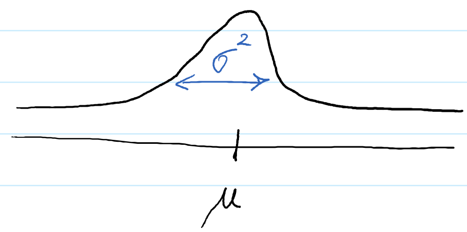
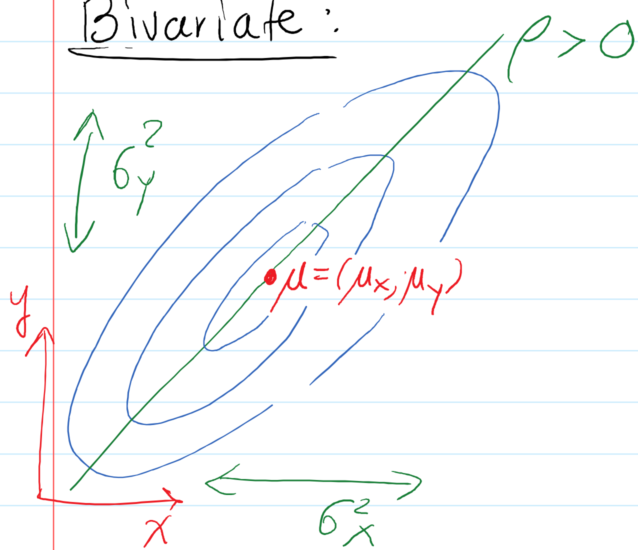
Bivariate Normal

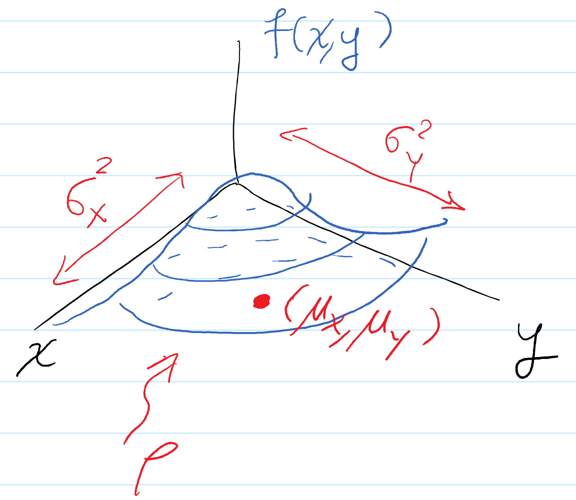
Univariate: $N(\mu, \sigma^2)$



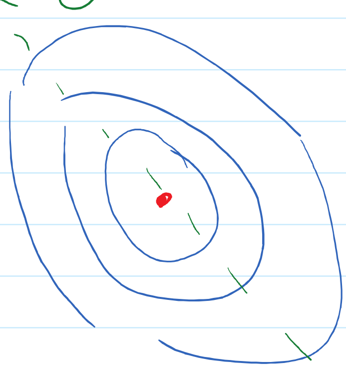
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad \forall x \in \mathbb{R}$$

Bivariate:

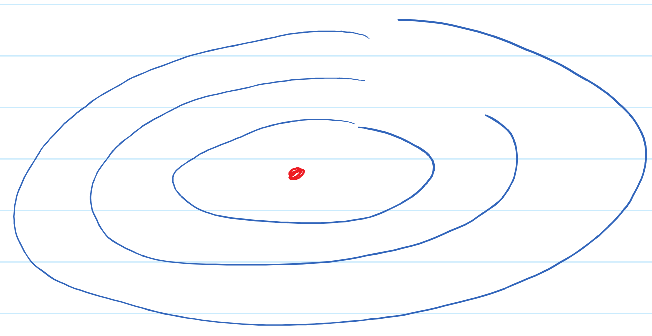




$\rho < 0$



$\rho \approx 0$



PDF: $(X, Y) \sim \mathrm{BivN}(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

$$f(x,y) = \frac{1}{2\pi\,\sigma_x \sigma_y \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2}\frac{1}{\sqrt{1-\rho^2}}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]\right\}$$

$\mu = (\mu_x, \mu_y)$ — mean vector

covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Var}(Y) \end{bmatrix}$$

$z = (x,y)$

then <u>PDF</u>:

$$f(z) = \frac{1}{2\pi}\frac{1}{\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}(z-\mu)^T \Sigma^{-1}(z-\mu)\right)$$

