

we denote this as

$$X \overset{d}{=} Y.$$

This doesn't mean $X = Y$.

Ex. 3 coin flips

X = # heads

Y = # tails

these are different RVs,

$$X(HTT) = 1 \quad \text{but} \quad Y(HTT) = 2$$

However, $\boxed{X \overset{d}{=} Y.}$

$$\mathbb{P}(X = 0) = 1/8 = \mathbb{P}(Y = 0)$$

$$\mathbb{P}(X = 1) = 3/8 = \mathbb{P}(Y = 1)$$

Theorem:

$$X \stackrel{d}{=} Y \quad \text{iff} \quad F_X = F_Y.$$

CDF of X CDF of Y

Ex. Toss a coin (independently) until a H appears.

$$S = \{H, TH, TTH, \ldots\}$$

Let p be the prob. I get a H on any flip.

X = # flips until I get a H

$s \in S$	$X(s)$
H	1
TH	2
TTH	3
$\vdots$	$\vdots$

Q: What is the CDF of X?

$$F(x) = \mathbb{P}(X \le x)$$

$F(x) = \mathbb{P}(X = x)$

We'll look at $\mathbb{P}(X = x)$

Let H_i = i^{th} toss is a H, $T_i = H_i^c$

$$\text{“}X = i\text{”} = T_1 T_2 \cdots T_{i-1} H_i$$

independent

So

$$\mathbb{P}(X=i) = \mathbb{P}(T_1 T_2 \cdots T_{i-1} H_i)$$
$$= \mathbb{P}(T_1)\mathbb{P}(T_2) \cdots \mathbb{P}(T_{i-1})\mathbb{P}(H_i)$$
$$= (1-p)(1-p) \cdots (1-p)\,p$$
$$= \boxed{(1-p)^{i-1} p}$$

$$\text{“}X \leq x\text{”} = \text{“}X=1\text{”} \cup \text{“}X=2\text{”} \cup \text{“}X=3\text{”} \cup \cdots \cup \text{“}X=x\text{”}$$

disjoint

$$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X=1) + \mathbb{P}(X=2) + \cdots + \mathbb{P}(X=x)$$
$$= \sum_{i=1}^{x} \mathbb{P}(X=i)$$
$$= \sum_{i=1}^{x} (1-p)^{i-1} p$$
$$= p \sum_{i=1}^{x} (1-p)^{i-1}$$

