Lecture 7: Random Variables

Defn: Support

If X is a RV its support is the collection of possible values it can take on.

Ex. Flip coin 3 times, $\underbrace{\text{count \# heads}}_{X}$

$$\text{Support}(X) = \{0,1,2,3\}$$

Notice $\mathbb{P}(X=5) = 0$

More generally, if $A \subset \mathbb{R}$, and
$$\text{Support}(X) \cap A = \emptyset$$
then $\mathbb{P}(X \in A) = 0$

$\hookrightarrow \mathbb{P}(X^{-1}(A)) = 0$

Defn: Discrete/Continuous RVs

(1) discrete RV: support is finite/countable

E.g. X = sum of two dice

E.g. X = # customers arriving in shop

(2) continuous RV : Support is uncountable

E.g. time or space

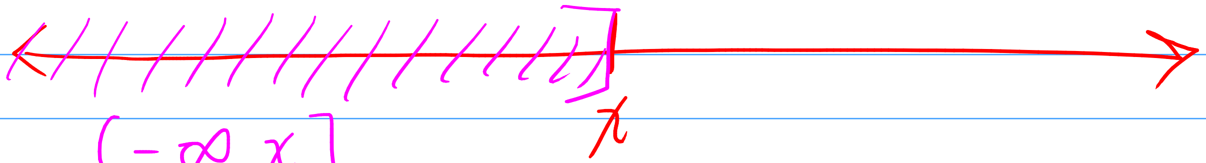
Defn: Cumulative Distribution Function (CDF)

If X is a RV then its CDF is a function,

$$F : \mathbb{R} \to \mathbb{R}$$

defined for $x \in \mathbb{R}$ as

$$F(x) = \mathbb{P}(\underbrace{X}_{\text{a RV}} \le \underbrace{x}_{\text{a number}})$$



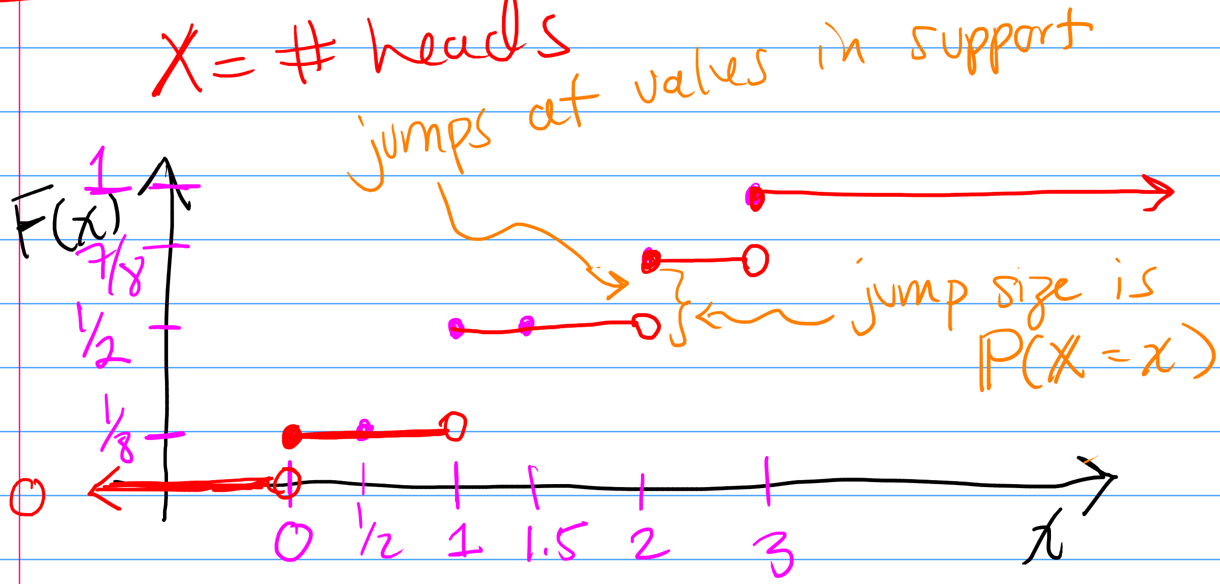
what's the prob that X is here

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}(X \in (-\infty, x])$$

$$= \mathbb{P}(\underbrace{X^{-1}((-\infty, x])}_{\text{event}})$$

Ex. Toss a coin 3 times

$X = \#$ heads



$F(0) = P(X \le 0) = P(X=0) = 1/8$

$F(1/2) = P(X \le 1/2) = P(X=0) = 1/8$

$F(1) = P(X \le 1) = 4/8 = 1/2$

$F(1.5) = P(X \le 1.5) = P(X \le 1) = 1/2$

$F(2) = P(X \le 2) = 7/8$

$F(3) = P(X \le 3) = 1$

$F(-1) = P(X \le -1) = 0$

$F(4) = P(X \le 4) = 1$

Facts :

