4 options: Sampling $r$ from $n$

| | w/o repl. | w/ repl. |
|---|---|---|
| ordered | 1 $\frac{n!}{(n-r)!}$ | 2 $n^r$ |
| unordered | 3 $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ | 4 $\binom{n+r-1}{r}$ |

Theorem:

If I have $n$ items and I sample $r$ of them w/o replacement but w/ ordering then the number of ways to do this is

$$\frac{n!}{(n-r)!}$$

Pf. Use FTC

| task # | task | # ways |
|---|---|---|
| 1 | sample 1st | $n$ |
| 2 | " 2nd | $n-1$ |
| 3 | " 3rd | $n-2$ |
| ⋮ | ⋮ | ⋮ |
| r | " rth | $n-r+1$ |

mutiply

$$n(n-1)(n-2)\cdots(n-r+1) = \boxed{\frac{n!}{(n-r)!}}$$

$$\frac{\boxed{n(n-1)\cdots(n-r+1)}\cancel{(n-r)}\cancel{\cdots 3\cdot 2\cdot 1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cancel{\cdots 3\cdot 2\cdot 1}}$$

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Ex. I form a committee from $n=10$ student of size $r=3$

where the committee's 3 members are

Pres, VP, treasurer

How many ways can I form this committee?

Sample w/o replacement

b/c can't have same student twice

Sample w/ order

$1^{st}$ = Pres, $2^{nd}$ = VP, $3^{rd}$ = treasurer

Apply formula :

$$\frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10\cdot 9\cdot 8\cdot \cancel{7!}}{\cancel{7!}} = 10\cdot 9\cdot 8$$
$$= 720$$

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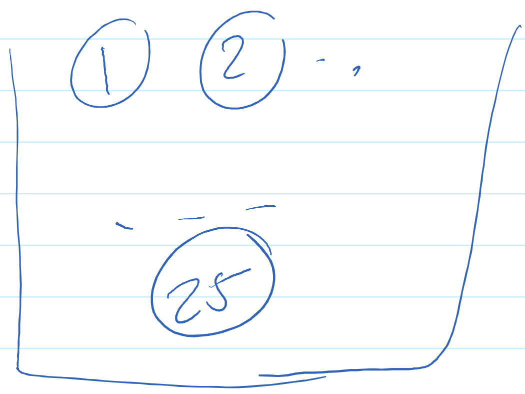
Ex. Lotto.

I have a basket w/ 25 numbered balls




Draw 4 of them. all draws equally likely

Guess: 1 3 22 7

What's the prob. I correctly guess?

$$P(E) = \frac{|E|}{|S|}$$

$$P(E) = \frac{|E|}{|S|}$$

$E = \{ 1\ 3\ 22\ 7 \}$ so $|E| = 1$

$S = \{$ all possible draws $\}$

$$|S| = \frac{25!}{(25-4)!} = \frac{25!}{21!} = 25 \cdot 24 \cdot 23 \cdot 22$$

$$\boxed{P(E) = \frac{1}{25 \cdot 24 \cdot 23 \cdot 22}}$$

---

Theorem: Sampling w/ repl. and w/ ordering

The number of ways to sample r from n w/ repl. and w/ ordering is

$$n^r.$$

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pf. Use FTC

| task # | task | # ways |
|---|---|---|
| 1 | sample 1st | $n$ |
| 2 | " 2nd | $n$ |
| 3 | " 3rd | $n$ |

product

| | | |
|---|---|---|
| 2 | " $2^{nd}$ | $n$ |
| 3 | " $3^{rd}$ | $n$ |
| $\vdots$ | | $\vdots$ |
| $r$ | " $r^{th}$ | $n$ |

product

$n^r$

---

## Ex. Braille Alphabet

6 spot each can be raised or not

Q: how many braille letters can I make?

Sampling $r=6$ from $n=2$

So I have

$2^6 = 64$.

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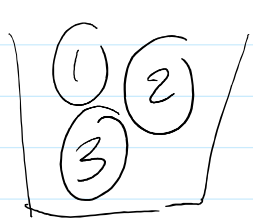
## Sampling w/o replacement and w/o ordering

Ex.



draw $r=2$ from $n=3$

If order matters

3!

