

4 options: Sampling  $r$  from  $n$

	w/o repl.	w/ repl.
Ordered	① $\frac{n!}{(n-r)!}$	② $n^r$
Unordered	③ $\binom{n}{r} = \frac{n!}{r!(n-r)!}$	④ $\binom{n+r-1}{r}$

Theorem:

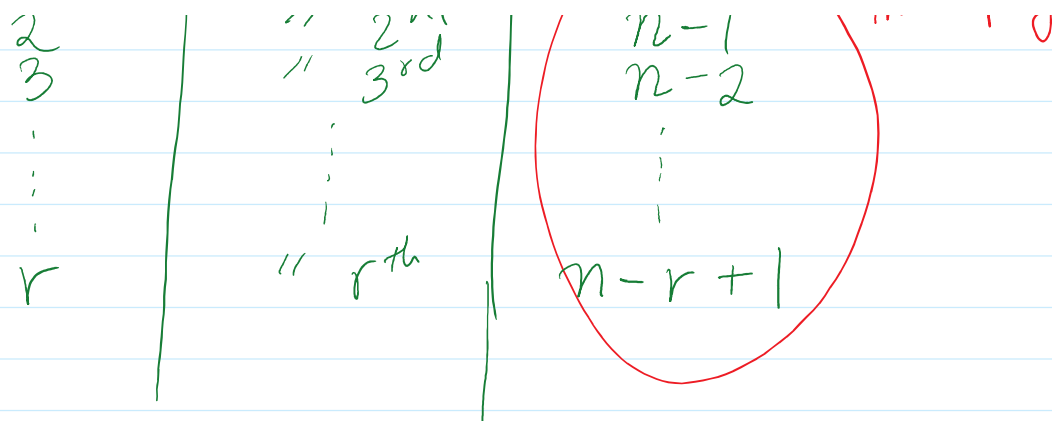
If I have  $n$  items and I sample  $r$  of them w/o replacement but w/ ordering then the number of ways to do this is

$$\frac{n!}{(n-r)!}$$

Pf. Use FTC

task #	task	# ways
1	sample 1 <sup>st</sup>	$n$
2	" 2 <sup>nd</sup>	$n-1$
3	" 3 <sup>rd</sup>	$n-2$

multiply



$$n(n-1)(n-2) \dots (n-r+1) = \boxed{\frac{n!}{(n-r)!}}$$

$$\rightarrow \frac{n(n-1) \dots (n-r+1) \cancel{(n-r)} \dots \cancel{3 \cdot 2 \cdot 1}}{\cancel{(n-r)} \cancel{(n-r-1)} \dots \cancel{3 \cdot 2 \cdot 1}}$$

Ex. I form a committee from  $n=10$  student  
of size  $r=3$

where the committee's 3 members are  
Pres, VP, treasurer

How many ways can I form this committee?

Sample w/o replacement  
b/c can't have same student twice

Sample w/ order  
 $1^{\text{st}} = \text{Pres. } 2^{\text{nd}} = \text{VP. } 3^{\text{rd}} = \text{treasurer}$

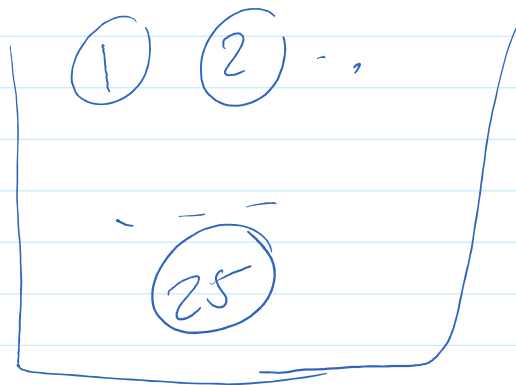
1<sup>st</sup> = Pres, 2<sup>nd</sup> = VP, 3<sup>rd</sup> = treasurer

Apply formula:

$$\frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} = 10 \cdot 9 \cdot 8 = 720$$

Ex. Lotto.

