Lecture 3 - Basic Theorems

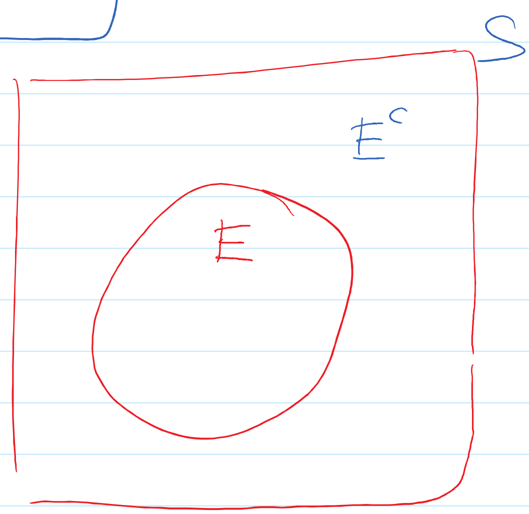
Ex. $E =$ "its raining"

$P(E) = 1/3$

$P(\text{not raining}) = 1 - 1/3$

$P(E^c) = 1 - 2/3$

Theorem: $\boxed{P(E^c) = 1 - P(E)}$



Pf. $S = E \cup E^c$

a partition

So

$1 = P(S) = P(E) + P(E^c)$

$1 = P(E) + P(E^c)$

rearrange

$P(E^c) = 1 - P(E).$

Theorem: $0 \le P(E) \le 1$

Pf. $P(E) \geq 0$ by Axiom 1

$$P(E^c) \geq 0$$

and so $1 - P(E) \geq 0$

rearrange to get $P(E) \leq 1$.

Theorem: If $E, F \subset S$ then

$$P(E \setminus F) = P(EF^c) = P(E) - P(EF)$$

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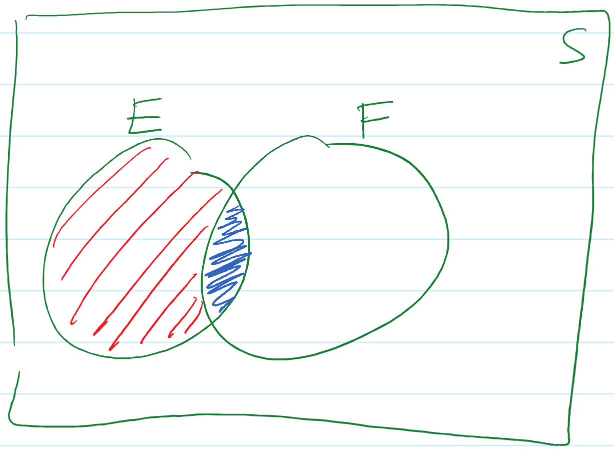
pf. $E = EF \cup EF^c$

partition of E



$$P(E) = P(EF) + P(EF^c)$$

rearrange

$$P(EF^c) = P(E) - P(EF).$$

Theorem: Let $E, F \subset S$

(maybe not disjoint)

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

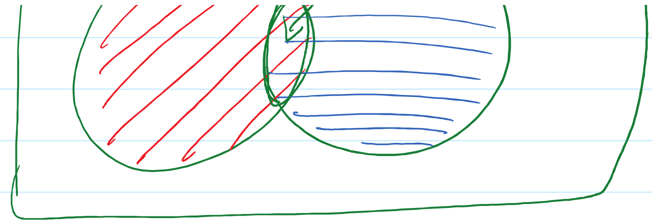
double counted

pf.