If order matters

| | | |
|---|---|---|
| $(1,2)$ | $(1,3)$ | $(2,3)$ |
| $(2,1)$ | $(3,1)$ | $(3,2)$ |

$\Big\}\ 6 = \dfrac{3!}{(3-2)!}$

If order doesn't matter

$\{1,2\},\ \{1,3\},\ \{2,3\}\ \Big\}\ 3$

General fact:

Each unordered sample of size r can be permuted in r! ways to make an ordered sample.

$$\{1,2,3\} \to (1,2,3),\ (2,1,3),\ (1,3,2),\ \vdots$$

$$\underbrace{(\#\text{ ordered})}_{1} = r!\,(\#\text{ unordered})$$

So

$$(\#\text{ unordered}) = \frac{1}{r!}(\#\text{ ordered})$$

$$= \frac{1}{r!}\,\frac{n!}{(n-r)!}$$

$(n-r)!$

## Theorem: Unordered w/o Repl.

I can sample $r$ from $n$ w/o repl. w/o order in

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Binomial coefficient

read: n choose r

Ex. I have $n=10$ professors, how many co-equal committees can I form of size $r=4$?

Sample w/o order b/c co-equal

w/o replacement b/c can't have same prof. twice

I can do this in

$$\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!\,6!}$$

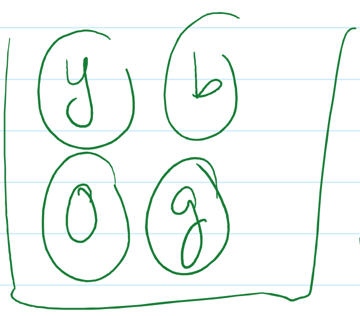
$$= \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot \cancel{6!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{6!}}$$

$$= 10 \cdot 3 \cdot 7 = 210.$$

Ex. How many 5-card poker hands can I form (from a deck of 52 cards)

$$\binom{52}{5} \approx 2.5 \text{ mil}$$

Ex. I have a jar w/ 4 marbles of colors yellow, blue, orange, green.




I choose 3 w/o repl. (all choices equally likely)

What is the prob I have a y and b in my choice?

$E = \{ y \text{ and } b \} = \{ \{y, b, o\}, \{y, b, g\} \}$

$|E| = 2$

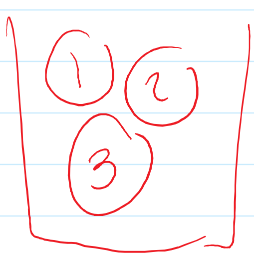
$S = \{\text{all selections}\}$

$$|S| = \binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = 4.$$

Hence $P(E) = |E|/|S| = 2/4 = 1/2.$

## Sampling Unordered w/ Repl.

Consider $n=3$, $r=2$




If ordered:

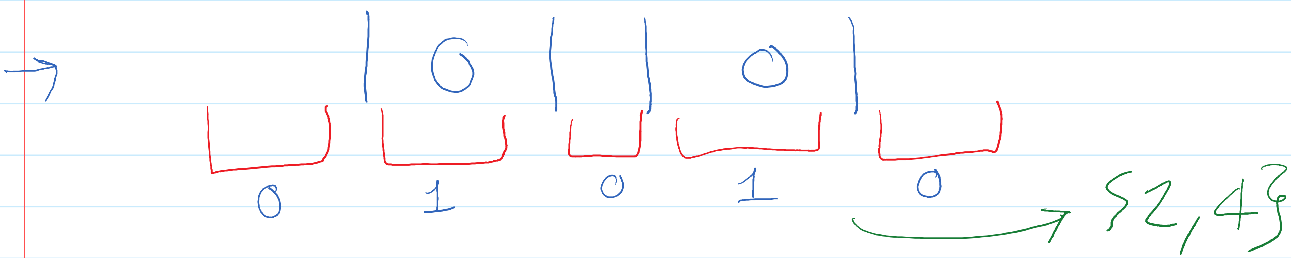
$(2,1)$ $(1,2)$ | $(3,1)$ $(1,3)$ | $(2,3)$ $(3,2)$ | $(1,1)$ | $(2,2)$ | $(3,3)$ $\Big\}\ 9$

