## Defn: Conditional PMF/PDF

Given $X$ and $Y$ then the conditional PMF/PDF of $X \mid Y = y$ is

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

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## Defn: Conditional Expectation

If $g: \mathbb{R} \to \mathbb{R}$ then the conditional expectation of $g(X)$ given $Y = y$ is

$$\mathbb{E}[g(X) \mid Y = y] = \begin{cases} \sum_x g(x) f(x \mid y) & \text{discrete} \\ \int_{\mathbb{R}} g(x) f(x \mid y)\, dx & \text{cts} \end{cases}$$

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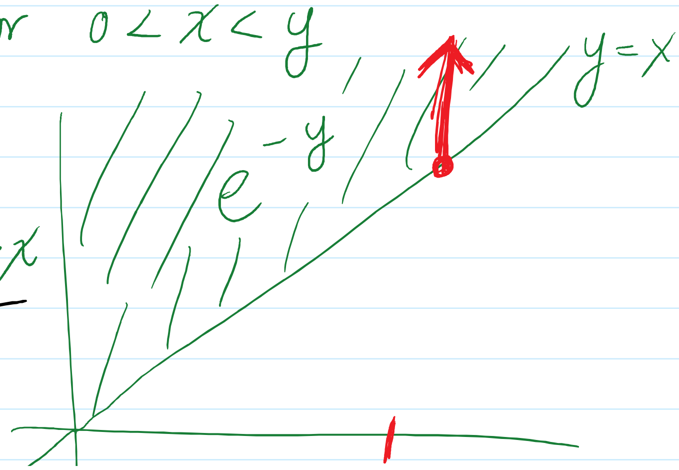
Ex. $f(x,y) = e^{-y}$ for $0 < x < y$

Had shown:

$f(y \mid x) = e^{-(y-x)}$ for $\underline{y > x}$




$\mathbb{E}[Y \mid X = x]$

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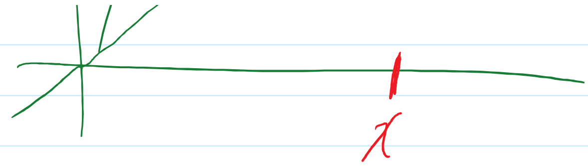


$$= \int_{\mathbb{R}} y f(y \mid x)\, dy$$

$$= \int_x^{\infty} y e^{-(y-x)}\, dy = \cdots = 1 + x$$

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Defn: Conditional Variance

$$\mathrm{Var}(Y \mid X = x) = \mathbb{E}\left[ \left( Y - \mathbb{E}[Y \mid X = x] \right)^2 \mid X = x \right]$$

Short-cut formula:

$$\mathrm{Var}(Y \mid X = x) = \mathbb{E}[Y^2 \mid X = x] - \mathbb{E}[Y \mid X = x]^2$$

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Ex. $\mathbb{E}[Y^2 \mid X = x] = \int_{\mathbb{R}} y^2 f(y \mid x)\, dy$

$$= \int_x^{\infty} y^2 e^{-(y-x)}\, dy = \cdots = x^2 + 2x + 2$$

$$\mathrm{Var}(Y \mid X = x) = \mathbb{E}[Y^2 \mid X = x] - \mathbb{E}[Y \mid X = x]^2$$

$$= (x^2+2x+2) - (1+x)^2$$

$$= \cancel{x^2} + \cancel{2x} + 2 - \cancel{x^2} - \cancel{2x} - 1$$

$$= 2 - 1$$

$$= 1$$

## Independence

**For events:** If $A, B \subset S$ then

$$A \perp\!\!\!\perp B \iff P(AB) = P(A)P(B)$$

**For RVs**

$$X \perp\!\!\!\perp Y \iff P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

