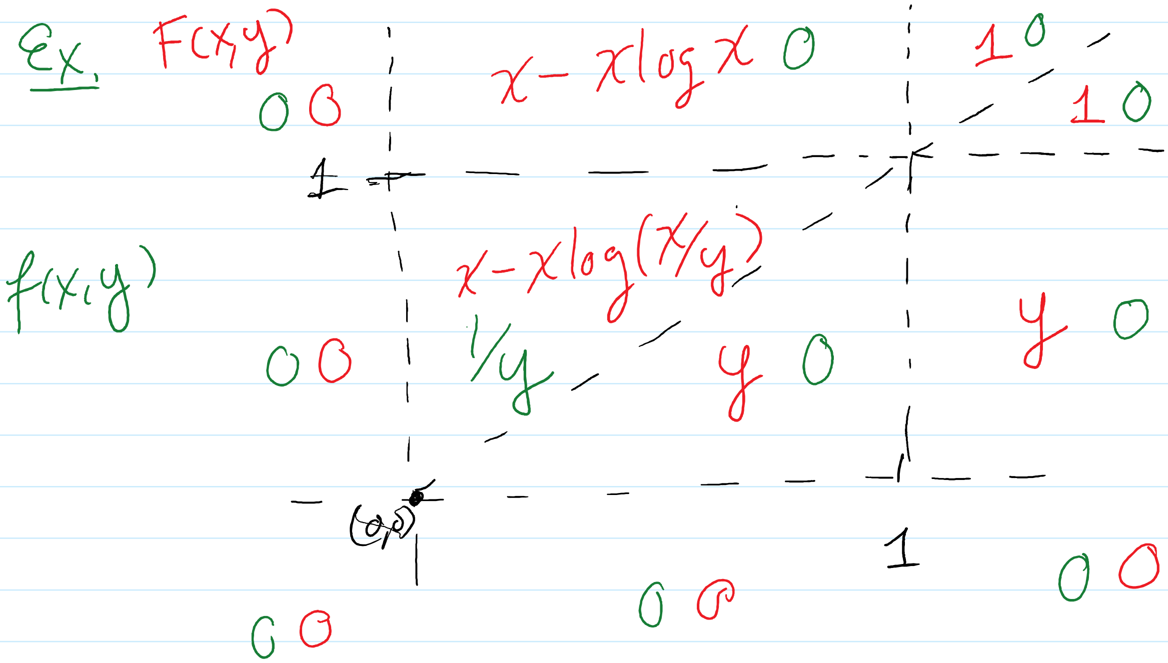
Ex.



What is the joint PDF?

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

For $0 < x < y < 1$

$$f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(x - x\log(x/y)\right)$$

$$= \frac{\partial}{\partial x}\left[x/y\right]$$

$$= 1/y$$

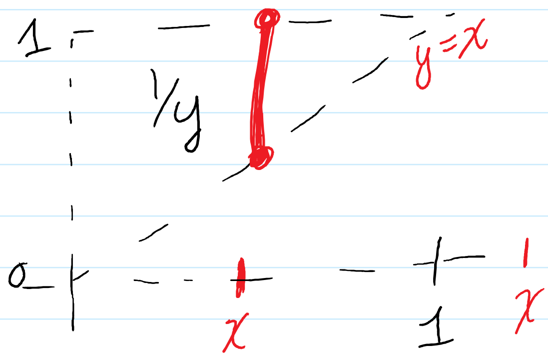
$$\frac{\partial}{\partial y}\log\left(\frac{x}{y}\right) = \frac{\frac{d}{dy}\left(\frac{x}{y}\right)}{x/y} = \frac{-\frac{x}{y^2}}{x/y} = -1/y$$

So...
$$\boxed{f(x,y) = \frac{1}{y} \quad \text{for } 0 < x < y < 1}$$

What is the marginal dist of X?

