Lecture 4: More Counting

Ex. I form a committee from 10 students ($n=$) of size 3 ($r=$).

Where the committee members are

Pres, VP, treasurer.

How many ways can I do this?

Claim: w/o replacement, b/c same person can't have multiple roles

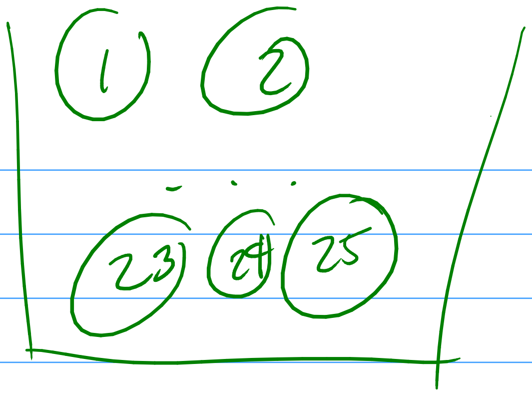
w/ order, 1^{st} = Pres, 2^{nd} = VP, 3^{rd} = treasurer

Then by our theorem there are

$$\frac{n!}{(n-r)!} = \frac{10!}{7!} = \frac{10\cdot 9\cdot 8\cdot 7!}{7!} = 10\cdot 9\cdot 8$$
$$= 720$$

Ex. Lotto.

Basket w/ 25 ($=n$) numbered balls



draw 4 ($=r$) of them (all such draws equally likely)

Guess! 1 3 22 7

what's prob I win?

E = I win,

$$\mathbb{P}(E) = \frac{|E| \leftarrow 1}{|S|}$$

$$|S| = \frac{25!}{(25-4)!} = \frac{25!}{21!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21!}}{\cancel{21!}}$$

thus

$$P(E) = \frac{1}{|S|} = \frac{1}{25 \cdot 24 \cdot 23 \cdot 22}$$

Theorem: Sample w/ replacement, w/ ordering

The num. ways to sample r from n w/ repl. and w/ order is

$$n^r$$

~~pf~~ FTC:

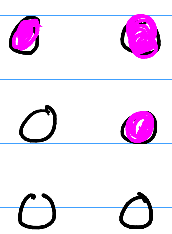
task #	task	# ways
1	Sample 1st	n
2	" 2nd	n
3	" 3rd	n
⋮	⋮	⋮
r	" rth	n

multiply

$= n^r$

Ex. Braille Alphabet

Six spots either raised or not



Q: How many braille letters do I have?

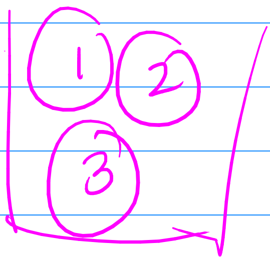
Sample $r=6$ spots from $n=2$ options (raised/not) w/ order, w/ replacement.

By theorem, $2^6 = 64$ options.

Sampling w/o replacement, w/o order.

Ex.

draw $r = 2$ from $n = 3$



If order matters

$(1,2)$ $(1,3)$ $(2,3)$

$(2,1)$ $(3,1)$ $(3,2)$

$\Big\}\ 6 = \dfrac{3!}{(3-2)!}$

w/o order

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\Big\}\ 3$

General fact:

Each unordered sample of size r can be permuted in $r!$ ways to make an ordered sample

$r=3$

$$\{1,2,3\} \to \left.\begin{array}{l}(1,2,3)\\(1,3,2)\\(3,1,2)\\ \vdots\end{array}\right\} 3! = 6$$

Fact: (w/o replacement)

$$\#\ \overset{\frac{n!}{(n-r)!}}{\text{ordered}} = r!\ \overbrace{(\#\ \text{unordered})}$$

Theorem: Sampling w/o repl, w/o ordering

I can sample r from n w/o repl, w/o ordering in

read: "n choose r"

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \text{binomial coefficient}$$

Ex. I have $\overset{=n}{10}$ proffessors, how many co-equal committees of size $\overset{=r}{4}$ can I assemble?

w/o replacement, w/o order (co-equal)

$$\binom{10}{4} = \frac{10!}{(10-4)!\,4!} = \frac{10!}{6!\,4!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!\,4!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{24} = 210$$

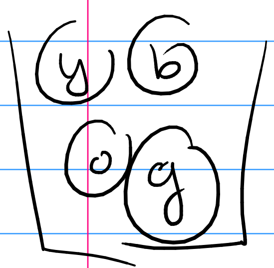
Ex. How many 5-card poker hands are there?

Sampling $r = 5$, from $n = 52$
w/o order, w/o replacement

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!}\,5!}$$

$$\cong 2.5 \text{ mil}$$

Ex. Jar w/ 4 marbles of colors
yellow, blue, orange, green.



I choose 3 from jar w/o replacement,
[all such choices equally likely]

Q: What's the prob I choose a y and b among the 3.

