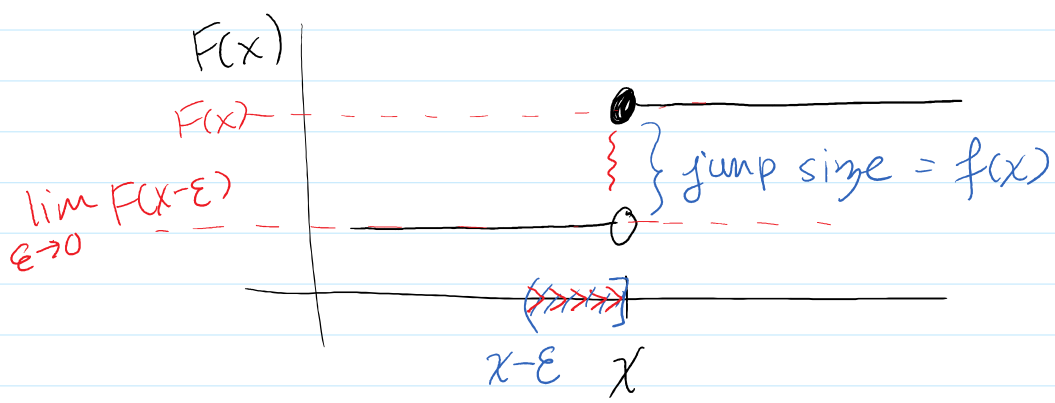
Lecture 9 - PDFs

PMF : $f(x) = \mathbb{P}(X = x)$

CDF : $F(x) = \mathbb{P}(X \le x)$

$$F(x) = \sum_{i \le x} f(i)$$



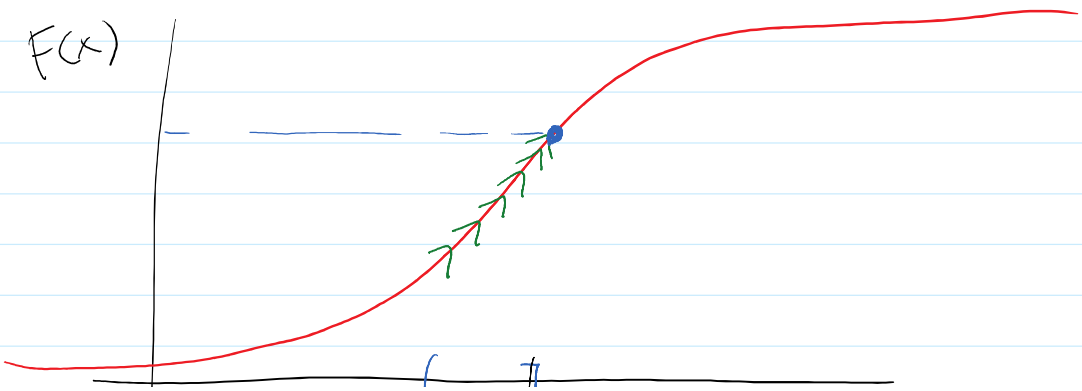

$$\mathbb{P}(a < X \le b) = F(b) - F(a)$$

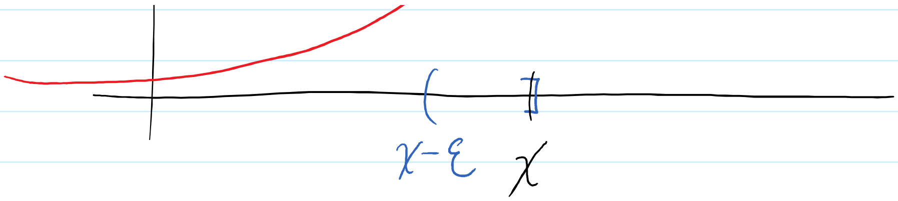
$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(x - \varepsilon < X \le x) = \lim_{\varepsilon \downarrow 0} F(x) - F(x - \varepsilon)$$

