What about PDFs?

Theorem: If X is continuous and $Y = g(X)$

and if

1. g is invertible
2. g^{-1} is differentiable

then

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$

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pf. Case 1: g increasing

Our previous CDF theorem said

$$F_Y(y) = F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{dF_Y}{dy} = \frac{d}{dy} F_X(g^{-1}(y))$$

$$= f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

g increasing

g^{-1} increasing

So $\frac{dg^{-1}}{dy} > 0$

Case 2: g decreasing

Case 2: g decreasing

By our previous CDF theorem

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

→ g decreasing

→ g^{-1} decreasing

→ $\frac{dg^{-1}}{dy} < 0$

and so

$$f_Y(y) = \frac{dF_Y}{dy} = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right)$$

$$= -f_X(g^{-1}(y))\,\frac{dg^{-1}}{dy}$$

$$= f_X(g^{-1}(y))\,-\frac{dg^{-1}}{dy}$$

$-(-5) = |5|$

$$= f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

Ex. Let $X \sim \text{Gamma}(k, \lambda)$

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)} \quad \text{for } x > 0$$

Let $Y = \frac{1}{X}$ i.e. $y = g(x) = \frac{1}{x} \Leftrightarrow y = 1/x \Rightarrow x = \frac{1}{y} = g^{-1}(y)$

So $\frac{dg^{-1}}{dy} = -\frac{1}{y^2}$

Our PDF theorem says

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

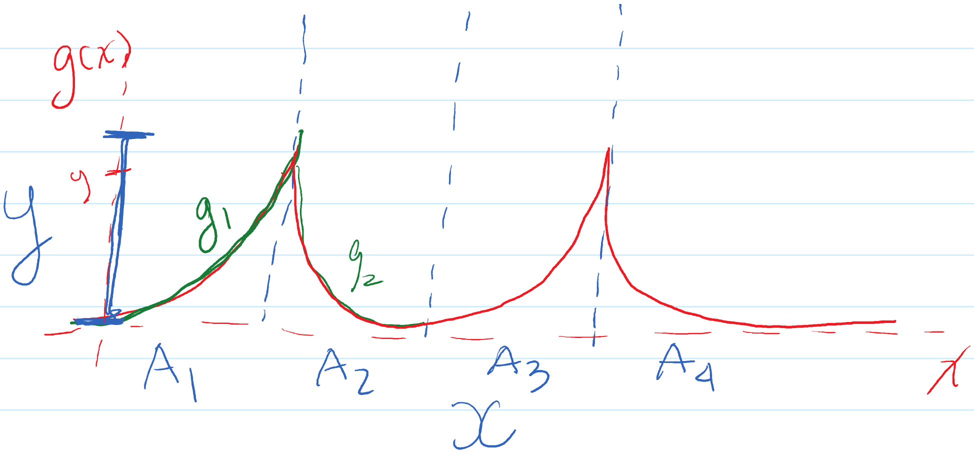
$$= f_X\left(\frac{1}{y}\right) \left| -\frac{1}{y^2} \right|$$

$$= \boxed{\frac{\lambda e^{-\frac{\lambda}{y}} (\lambda/y)^{k-1}}{\Gamma(k)} \frac{1}{y^2} \quad \text{for } y > 0}$$

called the Inverse Gamma dist.

What about non-invertible g?

Theorem: Let X is a continuous RV w/ support $\mathcal{X}$ and for $i=1, \dots, K$ let A_i partition $\mathcal{X}$



Let g_i to be g restricted to A_i.

If ① my prev. theorem applies on each part of the partition separately

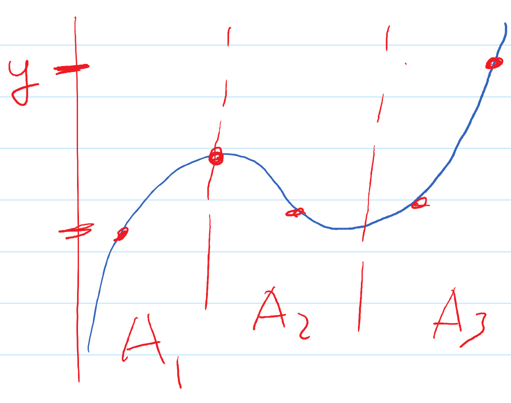
[g_i invertible on A_i
g_i^{-1} is differentiable]

② The image of A_i under each g_i is the same.

