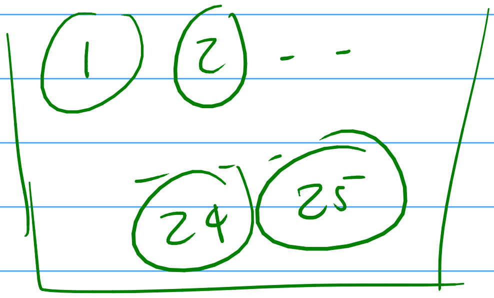
Lecture 4: more counting

Theorem: Sample r items from n, w/o replacement, w/ order, I can do this in

$$\frac{n!}{(n-r)!} \text{ ways.}$$

Ex. Lotto.

Basket w/ $\overset{=n}{25}$ numbered balls



Draw $\overset{=r}{4}$ of them, in some order [all such draws equally likely]

Guess: 1 3 22 7

What's the prob of winning?

$E =$ win, $|E| = 1$

$$P(E) = \frac{|E|}{|S|}$$

$$|S| = \frac{n!}{(n-r)!} = \frac{25!}{(25-4)!}$$

$$= \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21!}}{\cancel{21!}} = 25 \cdot 24 \cdot 23 \cdot 22$$

then $\mathbb{P}(E) = 1/25 \cdot 24 \cdot 23 \cdot 22$

Theorem! Sampling w/ repl., w/ order

The number of ways to draw r items from n w/ repl. and w/ ordering is

$$n^r.$$

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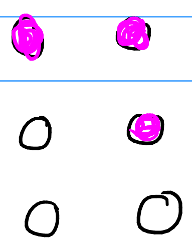
pf FTC

task #	task	# ways
1	draw 1st	n
2	draw 2nd	n
3	" 3rd	n
⋮		
r	" rth	n

multiply = n^r

Ex. Braille alphabet

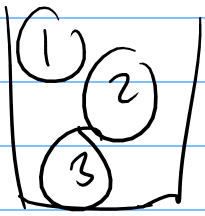


Q! how many braille letters are there?

Sample $\overbrace{\text{bump/ no bump}}^{n=2}$ $\overbrace{\text{six}}^{r=6}$ times in some order.

Formula says: $2^6 = 64$

Sampling w/o order, w/o replacement.

Ex  draw $r=2$ from $n=3$

Order:

$(1,2)$	$(1,3)$	$(2,3)$
$(2,1)$	$(3,1)$	$(3,2)$

$\Big\} \; 6 = \frac{3!}{(3-2)!} = 6$

Unordered:

$\{1,2\} \quad \{1,3\} \quad \{3,2\} \quad \Big\} \; 3$

General fact

Each $\overbrace{\text{unordered samples}}^{\text{of size } r}$ can be permuted in $r!$ ways to create ordered sample

$r=3$

$$\{1,2,3\} \to \begin{cases} (1,2,3) \\ (3,2,1) \\ (2,1,3) \\ \vdots \end{cases} \quad 6 \text{ ordered}$$

$$(\#\text{ ordered}) = r!\,(\#\text{ unordered})$$

So $\#\text{ unordered} = \frac{1}{r!}(\#\text{ordered})$

$$= \frac{1}{r!}\frac{n!}{(n-r)!}$$

Theorem: Sample r items from n w/o repl., w/o order, can be done in

$$\frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$ ← binomial coefficient

read: "n choose r"

ways.

Ex. I have $n=10$ profs, how many co-equal committees can I form of size 4? (r)

unordered

I can do this in $\binom{10}{4} = \frac{10!}{(10-4)!\,4!} = \frac{10!}{6!\,4!}$

$$= \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot \cancel{6!}}{\cancel{6!}\; 4!} = 10 \cdot 3 \cdot 7 = 210$$

$$4! = \cancel{4} \cdot \cancel{3} \cdot \cancel{2}$$

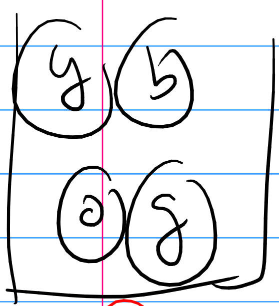
Ex. How many 5 card poker hands can I form?

$n = 52$, $r = 5$, unordered, w/o replacement

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!}\; 5!}$$

$$\approx 2.5 \text{ mil}$$

Ex. I have a jar w/ 4 marbles, of colors yellow, blue, orange, green.



I choose 3 w/o replacement

(all such samples are equally likely)

Q: what is the prob I have a y and b in my choice?

$E = y$ and b,

then $\mathbb{P}(E) = \frac{|E|}{|S|}$

$E = \{\{y, b, o\}, \{y, b, g\}\}$, $|E| = 2$

S = all possible samples,

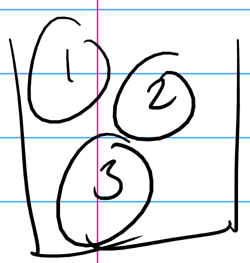
$n=4, r=3$

$$|S| = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{4!}{3!\,1!} = \frac{4\cdot3\cdot2}{3\cdot2} = 4$$

So $\mathbb{P}(E) = \frac{|E|}{|S|} = 2/4 = 1/2$.

