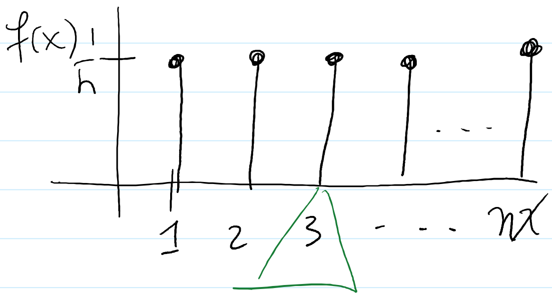
Lecture 12 - Common Distributions

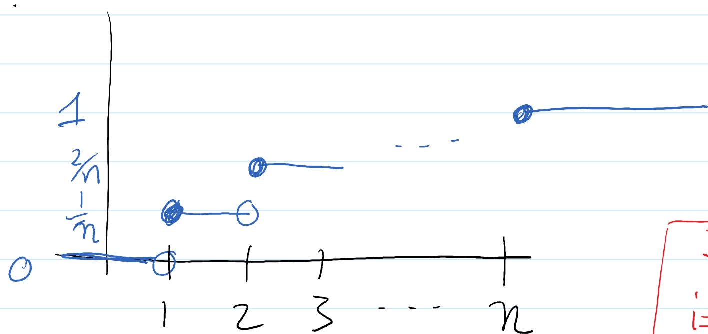
Discrete Uniform Distribution

$$X \sim \mathcal{U}(\{1, \ldots, n\})$$

PMF: $f(x) = \frac{1}{n}$ for $x = 1, \ldots, n$



CDF:



$$F(x) = \begin{cases} 0 & x < 1 \\ \lfloor x \rfloor / n & 1 \le x \le n \\ 1 & x \ge n \end{cases}$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

Expectation

$$E[X] = \sum_{x=1}^{n} x f(x) = \sum_{x=1}^{n} x \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x$$

$$= \frac{1}{n} \frac{n(n+1)}{2} = \boxed{\frac{n+1}{2}}$$

$$E[X^2] = \frac{1}{n} \sum_{x=1}^{n} x^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

Variance:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \cdots = \boxed{\frac{n^2-1}{12}}$$

MGF:

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=1}^{n} e^{tx}\frac{1}{n} = \frac{1}{n}\sum_{x=1}^{n}(e^t)^x \quad (*)$$

recall: geometric sum

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r} \qquad r \neq 1$$

$r = e^t$ then

$$(*) = \frac{1}{n}\sum_{x=0}^{n-1}(e^t)^{x+1} = \frac{e^t}{n}\sum_{x=0}^{n-1}(e^t)^x = \frac{e^t}{n}\,\frac{1-(e^t)^n}{1-e^t}$$

$$\boxed{= \frac{e^t - e^{t(n+1)}}{n(1-e^t)}}$$

$r = e^t \neq 1$

$\Rightarrow t \neq 0$

$M(t) = 1$ when $t = 0$

Consider $X \sim U(\{a, \ldots, b\})$ total

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Note: $n = b - a + 1$ ← total size of support

$X = (a-1) + Y$ ←

where $Y \sim U(\{1, \ldots, n\})$

$$f(x) = \frac{1}{b-a+1} \quad \text{for } x = a, \ldots, b$$

$$\begin{aligned} \mathbb{E}[X] = \mathbb{E}[(a-1) + Y] &= (a-1) + \mathbb{E}[Y] \\ &= (a-1) + \frac{n+1}{2} \\ &= (a-1) + \frac{(b-a+1)+1}{2} \\ &= \frac{a+b}{2} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) = \mathrm{Var}((a-1)+Y) = \mathrm{Var}(Y) &= \frac{n^2-1}{12} \\ &= \frac{(b-a+1)^2 - 1}{12} \end{aligned}$$

$X = (a-1) + Y$

$M(t) = e^{(a-1)t} M_Y(t)$

$n = b - a + 1$

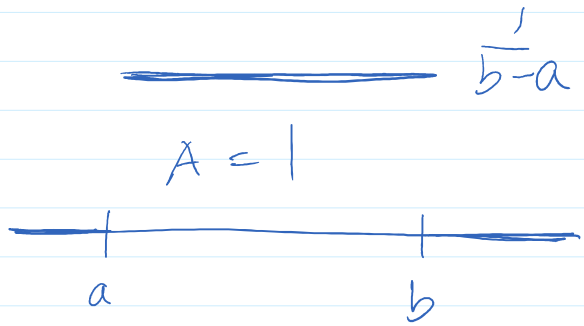
$$= e^{(a-1)} \frac{e^{t} - e^{t(n+1)}}{n(1-e^{t})}$$