I have a basket w/ 25 numbered balls



Draw 4 of them. *all draws equally likely*

Guess: ① ③ ②② ⑦

What's the prob. I correctly guess?

$$P(E) = \frac{|E|}{|S|}$$

$$E = \{ \textcircled{1} \textcircled{3} \textcircled{22} \textcircled{7} \} \quad \text{so } |E| = 1$$

$S = \{ \text{all possible draws} \}$

$$|S| = \frac{25!}{(25-4)!} = \frac{25!}{21!} = 25 \cdot 24 \cdot 23 \cdot 22$$

$$P(E) = \frac{1}{25 \cdot 24 \cdot 23 \cdot 22}$$

Theorem: Sampling w/ repl. and w/ ordering

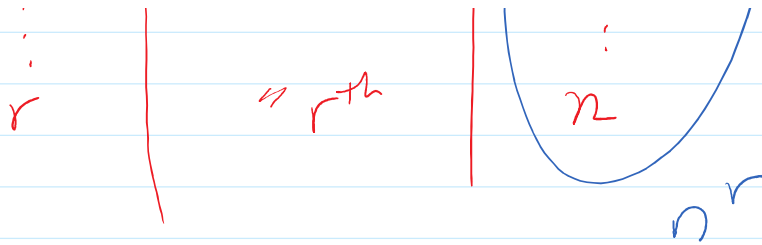
The number of ways to sample  $r$  from  $n$  w/ repl. and w/ ordering is

$$n^r$$

pf. Use FTC

task #	task	#ways
1	sample 1 <sup>st</sup>	$n$
2	" 2 <sup>nd</sup>	$n$
3	" 3 <sup>rd</sup>	$n$
$\vdots$		$\vdots$
$r$	" $r$ -th	$n$

product



## Ex. Braille Alphabet

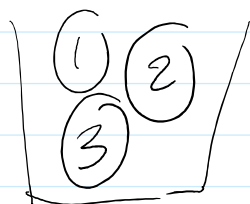
6 spot each  
can be raised or not

Q: how many braille letters  
can I make?

Sampling  $r=6$  from  $n=2$   
so I have  
 $2^6 = 64$

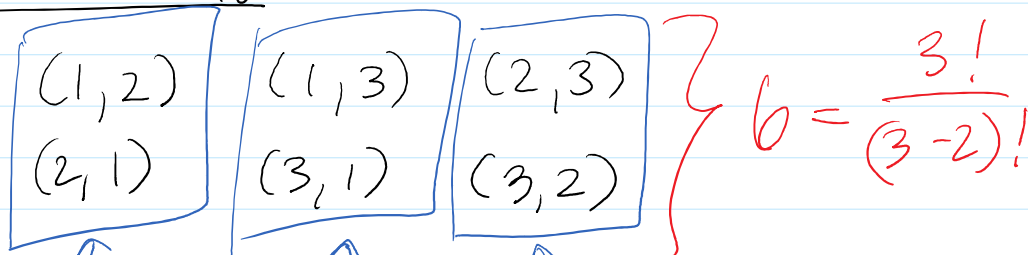
## Sampling w/o replacement and w/o ordering

EX.

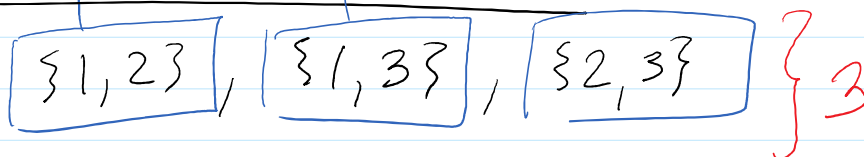


draw  $r = 2$  from  $n = 3$

If order matters

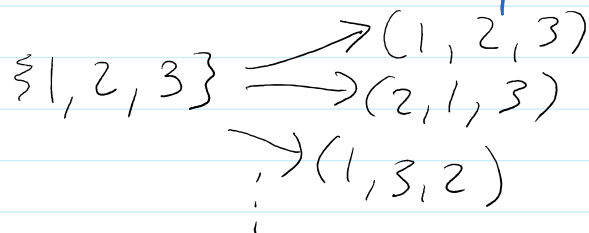


If order doesn't matter



General fact:

