Lecture 22 - Bivariate Transformations

Tuesday, November 30, 2021 2:01 PM

Uni: $g: \mathbb{R} \to \mathbb{R}$ what is dist $g(X)$?

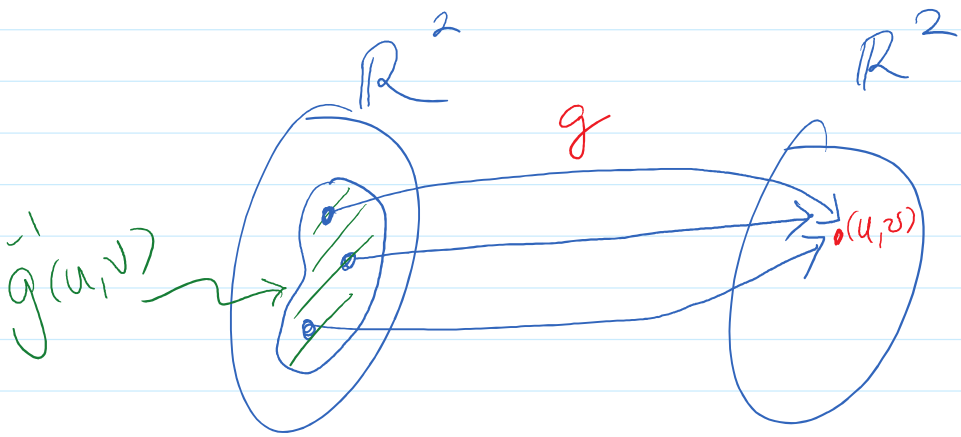
Biv: $g: \mathbb{R}^2 \to \mathbb{R}^2$ what is the dist of $g(X, Y)$?

notation: $(X, Y) \xrightarrow{g} (U, V)$

Ex. $(U, V) = (\underbrace{X^2 Y}_{g_1}, \underbrace{-\log Y}_{g_2})$

Discrete: $(U, V) = (g_1(X, Y), g_2(X, Y))$

Assume X and Y discrete.



$g^{-1}(u, v) = \{(x, y) \mid g_1(x, y) = u \text{ and } g_2(x, y) = v\}$

<u>Want</u>: PMF of (U, V) from PMF of (X, Y)

$$f_{U,V}(u,v) = P(U=u, V=v)$$

$$= P((U,V) \in \{(u,v)\})$$

$$= P(g(X,Y) \in \{(u,v)\})$$

$$= P((X,Y) \in g^{-1}(u,v))$$ same set

$$\boxed{= \sum_{(x,y) \in g^{-1}(u,v)} f_{X,Y}(x,y)}$$

<u>Ex.</u> let $X \perp\!\!\!\perp Y$

and $X \sim Pois(\theta)$, $Y \sim Pois(\lambda)$ } discrete

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \boxed{\frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^y e^{-\lambda}}{y!}}$$

let $U = X+Y$ and $V = Y$.

$u = g_1(x,y) = x+y$ and $\underline{v = g_2(x,y) = y}$

Solve for x,y in terms of u,v

$$\boxed{\begin{aligned} y &= v = g_2^{-1}(u,v) \\ x &= u-v = g_1^{-1}(u,v) \end{aligned}}$$

$u = x+y$

$x = u-y = u-v$

So

$$f_{U,V}(u,v) = f_{X,Y}(\underline{u-v}, \underline{v})$$

$$\boxed{= \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^{v} e^{-\lambda}}{v!}}$$

$U = X+Y$

$V = Y$ so $\underline{U \geq V}$

what is dist of U?

$$f_U(u) = \sum_{v=0}^{u} f(u,v)$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^{u} \theta^{u-v} \lambda^{v} \frac{u!}{(u-v)!\,v!} \quad \binom{u}{v}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \underbrace{\sum_{v=0}^{u} \binom{u}{v} \lambda^{v} \theta^{u-v}}$$

Binomial Theorem

$(\theta+\lambda)^{u}$

$$f(u) = \frac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u$$

$\sim \text{Pois}(\theta+\lambda)$

<u>Theorem:</u> $X \perp\!\!\!\perp Y$ and $X \sim \text{Pois}(\theta)$

$Y \sim \text{Pois}(\lambda)$

then $X+Y \sim \text{Pois}(\theta+\lambda)$.

What about cts?

