Lecture 16: Bivariate RVs

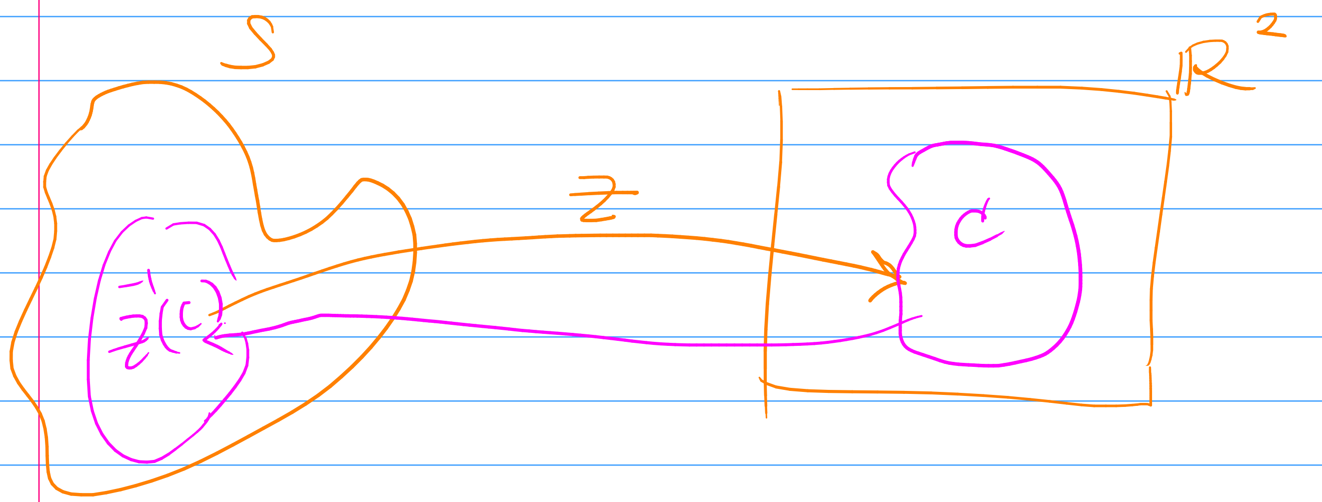
If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$

then $Z = (X, Y)$ is called a bivariate RV.

So $Z : S \to \mathbb{R}^2$ so that $Z(s) = (X(s), Y(s))$

Say: $\mathbb{P}(Z \in C) = \mathbb{P}((X, Y) \in C)$ $\quad C \subset \mathbb{R}^2$

$$= \mathbb{P}(Z^{-1}(C))$$



Often $C = A \times B$ where $A, B \subset \mathbb{R}$

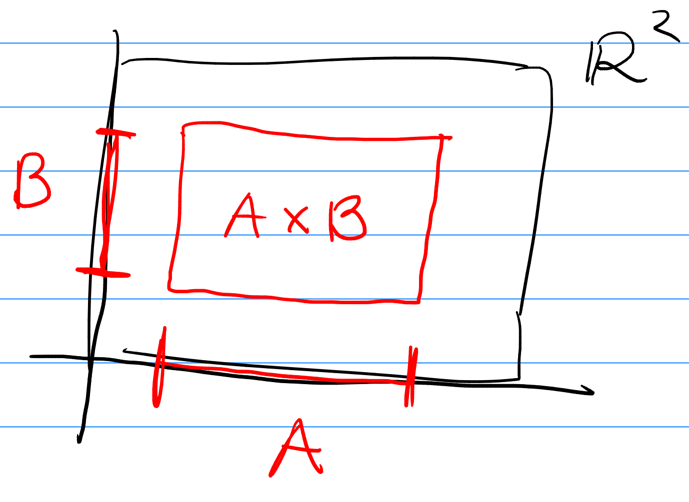
Write

$\mathbb{P}((X, Y) \in C)$

or be lazy

$\mathbb{P}(X \in A, Y \in B)$

"and"



Ex. Consider flipping a coin 3 times.

$$X = \begin{cases} 0 & \text{if last flip is } T \\ 1 & \text{if last flip is } H \end{cases}$$

$$Y = \#\text{ heads among 3 flips}$$

$$Z = (X, Y)$$

$s \in S$	$Z(s) \in \mathbb{R}^2$
H H H	$(1, 3)$
H H T	$(0, 2)$
H T H	$(1, 2)$
H T T	$(0, 1)$
T H H	$(1, 2)$
T H T	$(0, 1)$
T T H	$(1, 1)$
T T T	$(0, 0)$

Defn: Bivariate CDF (Joint CDF)

The joint CDF is a function

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

so that for any $(x,y) \in \mathbb{R}^2$

$$F(x,y) = \mathbb{P}(X \le x, Y \le y)$$

[Univariate: $F(x) = \mathbb{P}(X \le x)$]

Properties of Joint CDF

① $F(x,y) \geq 0$

② $\lim_{x,y \to \infty} F(x,y) = 1$ [Uni: $\lim_{x \to \infty} F(x) = 1$]

③ $\lim_{x \to -\infty} F(x,y) = 0$ [Uni: $\lim_{x \to -\infty} F(x) = 0$]

$\lim_{y \to -\infty} F(x,y) = 0$

④ F is non-decreasing and right-cts in each argument (x,y)

Defn: Marginal Properties

If (X, Y) is a bivar RV then X and Y individually are called the marginal RVs — and their corresp. properties are called marginal props.

e.g. marginal CDFs, marginal PMF, ...