Calc II: geometric sum,

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$$

$$= p \sum_{i=1} (1-p)$$

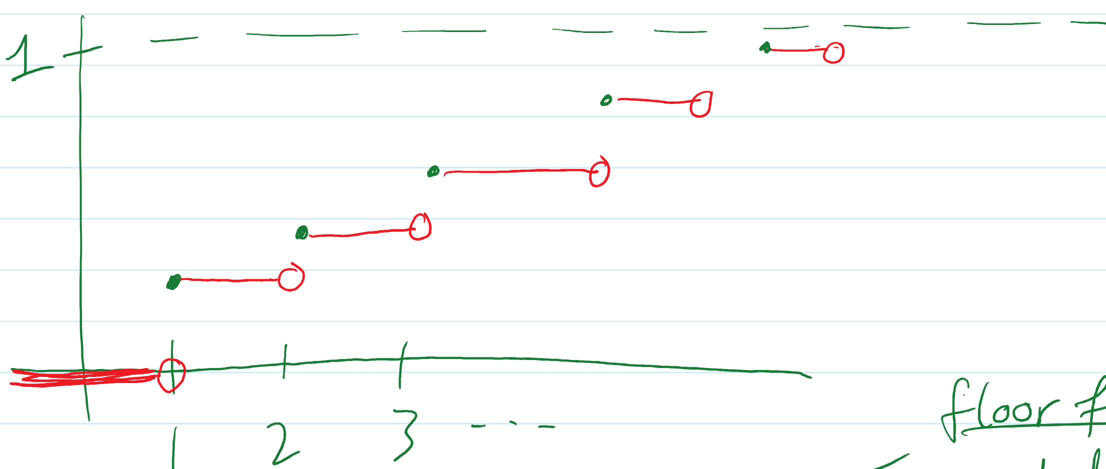
$$= p \sum_{i=0}^{x-1} (1-p)^i$$

$r = 1-p$

$n = x$

$$= \cancel{p} \, \frac{1-(1-p)^x}{\cancel{1-(1-p)}}$$

$$F(x) = 1-(1-p)^x$$



floor function: round down

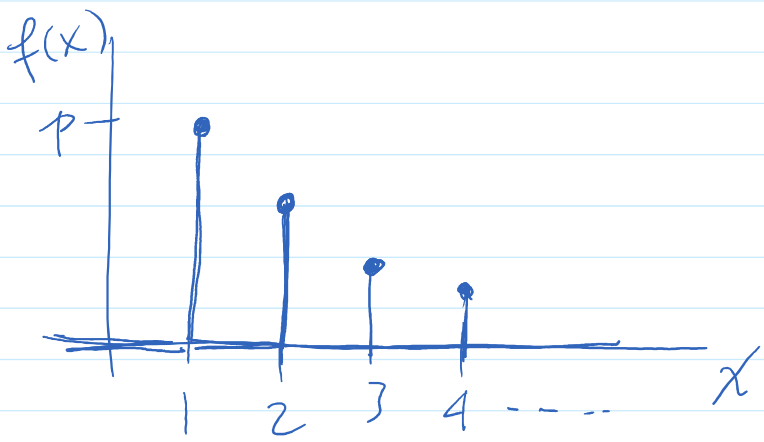
$$F(x) = \begin{cases} 0, & x < 1 \\ 1-(1-p)^{\lfloor x \rfloor}, & x \geq 1 \end{cases}$$

Defn : Discrete/Continuous RVs

A discrete RV is a RV whose CDF is a step function.

A continuous RV is a RV whose CDF is continuous.

Plot of $\mathbb{P}(X=x) = (1-p)^{x-1}p = f(x)$



Defn: Probability Mass Function (PMF)

For a discrete RV X, the PMF is a function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \mathbb{P}(X = x) \quad \forall x \in \mathbb{R}$$

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Also sometimes called the distribution of X.

Theorem: For discrete RVs

$$F(x) = \sum_{i \le x} f(i)$$

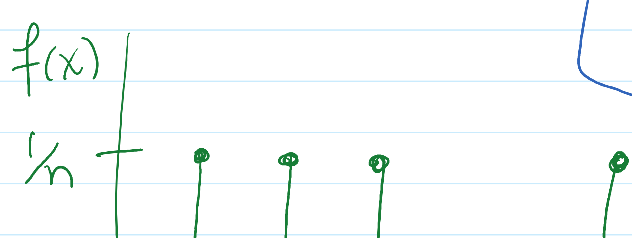
Pf. $$``X \le x" = \bigcup_{i \le x} ``X = i"$$

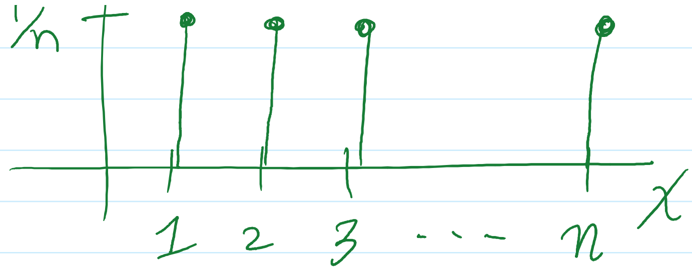
disjoint union

then
$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}\Big(\bigcup_{i \le x} ``X = i"\Big)$$
$$= \sum_{i \le x} \mathbb{P}(X = i)$$
$$= \sum_{i \le x} f(i).$$

