Lecture 16: Bivariate RVs

If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$

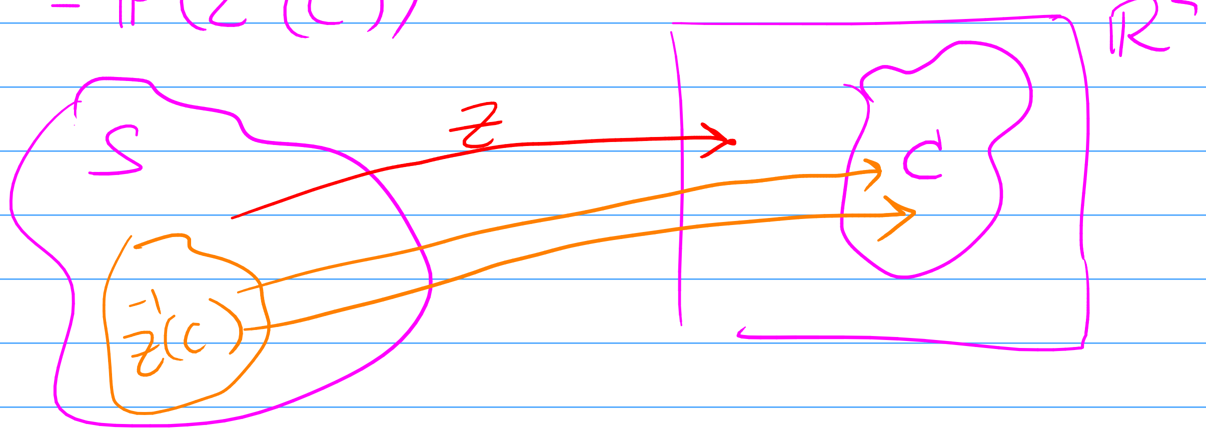
then $Z = (X, Y)$ is called a bivariate RV.

So $Z : S \to \mathbb{R}^2$ such that

$$Z(s) = (X(s), Y(s)).$$

Say: $P(Z \in C)$ where $C \subset \mathbb{R}^2$

$$= P(Z^{-1}(C))$$



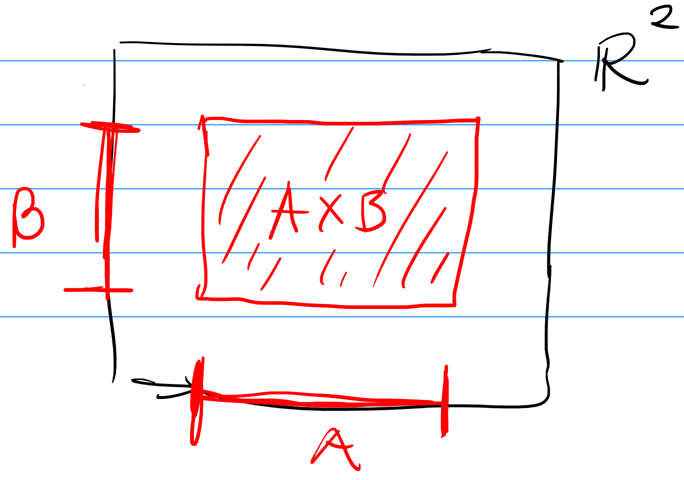
Often, $C = A \times B$ when $A, B \subset \mathbb{R}$

May write

$$P((X, Y) \in A \times B)$$

or be lazy

$$P(X \in A, Y \in B)$$ "and"



Ex. Consider flipping a coin 3 times

$$X = \begin{cases} 0 & \text{if last flip is } T \\ 1 & \text{if last flip is } H \end{cases}$$

$$Y = \#\text{ heads}$$

$Z = (X, Y)$

$s \in S$	$Z(s) \in \mathbb{R}^2$
H H H	(1, 3)
H H T	(0, 2)
H T H	(1, 2)
H T T	(0, 1)
T H H	(1, 2)
T H T	(0, 1)
T T H	(1, 1)
T T T	(0, 0)

Defn: Bivariate CDF

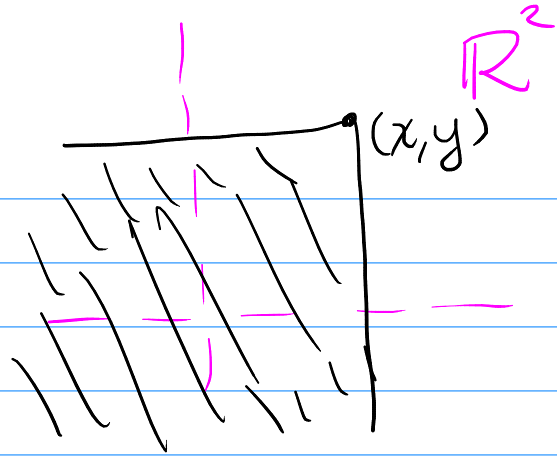
The bivariate (joint) CDF is a function

$$F : \mathbb{R}^2 \to \mathbb{R}$$

So that for $(x, y) \in \mathbb{R}^2$ then

$$F(x,y) = P(X \le x, Y \le y)$$

[uni: $F(x) = P(X \le x)$]



