Theorem: $Cov(aX+b, Y) = a\, Cov(X, Y)$

Recall: $Var(aX+b) = a^2\, Var(X)$

Pf. $Cov(aX+b, Y) = \mathbb{E}\left[\left((aX+b) - \mathbb{E}[aX+b]\right)(Y - \mathbb{E}Y)\right]$

$$= \mathbb{E}\left[(aX + \cancel{b} - a\mathbb{E}X - \cancel{b})(Y - \mathbb{E}Y)\right]$$

$$= \mathbb{E}\left[a(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$$

$$= a\,\mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$$

$$= a\, Cov(X, Y)$$

Corollaries: ① $Cov(X, cY+d) = c\, Cov(X, Y)$

② $Cov(aX+b, cY+d) = ac\, Cov(X, Y)$.

Theorem:

$$\underline{\mathrm{Cor}(aX+b,\ cY+d)} = \underline{\mathrm{Sign}(a)}\,\underline{\mathrm{Sign}(c)}\,\underline{\mathrm{Cor}(X,Y)}$$

$$\mathrm{Sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Ex. $\mathrm{Cor}(-5X, Y) = -\mathrm{Cor}(X,Y)$

pf. $a, c \neq 0$

if $x \neq 0$ then $\mathrm{sign}(x) = \dfrac{x}{|x|}$

$$\mathrm{Cor}(aX+b,\ cY+d) = \frac{\mathrm{Cov}(aX+b,\ cY+d)}{\sqrt{\mathrm{Var}(aX+b)\,\mathrm{Var}(cY+d)}}$$

$$= \frac{ac\ \mathrm{cov}(X,Y)}{\sqrt{a^2\,\mathrm{Var}(X)\ c^2\,\mathrm{Var}(Y)}}$$

$$= \frac{a\ \ c}{|a|\,|c|}\ \boxed{\frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}}$$

$$= \mathrm{Sign}(a)\ \mathrm{Sign}(c)\ \mathrm{Cor}(X,Y).$$

Claim: $\boxed{-1 \le Cor(X,Y) \le 1}$

pf. $\tilde{X} = \dfrac{X - EX}{\sqrt{Var(X)}} = \underbrace{\dfrac{1}{\sqrt{Var(X)}}}_{a>0} X - \underbrace{\dfrac{E[X]}{\sqrt{Var(X)}}}_{b} = aX + b$

$$E[\tilde{X}] = \frac{EX - EX}{\sqrt{Var(X)}} = 0$$

$$Var(\tilde{X}) = \left(\frac{1}{\sqrt{Var(X)}}\right)^2 Var(X) = 1$$

$$\tilde{Y} = \frac{Y - EY}{\sqrt{Var(Y)}}$$

Notice: $Cor(\tilde{X}, \tilde{Y}) = Cor(X, Y)$ ⊛

Also: $Cor(\tilde{X}, \tilde{Y}) = \dfrac{Cov(\tilde{X}, \tilde{Y})}{\sqrt{\underbrace{Var(\tilde{X})}_{1}\underbrace{Var(\tilde{Y})}_{1}}} = Cov(\tilde{X}, \tilde{Y})$

Chain: $Cor(X,Y) = Cor(\tilde{X}, \tilde{Y}) = Cov(\tilde{X}, \tilde{Y})$

Consider

$$Var(\tilde{X} \pm \tilde{Y}) = Var(\tilde{X}) + Var(\tilde{Y}) \pm 2Cov(\tilde{X}, \tilde{Y})$$

Consider

$$\underbrace{Var(\tilde{X} \pm \tilde{Y})} = \underbrace{Var(\tilde{X})}_{1} + \underbrace{Var(\tilde{Y})}_{1} \pm 2\underbrace{Cov(\tilde{X}, \tilde{Y})}_{Cor(X,Y)}$$

$$= \underbrace{2 \pm 2\,Cor(X,Y) \geq 0}$$

So $1 \pm Cor(X,Y) \geq 0$

$1 + Cor(X,Y) \geq 0 \quad\quad\quad 1 - Cor(X,Y) \geq 0$

$\downarrow \quad\quad\quad \downarrow$

$\boxed{-1 \leq Cor(X,Y)} \quad\quad\quad \boxed{Cor(X,Y) \leq 1}$

Theorem: Short-Cut formula for Covariance

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Analogy: $Var(X) = E[X^2] - (EX)^2$