If g is nice in uni case:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

<u>Bivariate Case:</u>

- X and Y continuous
- $(u, v) = (g_1(X,Y), g_2(X,Y))$

g is invertible and g^{-1} is differentiable

$$\boxed{f_{U,V}(u,v) = f_{X,Y}\left(g_1^{-1}(u,v), g_2^{-1}(u,v)\right)\left|\det J\right|}$$

Jacobian of g^{-1}

Jacobians

$h: \mathbb{R}^2 \to \mathbb{R}^2$

$$h(x,y) = \left(h_1(x,y), h_2(x,y)\right)$$

then the Jacobian is

$$J = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix}$$

in our case we want Jacobian of g^{-1}

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix}$$

For a 2×2 Mtx

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ then $\det A = ad - cb$

Process:

① get g_1^{-1}, g_2^{-1}

② Find J and $\det J$

③ plug in formula

Ex.

$$\begin{cases} (u,v) = (x+y,\ x-y) \\ u = g_1(x,y) = x+y \\ v = g_2(x,y) = x-y \end{cases}$$

notice $u+v = 2x$

so $x = \frac{u+v}{2}$

and $u-v = 2y$

so $y = \frac{u-v}{2}$

① get inverses

$$x = g_1^{-1}(u,v) = \frac{u+v}{2}$$

$$y = g_2^{-1}(u,v) = \frac{u-v}{2}$$

② get J and $\det J$

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

so $|\det J| = \left|\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right|$

$= \left|-\frac{1}{2}\right| = \frac{1}{2}$

③ Plug in formula

$$f_{U,V}(u,v) = \underbrace{f_{X,Y}}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)\underline{\frac{1}{2}}$$

let $X, Y \overset{iid}{\sim} N(0,1)$ ← independent and identically distributed

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u+v}{2}\right)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-v}{2}\right)^2} \frac{1}{2}$$

$$-\frac{1}{2}\left[\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right]$$

$$= -\frac{1}{2}\left(\frac{1}{4}(u^2 \cancel{+2uv} + v^2) + \frac{1}{4}(u^2 \cancel{-2uv} + v^2)\right)$$

$$= -\frac{1}{2}\left(\frac{1}{4}\right)(2u^2 + 2v^2)$$

$$= -\frac{1}{2}\left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right)$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right)}$$

$$f(u,v) = \underbrace{\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u^2}{2}\right)}}_{N(0,2)} \underbrace{\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{v^2}{2}\right)}}_{N(0,2)}$$

only involves u only v

$$U, V \overset{iid}{\sim} N(0,2)$$

<u>ex,</u>

$X \sim \text{Gamma}(\alpha, \lambda)$

$Y \sim \text{Gamma}(\beta, \lambda)$ and $X \perp Y$

$$U = \underbrace{X+Y}_{g_1} \quad \text{and} \quad V = \underbrace{X/(X+Y)}_{g_2}$$

(1) get inverses

$$uv = (x+y)\left(x/(x+y)\right) = x$$

so $\boxed{x = g_1^{-1}(u,v) = uv}$

$u = x+y$ so $y = u - x = u - uv = u(1-v)$

so $\boxed{y = g_2^{-1}(u,v) = u(1-v)}$

So $\boxed{y = g_2^{-1}(u,v) = u(1-v)}$

② Get J and $\det J$

$$J = \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} \Rightarrow \det J = -uv - u(1-v)$$

$$= -u$$

$$\Rightarrow |\det J| = u$$

③ plug-in

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)}$$

$$f(u,v) = f_{X,Y}\big(uv, u(1-v)\big)\, u$$

$$\boxed{= \frac{\lambda e^{-\lambda uv}(\lambda uv)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda u(1-v)}(u(1-v)\lambda)^{\beta-1}}{\Gamma(\beta)} u}$$

$= \cdots$ algebra

$$f(u,v) = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{u^{\alpha+\beta-1} e^{-\lambda u}}_{\substack{\text{only } u \\ \text{(some Gamma)}}} \underbrace{v^{\alpha-1}(1-v)^{\beta-1}}_{\substack{\text{only } v \\ \text{(some Beta)}}}$$

So $U \perp\!\!\!\perp V$.

and $U \sim \text{Gamma}(\alpha+\beta, \lambda)$

$V \sim \text{Beta}(\alpha, \beta)$

Theorem: Independence and Transformations

If $X \perp\!\!\!\perp Y$ and

$U = g(X)$ ← only X

$V = h(Y)$ ← only Y

then $U \perp\!\!\!\perp V$.

Ex. $U = X^2$ and $V = \log Y$.

Ex. $U = XY$ and $V = X$

what is dist of (U,V)?

① get inverses

$u = xy$ and $v = x$

$$\boxed{x = g_1^{-1}(u,v) = v}$$

$$\boxed{y = g_2^{-1}(u,v) = u/x = u/v}$$

② get J and $\det J$

$$J = \begin{bmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{bmatrix}$$

then $\det J = (0)(-u/v^2) - (1/v)(1)$

$= -1/v$

hence $|\det J| = 1/v$ (assuming $v > 0$)

③ plug-in

$$f(u,v) = f_{X,Y}(v, u/v)\frac{1}{v}$$