Lecture 10: Expectation

General trick: PMF/PDF trick

Often I can recognize some part of a calculation as either

$$\underbrace{\sum_x f(x)}_{=1} \quad \text{or} \quad \underbrace{\int_{\mathbb{R}} f(x)dx}_{=1}$$

Functions of RVs

A fn of a RV is another RV

e.g. if X is a RV then so is X^2, $\log X$, $\sqrt{X}$ etc...

Theorem: Law of the Unconscious Statistician

If $g: \mathbb{R} \to \mathbb{R}$ and X is a RV then

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x) f(x) & \text{discrete} \\ \int g(x) f(x) dx & \text{cts} \end{cases}$$

Ex, Let $X \sim Exp(\lambda)$ $\lambda > 0$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

$$E[X] = 1/\lambda$$

$$E[X^2] = \int_{\mathbb{R}} x^2 f(x)dx = \int_0^\infty \underbrace{x^2}_{u} \underbrace{\lambda e^{-\lambda x} dx}_{dv}$$

$$u = x^2 \qquad v = -e^{-\lambda x}$$

$$du = 2x\,dx \qquad dv = \lambda e^{-\lambda x} dx$$

$$= uv - \int v\,du = (x^2)(-e^{-\lambda x})\Big|_0^\infty + 2\int_0^\infty e^{-\lambda x} x\,dx$$

$$= 0 - 0 + 2\int_0^\infty x e^{-\lambda x} dx$$

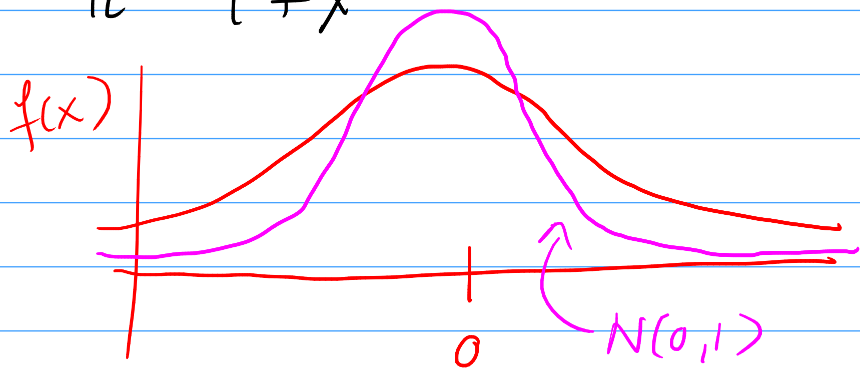
$$= \frac{2}{\lambda}\int_0^\infty x \lambda e^{-\lambda x} dx$$

$$= \frac{2}{\lambda} E[X]$$

$$\boxed{E[X^2] = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = 2/\lambda^2}$$

Ex. Cauchy Distribution

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$



$$EX = \int_{\mathbb{R}} x f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$$

$$\frac{x}{1+x^2} \sim \frac{x}{x^2} = \frac{1}{x}$$

$$\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty, \quad \int_0^{\infty} \frac{1}{x^2}\,dx < \infty$$

$$\sum_{i=1}^{\infty} \frac{1}{i} = \infty, \quad \int_0^{\infty} \frac{1}{x}\,dx = \infty$$

$$\sim \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x}\,dx$$

$$= \infty$$

Has no (finite) expected value.

Theorem: Properties of Expectation

① Expectation is linear

$$E[aX+b] = aE[X]+b$$

pf. (cts)

$$E[aX+b] = \int (ax+b)f(x)dx$$

$$= \int axf(x)\,dx + \int b f(x)dx$$

$$= a\underbrace{\int x f(x)\,dx}_{E[X]} + b\underbrace{\int f(x)dx}_{1}$$

$$= aEX + b$$

② If $\underbrace{X \geq 0}_{\text{Support is positive } (\geq 0)}$ then $E[X] \geq 0$

pf (cts)

$$E[X] = \int_{0}^{\infty} \underbrace{\underbrace{x}_{0\geq 0}\underbrace{f(x)}_{\geq 0}}_{\geq 0}dx \geq 0$$

