4 options: Choose r from n

	w/o replacement	w/ replacement
ordered	1 $\frac{n!}{(n-r)!}$	2 n^r
un-ordered	4	3 $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Theorem:

If I have n items and I sample r of them $(r \leq n)$ w/o replacement but w/ ordering.

I can do this in $\frac{n!}{(n-r)!}$ ways.

pf. By FTC

task #	task	# ways
1	choose 1st	n
2	choose 2nd	$n-1$

1	choose 1^{st}	n
2	choose 2^{nd}	$n-1$
3	" 3^{rd}	$n-2$
$\vdots$	$\vdots$	$\vdots$
r	" r^{th}	$n-r+1$

multiply

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

$$\frac{\boxed{n(n-1)(n-2)\cdots(n-r+1)}\cancel{(n-r)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

Ex. I form a committee of $n=10$ students where the committee has $r=3$ members:

Pres., VP, treasurer

How many committees are possible?

Sample $r=3$ from $n=10$

w/o replacement (diff. people diff. roles)

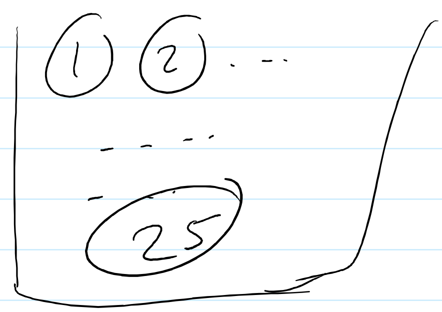
w/ ordering (1^{st} = Pres, 2^{nd} = VP, 3^{rd} = treasurer)

So I can do this in

$$\frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10\cdot 9\cdot 8\cdot \cancel{7!}}{\cancel{7!}} = 10\cdot 9\cdot 8 = 720$$

Ex. Lotto.

I have 25 balls in a basket



Draw 4 w/o repl.

Assuming each draw is equally likely and I guess the draw is

1 3 22 7

what is the prob. I am correct?

Soln: $E = \{ 1\ 3\ 22\ 7 \}$

$S = \{$ all possible draws $\}$

then

$$P(E) = \frac{|E|}{|S|} = \frac{1}{|S|}$$

$n = 25$ drawing $r = 4$

So $$|S| = \frac{25!}{(25-4)!} = \frac{25!}{21!} = 25 \cdot 24 \cdot 23 \cdot 22$$

Finally,

$$P(E) = \frac{1}{25 \cdot 24 \cdot 23 \cdot 22}$$

<u>Theorem:</u> <u>Sampling w/ repl. and w/ ordering</u>

The number of ways to draw a sample of r from n (w/ repl., w/ ordering) is

$$n^r.$$

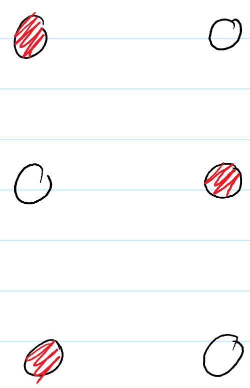
<u>pf.</u> Use FTC

task #	task	# ways
1	choose 1st	n
2	" 2nd	n
$\vdots$	" 3rd	n
$\vdots$		$\vdots$

multiply to get

⋮	" 3rd	n	multiply to get
⋮	⋮	⋮	
r	" r th	n	$n \cdots n = n^r$

Ex. **Braille Alphabet**



6 possible locs. can either be raised or not

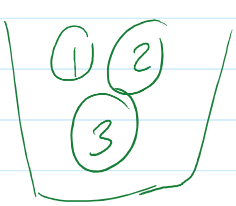
Q: How many possible braille letters?

Ans: $r=6$ from $n=2$

so we have 2^6.

Unordered sampling: I don't care about order of draws

Ex.



draw $r=2$ from $n=3$

If order matters: $\frac{n!}{(n-r)!} = \frac{3!}{1!} = 6$

$(1,2)$ $(1,3)$ $(2,3)$

(1,2) (1,3) (2,3)

(2,1) (3,1) (3,2)

If order doesn't matter:

{1,2}, {1,3}, {3,2} 3 possibilities

General rule: for each unordered sample of size r I can permute it in r! to create an ordered sample

$\{1,2,5\}$ → (1,2,5), (2,1,5), (1,5,2), ...

$$(\#\text{ ordered samples}) = r!\,(\#\text{ unordered})$$

or

$$(\#\text{ unordered}) = \frac{1}{r!}(\#\text{ ordered})$$

$$= \frac{1}{r!}\,\frac{n!}{(n-r)!}$$

Theorem: Unordered w/o replacement

The number of ways to sample r from n (unordered w/o repl.) is

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Binomial coefficient

read: "n choose r"

Ex. I have $10 = n$ profs, how many co-equal committees of size $4 = r$ can I make?

