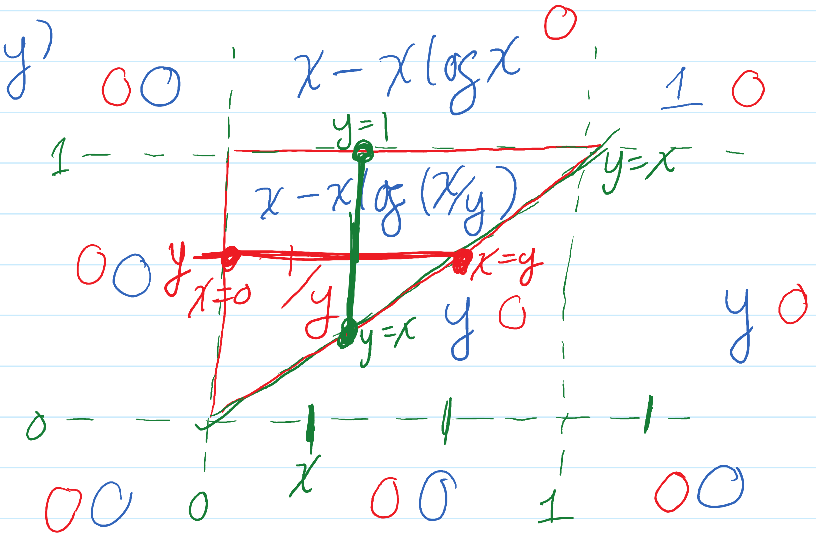
Ex. $F(x,y)$




What is the joint PDF?

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

For $0 < x < y < 1$, $F(x,y) = x - x\log(x/y)$

$$f(x,y) = \frac{\partial F}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(x - x\log(x/y)\right)$$

$$= \frac{\partial}{\partial x}\left[x/y\right]$$

$$= 1/y$$

$$\frac{\partial}{\partial y}\left(x - x\log(x/y)\right) = -x\,\frac{-x/y^2}{x/y} = x/y$$

All together: $\boxed{f(x,y) = \frac{1}{y} \text{ for } 0 < x < y < 1}$

What is the marginal PDF of $X$?

What is the marginal PDF of $X$?

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_x^1 \frac{1}{y}\,dy = \log(y)\Big|_x^1$$

$$= \log(1) - \log(x)$$

$$= -\log(x)$$

$$\boxed{f_X(x) = -\log(x) \quad \text{for} \quad 0 < x < 1}$$

What is the marginal of $Y$?

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \int_0^y \frac{1}{y}\,dx = \frac{1}{y}\int_0^y dx$$

$$= \frac{1}{y}\,x\Big|_0^y = \frac{1}{y}[y-0]$$

$$= 1$$

for $0 < y < 1$

So $Y \sim U(0,1)$

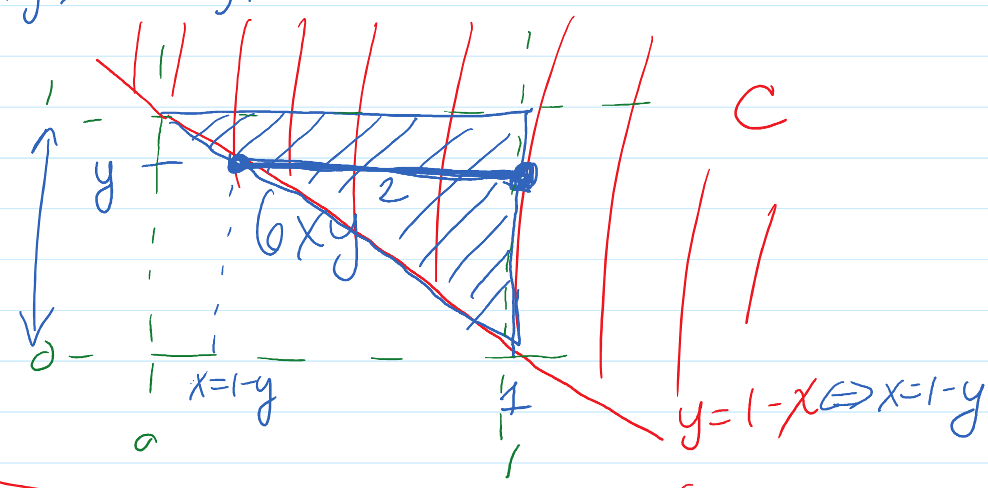
Ex. Let $f(x,y) = 6xy^2$ for $0 < x < 1$, $0 < y < 1$