[all g_i have same range $\mathcal{Y}$]

then

$$\boxed{f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{d g_i^{-1}}{dy} \right|} \text{ for } y \in \mathcal{Y}$$



Ex. Chi-squared Distribution

If $X \sim N(0,1)$ and $Y = X^2$

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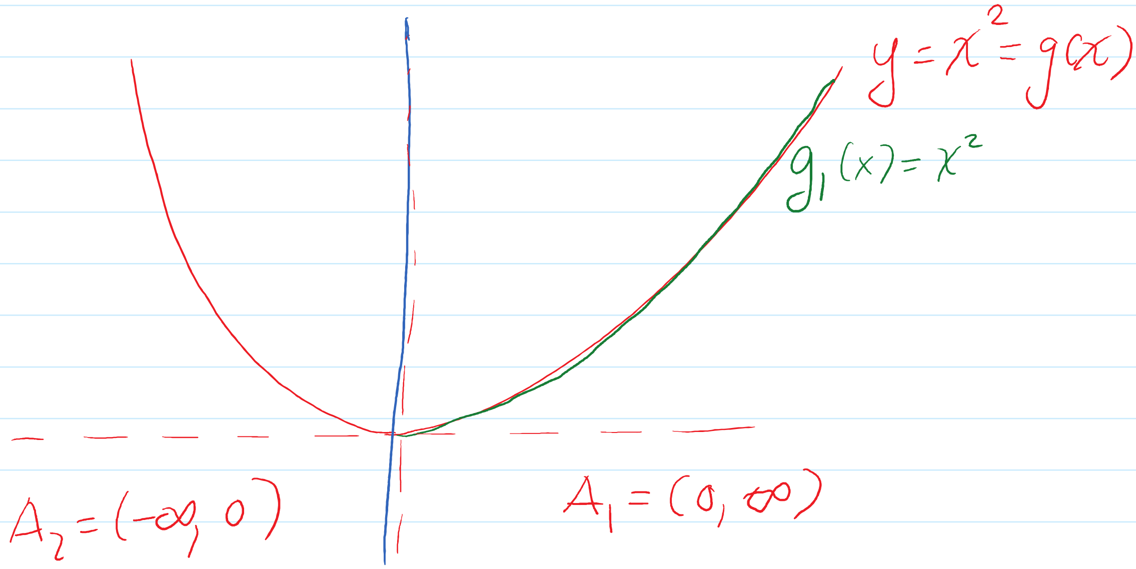
then we say Y has a chi-sq. dist.

w/ one degree of freedom,

denoted $Y \sim \chi^2(1)$

What is the PDF of a Chi-Sq?



$A_1 = (0,\infty)$, $g_1(x) = x^2$, $g_1^{-1}(y) = \sqrt{y}$; $\frac{dg_1^{-1}}{dy} = \frac{1}{2\sqrt{y}}$

$A_2 = (-\infty, 0)$, $g_2(x) = x^2$, $g_2^{-1}(y) = -\sqrt{y}$; $\frac{dg_2^{-1}}{dy} = \frac{-1}{2\sqrt{y}}$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2} \text{ for } x \in \mathbb{R}$$

$$f_Y(y) = f_X(g_1^{-1}(y)) \left| \frac{dg_1^{-1}}{dy} \right| + f_X(g_2^{-1}(y)) \left| \frac{dg_2^{-1}}{dy} \right|$$

$$f_Y(y) = f_X(g_1^{-1}(y))\left|\frac{dg_1^{-1}}{dy}\right| + f_X(g_2^{-1}(y))\left|\frac{dg_2^{-1}}{dy}\right|$$

$$= f_X(\sqrt{y})\left|\frac{1}{2\sqrt{y}}\right| + f_X(-\sqrt{y})\left|\frac{-1}{2\sqrt{y}}\right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2} \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} \left(e^{-y} + e^{-y}\right)$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y}} \quad \text{for } y > 0$$

