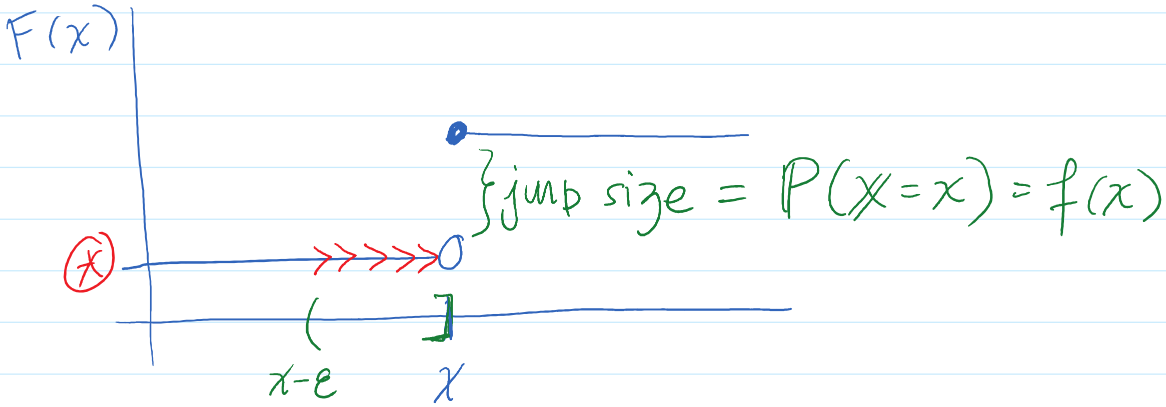
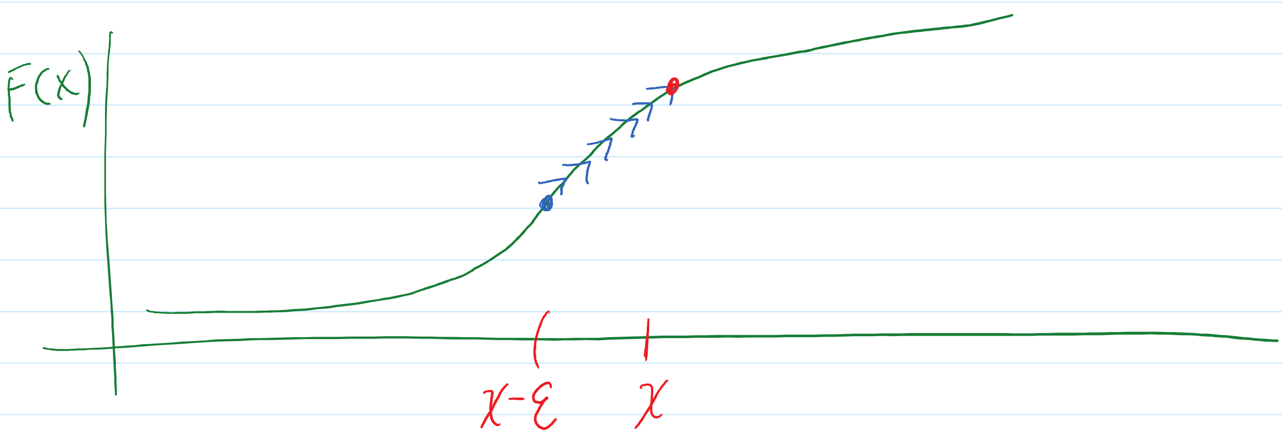


$$\lim_{\varepsilon \to 0} P(x-\varepsilon < X \leq x) = \lim_{\varepsilon \to 0} \left(F(x) - F(x-\varepsilon)\right)$$

$$= F(x) - \lim_{\varepsilon \to 0} F(x-\varepsilon)$$

$$= \text{jump size} = f(x)$$

Continuous Case:



$$\lim_{\varepsilon \to 0} P(x-\varepsilon < X \leq \varepsilon) = \ldots$$

$$= F(x) - \lim_{\varepsilon \to 0} F(x-\varepsilon)$$

$$= F(x) - F(x) = 0$$

Punchline: PMF formulation not as useful for cts RVs

Want: something for cts RVs that behaves like PMF

$$F(x) = \sum_{i \leq x} f(i)$$

Defn: Probability Density Function (PDF)

Continuous analog of PMF.

