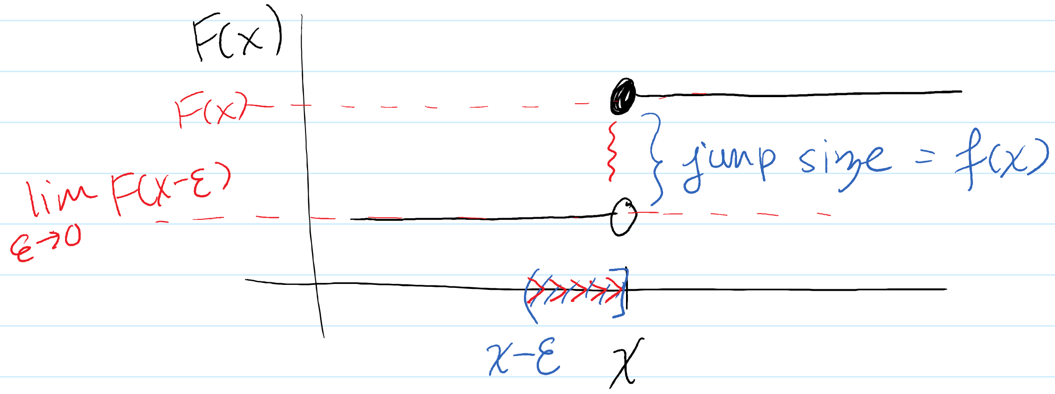
$$\left.\begin{array}{l}\text{PMF}: \; f(x) = \mathbb{P}(X = x) \\ \text{CDF}: \; F(x) = \mathbb{P}(X \le x)\end{array}\right\} \quad F(x) = \sum_{i \le x} f(i)$$



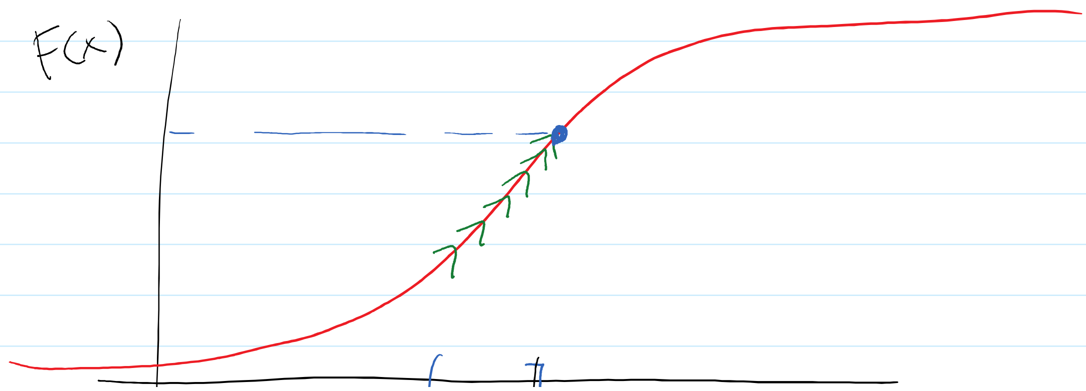
$$\mathbb{P}(a < X \le b) = F(b) - F(a)$$

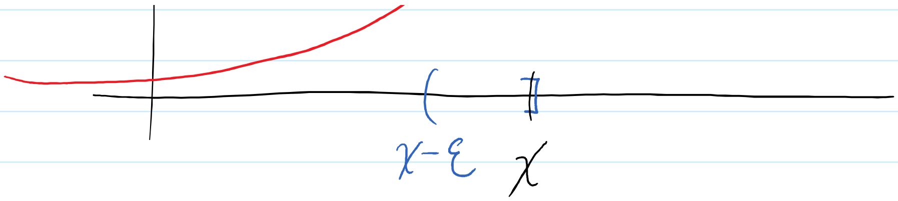
$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(x - \varepsilon < X \le x) = \lim_{\varepsilon \downarrow 0} F(x) - F(x - \varepsilon)$$

$$\mathbb{P}(X = x) = F(x) - \lim_{\varepsilon \downarrow 0} F(x - \varepsilon)$$

$$f(x) = \text{jump size}$$

What about a cts RV?