$$\forall A, B \subset \mathbb{R}$$

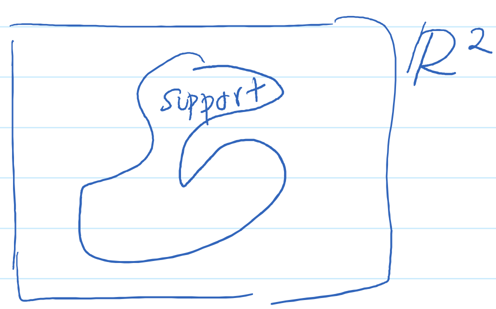
## Product Spaces

$$\text{Support}(X,Y) = \{(x,y) \mid f(x,y) > 0\}$$

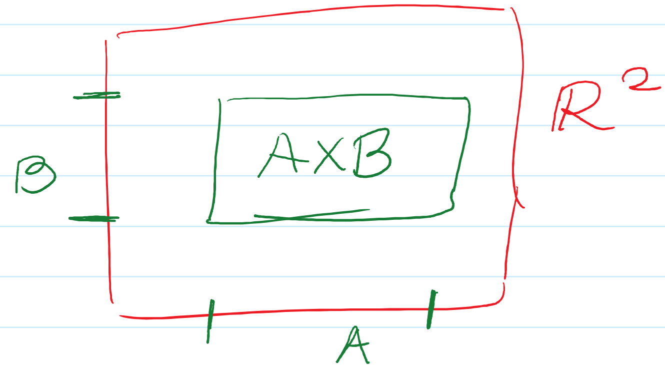
If $f(x,y) =$

for $x \in A$ and $y \in B$

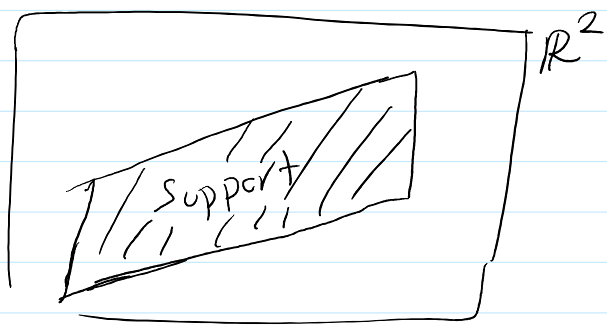
doesn't depend on $y$     doesn't depend on $x$

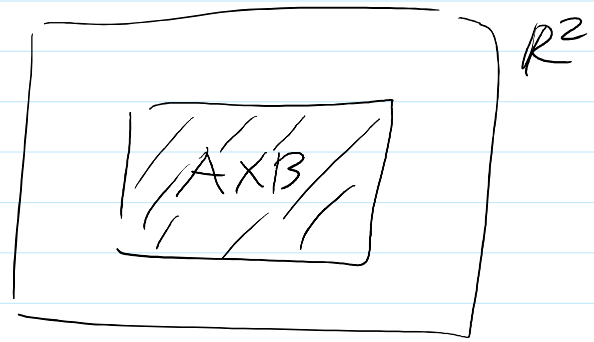


then the support is a product space $A \times B$




Two different support examples




Theorem: Factorization Theorem

$X \perp\!\!\!\perp Y$

iff

① support of $X$ and $Y$ is a product space

② either $F(x,y) = F_X(x) F_Y(y)$

or $f(x,y) = f_X(x) f_Y(y)$

**Ex.**

| $Y$ \ $X$ | 10 | 20 | $f_Y$ |
|---|---|---|---|
| $f_X$ | 1/2 | 1/2 | |
| 3 | 1/5 | 3/10 | 5/10 |
| 2 | 1/5 | 1/10 | 3/10 |
| 1 | 1/10 | 1/10 | 2/10 |

**Q:** $X \perp\!\!\!\perp Y$?

① Product space? Yes: $A = \{10, 20\}$, $B = \{1, 2, 3\}$
then $A \times B$ is my support

② $f(x,y) = f_X(x) f_Y(y)$?

Ex. $f((10,3)) = \frac{1}{5} \neq f_X(10) f_Y(3) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

So $X \not\perp\!\!\!\perp Y$.

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**Corollary:**

$X \perp\!\!\!\perp Y$

iff

① support is a product space

② $f(x,y) = g(x)h(y)$

h fn only of y

some fn only of x

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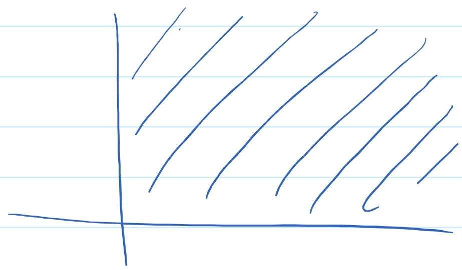
Ex.

$$f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)} \qquad x>0,\ y>0$$

$X \perp\!\!\!\perp Y$?