$$\mathbb{P}(X = x) = F(x) - \lim_{\varepsilon \downarrow 0} F(x - \varepsilon)$$

$$f(x) = \text{jump size}$$

## What about a cts RV?






$$P(X=x) = \lim_{\varepsilon \downarrow 0} P(x-\varepsilon < X \le x)$$

$$= \cdots = F(x) - \underbrace{\lim_{\varepsilon \downarrow 0} F(x-\varepsilon)}_{F(x)}$$

$$= 0$$

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Want: cts analog for PMF:

$$F(x) = \sum_{i \le x} f(i)$$

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Defn: Probability Density Function (PDF)

Cts version of PMF

The PDF for a cts RV is a function $f: \mathbb{R} \to \mathbb{R}$, defined for $x \in \mathbb{R}$, as the function that satisfies

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

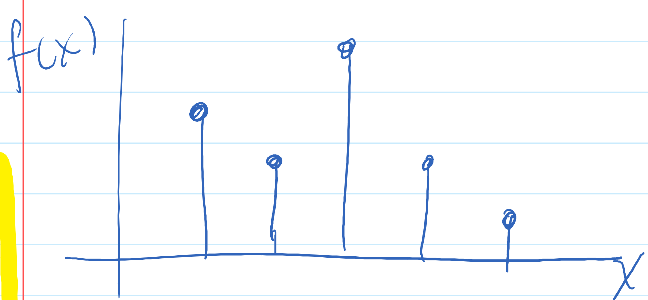
$-\infty$

**Note:** by the Fund. Theorem of Calc,

$$\frac{dF}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)$$

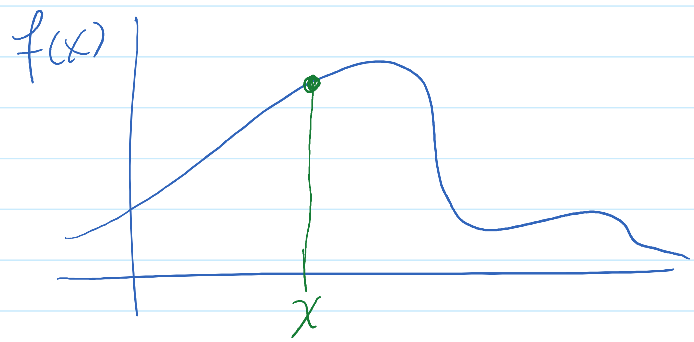
So $f(x) = \dfrac{dF}{dx}$. (PDF = deriv. of CDF)

discrete PMF




$$f(x) = P(X = x)$$

Continuous PDF

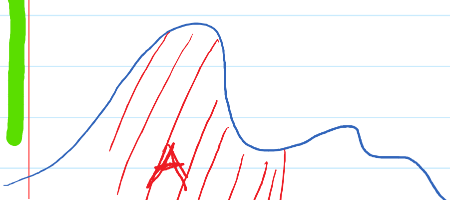



$$f(x) \neq P(X = x)$$

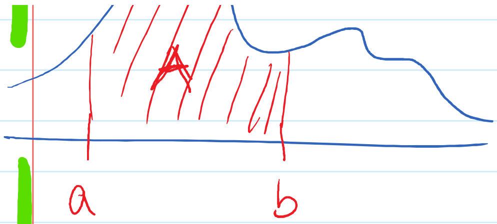
## Properties of PDFs

$$P(a < X \leq b) = F(b) - F(a)$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$




$\int^{b}$




$$= \int_a^b f(t)\,dt$$

$$\mathbb{P}(X=a) = \mathbb{P}(X=b) = 0$$

$$\left.\begin{aligned} \mathbb{P}(a < X \le b) &= \mathbb{P}(a \le X \le b) \\ &= \mathbb{P}(a < X < b) \\ &= \mathbb{P}(a \le X < b) \end{aligned}\right\} \text{for cts RV}$$