Each unordered sample of size  $r$   
can be permuted in  $r!$  ways to  
make an ordered sample.



$$(\# \text{ ordered}) = r! (\# \text{ unordered})$$

$$\underbrace{(\# \text{ ordered})}_{(i)} = r! \underbrace{(\# \text{ unordered})}$$

$$\begin{aligned} \text{so } (\# \text{ unordered}) &= \frac{1}{r!} (\# \text{ ordered}) \\ &= \frac{1}{r!} \frac{n!}{(n-r)!} \end{aligned}$$

Theorem: Unordered w/o Repl.

I can sample  $r$  from  $n$  w/o repl.  
w/o order in

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Binomial coefficient  
reads:  $n$  choose  $r$

Ex. I have  $n=10$  professors, how many  
Co-equal committees can I form of  
size  $r=4$ ?

✓      ∪

I can do this in

$$\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!} \leftarrow$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{4 \cdot 3 \cdot 2 \cdot 6!}$$

$$= 10, 3, 7 = 210,$$

Ex. How many 5-card poker hands  
can I form (from a deck of 52 cards)

$$\binom{52}{5} \approx 2.5 \text{ mil}$$

Ex. I have a jar w/ 4 marbles of colors yellow, blue, orange, green.



$\left( \begin{array}{cc} (y) & (b) \\ (o) & (g) \end{array} \right) \quad \left| \quad \begin{array}{l} \text{I choose 3 w/o repl.} \\ \text{(all choices equally likely)} \end{array} \right.$   
 What is the prob I have  
 a  $(y)$  and  $(b)$  in my choice?

$$E = \{(y) \text{ and } (b)\} = \{\{y, b, o\}, \{y, b, g\}\}$$

$$|E| = 2$$

$S = \{\text{all selections}\}$

$$|S| = \binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = 4.$$

Hence  $P(E) = \frac{|E|}{|S|} = \frac{2}{4} = \frac{1}{2}.$

Sampling Unordered w/ Repl.

Consider  $n=3, r=2$

$\left( \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \right)$

If ordered:

$\left\{ \begin{array}{c} (2,1) \\ (1,2) \end{array} \right\} \left\{ \begin{array}{c} (3,1) \\ (1,3) \end{array} \right\} \left\{ \begin{array}{c} (2,3) \\ (3,2) \end{array} \right\} \left\{ \begin{array}{c} (1,1) \\ (1,1) \end{array} \right\} \left\{ \begin{array}{c} (3,3) \\ (3,3) \end{array} \right\}$

(3)

(2,1)  
(1,2)

(3,1)  
(1,3)

(2,3)  
(3,2)

(1,1)  
(2,2)

(3,3)

Unordered:

