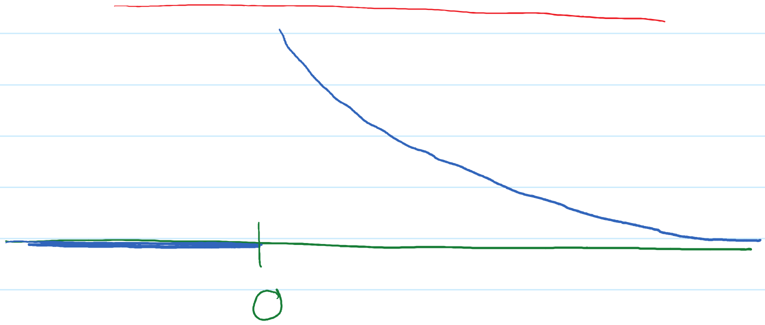
# Lecture 10 - Expectation

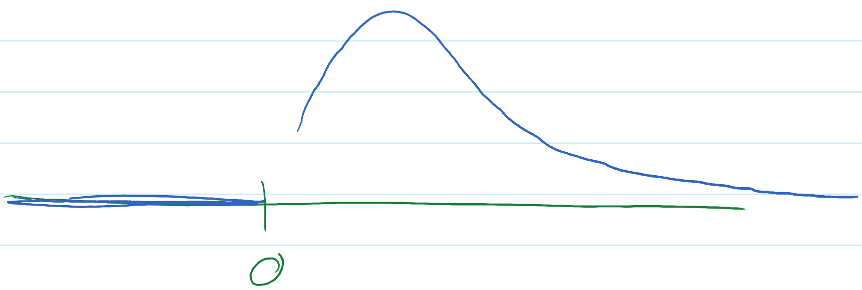
an exponential dist

<u>Ex.</u> let $X \sim \text{Exp}(\lambda)$  $\lambda > 0$

means

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$




$\mathbb{E}[X]$?

According to defn

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \underbrace{x}_{u}\, \underbrace{\lambda e^{-\lambda x} dx}_{dv} \quad \circledast$$

<u>integration by parts:</u>  $u = x$  $du = dx$  $v = -e^{-\lambda x}$  $dv = \lambda e^{-\lambda x} dx$

$$\int u\,dv = uv - \int v\,du.$$

$$(*) = \int u\,dv = \underbrace{x(-e^{-\lambda x})\Big|_0^\infty}_{} - \int_0^\infty (-e^{-\lambda x})\,dx$$

$$= (0 - 0) + \int_0^\infty e^{-\lambda x}\,dx$$

$$= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty$$

$$= 0 - \left(-\frac{1}{\lambda}\right) = \boxed{1/\lambda = \mathbb{E}[X]}$$

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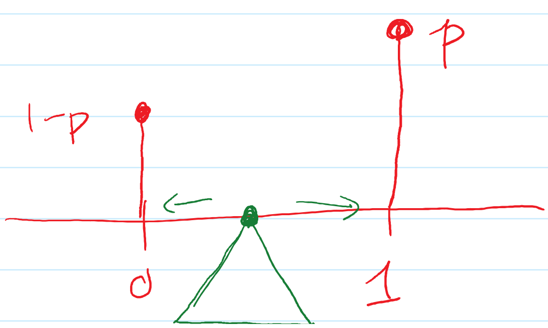
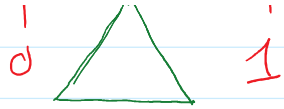
<u>Ex.</u> $X \sim \text{Bernoulli}(p)$

$0 \le p \le 1$

Any binary experiment w/ outcome 0 or 1

$$f(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \end{cases}$$






$$\mathbb{E}[X] = \sum_x x f(x) = \sum_{x=0,1} x f(x) = (0)f(0) + (1)f(1)$$
$$= (0)(1-p) + (1)p$$
$$= p$$

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Ex. Binomial

$p \in [0,1]$

$$X \sim \text{Bin}(n, p)$$

$n \in \mathbb{N}$

$X$ = # heads among $n$ independent coin flips w/ a prob. $p$ of H on each

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x = 0, 1, \ldots, n$$

**Binomial Theorem**

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{x} x^i y^{n-i}$$

$$(x+y)^2 = \underset{\binom{2}{0}}{\underline{\ }}x^2 + \underset{\binom{2}{1}}{\underline{2}}xy + \underset{\binom{2}{2}}{\underline{\ }}y^2$$

justify that $\sum_x f(x) = 1$ w/ Bin. theorem using $x = p$ and $y = 1-p$

$$\mathbb{E}[X] = \sum^{n} x f(x) = \sum^{n} x \binom{n}{\cdot} p^x (1-p)^{n-x}$$

$$\mathbb{E}[X] = \sum_{x=\cancel{0}1}^{n} x f(x) = \sum_{x=1}^{n} x \underbrace{\binom{n}{x}} p^{x}(1-p)^{n-x}$$

$$\underbrace{x\binom{n}{x}} = x \frac{n!}{x!(n-x!)}$$

$$= \cancel{x} \frac{n(n-1)!}{\cancel{x}(x-1)!((n-1)-(x-1))!}$$

$$= n \frac{(n-1)!}{(x-1)!((n-1)-(x-1))!}$$

$$= n \binom{n-1}{x-1}$$

$$= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^{x}(1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1}(1-p)^{(n-1)-(x-1)}$$

$$y = x-1 \iff x = y+1$$

$$= np \underbrace{\sum_{y=0}^{n-1} \binom{n-1}{y} p^{y}(1-p)^{(n-1)-y}}_{\substack{\text{Sum of} \\ \text{PMF of a } Bin(n-1,p) \\ \text{over its support} \\ = \\ 1}}$$

$$= np$$

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General trick:

Often I can recognize either

$$\sum f(x) \quad \text{or} \quad \int f(x)\,dx \quad \text{(over entire support)}$$

in a calculation and replace it w/ 1.