$E = \{ y \text{ and } b \}$

$$P(E) = \frac{|E|}{|S|}$$

$$E = \{\{y, b, o\}, \{y, b, g\}\}, \quad |E| = 2$$

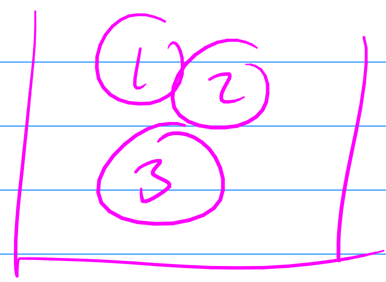
S = all such samples,

$$|S| = \binom{4}{3} = \frac{4!}{(4-3)!\,3!} = \frac{4!}{1!\,3!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 4$$

So $P(E) = \dfrac{|E|}{|S|} = 2/4 = 1/2$

<u>Sampling w/ replacement</u>

Consider $n = 3$, $r = 2$



<u>Ordered:</u> $n^r = 9$

$(2,1)$, $(1,2)$ | $(3,1)$, $(1,3)$ | $(2,3)$, $(3,2)$ | $(1,1)$ | $(2,2)$ | $(3,3)$

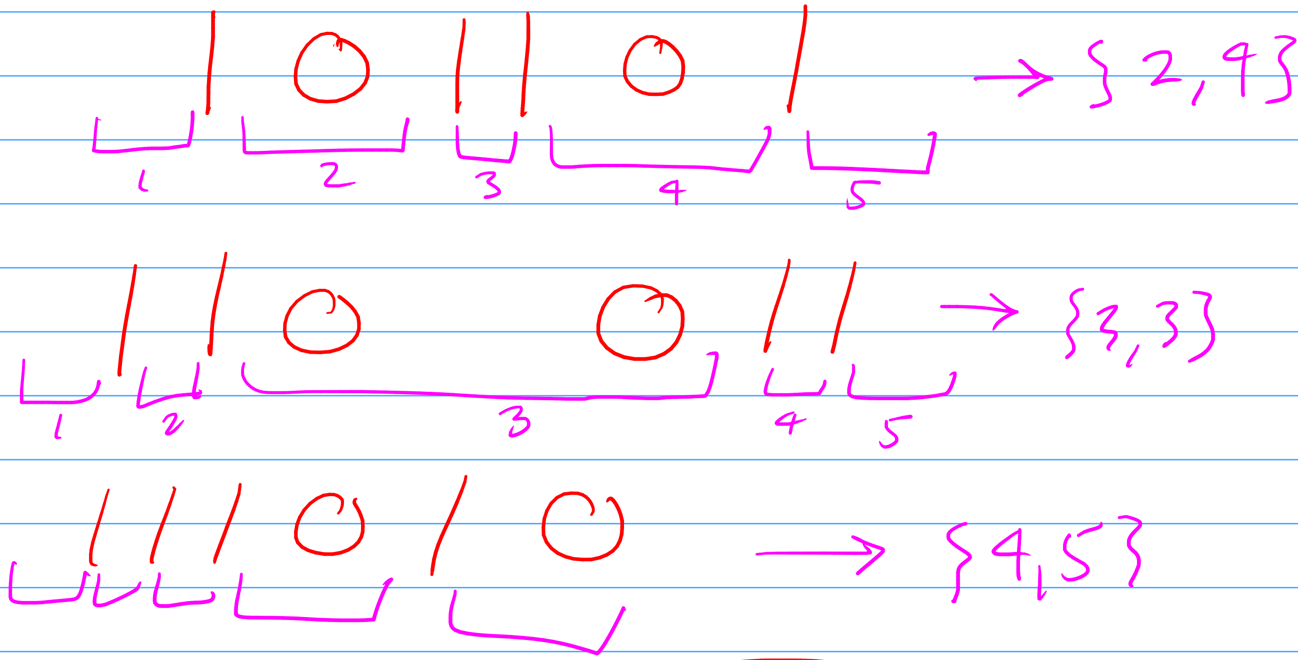
<u>Unordered</u>:

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,1\}$ $\{2,2\}$ $\{3,3\}$

6 options

Game of Partitioning

How many ways can I partition $r = 2$ objects using $n-1 = 4$ walls



1-1 corresp btwn ways to partition and samples of size r from n, w/ repl. w/o order.

How many ways to play game?

have $n-1+r$ symbols

So I can order in $(n-1+r)!$ ways

need to divide by non-distinct arrangemts,

— switch any of $n-1$ walls in $(n-1)!$

— " " r obj's in $r!$

So I get

$$\frac{(n+r-1)!}{(n-1)!\, r!}$$

distinct options.

<u>Theorem:</u>

The number of ways to sample r from n w/ repl., w/o order is

$$\frac{(n+r-1)!}{(n-1)!\, r!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

<u>Ex.</u> 10 passengers on a bus route w/ 5 stops.

Driver records # people that get off at each stop.

<u>Q</u>: How many possible records are there?

Stop	#
1	0
2	3
3	1
4	2
5	4

$\rightarrow \{2,2,2,3,4,4,5,5,5,5\}$

Sampling $r = 10$ from $n = 5$

ways : $\binom{n+r-1}{r} = \binom{5+10-1}{10} = 1001$

Ex. Jar w/ 4 marbles, y, b, o, g

Draw $r = 3$ from $n = 4$

(w/ replacement, w/o order)

Q: prob. I $\underbrace{\text{get a y and b}}_{E}$?

$$P(E) = \frac{|E|}{|S|}$$

$$E = \{\{y,b,o\},\{y,b,g\},\{y,b,b\},\{y,b,y\}\}$$

$|E| = 4$

S = all such samples,

$$|S| = \binom{n+r-1}{r} = \binom{4+3-1}{3} = \binom{6}{3} = 20$$

hence

$$P(E) = \frac{|E|}{|S|} = 4/20 = 1/5$$

---

	w/o repl	w/ repl.
Ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered	$\binom{n}{r} = \frac{n!}{(n-r)!r!}$	$\binom{n+r-1}{r}$