$\to x + y \geq 1 \iff y \geq 1 - x$

$P(X+Y \geq 1)$ → $x+y \geq 1 \iff y \geq 1-x$

$= \iint_C f(x,y)\,dx\,dy$

$= \int_0^1 \int_{x=1-y}^{x=1} 6xy^2\,dx\,dy$




Uni: $P(X \in A) = \int_A f_X(x)\,dx$

$= \dots$

$= 9/10$

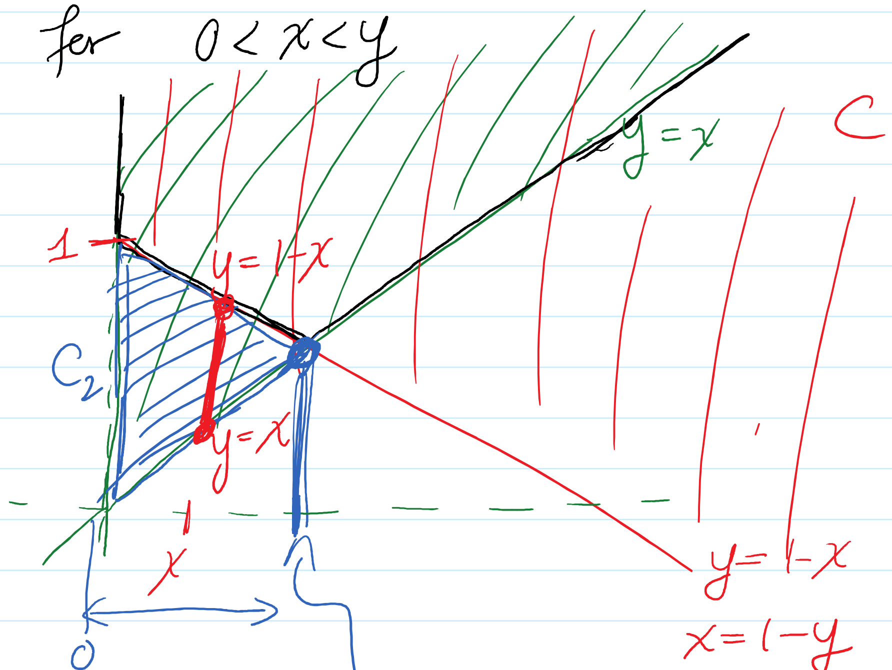
biv: $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$

---

Ex. $f(x,y) = e^{-y}$ for $0 < x < y$

$P(X+Y \geq 1)$

$= \iint_C f(x,y)\,dx\,dy$




$= 1 - P(X+Y<1)$

$= 1 - \iint_{C_2} f(x,y)\,dx\,dy$

$= 1 - \int^{1/2} \int^{y=1-x} e^{-y}\,dy\,dx$

intersection of $y=x$ and $y=1-x$

$x = 1-x$

$\Rightarrow x = 1/2$

$$= 1 - \int_{0} \int_{y=x} e^{-y} \, dy \, dx$$

$x = 1 - x$

i.e. $x = 1/2$

$$= \cdots = \boxed{2e^{-1/2} - e^{-1}}$$

---

## Defn: Bivariate Expectation

If $(X, Y)$ is a biv. RV and $g: \mathbb{R}^2 \to \mathbb{R}$ then

$$\mathbb{E}[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) & \text{(discrete)} \\ \iint_{\mathbb{R}^2} g(x,y) f(x,y) \, dx\,dy & \text{(cts)} \end{cases}$$

