Theorem: Cov/Cor of Independent

If $X \perp Y$ then $\text{cov}(X, Y) = \text{Cor}(X, Y) = 0$.

Pf.

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

independence

$$= \mathbb{E}[X]\mathbb{E}[Y] - \cdots = 0$$

Cor is just re-scaled cov, so it is also zero.

Converse is generally false

If $\text{cor}(X, Y) = 0$ then X and Y may or may not be independent.

Ex. $X \sim N(0,1)$ and $Y = X^2$.

not independent

however

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y$$

$$= \mathbb{E}[X X^2] - \mathbb{E}X\,\mathbb{E}X^2$$

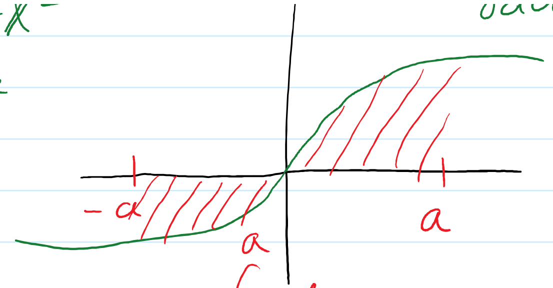
odd fn

$= \mathbb{E}[X X^2] - \mathbb{E}X\,\mathbb{E}X^2$

$= \mathbb{E}[X^3] - \underbrace{\cancel{\mathbb{E}X}}_{0}\,\mathbb{E}X^2$

$= \mathbb{E}[X^3]$

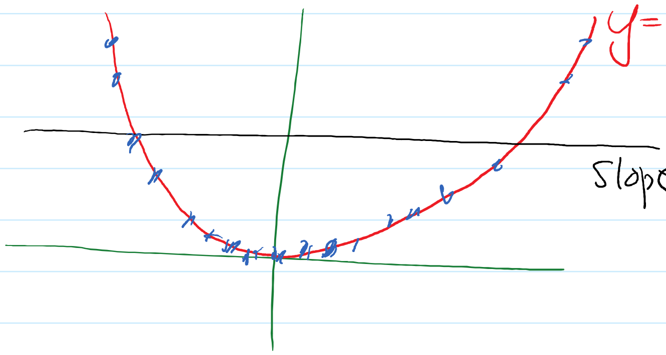
odd fn



$\int_{-a}^{a} \text{odd fn} = 0$

$$\mathbb{E}[X^3] = \int_{\mathbb{R}} \underbrace{\underbrace{x^3}_{\text{odd}} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-x^2}}_{\text{even fn}}}_{\text{odd fn}} dx = 0$$

$= 0$ So $\mathrm{Cor}(X, Y) = 0$.



Bayes' Theorem:

Events: $\mathbb{P}(A|B) = \dfrac{P(B|A)\,P(A)}{P(B)}$

RV: $f(x|y) = \dfrac{f(y|x)\,f_X(x)}{f_Y(y)}$

Law of Total Prob:

Events: (C_i) partition S then

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A|C_i)\mathbb{P}(C_i)$$

RVs:

$$\boxed{\begin{aligned} &(\text{discrete}) \quad f(y) = \sum_x f(y|x) f(x) \\ &(\text{cts}) \quad f(y) = \int_{\mathbb{R}} f(y|x) f(x)\, dx \end{aligned}}$$

Pf: (cts case)

1. $f(y|x) = \dfrac{f(x,y)}{f(x)}$

$$\Updownarrow$$

$$\underline{f(y|x) f(x)} = f(x,y)$$

2. $\boxed{f(y)} = \int_{\mathbb{R}} f(x,y)\,dx = \boxed{\int_{\mathbb{R}} f(y|x) f(x)\, dx}.$

Ex. $\begin{cases} X \sim \text{Exp}(\lambda) & \to f(x) = \lambda e^{-\lambda x} \\ Y | X = x \sim \text{Pois}(x) & \to f(y|x) = \dfrac{x^y e^{-x}}{y!} \end{cases}$

$(Y \mid X = x \sim \text{Pois}(x) \rightarrow f(y|x) = \frac{x^y e^{-x}}{y!})$

What is the dist of Y?

Law of Tot. Prob.

The form $x^{\sim} e^{-x}$ is basically a Gamma RV PDF

$$\text{Gama}(a,b) \text{ PDF } \frac{(bx)^{a-1} b e^{-bx}}{\Gamma(a)}$$

$$f(y) = \int_R f(y|x) f(x)\, dx$$

$$= \int_0^\infty \frac{x^y e^{-x}}{y!} \lambda e^{-\lambda x}\, dx$$

$$= \frac{\lambda}{y!} \int_0^\infty x^{\overset{a-1}{y}} e^{-\overset{b}{(\lambda+1)} x}\, dx$$

$a - 1 = y$

$a = y + 1$

$b = \lambda + 1$

$$= \frac{\lambda}{y!} \frac{\Gamma(a)}{b^{a-1}} \underbrace{\int_0^\infty \frac{x^{a-1} e^{-bx} b^{a-1}}{\Gamma(a)}\, dx}_{\text{integrate to } 1}$$

$$= \frac{\lambda}{y!} \frac{\Gamma(y+1)}{(\lambda+1)^y}$$

$$= \frac{\lambda}{\cancel{y!}} \frac{\cancel{y!}}{(\lambda+1)^y} = \frac{\lambda}{(\lambda+1)^y} \quad \text{for } y = 0, 1, 2, \ldots$$

$$= f(y)$$

Ex.