What is the marginal dist of X?

$$f_X(x) = \int f(x,y)\,dy = \int_{y=x}^{y=1} \frac{1}{y}\,dy = \log(y)\Big|_x^1$$



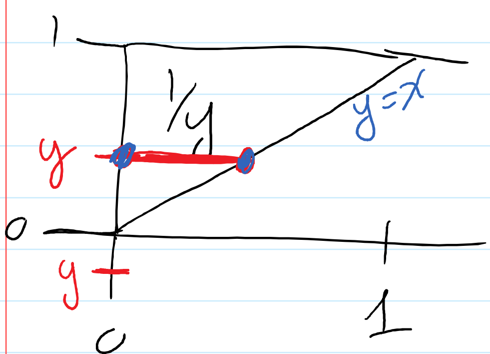
$$= \log(1) - \log(x)$$

$$\boxed{f_X(x) = -\log(x) \quad \text{for } 0<x<1}$$

What is the marginal PDF of Y?

$$f_Y(y) = \int f(x,y)\,dx = \int_{x=0}^{x=y} 1/y\,dx = \frac{1}{y}\int_0^y dx = \frac{1}{y}[x]_0^y$$

$$= 1$$



So $\boxed{f_Y(y) = 1 \quad \text{for } 0<y<1}$

i.e. $Y \sim U(0,1)$

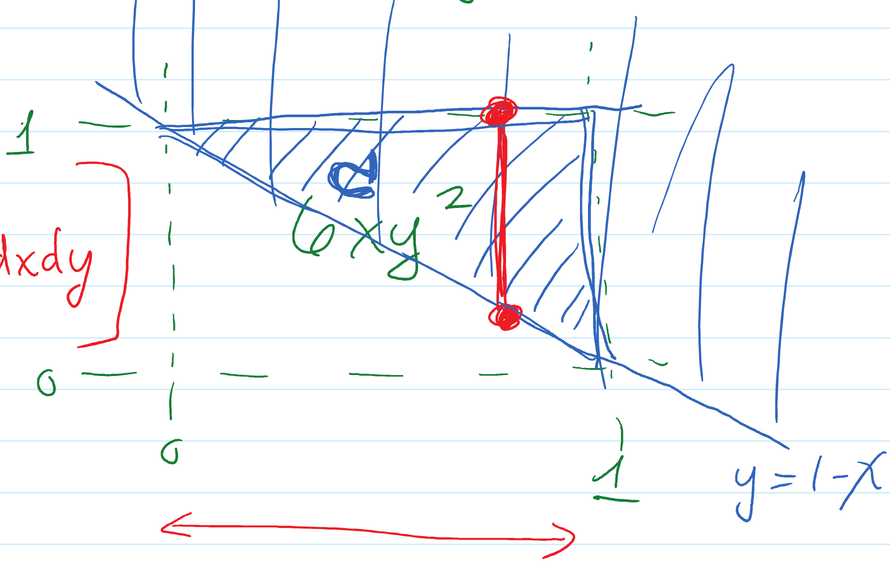
Ex.

