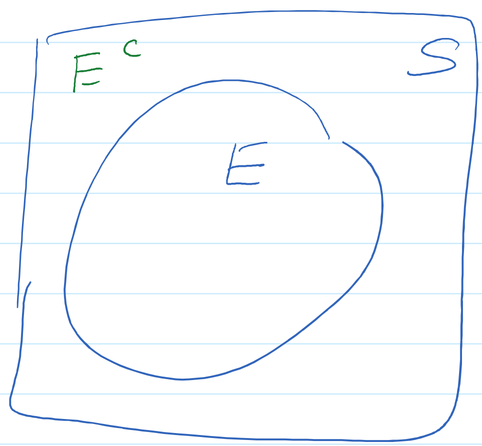
Ex. If $E=$ "its raining"
and $P(E)=1/3$

$P($"Its not raining"$)$?
$=2/3$

$P(E^c)=2/3=1-1/3$

Theorem: $\boxed{P(E^c)=1-P(E)}$



Pf.
$S=E\cup E^c$
partition

So
$1=P(S)=P(E)+P(E^c)$

So rearrange to get $P(E)=1-P(E^c)$.

$\boxed{\text{Theorem: } 0\le P(E)\le 1}$

Pf. $P(E)\ge 0$ by Axiom 1.

Similarly $P(E^c)\ge 0$

by prev. result,

$$1 - P(E) \geq 0$$

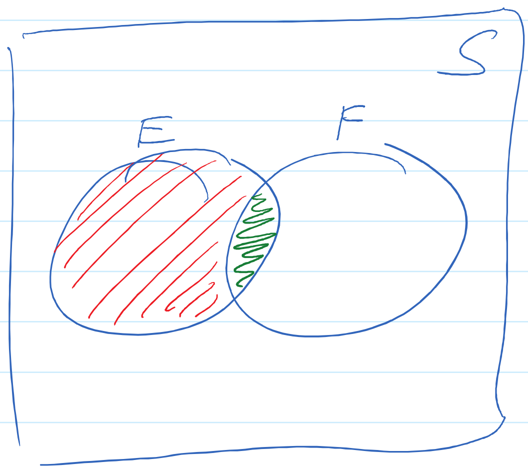
or, rearrange,

$$P(E) \leq 1.$$

<u>Theorem:</u> If $E, F \subset S$ then

$$P(E \setminus F) = P(E) - P(EF)$$

<u>Pf.</u> $E = \underbrace{EF}_{\text{green}} \cup \underbrace{EF^c}_{\text{red}}$



$$P(E) = P(EF) + P(EF^c)$$

rearrange

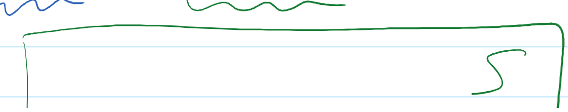
$$\underbrace{P(EF^c)}_{=\,P(E \setminus F)} = P(E) - P(EF)$$

<u>Theorem: Non-disjoint Unions</u>

$E, F \subset S$, then

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

pf.



pf.

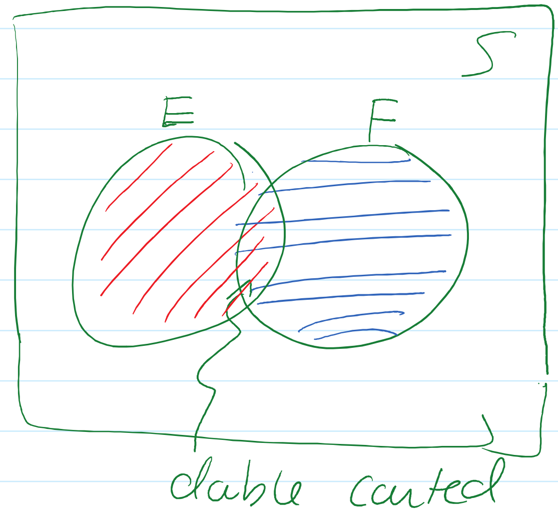
$$E \cup F = E \cup FE^c$$

↑ partition

So

$$P(E \cup F) = P(E) + P(FE^c)$$

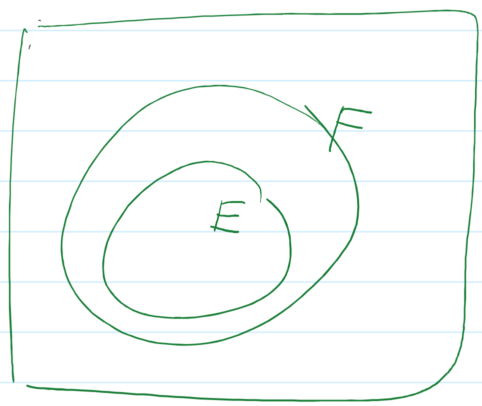
$$= P(E) + P(F) - P(\underbrace{FE^{cc}}_{FE})$$



