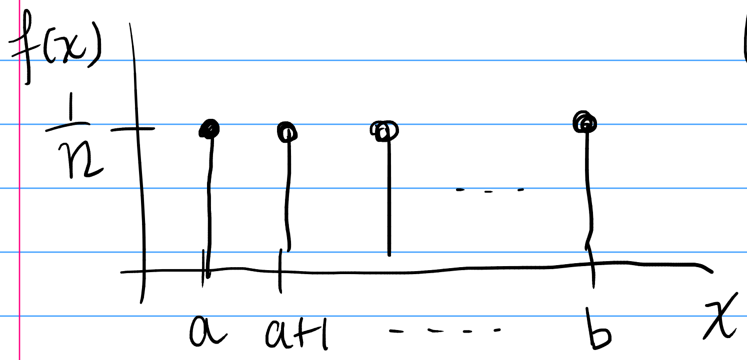
Lecture 12

Discrete Uniform

$Y \sim U(\{a, \dots, b\})$



Let $n = b - a + 1$

$X \sim U(\{1, \dots, \underbrace{n}_{b-a+1}\})$

then

$$Y = X + (a-1)$$
$$\sim U(\{a, \dots, b\})$$

$$f(y) = \frac{1}{b-a+1} \quad \text{for } y = a, \dots, b$$

Expectation

$$E[Y] = E[X + (a-1)] = E[X] + a - 1$$
$$= \frac{n+1}{2} + a - 1$$
$$= \frac{b-a+1+1}{2} + a - 1$$
$$= \dots$$
$$= \frac{a+b}{2}$$

$$Var(Y) = Var(X + a - 1)$$
$$= Var(X)$$
$$= \frac{n^2 - 1}{12} = \frac{(b-a+1)^2 - 1}{12}$$

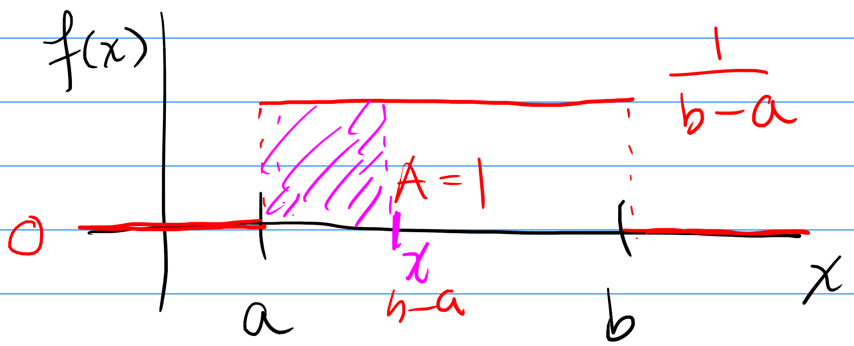
MGF $\quad Y = X + a - 1$

$$M_Y(t) = e^{(a-1)t} M_X(t)$$
$$= e^{(a-1)t} \frac{e^t - e^{(n+1)t}}{n(1-e^t)}$$
$$= \dots$$

$$M_{\alpha X + \beta}(t) = e^{t\beta} M_X(\alpha t)$$

$$\boxed{M(t) = \frac{e^{at} - e^{(b+1)t}}{(b-a+1)(1-e^t)}}$$

Continuous Uniform $\quad X \sim U(a,b)$



PDF

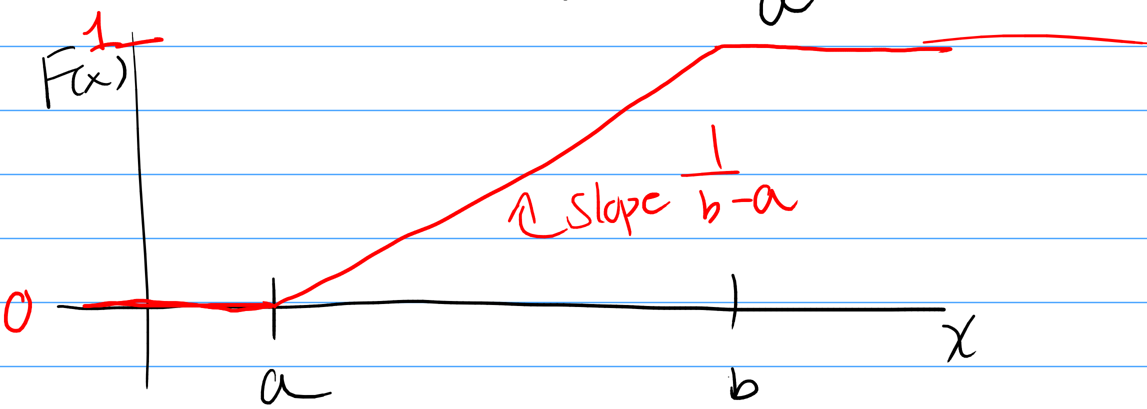
$$f(x) = \frac{1}{b-a} \qquad a < x < b$$