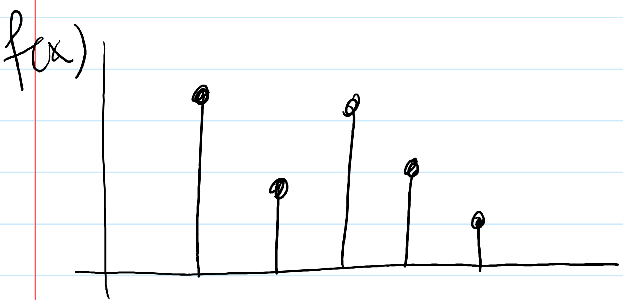
For a cts RV X the PDF is a function $f: \mathbb{R} \to \mathbb{R}$ so that $\forall x \in \mathbb{R}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Note: by Fund. Thrm. of Calc.

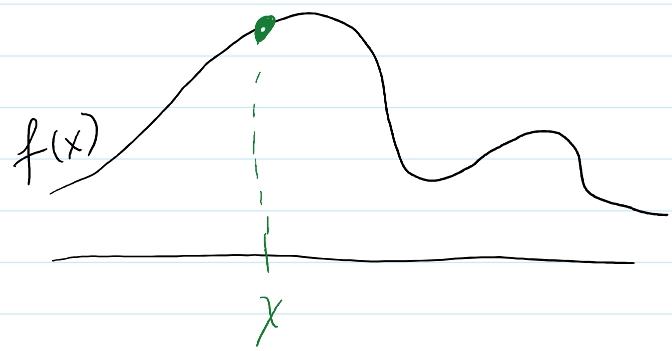
$$\frac{dF}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)$$

discrete PMF



$$f(x) = \mathbb{P}(X = x)$$

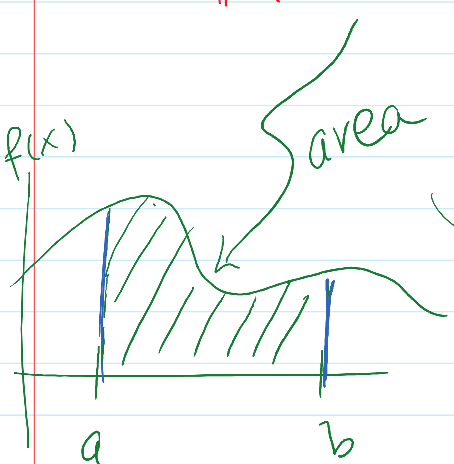
Continuous PDF



$$f(x) \neq \mathbb{P}(X = x)$$

Properties:

$$\mathbb{P}(a < X \le b) = F(b) - F(a)$$



$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$

$$= \int_{a}^{b} f(t)\,dt$$

Said $\mathbb{P}(X=a) = \mathbb{P}(X=b) = 0$

be sloppy w/ end points

$$\left.\begin{aligned} \mathbb{P}(a < X \le b) &= \mathbb{P}(a \le X \le b) \\ &= \mathbb{P}(a \le X < b) \\ &= \mathbb{P}(a < X < b) \end{aligned}\right\} \text{only for cts}$$

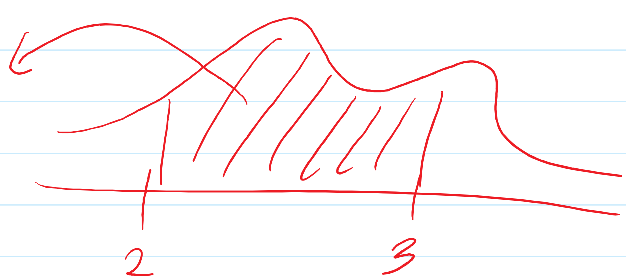
<u>More general fact!</u>

(discrete): $\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$

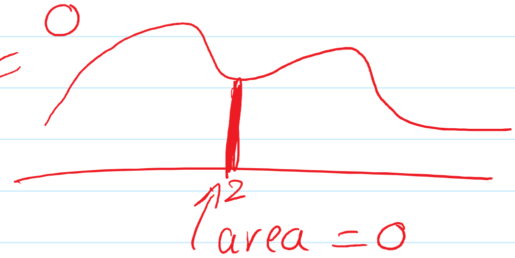
(cts): $\mathbb{P}(X \in A) = \int_A f(x)\,dx$

<u>Ex.</u>

$\mathbb{P}(X \in (2,3]) = \int_2^3 f(x)\,dx$



$\mathbb{P}(X = 2) = \mathbb{P}(X \in \{2\}) = \int_2^2 f(x)\,dx = 0$



Ex.

$$F(x) = \frac{1}{1+e^{-x}}$$

What is the assoc. PDF?