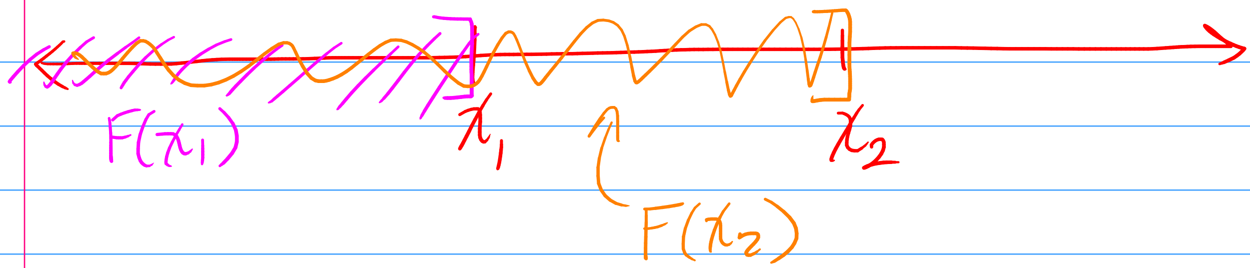
① $\boxed{0 \le F(x) \le 1}$

pf. $F(x) = \mathbb{P}(\sim\sim) \in [0,1]$

② $\boxed{\lim_{x\to-\infty} F(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} F(x) = 1}$

③ $\boxed{F \text{ is non-decreasing}}$

pf. If $x_1 < x_2$ then $F(x_1) \le F(x_2)$



$F(x_1) \quad \le \quad F(x_2)$

$= \mathbb{P}(X \le x_1)$

$= \mathbb{P}(X \in (-\infty, x_1])$

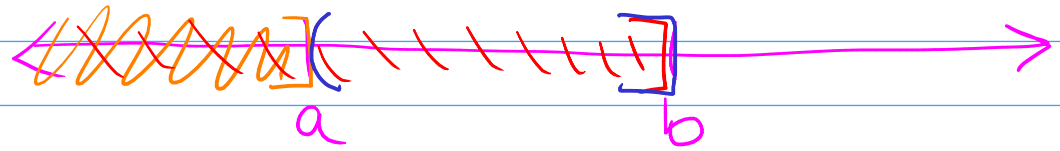
$= \mathbb{P}(\underbrace{X^{-1}((-\infty, x_1])}_{E_1}) \le \mathbb{P}(\underbrace{X^{-1}((-\infty, x_2])}_{E_2})$

If $E_1 \subset E_2$ then $\mathbb{P}(E_1) \le \mathbb{P}(E_2)$

note that $(-\infty, x_1] \subset (-\infty, x_2]$

note inverse images preserve subset relation.