Let $f(x,y) = 6xy^2$ for $0<x<1$
$0<y<1$

$0<y<1$

$P(X+Y \geq 1)$

$$P((X,Y)\in C) = \iint_C f(x,y)\,dx\,dy$$



draw line $x+y=1$

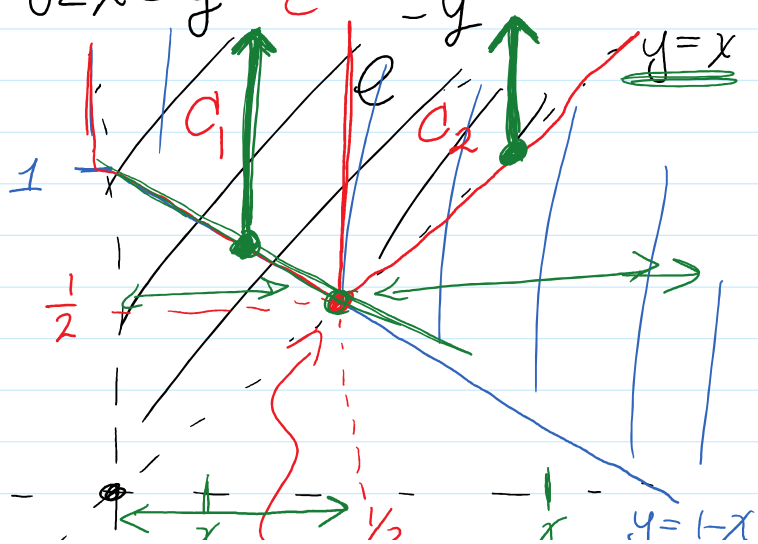
$y=1-x$

$x=1-y$

$$P(X+Y\geq 1) = \int_0^1\int_{y=1-x}^{1} 6xy^2\,dy\,dx$$

$$= \dots = 9/10$$

Ex. $f(x,y) = e^{-y}$ for $0<x<y$

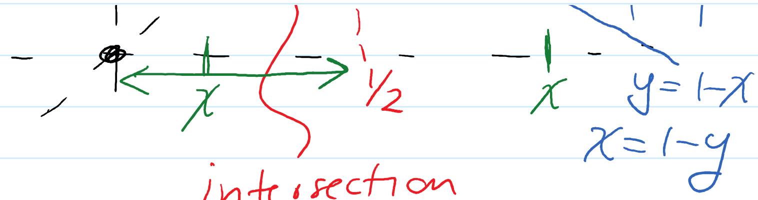


$P(X+Y\geq 1)$

$$= \iint_C e^{-y}\,dx\,dy$$

$$= \iint e^{-y}\,dx\,dy + \iint e^{-y} \dots$$

$$= \iint_{C_1} e^{-y}\,dx\,dy + \iint_{C_2} e^{-y}\,dx\,dy$$



intersection of $y = 1 - x$ and $y = x$

$1 - x = x$

$\Rightarrow x = 1/2$

$$= \int_0^{1/2}\int_{y=1-x}^{\infty} e^{-y}\,dy\,dx + \int_{1/2}^{\infty}\int_{y=x}^{\infty} e^{-y}\,dy\,dx$$

$$= \cdots = \boxed{2e^{-1/2} - e^{-1}}$$

Defn: Bivariate Expectation

If (X, Y) is a BiRV and $g: \mathbb{R}^2 \longrightarrow \mathbb{R}$ then

$$\mathbb{E}[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) & \text{(discrete)} \\ \iint_{\mathbb{R}^2} g(x,y) f(x,y)\,dx\,dy & \text{(cts)} \end{cases}$$

[Uni: $\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f(x)\,dx$]

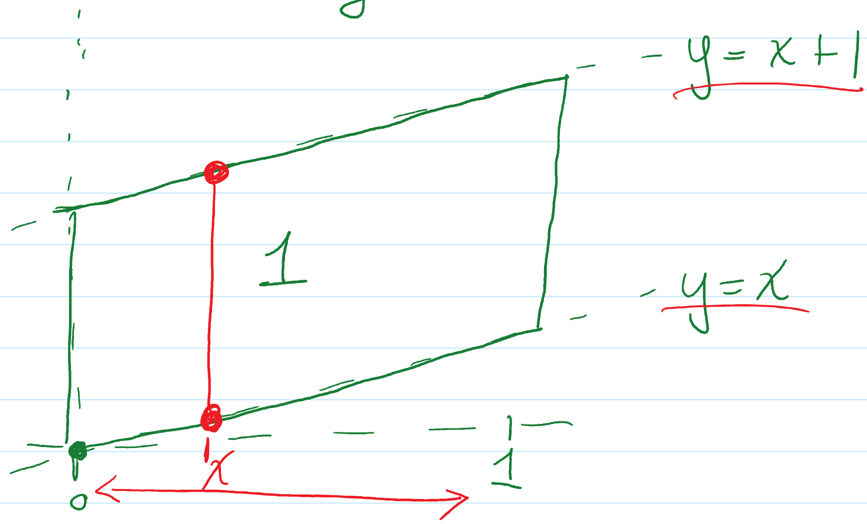
Ex. let $f(x,y) = 1$ for $0 < x < 1$
$x < y < x+1$