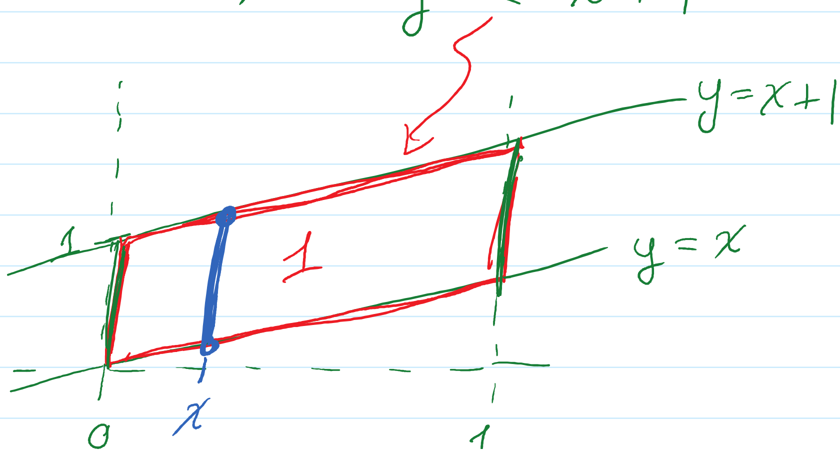
[ Uni: $\mathbb{E}[g(X)] = \int g(x) f(x) \, dx$ ]

---

Ex. let $f(x,y) = 1$ for $0 < x < 1$

$x < y < x+1$

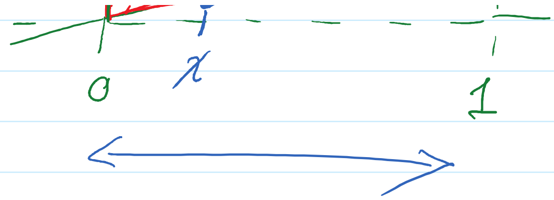
$\mathbb{E}[XY]$ $g(x,y) = xy$




$$= \iint_{\mathbb{R}^2} g(x,y) f(x,y) \, dx\,dy$$

$$= \int^{1} \int^{y=x+1} xy \, (1) \, dy \, dx$$

$$= \int_{0} \int_{y=x} xy\,(1)\,dy\,dx$$




$$= \cdots = 7/12$$

## Theorem: Bivariate Expectation is Linear

If $g_1 : \mathbb{R}^2 \to \mathbb{R}$, $g_2 : \mathbb{R}^2 \to \mathbb{R}$ and $a, b \in \mathbb{R}$

then

$$\mathbb{E}\left[a g_1(X,Y) + b\, g_2(X,Y)\right] = a\,\mathbb{E}[g_1(X,Y)] + b\,\mathbb{E}[g_2(X,Y)]$$

