Lecture 5

Ex. Flip a coin twice,

What's the prob of $\underbrace{\text{getting a H and T}}_{E}$.

Option 1: S unordered

$$S = \{HH, TT, HT\}$$

$E = \{HT\}$ so $\mathbb{P}(E) = |E|/|S| = 1/3$

Option 2: S ordered

$$S = \{HH, TT, HT, TH\}$$

$E = \{HT, TH\}$
so $\mathbb{P}(E) = 2/4 = 1/2$

Point of counting is that

$$\mathbb{P}(E) = |E|/|S|$$

if outcomes in S are equally likely

General rule:

If I build S through a seq. of independent actions then typically an ordered S makes sense.

When sampling w/ repl. this matters,
Sampling w/o replacement

$$\mathbb{P}(E) = \frac{|E|\, r!}{|S|\, r!}$$

Ex. Survey W&M students, ask about political afil.

gender	A	B	
men	501	238	739
women	782	123	905
		361	1644

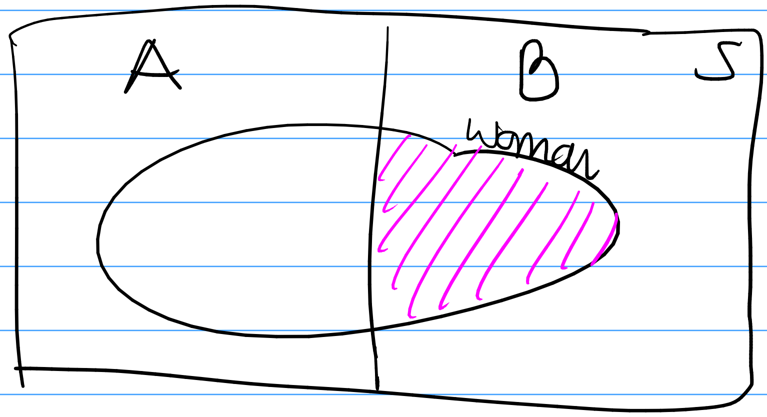
Q1: If I randomly select student, what's the prob they are a woman?

$$\mathbb{P}(\text{women}) = 905/1644$$

Q2: Given student is in B, what's the prob. they are woman?

$$P(\text{woman } \underline{\text{given}}\ B) = 123/361$$

Venn Diagram



Q1:

$$P(\text{woman}) = \frac{\bigcirc}{\square}$$

Q2:

$$P(\text{woman given } B) = \frac{D}{\square} = \frac{\bigcirc \cap \boxed{B}}{\boxed{B}}$$

Defn: Conditional Prob.

If $A, B \subset S$, and $\mathbb{P}(B) > 0$ then the conditional prob. of A given B is

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$$

"given"

Facts: ① $\mathbb{P}(B|B) = 1$

pf. $\mathbb{P}(B|B) = \frac{\mathbb{P}(BB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$

② If $AB = \emptyset$ then $\mathbb{P}(A|B) = 0$

pf $\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(B)} = \frac{0}{\mathbb{P}(B)}$

$= 0$

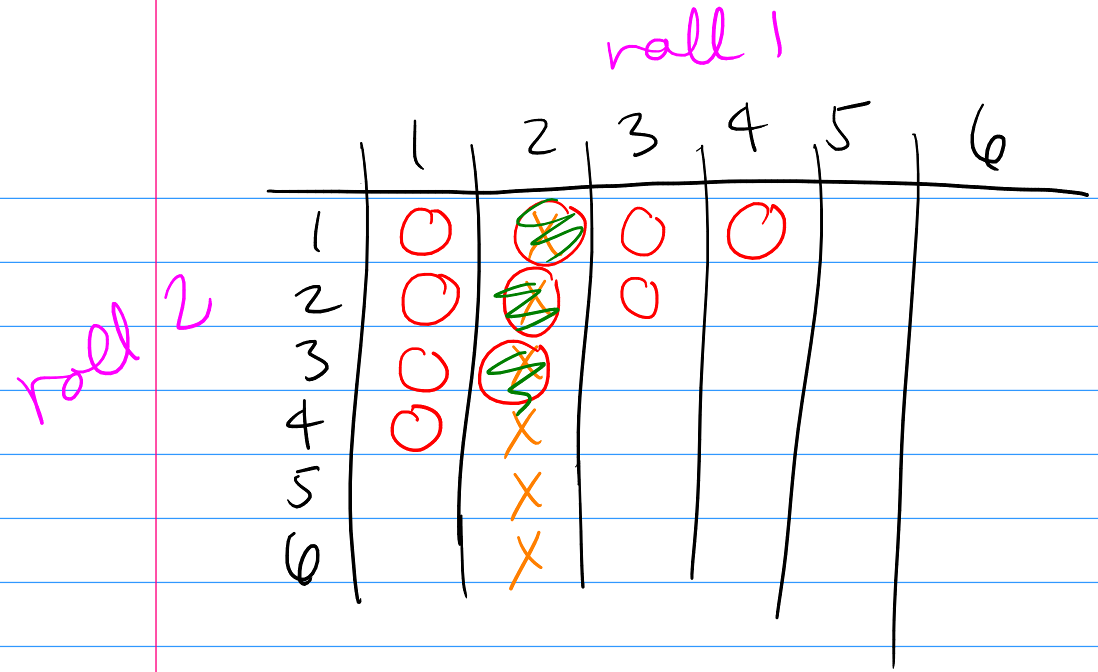
Ex. Roll two dice.

