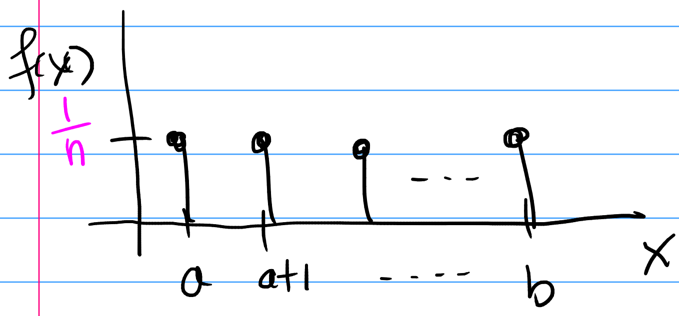
Lecture 12

Discrete Uniform

$Y \sim U(\{a, \ldots, b\})$

Let $n = b - a + 1$



$Y = X + (a-1)$

$X \sim U(\{1, \ldots, n\})$

PMF

$$f(x) = \frac{1}{n} = \frac{1}{b-a+1} \quad \text{for } x = a, \ldots, b$$

Expectation

$$E[Y] = E[X + (a-1)] = E[X] + a - 1$$

$$= \frac{n+1}{2} + a - 1$$

$$= \frac{(b-a+1)+1}{2} + a - 1$$

$$= \cdots = \boxed{\frac{a+b}{2}}$$

Variance

$$Var(Y) = Var(X + (a-1))$$

$$= Var(X)$$

$$= \frac{n^2-1}{12}$$

$$\boxed{= \frac{(b-a+1)^2-1}{12}}$$

$$M_{aX+b}(t) = e^{tb} M_X(at)$$

MGF

$$M_Y(t) = M_{X+(a-1)}(t)$$

$$= e^{(a-1)t} M_X(t)$$

$$= e^{(a-1)t} \frac{e^t - e^{t(n+1)}}{n(1-e^t)}$$

$$= e^{(a-1)t} \frac{e^t - e^{t(b-a+2)}}{(b-a+1)(1-e^t)}$$

$$\boxed{M_Y(t) = \frac{e^{at} - e^{(b+1)t}}{(b-a+1)(1-e^t)}}$$

Continuous Uniform

$X \sim U(a, b)$

PDF:

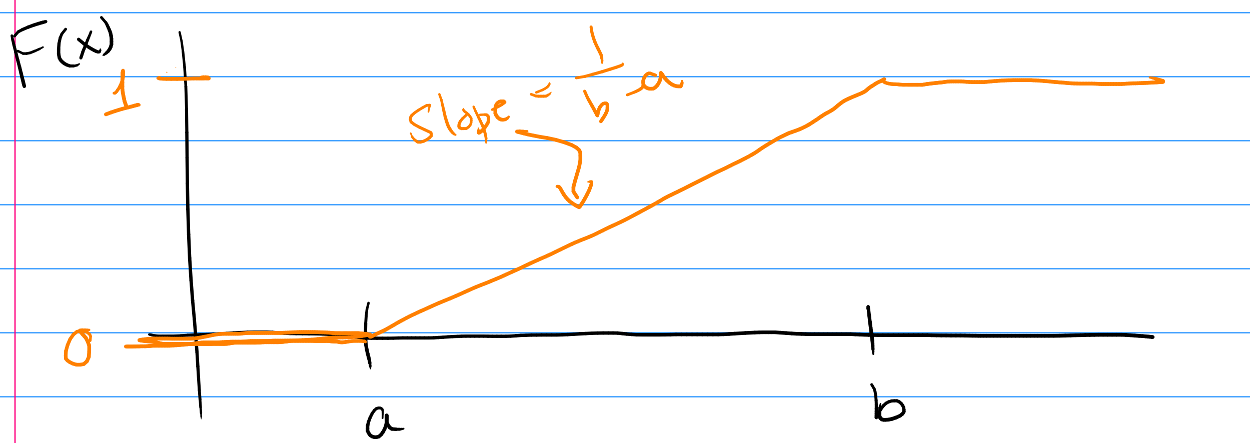
$$f(x) = \frac{1}{b-a} \qquad a < x < b$$

CDF:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt$$

$a < x < b$

$$= \frac{t}{b-a}\Big|_{a}^{x} = \frac{x-a}{b-a}$$



Expectation

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{x^2}{2}\frac{1}{b-a}\Bigg|_a^b$$

$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{(a+b)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$\boxed{\mathbb{E}[X] = \frac{a+b}{2}}$$

$$\mathbb{E}[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{x^3}{3}\frac{1}{b-a}\Bigg|_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(a^2+ab+b^2)}{3(b-a)}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \cdots \boxed{= \frac{(b-a)^2}{12}}$$