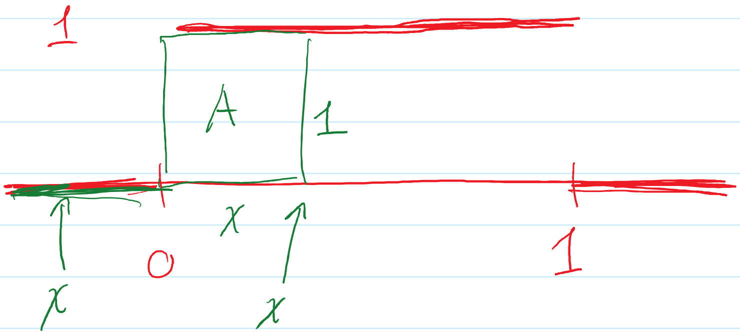
$$f(x) = \frac{dF}{dx} = \cdots = \frac{e^{-x}}{(1+e^{-x})^2}$$

Ex. **Continuous Unif. Dist.**

$$X \sim U(0,1)$$

means:

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$



What is the assoc. CDF?

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$x<0$: $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$

$0<x<1$: $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 1\,dt = x$

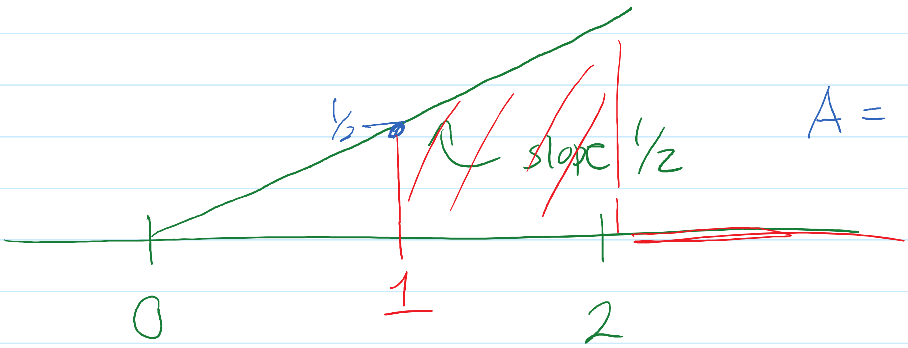
$x>1$: $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{1} dt = 1$

$$F(x) = \begin{cases} 0 & x<0 \\ x & 0<x<1 \\ 1 & x>1 \end{cases}$$

Ex.

Ex.

$$f(x) = \begin{cases} x/2 & 0 < x < 2 \\ 0 & \text{else} \end{cases}$$

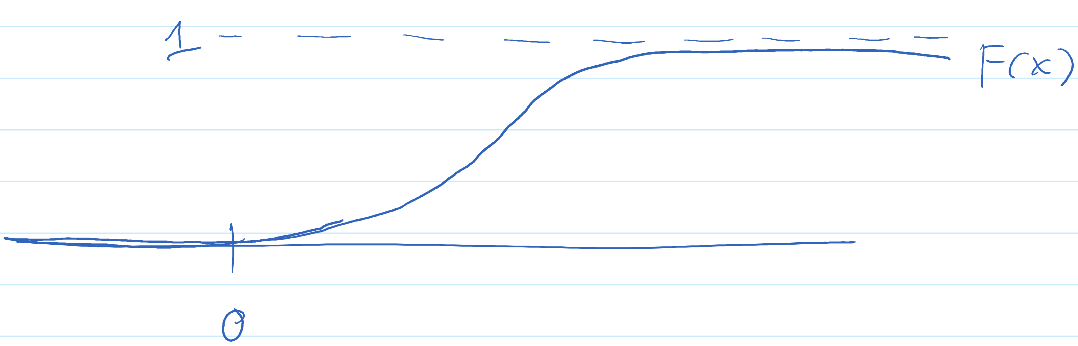


$$A = \frac{\left(\frac{1}{2}+1\right)(1)}{2} = 3/4$$

$$P(X > 1) = \int_1^\infty f(x)\,dx = \int_1^2 \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_1^2 = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

Ex.

$$F(x) = 1 - e^{-x} \quad \text{for } x > 0$$



Q: $P(1 < X < 2)$?