CDF:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt$$

$a < x < b$

$$= \frac{t}{b-a}\Big|_{a}^{x} = \frac{x-a}{b-a}$$



Expectation

$$\mathbb{E}X = \int_{\mathbb{R}} x f(x)\,dx = \int_{a}^{b} x \frac{1}{b-a}\,dx$$

$$= \frac{x^2}{2}\frac{1}{b-a}\Big|_{a}^{b}$$

$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2}$$

$$\mathbb{E}[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \frac{x^3}{3}\frac{1}{b-a}\Big|_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \ldots$$

$$= \frac{(b-a)^2}{12}$$

<u>MGF:</u>

$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

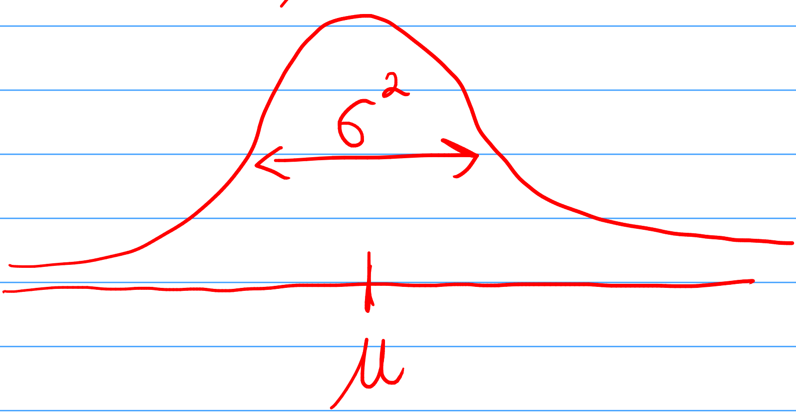
$$= \int_a^b e^{tx}\frac{1}{b-a}\,dx$$

$$= \frac{1}{t} e^{tx} \frac{1}{b-a} \Big|_a^b$$

$$\boxed{M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}}$$

Normal Distribution $X \sim N(\mu, \sigma^2)$

$\mu \in \mathbb{R}$, $\sigma^2 > 0$



PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

for all $x \in \mathbb{R}$

CDF:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$ = no simple formula

Claim: $\mathbb{E}X = \mu$, $Var(X) = \sigma^2$

$$M(t) = \mathbb{E}[e^{tX}]$$

$$= \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

$$tx - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$= tx - \frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)$$

$$= -\frac{1}{2\sigma^2}\left(-2\sigma^2 tx + x^2 - 2x\mu + \mu^2\right)$$

$$= -\frac{1}{2\sigma^2}\left(\underbrace{x^2 - 2x(\sigma^2 t + \mu)}_{\text{looks like first two terms from } (x-(\mu+\sigma^2 t))^2} + \mu^2\right)$$

$$= -\frac{1}{2\sigma^2}\left(x^2 - 2x(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2 - (\mu+\sigma^2 t)^2 + \mu^2\right)$$

$$= -\frac{1}{2\sigma^2}\left([x-(\mu+\sigma^2 t)]^2 - (\mu+\sigma^2 t)^2 + \mu^2\right)$$

$$M(t) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\quad) \, dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2 t)]^2\right) \cdot \underbrace{\exp\left(-\frac{1}{2\sigma^2}[-(\mu+\sigma^2 t)^2+\mu^2]\right)}_{\text{no } x} dx$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \underbrace{\underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2 t)]^2\right) dx}_{\text{PDF of } N(\mu+\sigma^2 t, \sigma^2)}}_{1}$$

$$\boxed{M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}$$

$M(t) = \exp(\mu t + \sigma^2 t^2/2)$

$$E[X] = \left.\frac{dM}{dt}\right|_{t=0}$$

$$= \left.(\mu + \sigma^2 t)\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\right|_{t=0}$$

$$= (\mu + 0)\exp(0) = \mu$$

$$E[X^2] = \left.\frac{d^2 M}{dt^2}\right|_{t=0} = \sigma^2 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) + (\mu + \sigma^2 t)^2 \cdot \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

plug in $t = 0$

$$= \sigma^2(1) + (\mu + 0)^2(1)$$

$$= \mu^2 + \sigma^2$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \mu^2 + \sigma^2 - (\mu)^2$$

$$= \sigma^2.$$

Theorem:

If $X \sim N(\mu, \sigma^2)$

and $Y = aX + b$

then

$$Y \sim N(a\mu + b, a^2\sigma^2)$$

$$E[Y] = aEX + b = a\mu + b$$

$$Var(Y) = Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

Pf

① $M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

② $M_{aX+b}(t) = e^{tb} M_X(at)$

$$M_Y(t) = M_{aX+b}(t) = e^{tb} M_X(at)$$

$$= e^{tb} \exp\left(\mu(at) + \frac{\sigma^2 (at)^2}{2}\right)$$

$$= \underbrace{\exp\left((a\mu + b)t + a^2\sigma^2 \frac{t^2}{2}\right)}_{\text{MGF of } N(a\mu+b,\, a^2\sigma^2)}$$

Poisson Distribution

- discrete RV
- Support is $\{0, 1, 2, 3, \ldots\}$

Canonical Experiment:

Count the number of "events" that occur in some time period

Ex. - capture fish in a river
- Count # mRNA in a cell
- Count radioactive decay

$X \sim \text{Pois}(\lambda)$ $\lambda > 0$, rate of occurence per time interval

events

PMF: $f(x) = \dfrac{e^{-\lambda} \lambda^{x}}{x!}$ for $x = 0, 1, 2, 3, \ldots$

Expectation

$\mathbb{E}X = \displaystyle\sum_{x=\cancel{0}\,1}^{\infty} x \frac{e^{-\lambda} \lambda^{x}}{x!}$ $\quad \dfrac{\cancel{x}}{\cancel{x}(x-1)!} = \dfrac{1}{(x-1)!}$

$\downarrow$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^{x}}{(x-1)!}$$

$$e^{y} = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

$$= \sum_{i=0}^{\infty} y^{i}/i!$$

$$= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \lambda \underbrace{\sum_{x=0}^{\infty} \lambda^{x}/x!}_{e^{\lambda}}$$

$$= e^{-\lambda} \lambda e^{\lambda}$$

$$\boxed{\mathbb{E}X = \lambda}$$