$$P(X=x) = \lim_{\varepsilon \downarrow 0} P(x-\varepsilon < X \le x)$$

$$= \cdots = F(x) - \underbrace{\lim_{\varepsilon \downarrow 0} F(x-\varepsilon)}_{F(x)}$$

$$= 0$$

Want: cts analog for PMF:

$$F(x) = \sum_{i \le x} f(i)$$

<u>Defn</u>: Probability Density Function (PDF)

Cts version of PMF

The PDF for a cts RV is a function $f: \mathbb{R} \to \mathbb{R}$, defined for $x \in \mathbb{R}$, as the function that satisfies

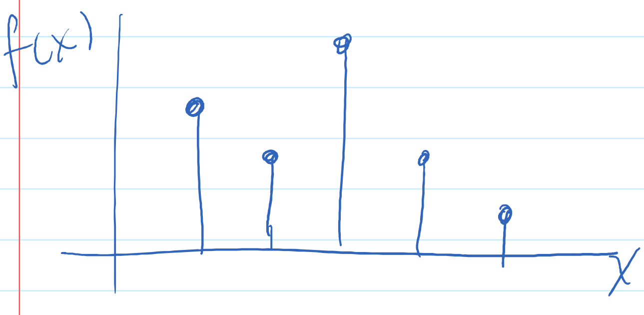
$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Note: by the Fund. Theorem of Calc,

$$\frac{dF}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)$$

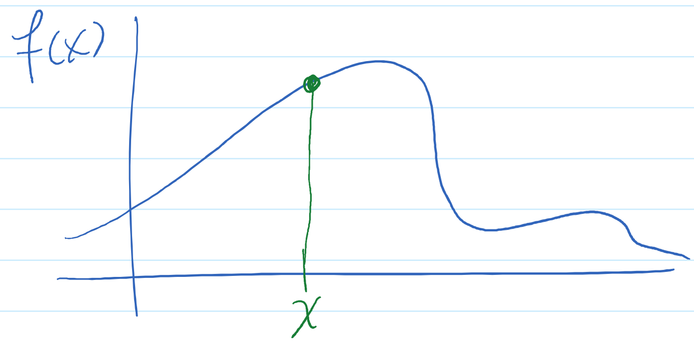
So $f(x) = \frac{dF}{dx}$. (PDF = deriv. of CDF)

discrete PMF



$f(x) = \mathbb{P}(X = x)$

Continuous PDF

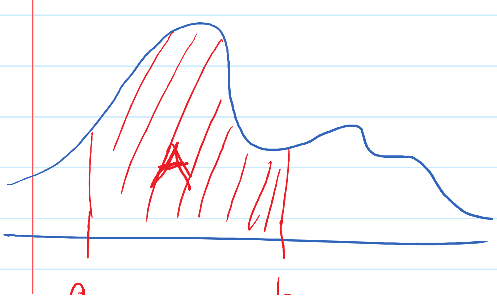


$f(x) \neq \mathbb{P}(X = x)$

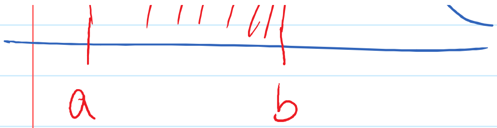
Properties of PDFs

$$\mathbb{P}(a < X \leq b) = F(b) - F(a)$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$



