Lecture 8: Probability Mass Function (PMF)

Ex from last time

Saw $\mathbb{P}(X=x) = (1-p)^{x-1}\, p$

for $x = 1, 2, 3, 4, \ldots$

Defn: Probability Mass Function (PMF)

For a discrete RV X the PMF is the fn

$$f: \mathbb{R} \to \mathbb{R}$$

So that $x \in \mathbb{R}$

$$f(x) = \mathbb{P}(X=x)$$

Also called the distribution of X.

Theorem: For discrete RVs

$$F(x) = \sum_{i \le x} f(i)$$

Pf. $$\text{“}X \le x\text{”} = \bigcup_{i \le x} \text{“}X = i\text{”}$$

disjoint union

So

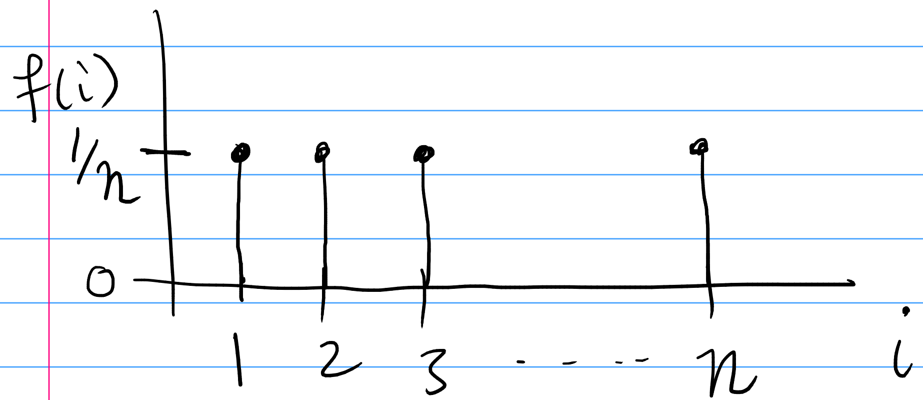
$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}\Big(\bigcup_{i \le x} \text{“}X=i\text{”}\Big) = \sum_{i \le x} \overbrace{\mathbb{P}(X=i)}^{f(i)}$$

Ex. We say X has a discrete Uniform dist. over $1, \dots, n$

Notation: $X \sim U(\{1, 2, \dots, n\})$

$\sim$: distributed as

if



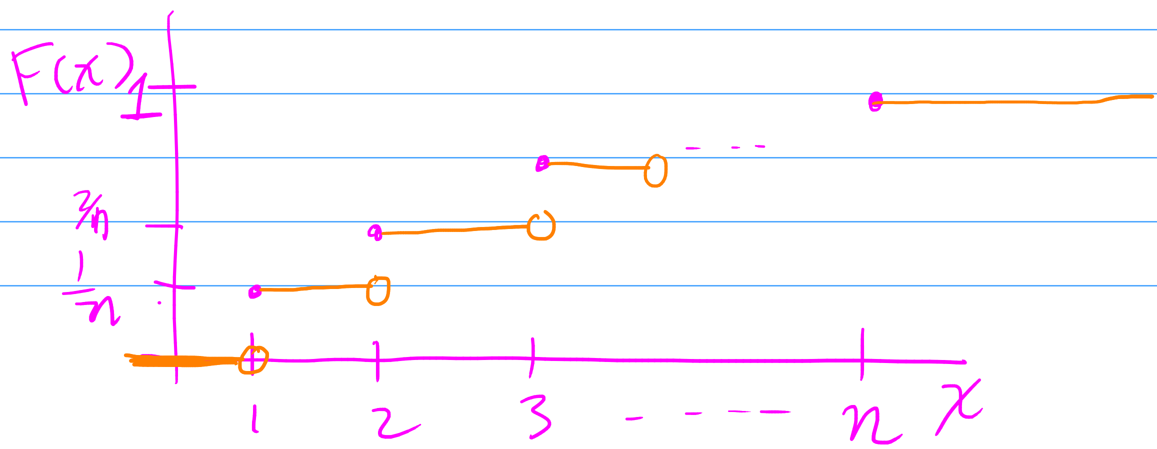
algebraically:

$$f(x) = \begin{cases} \frac{1}{n}, & \text{for } x = 1, 2, 3, \dots, n \\ 0, & \text{else} \end{cases}$$

Q: What's the CDF of X?

$$F(x) = \sum_{i \le x} f(i) = \sum_{i=1}^{x} \frac{1}{n} = \frac{x}{n}$$

for $x = 1, 2, 3, 4, \dots$



$$F(x) = \begin{cases} 0 & , \ x < 1 \\ \frac{\lfloor x \rfloor}{n} & , \ 1 \le x \le n \\ 1 & , \ x > n \end{cases}$$

More generally, (for discrete RVs)

$$\mathbb{P}(X \in A) = \sum_{i \in A} f(i)$$

Ex. $X \sim U(\{1, \ldots, 7\})$

$$\mathbb{P}(2 \le X \le 5) = \mathbb{P}(X \in \{2,3,4,5\})$$

$$= \sum_{i=2}^{5} f(i)$$

$$= \sum_{i=2}^{5} 1/7 = 4/7$$

Ex. Roll a die 60 times (independently)

$X = \#$ 6s I roll.

