Lecture 6 : Independence

Ex. Consider rolling two dice (independently)

$\mathbb{P}$(at least one 6)

$= 1 - \mathbb{P}(\text{no } 6\text{s})$

A_1 = no 6 on first roll
A_2 = " " " second roll

$= 1 - \mathbb{P}(A_1 A_2)$

$= 1 - \mathbb{P}(A_1)\mathbb{P}(A_2)$ (indep)

$= 1 - (5/6)(5/6)$

$= 11/36$

Counting perspective.

Sample twice $(r=2)$ from $\{1, \ldots, 6\}$ $(n=6)$ w/ replacement

Ordered:

$|S| = n^r = 6^2 = 36$

E = "at least one 6"

$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$

$|E| = 11$

So $P(E) = \frac{|E|}{|S|} = 11/36$

Unordered:

$|S| = \binom{n+r-1}{r} = \binom{6+2-1}{2} = \binom{7}{2} = 21$

$E = \{\{1,6\},\{2,6\},\{3,6\},\{4,6\},\{5,6\},\{6,6\}\}$

$|E| = 6$

so $P(E) = 6/21$

Ex. Roll two dice (independently)

$E = \{$1 or 2 on first, 3,4,5 on second$\}$

Ordered counting: $n=6, r=2$

$|S| = n^r = 6^2 = 6 \cdot 6$

$E = \left\{ \begin{matrix} (1,3),(1,4),(1,5) \\ (2,3),(2,4),(2,5) \end{matrix} \right\} = \{1,2\} \times \{3,4,5\}$

$|E| = |\{1,2\}| \cdot |\{3,4,5\}| = 2 \cdot 3$

Overall:

$$P(E) = \frac{2 \cdot 3}{6 \cdot 6} = \left(\frac{2}{6}\right)\left(\frac{3}{6}\right)$$

prob of 1 or 2 on first

3,4,5 on second roll

Theorem: Complements / Independence

If $A \perp B$ then

① $A \perp B^c$

② $A^c \perp B$

③ $A^c \perp B^c$

Pf: ① $P(AB^c)$

$= P(A) - P(AB)$

$= P(A) - P(A)P(B)$

$= P(A)(1 - P(B))$

$= P(A)P(B^c)$

Defn: Mutual Independence

Generalize indep to multi events

If (A_i) are seq of events, we say they are (mutually) independent if

for any subsequence $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$

$$P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

$$= \prod_{j=1}^{k} P(A_{i_j})$$

Q: Do I really need to check *all* subsequences? Yes

Can I just check

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_n)?$$

No :(

Ex. Roll two dice

A = "doubles" $= \{(1,1),(2,2),\dots,(6,6)\}$

$|A| = 6$

B = sum is between 7 and 10

$= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$
$(2,6),(3,5),(4,4),(5,3),(6,2)$
$(3,6),(4,5),(5,4),(6,3),$
$(4,6),(5,5),(6,4)\}$

$|B| = 18$

C = sum is 2, 7, or 8

$= \{(1,1),$

$|C| = 12$

Mutually indep?

✓ $$\underbrace{P(ABC)}_{(4,9)} \stackrel{?}{=} \underbrace{P(A)}_{6/36}\underbrace{P(B)}_{18/36}\underbrace{P(C)}_{12/36}$$

$$1/36 \quad = \quad \underbrace{1/6 \quad 1/2 \quad 1/3}_{1/36}$$

$$\underbrace{P(BC)}_{\text{sum is 7 or 8}} \stackrel{?}{=} \underbrace{P(B)}_{1/2}\underbrace{P(C)}_{1/3}$$

$$11/36 \quad = \quad 1/6$$

Defn: Pairwise Independence

If (A_i) is a seq of events we say they are pairwise indep if

$$P(A_iA_j) = P(A_i)P(A_j) \quad \forall i \neq j$$

Can $A \perp A$?

$$P(A) = P(AA) = P(A)P(A) = P(A)^2$$

$P(A) \in [0,1]$

Works only if $P(A) = 0$ or 1

Mutual Independence $\neq$ Pairwise Indep.

Ex. $S = \{abc, acb, bac, bca, cab, cba, aaa, bbb, ccc\}$

↑ all outcomes equally likely

$|S| = 9$

A_i = i^{th} place in string is an "a"

$A_1 = \{abc, acb, aaa\}$

$A_2 = \{bac, cab, aaa\}$ $\quad |A_i| = 3$

$A_3 = \{cba, bca, aaa\}$

Pairwise Independent? ✓

$$\mathbb{P}(\underbrace{A_i A_j}_{\substack{aaa \\ 1/9}}) = \underbrace{\underbrace{\mathbb{P}(A_i)}_{3/9} \cdot \underbrace{\mathbb{P}(A_j)}_{3/9}}_{=\ 1/9}, \quad i \neq j$$

Mutual Independence?

$$\mathbb{P}(\underbrace{A_1 A_2 A_3}_{\substack{aaa \\ 1/9}}) \cancel{=} \underbrace{\underbrace{\mathbb{P}(A_1)}_{1/3} \underbrace{\mathbb{P}(A_2)}_{1/3} \underbrace{\mathbb{P}(A_3)}_{1/3}}_{1/27}$$

Ex. JWST, had 400 points of failure

$$\boxed{T1} \to \boxed{T2} \to \boxed{T3} \to \cdots \to \boxed{T400}$$

JSWT fails if any fail.