$$= \int^{b} f(t)\,dt$$



$$= \int_a f(t)\,dt$$

$$\mathbb{P}(X=a) = \mathbb{P}(X=b) = 0$$

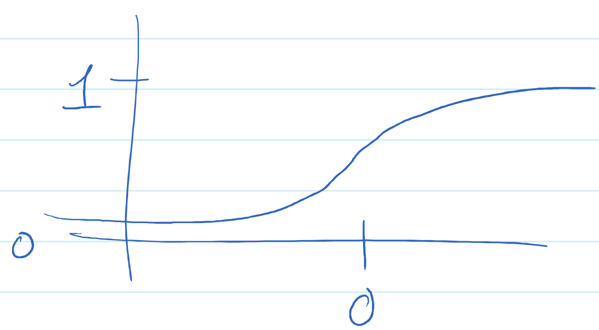
$$\left.\begin{aligned} \mathbb{P}(a < X \le b) &= \mathbb{P}(a \le X \le b) \\ &= \mathbb{P}(a < X < b) \\ &= \mathbb{P}(a \le X < b) \end{aligned}\right\} \text{ for cts RV}$$

Generally:

(discrete) $\mathbb{P}(X \in A) = \sum_{i \in A} f(i)$

(cts) $\mathbb{P}(X \in A) = \int_A f(x)\,dx$

Ex. $F(x) = \dfrac{1}{1+e^{-x}}$



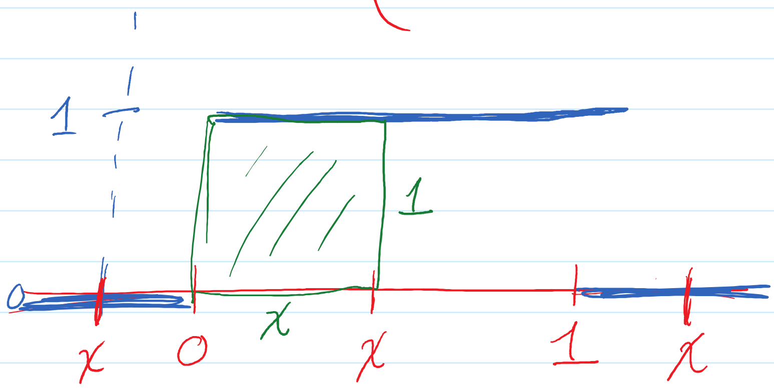
Q: What is its PDF?

$$f(x) = \frac{dF}{dx} = \cdots = \frac{e^{-x}}{(1+e^{-x})^2}$$

Ex. Continuous Unif. Dist.

$$X \sim U(0,1)$$

means $$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$



what's the CDF? $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$x < 0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

$0 < x < 1$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 1\,dt = x$$

$x > 1$

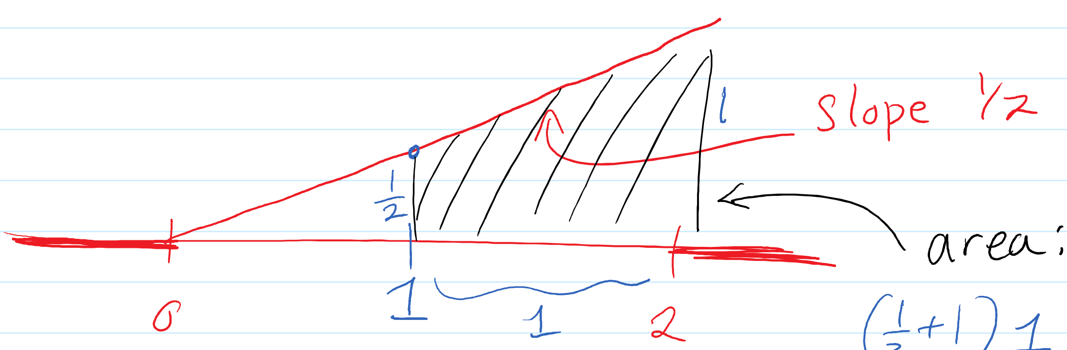
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{1} 1\,dt = 1$$

$$F(x) = \int_{-\infty} f(t)\,dt = \int_0 1\,dt = 1$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

