What about PDF?

Theorem: If X is continuous and $Y = g(X)$ and if

① g is invertible

② g^{-1} is differentiable

then

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$

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pf. Case 1: g is increasing — g^{-1} is inc. So $\frac{dg^{-1}}{dy} > 0$

Our prev. result for CDFs said:

$$F_Y(y) = F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{dF_Y}{dy} = \frac{d}{dy} F_X(g^{-1}(y)) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Case 2: g is decreasing

So by our CDF theorem

So by our CDF theorem

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

g^{-1} is decreasing and so $\frac{dg^{-1}}{dy} < 0$

So

$$f_Y(y) = \frac{dF_Y}{dy} = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right) = -f_X(g^{-1}(y))\frac{dg^{-1}}{dy}$$

$$= f_X(g^{-1}(y))\left(-\frac{dg^{-1}}{dy}\right)$$

$$= f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

$$-(-5) = |5|$$

Ex. Let $X \sim \text{Gamma}(k, \lambda)$

$$\rightarrow f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)} \quad \text{for } x > 0$$

Let $\boxed{Y = 1/X}$ so $g(x) = 1/x = y$

$$\Updownarrow$$

$$g^{-1}(y) = x = \frac{1}{y}$$

and so $\frac{dg^{-1}}{dy} = \frac{d}{dy}\left(\frac{1}{y}\right) = -1/y^2$

So then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

$$= f_X\left(\frac{1}{y}\right) \left| -\frac{1}{y^2} \right|$$

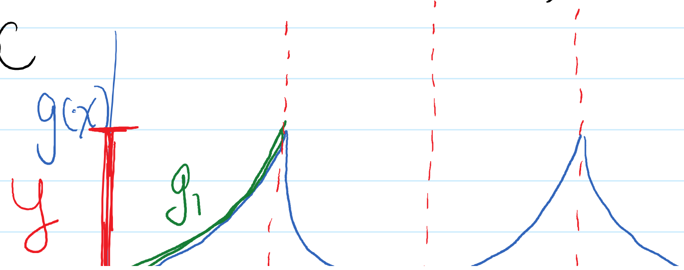
$$= \frac{\lambda e^{-\lambda/y} (\lambda/y)^{k-1}}{\Gamma(k)} \frac{1}{y^2} \text{ for } y > 0$$

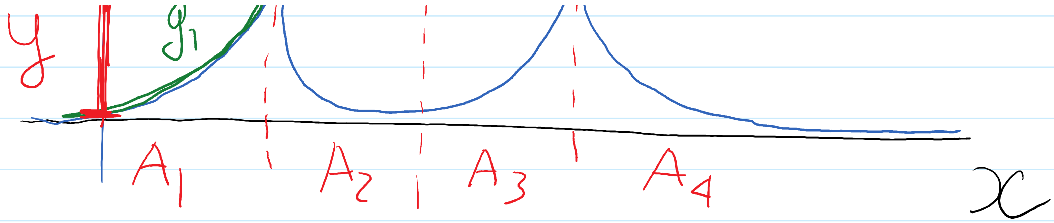
PDF of Y

Called an Inverse Gamma dist.

What if g isn't invertible?

<u>Theorem</u>: Let X be a cts RV w/ support $\mathcal{X}$ and let $A_1, \ldots, A_K$ be a partition of $\mathcal{X}$





so that g_i to be g restricted to A_i

If ① my prev. theorem applies to each g_i separately [g_i invertible, g_i^{-1} diffable]

② the image of A_i under g_i is the same for all i
[Call this $\mathcal{Y}$]

then

$$\boxed{f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{d g_i^{-1}}{dy} \right|} \text{ for } y \in \mathcal{Y}$$