PDF of a $\chi^2(1)$

Probability Integral Transformation

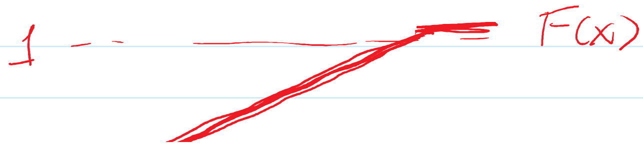
If X is a continuous RV w/ CDF F_X

$$F_X(X) \sim U(0,1).$$

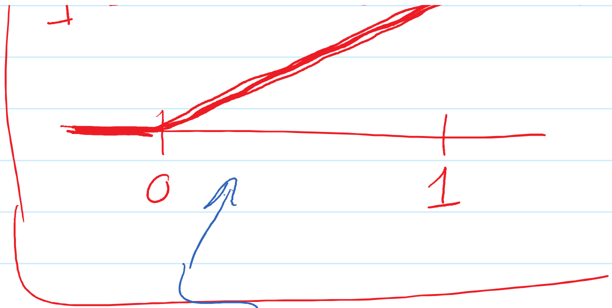
Pf. Assume F_X is strictly increasing

then F^{-1} exists.

CDF of a $U(0,1)$



then F_X^{-1} exists.



Our CDF theorem says

$$Y = F_X(X) = g(X)$$

Then for $0<y<1$

$$F_Y(y) = F_X(g^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

So the CDF of Y is the CDF of a $U(0,1)$

and so $Y \sim U(0,1)$.

Converse:

$$g(X) \sim U(0,1) \Leftrightarrow g = F_X$$

Know how to generate $U \sim U(0,1)$

Want: generate some RV w/ CDF F_X

Let $Z = F_X^{-1}(U)$

CDF of $U(0,1)$ is $F(x)=x$

then $F_Z(z) = P(Z \le z) = P(F_X^{-1}(U) \le z)$

$= P(U \le F_X(z))$

$$= F_X(z)$$

So Z follows dist w/ CDF F_X.

Algorithm: Want $Z \sim F_X$

① $U \sim U(0,1)$

② $Z = F_X^{-1}(U)$

this has correct dist.

Ex. Want $X \sim \text{Exp}(1)$

CDF of Exp(1) is $F(x) = 1 - e^{-x}$

$$y = 1 - e^{-x}$$
$$\Updownarrow$$
$$1 - y = e^{-x}$$
$$\Updownarrow$$
$$\log(1-y) = -x \implies \boxed{x = -\log(1-y) = F_X^{-1}(y)}$$

Bivariate RVs

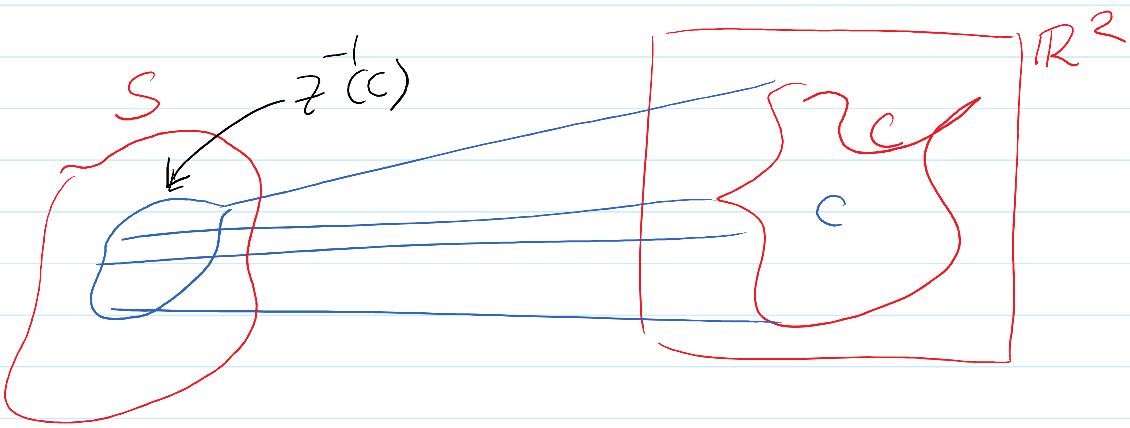
If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$

then $Z = (X, Y)$ is called a bivariate RV

so $Z : S \to \mathbb{R}^2$ so that $Z(s) = (X(s), Y(s))$

Say: $P(Z \in C) = P((X,Y) \in C)$ $C \subset \mathbb{R}^2$

$= P(Z^{-1}(C))$



Often, $C = A \times B$ where $A, B \subset \mathbb{R}$

Write
$P((X,Y) \in C)$
or be lazy

$P(X \in A, Y \in B)$
"and"