(4) $\boxed{\mathbb{P}(a < X \le b) = F(b) - F(a)}$ $a<b$

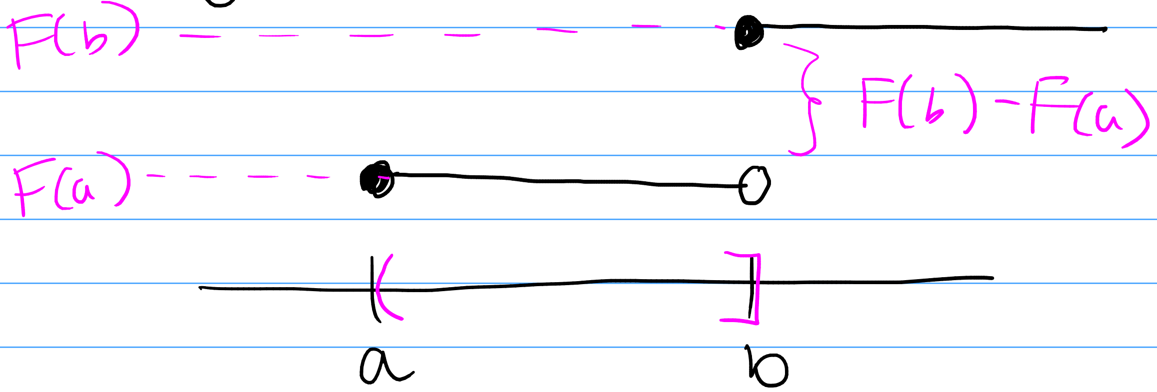


$(-\infty, b] \setminus (-\infty, a] = (a, b]$

so intuition

$F(b) - F(a) = \mathbb{P}(X \in (a, b])$

practically



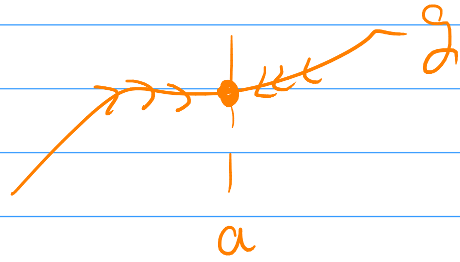
$\mathbb{P}(X = b) = \mathbb{P}(a < X \le b)$

$= F(b) - F(a)$

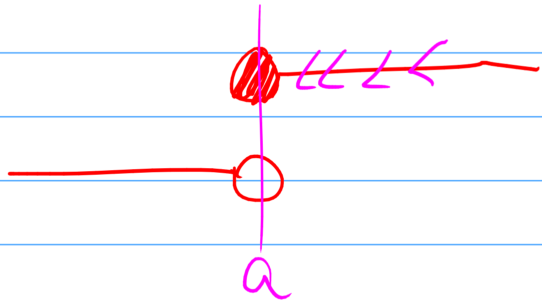
$=$ jump size

5 $\boxed{\text{F is right continuous}}$

recall cts fn g: $\lim_{x \to a} g(x) = g(a)$



$$\lim_{x \to a^+} F(x) = F(a)$$



Note: if F is cts, it right cts

Theorem: F is the CDF of some RV

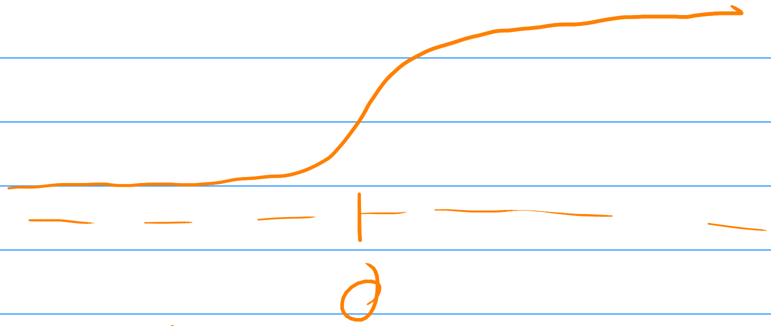
iff

1. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$
2. F is non-decreasing
3. F is right cts

Ex, Consider

$$F(x) = \frac{1}{1+e^{-x}}$$

Q: Is this a valid CDF?



Check 3 conditions.

① $$\lim_{x \to \infty} F(x) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$$\lim_{x \to -\infty} F(x) = \frac{1}{1+e^{-(-\infty)}} = 0$$

② Non-decreasing

$$\frac{dF}{dx} = \frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$$

③ Right cts?

Yes, its cts.

So Yes this is a valid CDF.

Defn: Equal in Dist

We say two RVs X and Y are equal in distribution if

$\forall A \subset \mathbb{R}$, we have

$$P(X \in A) = P(Y \in A)$$

We denote this as

$$X \overset{d}{=} Y.$$

This doesn't mean that $X = Y$.