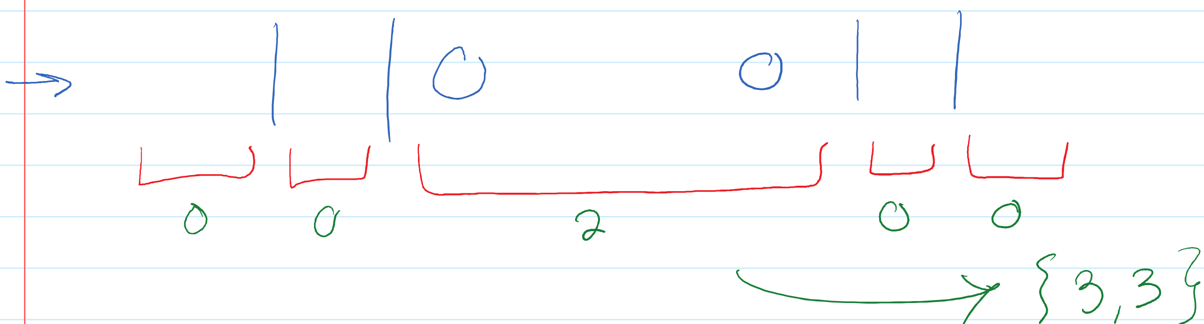
Unordered:

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,1\}$ $\{2,2\}$ $\{3,3\}$

## Game of Partitioning

How many ways can I partition $r=2$ objects using $n-1=4$ walls




In total I have $n-1+r$ things:

- $r$ objects
- $n-1$ walls

I have

$$\frac{(n+r-1)!}{(n-1)!\,r!}$$

ways to play this game

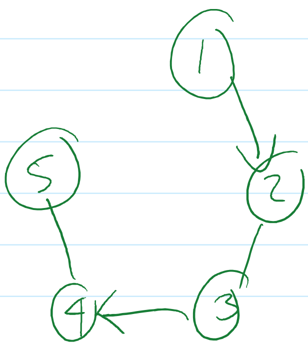
---

Theorem: w/ repl. w/o ordering

The number of ways to sample $r$ from $n$ w/ repl. w/o ordering is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Ex. 10 passengers on a bus route w/ 5 stops. The driver records the number of people that get off at each stop.




Q: How many possible records are there?

| Stop | # |
|---|---|
| 1 | 0 |
| 2 | 3 |
| 3 | 1 |
| 4 | 2 |
| 5 | 4 |

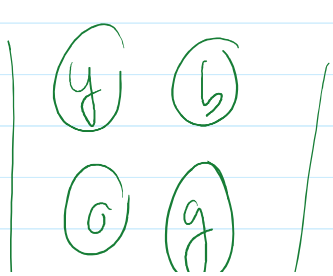
$\longrightarrow \{2,2,2,3,4,4,5,5,5,5\}$

Our theorem says:

$$\binom{10+5-1}{10} = \binom{14}{10} = 1001.$$

Ex. Jar w/ 4 marbles: y, b, o, g

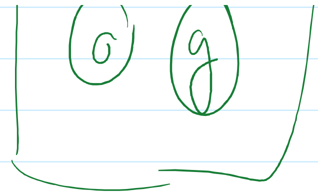
w/ replacement




Draw $r=3$ from $n=4$

(all draws are equally likely)

Q: What's the prob. my sample



Q: What's the prob. my sample includes both $y$ and $b$?

$$\mathbb{P}(E) = \frac{|E|}{|S|}$$

$$E = \{ y \text{ and } b \} = \{ \{y,b,o\}, \{y,b,g\}, \{y,b,y\}, \{y,b,b\} \}$$

So $|E| = 4$

and $|S| = \binom{3+4-1}{3} = \binom{6}{3} = 20$

So $\mathbb{P}(E) = 4/20 = 1/5$