Theorem: Relation between Joint/Marginal CDFs

① $F_X(x) = \lim_{y \to \infty} F(x,y)$

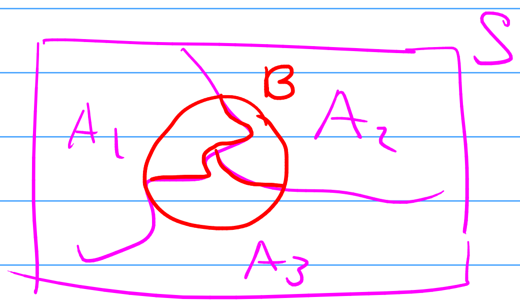
F_X: marginal CDF of X

$F(x,y)$: joint CDF

② $F_Y(y) = \lim_{x \to \infty} F(x,y)$

Recall: If A_i partition S then

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B \cap A_i)$$



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pf:

$$
\begin{aligned}
F_X(x) &= \mathbb{P}(X \le x) \\
&= \mathbb{P}(X \le x,\ Y = \text{anything}) \\
&= \mathbb{P}(X \le x,\ Y < \infty) \\
&= \lim_{y \to \infty} \mathbb{P}(X \le x,\ Y \le y)
\end{aligned}
$$

$$= \lim_{y \to \infty} F(x,y)$$

[Uni: $\mathbb{P}(X > x) = 1 - F(x)$]

Bivariate:

$$\mathbb{P}(X > x, Y > y) = 1 - \underbrace{F_X(x)} - \underbrace{F_Y(y)} + F(x,y)$$

Defn: Joint PMF

If X and Y are discrete RVs then the joint PMF is defined as

$$f(x,y) = \mathbb{P}(X = x, Y = y)$$

[Uni: $f(x) = \mathbb{P}(X = x)$]

__Theorem__: Valid PMF

A function f is a valid joint PMF

iff

(1) $f(x,y) \geq 0 \quad \forall x,y$

(2) $\sum_x \sum_y f(x,y) = 1$

__Theorem__: Rel. btwn joint/marginal PMFs

(1) $f_X(x) = \sum_y f(x,y)$

marginal PMF of X

joint PMF

(2) $f_Y(y) = \sum_x f(x,y)$

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pf: Let $A_y = \{s : Y(s) = y\}^{CS}$ for all y

Notice that A_y partition S

Let $B = $ "$X = x$" $\subset S$

then

$$f_X(x) = P(X=x) = P(B)$$

$$= \sum_y P(B \cap A_y)$$

$$= \sum_y P(\text{"}X = x\text{"} \cap \text{"}Y = y\text{"})$$

$$= \sum_y P(X = x, Y = y)$$

$$\boxed{f_X(x) = \sum_y f(x,y)}$$

Ex.

$$X = \begin{cases} 0 & \text{if last flip } T \\ 1 & \text{'' } H \end{cases}$$

$Y = \#$ heads

sum of first row

X \ Y	0	1	2	3	
0	$f(0,0) = 1/8$	$f(0,1) = 2/8$	$f(0,2) = 1/8$	$f(0,3) = 0$	$f_X(0) = 1/2$
1	0	$1/8$	$2/8$	$1/8$	$f_X(1) = 1/2$

$f_Y(0) = 1/8$ $f_Y(1) = 3/8$ $f_Y(2) = 3/8$ $f_Y(3) = 1/8$

Defn: Joint PDF

If X and Y are cts RVs then we call the function

$$f: \mathbb{R}^2 \to \mathbb{R}$$

the joint PDF if for all $C \subset \mathbb{R}^2$

$$\mathbb{P}((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$$

[Uni: $\mathbb{P}(X \in A) = \int_A f(x)\,dx$]

Facts:

1. $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,du\,dv$

[Uni: $F(x) = \int_{-\infty}^{x} f(t)\,dt$]

2. $f(x,y) = \dfrac{\partial F}{\partial x \partial y}$ [Uni: $f(x) = \dfrac{dF}{dx}$]

③ f is a valid joint density iff

① $f(x,y) \geq 0 \ \forall x,y$

② $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$

Theorem: Rel. btwn joint/marginal PDFs

① $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$

marginal PDF

joint PDF

② $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$

Ex.

$$F(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ xy, & 0<x<1,\ 0<y<1 \\ x, & 0<x<1,\ y>1 \\ y, & 0<y<1,\ x>1 \\ 1, & x>1 \text{ and } y>1 \end{cases}$$

$F(x,y)$ $f(x,y)$

	0 0	0 x	0 1
1			
	0 0	1 xy	0 y
0			
	0 0	0 0	0 0
	0		1

what's the joint PDF? $f(x,y) = \frac{\partial F}{\partial x \partial y}$

$$f(x,y) = \begin{cases} 1 & \text{for } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

(Uniformly dist over unit square)