

Laymen's defn:

→ things don't affect each other

→ events are indep. if the occurrence (or not) of one doesn't affect the prob. of other.

Defn: Independence (of Events)

If $A, B \in \mathcal{S}$ we say "A is independent of B" denoted $A \perp B$, if

$$P(AB) = P(A)P(B).$$

→ independence is a distributive law

→ intuition for product notation for intersection.

Theorem: If $A \perp B$ then

$$P(A|B) = P(A).$$

Pf.

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

Ex. Consider rolling two dice (independently)

$$P(\text{at least one } 6)$$

$$= 1 - P(\text{"at least one } 6" ^c)$$

$$= 1 - P(\text{no } 6\text{'s})$$

$A_1 = \text{no } 6 \text{ on first roll}$
 $A_2 = \text{"second"}$

$$= 1 - P(A_1, A_2) \quad \leftarrow \text{by independence}$$

$$= 1 - P(A_1)P(A_2)$$

$$= 1 - (5/6)(5/6) = 11/36$$

Counting perspective:

Sampling from $\{1, \dots, 6\}$ ($n=6$)

two times ($r=2$) w/ replacement

Ordered: $|S| = 6^2 = 36$

$$E = \text{"at least one } 6\text{'s"}$$

$$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5)\}$$

$$|E| = 11$$

$$P(E) = 11/36$$

Unordered: $|S| = \binom{n+r-1}{r} = \binom{6+2-1}{2} = \binom{7}{2}$

$$= 21$$

and

$$E = \{\{1,6\}, \{2,6\}, \{3,6\}, \{4,6\}, \{5,6\}, \{6,6\}\}$$

So

$$P(E) = 6/21$$

Ex. why does ordered counting encode independence?

Roll two dice (independently)

$$E = \{1 \text{ or } 2 \text{ on first roll},$$

3, 4, 5 on second}

Solve w/ counting (in ordered way)

$$|S| = n^r = 6^2 = 36 = 6 \cdot 6$$

$$E = \{ (1, 3), (1, 4), (1, 5), \\ (2, 3), (2, 4), (2, 5) \}$$

$$= \{1, 2\} \times \{3, 4, 5\}$$

↑ Cartesian product

$$|E| = |\{1, 2\} \times \{3, 4, 5\}|$$

$$= 2 \cdot 3$$

$$P(E) = \frac{|E|}{|S|} = \frac{2 \cdot 3}{6 \cdot 6} = \left(\frac{2}{6}\right) \left(\frac{3}{6}\right)$$

↑
prob. first
roll is 1 or 2

↑
second roll
is 3, 4, 5

Theorem: Independence and Complements

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If $A \perp B$ then pf. case 1:

$$(1) A \perp B^c$$

$$P(AB^c) = P(A) - P(AB)$$

$$(2) A^c \perp B$$

$$= P(A) - P(A)P(B)$$

$$(3) A^c \perp B^c$$

$$= P(A)(1 - P(B))$$

$$= P(A)P(B^c) \quad \swarrow$$

Defn: Mutual Independence

Generalize independence to multiple events.

If $(A_i)_{i=1}^n$ is a seq of events, we say they are (mutually) independent if

(all)
for any subsequence $A_{i_1}, A_{i_2}, \dots, A_{i_k}$

$$P\left(\bigcap_{j=1}^k A_{i_j}\right) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

$$= \prod_{j=1}^k P(A_{i_j})$$

Q: Do I really have to check all possible subsequences?

Can I just check

$$P(A_1 A_2 \dots A_n) = P(A_1) P(A_2) \dots P(A_n) ?$$

No.