Note: functions of RVs

a function of a RV is also a RV.

Ex. If $X$ is a RV then so is

$$X^2 \quad \text{or} \quad \log X \quad \text{or} \quad \sqrt{X} \quad \ldots$$

Theorem: Law of the Unconscious Statistician

If $g:\mathbb{R}\to\mathbb{R}$ and $X$ is a RV then

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x) f(x) & \text{(discrete)} \\ \int g(x) f(x)\,dx & \text{(cts)} \end{cases}$$

Ex.

$X \sim \text{Exp}(\lambda)$

$f(x) = \lambda e^{-\lambda x}$ for $x > 0$

$\mathbb{E}[X] = 1/\lambda$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_x^{\infty} x^2 \lambda e^{-\lambda x}\,dx$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^\infty \underbrace{x^2}_{u} \underbrace{\lambda e^{-\lambda x}\,dx}_{dv}$$

$u = x^2$ $\quad dv = \lambda e^{-\lambda x} dx$

$du = 2x\,dx$ $\quad v = -e^{-\lambda x}$

$$= \int u\,dv = uv - \int v\,du = \underbrace{(x^2)(-e^{-\lambda x})\Big|_0^\infty}_{0} - \int_0^\infty (-e^{-\lambda x})(2x\,dx)$$

$$= \frac{2}{\lambda} \underbrace{\int_0^\infty x \lambda e^{-\lambda x}\,dx}_{\mathbb{E}[X]}$$

$$= \frac{2}{\lambda}\left(\frac{1}{\lambda}\right) = \boxed{\frac{2}{\lambda^2} = \mathbb{E}[X^2]}$$

Notice:

$$\mathbb{E}[X]^2 = \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} \neq \mathbb{E}[X^2]$$

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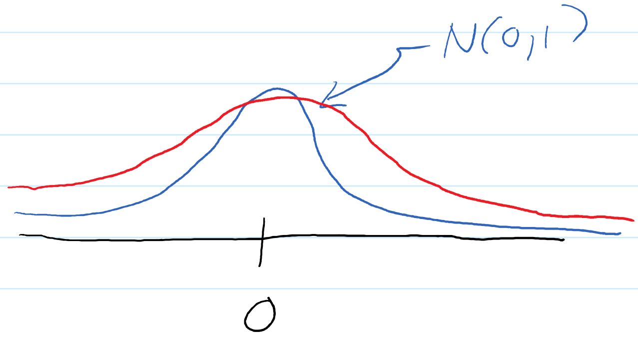
Ex. Cauchy Distribution

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$

$\sim N(0,1)$

$\infty$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$




$$= \int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\,dx$$

$$= 2\int_{0}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\,dx$$

looks like $\int_0^\infty \frac{x}{x^2}\,dx = \int_0^\infty \frac{1}{x}\,dx$

$= \infty$

$$= \infty$$

Recall:

$\sum \frac{1}{i^2} < \infty$ but $\sum \frac{1}{i} = \infty$

$\int \frac{1}{x^2}\,dx < \infty$ but $\int \frac{1}{x}\,dx = \infty$

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Punchline: expected value might not exist.

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Theorem: Properties of Expectation

1. Expectation is linear.

$$\mathbb{E}[aX+b] = a\mathbb{E}[X] + b$$

Pf. (cts)

$$E[aX+b] = \int (ax+b) f(x)\,dx$$

$$E[aX+b] = \int (ax+b) f(x)\,dx$$

$$= \int [a x f(x) + b f(x)]\,dx$$

$$= \int a x f(x)\,dx + \int b f(x)\,dx$$

$$= a \underbrace{\int x f(x)\,dx}_{\mathbb{E}X} + b \underbrace{\int f(x)\,dx}_{1}$$

$\boxed{\text{2. If } X \geq 0 \text{ then } \mathbb{E}[X] \geq 0.}$

Support $\subset (0, \infty)$

Pf. (cts)

$$\mathbb{E}X = \int_0^\infty \underbrace{x}_{\geq 0} \underbrace{f(x)}_{\geq 0}\,dx \geq 0$$

3. If $g_1$ and $g_2$ are functions then

i. $E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)]$.

ii. If $g_1(x) \leq g_2(x)$ $\forall x$ then

$$E[g_1(X)] \leq E[g_2(X)]$$

④ If $a \le X \le b$ then $a \le \mathbb{E}X \le b$.

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<u>Variance</u>: $\mu = \mathbb{E}X$


Variance $\approx$ how spread values are arond the mean

$$\mathrm{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right]$$

$$= \mathbb{E}\left[(X - \mathbb{E}X)^2\right]$$

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PDF of $X$ PDF of $Y$

$\mu = E[X]$ let $Y = X - \mu$ 0

$$\mathbb{E}[Y] = \mathbb{E}[X - \mu]$$

$$\mathbb{E}[Y] = \mathbb{E}[X - \mu]$$
$$= \mathbb{E}[X] - \mu$$
$$= \mu - \mu = 0$$

then

$$\operatorname{Var} X = \mathbb{E}[Y^2].$$