Way 1: $\mathbb{P}(1 < X < 2) = F(2) - F(1)$

$$= (1 - e^{-2}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-2}$$

Way 2: $f(x) = \frac{dF}{dx} = \frac{d}{dx}(1 - e^{-x}) = e^{-x}$ for $x > 0$

$$\mathbb{P}(1 < X < 2) = \int_1^2 e^{-x} dx = -e^{-x}\Big|_1^2 = e^{-1} - e^{-2}$$

Theorem: PMF/PDF characterization

A function f is the PMF/PDF of some RV

iff

(1) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

(2) (discrete) $\sum_{x \in \mathbb{R}} f(x) = 1$

(cts) $\int_{\mathbb{R}} f(x) dx = 1$

Fact: If $g(x) \geq 0$

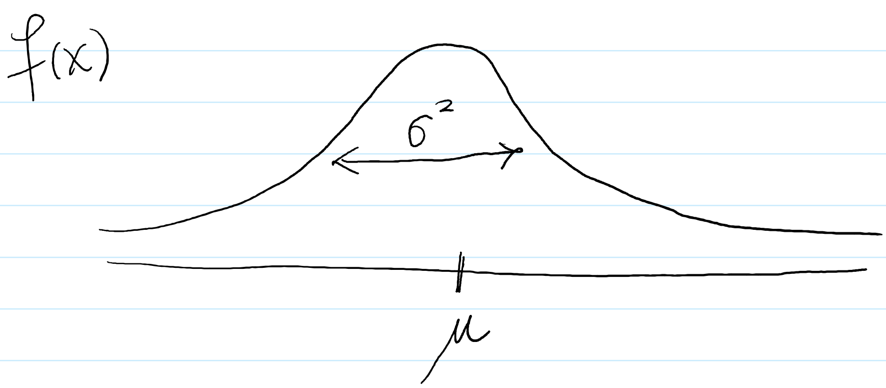
and $\displaystyle\int_{\mathbb{R}} g(x)\,dx = c < \infty$

If $f(x) = \frac{1}{c} g(x)$ then f is a PDF.

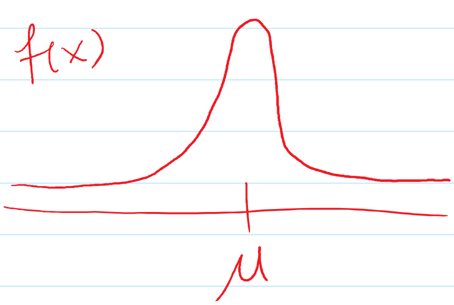
Ex. Normal Distribution (Gaussian)

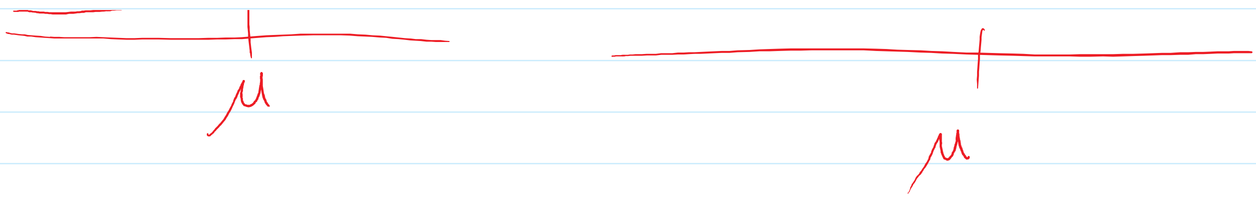
notation: $X \sim N(\mu, \sigma^2)$

mean: $\mu \in \mathbb{R}$ variance: $\sigma^2 > 0$



small σ^2 large σ^2





Special Case: Standard Normal, $\mu = 0$, $\sigma^2 = 1$

$$X \sim N(0,1)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

$\exp(a) = e^a$

Q: Is this a valid PDF?

1. $f(x) \geq 0 \quad \forall x$ ✓

2. $\displaystyle\int_{\mathbb{R}} f(x)\,dx = 1$

$$\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx}_{I} = 1$$

Want: $I = 1 \iff \boxed{I^2 = 1}$

$$I^2 = I \cdot I = \int^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \int^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

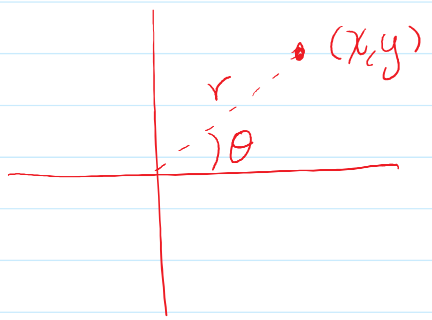
$$I^2 = I \cdot I = \int_{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \int_{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2} dx\,dy$$