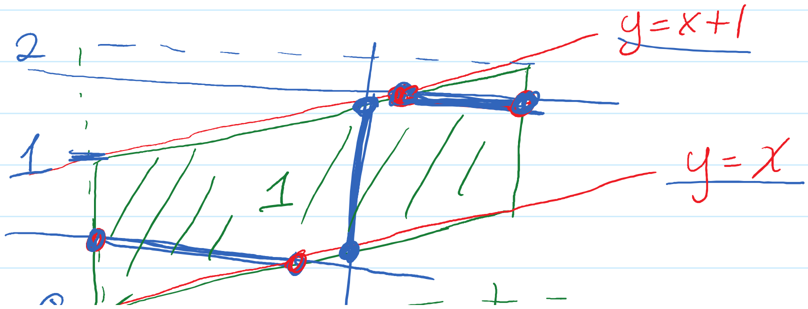
Ex. $f(x,y) = 1$ for $0 < x < 1$

$x < y < x+1$

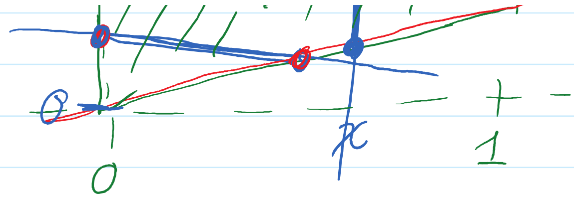
We had calculated

$$\boxed{\mathbb{E}[XY] = 7/12}$$

What is Cov/Cor?



What is Cov / Cor ?



Marginal of X

$0 < x < 1$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_{y=x}^{y=x+1} 1\,dy = (x+1) - x = 1$$

So $f_X(x) = 1$ for $0 < x < 1$

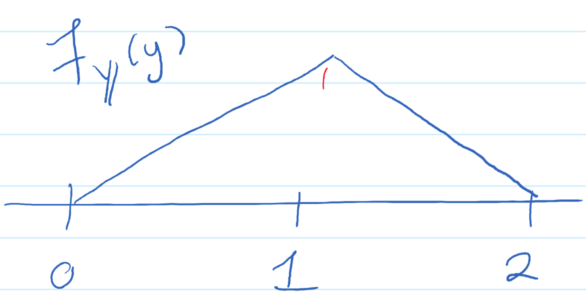
i.e. $X \sim U(0,1)$

So $\boxed{E[X] = 1/2 \quad \text{and} \quad Var(X) = 1/12}$

Marginal of Y

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \begin{cases} \int_0^y 1\,dx & 0 < y < 1 \\ \int_{y-1}^{1} 1\,dx & 1 < y < 2 \end{cases}$$

$$= \begin{cases} y, & 0 < y < 1 \\ 2-y, & 1 < y < 2 \end{cases}$$



0 1 2

$$\boxed{E[Y] = 1 \quad \text{and} \quad Var(Y) = 1/6}$$

So

$$Cov(X, Y) = E[XY] - EX\,EY$$

$$= 7/12 - (1/2)(1) = 1/12$$

$$Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{1/12}{\sqrt{\frac{1}{12} \cdot \frac{1}{6}}}$$

Conditional Probability

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

If X and Y are discrete. Consider

$$A = \{X = x\} \quad \text{and} \quad B = \{Y = y\}$$

$$P(X = x \mid Y = y) = P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(X = x, Y = y)}{P(Y = y)}$$

$\mathbb{P}(X=x \mid Y=y)$

$$\boxed{= \frac{f(x,y)}{f_Y(y)}}$$

Defn: Conditional PMF

If X and Y are discrete then the conditional PMF of X given $Y=y$ is

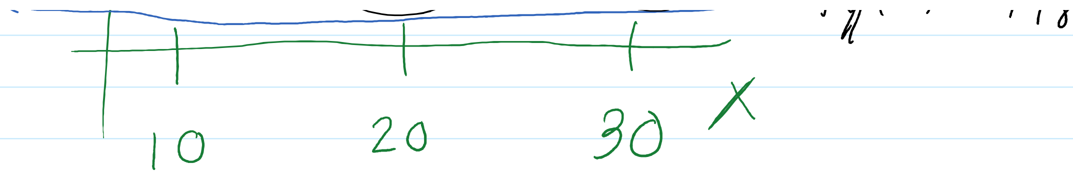
$$f(x \mid y) = f_{X \mid Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

Basically think of $Z =$ "$X \mid Y=y$" as a univariate RV.