<u>Ex.</u>

$$f(x) = \frac{x}{2} \quad \text{for} \quad 0 < x < 2$$



$$\frac{\left(\frac{1}{2}+1\right)}{2}\,1 = \frac{3}{4}$$

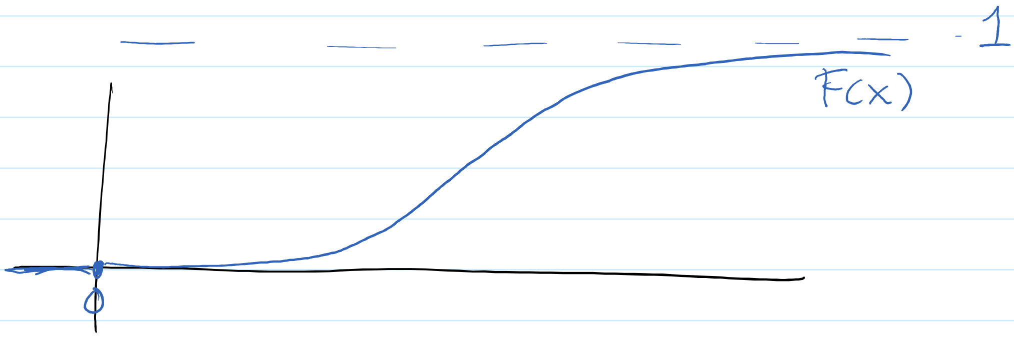
$$P(X>1) = \int_1^{\infty} f(x)\,dx = \int_1^2 \frac{x}{2}\,dx$$

$$= \frac{x^2}{4}\Bigg|_1^2 = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

$$= \frac{x^2}{4} = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

Ex.

$$F(x) = 1 - e^{-x} \text{ for } x > 0$$



Q: $\mathbb{P}(1 < X < 2)$?

Way 1: $\mathbb{P}(1 < X < 2) = F(2) - F(1)$

$$= (1-e^{-2}) - (1-e^{-1})$$

$$= e^{-1} - e^{-2}$$

Way 2: $f(x) = \frac{dF}{dx} = \frac{d}{dx}(1-e^{-x}) = e^{-x}$ for $x > 0$

$$P(1 < X < 2) = \int_1^2 f(x)\,dx = \int_1^2 e^{-x}\,dx = -e^{-x}\Big|_1^2$$

$$= e^{-1} - e^{-2}$$

Theorem: PMF/ PDF characterization

A function f is the PMF/PDF of some RV

iff

① $f(x) \geq 0 \quad \forall x$

② (discrete) $\sum_{x \in \mathbb{R}} f(x) = 1$

(cts) $\int_{\mathbb{R}} f(x)\,dx = 1$

If $g(x) \geq 0$ and $\int_{\mathbb{R}} g(x)\,dx = c < \infty$

$$f(x) = \frac{1}{c} g(x)$$

then f is a PDF.

Ex. Normal Distribution (Gaussian)

