Lecture 13 : More Common Dists

$X \sim \text{Pois}(\lambda)$

$$E[X(X-1)] = \sum_{x=\cancel{0}\,2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\frac{x(x-1)}{x!} = \frac{\cancel{x(x-1)}}{\cancel{x(x-1)}(x-2)!} = \frac{1}{(x-2)!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-2)!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x+2}}{x!}$$

$$= \lambda^2 e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}}$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} = \boxed{\lambda^2 = E[X(X-1)]}$$

$$= E[X^2 - X]$$

$$= E[X^2] - E[X]$$

$$\boxed{\begin{aligned} EX^2 &= \lambda^2 + EX \\ &= \lambda^2 + \lambda \end{aligned}}$$

$$Var(X) = E[X^2] - (EX)^2$$
$$= \lambda^2 + \lambda - (\lambda)^2$$
$$= \lambda$$

MGF:

$$M(t) = E[e^{tX}]$$
$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$e^y = \sum_{i=0}^{\infty} y^i / i!$$

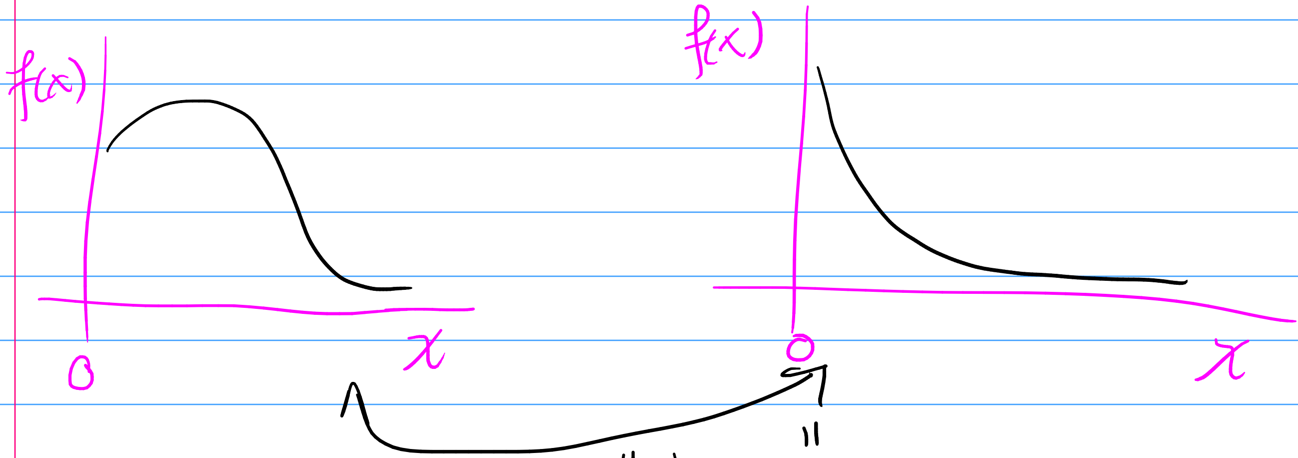
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$
$$= e^{-\lambda} \exp(\lambda e^t)$$
$$= \boxed{\exp(\lambda(e^t - 1)) = M(t)}$$

Gamma Distribution

- cts dist w/ support $(0, \infty)$
- generalization of $Exp(\lambda)$

$X \sim \text{Gamma}(k, \lambda)$

shape (k) rate (λ)

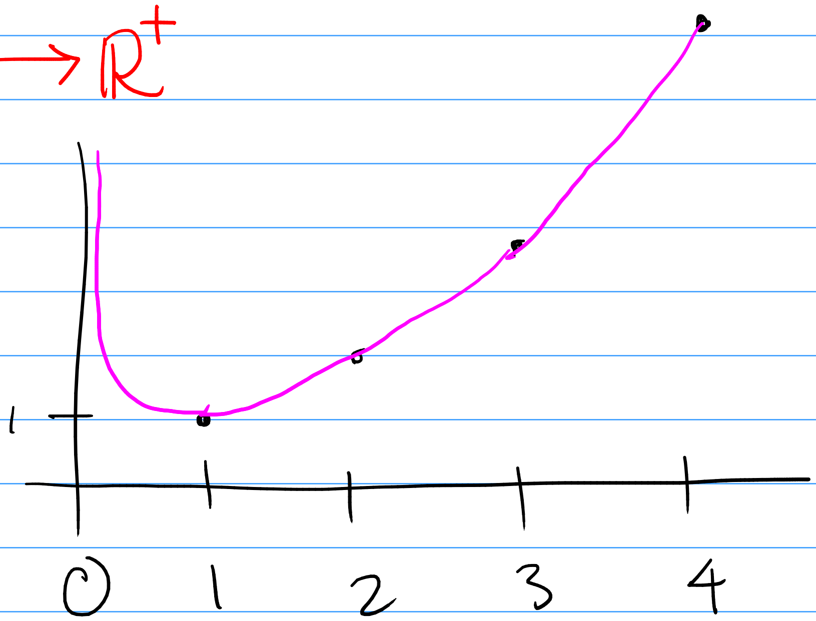


diffrent "shapes" but same λ

Gamma Function

Extend factorial function

$\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$



For $k > 0$ we define

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} \, dx$$

Properties:

(1) If $k>0$ is an integer then

$$\boxed{\Gamma(k) = (k-1)! \quad \text{or} \quad \Gamma(k+1) = k!}$$

Notice: $\Gamma(k) = (k-1)! = (k-1)(k-2)!$

$$= (k-1)\Gamma(k-1)$$

(2) This is true for all k,

For $k>0$

$$\boxed{\Gamma(k) = (k-1)\Gamma(k-1) \quad \text{or} \quad \Gamma(k+1) = k\Gamma(k)}$$

Let $X \sim \text{Gamma}(k, \lambda)$

PDF: $f(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}$ for $x>0$

Notice: If $k=1$ then this is $\text{Exp}(\lambda)$

$$\underbrace{c\int \text{Gamma PDF}}_{1}$$

Expectation:

$$EX = \int_0^\infty x \, \frac{\lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma(k)} \, dx$$

$$= \int_0^\infty \frac{\lambda e^{-\lambda x} \lambda^{k-1} x^{k}}{\Gamma(k)} \, dx$$

Claim: looks like $\boxed{\text{Gamma}(k+1, \lambda)}$

$$\frac{\lambda e^{-\lambda x} \lambda^{k} x^{k}}{\Gamma(k+1)}$$

$$= \frac{\Gamma(k+1)}{\Gamma(k)} \frac{1}{\lambda} \underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x} \lambda^{k} x^{k}}{\Gamma(k+1)} \, dx}_{1}$$

$$= \frac{\Gamma(k+1)}{\Gamma(k)} \frac{1}{\lambda}$$

Recall: $\Gamma(k+1) = k\Gamma(k)$

$$= \frac{k\cancel{\Gamma(k)}}{\cancel{\Gamma(k)}} \frac{1}{\lambda} = \boxed{k/\lambda = EX}$$

$$E[X^r] = \int_0^\infty \frac{x^r \lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma(k)} dx$$

$e^{-mx} x^{m}$

$$= \frac{\lambda^{k-1}}{\Gamma(k)} \int_0^\infty \lambda e^{-\lambda x} x^{r+k-1} dx$$

looks like PDF of Gamma$(k+r, \lambda)$

$$\frac{\lambda e^{-\lambda x} x^{k+r-1} \lambda^{k+r-1}}{\Gamma(k+r)}$$

$$= \frac{\lambda^{k-1}}{\Gamma(k)} \frac{\Gamma(k+r)}{\lambda^{k+r-1}} \underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x} x^{k+r-1} \lambda^{k+r-1}}{\Gamma(k+r)} dx}_{1}$$

$$= \boxed{\frac{\Gamma(k+r)}{\Gamma(k)} \frac{1}{\lambda^r} = E[X^r]}$$

$$E[X^2] = \frac{\Gamma(k+2)}{\Gamma(k)} \frac{1}{\lambda^2}$$

$$= \frac{(k+1)\Gamma(k+1)}{\Gamma(k)} \frac{1}{\lambda^2}$$

$$= \frac{(k+1)k\cancel{\Gamma(k)}}{\cancel{\Gamma(k)}} \frac{1}{\lambda^2}$$

$$= (k+1)k / \lambda^2$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

$$= \frac{(k+1)k}{\lambda^2} - \left(\frac{k}{\lambda}\right)^2$$

$$= \dots$$

$$\boxed{\mathrm{Var}(X) = k/\lambda^2}$$

Geometric Distribution

Canonical experiment:

Flip coins (independently), each w/ a prob p of a H, until I get my first H.

X = # flips until first H

Support: $1, 2, 3, 4, \dots$

$$X \sim \mathrm{Geometric}(p)$$

PMF: $f(x) = (1-p)^{x-1} p$ for $x = 1, 2, 3, \ldots$

CDF: $F(x) = 1 - (1-p)^{\lfloor x \rfloor}$ for $x \geq 1$

Recall: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $|r| < 1$

Geometric Series

Expectation:

$$EX = \sum_{x=1}^{\infty} x (1-p)^{x-1} p$$

$$= p \sum_{x=1}^{\infty} x (1-p)^{x-1}$$

looks like $-\frac{d}{dp} (1-p)^x$

$$= p \sum_{x=1}^{\infty} -\frac{d}{dp} (1-p)^x$$

$$= -p \frac{d}{dp} \left[\sum_{x=1}^{\infty} (1-p)^x \right]$$

geo series $r = 1-p$

$$= -p \frac{d}{dp} \left[\sum_{x=0}^{\infty} (1-p)^{x+1} \right]$$

$$(1-p) \sum_{x=0}^{\infty} (1-p)^x$$

$$(1-p) \frac{1}{1-(1-p)} = \frac{1-p}{p}$$

$$= -p \frac{d}{dp}\left[\frac{1-p}{p}\right]$$

$$= -p\left(-\frac{1}{p^2}\right)$$

$$\boxed{\mathbb{E}X = \frac{1}{p}}$$