Ex. Chi-Squared Distribution

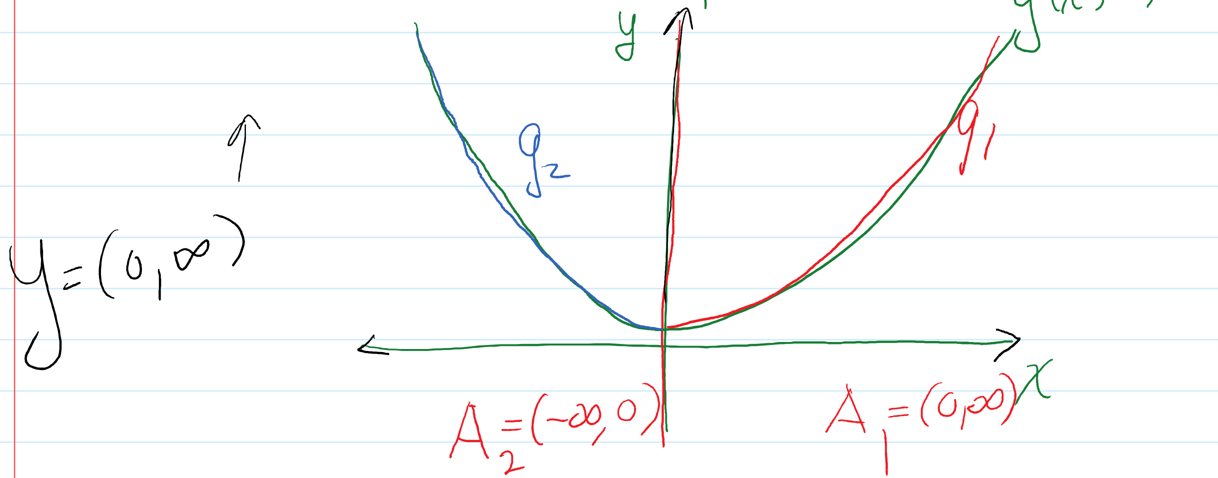
If $X \sim N(0,1)$ and $Y = X^2$

then Y has a Chi-Squared distribution

w/ one degree of freedom,

denoted $Y \sim \chi^2(1)$

What is the PDF of Y?



$A_1 = (0, \infty)$, $g_1(x) = x^2$, $g_1^{-1}(y) = \sqrt{y}$; $\dfrac{dg_1^{-1}}{dy} = \dfrac{1}{2\sqrt{y}}$

$A_2 = (-\infty, 0)$; $g_2(x) = x^2$, $g_2^{-1}(y) = -\sqrt{y}$; $\dfrac{dg_2^{-1}}{dy} = \dfrac{-1}{2\sqrt{y}}$

$$f_Y(y) = f_X(g_1^{-1}(y)) \left| \frac{dg_1^{-1}}{dy} \right| + f_X(g_2^{-1}(y)) \left| \frac{dg_2^{-1}}{dy} \right|$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}$$

$$= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2} \frac{1}{\quad} + \frac{1}{\quad} e^{-(-\sqrt{y})^2} \frac{1}{\quad}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2} \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} \left(e^{-(\sqrt{y})^2} + e^{-(-\sqrt{y})^2} \right)$$

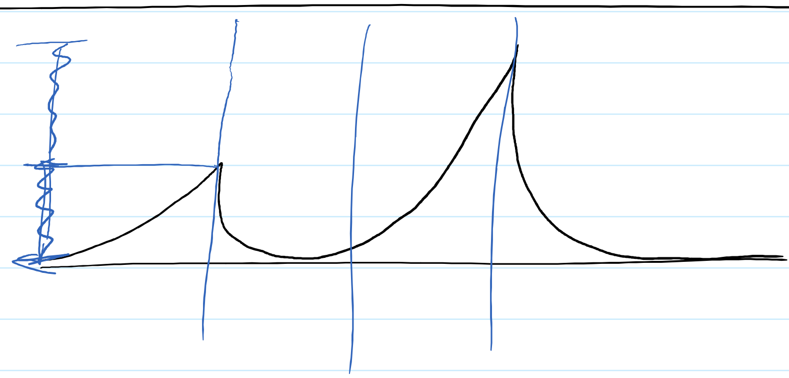
$$(\sqrt{y})^2 = y$$

$$\sqrt{y^2} = |y|$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} \left(e^{-y} + e^{-y} \right)$$

$$= \boxed{\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y} = f_Y(y)}$$

for $y > 0$



Probability Integral Transformation

If X is a continuous RV w/ CDF F_X then

$$F_X(X) \sim U(0,1)$$

Pf: Let $Y = g(X)$ where

$g = F_X$

(assume F_X strictly increasing)

So F_X is invertible

CDF of $U(0,1)$

Then our CDF theorem says, $0 < y < 1$

$$F_Y(y) = F_X(g^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

CDF of a $U(0,1)$

So $Y \sim U(0,1)$.