W_i = i^{th} task succeeds

W_i^c = i^{th} " fails

Assume all tasks are independent.

and assume $\mathbb{P}(W_i^c) = 1/1000$

$$
\begin{aligned}
\mathbb{P}(\text{JWST works}) &= \mathbb{P}(W_1 W_2 W_3 \cdots W_{400}) \\
&= \mathbb{P}\left(\bigcap_{i=1}^{400} W_i\right) \\
&= \prod_{i=1}^{400} \mathbb{P}(W_i) \\
&= \prod_{i=1}^{400} 1 - \mathbb{P}(W_i^c) \\
&= \prod_{i=1}^{400} 1 - \frac{1}{1000} = \left(1 - \frac{1}{1000}\right)^{400} \approx .67
\end{aligned}
$$

Random Variables

Ex. Flip a coin 3 times.

$X = \#$ heads in 3 flips

$s \in S$	$X(s)$
H H H	3
H H T	2
H T H	2
H T T	1
T H H	2
T H T	1
T T H	1
T T T	0

a function

Defn: Random Variable

A random variable (RV) X is a function

$$X : S \to \mathbb{R}$$

also called a random variate,

a real-valued RV

a univariate RV

co-domain $\mathbb{R}$ not $\mathbb{R}^n$

Ex. (1) toss two dice

X = sum of dice

(2) toss a coin 25 times

X = length of longest run of Hs

(3) observe rainfall

X = crop yield

We'd like to say

$$P(\underbrace{X = 1}) \qquad \text{abuse of notation}$$

recall! $\mathbb{P} : 2^S \to \mathbb{R}$

X = # heads, 3 flips

$$\mathbb{P}(X=1) = \mathbb{P}(\underbrace{\{HTT, THT, TTH\}}) = 3/8$$

$$\text{"}X=1\text{"} \longrightarrow \{s \in S \mid X(s) = 1\}$$

short-hand

inverse-image of $\{1\}$ under X

Review

~~Image~~

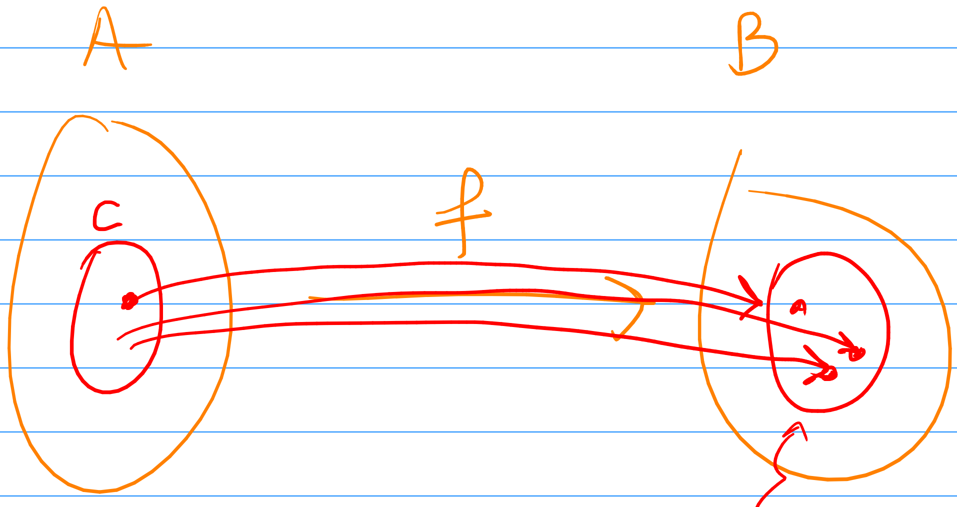
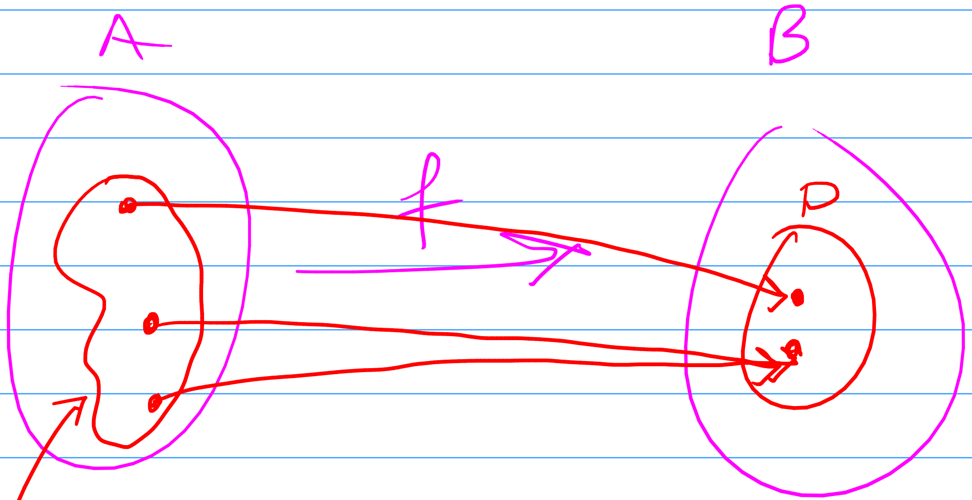


Image of C under f

$$f(C) = \{ f(x) \text{ for } x \in C \}$$

~~Inverse Image~~



inverse image

$$f^{-1}(D) = \{ a \in A : f(a) \in D \}$$

We write
$$\mathbb{P}(X \in A), \quad A \subset \mathbb{R}$$

<u>means</u>
$$\mathbb{P}(\underbrace{X^{-1}(A)}_{\text{an event } \subset S})$$