Ex. Flip 3 coins

$$X = \#\text{ heads}$$
$$Y = \#\text{ tails}$$

There are diffent RVs

$$X(HTT) = 1, \quad Y(HTT) = 2$$

Nonetheless, $X \overset{d}{=} Y$.

$$P(X=0) = 1/8 = P(Y=0)$$
$$P(X=1) = 3/8 = P(Y=1)$$
$$\vdots$$

Theorem: $X \overset{d}{=} Y$ iff $F_X = F_Y$ (as fns)

F_X: CDF X $\quad$ F_Y: CDF Y

Ex. Toss a coin (independently) until a H appears.

$$S = \{H, TH, TTH, TTTH, TTTTH, \dots\}$$

Let p be the prob of a H and let

X = # flips until I get H

$s \in S$	$X(s)$
H	1
TH	2
TTH	3
TTTH	4
⋮	⋮

support X = $1, 2, 3, 4, \dots$

Q: What's the CDF?

$$F(x) = \mathbb{P}(X \le x)$$

Let's work with $\mathbb{P}(X = i)$

Let H_i = i^{th} flip is a H, $T_i = H_i^c$

$$\text{"}X = i\text{"} = T_1 T_2 T_3 \cdots T_{i-1} H_i$$

$$P(X=i) = P(T_1 T_2 \cdots T_{i-1} H_i)$$

$$= P(T_1)P(T_2)\cdots P(T_{i-1})P(H_i)$$

$x = 1, 2, 3, \ldots$

$$= (1-p)(1-p)\cdots(1-p)\,p$$

$$= (1-p)^{i-1}\,p$$

$$\text{"}X \le x\text{"} = \text{"}X=1\text{"} \cup \text{"}X=2\text{"} \cup \cdots \cup \text{"}X=x\text{"}$$

$$P(X \le x) = P(X=1) + P(X=2) + \cdots + P(X=x)$$

$$= \sum_{i=1}^{x} P(X=i)$$

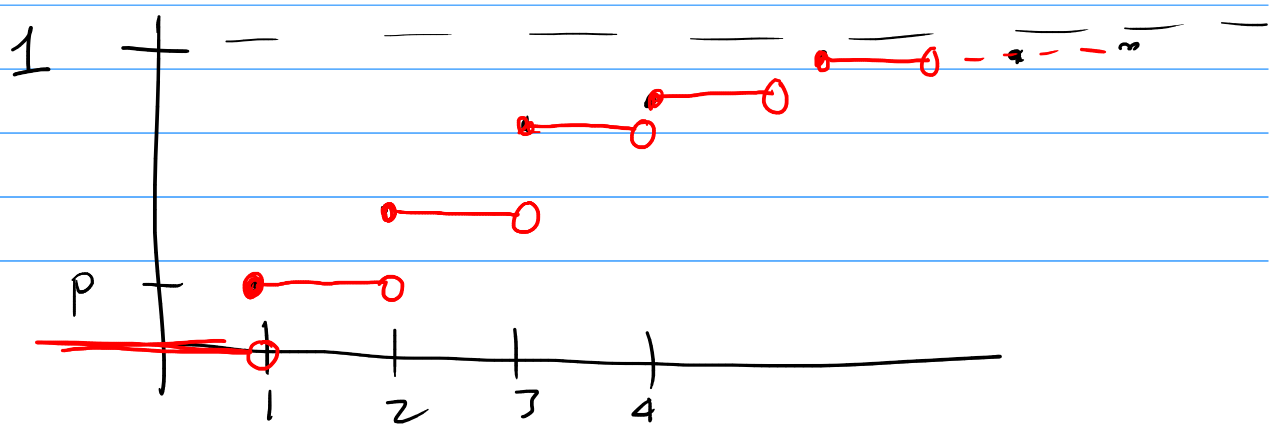
$$= \sum_{i=1}^{x} (1-p)^{i-1}\,p$$

$$\boxed{\text{Geometric Sum} \quad \sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}}$$

$$= p \sum_{i=0}^{x-1} (1-p)^i \qquad r = 1-p \quad x = n$$

$$= \cancel{p}\,\frac{1-(1-p)^x}{\cancel{1-(1-p)}}$$

$$F(x) = P(X \le x) = 1-(1-p)^x \quad \text{for } x = 1, 2, 3, \ldots$$



$$F(x) = \begin{cases} 0, & x < 1 \\ 1-(1-p)^{\lfloor x \rfloor}, & x \geq 1 \end{cases}$$

$\lfloor x \rfloor$ = round down

Defn: Discrete/Cts RVs

A discrete RV has a step-fn CDF

A cts RV is one whose CDF is a continuous fn