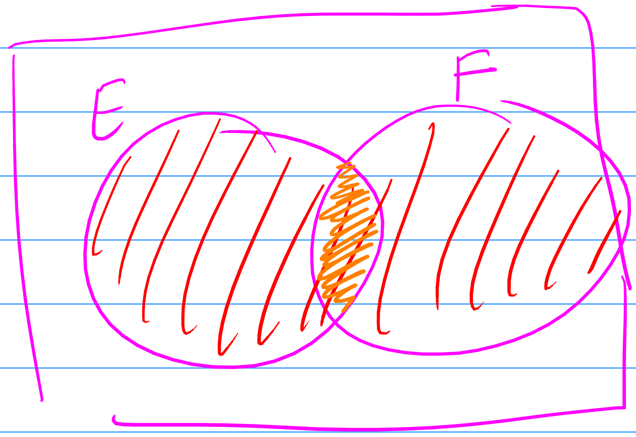
Lecture 3: More Basic Theorems

Theorem: Let $E, F \subset S$ (may not be disjoint)

then
$$P(E \cup F) = P(E) + P(F) - P(EF)$$



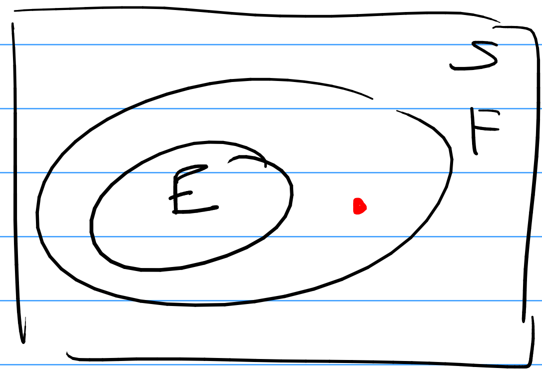
pf. $E \cup F = \underbrace{E \cup FE^c}_{\text{disjoint}}$

by additivity:

$$P(E \cup F) = P(E) + P(FE^c)$$
$$= P(E) + P(F) - P(EF)$$

Theorem: If $E \subset F \subset S$

$P(E) \le P(F)$.



pf. By axiom 1
$P(FE^c) \geq 0$

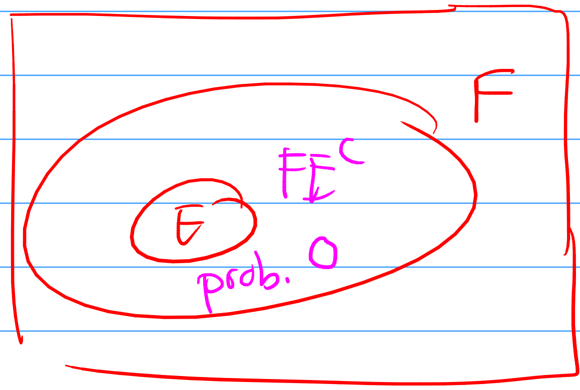
and so $P(F) - P(EF) \geq 0$

thus $P(F) - P(E) \geq 0$

all together, $P(E) \le P(F)$.

Consider $E \subset F$ but $E \neq F$
(proper subset)

~~is it true that $P(E) < P(F)$?~~



<u>Said:</u> $P(E \cup F) = P(E) + P(F) - \underbrace{P(EF)}_{\geq 0}$

$$\leq P(E) + P(F)$$

<u>Generalize!</u> <u>Boole's Inequality</u>

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i)$$

<u>pf sketch</u>

Replace E_i w/ B_i where

(1) $\bigcup_i E_i = \bigcup_i B_i$

(2) B_i are disjoint

$B_1 = E_1$, $B_2 = E_2 E_1^c$, $B_3 = E_3 E_2^c E_1^c$, ...

① and ② are true.

