The point of counting

If I have a S w/ equally likely outcomes

then

$$P(E) = \frac{|E|}{|S|}$$

need to count

The most important fact:

we assume all outcomes equally likely

Q: Ordering? replacement?

needs to respect this fact.

Ex. Flip a coin twice.

What is the prob. of getting a H and a T?

Option 1: Unordered Sample Space

$$S = \{HH, TT, HT\} \quad \text{so} \quad |S| = 3$$

and $E = \{HT\}$

so $P(E) = 1/3$

Option: (ordered)

$$\underbrace{H}_{1/2}\cdot\underbrace{T}_{1/2} \quad \text{or} \quad \underbrace{T}_{1/2}\cdot\underbrace{H}_{1/2}$$

$$\tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{2}$$

$S = \{HH, TH, HT, TT\}$ so $|S| = 4$

and $E = \{TH, HT\}$

hence $P(E) = 2/4 = 1/2$.

General Rule:

If I assume my sampling comes about from a seq of independent actions the counting S in an ordered way is typically correct.

Typically only need to be careful when sampling w/ repl.

Not so much when sampling w/o repl:

$$P(E) = \frac{|E|r!}{|S|r!}$$

← w/ ordering I introduce a $r!$ as comp. w/o order.

Conditional Prob.

Ex. Survey WM students, ask about political afil.

gender	A	B	
men	501	238	739
women	782	123	905
		361	1644

Q1: If I randomly select a student what is the prob. they are a woman?

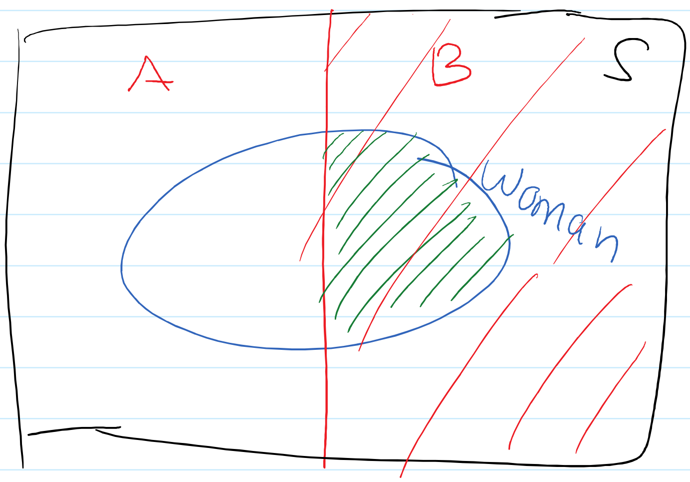
$$P(\text{woman}) = 905/1644 \approx 55\%$$

Q2: Given that a student is in party B,

what is the prob they are a woman?

$$P(\text{woman given } B) = \frac{123}{361}$$

Venn Diagram



Q1: $P(\text{woman}) = \dfrac{\text{area } \bigcirc}{\text{area } \square}$

Q2: $P(\text{woman given } B)$

$$= \frac{\text{area}}{\text{area } \boxed{B}}$$

$$= \frac{\text{area of } \text{woman} \cap \boxed{B}}{\text{area } \boxed{B}}$$

Defn: Conditional Prob.

If $A, B \subset S$ and $P(B) > 0$ then the conditional prob. of A given B is

given

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

Facts: Assume $\mathbb{P}(B) > 0$

① $\boxed{P(B|B) = 1}$

pf $P(B|B) = \dfrac{P(BB)}{\mathbb{P}(B)} = \dfrac{P(B)}{P(B)} = 1$

② $\boxed{\text{If } AB = \emptyset \text{ then } P(A|B) = 0.}$

pf $\mathbb{P}(A|B) = \dfrac{P(AB)}{\mathbb{P}(B)} = \dfrac{\mathbb{P}(\emptyset)}{\mathbb{P}(B)} = \dfrac{0}{\mathbb{P}(B)} = 0$

Ex, Roll two dice.

Q: what is the prob the first roll is a 2 given the sum is ≤ 5.

A — first roll is a 2

B — sum is ≤ 5

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{|AB|/|S|}{|B|/|S|} = \frac{|AB|}{|B|} = \frac{3}{10}$$

roll 1

	1	2	3	4	5	6
1						

10

roll 2

	1	2	3	4	5	6
1	△	△ ○	△	△		
2	△	△ ○	△			
3	△	△ ○				
4	△	○				
5		○				
6		○				

Theorem: Compound Prob. $\quad P(A), P(B) > 0$

$$\mathbb{P}(AB) = \mathbb{P}(A|B)P(B) = \mathbb{P}(B|A)P(A)$$

~~Pf~~. $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(AB)}{P(B)}$

and rearrange.

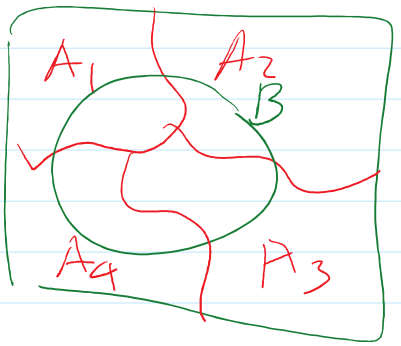
Recall: (A_i) partition S then

$$\mathbb{P}(B) = \sum_i P(BA_i)$$

Theorem: Law of Total Prob.