$$\boxed{= \frac{e^{at} - e^{t(b+1)}}{(b-a+1)(1-e^{t})}}$$

Continuous Uniform Distribution

$$X \sim U(a,b)$$

$$f(x) = \frac{1}{b-a} \text{ for } a<x<b$$



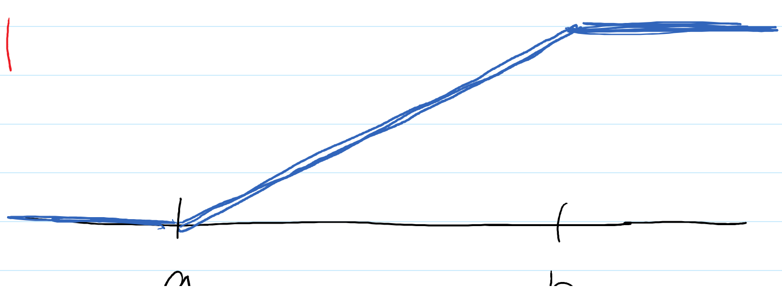
CDF: For $a<x<b$

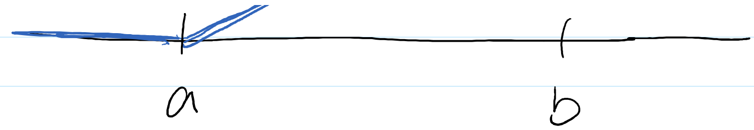
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt = \frac{t}{b-a}\Bigg|_{a}^{x}$$

$$= \frac{x-a}{b-a}$$

If $x<a$ $F(x) = 0$

$x \geq b$ $F(x) = 1$





Expectation

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx \quad \int_a^b x \frac{1}{b-a}\,dx = \frac{x^2}{2(b-a)}\Big|_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$\boxed{= \frac{a+b}{2}}$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{x^3}{3(b-a)}\Big|_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \cdots = \boxed{\frac{(b-a)^2}{12}}$$

MGF:

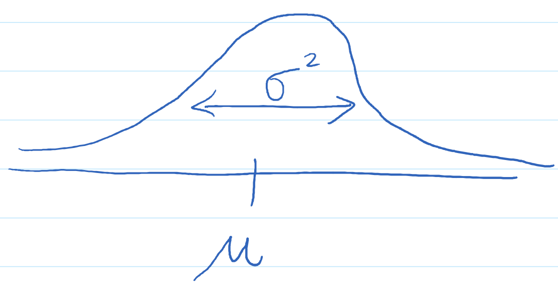
$$M(t) = \mathbb{E}[e^{tX}] = \int_a^b e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx}/_{b-a}\,dx$$

$$= \frac{e^{tx}}{t(b-a)}\Bigg|_a^b = \boxed{\frac{e^{tb} - e^{ta}}{t(b-a)}}$$

Normal Distribution

$$X \sim N(\mu, \sigma^2)$$



PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad \forall x \in \mathbb{R}$$

special case: $\mu = 0$ and $\sigma^2 = 1$ (standard normal)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

CDF: no simple form $F(x) = \int_{-\infty}^{x} f(t)\,dt$

Claim: $\mathbb{E}[X] = \mu$ and $Var(X) = \sigma^2$

MGF:

$$M(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx}$$

$\left[e^a e^b = e^{a+b} \right.$

$$tx - \frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)$$

$$= -\frac{1}{2\sigma^2}\left(-2\sigma^2 tx + x^2 - 2x\mu + \mu^2\right)$$

$$= \frac{-}{2\sigma^2}\left(x^2 \underbrace{2x\mu - 2\sigma^2 tx}_{\downarrow} + \mu^2\right)$$

$$= -\frac{1}{2\sigma^2}\left(\underbrace{x^2 - 2x(\mu + \sigma^2 t)} + \mu^2\right)$$

looks almost like $(x - (\mu + \sigma^2 t))^2$

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$$= -\frac{1}{2\sigma^2}\Big(\underbrace{x^2 - 2x(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2}_{} - (\mu+\sigma^2 t)^2 + \mu^2\Big)$$

$$= -\frac{1}{2\sigma^2}\Big((x-(\mu+\sigma^2 t))^2 - (\mu+\sigma^2 t)^2 + \mu^2\Big)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(\underbrace{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2}_{e^a e^b = e^{a+b}} + \underbrace{\frac{1}{2\sigma^2}(\mu+\sigma^2 t)^2 - \frac{\mu^2}{2\sigma^2}}_{\text{doesn't depend on } x}\Big)dx$$

$$= \exp\Big(\frac{1}{2\sigma^2}(\mu+\sigma^2 t)^2 - \frac{\mu^2}{2\sigma^2}\Big)\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2\Big)dx}_{\substack{\text{PDF of } N(\mu+\sigma^2 t,\ \sigma^2) \\ \text{So integrates to } 1}}$$

$= .$

$$= \overset{\text{algebra}}{\cdots} = \boxed{\exp\Big(\mu t + \frac{\sigma^2 t^2}{2}\Big) = M(t).}$$