<u>MGF:</u>

$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

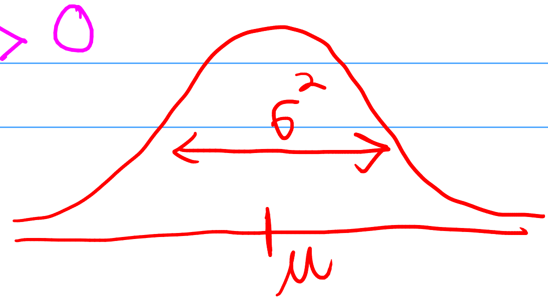
$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{t} e^{tx} \frac{1}{b-a} \Big|_a^b$$

$$\boxed{M(t) = \frac{e^{bt} - e^{at}}{t(b-a)}}$$

<u>Normal Distribution</u> $X \sim N(\mu, \sigma^2)$

$\mu \in \mathbb{R}$, $\sigma^2 > 0$



PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

for all $x \in \mathbb{R}$

CDF: $F(x) = \int_{-\infty}^{x} f(t)\,dt$ = no simple formula

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Claim: $\mathbb{E}X = \mu$ and $\mathrm{Var}(X) = \sigma^2$

MGF:

$$M(t) = \mathbb{E}[e^{tX}]$$

$$= \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

exponent:

$$tx - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$= tx - \frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)$$

$$= -\frac{1}{2\sigma^2}(-2\sigma^2 tx + x^2 - 2\mu x + \mu^2)$$

$$= -\frac{1}{2\sigma^2}\Big(\underbrace{x^2 - 2x(\mu+\sigma^2 t)}_{\text{looks like first two terms of } (x-(\mu+\sigma^2 t))^2} + \mu^2\Big)$$

$$= -\frac{1}{2\sigma^2}\left(x^2 - 2x(\mu+\sigma^2 t) + \underline{(\mu+\sigma^2 t)^2} - \underline{(\mu+\sigma^2 t)^2} + \mu^2\right)$$

$$= -\frac{1}{2\sigma^2}\left([x-(\mu+\sigma^2 t)]^2 - (\mu+\sigma^2 t)^2 + \mu^2\right)$$

$$M(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\quad) \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2 t)]^2\right) \underbrace{\exp\left(\frac{(\mu+\sigma^2 t)^2+\mu^2}{-2\sigma^2}\right)}_{\text{no } x} dx$$

$$= \exp\left(\frac{(\mu+\sigma^2 t)^2+\mu^2}{-2\sigma^2}\right) \underbrace{\int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2 t)]^2\right)}_{\text{PDF of a } N(\mu+\sigma^2 t,\, \sigma^2)} dx}_{1}$$

$= \ldots$

$$\boxed{M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}$$

$$E[X] = \left.\frac{dM}{dt}\right|_{t=0} = \left.(\mu + \sigma^2 t)\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\right|_{t=0}$$

$$= (\mu + \sigma^2(0))e^0$$

$$\boxed{E[X] = \mu}$$

$$E[X^2] = \left.\frac{d^2M}{dt^2}\right|_{t=0} =$$

$$= \left.\sigma^2 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) + (\mu + \sigma^2 t)^2 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\right|_{t=0}$$

$$= \sigma^2(1) + (\mu)^2(1)$$

$$E[X^2] = \mu^2 + \sigma^2$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$
$$= \mu^2 + \sigma^2 - (\mu)^2$$
$$= \sigma^2.$$

Theorem: Linear Transf of Normal

Let $X \sim N(\mu, \sigma^2)$ and

$$Y = aX + b$$

then $Y \sim N(a\mu + b,\ a^2\sigma^2).$

$$E[Y] = aEX + b = a\mu + b$$
$$\mathrm{Var}(Y) = a^2\,\mathrm{Var}(X) = a^2\sigma^2$$

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pf Recall ① $M_X(t) = \exp(\mu t + \sigma^2 t^2/2)$

② $M_{aX+b}(t) = e^{tb} M_X(at)$

$$M_Y(t) = e^{tb} M_X(at)$$

$$= e^{tb} \exp\left(\mu(at) + \sigma^2 (at)^2/2\right)$$

$$= \exp\left(\underbrace{(a\mu + b)}\, t + \underbrace{(a^2\sigma^2)}\, t^2/2\right)$$

Claim: this is the MGF of a $N(a\mu + b,\ a^2\sigma^2)$

Poisson Distribution

- discrete RV
- Support $\{0, 1, 2, 3, \dots\}$

Canonical experiment:

Counting the number of events that happen in some time period

Ex.
- capture fish in a river in 1 hr
- Count # mRNA molecules in a cell
- radioactive decay

$X \sim \text{Pois}(\lambda)$

$\lambda > 0$, rate of occurence of events

events

PMF:

$$f(x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots$$

Expectation

$$E[X] = \sum_{x=\cancel{0}\,1}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\frac{x}{x!} = \frac{x}{x(x-1)!} = \frac{1}{(x-1)!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-1)!}$$

$$e^{y} = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

$$= \sum_{i=0}^{\infty} y^{i}/i!$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\lambda \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}}$$

$$= e^{-\lambda}\lambda e^{\lambda}$$

$$\boxed{\mathbb{E}X = \lambda}$$