What is the PMF of X?

$$f(0) = \mathbb{P}(X = 0) = \underbrace{(5/6)(5/6)(5/6) \cdots (5/6)}_{60 \text{ times}}$$

$$= (5/6)^{60}$$

$$f(1) = P(X=1) = \binom{60}{1}\left(\tfrac{1}{6}\right)\underbrace{\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)}_{59 \text{ times}}$$

$$= \binom{60}{1}\left(\frac{1}{6}\right)\left(\tfrac{5}{6}\right)^{59}$$

$$f(2) = P(X=2) = \binom{60}{2}\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)$$

$$= \binom{60}{2}\left(\frac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^{60-2}$$

General pattern

$$\boxed{f(x) = \binom{60}{x}\left(\frac{1}{6}\right)^x\left(\tfrac{5}{6}\right)^{60-x}}$$

X	X	$\cancel{0}$	_	0	- - - - - -	0
$5/6$	$5/6$	$1/6$	$5/6$	$1/6$		$1/6$

This is generically called a Binomial RV

I do a series of n independent experiments

each w/ a binary/yes-no/success-fail outcome
0, 1

the prob of a 1 is p $\in[0,1]$ for each experiment

Then let X = # of 1s among n experiments.

then X has a Binomial dist

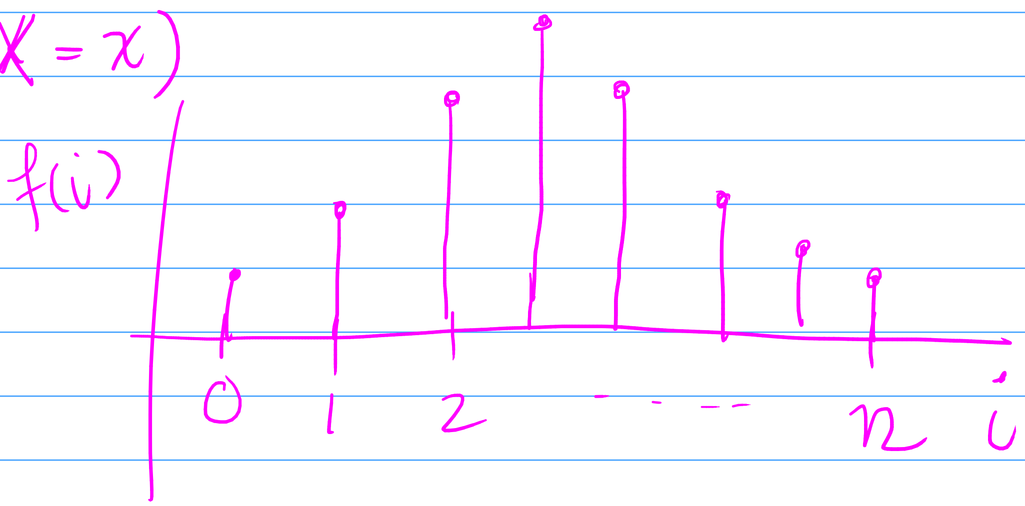
notation: $X \sim Bin(n, p)$