Q: what's the prob. $\underbrace{\text{the first is a 2}}_{A}$, GIVEN the $\underbrace{\text{sum of them is} \le 5}_{B}$.

$$\mathbb{P}(A|B) = \mathbb{P}(AB) / \mathbb{P}(B)$$

$$= \frac{|AB| / |S|}{|B| / |S|}$$

$$= |AB| / |B|$$



So $|AB| = 3$, $|B| = 10$

So $\mathbb{P}(A|B) = 3/10$

Theorem: Compound Prob.

Let $\mathbb{P}(A), \mathbb{P}(B) > 0$

then
$$\mathbb{P}(AB) = \mathbb{P}(A|B)\,\mathbb{P}(B)$$
$$= \mathbb{P}(B|A)\,\mathbb{P}(A)$$

Pf: $\mathbb{P}(A|B) = \mathbb{P}(AB)/\mathbb{P}(B)$

so rearrange to get

$$\mathbb{P}(AB) = \mathbb{P}(A|B)\,\mathbb{P}(B).$$

Recall: partitioning theorem,

If (A_i) partition S then

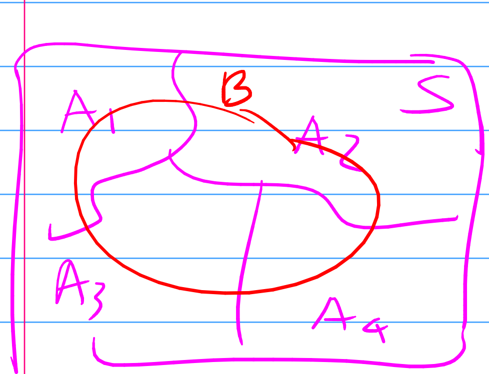
$$\mathbb{P}(B) = \sum_i \mathbb{P}(BA_i)$$

Theorem: Law of Total Prob.

If (A_i) partition S and $\mathbb{P}(A_i) > 0$ then for $B \subset S$, we have

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)$$

pf



$$\mathbb{P}(B) = \sum_i \mathbb{P}(BA_i)$$

$$= \sum_i \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)$$

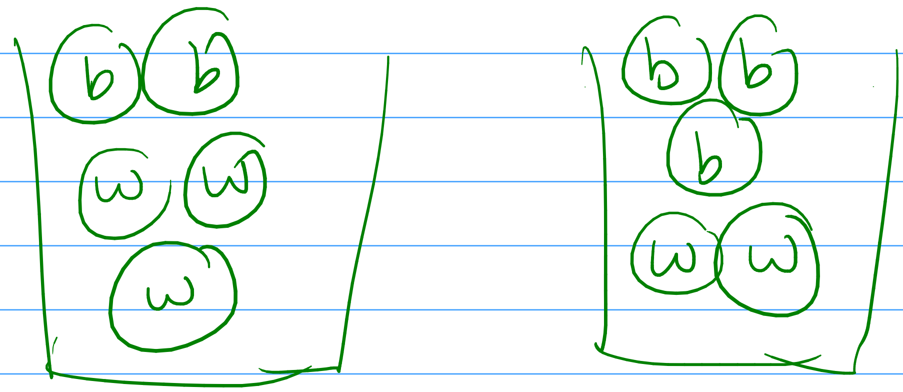
Special Case: A and A^c partition S

so Law of Total prob says

$$\mathbb{P}(B) = \mathbb{P}(B \mid A)\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\mathbb{P}(A^c)$$

Ex.