Further, $B_i \subset E_i$. $[P(B_i) \le P(E_i)]$

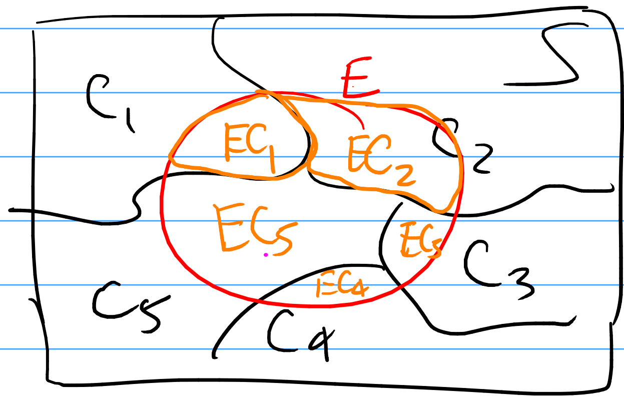
Thus,

$$P\left(\bigcup_i E_i\right) = P\left(\bigcup_i B_i\right) = \sum_{i=1}^{\infty} P(B_i) \le \sum_{i=1}^{\infty} P(E_i).$$

<u>Theorem</u>: If (C_i) partition S

if $E \subset S$ then

$$P(E) = \sum_i P(EC_i).$$



Pf. ① EC_i partition E

i.e. $E = \bigcup_i EC_i$

② by additivity,

$$P(E) = P\left(\bigcup_i EC_i\right) = \sum_i P(EC_i).$$

Equally likely outcomes

Consider a sample space S

$$S = \{s_1, s_2, \dots, s_n\}, \quad |S| = n.$$

and assume that

$$\mathbb{P}(\{s_i\}) = \mathbb{P}(\{s_j\}) \quad \forall i, j$$

then it must be that

$$\mathbb{P}(\{s_i\}) = 1/n \quad \forall i.$$

True because:

$$1 = \mathbb{P}(S) = \mathbb{P}\Big(\bigcup_i \{s_i\}\Big)$$
$$= \sum_{i=1}^{n} \mathbb{P}(\{s_i\})$$
$$= n\,\mathbb{P}(\{s_i\})$$

so $\mathbb{P}(\{s_i\}) = 1/n$

More generally,

$$\mathbb{P}(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S} = \frac{|E|}{|S|}$$

Ex. Roll a six-sided die,

$$S = \{1, 2, \ldots, 6\}$$

and $E = \{1, 2\}$

then $P(E) = \frac{|E|}{|S|} = 2/6 = 1/3$

Counting

Ex. An experiment has 3 factors

1. 2 temp settings
2. 2 pressure settings
3. 4 humidity settings

Q: How many possible experiments are there?

$$16 = 2 \cdot 2 \cdot 4$$

Fundamental Theorem of Counting (FTC)

If I have a task that consists of k sub-tasks where sub-task i can be done in n_i ways.

Then the total number of ways to do the task is

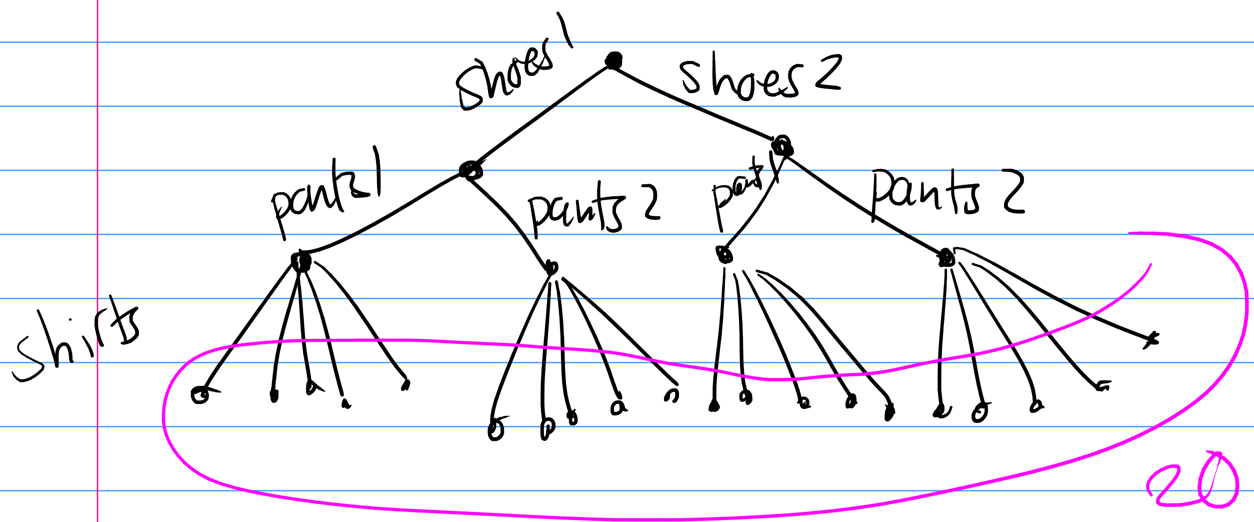
$$n_1 \cdot n_2 \cdot n_3 \cdot \dots \cdot n_k$$