Ex. Joint PMF $\rightarrow f(x,y)$

$Y \backslash X$	10	20	30
2	0	0	4/18
1	3/18	4/18	3/18
0	2/18	2/18	0

$f_Y(0) = 4/18$



Lets get dist of $X \mid Y = 0$

$$f(x|0) = \frac{f(x,0)}{f_Y(0)} = \begin{cases} \frac{2/18}{4/18} = 1/2 & x = 10 \\ \frac{2/18}{4/18} = 1/2 & x = 20 \\ 0 & x = 30 \end{cases}$$

$X \mid Y = 0$ equally split btwn 10, 20

Defn: Conditional PDF

If X and Y are continuous then the conditional PDF of X given $Y = y$ is

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

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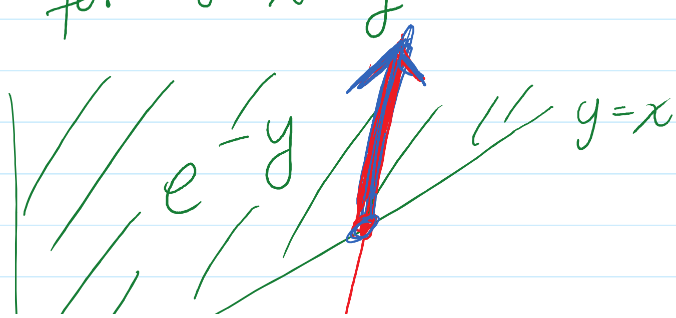
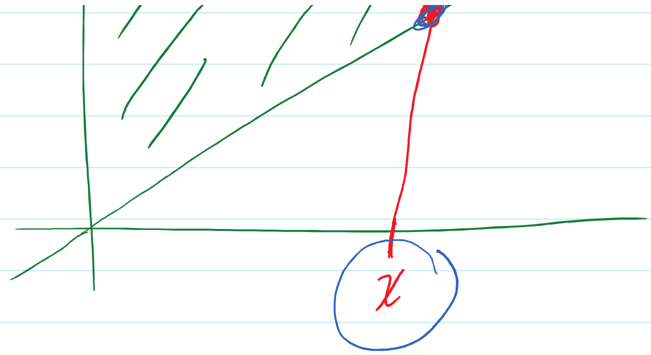
ex. $f(x,y) = e^{-y}$ for $0 < x < y$

What is the PDF of $Y \mid X = x$?



of $Y \mid X = x$:



Marginal of X

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_{y=x}^{\infty} e^{-y}\,dy = -e^{-y}\Big|_x^{\infty}$$

$$= 0 - (-e^{-x})$$

$$\boxed{\text{for } x > 0 \qquad = e^{-x}}$$

$X \sim \text{Exp}(\lambda = 1)$

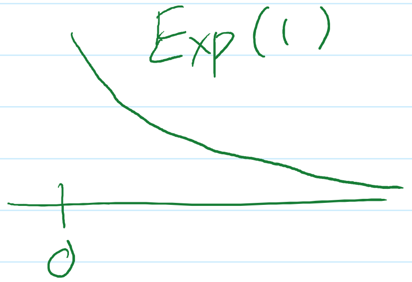
So

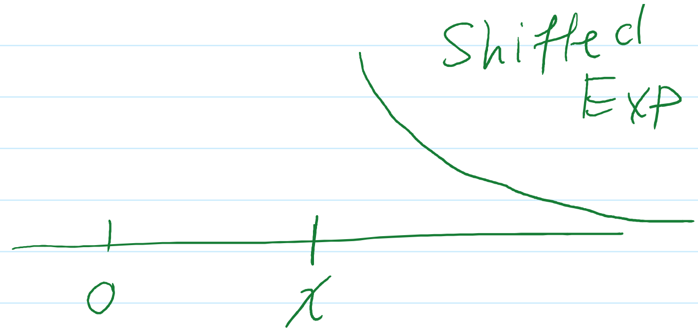
$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} \quad \text{for } x < y$$

$$\boxed{= e^{-(y-x)} \quad \text{for } y > x}$$

Called Shifted Exponential dist

Exp(1) Shifted Exp


Exp(1)
0


Shifted Exp
0
x