Theorem: If $E \subset F$

then

$$P(E) \leq P(F)$$



pf. Axiom 1 says

$$P(FE^c) \geq 0$$

then

$$P(F) - P(EF) \geq 0$$

so $$P(EF) \leq P(F)$$

but $E \subset F$ so $EF = E$

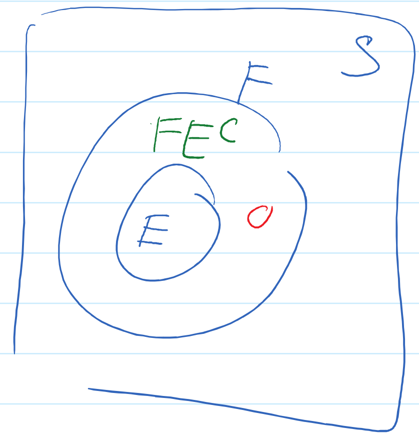
hence

$\mathbb{P}(E) \leq \mathbb{P}(F)$.

Consider $E \subset F$ but $E \neq F$.
(proper subset)

$$\cancel{\mathbb{P}(E) < \mathbb{P}(F)} \quad ?$$

It could be that FE^c has zero prob.



We said that

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \underbrace{\mathbb{P}(EF)}_{\geq 0}$$

$$\leq \mathbb{P}(E) + \mathbb{P}(F)$$

Generalized: Boole's Inequality

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

don't require disjoint

<u>pf.</u> Replace E_i w/ B_i where

1. $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} E_i$

(2) B_i disjoint

define:

$B_1 = E_1$

$B_2 = E_2 \setminus B_1$

$B_3 = E_3 \setminus B_2$

$B_4 = E_4 \setminus B_3$

$\vdots$

Convince yourself that the above 2 facts are true

Notice $B_i \subset E_i$

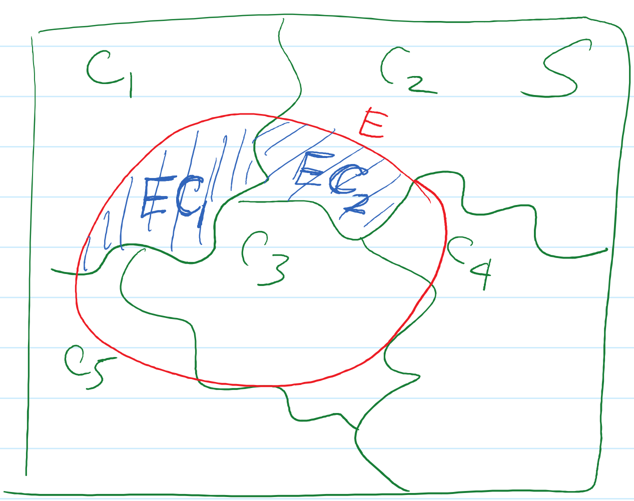
$P(B_i) \leq P(E_i)$

$$P\left(\bigcup_i E_i\right) = P\left(\bigcup_i B_i\right) = \sum_i P(B_i) \leq \sum_i P(E_i)$$

Theorem:

If (C_i) partitions S

$$P(E) = \sum_i P(EC_i).$$



Pf. (1) (EC_i) partitions E

(2) By additivity

$$P(E) = P\left(\bigcup_i EC_i\right)$$

$$= \sum_i P(EC_i)$$

Equally Likely Outcomes

I have a sample space S

$$S = \{s_1, \dots, s_n\} \quad \text{so that } |S| = n$$

and assume that

$$\frac{1}{n} = \mathbb{P}(\{s_i\}) = \mathbb{P}(\{s_j\}) \quad \forall i, j$$

Rationale:

$$1 = \mathbb{P}(S) = \sum_{i=1}^{n} \mathbb{P}(\{s_i\})$$

only works if each have prob $1/n$.

More generally:

If $E \subset S$ then

$$\mathbb{P}(E) = \frac{\#\text{ elements in } E}{\#\text{ elements in } S} = \frac{|E|}{|S|}$$

Ex. Roll six-sided die

$$S = \{1, 2, \dots, 6\}$$

If all rolls are equally likely then

$$E = \{2, 6\}$$

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{2}{6} = 1/3.$$

Counting

<u>Ex.</u> An experiment consists of 3 factors

1. 2 temp. settings
2. 2 pressure settings
3. 4 humidity settings

<u>Q</u>: How many possible experiments? 16!