<u>Univ</u>:

$$f(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\sigma^2}} \exp\left(-\frac{1}{2}(x-\mu)(\sigma^2)^{-1}(x-\mu)\right)$$

<u>Facts</u>: (1) $X \sim N(\mu_x, \sigma_x^2)$

$Y \sim N(\mu_y, \sigma_y^2)$

(2) $\mathrm{Cor}(X,Y) = \rho$

(3) $aX + bY \sim N(a\mu_x + b\mu_y,\ a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_x\sigma_y\rho)$

(4) $(X, Y) \sim BivN \iff \forall a,b \quad aX + bY \sim N$

we had a theorem that said
if $X \perp Y$ then $Cor(X, Y) = 0$
(but generally converse is false)

(5) If $(X, Y) \sim BivN$ and $\rho = 0$ then $X \perp Y$

Bivariate Transformations

Uni: $g: \mathbb{R} \to \mathbb{R}$ what is the dist of $g(X)$?

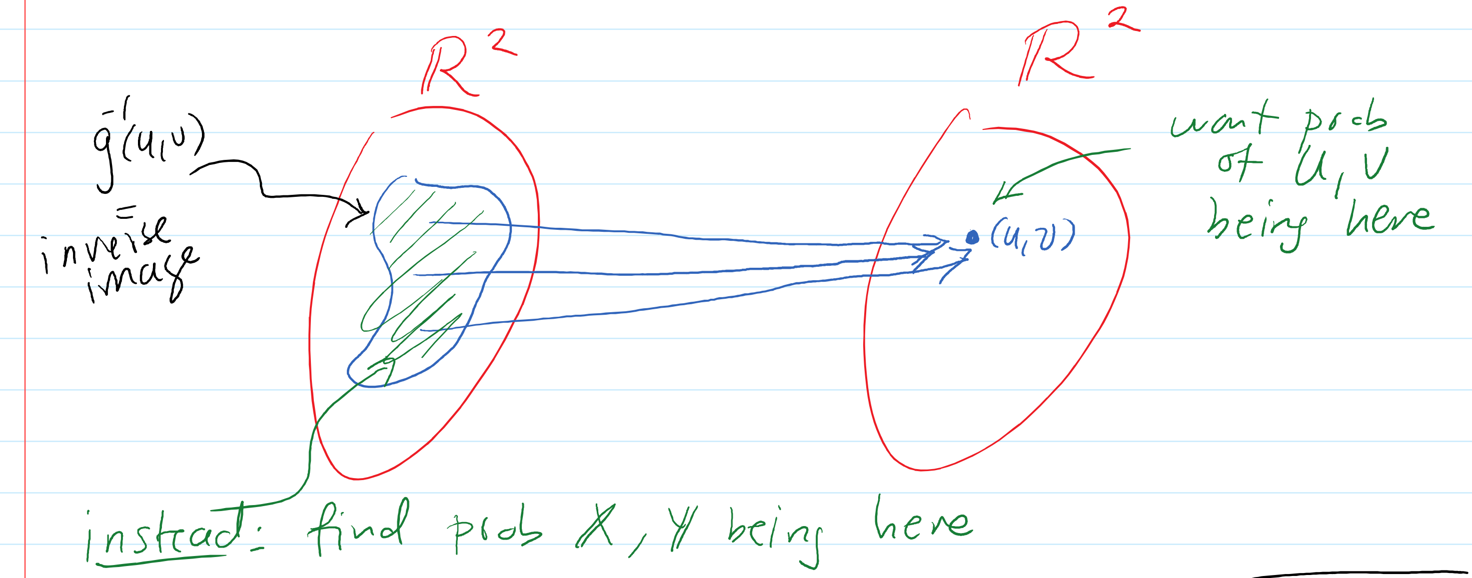
Biv: $g: \mathbb{R}^2 \to \mathbb{R}^2$ what is the dist of $g(X, Y)$?

Notation: $(X, Y) \xrightarrow{g} (U, V)$

Ex. $(U, V) = (\underbrace{X^2 Y}_{g_1(X,Y)},\ \underbrace{-\log Y}_{g_2(X,Y)}) = g(X, Y)$

Discrete: $(U, V) = (g_1(X,Y), g_2(X,Y))$

Assume X and Y are discrete.