$$Y \sim \text{Pois}(\lambda)$$

$$X \mid Y = y \sim \text{Bin}(y, p) \quad 0 < p < 1$$

Notice: $0 \le X \le Y$

What is the dist of Y?

$$\binom{y}{x}\frac{1}{y!} = \frac{\cancel{y!}}{x!(y-x)!}\frac{1}{\cancel{y!}} = \frac{1}{x!(y-x)!}$$

$$f(x) = \sum_{y} \underbrace{f(x|y)}_{\text{Bin}(y,p)}\underbrace{f(y)}_{\text{Pois}(\lambda)} = \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{\lambda^y e^{-\lambda}}{y!}$$

$$= \sum_{y=x}^{\infty} \frac{1}{x!(y-x)!} p^x (1-p)^{y-x} \lambda^y e^{-\lambda}$$

$$= \frac{e^{-\lambda} p^x \lambda^x}{x!} \sum_{y=x}^{\infty} \frac{1}{(y-x)!} (1-p)^{y-x} \lambda^{y-x}$$

$$= \frac{e^{-\lambda} p^x \lambda^x}{x!} \underbrace{\sum_{y=0}^{\infty} \frac{1}{y!} [\underbrace{(1-p)\lambda}_{z}]^y}_{e^{(1-p)\lambda}}$$

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!}$$

$$= \frac{\cancel{e^{-\lambda}} p^x \lambda^x}{x!} e^{(\cancel{1}-p)\lambda}$$

$$\frac{\quad}{x!} e$$

$$\boxed{f(x) = \frac{(p\lambda)^x e^{-(p\lambda)}}{x!}}$$

$$X \sim \text{Pois}(p\lambda)$$

Theorem: Iterated Expectation

If X and Y are RVs then

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$$

$$\mathbb{E}[X \mid Y = y] = \int_{\mathbb{R}} x f(x \mid y)\, dx \leftarrow \text{depend on } y$$

For each $y \in \mathbb{R}$ we get some value

This defines a function

$$g(y) = \mathbb{E}[X \mid Y = y]$$

a number

We can plug Y into g

$$g(Y) = \mathbb{E}[X \mid \cancel{Y =} Y]$$

weird notation

$g(y) = y^2 \leftarrow$ a number

$g(Y) = Y^2$

a RV

weird notation

$$= \mathbb{E}[X \mid Y]$$

a RV.

Basically:

1. $\mathbb{E}[X \mid Y = y]$ is a number
2. $\mathbb{E}[X \mid Y]$ is a RV

Ex. $\mathbb{E}[X \mid Y = y] = \log(y)$

$$\mathbb{E}[X \mid Y] = \log(Y)$$

Ex. $Y \sim \mathrm{Pois}(\lambda)$

$$X \mid Y = y \sim \mathrm{Bin}(y, p)$$

What is $\mathbb{E}[X]$? $(= \lambda p)$

1. get $\mathbb{E}[X \mid Y = y] = yp$
2. $\mathbb{E}[X \mid Y] = Yp$
3. $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[Yp]$

$= p\, \mathbb{E}Y$

$= p\lambda$

---

pf. $\mathbb{E}[X] = E[E[X|Y]]$ [cts case]

① $f(x) = \int f(x,y)\,dy$

② $f(x|y) = \dfrac{f(x,y)}{f(y)} \iff \underline{f(x,y) = f(x|y)f(y)}$

③ $\mathbb{E}[X|Y=y] = \int x f(x|y)\,dx$

$$\mathbb{E}[X] = \int x f(x)\,dx \overset{①}{=} \int x \int f(x,y)\,dy\,dx$$

$$\overset{②}{=} \int x \int f(x|y) f(y)\,dy\,dx$$

$$= \int \underbrace{\int x f(x|y)\,dx}_{\mathbb{E}[X|Y=y] \,=\, g(y)} f(y)\,dy$$

$$= \int g(y) f(y)\,dy$$

$$= \mathbb{E}[g(Y)]$$ called this $E[X|Y]$

$$\boxed{E[X] = E[E[X|Y]]}$$

Ex. $P \sim \text{Beta}(\alpha, \beta)$

$X \mid P = p \sim \text{Bin}(n, p)$

Known

$\mathbb{E}[X]$?

① $\mathbb{E}[X \mid P = p] = np$

② $\mathbb{E}[X \mid P] = nP$

③ $\mathbb{E}X = \mathbb{E}[\mathbb{E}[X \mid P]] = \mathbb{E}[nP]$

$= n\mathbb{E}P$

$$\boxed{= n\frac{\alpha}{\alpha+\beta}}$$

Theorem: Law of Total Variance

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X \mid Y)] + \text{Var}(\mathbb{E}[X \mid Y])$$

Ex. Continue prev.

$\text{Var}(X)$?

① $\mathbb{E}[X \mid P = p] = np$

$\text{Var}(X \mid P = p) = np(1-p)$

② $\mathbb{E}[X \mid P] = nP$

$Var(X \mid P) = nP(1-P)$

③ $Var(X) = \mathbb{E}[Var(X \mid P)] + Var(\mathbb{E}[X \mid P])$

$= \mathbb{E}[nP(1-P)] + Var(nP)$

$= n\left(\underline{\mathbb{E}[P]} - \underline{\mathbb{E}[P^2]}\right) + n^2 \underline{Var(P)}$

$P \sim Beta(\alpha, \beta)$ plug in

$= \cdots = n \dfrac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)} + \dfrac{n^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$