Defn: Sample Space

The sample space S is the set of all possible outcomes.

Ex. Flip a coin

$$S = \{H, T\}$$

Ex. Roll a six-sided die

$$S = \{1, 2, 3, 4, 5, 6\}$$

Ex. Roll two dice

$$S = \{(1,1), (1,2), (1,3), \ldots, (3,4), \ldots\}$$

Ex. Waiting for a bus to arrive

$$S = [0, \infty) \subset \mathbb{R}$$

Ex. Number of customers that arrive in my restaurant

$S = \mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

Types of sample spaces:

1. finite $(|S| < \infty)$ (roll dice)

2. infinte $(|S| \geq \infty)$
   - i) cantable $(\mathbb{N}_0)$
   - ii) uncantable $[0, \infty)$

Defn: Outcomes

We call elements of S outcomes:

outcome $s \in S$ sample space

Ex. $S = \{1, \dots, 6\}$

then $1 \in S$ so 1 is an outcome.

Defn: Event

An event E is a subset of S.

↳ collections of outcomes.

Ex. $S = \{1, \ldots, 6\}$ then

$$E = \{\text{even numbered roll}\}$$
$$= \{2, 4, 6\} \subset S$$

Ex. $S = \{(i, j) \mid 1 \leq i \leq 6,\ 1 \leq j \leq 6\}$

$$E = \{(2,1), (3,2)\}$$
$$F = \{(1,2), (2,3)\}$$

different events

We say an event E "happens" if the observed outcome is in E.

Ex. $\emptyset \subset S$ so $\emptyset$ is an event.

nothing happens ???

Ex. $S \subset S$ so S is an event

event that something happens.

Axiomatic Probability

Given: an experiment and assoc.

Given: an experiment and assoc. sample space S

Want: For any $E \subset S$ want to assign some measure of probability of E occurring

Mathematically:

For each $E \subset S$ assign a prob. $\mathbb{P}(E)$

What are the rules for building $\mathbb{P}$?

1. mathematically consistent
2. encode some intuition about probability.

Defn: Probability Function

Given a sample space S a prob. fn. $\mathbb{P}$ is a function

$$\mathbb{P} : 2^S \longrightarrow \mathbb{R}$$

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that satisfies the Kolmogorov Axioms

1. non-negativity

$$P(E) \geq 0 \quad \forall E \subset S$$

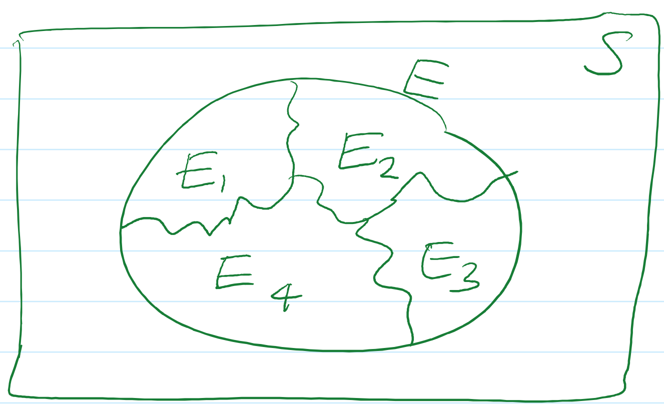
2. unit-measure

$$P(S) = 1$$

3. countable additivity

If $(E_i)_{i=1}^{\infty}$ is a countable partition of E.

$\left(\bigcup_i E_i = E \text{ and } E_i E_j = \emptyset \right)$



then

$$P(E) = \sum_{i=1}^{\infty} P(E_i)$$

notes: 1. $E = \bigcup_{i=1}^{\infty} E_i$

then this axiom says

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

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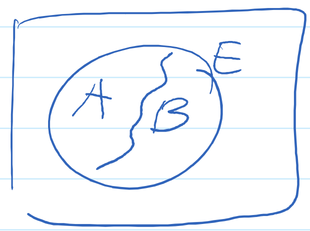
basically: can distribute $\mathbb{P}$ over disjoint unions

(2) this implies same for finite partitions.

$E = \bigcup_{i=1}^{n} E_i$ then $\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E_i)$.

Famous example: $E = A \cup B$ and $AB = \emptyset$.

$\mathbb{P}(E) = \mathbb{P}(A) + \mathbb{P}(B)$



Ex. Flip a coin.

$S = \{H, T\}$

what is a valid prob. fn?

$\mathbb{P}(\{H\}) = \frac{1}{2}$ $\qquad$ $P(\underbrace{\{H, T\}}_{S}) = 1$

$\mathbb{P}(\{T\}) = \frac{1}{2}$ $\qquad$ $\mathbb{P}(\emptyset) = 0$

Is this a valid $\mathbb{P}$?

