Lecture 5: Conditional Probability & Independence

Ex. Survey W&M students, ask about political afil.

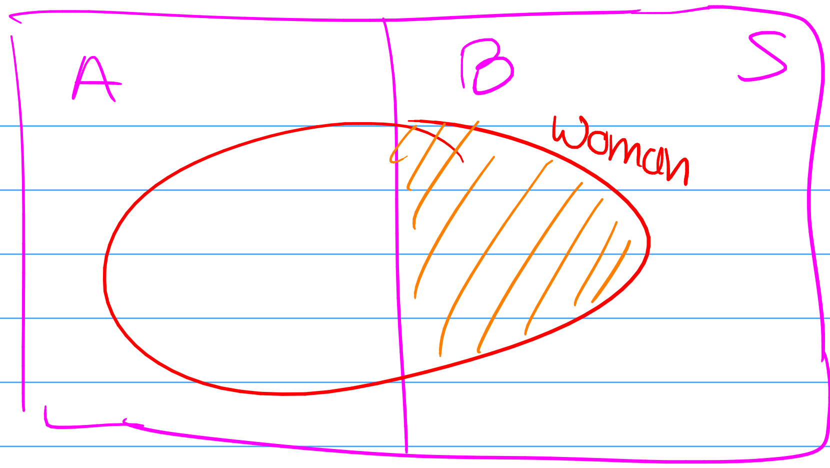
	A	B	
men	507	238	739
woman	782	123	905
		361	1644

Q1: Randomly select a W&M student, what's prob they are a woman?

$$P(\text{woman}) = 905/1644$$

Q2: Given that they are in party B, what's the prob they are a woman?

$$P(\text{woman } \underline{\text{given}} \text{ B}) = \frac{123}{361}$$



Q1: $P(\text{woman}) = \dfrac{\bigcirc}{\square}$

Q2: $P(\text{woman } \underline{\text{given}} \text{ } B) = \dfrac{D}{\boxed{B}} = \dfrac{\bigcirc \cap \boxed{B}}{\boxed{B}}$

Defn: Conditional Prob

If $A, B \subset S$, and $P(B) > 0$ then the conditional prob. of A given B is

$$P(A|B) = \frac{P(AB)}{P(B)}$$

"given"

Facts: $\boxed{(1)\ P(B|B) = 1}$

Pf

$$P(B|B) = \frac{P(BB)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

② If $AB = \emptyset$

then $P(A|B) = 0$

pf: $$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0$$

Ex Roll two dice.

Q: What's the prob the first is a 2 given the sum is ≤ 5.

A: the first is a 2

B: the sum is ≤ 5

Want: $P(A|B)$

$$= P(AB)/P(B)$$

$$= \frac{|AB|/|S|}{|B|/|S|} = \frac{|AB|}{|B|} = \frac{3}{10}$$

first roll

second roll	1	2	3	4	5	6
1	B	AB	B	B		
2	B	AB	B			
3	B	AB				
4	B	A				
5		A				
6		A				

Theorem: Compound Probability

If $P(A), P(B) > 0$ then

$$\mathbb{P}(AB) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A)$$

pf: $$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$$

multiply by $\mathbb{P}(B)$ on both sides

$$\mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(AB).$$

Recall: partitioning theorem

(A_i) partition S then

$$\mathbb{P}(B) = \sum_i \mathbb{P}(BA_i)$$

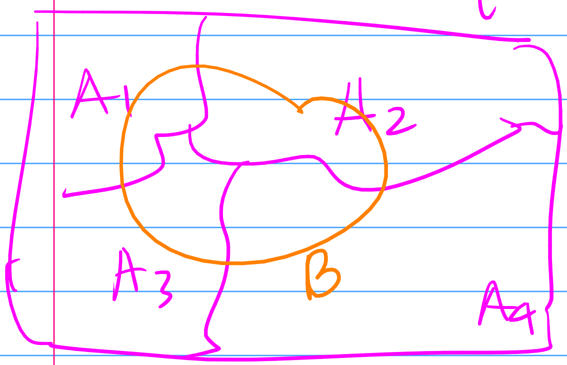
Theorem: Law of Total Prob.

If (A_i) that partition S then if $\mathbb{P}(A_i) > 0$ and $B \subset S$,

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$

pf. $\mathbb{P}(B) = \sum_i \mathbb{P}(BA_i)$ [partitioning theorem]

$= \sum_i \mathbb{P}(B|A_i)\mathbb{P}(A_i)$ [compound prob.]