$$= \prod_{i=1}^{k} n_i$$

Ex. A man has 5 shirts, 2 pair pants, 2 pair shoes.

How many outfits does he have?

By FTC there are $20 = 5 \cdot 2 \cdot 2$ outfits.



Ex. I have a deck of 52 cards.

I shuffle them so that each ordering is equally likely

Q: What's the prob. that the cards are in order?

→ A-K, C, D, H, S

E = in order

S = all possible shuffles

$$P(E) = \frac{|E|}{|S|} \quad |E| = 1$$

How to count $|S|$?

Use FTC:

task #	task	# ways
1	choose 1st	52
2	" 2nd	51
3	" 3rd	50
⋮	⋮	⋮
52	" 52nd	1

multiply

So $|S| = 52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1$

and hence

$$P(E) = 1/52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1$$

Defn **Factorial**

For any non-neg. integer n, we define n factorial as

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$
$$= \prod_{i=1}^{n} i$$

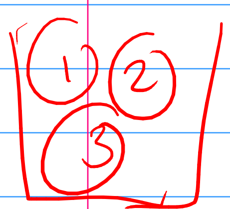
$0! = 1$

In prev. example,

$$P(E) = 1/52!$$

Sampling w/ and w/o replacement/ordering

Ordering



draw 1	draw 2
1 3 2	2 1 3

Q: are these two samples different?

w/ ordering: Yes.

w/o ordering: No.

Replacement

Can I draw the sample 1 1 2 ?

w/ replacement: Yes.

w/o replacement: No.

4 options for sampling

	w/o repl.	w/ repl.
ordered	1	2
un-ordered	4	3

Defn: Permutation

A permutation is an ordering of objects.

Ex, 1 2 3

permutations

1 2 3 1 3 2
2 1 3 2 3 1
3 2 1 3 1 2

$\Big\}\ 6 = 3!$

Theorem: The number of ways to permute n items is $n!$

pf: Use FTC

task#	task	# ways
1	choose 1st	n
2	" 2nd	$n-1$
3	" 3rd	$n-2$
⋮	⋮	⋮
n	" nth	1

multiply $= n!$

Theorem: If I have n items and I draw a sample of size r w/o replacement and w/ ordering.
The number of ways to do this is

$$\frac{n!}{(n-r)!}$$

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pf. FTC

task#	task	# ways
1	draw 1st	n
2	" 2nd	$n-1$
⋮	⋮	⋮
r	" rth	$n-r+1$

by FTC, multiply to get

$$n(n-1)(n-2)\cdots(n-r+1)$$

$$\frac{n!}{(n-r)!} = \frac{\boxed{n(n-1)(n-2)\cdots(n-r+1)}\cancel{(n-r)}\cancel{\cdots}\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

<u>Ex.</u> I form a committee from $n = 10$ students of size $\boxed{r=3}$.

where the committee consists of

Pres, VP, treasurer

How many ways can I do this?

$$10!/(10-3)! = \frac{10\cdot 9\cdot 8\cdot \cancel{7\cdots 3\cdot 2\cdot 1}}{\cancel{7\cdot 6\cdot 5}\cdots}$$

$$= 10\cdot 9\cdot 8 = 720$$