$$\mathbb{E}[XY] \quad g(x,y) = xy$$

$$= \iint_{\mathbb{R}^2} g(x,y) f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_{y=x}^{y=x+1} xy\,(1)\, dy\, dx$$

$$= \cdots = 7/12$$



Theorem: Bivariate Expectation is Linear

If $g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ and $a, b \in \mathbb{R}$

then

$$\mathbb{E}[a\, g_1(X,Y) + b\, g_2(X,Y)]$$
$$= a\,\mathbb{E}[g_1(X,Y)] + b\,\mathbb{E}[g_2(X,Y)]$$

Defn: Covariance

We define the covariance between two RVs X, Y as

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[\underbrace{(X-\mathbb{E}X)}\underbrace{(Y-\mathbb{E}Y)}\right]$$

$\mu_X = \mathbb{E}X$ and $\mu_Y = \mathbb{E}Y$

$$= \mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right]$$

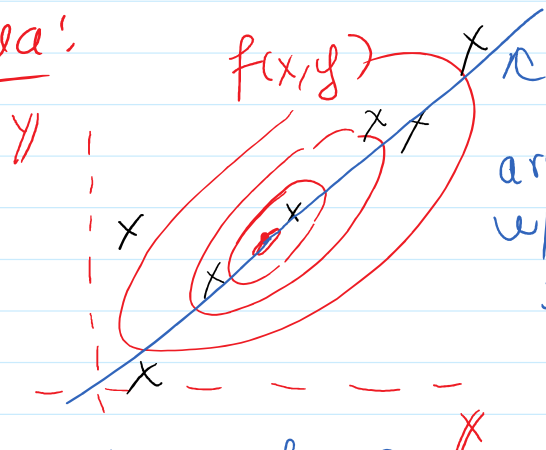
$g(x,y) = (x-\mu_x)(y-\mu_y)$

$$= \mathbb{E}\left[g(X,Y)\right]$$

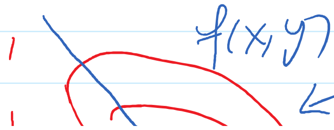
<u>Recall</u>: $\mathrm{Var}(X) = \mathbb{E}\left[(X-\mathbb{E}X)^2\right]$

<u>note</u>: $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

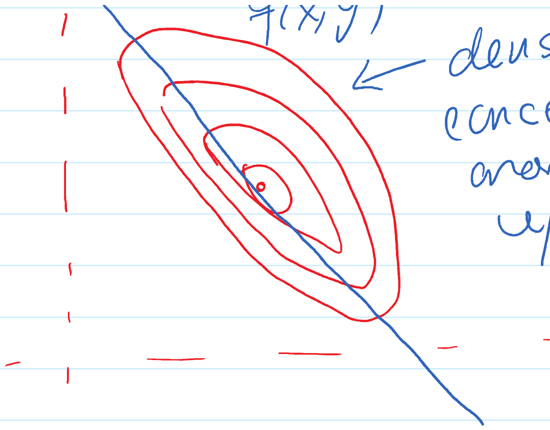
<u>Idea</u>:



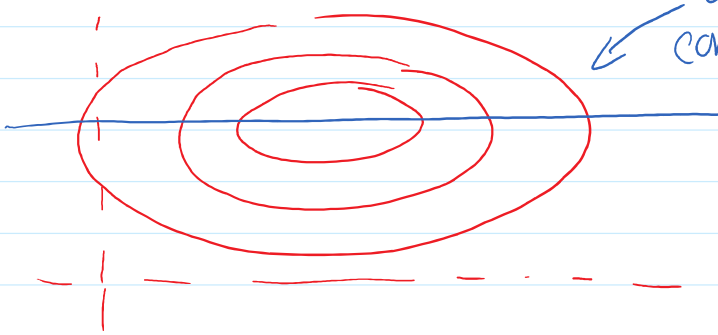
density concentrated around line w/ pos slope $\Rightarrow \mathrm{Cov}(X,Y) >> 0$



density



$\Rightarrow Cov(X,Y) \ll 0$



$\Rightarrow Cov(X,Y) \approx 0$

Cov: measuring strength of linear rel. btween X and Y

Defn: Corelation

Re-scaled covariance so that it is between -1 and 1.

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

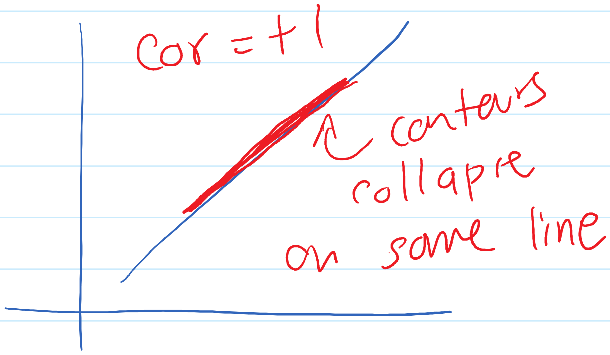
$sd(X) = \sqrt{Var(X)}$

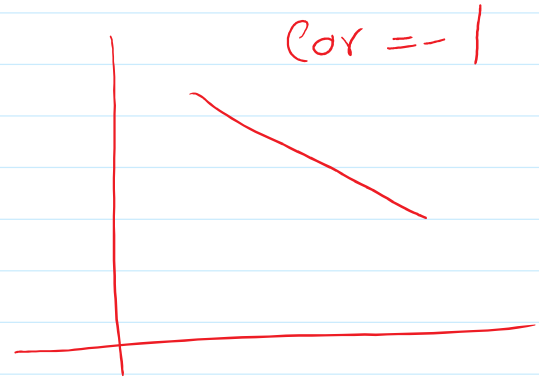
$$= \frac{Cov(X,Y)}{\quad}$$

$sd(X)$

$$= \frac{Cov(X,Y)}{sd(X)\,sd(Y)}$$

Note: $Cor = \pm 1$ then we have a "perfect" lin. rel. btwn X and Y





Theorem: If $a, b \in \mathbb{R}$

$$\boxed{Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X,Y)}$$

pf. $Z = aX + bY$

$$Var(Z) = \mathbb{E}\left[(Z - \mathbb{E}Z)^2\right]$$

$$= \mathbb{E}\left[(aX + bY - \mathbb{E}[aX+bY])^2\right]$$

$$= \mathbb{E}\left[\left(aX + bY - (a\mathbb{E}X + b\mathbb{E}Y)\right)^2\right]$$

$$= \mathbb{E}\left[\Big(\underbrace{\underbrace{a(X - \mathbb{E}X)}_{\alpha} + \underbrace{b(Y - \mathbb{E}Y)}_{\beta}}_{(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta}\Big)^2\right]$$

$$= \mathbb{E}\left[a^2(X - \mathbb{E}X)^2 + b^2(Y - \mathbb{E}Y)^2 + 2ab(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$$

$$= a^2\,\mathbb{E}\left[(X - \mathbb{E}X)^2\right] + b^2\,\mathbb{E}\left[(Y - \mathbb{E}Y)^2\right] + 2ab\,\mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$$

$$= a^2\,\mathrm{Var}(X) + b\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X, Y).$$