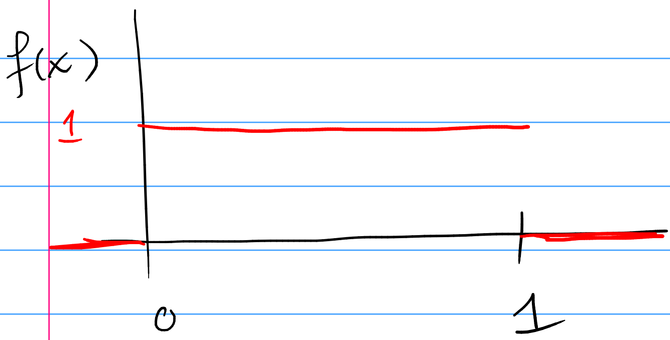
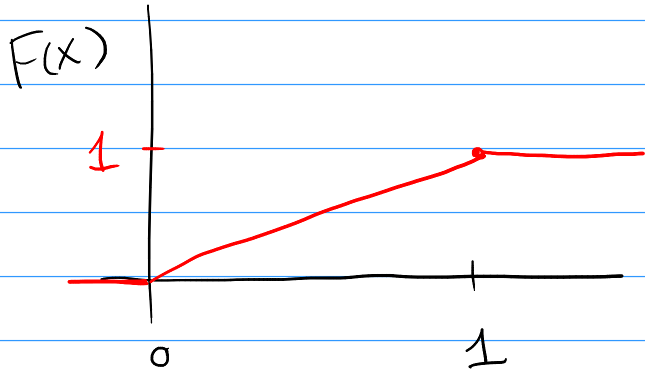
Lecture 15

OHs: Today 2:30 - 3:30
Weds 1-2 pm
Mon 10-11 am, 3-4 pm

Ex. $X \sim U(0,1)$





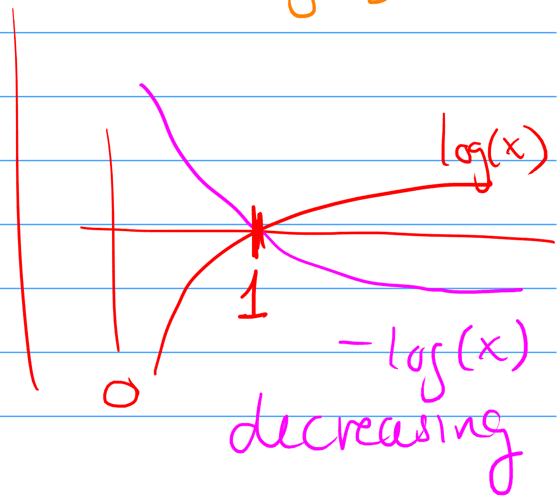
$F_X(x) = x$ for $0 < x < 1$

Let $Y = -\log X$

$g(x) = -\log(x)$

$y = -\log(x) \Rightarrow -y = \log(x) \Rightarrow e^{-y} = x$
$= g^{-1}(y)$

Apply theorem:



$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

$$= 1 - F_X(e^{-y})$$

If $0 < x < 1$ then $\log(x) < 0$

$y = -\log(x) > 0$

So $0 < e^{-y} = \frac{1}{e^y} \leq 1$

$= 1 - e^{-y} = F_Y(y)$

CDF of Exp(1)

So $Y \sim \text{Exp}(1)$

$Z \sim \text{Exp}(1)$
then
$f(z) = e^{-z}$ for $z > 0$

$F(z) = \int_0^z e^{-t}\,dt = 1 - e^{-z}$

What about PDFs?

<u>Theorem</u>: If X is cts and $Y = g(X)$

and if

1. g is invertible
2. g^{-1} is differentiable

then

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$

pf: Case 1: g increasing

Prev. CDF theorem said

- g inc
- g^{-1} inc
- $\frac{dg^{-1}}{dy} > 0$

$$F_Y(y) = F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{dF_Y}{dy} = f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

Case 2: g is decreasing $\to g^{-1}$ is dec, $\frac{dg^{-1}}{dy} < 0$

Prev. thm said $F_Y(y) = 1 - F_X(g^{-1}(y))$

$$f_Y(y) = \frac{dF_Y}{dy} = -f_X(g^{-1}(y))\frac{dg^{-1}}{dy}$$

$$= f_X(g^{-1}(y))\left(-\frac{dg^{-1}}{dy}\right)$$

$$= f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

Ex. Let $X \sim \text{Gamma}(k, \lambda)$

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}, \quad x > 0$$

Let $Y = 1/X$

$g(x) = 1/x$

$y = 1/x \Rightarrow x = 1/y = g^{-1}(y)$

so $\dfrac{dg^{-1}}{dy} = \dfrac{-1}{y^2}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