{1,2}

{1,3}

{2,3}

{1,1}

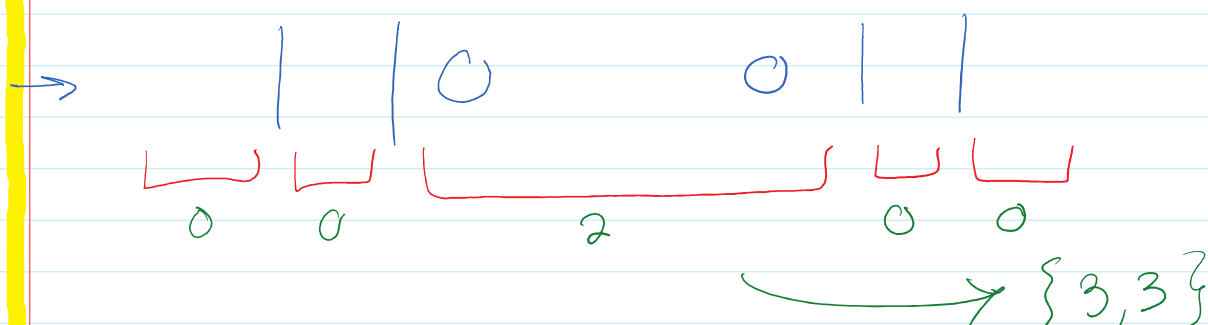
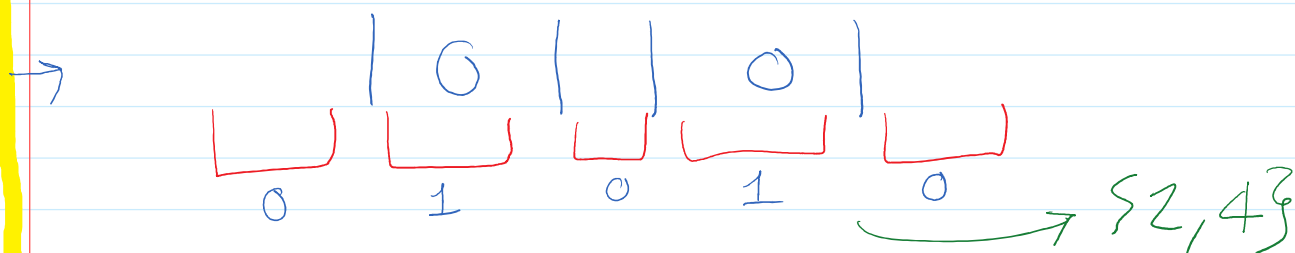
{2,2}

{3,3}

# Game of Partitioning

How many ways can I partition

$r=2$  objects using  $n-1=4$  walls



In total I have  $n-1+r$  things:

- $r$  objects
- $n-1$  walls

I have

$$\frac{(n+r-1)!}{(n-1)! r!}$$

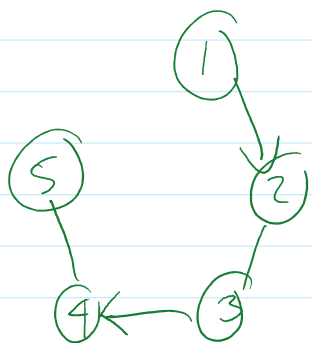
ways to play this game

Theorem: w/ repl. w/o ordering

The number of ways to sample  $r$  from  $n$   
w/ repl. w/o ordering is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Ex. 10 passengers on a bus route w/ 5 stops. The driver records the number of people that get off at each stop.



Q: How many possible records are there?

Stop	#
1	0
2	3
3	1
4	2
5	4

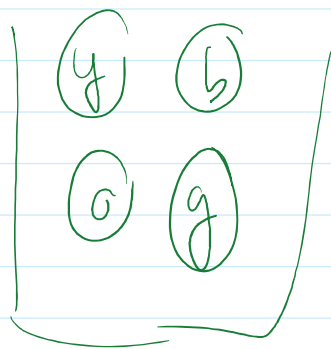
→ {2, 2, 2, 3, 4, 4, 5, 5, 5, 5}

Our theorem says:

$$\binom{10+5-1}{10} = \binom{14}{10} = 1001.$$

Ex. Jar ... 4 marbles: 4 b, 0 g ...

Ex. Jar w/ 4 marbles: y, b, o, g w/ replacement



Draw  $r=3$  from  $n=4$   
(all draws are equally likely)

Q: what's the prob. my sample includes both y and b?

$$P(E) = \frac{|E|}{|S|}$$

$$E = \{y \text{ and } b\} = \{\{y, b, o\}, \{y, b, g\}, \{y, b, y\}, \{y, b, b\}\}$$

So

$$|E| = 4$$

and

$$|S| = \binom{3+4-1}{3} = \binom{6}{3} = 20$$

So

$$P(E) = \frac{4}{20} = \frac{1}{5}$$