Generally:

(discrete) $\mathbb{P}(X \in A) = \sum_{i \in A} f(i)$

(cts) $\mathbb{P}(X \in A) = \int_A f(x)\,dx$

Ex. $F(x) = \dfrac{1}{1+e^{-x}}$

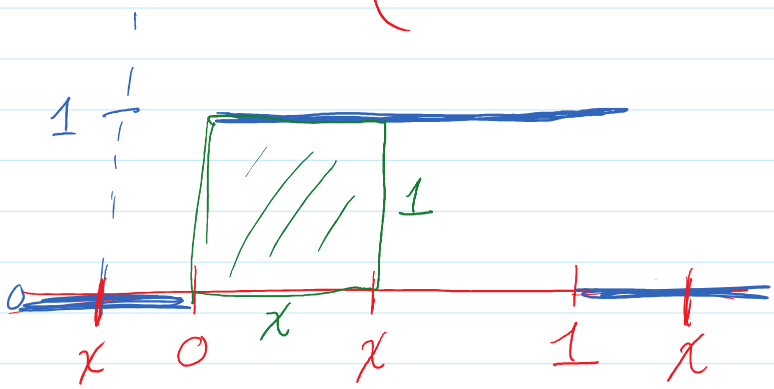

Q: What is its PDF?

$$f(x) = \frac{dF}{dx} = \cdots = \frac{e^{-x}}{(1+e^{-x})^2}$$

Ex. Continuous Unif. Dist.

$X \sim U(0,1)$

means $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{else} \end{cases}$




what's the CDF? $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$x < 0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

$0 < x < 1$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 1\,dt = x$$

$x > 1$

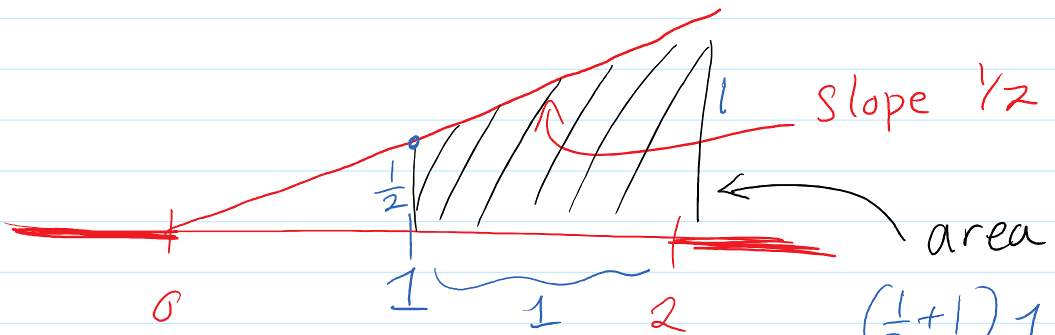
$$F(x) = \int^{x} f(t)\,dt = \int^{1} 1\,dt = 1$$

$$F(x) = \int_{-\infty}^{1} f(t)\,dt = \int_{0}^{1} 1\,dt = 1$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

## Ex.

$$f(x) = \frac{x}{2} \quad \text{for} \quad 0 < x < 2$$




area: $\frac{\left(\frac{1}{2}+1\right)}{2} 1 = \frac{3}{4}$

$$P(X > 1) = \int_1^{\infty} f(x)\,dx = \int_1^{2} \frac{x}{2}\,dx$$

$$= \left.\frac{x^2}{4}\right|_1^2 = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