③ If g_1 and g_2 are functions then

(i) $E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)]$

(ii) If $g_1(x) \le g_2(x)$

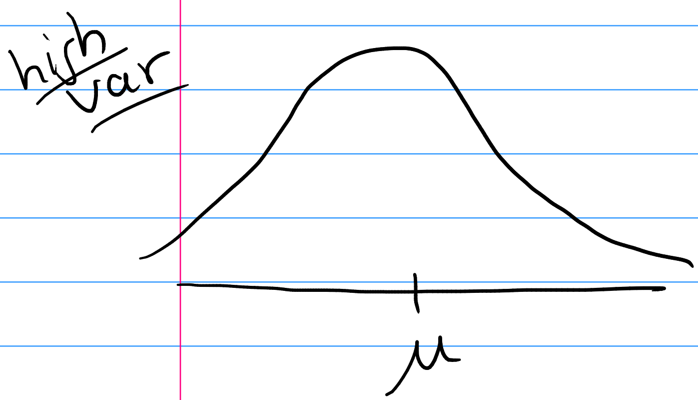
then $E[g_1(X)] \le E[g_2(X)]$

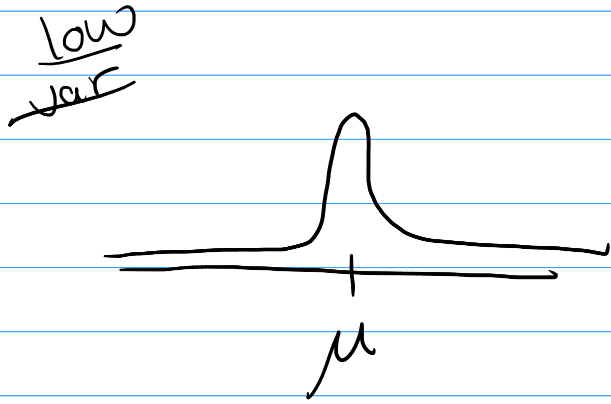
④ If $a \le X \le b$ then $a \le EX \le b$.

Defn: Variance

$\mu = EX =$ loc. of dist.

$\sigma^2 = Var(X) =$ spread around mean





Defn:

$$Var(X) = E[(X-\mu)^2]$$

$$= E[(X - E[X])^2]$$

Ex. $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$\mu = EX = 1/\lambda, \quad E[X^2] = 2/\lambda^2.$$

$$Var(X) = E[\underbrace{(X-\mu)^2}_{g(x)=(x-\mu)^2}] = \int g(x) f(x)\,dx$$

$$= \int (x-\mu)^2 \lambda e^{-\lambda x}\,dx$$

correct, but labor intensive

$$= E[X^2 - 2\mu X + \mu^2]$$

(*) $E[aX+b] = a\,EX + b$

$$= E[X^2] - 2\mu E[X] + \mu^2$$

$$= \frac{2}{\lambda^2} - 2\mu\left(\frac{1}{\lambda}\right) + \mu^2$$

$\mu = EX = 1/\lambda$

$$= \frac{2}{\lambda^2} - 2\left(\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}\right) + \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{2}{\lambda^2} + \frac{1}{\lambda^2} = \boxed{1/\lambda^2} = Var(X)$$

Standard deviation

$$sd(X) = \sqrt{Var(X)}$$

Theorem: Short-cut formula for Variance

$$Var(X) = E[X^2] - E[X]^2$$

Pf.

$$Var(X) = E[(X-\mu)^2]$$

$$= E[X^2 - 2\mu X + \mu^2]$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2$$

$$= E[X^2] - 2E[X]^2 + E[X]^2$$

$$= E[X^2] - E[X]^2.$$

Ex. $X \sim Exp(\lambda)$

$EX = 1/\lambda$, $E[X^2] = 2/\lambda^2$

$$Var(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Ex, $X \sim Bin(n,p)$

$$\mathbb{E}[X] = np$$

$$\mathbb{E}[X^2] = np(np - p + 1)$$

$$\begin{aligned} Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= np(np-p+1) - (np)^2 \\ &= \cdots \\ &= np(1-p) \end{aligned}$$

$$Sd(X) = \sqrt{Var(X)} = \sqrt{np(1-p)}$$

Theorem: $Var(aX + b) = a^2 Var(X)$

two rules:

(1) multiply X by a:
var gets mult. by a^2

(2) ignore additive consts b.

pf.

$$\mathrm{Var}(\overbrace{aX+b}^{Y})$$

$$= \mathbb{E}[\underbrace{(aX+b)^2}_{Y^2}] - \mathbb{E}[\underbrace{aX+b}_{Y}]^2$$

$$= \mathbb{E}[a^2X^2 + 2aXb + b^2] - (a\,\mathbb{E}X + b)^2$$

$$= a^2\,\mathbb{E}[X^2] + \cancel{2ab\,\mathbb{E}X} + \cancel{b^2} - (a^2(\mathbb{E}X)^2 + \cancel{2ab\,\mathbb{E}X} + \cancel{b^2})$$

$$= a^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2)$$

$$= a^2\,\mathrm{Var}(X).$$