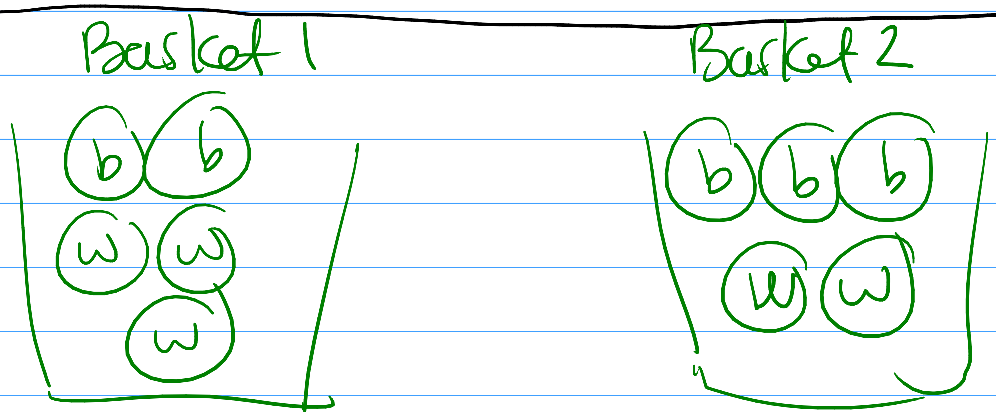
Special Case:

A and A^c always partition S

So theorem says

$$\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)$$

Ex.



Game: (1) randomly select ball from basket 1 and place in basket 2

(2) randomly draw ball from basket 2

Q: What's the prob. I draw a black ball on step 2?

$$\left.\begin{aligned} W &= \text{choose } (w) \text{ step 1} \\ W^c &= \text{'' } (b) \text{ ''} \end{aligned}\right\} \text{partition/condition}$$

B = choose (b) step 2

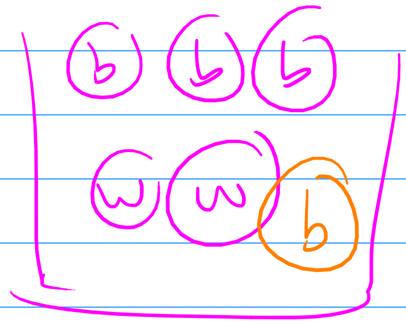
B^c = " (w) "

Law of total prob says

$$P(B) = P(B|W)P(W) + P(B|W^c)P(W^c)$$

$$\left(\tfrac{1}{2}\right)\left(\tfrac{3}{5}\right) + \left(\tfrac{2}{3}\right)\left(1-\tfrac{3}{5}\right)$$

Given W, $P(B|W) = 1/2$ | Given W^c, $P(B|W^c) = 4/6 = 2/3$



So $P(B) = 17/30$

Theorem: Bayes' Theorem

How to calc $P(A|B)$ from $P(B|A)$?

If $A, B \subset S$, $P(A), P(B) > 0$ then

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}$$

pf.

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}.$$

Ex. Prev. example,

Given I chose a black ball on second step, what's the prob. I chose a white on first?

$$P(W|B) = \frac{P(B|W)P(W)}{P(B)}$$

$$= \frac{(1/2)(3/5)}{(17/30)}$$

Theorem: Law of Tot Prob + Bayes'

If (A_i) partition S and $P(A_i) > 0$, $P(B) > 0$ then

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)}$$

pf.

$$P(A_i|B) \overset{\text{Bayes'}}{=} \frac{P(B|A_i)P(A_i)}{P(B)}$$

$$\overset{\text{Law of Tot Prob}}{=} \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)}$$

Special Case: A and A^c partition S so

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

Ex. Covid has a prevalence rate of 1%

D = have covid, D^c = no covid, $P(D) = .01$, $P(D^c) = .99$

We test for covid and get a + or −.

$+$ = pos. test result, $- = +^c$

The test accurately reports a + 95% (sensitivity) $P(+|D) = .95, P(-|D) = .05$

The test accurately reports a − 99% (specificity) $P(-|D^c) = .99, P(+|D^c) = .01$

Q: I get a test and its +

what is the prob. I have covid?

$P(\text{covid} \mid \text{pos test})$

$$= P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)}$$

$$= \frac{(.95)(.01)}{(.95)(.01) + (.01)(.99)} \approx .49$$

Laymen's idea of independence

→ things don't affect each other

→ prob. of one happening doesn't depend on occurence of other

Defn: Independence of events

If $A, B \subseteq S$ we say that "A is independent of B" denoted $A \perp B$ if

$$P(AB) = P(A)P(B).$$

→ distributive law for intersections
→ justification for intersection notation

Theorem: If $A \perp B$ then

$$P(A|B) = P(A).$$

pf.

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A).$$