Want: Joint PMF of (U,V) from PMF of (X,Y)

$$f_{U,V}(u,v) = \mathbb{P}(U=u, V=v)$$
$$= \mathbb{P}((U,V) \in \{(u,v)\})$$
$$= \mathbb{P}(g(X,Y) \in \{(u,v)\})$$

$\{(x,y) : g(x,y) = (u,v)\}$

$$= \mathbb{P}((X,Y) \in g^{-1}(u,v))$$
$$= \sum_{(x,y) \in g^{-1}(u,v)} f_{X,Y}(x,y)$$
$$\boxed{= \sum_{x,y\,:\,g(x,y)=u,v} f_{X,Y}(x,y)}$$

If g is invertible then

$$\boxed{= f_{X,Y}(g^{-1}(u,v))} = f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v))$$

Ex. Let $X \perp Y$

and $\left.\begin{array}{l} X \sim Pois(\theta) \\ Y \sim Pois(\lambda) \end{array}\right\}$ discrete

$U = V + X$

$\boxed{U \geq V}$

Consider: $U = X + Y$ and $V = Y$

We know:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^y e^{-\lambda}}{y!}$$

$u = x + y$, $\boxed{v = y}$

Solve for x, y in terms of u, v

notice $u - v = (x + y) - y = x$

$\boxed{x = u - v}$

$y = u - x = u - (u - v) = v$

$$\boxed{\begin{aligned} x &= g_1^{-1}(u,v) = u-v \\ y &= g_2^{-1}(u,v) = v \end{aligned}}$$

$$f_{U,V}(u,v) = f_{X,Y}\left(g_1^{-1}(u,v), g_2^{-1}(u,v)\right)$$

$$= f_{X,Y}(u-v, v)$$

$$= \boxed{\frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^{v} e^{-\lambda}}{v!}} \quad \text{for } \underline{u \geq v \geq 0}$$

lets get the marginal of $U = X + Y$

$$\underline{f_U(u)} = \sum_{v=0}^{u} f(u,v)$$

$$= \sum_{v=0}^{u} \frac{\theta^{u-v} e^{-\theta} \lambda^{v} e^{-\lambda}}{(u-v)!\, v!}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^{u} \frac{u!}{(u-v)!\, v!} \theta^{u-v} \lambda^{v}$$

$\binom{u}{v}$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \underbrace{\sum_{v=0}^{u} \binom{u}{v} \lambda^{v} \theta^{u-v}}_{(\lambda+\theta)^{u}}$$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

Binomial Formula

$(\lambda + \theta)^{u}$

Formula

$$f_u(u) = \frac{e^{-(\theta+\lambda)}(\lambda+\theta)^u}{u!}$$

$Z \sim Pois(\lambda)$

$$\frac{e^{-\lambda}\lambda^3}{3!}$$

$u \sim Pois(\theta+\lambda)$

Theorem: $X \perp\!\!\!\perp Y$ and
$X \sim Pois(\theta)$
$Y \sim Pois(\lambda)$

then $X + Y \sim Pois(\theta+\lambda)$

What about cts RVs?

Uni: If g is nice: $Y = g(X)$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Bivariate Case:

Assume X and Y are cts and

$$(u, v) = (g_1(x, y), g_2(x, y))$$

and ① g is invertible

② g^{-1} is differentiable

then

$$f_{u,v}(u,v) = f_{x,y}\left(g_1^{-1}(u,v), g_2^{-1}(u,v)\right) \left| \det J \right|$$

Jacobian of g^{-1}

Jacobian matrix: $h : \mathbb{R}^2 \to \mathbb{R}^2$

$$h(x,y) = (h_1(x,y), h_2(x,y))$$

$$J = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix}$$

in our case we want the Jacobian of $g^{-1} = g^{-1}(u,v)$

$$\boxed{J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix}}$$

$$\boxed{J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix}}$$

For a 2X2 mtx

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $\det A = ad - cb$

Process:

① get g_1^{-1}, g_2^{-1}
(solve for x, y in terms of u, v)

② Find J, $\det J$

③ plug in formula

Ex. $(u, v) = (x+y, x-y)$

$u = g_1(x,y) = x+y$

$u = g_1(x,y) = x+y$

$v = g_2(x,y) = x-y$

(1) get inverses

notice: $u+v = x+y+x-y = 2x$

so $\boxed{x = \dfrac{u+v}{2}} = g_1^{-1}(u,v)$

similarly $u-v = x+y-(x-y) = 2y$

so $\boxed{y = \dfrac{u-v}{2}} = g_2^{-1}(u,v)$

(2) get J

$$J = \begin{bmatrix} \dfrac{\partial g_1^{-1}}{\partial u} & \dfrac{\partial g_1^{-1}}{\partial v} \\[2mm] \dfrac{\partial g_2^{-1}}{\partial u} & \dfrac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\[2mm] \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

so $\det J = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$