disjoint

$$E \cup F = E \cup FE^c$$

$E \cup F = E \cup FE^c$



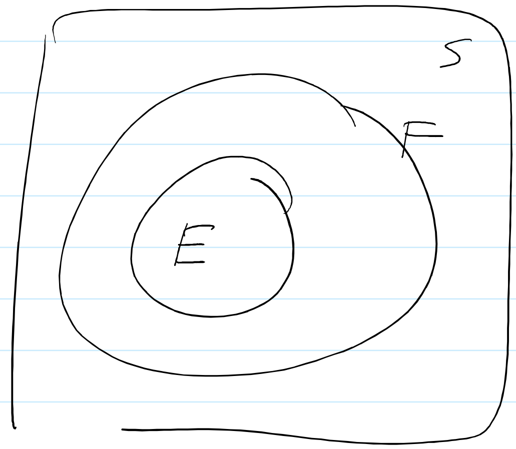
So

$$P(E \cup F) = P(E) + P(FE^c)$$
$$= P(E) + P(F) - P(FE)$$

Theorem: If $E \subset F$

then

$$P(E) \leq P(F).$$



Pf. By Axiom 1,

$$P(FE^c) \geq 0$$

so $\quad P(F) - P(EF) \geq 0$ $\qquad$ $E \subset F$ so $EF = E$

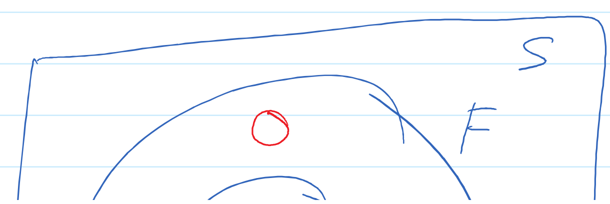
hence $\quad P(EF) \leq P(F)$

so $\quad P(E) \leq P(F).$

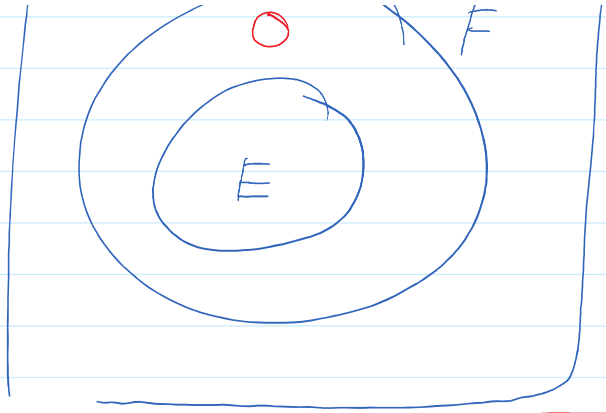
Consider $E \subset F$ but $E \neq F$.

$$P(E) \cancel{<} P(F) \; ?$$

Generally can't say.



Generally can't say.



Said:

$$P(E \cup F) = P(E) + P(F) - \overbrace{P(EF)}^{\geq 0}$$

$$\leq P(E) + P(F)$$

Generalize this: Boole's Inequality

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i).$$

Pf. Replace E_i w/ B_i where

1. $\bigcup_i B_i = \bigcup_i E_i$.
2. B_i are disjoint.

defn:

$B_1 = E_1$

$B_2 = E_2 E_1^c$

Convince yourself that this satisfies 1 & 2.

$B_2 = E_2 E_1^c$

$B_3 = E_3 E_2^c E_1^c$

$B_4 = E_4 E_3^c E_2^c E_1^c$

this satisfies (1) & (2).

<u>Notice:</u> $B_i \subset E_i$

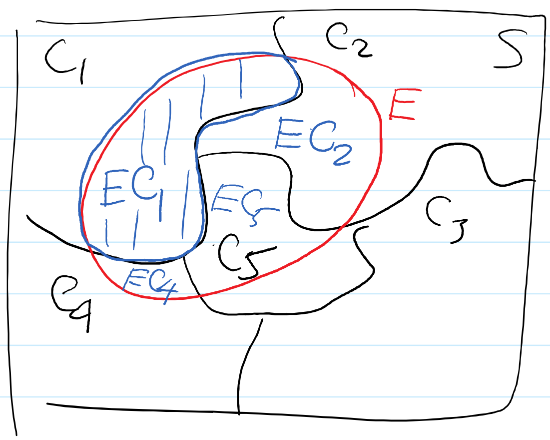
$P(B_i) \leq P(E_i)$

then

$$P\left(\bigcup_i E_i\right) = P\left(\bigcup_i B_i\right) = \sum_i P(B_i) \leq \sum_{i=1}^{\infty} P(E_i)$$

<u>Theorem:</u> If (C_i) are a partition of S and $E \subset S$

$$P(E) = \sum_i P(EC_i).$$



Pf. (1) (EC_i) partitions E

(2) by Additivity

$$P(E) = \sum_i P(EC_i).$$

<u>Equally Likely Outcomes in a Finite Sample Space</u>

I have a sample space

$$S = \{s_1, \ldots, s_n\} \text{ so that } |S| = n$$

$S = \{s_1, \dots, s_n\}$ so that $|S| = n$

assume that

$$\frac{1}{n} = \underline{\mathbb{P}(\{s_i\}) = \mathbb{P}(\{s_j\})} \quad \forall\, i, j$$

Reason:

$$1 = \mathbb{P}(S) = \sum_{i=1}^{n} \underline{\mathbb{P}(\{s_i\})}$$

the only way this works is if $\mathbb{P}(\{s_i\}) = \frac{1}{n}$

More generally:

$$\mathbb{P}(E) = \frac{\#\text{ elements in } E}{\#\text{ elements in } S} = \frac{|E|}{|S|}$$

Ex. Roll a six-sided die.

$$S = \{1, \dots, 6\}$$

and all rolls are equally likely

and

$$E = \{2, 6\}$$

then

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{2}{6} = 1/3.$$

$\mathbb{P}(E) = \dfrac{\quad}{|S|} \quad 6 \quad 3.$

Counting

Ex, An experiment has 3 factors

1. 2 temp. settings
2. 2 pressure settings
3. 4 humidity settings.

Q! How many experiments possible?

