Lecture 6 : Independence

Ex. Roll two dice (independently)

$\mathbb{P}(\text{at least one } 6)$

$= 1 - \mathbb{P}(\text{no } 6\text{s})$

$A_1 = \text{no } 6 \text{ on first roll}$

$A_2 = \quad " \quad \text{second roll}$

$= 1 - \mathbb{P}(A_1 A_2)$

$= 1 - \mathbb{P}(A_1)\mathbb{P}(A_2)$

$= 1 - (5/6)(5/6)$

$= 11/36$

Counting Perspective

Sampling twice $(r=2)$ from $\{1, \ldots, 6\}$
$(n=6)$

w/ replacement.

Ordered : $|S| = 6^2 = 36$

$E =$ "at least one 6"

$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5)\}$

$|E| = 11$

$\mathbb{P}(E) = 11/36$

Unordered:

$|S| = \binom{n+r-1}{r} = \binom{6+2-1}{2} = \binom{7}{2} = 21$

$E = \{\{1,6\}, \{2,6\}, \{3,6\}, \{4,6\}, \{5,6\}, \{6,6\}\}$

$|E| = 6$

So $\mathbb{P}(E) = 6/21 \neq 11/36$

Ex. Roll two dice (independently)

$E = \{$ 1 or 2 on first,
3, 4, 5 on second $\}$

Ordered: $|S| = n^r = 6^2 = 6 \cdot 6$

$E = \{1,2\} \times \{3,4,5\}$

$|E| = |\{1,2\}| \cdot |\{3,4,5\}| = 2 \cdot 3$

Overall:

$$\mathbb{P}(E) = \frac{2 \cdot 3}{6 \cdot 6} = \left(\frac{2}{6}\right)\left(\frac{3}{6}\right)$$

prob of 1 or 2 on first

prob. of 3, 4 or 5 second

Theorem: Independence and Complements

If $A \perp B$ then

1. $A \perp B^c$
2. $A^c \perp B$
3. $A^c \perp B^c$

pf: 1. $\mathbb{P}(AB^c)$

$$= \mathbb{P}(A) - \mathbb{P}(AB)$$
$$= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)$$
$$= \mathbb{P}(A)(1 - \mathbb{P}(B))$$
$$= \mathbb{P}(A)\mathbb{P}(B^c).$$

Defn: Mutual Independence

Generalize indep. to multiple events

If (A_i) is a seq of events, we say they are (mutually) independent if for all subsequences $A_{i_1}, A_{i_2}, \dots, A_{i_k}$

$$\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} \mathbb{P}(A_{i_j})$$

Q: Do I really need $\underline{\underline{\text{all}}}$ subsequences? Yes

Could I just check

$$\mathbb{P}(A_1 A_2 A_3 \cdots A_n) = \mathbb{P}(A_1) \cdots \mathbb{P}(A_n)\,?$$

No.

Ex. Roll two dice

$|A| = 6$

A = "doubles" $= \{(1,1), (2,2), \ldots, (6,6)\}$

B = "Sum is between 7 and 10"

$= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$
$(2,6), (3,5), (4,4), (5,3), (6,2),$
$(3,6), (4,5), (5,4), (6,3),$
$(4,6), (5,5), (6,4)\}$

$|B| = 18$

C = "Sum is 2, 7 or 8"

$= \{(1,1), \quad \}$

$|C| = 12$

Are these events mutually independent?

✓ $\mathbb{P}(ABC) = \mathbb{P}(\{(4,4)\}) = 1/36$

$= \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$

$= (1/6)(\frac{1}{2})(1/3) = 1/36$

Consider B and C,

$$P(BC) = 11/36 \neq P(B)P(C)$$
$$(\tfrac{1}{2})(1/3) = 1/6$$

Not mutually independent.