Solve unordered (co-equal) w/o repl.

total of $\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!}$

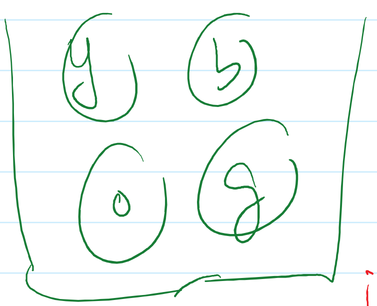
$$= \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{4 \cdot \cancel{3} \cdot \cancel{2}}$$

$$= 10 \cdot 3 \cdot 7 = 210$$

Ex. How many 5 card hands can I get from a deck of cards

Theorem says: $\binom{52}{5} \approx 2.5$ mil

Ex. I have a jar w/ 4 marbles of colors yellow, blue, orange, green



I choose w/o repl. 3 marbles (all such choices are equally likely) what is the prob I have a yellow and blue in my selection.

$$\mathbb{P}(E) = \frac{|E|}{|S|}$$

$E = \{y \text{ and } b \text{ in sample}\}$

$= \{\{y,b,o\}, \{y,b,g\}\} \Rightarrow |E| = 2$

$S = \{\text{all possible draws}\}$

$$|S| = \binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!\,1!} = \frac{4\cdot 3\cdot 2}{3\cdot 2} = 4$$

hence

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{2}{4} = 1/2.$$

Last Case: Sampling w/ replacement w/o ordering

Ex. $n=3$, $r=2$

Ordered: $n^r = 3^2 = 9$

$\boxed{(2,1)\ (1,2)}$ $\boxed{(3,1)\ (1,3)}$ $\boxed{(2,3)\ (3,2)}$ $\boxed{(2,2)}$ $\boxed{(3,3)}$ $\boxed{(1,1)}$

Unordered:

$\boxed{\{1,2\}}$ $\boxed{\{3,1\}}$ $\boxed{\{2,3\}}$ $\boxed{\{1,1\}}$, $\boxed{\{2,2\}}$, $\boxed{\{3,3\}}$

No $r!$ corresp. so can't use same trick as last time.

Game of partitioning

Consider dividing $r=2$ objects using $n-1=4$ walls

Ex,

$$| \; O \; | \; | \; O \; |$$

$$0 \quad 1 \quad 0 \quad 1 \quad 0 \rightsquigarrow \{2,4\}$$

$$| \; | \; | \; O \; O \; |$$

$$0 \quad 0 \quad 0 \quad 2 \quad 0 \rightsquigarrow \{4,4\}$$

Each such partition corresp. to a certain unordered sample of r from n w/ replacement.

Ex, $r=4$, $n-1=5$ walls

$$O \; | \; | \; O \; O \; | \; O \; | \; |$$

$$1 \quad 0 \quad 2 \quad 1 \quad 0 \quad 0$$

$$\{1,3,3,4\}$$

1-1 corresp. btwn partition arrangements and samples.

Lets count how many ways to build these partitions.

In total I have $n-1+r = n+r-1$ things.

I can form a partion by permuting these.

However we can permute the walls in $(n-1)!$ ways w/o changing the partition.

Similarly permute objects in $r!$ ways.

In total we have

$$\frac{(n+r-1)!}{(n-1)!\,r!} = \binom{n+r-1}{r}$$

$$= \binom{n+r-1}{n-1}$$

Theorem: Sampling w/o ordering, w/ repl.

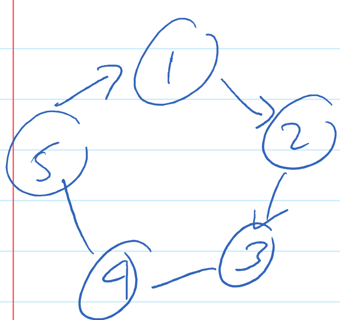
The number of ways I can draw r from n (w/o order, w/ repl.) is

$$\frac{(n+r-1)!}{(n-1)!\,r!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Ex. 10 passengers on a bus route w/ 5 hotels.

The driver records how may get off at each stop.

How many possible records are there?



hotel	# people
1	0
2	3
3	1
4	2
5	4

$\longrightarrow \{2,2,2,3,4,4,5,5,5,5\}$

Solve as sampling $r=10$ from $n=5$ w/ repl. and w/ ordering.

$$) = \binom{14}{\ } = \frac{14!}{\ } \qquad \frac{14\cdot 13\cdot 12\cdot 11}{\ }$$

$$\binom{n+r-1}{r} = \binom{14}{10} = \frac{14!}{10!(14-10)!} = \frac{14\cdot 13\cdot 12\cdot 11}{4!}$$

$$= 1001$$

Ex, Jar w/ 4 marbles, y, b, o, g.

Draw a sample of size $r=3$ w/ replacement w/o ordering

(all such samples are equally likely)

Q: What is the prob. my sample contains a y and b?

$E = \{\text{sample has } y \text{ and } b\}$

$= \{\{y,b,o\}, \{y,b,g\}, \{y,b,b\}, \{y,b,y\}\}$

$|E| = 4$

$S = \{\text{all poss. samples}\}$.

$$|S| = \binom{n+r-1}{r} = \binom{6}{3} = 20$$

hence

$$\mathbb{P}(E) = 4/20 = 1/5.$$