Theorem: The Fundamental Theorem of Counting. (FTC)

If I have a task consisting of k subtasks where the i^{th} subtask can be done in n_i possible ways.

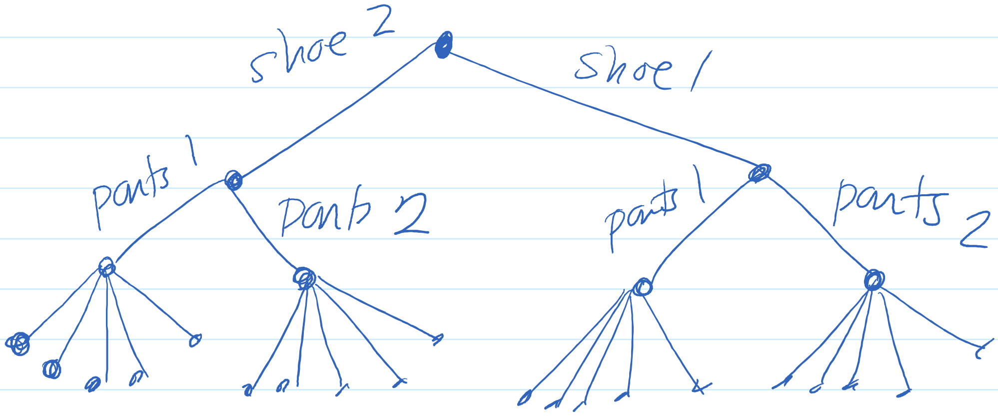
I can complete the overall task in

$$N = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$
$$= \prod_{i=1}^{n} n_i$$

<u>Ex.</u> A man has 5 shirts, 2 pair pants, 2 pair shoes.

How many outfits does he have?

<u>Ans.</u> By FTC he has $5 \cdot 2 \cdot 2 = 20$ outfits.



20 leaf nodes

<u>Ex.</u> I have a deck of 52 cards.

I shuffle so that each ordering is equally likely

Q: what is the prob. after shuffle they are "in order": A–K, C, D, H, S

E = in order

S = all possible orderings

then
$$P(E) = \frac{|E| \leftarrow 1}{|S| \leftarrow ???}$$

By FTC let $k = 52$

task #	task	# ways
1	choose card 1	52
2	" 2	51
3	" 3	50
⋮	⋮	⋮

multiply

so $|S| = 52 \cdot 51 \cdot 50 \cdot 49 \cdots 1$

hence $P(E) = 1/(52 \cdot 51 \cdot 50 \cdots 1)$.

Defn: Factorial

For any non-neg. integer n we define n factorial as

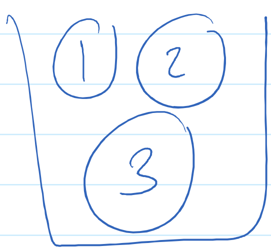
$$n! = (n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$
$$= \prod_{i=1}^{n} i$$

note: $0! = 1$

Prev. Ex $\quad \mathbb{P}(E) = 1/52!$

Sampling w/ and w/o Ordering and Replacement

Ordering

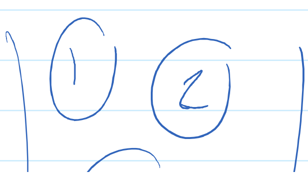


draw 1: $\quad 1 \; 3 \; 2$

draw 2: $\quad 2 \; 1 \; 3$

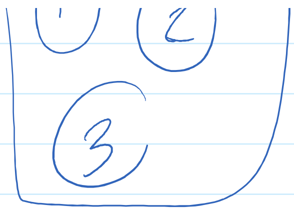
are these different?

Replacement



Can I draw a sample

$1 \; 1 \; 2$?



1 1 2 ?

<u>Yes!</u> w/ replacement

<u>No!</u> w/o replacement

<u>4 scenarios:</u>

	w/o repl.	w/ repl.
ordered	1	2
un-ordered	4	3

<u>Permutation:</u>

A permutation is an ordering of a collection of objects

<u>Ex.</u> Objects A_1, A_2, A_3 ← 3 objects

then my permutations are

Then my permutations are

$A_1 A_2 A_3 \quad A_1 A_3 A_2 \quad A_2 A_3 A_1$
$A_2 A_1 A_3 \quad A_3 A_2 A_1 \quad A_3 A_1 A_2$ $\Big\} \; 6 = 3!$

<u>Theorem</u>: The number of ways to permute n items is $n!$

<u>Pf.</u> Use FTC w/ $k = n$ subtasks

task #	task	# ways
1	choose item 1	n
2	" 2	$n-1$
3	" 3	$n-2$
⋮	⋮	⋮

FTC say multiply to get

$n \cdot (n-1)(n-2) \cdots 1 = n!$