Properties of Joint CDF

1. $F(x,y) \geq 0$

2. $\lim_{x,y \to \infty} F(x,y) = 1$ [uni: $\lim_{x \to \infty} F(x) = 1$]

3. $\lim_{x \to -\infty} F(x,y) = 0$ (uni: $\lim_{x \to -\infty} F(x) = 0$)

   $\lim_{y \to -\infty} F(x,y) = 0$

4. F is non-decreasing and right-cts in each argument (x and y)

Defn: Marginal RVs/properties

If (X, Y) is a bivariate RV then we call X and Y individually, the marginal RVs. Their properties are also called marginal.

Theorem: Rel. btwn Joint/Marginal CDFs

① $F_X(x) = \lim_{y\to\infty} F(x,y)$

marginal CDF of X — joint CDF

② $F_Y(y) = \lim_{x\to\infty} F(x,y)$

Pf.

$$F_X(x) = P(X \le x) = P(X \le x, \overbrace{Y = \text{anything}}^{S})$$
$$= P(X \le x, Y < \infty)$$
$$= \lim_{y\to\infty} P(X \le x, Y \le y)$$
$$= \lim_{y\to\infty} F(x,y)$$

[Uni: $P(X > x) = 1 - F(x)$]

$$P(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F(x,y)$$

Defn: Joint PMF

If X and Y are discrete RVs then the joint PMF is defined as

$$f(x,y) = \mathbb{P}(X = x, Y = y)$$

[uni: $f(x) = \mathbb{P}(X = x)$]

Theorem: Valid PMF

A function f is a valid PMF iff

1. $f(x,y) \geq 0 \quad \forall x, y$

2. $\sum_x \sum_y f(x,y) = 1$

Theorem: Rel. btwn joint/marginal PMFs

(1) $f_X(x) = \sum_y f(x,y)$

marginal PMF of X

joint PMF

(2) $f_Y(y) = \sum_x f(x,y)$

Recall: A_i partition S then

$$P(B) = \sum_i P(B \cap A_i)$$

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pf: $A_y = $ "$Y = y$" $\subset S$

$B = $ "$X = x$" $\subset S$

$$f_X(x) = \mathbb{P}(X = x) = \mathbb{P}(B)$$

$$= \sum_y \mathbb{P}(B \cap A_y)$$

$$= \sum_y \mathbb{P}(\text{"}X = x\text{" and "}Y = y\text{"})$$

$$= \sum_y \underbrace{\mathbb{P}(X = x, Y = y)}_{f(x,y)}$$

Ex. Flip 3 coins

$$X = \begin{cases} 0 & \text{if last } T \\ 1 & \text{" } H \end{cases}$$

$Y = \#$ of H

	Y: 0	1	2	3	
X: 0	$f(0,0) = 1/8$	$f(0,1) = 2/8$	$f(0,2) = 1/8$	$f(0,3) = 0$	$f_X(0) = 1/2$
1	0	1/8	2/8	1/8	$f_X(1) = 1/2$
	$f_Y(0) = \frac{1}{8}$	$f_Y(1) = 3/8$	$f_Y(2) = 3/8$	$f_Y(3) = 1/8$	

Defn: Joint PDF

If X and Y are continuous RVs we call the function $f : \mathbb{R}^2 \to \mathbb{R}$ the joint density of X and Y if $\forall C \subset \mathbb{R}^2$

$$P((X,Y) \in C) = \iint_C f(x,y)\, dx\, dy$$

Facts:

① $F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,du\,dv$

[uni: $F(x) = \int_{-\infty}^{x} f(t)\,dt$]

② $f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$

[uni: $f(x) = \frac{dF}{dx}$]

③ f is a valid Joint Density iff

① $f(x,y) \geq 0 \quad \forall x,y$

② $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$

Theorem: Rel. btwn joint/marginal densities

① $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$

② $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$

Ex.

$$F(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ xy, & 0<x<1,\ 0<y<1 \\ x, & 0<x<1,\ y>1 \\ y, & 0<y<1,\ x>1 \\ 1, & x>1 \text{ and } y>1 \end{cases}$$

F(x,y) f(x,y)

1	0 0	0 x	0 1
0	0 0	1 xy	0 y
	0 0	0 0	0 0
		0	1

What's the joint density?

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$f(x,y) = \begin{cases} 1 & \text{for } 0<x<1 \text{ and } 0<y<1 \\ 0 & \text{else} \end{cases}$$