Generalize:

$$g(X) \sim U(0,1) \iff g = F_X$$

Have: $U \sim U(0,1)$

Want: g so that $X = g(U) \sim F_X$

$$\underbrace{F_X(x)} = \underbrace{P(X \le x)} = P(g(U) \le x)$$

$$= P(U \le g^{-1}(x))$$

$$= g^{-1}(x)$$

So $g^{-1} = F_X$ or $\boxed{g = F_X^{-1}}$

Procedure for generating RV following F_X

1. generate $U \sim U(0,1)$
2. let $X = F_X^{-1}(U)$

then this has req. dist.

Ex. Want $X \sim \text{Exp}(1)$

then $F_X(x) = 1 - e^{-x} = y$

$$\Rightarrow 1 - y = e^{-x}$$

$$\Rightarrow \log(1-y) = -x$$

$$\Rightarrow \boxed{-\log(1-y) = x = F_X^{-1}(y)}$$

If $U \sim U(0,1)$ then $-\log(1-U) \sim Exp(1)$

Bivariate RVs

If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$

then

$$Z = (X, Y)$$

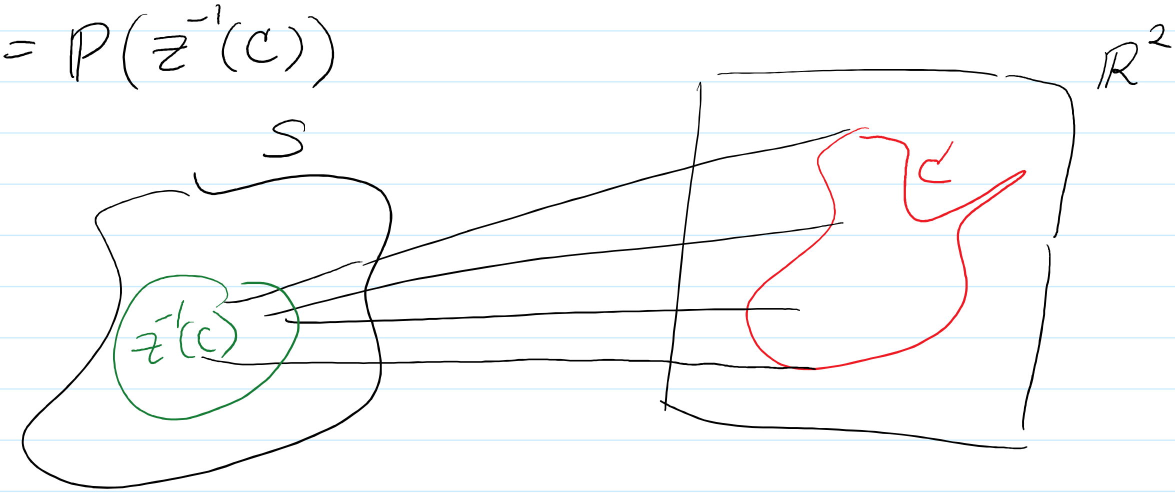
is called a bivariate RV.

So $Z : S \to \mathbb{R}^2$, $Z(s) = (X(s), Y(s))$

Say: $P((X,Y) \in C)$ where $C \subset \mathbb{R}^2$

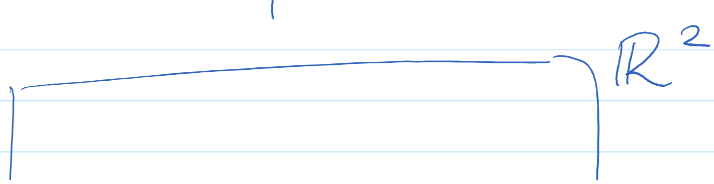
$= P(Z \in C)$

$= P(Z^{-1}(C))$



often $C = A \times B$ when $A, B \subset \mathbb{R}$

Could say



Could say

$P((X,Y) \in C)$

lazy notation

$= P(X \in A, Y \in B)$

"and"