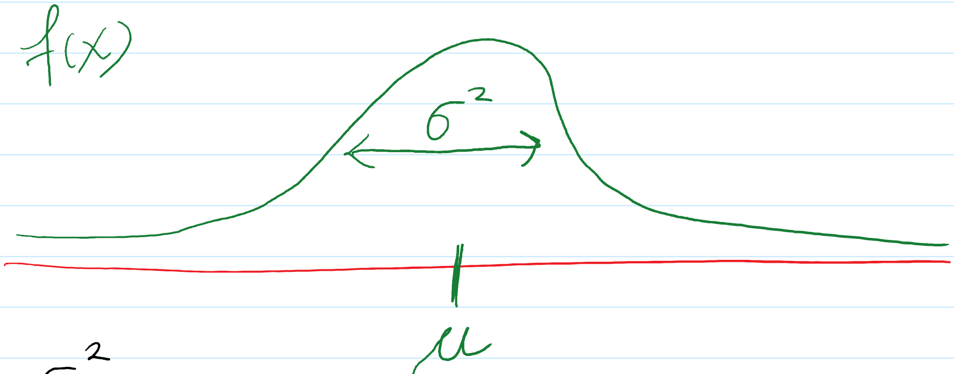
notation: $X \sim N(\mu, \sigma^2)$

variance: $\sigma^2 > 0$

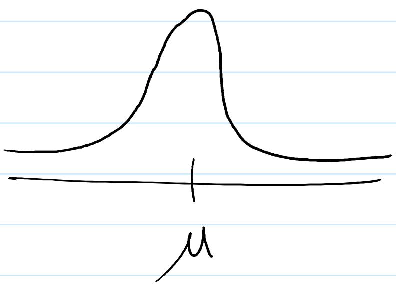
$X \sim N(\mu, \sigma^2)$

mean: $\mu \in \mathbb{R}$

variance: $\sigma^2 > 0$



Small σ^2



large σ^2

Special Case: Standard normal $\mu = 0$, $\sigma^2 = 1$

$X \sim N(0,1)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

$\exp(a) = e^a$

Q: Is this a valid PDF?

1. $f(x) \geq 0$ ✓

② $$\int_{\mathbb{R}} f(x)\,dx = 1$$

$$\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx}_{I} = 1$$

Want: $I = 1 \iff I^2 = 1$.

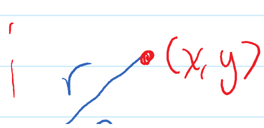
$$I^2 = I \cdot I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}x^2\right) \exp\left(-\frac{1}{2}y^2\right) dx\,dy$$

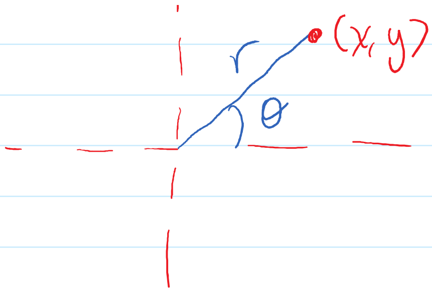
$$e^a e^b = e^{a+b}$$

$$= \iint_{\mathbb{R}^2} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right) dx\,dy$$

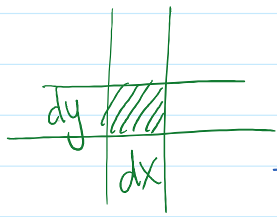
Polar Coordinates



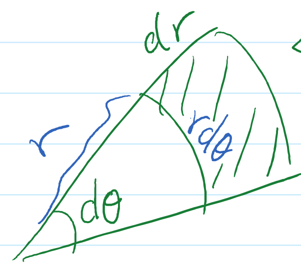
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



$$\begin{cases} y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$$



$\boxed{dxdy}$



$\boxed{rdrd\theta}$

$$\frac{1}{2\pi}\iint_{\mathbb{R}^2} \exp\left(-\tfrac{1}{2}(x^2+y^2)\right)dxdy$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\underbrace{\int_0^{\infty} \exp\left(-\tfrac{1}{2}r^2\right) r\,dr}_{\text{u-substitution}}\,d\theta$$

$u = \frac{1}{2}r^2$

$du = r\,dr$

$$\int_0^{\infty} e^{-u}\,du = -e^{-u}\Big|_0^{\infty} = 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1 = I^2 \Rightarrow I = 1.$$

Expected Value

If X is a RV then the mean or

expected value of X

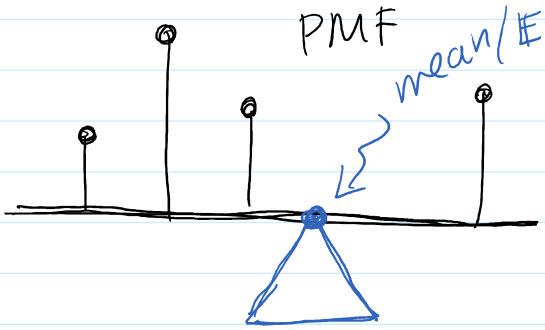
denoted $\mathbb{E}[X]$.

is defined as

① discrete

PMF

$$\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x f(x)$$



$$= \sum_{x \in \text{Support}(X)} x f(x)$$

② continuous:

PDF

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\, dx$$