$16 = 2 \cdot 2 \cdot 4$

Fundamental Theorem of Counting (FTC)

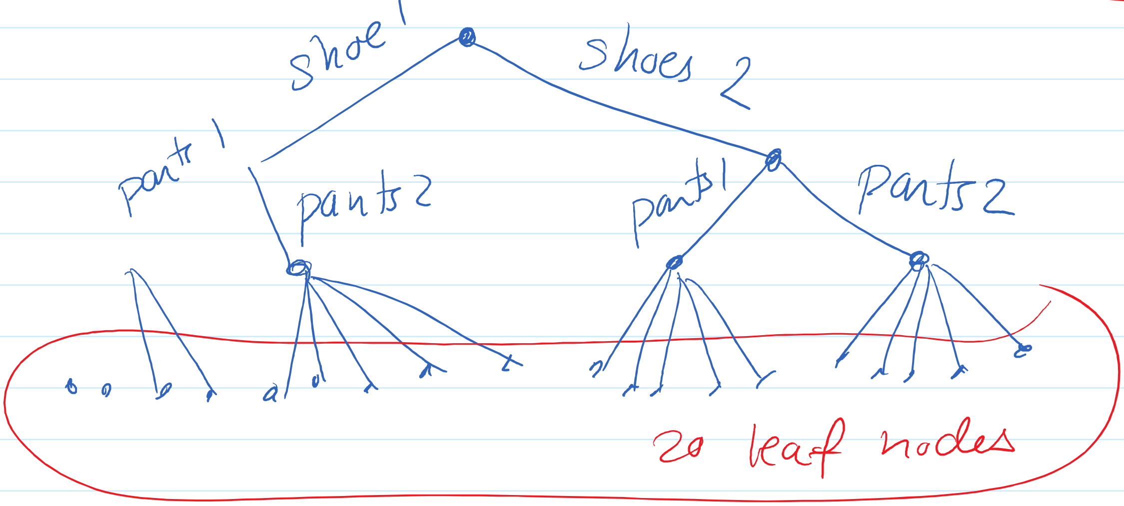
If I have a task that consists of k sub-tasks — where subtask i has n_i ways of being completed. Then the total number of ways to complete the task is

$$N = n_1 n_2 n_3 n_4 \cdots n_k$$

$$= \prod_{i=1}^{k} n_i$$

Ex. A man has 5 shirts, 2 pair pants, 2 pair shoes. How many outfits does he have?

By FTC he has $5 \cdot 2 \cdot 2 = 20$ outfits.



Ex. I have a deck of 52 cards

I shuffle them so each ordering is equally likely.

Q: What is the prob (after shuffle) that the cards are "in order"

↳ A–K, C, D, H, S

E = in order

E = in order

S = all possible shuffles

$$P(E) = \frac{|E|}{|S|} \qquad |E| = 1$$

Use FTC w/ $k = 52$

task #	task	# ways
1	choose 1^{st} card	52
2	" 2^{nd} "	51
3	" 3^{rd} "	50
⋮	⋮	⋮
52	" 52^{nd} "	1

multiply

$$|S| = 52 \cdot 51 \cdot 50 \cdot \cdots 3 \cdot 2 \cdot 1$$

so $P(E) = 1/(52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1)$

Defn: Factorial

For any non-neg. integer n we define n factorial as define n factorial as

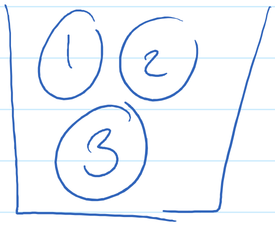
$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$
$$= \prod_{i=1}^{n} i$$

In prev. example:

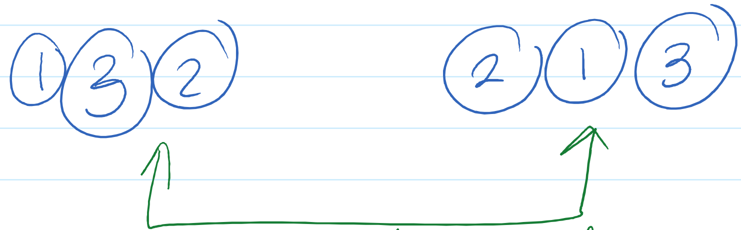
$$\mathbb{P}(E) = 1/52!$$

Sampling w/ and w/o Replacement / Ordering

Ordering

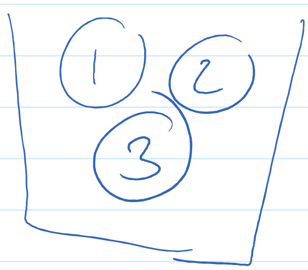


draw 1: draw 2:



are these different?

Replacement



Can I draw 1 1 2 ?

w/ replacement: Yes

w/o replacement: No

4 options:

	w/o repl.	w/ repl.
ordered	1	2
unordered	4	3

unordered | 1 | 3

Defn: Permutation

A permutation is an ordering of objects.

Ex. 1 2 3

permutations:

1 2 3   1 3 2
2 1 3   2 3 1
3 1 2   3 2 1

} 6 perms = 3!

Theorem: The number of ways to permute n items is $n!$

pf. FTC w/ $k = n$ tasks.

task #	task	# ways
1	choose 1st	n
2	" 2nd	$n-1$
3	" 3rd	$n-2$
⋮	⋮	

multiply

$\vdots$	$\vdots$	
n	n^{th}	1

So total num. ways is $n(n-1)\cdots 3\cdot 2\cdot 1$

$$= n!$$