If (A_i) partition S and $P(A_i) > 0$
then $\forall B \subset S$

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B|A_i)\,\mathbb{P}(A_i)$$



pf.

partitioning theorem

$$P(B) = \sum_i \underbrace{P(B A_i)}$$

compound prob. rule

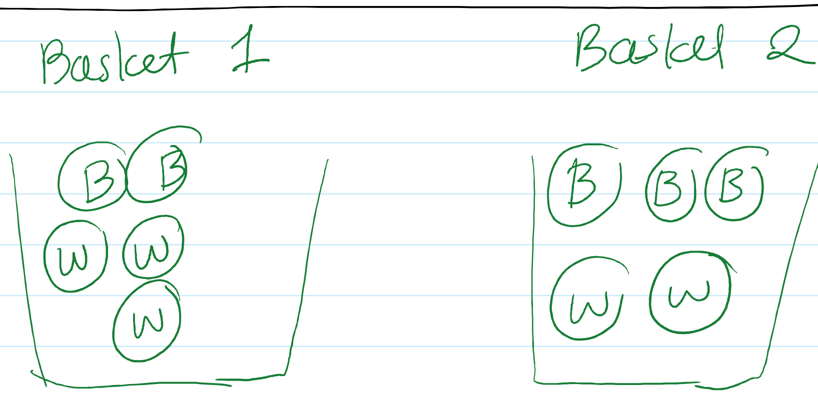
$$= \sum_i \mathbb{P}(B|A_i)\,\mathbb{P}(A)$$

Note: For any event A, A and A^c partition S

so by our law

$$\mathbb{P}(B) = \mathbb{P}(B|A)\,\mathbb{P}(A) + \mathbb{P}(B|A^c)\,\mathbb{P}(A^c).$$

Ex.



Game: ① randomly select ball from basket 1 and put in basket 2

(2) randomly select ball from basket 2.

Q: what is the prob I choose a (b) on second step.

Let W = choose white on step 1

W^c = " black "

B = choose black on step 2

B^c = " white "

Want: $\mathbb{P}(B)$?

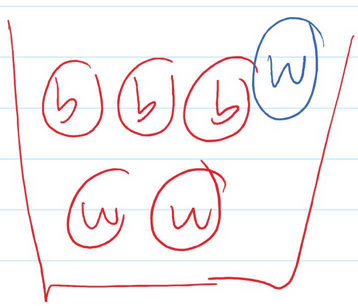
Use Law of Tot. Prob partitioning on $\{W, W^c\}$.

$$\mathbb{P}(B) = \mathbb{P}(B|W)\mathbb{P}(W) + \mathbb{P}(B|W^c)\mathbb{P}(W^c)$$

$$\left(\tfrac{3}{6}\right)\left(\tfrac{3}{5}\right) + \left(\tfrac{4}{6}\right)\left(\tfrac{2}{5}\right) = \tfrac{17}{30}$$

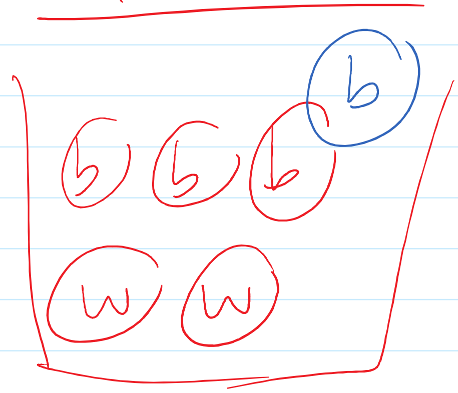
Given W

Basket 2



$\mathbb{P}(B|W) = 3/6$

Given W^c



$\mathbb{P}(B|W^c) = 4/6$

$P(B|W) = 3/6$ $\qquad$ $P(B|W^c) = 4/6$

Theorem: Bayes' Theorem

Want way to calc $P(A|B)$ from $P(B|A)$.

If $A, B \subset S$ and $P(A), P(B) > 0$ then

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}$$

Pf. $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$.

defn of conditional

compound prob. law

Ex Continue prev.

Q: Given that I chose a b on second, what is the prob. I chose a w on first.

Bayes' Says

$$P(W|B) = \frac{P(B|W)P(W)}{P(B)}$$

$$= \frac{(1/2)(3/5)}{(17/30)}$$

Theorem: Law of Tot. Prob + Bayes'

If (A_i) partition S and $P(A_i) > 0$

and $B \subset S$ where $P(B) > 0$ then

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)}$$

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pf.

Bayes' say

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$$

expand w/ law of Tot prob

$$= \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)}$$

Often apply to partition of two events: A, A^c

in this case our formula simplifies to

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}}$$

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ex. COVID has a prevalence rate of 1%. D = disease $P(D) = .01$

D^c = don't $P(D^c) = .99$

We test for COVID and get + or −.

→ The test accurately reports a + 95% of time (sensitivity) $P(+|D) = .95$

$P(-|D) = .05$

→ The test acc. reports a − 99% of time (specificity) $P(-|D^c) = .99$

$P(+|D^c) = .01$

Q: I get a COVID test, I get a +,

What is the prob. I have COVID?

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)}$$

$$= \frac{(.95)(.01)}{(.95)(.01) + (.01)(.99)}$$

$$\approx .49$$