<u>Defn</u>: Pairwise Independent

A seq (A_i) are pairwise independent if

$$P(A_iA_j) = P(A_i)P(A_j) \text{ for } i \neq j.$$

Can $A \perp A$?

$$P(A) = P(AA) = P(A)P(A) = P(A)^2$$

$P(A) \in [0,1]$

Works if $P(A) = 0$ or 1

Pairwise Independent $\neq$ Mutual Independence

<u>Ex.</u>

$S = \{abc, acb, bac, cab, cba, bca, aaa, bbb, ccc\}$

$|S| = 9$, all equally likely

$A_i = i^{th}$ letter is an "a"

$A_1 = \{abc, acb, aaa\}$

$A_2 = \{bac, cab, aaa\}$ $\qquad |A_i| = 3$

$A_3 = \{bca, cba, aaa\}$

Pairwise Indep?

$$\underbrace{P(A_iA_j)}_{P(\{aaa\}) = 1/9 = \frac{1}{3}\cdot\frac{1}{3}} = \overbrace{P(A_i)}^{3/9}\overbrace{P(A_j)}^{3/9} \quad i \neq j$$

Mutually Independent?

$$P(A_1A_2A_3) = P(A_1)P(A_2)P(A_3)$$

$$\underbrace{P(\{aaa\})}_{1/9} \neq \underbrace{\tfrac{1}{3}}\cdot\underbrace{\tfrac{1}{3}}\cdot\underbrace{\tfrac{1}{3}} = 1/27$$

Ex. JWST had ~400 points of failure

$$\boxed{T1} \to \boxed{T2} \to \boxed{T3} \to \cdots \to \boxed{T400}$$

JWST fails if any task fails.

W_i = i^{th} task succeeds

W_i^c = i^{th} task fails

Assume tasks independent and $P(W_i^c) = 1/1000$

$$\mathbb{P}(\text{JWST works})$$

$$= P\left(\bigcap_{i=1}^{400} W_i\right) = \prod_{i=1}^{400} P(W_i)$$

$$= \prod_{i=1}^{400} \left(1 - P(W_i^c)\right)$$

$$= \prod_{i=1}^{400} \left(1 - \frac{1}{1000}\right)$$

$$= \left(1 - \frac{1}{1000}\right)^{400}$$

$$\approx .67$$

<u>Ex.</u> Flip a coin 3 times.

X = # heads

$s \in S$	$X(s)$
H H H	3
H H T	2
H T H	2
H T T	1
T H H	2
T H T	1
T T H	1
T T T	0

Defn: Random Variable

A random variable (RV) X is a fn

$$X : S \to \mathbb{R}$$

also called a random variate,
real-valued RV,
univariate RV

$\mathbb{R}$ not $\mathbb{R}^n$

Ex. (1) toss two dice,

X = sum of dice

(2) toss a coin 25 times

X = length of longest chain of consecutive Hs

(3) observe rainfall

X = crop yield

We'd like to say,

$$\mathbb{P}(X = 1)$$

abuse of notation

Recall: $\mathbb{P} : 2^S \longrightarrow \mathbb{R}$

What we really mean

X = # heads among 3 flips

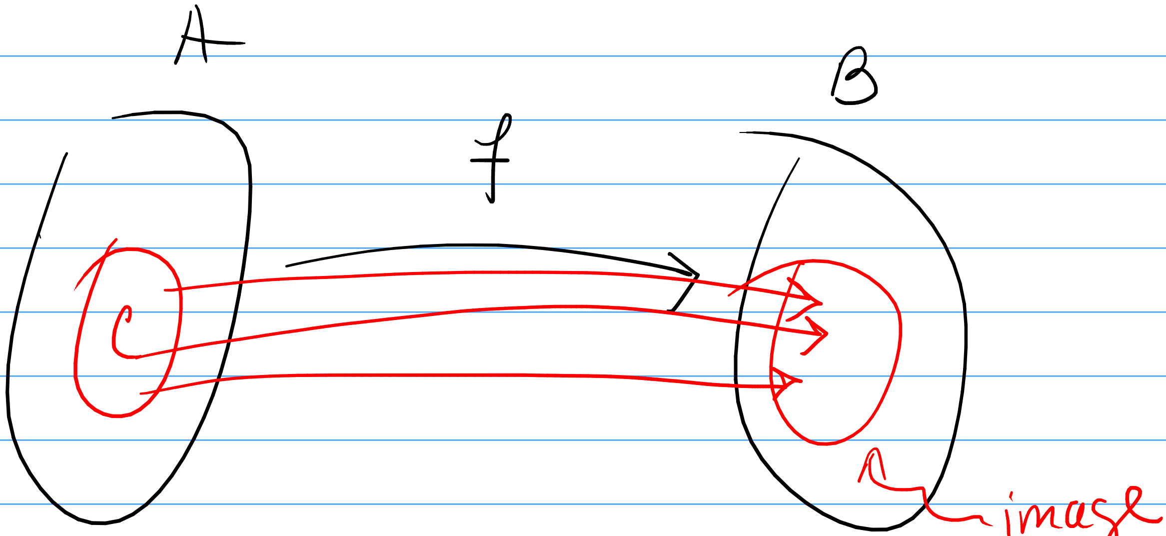
$$\mathbb{P}(X=1) = \mathbb{P}(\{HTT, THT, TTH\}) = 3/8$$