Sampling Unordered w/ replacement

Consider $n=3$, $r=2$



$9 = 3^2 = n^r = 9$

Ordered:

$(1,2)$ $(2,1)$, $(1,3)$ $(3,1)$, $(2,3)$ $(3,2)$, $(1,1)$, $(2,2)$, $(3,3)$

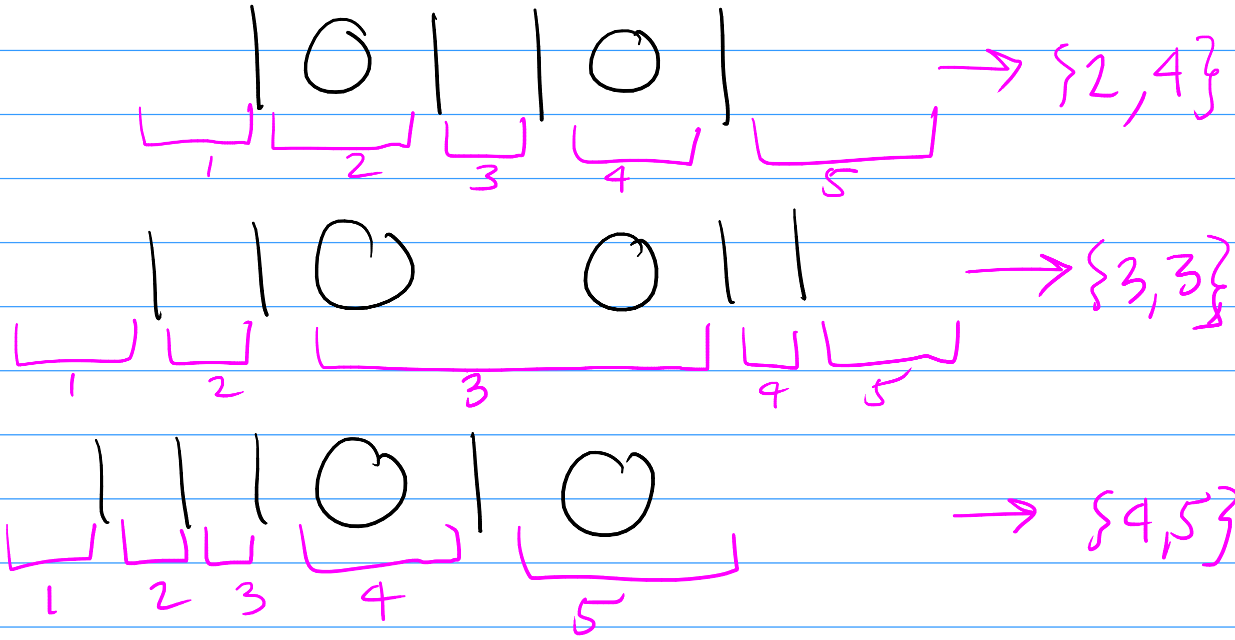
Unordered:

$\{1,2\}$, $\{1,3\}$, $\{2,3\}$ $\{1,1\}$ $\{2,2\}$ $\{3,3\}$

6 possibilities

Game of partitioning

Partition $r=2$ objects using $n-1=4$ walls



How many ways can I play this?

Have $n-1+r$ symbols

I can rearrange (permute) in $(n-1+r)!$

but, I can permute $n-1$ walls in $(n-1)!$ ways

or permute r objs in $r!$ ways

and get the same partitioning.

So in total I have

$$\frac{(n+r-1)!}{r!\,(n-1)!}$$

distinct partitionings.

I can

Theorem: Sample r items from n w/ repl., w/o order in

$$\frac{(n+r-1)!}{(n-1)!\,r!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$

Ex. 10 passengers on a bus route w/ 5 stops. Driver records num. people that get off at each stop.

Q: How many possible records are there?

Stop	# people
1	0
2	3
3	1
4	2
5	4

Sample of size 10 from 5

$\rightarrow \{2,2,2,3,4,4,5,5,5,5\}$

So we do in

$$\binom{n+r-1}{r} = \binom{5+10-1}{10} = \binom{14}{10}$$

Ex. Jar w/ 4 marbles: y, b, o, g

Draw $r=3$ from $n=4$ allow replacement

(all such draws equally likely)

Q: Prob that sample has a y and b

$E =$ y or b

then $P(E) = \dfrac{|E|}{|S|}$

$E = \{\{y,b,o\}, \{y,b,g\}, \{y,b,b\}, \{y,b,y\}\}$

$|E| = 4$

S = all possible draws,

$$|S| = \binom{n+r-1}{r} = \binom{4+3-1}{3} = \binom{6}{3} = 20$$

So $P(E) = 4/20 = 1/5$

Sampling r from n

	w/o repl.	w/ repl.
ordered	$\frac{n!}{(n-r)!}$	n^r
unordered	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	$\binom{n+r-1}{r}$

The point:

If I have S w/ equally likely outcomes

then $P(E) = \frac{|E|}{|S|}$ ← need to count

Q: Ordering? Replacement?

A: all outcomes must be equally likely.

Ex. Flip a coin twice.

What's the prob of getting a H and T.

Option 1: Unordered sample space

$$S = \{HH, TT, HT\}$$

$E = \{HT\}$ so $P(E) = \frac{|E|}{|S|} = 1/3$.

Option 2: Ordered sample space

$$S = \{HH, TT, HT, TH\}$$

$E = \{HT, TH\}$

So $\mathbb{P}(E) = 2/4 = 1/2$.

General rule:

If I build S through a seq of independent actions typically counting in an ordered way is correct.