Ex. We say X has a discrete uniform distribution over $1, \dots, n$

Notation:
$$X \sim \mathcal{U}(\{1, \dots, n\})$$





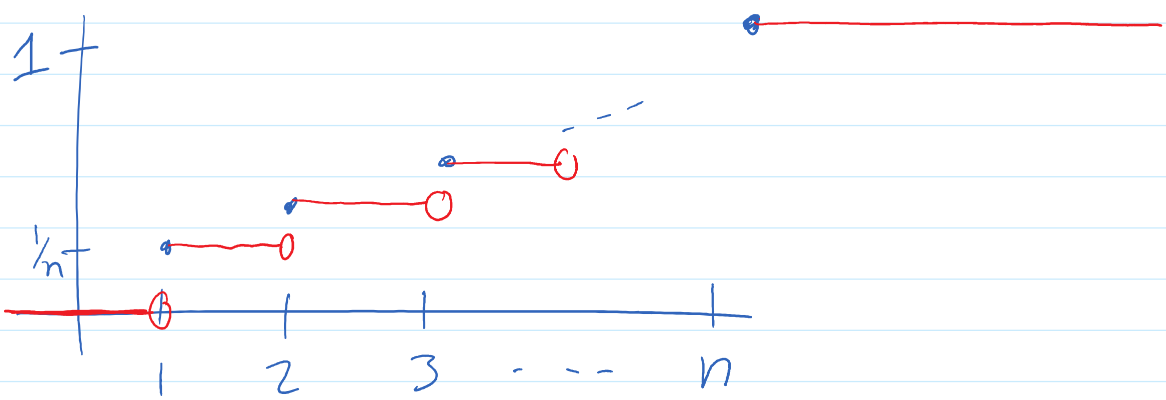
in math:

$$f(x) = \begin{cases} \frac{1}{n} & x = 1, 2, 3, \ldots \\ 0 & \text{else} \end{cases}$$

Q: What's the CDF?

$$F(x) = \sum_{i \le x} f(i) = \sum_{i=1}^{x} \frac{1}{n} = \frac{x}{n}$$

if $x = 1, 2, \ldots, n$



$$F(x) = \begin{cases} 0, & x < 0 \\ \lfloor x \rfloor / n & \\ 1, & x \geq n \end{cases}$$

Saw: $P(X \le x) = F(x) = \sum_{i \le x} f(i)$

More generally:

$$P(X \in A) = \sum_{i \in A} f(i)$$

Ex. $X \sim U(\{1, \dots, 7\})$

$$P(2 \le X \le 5) = P(X \in \{2,3,4,5\})$$
$$= \sum_{i=2}^{5} f(i)$$
$$= \sum_{i=2}^{5} 1/7 = 4/7.$$

Ex. Roll a die 60 times. (independently)

X = # of 6s I roll

What is PMF of X?

$$f(0) = P(X=0) = \underbrace{(5/6)(5/6)(5/6)\cdots(5/6)}_{60}$$
$$= (5/6)^{60}$$

$$= \left(\tfrac{5}{6}\right)^{60}$$

$$f(1) = P(X=1) = \binom{60}{1}\left(\tfrac{1}{6}\right)\underbrace{\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)}_{59}$$

$$= \binom{60}{1}\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)^{59}$$

$$f(2) = P(X=2) = \binom{60}{2}\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)\underbrace{\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)}_{58}$$

$$= \binom{60}{2}\left(\tfrac{1}{6}\right)^{2}\left(\tfrac{5}{6}\right)^{58}$$

$$\vdots$$

General pattern:

$$\boxed{f(x) = P(X=x) = \binom{60}{x}\left(\tfrac{1}{6}\right)^{x}\left(\tfrac{5}{6}\right)^{60-x}}$$

We call this type of RV a Binomial RV.

If I do a series of n independent experiments each with a binary outcome:

Yes/No, $\boxed{0/1}$, Success/Failure etc.

Yes/No , $\boxed{0/1}$, Success/Failure etc.

and the prob. of a 1 is p for each experiment.

Let X = # of 1s

then X has a Binomial dist,

$$X \sim Bin(n, p)$$

and

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$