and PMF

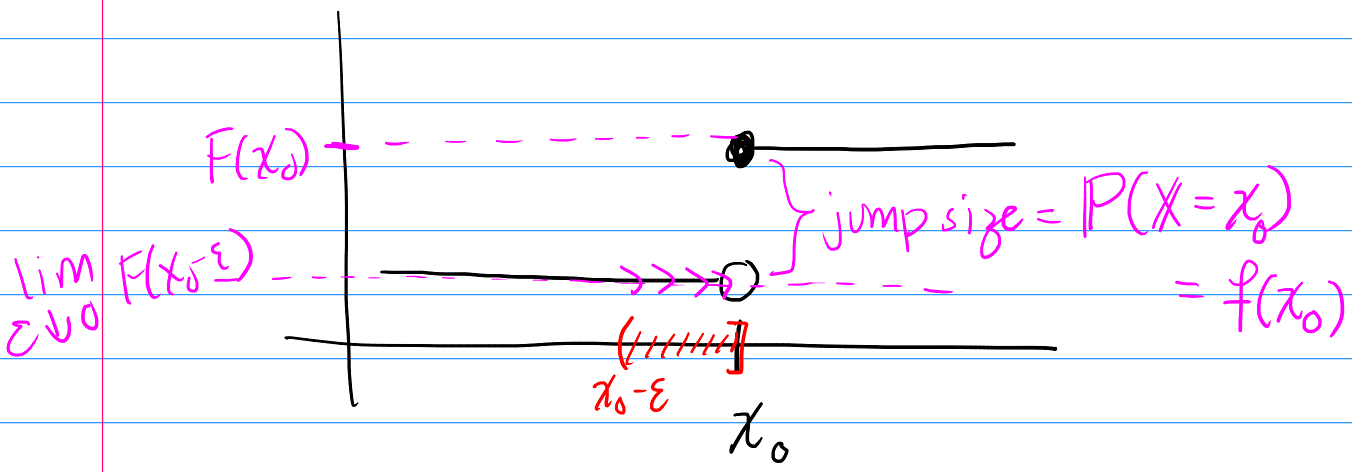
$$f(x) = \binom{n}{x} p^{x} (1-p)^{n-x}, \text{ for } x = 0, 1, 2, \dots, n$$

$$\parallel$$

$$P(X = x)$$



Discrete CDF



rule: $P(a < X \le b) = F(b) - F(a)$

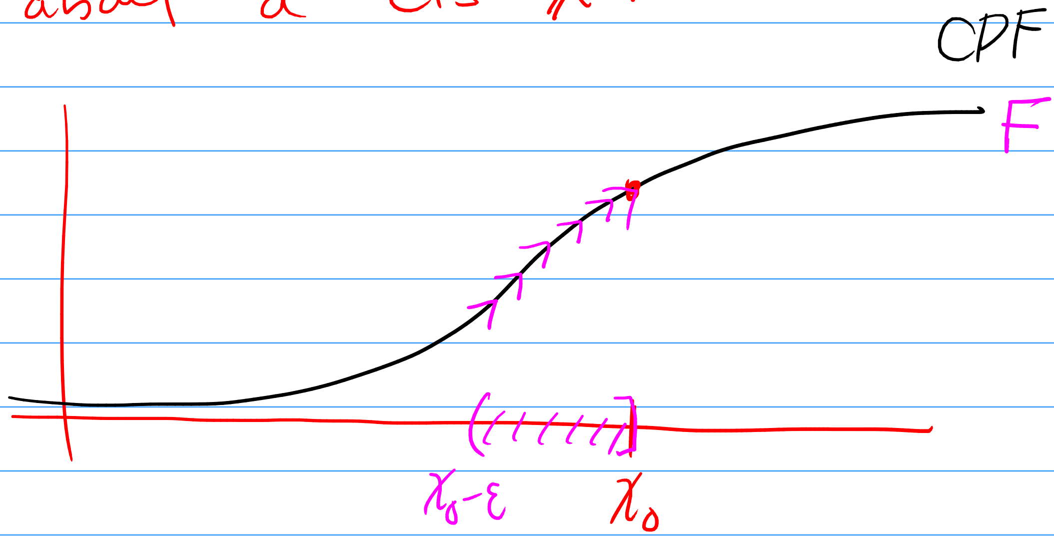
$$\underbrace{\lim_{\varepsilon \downarrow 0} P(X_0 - \varepsilon < X \le x_0)}_{\parallel} = \lim_{\varepsilon \downarrow 0} F(x_0) - F(x_0 - \varepsilon)$$

$$P(X = x_0) = f(x_0)$$

$$= F(x_0) - \lim_{\varepsilon \downarrow 0} F(x_0 - \varepsilon)$$

$$= \text{jump size}$$

What about a cts X?



$$P(X=x_0) = \cdots = F(x_0) - \underbrace{\lim_{\varepsilon \downarrow 0} F(x_0-\varepsilon)}_{F(x_0)}$$

$$= F(x_0) - F(x_0) = 0$$

Want cts analogy for PMF

$$F(x) = \sum_{i \le x} f(i)$$

<u>Defn</u>: Probability Density Function (PDF)
(cts ver of a pmf)

The PDF of a cts RV is a function

$$f : \mathbb{R} \to \mathbb{R}$$

defined for $x \in \mathbb{R}$ such that

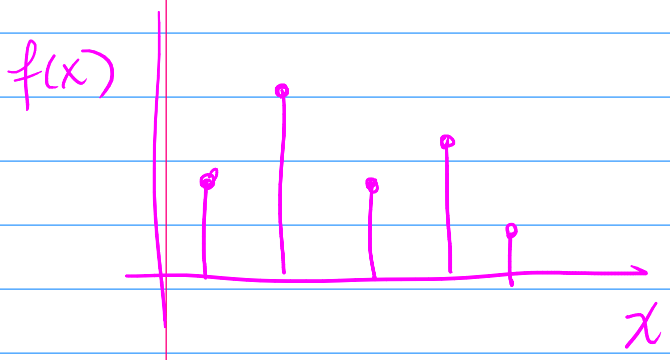
$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

