Lecture 23 - Multivariate Random Variables

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If $X_1, \dots, X_n$ are RVs then

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

is called a <u>multivariate random variable</u> or a <u>random vector</u>.

<u>Defn: PMF/PDFs</u>

If X_i's are discrete then the <u>joint PMF</u> as

$$f(\underset{\sim}{x}) = f(x_1, x_2, \dots, x_n) = \mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

$$\underset{\sim}{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$$

If X_is are continuous then the <u>joint PDF</u> is the function $f: \mathbb{R}^n \to \mathbb{R}$ so that for $A \subset \mathbb{R}^n$ then

$$\mathbb{P}(\underset{\sim}{X} \in A) = \int_A f(\underset{\sim}{x})\, d\underset{\sim}{x} = \underbrace{\int \cdots \int_A}_{n \text{ integrals}} f(x_1, \ldots, x_n)\, dx_1\, dx_2\, dx_3 \cdots dx_n$$

<u>Expectation</u>

If $g : \mathbb{R}^n \to \mathbb{R}$ then

$$\mathbb{E}[g(\underset{\sim}{X})] = \begin{cases} \sum_{x_1} \sum_{x_2} \sum_{x_3} \cdots \sum_{x_n} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n) & \text{(discrete)} \\ \int \cdots \int g(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n & \text{(cts)} \end{cases}$$

<u>Defn</u>: The marginal dist of X_i can be found as

$$f_{X_i}(x_i) = \begin{cases} \sum_{x_1} \sum_{x_2} \sum_{x_3} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_n} f(x_1, \ldots, x_n) & \text{(discrete)} \\ \int \cdots \int f(x_1, \ldots, x_n)\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n & \text{(cts)} \end{cases}$$

$$\int \cdots \int f(x_1, \ldots, x_n)\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n$$

We can get the joint/marginal of some subsequence $X_{i_1}, X_{i_2}, \ldots, X_{i_m}$ by summing or integrating the joint over all vars. but $X_{i_1}, \ldots, X_{i_m}$.

Conditional Distributions

If I have two sets of RVs

$$X_1, \ldots, X_n \text{ and } Y_1, \ldots, Y_m$$

the conditional dist. of the Xs given Ys is

$$f(x_1, \ldots, x_n \mid y_1, \ldots, y_m) = \frac{f(x_1, \ldots, x_n, y_1, \ldots, y_m)}{f(y_1, \ldots, y_m)}$$

Ex. Let $X_1, \ldots, X_4$ have a joint PDF of

$$f(x_1, \ldots, x_4) = \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$$

for $0 < x_i < 1$

for $\underline{0 < x_i < 1}$

ⓐ $P(\overbrace{X_1 < 1/2, X_2 < 3/4, X_4 > 1/2}^{A})$

$$= \int_A \cdots \int f(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$= \int_{1/2}^{1}\int_0^1\int_0^{3/4}\int_0^{1/2} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_1\, dx_2\, dx_3\, dx_4$$

$$= \cdots = 3/256$$

ⓑ What is dist of X_1 and X_2?

$$f(x_1, x_2) = \int_0^1\int_0^1 f(x_1, \ldots, x_4)\, dx_3\, dx_4$$

$$= \int_0^1\int_0^1 \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_3\, dx_4$$

$$= \cdots = \underline{\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)}$$

ⓒ <u>Expectation</u>

$$E[X_1 X_2] = \int \cdots \int x_1 x_2 f(x_1, \ldots, x_4)\, dx_1 \cdots dx_4$$

$$= \int_0^1\int_0^1\int_0^1\int_0^1 x_1 x_2 \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_3\, dx_4\, dx_1\, dx_2$$

$$= \int_0^1\int_0^1 x_1 x_2 \left(\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)\right) dx_1\, dx_2$$

$$= \cdots = 5/16$$

d) Conditional dists

what is the cond. of X_3, X_4 given X_1, X_2?

$$f(x_3, x_4 \mid x_1, x_2) = \frac{f(x_1, \ldots, x_4)}{f(x_1, x_2)} = \boxed{\frac{\frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)}{\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)}}$$

Mutual Independence

We say $X_1, \ldots, X_n$ are mutually independent if for any sets $A_1, \ldots, A_n \subset \mathbb{R}$

$$\mathbb{P}(X_1 \in A_1, X_2 \in A_2, X_3 \in A_3, \dots, X_n \in A_n)$$
$$= \mathbb{P}(X_1 \in A_1) \cdots \mathbb{P}(X_n \in A_n).$$

Theorem: Independence + Factorization

If the support of $\underset{\sim}{X}$ is a product space then the three following statements are equiv.