"$X=1$" short-hand for $\{s \in S : X(s)=1\} \subset S$

inverse image of $\{1\}$ under X

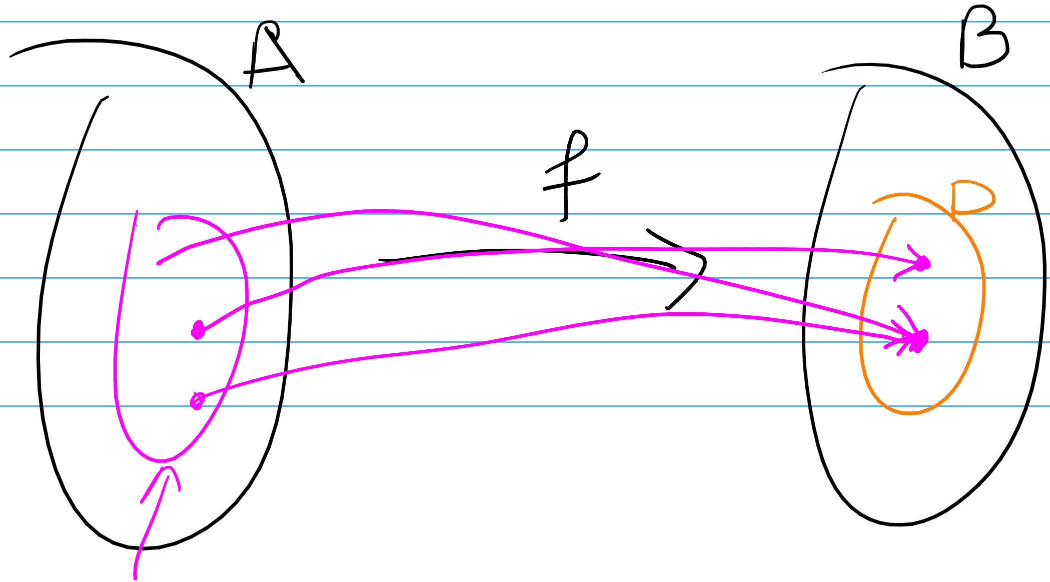
Review:

Image



$$f(C) = \{f(x),\ x \in C\}$$

Pre-image, Inverse Image



7

Inverse Image

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$

$$\text{"}X = 1\text{"} = X^{-1}(\{1\}) = \{s \in S \mid X(s) = 1\}$$

More generally, if X is a RV and $A \subset \mathbb{R}$ we can write

$$P(X \in A) \overset{\text{def}}{=} P(X^{-1}(A))$$

Ex.

$$P(X = 1 \text{ or } 2) \qquad X = \# \text{ heads } 3 \text{ flips}$$

$$= P(X \in \{1,2\})$$

$$= P(\underbrace{X^{-1}(\{1,2\})}_{\text{event} \subset S})$$

$$= P(\{HHT, THH, HTH, THT, TTH, HTT\})$$

$$= 6/8$$

Defn: Support of RV

If X is a RV its support is the set of possible values it can take on i.e. the image of S under X (range).

Ex. prev. ex.

$$\text{Support}(X) = \{0, 1, 2, 3\}$$

notice: $\mathbb{P}(X=5) = 0$

more generally, $A \subset \mathbb{R}$, $A \cap \text{Support}(X) = \emptyset$

then $\mathbb{P}(X \in A) = 0$