Basket 1 Basket 2



Game: (1) randomly select ball from basket 1 and place in basket 2

(2) randomly select ball from basket 2

Q: What's the prob I select a black ball on step 2?

W = choose (w) on step 1

W^c = " (b) "

B = choose (b) on step 2

B^c = " (w) "

Want: $\mathbb{P}(B)$

Solve: conditioning on W

Law of total prob says

$$\mathbb{P}(B) = \mathbb{P}(B|W)\mathbb{P}(W) + \mathbb{P}(B|W^c)\mathbb{P}(W^c)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{3}{5}\right) + \left(\tfrac{2}{3}\right)\left(1 - \tfrac{3}{5}\right) = \tfrac{17}{30}$$

$\mathbb{P}(B|W) = 1/2$ $\qquad$ $\mathbb{P}(B|W^c) = 4/6 = 2/3$

Given W

basket 2

Given W^c.

basket 2

<u>Theorem</u>: Bayes' Theorem

How to calculate $\mathbb{P}(A|B)$ from $\mathbb{P}(B|A)$?

If A, B CS, $\mathbb{P}(A), \mathbb{P}(B) > 0$ then

$$\mathbb{P}(A|B) = \mathbb{P}(B|A)\frac{\mathbb{P}(A)}{\mathbb{P}(B)}$$

Pf: $$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

Ex. Continue prev. example.

Given I choose a black ball on second step, what is the prob I chose a white on first?

By Bayes' we have that

$$\mathbb{P}(W|B) = \frac{\mathbb{P}(B|W)\,\mathbb{P}(W)}{\mathbb{P}(B)}$$

$$= \frac{(1/2)(3/5)}{17/30}$$

Theorem: Law of total prob + Bayes'

If (A_i) partition S, $\mathbb{P}(A_i) > 0$, $\mathbb{P}(B) > 0$

then

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\,\mathbb{P}(A_i)}{\sum_j \mathbb{P}(B|A_j)\,\mathbb{P}(A_j)}$$

pf. $$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\mathbb{P}(B)}$$ [bayes']

$$= \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_j \mathbb{P}(B|A_j)\mathbb{P}(A_j)}$$ [Law of tot. prob.]

Special Case:

A and A^c partition S so

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)}$$

Ex. COVID has prevalence rate of 1%

We test for COVID and get a + or −.

↳ Test accurately report a + 95% (sensitivity)

↳ Test accurately report a − 99% (specificity)

Q: I get a + test, what's the prob you have COVID?

D = have covid
D^c = don't have covid

$\mathbb{P}(D) = .01$
$\mathbb{P}(D^c) = .99$

$+$ = pos result
$-$ = $+^c$ = neg result

$\mathbb{P}(+ \mid D) = .95 \qquad \mathbb{P}(- \mid D) = .05$

$\mathbb{P}(- \mid D^c) = .99 \qquad \mathbb{P}(+ \mid D^c) = .01$

Want: $\mathbb{P}(D \mid +)$

according to rule! $\rightarrow$ $A = D$, $A^c = D^c$
$B = +$, $B^c = -$

$$\mathbb{P}(D \mid +) = \frac{P(+ \mid D) P(D)}{P(+ \mid D) P(D) + P(+ \mid D^c) P(D^c)}$$

$$= \frac{(.95)(.01)}{(.95)(.01) + (.01)(.99)} \approx .49$$

Independence

Laymen's defn:
→ things don't affect each other
→ knowing occurence of event doesn't change prob. of another

Defn: Independence

If $A, B \subseteq S$, we say "A is independent of B" denoted $A \perp B$ if

$$P(AB) = P(A)P(B).$$

→ distributive law for $\mathbb{P}$ over intersection

→ justifies product notation for intersection

Theorem: If $A \perp B$ then

$$P(A|B) = P(A).$$

Pf. $P(A|B) = \dfrac{P(AB)}{P(B)} = \dfrac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A).$