1. $X_1, \dots, X_n$ are independent
2. $f(x_1, \dots, x_n) = f(x_1) f(x_2) \cdots f(x_n) = \prod_{i=1}^{n} f(x_i)$
3. $F(x_1, \dots, x_n) = F(x_1) \cdots F(x_n)$

Theorem: Let $X_1, \dots, X_n$ be independent.

1. If $g_i : \mathbb{R} \to \mathbb{R}$ then $g_1(X_1), g_2(X_2), \dots, g_n(X_n)$ are independent.

2. $\mathbb{E}[X_1 X_2 X_3 \cdots X_n] = \mathbb{E}[X_1]\mathbb{E}[X_2] \cdots \mathbb{E}[X_n]$.

Corollary

If X_is are independent and

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$$Z = \sum_{i=1}^{n} X_i$$

then $$M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

Follow-on, let

$$Z = \sum_{i=1}^{n} (a_i X_i + b_i)$$

$$M_Z(t) = e^{t \sum_{i=1}^{n} b_i} \prod_{i=1}^{n} M_{X_i}(a_i t)$$

Ex: $X_i \sim N(\mu_i, \sigma_i^2)$ and they are independent

then

$$Y = \sum_{i=1}^{n} (a_i X_i + b_i) \sim N\left(\sum_{i=1}^{n} a_i \mu_i + b_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2 \right)$$

Multivariate Transformations

Let $\underset{\sim}{X} = (X_1, \ldots, X_n)^T$

and $g: \mathbb{R}^n \to \mathbb{R}^n$

Let $\underset{\sim}{U} = g(\underset{\sim}{X})$

$U_i = g_i(X_1, \ldots, X_n)$

If $\underset{\sim}{X}$ has cts components and

→ g is invertible

→ g^{-1} is differentiable

then

$$f_{\underset{\sim}{U}}(\underset{\sim}{u}) = f_{\underset{\sim}{X}}(g^{-1}(\underset{\sim}{u}))\,|\det J|$$

$$J_{ij} = \frac{\partial g_i^{-1}}{\partial u_j}$$

Means / Variances for MV-RVs

Uni: $E[X] \in \mathbb{R}$

$$Var(X) = \mathbb{E}[(X - \mathbb{E}X)^2] \in \mathbb{R}$$

MV: $\underset{\sim}{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

$$\mu = \mathbb{E}[\underset{\sim}{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix} \in \mathbb{R}^n$$

expected value of a vector

<u>covariance matrix</u>

$$\Sigma = Cov(\underset{\sim}{X}) \in \mathbb{R}^{n \times n}$$

where $\Sigma_{ij} = Cov(X_i, X_j)$

<u>notice:</u> $\Sigma_{ii} = Cov(X_i, X_i) = Var(X_i)$

$$\Sigma = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & \cdots \\ Cov(X_2, X_1) & Var(X_2) & \cdots \\ \vdots & & \ddots \\ & & & Var(X_n) \end{bmatrix}$$

$$Var(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$$

$$Cov(\underset{\sim}{X}) = \mathbb{E}[(\underset{\sim}{X} - \mathbb{E}\underset{\sim}{X})(\underset{\sim}{X} - \mathbb{E}\underset{\sim}{X})^T]$$

<u>Theorem:</u>

If $a \in \mathbb{R}^m$ and $B \in \mathbb{R}^{m \times n}$ and $\underset{\sim}{X}$ is a n-component Rand. vector then

(1) $\mathbb{E}[a + B\underset{\sim}{X}] = a + B\,\mathbb{E}[\underset{\sim}{X}]$

(2) $\mathrm{Cov}(a + B\underset{\sim}{X}) = B\,\mathrm{Cov}(\underset{\sim}{X})\,B^T \longleftarrow$

Multivariate Normal

$$\underset{\sim}{X} \sim N(\mu, \Sigma)$$

$\mu \in \mathbb{R}^n$ $\quad$ $\Sigma \in \mathbb{R}^{n \times n}$

then

$$f(\underset{\sim}{X}) = (2\pi)^{-n/2} (\det \Sigma)^{-1/2} \exp\left(-\frac{1}{2}(\underset{\sim}{X} - \mu)^T \Sigma^{-1} (\underset{\sim}{X} - \mu)\right)$$

Special case: $\mu = 0$ and $\Sigma = I$

we call this standard MV normal.

Theorem

If $\underset{\sim}{X} \sim N(\mu, \Sigma)$ and $a \in \mathbb{R}^m$ and $B \in \mathbb{R}^{n \times m}$

then

$$a + B\underset{\sim}{X} \sim N(a + B\mu, B\Sigma B^T)$$