Note that by Fund. Thrm. of Calc.

$$\frac{dF}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)$$

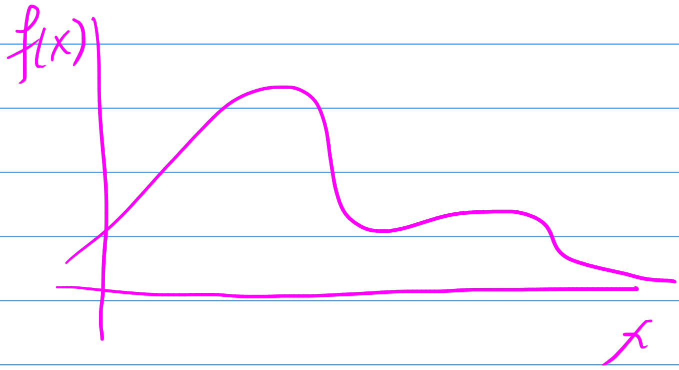
So $f(x) = \dfrac{dF}{dx}$ (PDF = deriv. of CDF)

discrete PMF



$f(x) = \mathbb{P}(X = x)$

Continuous PDF

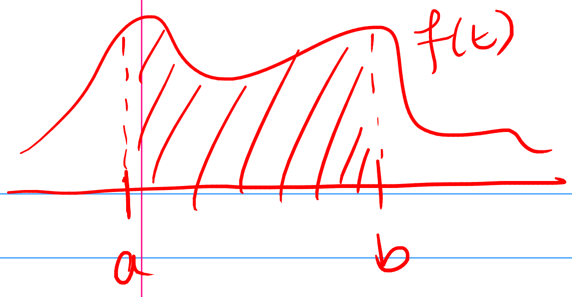


$f(x) \neq \mathbb{P}(X = x)$

Properties of PDFs

$$\mathbb{P}(a < X \leq b) = F(b) - F(a)$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$



$$= \int_a^b f(t)\,dt$$

Note that $P(X=a) = P(X=b) = 0$

$$\begin{aligned} P(a < X \le b) &= P(a \le X \le b) \\ &= P(a < X < b) \\ &= P(a \le X < b) \end{aligned}$$

all equal in cts case

Generally:

(discrete) $P(X \in A) = \sum_{i \in A} f(i)$

(cts) $P(X \in A) = \int_A f(t)\,dt$

Ex. $F(x) = \dfrac{1}{1+e^{-x}}$

Q: What's the corresp. PDF?

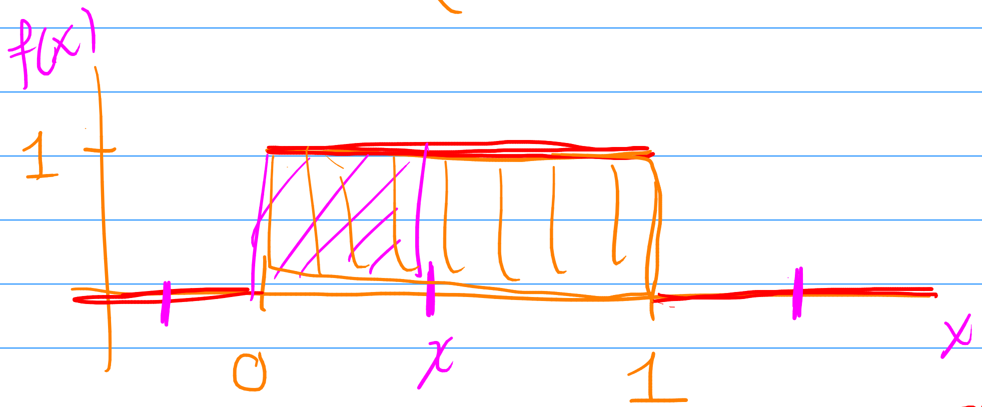
$$f(x) = \frac{dF}{dx} = \cdots = \frac{e^{-x}}{(1+e^{-x})^2}$$

Ex. Continuas uniform dist

$X \sim U(0,1)$

means

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{else} \end{cases}$$



What's the CDF of X? $F(x) = \int_{-\infty}^{x} f(t)dt$

$x < 0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

$0 < x < 1$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} 1\,dt = x$$

$x > 1$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{1} 1\,dt = 1$$

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 1, & x > 1 \end{cases}$$