Ex, Roll two dice.

$$A = \text{"doubles"} = \{(1,1), (2,2), \dots, (6,6)\},$$

$$|A| = 6$$

$$B = \text{"sum between 7 and 10"}$$

$$= \{ \underline{(1,6)}, \underline{(2,5)}, \underline{(3,4)}, \underline{(4,3)}, \underline{(5,2)}, \underline{(6,1)}, \\ \underline{(2,6)}, \underline{(3,5)}, \underline{(4,4)}, \underline{(5,3)}, \underline{(6,2)}, \\ (3,6), (4,5), (5,4), (6,3), \\ (4,6), (5,5), (6,4) \}$$

$$|B| = 18$$

$$C = \text{"sum is 2, 7, 8"}$$

$$C = \text{"sum is 2, 7, 8"} \\ = \{(1,1), \quad \quad \quad \}$$

$$|C| = 12$$

Mutually Independent?

$$P(ABC) = P(A)P(B)P(C). \quad \checkmark$$

$$\underbrace{\downarrow}_{\{(4,4)\}} \quad \underbrace{\quad}_{4/36} \quad \underbrace{\quad}_{18/36} \quad \underbrace{\quad}_{12/36} \\ \underbrace{1/36} = \underbrace{1/6} \quad \underbrace{1/2} \quad \underbrace{1/3}$$

"sum 7 or 8"

$$P(BC) \neq \underbrace{P(B)}_{1/2} \underbrace{P(C)}_{1/3}$$

↑
doesn't work.

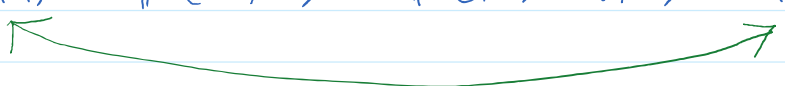
A, B, C not mutually independent.

Defn: Pairwise Independence

If (A_i) is a seq of events, we call them pairwise independent,

$$P(A_i A_j) = P(A_i)P(A_j) \quad \forall i \neq j.$$

Can $A \perp\!\!\!\perp A$? Yes, in certain cases.

$$P(A) = P(AA) = P(A)P(A) = P(A)^2$$


so $P(A) = P(A)^2$ recall, $P(A) \in [0, 1]$

hence either $P(A) = 0$ or $P(A) = 1$.

Ex. Pairwise Independence \neq Mutual Independence

$$S = \{abc, acb, bac, bca, cab, cba, \\ aaa, bbb, ccc\}$$

$|S| = 9$, all equally likely.

$$A_i = \{i^{\text{th}} \text{ place has an } a\}$$

$$A_1 = \{\text{first is an } a\} = \{abc, acb, aaa\}$$

$A_2 = \{\text{second}\} \quad \text{" } \} = \{\text{bac, cab, aab}\}$

$A_3 = \{\text{third}\} \quad \text{" } \} = \{\text{bca, cba, aab}\}$

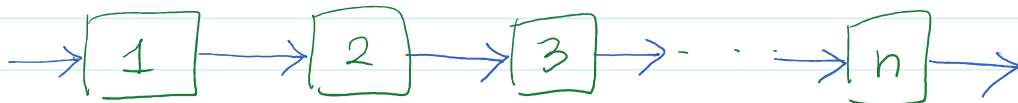
Pairwise independent? ✓

$$P(A_i A_j) = \underbrace{P(A_i)}_{1/3} \underbrace{P(A_j)}_{1/3}$$
$$1/9 = 1/3 \cdot 1/3$$

Mutually Independent? ✗

$$\underbrace{P(A_1 A_2 A_3)}_{1/9} \neq \underbrace{P(A_1) P(A_2) P(A_3)}_{1/27}$$
$$\underbrace{1/3 \cdot 1/3 \cdot 1/3}_{1/27}$$

Ex. Serial System



→ system only works if all components work

→ all components fail independently

$W_i = i^{\text{th}}$ component works

$W_i^c = i^{\text{th}}$ component fails; $P(W_i^c) = p_i$

$P(\text{system works})$

$$= P\left(\bigcap_{i=1}^n W_i\right)$$

$$= \prod_{i=1}^n P(W_i)$$

← defn of mutual independence

$$= \prod_{i=1}^n (1 - P(W_i^c))$$

$$= \prod_{i=1}^n (1 - p_i).$$

Random Variables

Ex. Flip a coin 3 times.

$X = \#$ heads among 3 flips

$$\underline{s \in S} \quad | \quad \underline{X(s)}$$

$s \in S$	$X(s)$
H H H	3
H H T	2
H T H	2
H T T	1
T H H	2
T H T	1
T T H	1
T T T	0

← a function