① product space? Yes

Support $= (0,\infty) \times (0,\infty)$



②

$$f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)}$$

$$= \frac{1}{384} x^2 e^{-y} e^{-(x/2)}$$

$$= \underbrace{\left(\frac{1}{384} x^2 e^{-x/2}\right)}_{g(x)} \underbrace{\left(e^{-y}\right)}_{h(y)}$$

So $X \perp\!\!\!\perp Y$.

w/o this theorem, need

$$f_X(x) = \int_0^\infty \frac{1}{384} x^2 e^{-(x/2)} e^{-y} \, dy$$

$$f_Y(y) = \int_0^\infty \frac{1}{384} x^2 e^{-x/2} e^{-y} \, dx$$

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<u>Fact:</u> $A, B \subset S$ and $A \perp B$ then

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)$$

<u>For RVs</u>: $X \perp\!\!\!\perp Y$ then

$$f(x \mid y) = f_X(x).$$

<u>Pf.</u> $f(x \mid y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{f_X(x) \cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)$

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<u>Theorem:</u> Expectation of Product of Independent

If $X \perp\!\!\!\perp Y$ and $g_1 : \mathbb{R} \to \mathbb{R}$, $g_2 : \mathbb{R} \to \mathbb{R}$

then

$$\mathbb{E}[g_1(X)g_2(Y)] = \mathbb{E}[g_1(X)]\,\mathbb{E}[g_2(Y)].$$

Pf. (cts)

$$\mathbb{E}[g_1(X)g_2(Y)] = \iint_{A\times B} g_1(x)g_2(y)f(x,y)\,dx\,dy$$

independence

$$= \iint_{A\times B} g_1(x)g_2(y)f_X(x)f_Y(y)\,dx\,dy$$

$$= \int_B \boxed{\int_A g_1(x)f_X(x)\,dx}\; g_2(y)f_Y(y)\,dy$$

$$= \underbrace{\int_A g_1(x)f_X(x)\,dx}_{\mathbb{E}[g_1(X)]}\;\underbrace{\int_B g_2(y)f_Y(y)\,dy}_{\mathbb{E}[g_2(Y)]}$$

Ex. $X, Y \overset{iid}{\sim} \mathrm{Exp}(1)$

independent identically distributed

means: $X \perp\!\!\!\perp Y$, $X \sim \mathrm{Exp}(1)$
$Y \sim \mathrm{Exp}(1)$

$$\mathbb{E}[X^2Y] = \underset{(1+1)}{\mathbb{E}[X^2]}\,\underset{(1)}{\mathbb{E}[Y]} = (2)(1) = 2$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

rearrange

$$\mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}X)^2$$

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## Theorem: MGF of Sum of Independent

If $X \perp Y$ then

$$M_{X+Y}(t) = M_X(t)\,M_Y(t).$$

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Pf:

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$$M_Y(t) = \mathbb{E}[e^{tY}]$$

$$M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}]$$

$$= \mathbb{E}[e^{tX}e^{tY}]$$

$$= \underbrace{\mathbb{E}[e^{tX}]}_{M_X(t)}\,\underbrace{\mathbb{E}[e^{tY}]}_{M_Y(t)}$$

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Ex. $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\gamma, \tau^2)$

assume $X \perp\!\!\!\perp Y$.

What is the dist. of $X+Y$?

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}\, e^{\gamma t + \frac{\tau^2 t^2}{2}}$$

$$= e^{\mu t + \gamma t + \frac{\sigma^2 t^2}{2} + \frac{\tau^2 t^2}{2}}$$

$$= e^{(\mu+\gamma)t + (\sigma^2+\tau^2)t^2/2}$$

MGF of a $N(\mu+\gamma, \sigma^2+\tau^2)$.

i.e. $\boxed{X+Y \sim N(\mu+\gamma, \sigma^2+\tau^2)}$