✓ (1) $P(E) \geq 0$

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✓ (2) $P(S) = 1$

(3) $P(E) = \sum_{i=1}^{\infty} P(E_i)$ when E_i partition E

<u>One example:</u>

$E = S$ and $E_1 = \{H\}$, $E_2 = \{T\}$

✓ $1 = P(S) = P(E) = P(E_1) + P(E_2) = \frac{1}{2} + \frac{1}{2}$

<u>Ex.</u> $S = \{H, T\}$

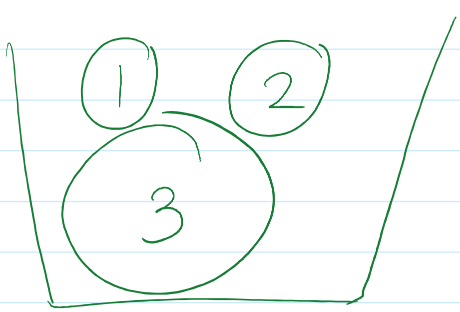
$P(S) = 1$ $\qquad$ $P(\{H\}) = \alpha$

$P(\emptyset) = 0$ $\qquad$ $P(\{T\}) = 1 - \alpha$

where $0 \leq \alpha \leq 1$

this should work.

<u>Ex.</u>



$S = \{1, 2, 3\}$

$1 \to \frac{1}{4}$, $2 \to \frac{1}{4}$, $3 \to \frac{1}{2}$

numbers are ≥ 0

and sum to 1.

$P(\{1\}) = \frac{1}{4}$

$P(\{3\}) = 1/2$

$P(\{2,3\}) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

<u>Theorem</u>: <u>Finite Sample Space Theorem</u>

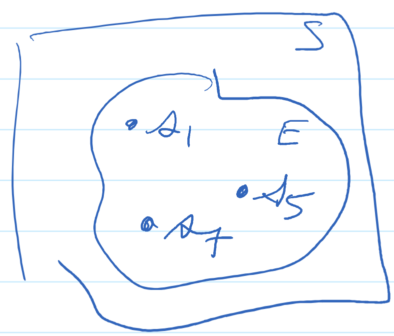
If $S = \{s_1, \ldots, s_n\}$ so that $|S| = n < \infty$

and we choose $p_1, p_2, \ldots, p_n$ so that

$$\text{(1)}\ p_i \geq 0 \ \forall i \quad \text{and} \quad \text{(2)}\ \sum_{i=1}^{n} p_i = 1$$

then if $E \subset S$

$P(E) =$ sum p_i assoc. w/ each $s_i \in E$



$$= \sum_{i : s_i \in E} p_i$$

$P(E) = p_1 + p_5 + p_7$

Then $\mathbb{P}$ is a valid prob. fn on S.

<u>Pf.</u> Show it satisfies Kolmogorov Axioms

① $P(E) \geq 0$

$P(E) = $ sum of some non-neg. p_i ≥ 0

② $P(S) = 1$

$$P(S) = \underline{\sum_{i: s_i \in S}} p_i = \sum_{i=1}^{n} p_i = 1$$

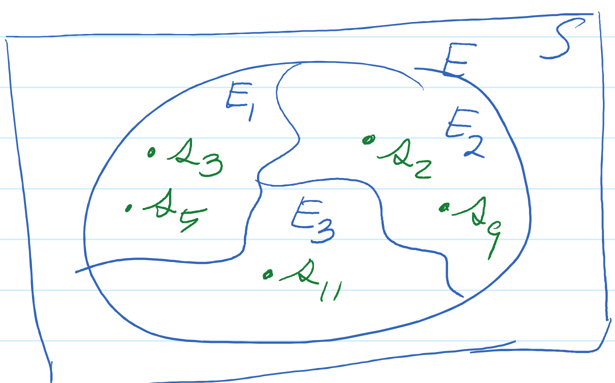
③ E_i partition E then $\underline{P(E) = \sum_{i=1}^{\infty} P(E_i)}$

$$P(E) = P\Big(\underline{\bigcup_{i=1}^{\infty}} E_i\Big)$$

$$= \sum_{j: s_j \in \underline{\cup E_i}} p_j \longleftarrow$$

$$= \sum_{i=1}^{\infty} \underline{\sum_{j: s_j \in E_i}} p_j \longleftarrow$$

$$= \sum_{i=1}^{\infty} P(E_i)$$



$$P(E) = p_3 + p_5 + p_2 + p_9 + p_{11}$$

$$\underline{P(E_1) + P(E_2) + P(E_3)}$$

Basic Theorems:

$$\boxed{\text{Theorem: } P(\emptyset)=0.}$$

pf

$$S = S \cup \underbrace{\emptyset \cup \emptyset \cup \emptyset \cup \cdots}_{\text{cantably infinte}}$$

$$\underbrace{P(S)}_{1} = P(S) + P(\emptyset) + P(\emptyset) + P(\emptyset) + \cdots$$

$$= \underbrace{P(S)}_{1} + \sum_{i=1}^{\infty} P(\emptyset)$$

So $\sum_{i=1}^{\infty} P(\emptyset) = 0$.

the only way this works is if $P(\emptyset)=0$.

Finite Additivity

$E = A \cup B \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots$ and $AB = \emptyset$.

Axiom 3:

$$P(E) = P(A) + P(B) + \underbrace{P(\emptyset) + P(\emptyset) + \cdots}_{0}$$
