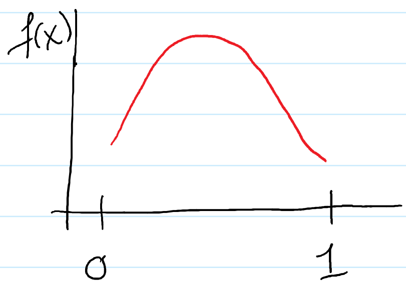
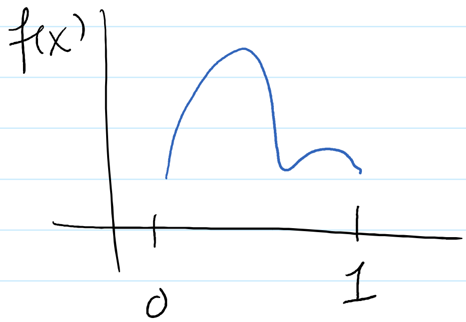
Beta Distribution

- continuous distribution w/ support on $[0, 1]$





Beta Function: $a, b \in \mathbb{R}^+$

then

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Beta distribution:

$$X \sim \text{Beta}(a,b)$$

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } 0 < x < 1$$

Expectation :

Expectation:

$$E[X] = \int_0^1 x \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+1,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}\,dx}_{\text{integrates to } 1}$$

looks like

$$\text{Beta}(a+1,\ b)$$

$$\downarrow$$

$$\frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}$$

$$= \frac{B(a+1,b)}{B(a,b)}$$

$$= \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} \Big/ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{a\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{(a+b)\cancel{\Gamma(a+b)}} \Big/ \frac{\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{\cancel{\Gamma(a+b)}} = \boxed{\frac{a}{a+b} = E[X]}$$

Moments:

$$E[X^r] = \int_0^1 x^r \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+r,b)}{B(a,b)} \int_0^1 x^{(a+r)-1}(1-x)^{b-1}\,dx$$

$$= \frac{B(a+r,b)}{B(a,b)} \int_0 \underbrace{\frac{x \qquad (1-x)}{B(a+r,b)}}_{\text{PDF of Beta}(a+r,b)} \, dx$$

... integrates to 1

$$= \frac{B(a+r,b)}{B(a,b)} = EX^r$$

So
$$EX^2 = \frac{B(a+2,b)}{B(a,b)} = \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)} \Big/ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{(a+1)a\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{(a+b+1)(a+b)\cancel{\Gamma(a+b)}} \Big/ \frac{\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{\cancel{\Gamma(a+b)}}$$

$$= \frac{a(a+1)}{(a+b)(a+b+1)}$$

$$\text{Var}(X) = E[X^2] - E[X]^2$$

$$= \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2$$

$$= \cdots = \frac{ab}{(a+b)^2(a+b+1)}$$

$(a+b)^2(a+b+1)$

Transformations

If I know something about X, what do I know about $Y = g(X)$?

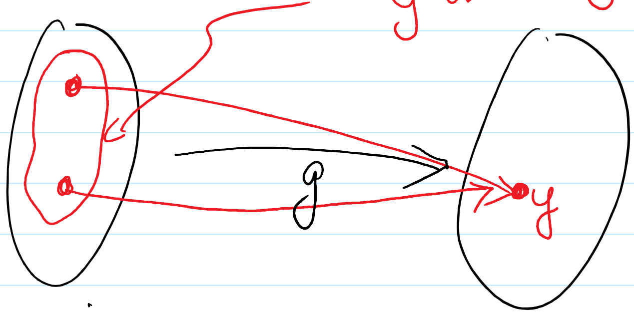
Discrete RVs

Q: If I know f_X can I get f_Y?

PMF of X — PMF of Y

Inverse Images

inverse image of y under g $= g^{-1}(\{y\})$



domain

co-domain

If g is invertible then g^{-1} is the true inverse.

Notice:

$$f_Y(y) = \mathbb{P}(Y = y) = \mathbb{P}(g(X) = y) \quad \circledast$$

$$f_Y(y) = P(Y=y) = \underline{P(g(X)=y)} \ (*)$$

If g is invertible then

$$(*) = P(X = g^{-1}(y)) = f_X(g^{-1}(y))$$

If g isn't invertible

$$(*) = P(X \in \underbrace{g^{-1}(y)}_{A})$$

$$= \sum_{x \in g^{-1}(y)} f_X(x)$$

$$\boxed{\begin{aligned} &P(X \in A) \\ &= \sum_{x \in A} f_X(x) \end{aligned}}$$