## Define: Covariance

We define the covariance between $X$ and $Y$ as

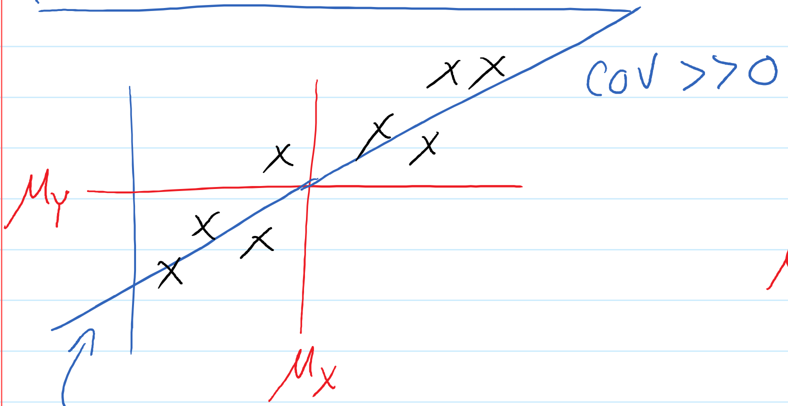
measuring lin-rel between $X$ and $Y$

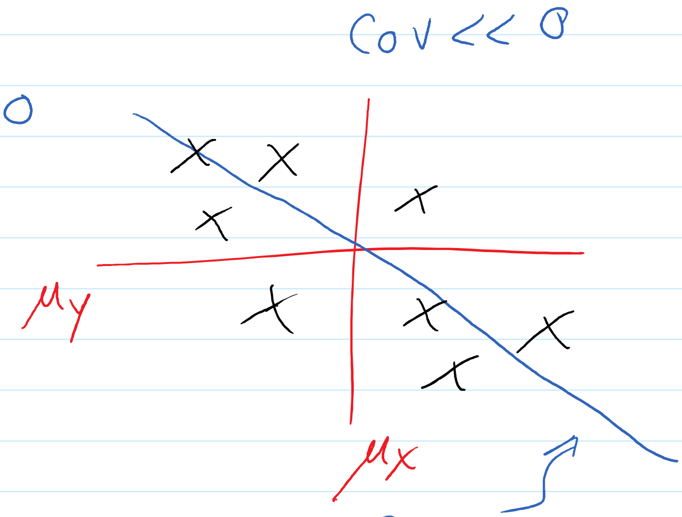
$$\mathrm{Cov}(X,Y) = E\left[(X - \underbrace{EX}_{\mu_X})(Y - \underbrace{EY}_{\mu_Y})\right]$$

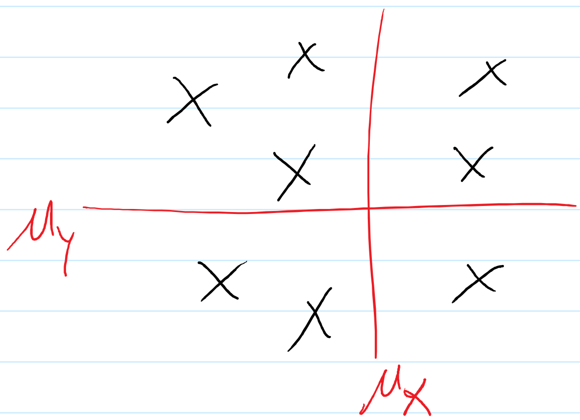
$$= \mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$g(x,y) = (x-\mu_X)(y-\mu_Y)$

## How to think about:




**Note:** $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$

$$\mathrm{Cov}(X, X) = \mathrm{Var}(X)$$

---

**Notice:**

$$\mathrm{Cov}(5X, Y) = \mathbb{E}[(5X - \mathbb{E}[5X])(Y - \mathbb{E}Y)]$$

$$= 5\,\mathrm{cov}(X, Y)$$

## Defn: Correlation

Re-scaled covariance so that it is between $-1$ and $1$

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

$$= \frac{\text{Cov}(X, Y)}{\text{sd}(X)\,\text{sd}(Y)}$$

Idea: $\text{corr} \approx 1 \Rightarrow$ strongly pos. lin. rel.

$\text{corr} \approx -1 \Rightarrow$ " neg. "

$\text{corr} \approx 0 \Rightarrow$ no lin. rel.

**Theorem:** If $a, b \in \mathbb{R}$

$$\boxed{\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2ab\,\text{Cov}(X, Y)}$$

pf. $Z = aX + bY$

$$\mathrm{Var}(Z) = \mathbb{E}[(Z - \mathbb{E}Z)^2]$$

$$= \mathbb{E}[(aX + bY - \mathbb{E}[aX + bY])^2]$$

$$= \mathbb{E}[(aX + bY - a\mathbb{E}X - b\mathbb{E}Y)^2]$$

$$= \mathbb{E}[(\underbrace{a(X - \mathbb{E}X)}_{\alpha} + \underbrace{b(Y - \mathbb{E}Y)}_{\beta})^2]$$

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$= \mathbb{E}[a^2(X - \mathbb{E}X)^2 + b^2(Y - \mathbb{E}Y)^2 + 2ab(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$$

$$= a^2 \underbrace{\mathbb{E}[(X - \mathbb{E}X)^2]}_{\mathrm{Var}\,X} + b^2 \overbrace{\mathbb{E}[(Y - \mathbb{E}Y)^2]}^{\mathrm{Var}(Y)}$$

$$+ 2ab \underbrace{\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]}_{\mathrm{Cov}(X, Y)}$$