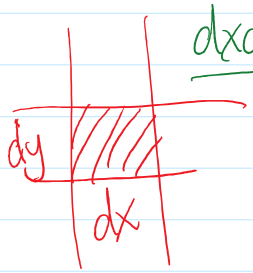
$e^a e^b = e^{a+b}$

$$= \iint_{\mathbb{R}^2} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right) dx\,dy$$

Polar coordinates:



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$$



dxdy

rdrdθ

$$\frac{1}{2\pi} \iint \exp\left(-\frac{1}{2}(x^2+y^2)\right) dx\,dy = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \underbrace{\int_{r=0}^{\infty} e^{-\frac{1}{2}r^2} r\,dr}_{1}\, d\theta \quad (*)$$

Solve inner integral w/ u-substitution

$$u = \frac{1}{2} r^2 \quad ; \quad du = r\,dr$$

$$\int_0^\infty e^{-u} du = -e^{-u} \Big|_0^\infty = 0 - (-1) = 1$$

$$(*) = \frac{1}{2\pi} \int_0^{2\pi} d\theta = \frac{1}{2\pi}(2\pi) = 1.$$

Expected Value:

If X is a RV then the mean or expected value of X

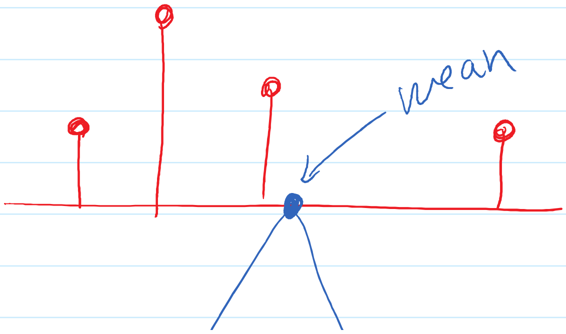
denoted $\mathbb{E}[X]$,

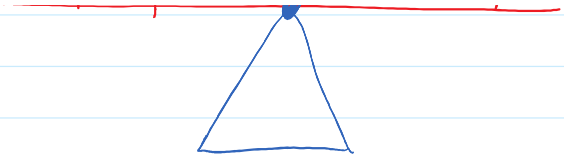
is defined

weighted avg

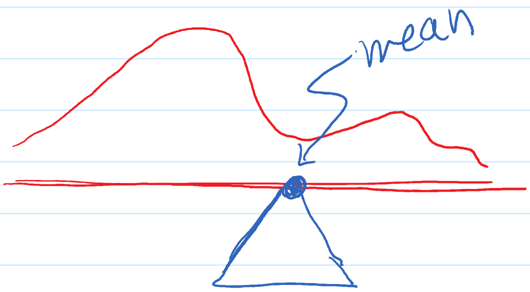
(1) discrete $\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x f(x)$

$$= \sum_{x \in \mathrm{Support}(X)} x f(x)$$





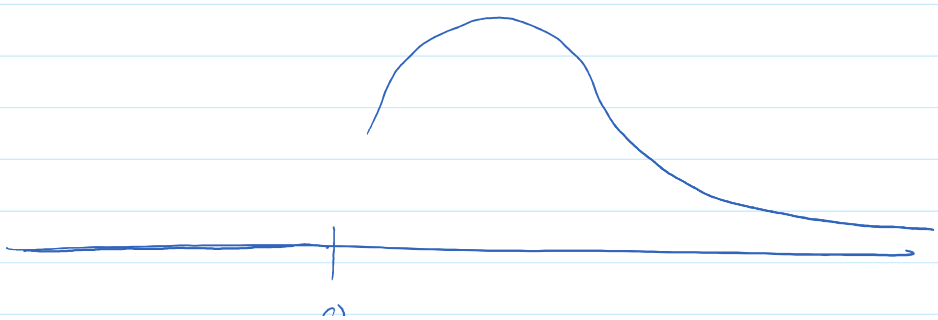
② Continuous: $E[X] = \int_{\mathbb{R}} x f(x)\,dx$



Ex. Let $X \sim Exp(\lambda)$ $\lambda > 0$

Exponential dist.

$f(x) = \lambda e^{-\lambda x}$ for $x > 0$



Q: what is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \underbrace{\int_0^\infty x \lambda e^{-\lambda x}\,dx}_{\text{integration by parts?}}$$