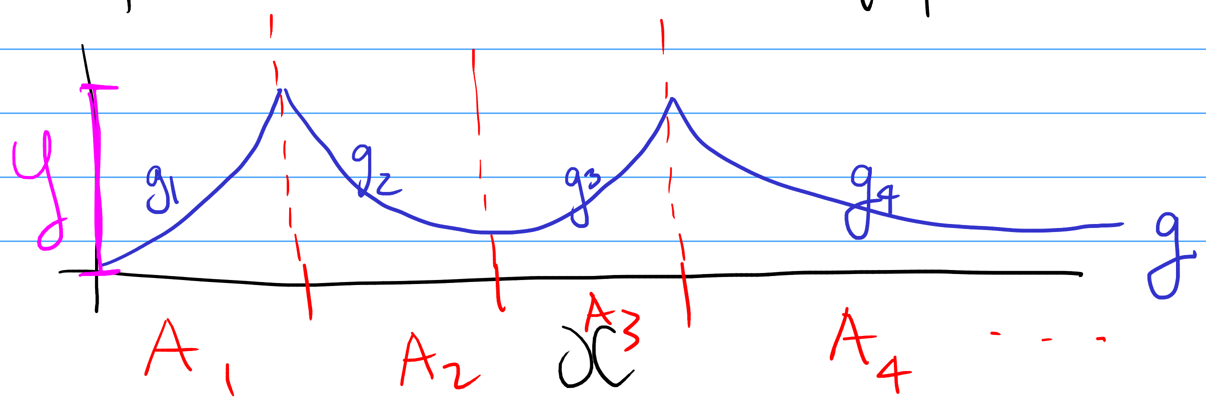
$$= f_X\left(\frac{1}{y}\right) \left| -\frac{1}{y^2} \right|$$

$$\boxed{= \frac{\lambda e^{-\lambda \frac{1}{y}} \left(\lambda \frac{1}{y}\right)^{k-1}}{\Gamma(k)} \frac{1}{y^2} \text{ for } y > 0}$$

called Inverse Gamma dist

What about non-invertible g?

Theorem: Let X be a cts RV w/ support $\mathcal{X}$ and for $i = 1, \ldots, K$ let A_i partition $\mathcal{X}$.



Let g_i to be g restricted to A_i

If ① my prev. theorem applies to each g_i restr. to A_i

[g_i invertible on A_i
g_i^{-1} is diff'able on A_i]

② The image of A_i under g_i is the same for all i

[all g_i have the range $\mathcal{Y}$]