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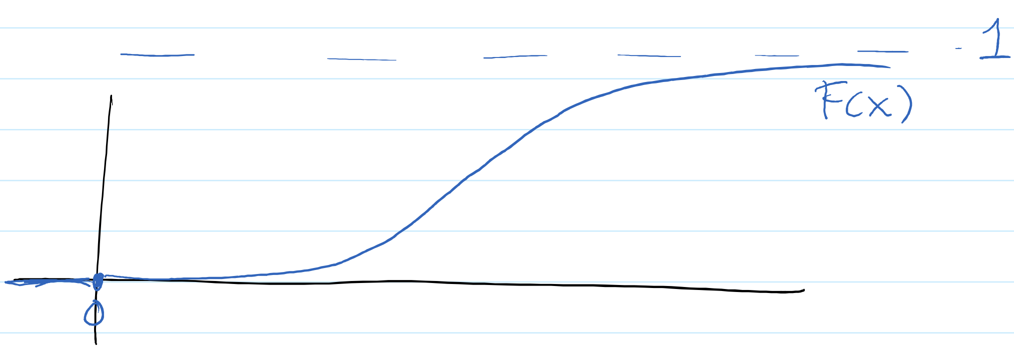
## Ex.

$$F(x) = 1 - e^{-x} \quad \text{for} \quad x > 0$$




Q: $P(1 < X < 2)$?

Way 1: $P(1 < X < 2) = F(2) - F(1)$

$$= (1-e^{-2}) - (1-e^{-1})$$

$$= e^{-1} - e^{-2}$$

Way 2: $f(x) = \frac{dF}{dx} = \frac{d}{dx}(1-e^{-x}) = e^{-x}$ for $x > 0$

$$P(1 < X < 2) = \int_1^2 f(x)\,dx = \int_1^2 e^{-x}\,dx = -e^{-x}\Big|_1^2$$

$$= e^{-1} - e^{-2}$$

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**Theorem: PMF/PDF characterization**

A function $f$ is the PMF/PDF of some RV iff

1. $f(x) \geq 0 \quad \forall x$

2. (discrete) $\sum_{x \in \mathbb{R}} f(x) = 1$

   (cts) $\int_{\mathbb{R}} f(x)\,dx = 1$

(cts) $\int_{\mathbb{R}} f(x)\,dx = 1$

If $g(x) \geq 0$ and $\int_{\mathbb{R}} g(x)\,dx = c < \infty$

$$f(x) = \frac{1}{c} g(x)$$

then $f$ is a PDF.

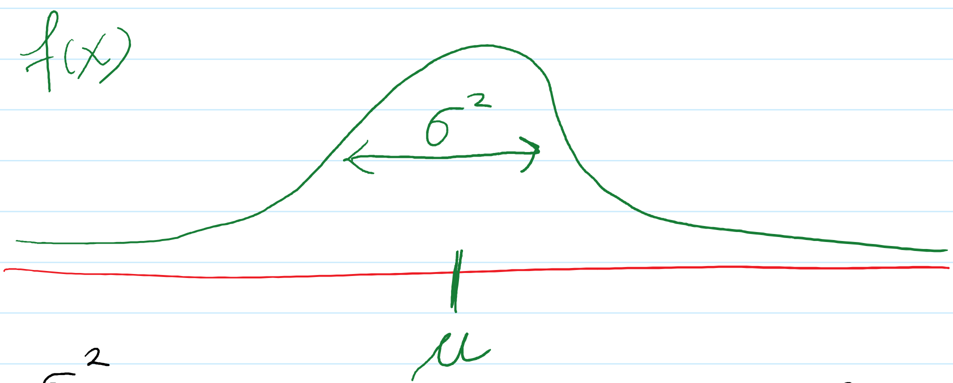
## Ex. Normal Distribution (Gaussian)

notation:

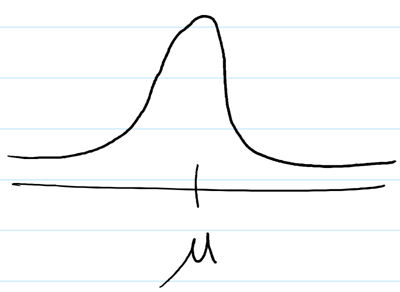
$$X \sim N(\mu, \sigma^2)$$

variance: $\sigma^2 > 0$

mean: $\mu \in \mathbb{R}$




Small $\sigma^2$




large $\sigma^2$


Special Case: Standard normal $\mu = 0$, $\sigma^2 = 1$

$$X \sim N(0,1).$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

$\exp(a) = e^a$

$\exp(a) = e$

Q: Is this a valid PDF?