<u>Theorem</u>: If X is discrete and $Y = g(X)$

then

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x)$$

$$= \sum_{x : g(x) = y} f_X(x)$$

<u>Ex</u>, Let $X \sim Bin(n, p)$

\# heads in n independent coin flips

each w/ a prob p of H

Consider: $\mathbb{Y} = n - \mathbb{X} \leftarrow$ # fails

$$y = g(x) = n - x \Leftrightarrow x = n - y = g^{-1}(y)$$

$$f_{\mathbb{Y}}(y) = \sum_{x : g(x) = y} f_{\mathbb{X}}(x) = \sum_{x = n-y} f_{\mathbb{X}}(x)$$

$$= f_{\mathbb{X}}(n-y)$$

Recall:

$$f_{\mathbb{X}}(x) = \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$

Note: $\binom{n}{n-y} = \binom{n}{y}$, let $q = 1-p$

$$f_{\mathbb{Y}}(y) = \underbrace{\binom{n}{y} q^{y} (1-q)^{n-y}}_{Bin(n,\, q=1-p)}$$

So $\boxed{\mathbb{Y} \sim Bin(n, 1-p)}$

So $\boxed{Y \sim Bin(n, 1-p)}$

Continuous RV Transformations (CDFs)

Theorem: If X is continuous and $Y = g(X)$ then

① If g is increasing then

$$F_Y(y) = F_X(g^{-1}(y))$$

② if g is decreasing then

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

Pf. Case 1: g inc.

g is increasing and so is g^{-1}

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y)$$

$$= \mathbb{P}(X \le g^{-1}(y))$$

$$= F_X(g^{-1}(y))$$

Case 2: g dec.

g, g^{-1} dec.

Case 2: g dec.

g, g^{-1} dec.

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

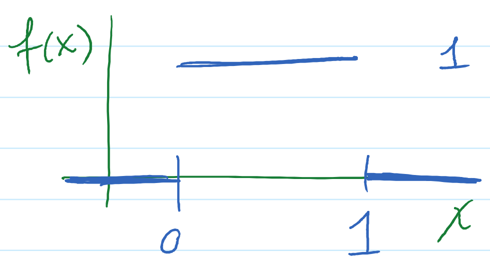
$$= P(X \ge g^{-1}(y))$$

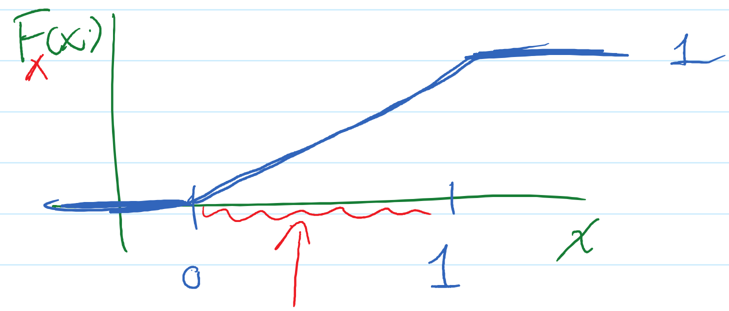
$$= 1 - P(X < g^{-1}(y))$$

reg. X cts.

$$= 1 - F_X(g^{-1}(y))$$

Ex. $X \sim U(0,1)$





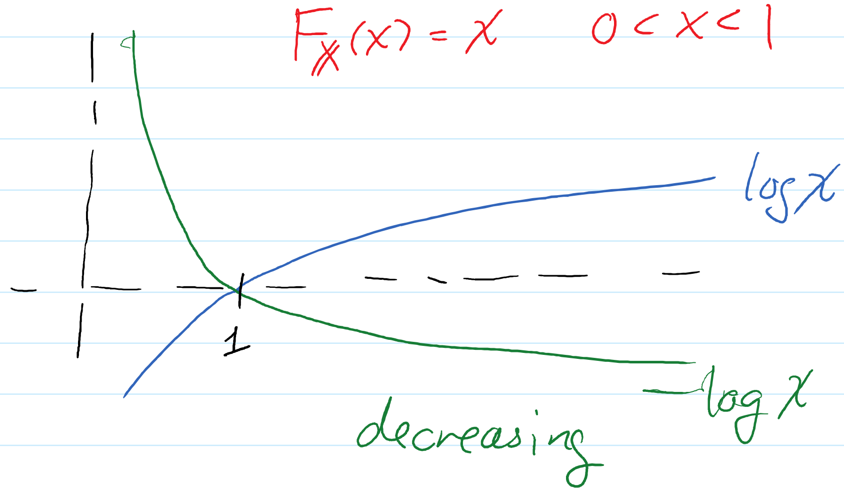
$F_X(x) = x \quad 0 < x < 1$

Let $Y = -\log X$

$g(x) = -\log x = y$

$\Updownarrow$

$x = e^{-y} = g^{-1}(y)$



$0 < x < 1$ then $\log x < 0$

$y = -\log x > 0$

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

$$= 1 - F_X(e^{-y})$$

So ...

$y = -\log x > 0$

$= 1 - F_X(e^{-y})$

so

$0 < e^{-y} = 1/e^{y} < 1$

$= \underbrace{1 - e^{-y}}_{\text{CDF of an Exp(1)}}$

If $Z \sim Exp(1)$

then $f(z) = e^{-z}$

$F(z) = \int_0^z e^{-t}\,dt = 1 - e^{-z}$

then

$\boxed{Y \sim Exp(1)}$