then

$$f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}}{dy} \right|$$

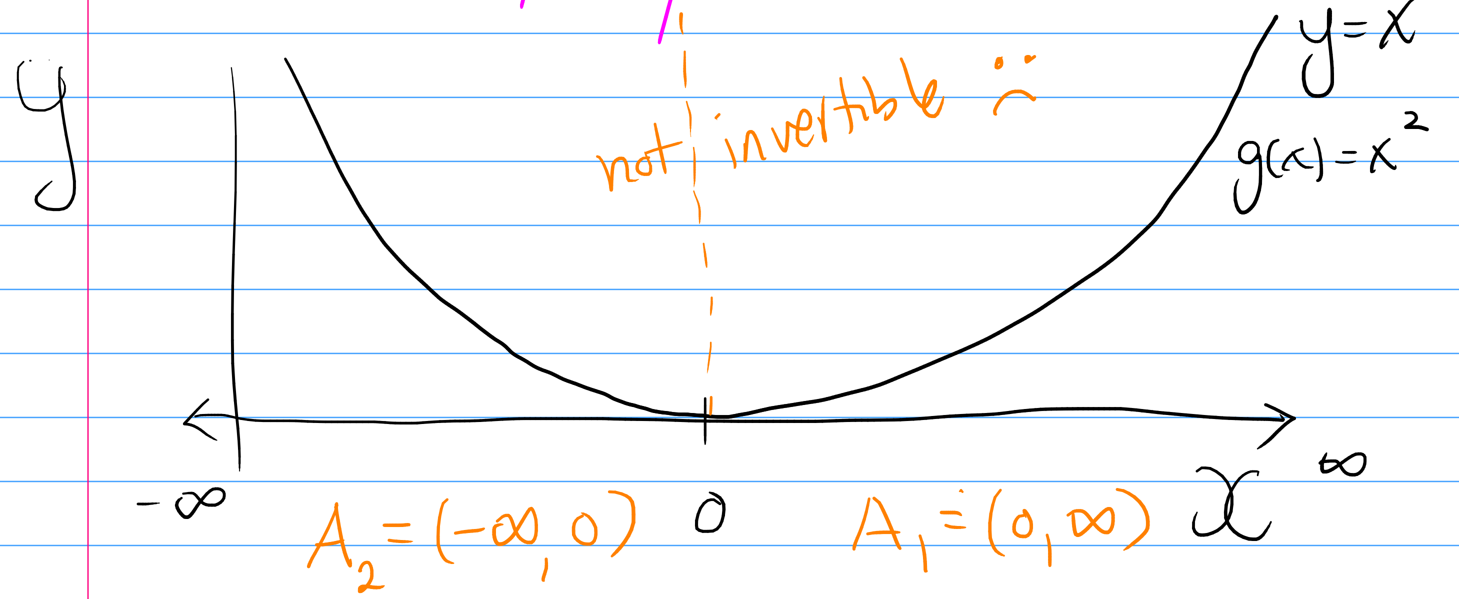
for $y \in \mathcal{Y}$

<u>Ex.</u> <u>Chi-Squared Dist</u>

If $X \sim N(0,1)$ and $Y = X^2$
then we say Y has a Chi-Sq dist

w/ one degree of freedom.

Notation: $Y \sim \chi^2(1)$



$A_1 = (0,\infty)$, $g_1(x) = x^2$, $g_1^{-1}(y) = \sqrt{y}$;

$$\frac{dg_1^{-1}(y)}{dy} = \frac{1}{2\sqrt{y}}$$

$A_2 = (-\infty, 0)$, $g_2(x) = x^2$, $g_2^{-1}(y) = -\sqrt{y}$

$$\frac{dg_2^{-1}}{dy} = \frac{-1}{2\sqrt{y}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \text{ for } x \in \mathbb{R}$$

Apply theorem:

$$f_Y(y) = f_X(g_1^{-1}(y))\left|\frac{dg_1^{-1}}{dy}\right| + f_X(g_2^{-1}(y))\left|\frac{dg_2^{-1}}{dy}\right|$$

$$= f_X(\sqrt{y})\left|\frac{1}{2\sqrt{y}}\right| + f_X(-\sqrt{y})\left|\frac{-1}{2\sqrt{y}}\right|$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\sqrt{y})^2\right)\frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(-\sqrt{y})^2\right) \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{2\sqrt{y}}\left[e^{-\frac{1}{2}y} + e^{-\frac{1}{2}y}\right]$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{y}}\,e^{-\frac{1}{2}y} \quad \text{for } y > 0}$$

↑ density of $\chi^2(1)$ [Type of Gamma]

Thrm Probability Integral Transf

If X is a cts RV w/ CDF F_X then

$$F_X(X) \sim U(0,1).$$

pf. Assume that F_X is strictly increasing so that it is invertible so F_X^{-1} exists.

Our CDF thrm says that if

$$Y = g(X), \ g \text{ inc, then } F_Y(y) = F_X(g^{-1}(y))$$

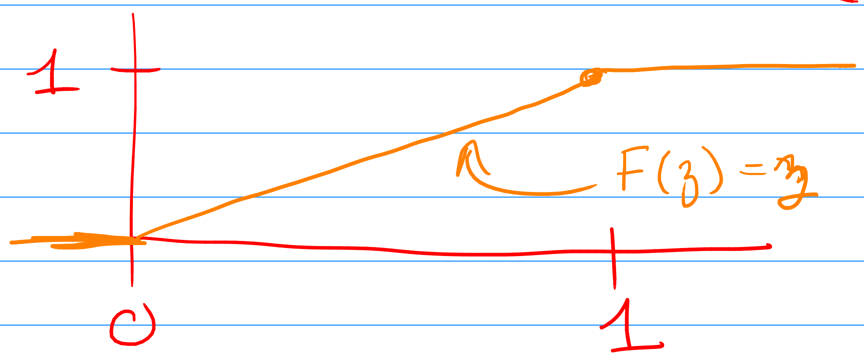
Let $Y = F_X(X)$ i.e. $g = F_X$

so $g^{-1} = F_X^{-1}$

hence

$$F_Y(y) = F_X(F_X^{-1}(y)) = y$$

CDF of $U(0,1)$



Know: how to generate a $U(0,1)$

Want: generate some RV w/ CDF F_X

Let $Z = F_X^{-1}(U)$ where $U \sim U(0,1)$

then $Z \sim F_X$

this happens b/c

$$F_Z(z) = \mathbb{P}(Z \le z)$$
$$= \mathbb{P}(F_X^{-1}(U) \le z)$$
$$= \mathbb{P}(U \le F_X(z))$$
$$= F_U(F_X(z))$$
$$= F_X(z)$$

Algorithm:

1. generate $U \sim U(0,1)$
2. $Z = F_X^{-1}(U)$

Ex. Want $X \sim \text{Exp}(1)$

CDF of Exp(1) is $F_X(x) = 1 - e^{-x}$

$$y = 1 - e^{-x}$$
$$\Rightarrow x = -\log(1-y) = F_X^{-1}(y)$$