① $f(x) \geq 0$ ✓

② $\int_{\mathbb{R}} f(x)\,dx = 1$

$$\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx}_{I} = 1$$

Want: $I = 1 \iff I^2 = 1$.

$$I^2 = I \cdot I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy$$

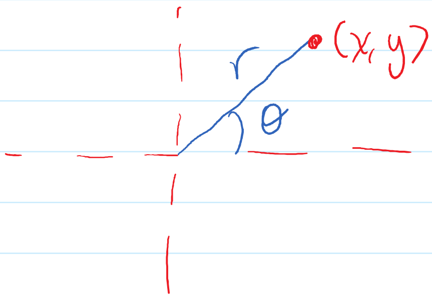
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}x^2\right) \exp\left(-\frac{1}{2}y^2\right) dx\,dy$$

$$e^a e^b = e^{a+b}$$

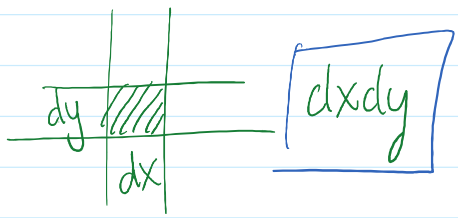
$$= \iint \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2 + y^2)\right) dx\,dy$$

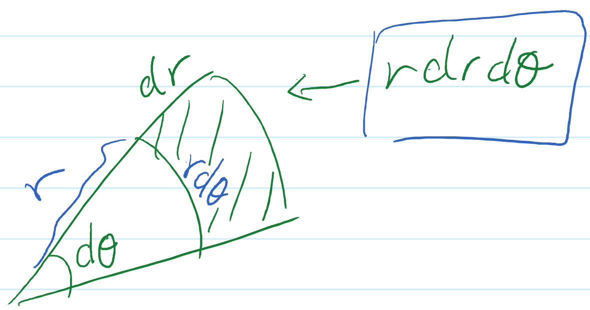
$$= \iint_{\mathbb{R}^2} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right) dx\,dy$$

## Polar Coordinates




$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$$




$$\frac{1}{2\pi} \iint_{\mathbb{R}^2} \exp\left(-\frac{1}{2}(x^2+y^2)\right) dx\,dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\int_0^{\infty} \exp\left(-\frac{1}{2}r^2\right) r\,dr}_{\text{u-substitution}}\, d\theta$$

$u = \frac{1}{2}r^2$

$du = r\,dr$

$$\int_0^{\infty} e^{-u}\,du = -e^{-u}\Big|_0^{\infty} = 1$$

$2\pi$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1 = I^2 \Rightarrow I = 1.$$

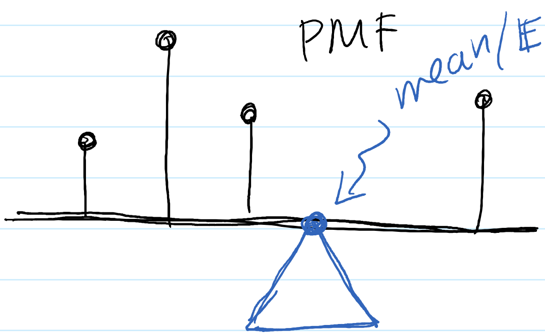
## Expected Value

If $X$ is a RV then the mean or expected value of $X$

denoted $\mathbb{E}[X]$.

is defined as

① discrete $\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x f(x)$ ← PMF




$$= \sum_{x \in \text{Support}(X)} x f(x)$$

② continuous: $\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\, dx$ ← PDF