Lecture 8: PMFs and PDFs

In prev. ex. had

$$f(x) = \mathbb{P}(X = x) = (1-p)^{x-1} p$$

$$F(x) = \sum_{i=1}^{x} \mathbb{P}(X = i).$$

Defn: Probability Mass Function (PMF)

For a discrete RV X, the PMF is a function $f: \mathbb{R} \to \mathbb{R}$ so that for $x \in \mathbb{R}$

$$f(x) = \mathbb{P}(X = x).$$

Theorem: For discrete RVs

$$F(x) = \sum_{i \le x} f(i)$$

Pf. "$X \le x$" $= \bigcup_{i \le x}$ "$X = i$" disjoint union

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}\left(\bigcup_{i \le x} \text{"}X = i\text{"}\right)$$

$$= \sum_{i \le x} \mathbb{P}(X = i) = \sum_{i \le x} f(i)$$

Ex. We say that X has a discrete uniform distribution over $1, \dots, n$

Notation: $X \sim U(\{1, \dots, n\})$

$\sim$ distributed as

If the PMF is as follows

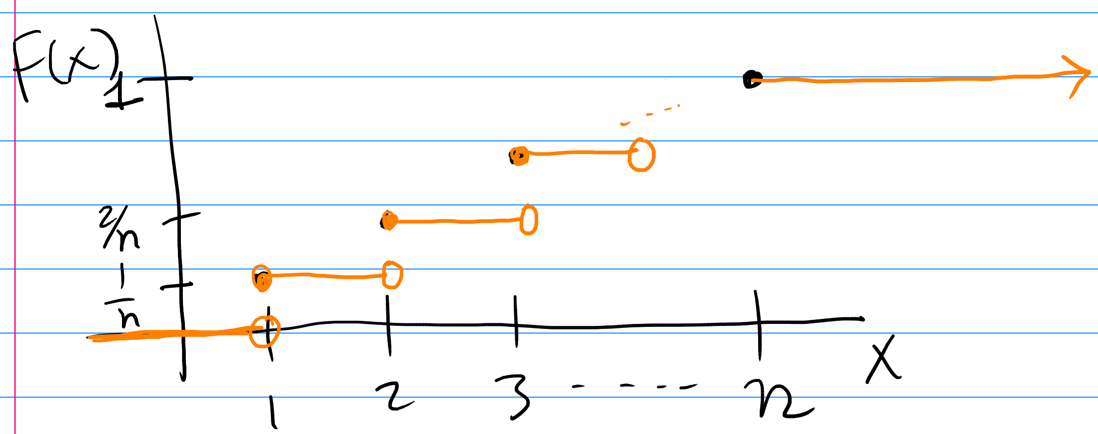
$$f(x) = \frac{1}{n}$$

for $x = 1, \dots, n$

What's the CDF?

$$F(x) = \sum_{i \le x} f(i) = \sum_{i=1}^{x} \frac{1}{n} = \frac{x}{n}$$

$x = 1, 2, 3, \dots, n$



$$F(x) = \begin{cases} 0, & x < 1 \\ \lfloor x \rfloor / n, & 1 \le x \le n \\ 1, & x > n \end{cases}$$

More generally,

$$P(X \in A) = \sum_{i \in A} f(i)$$

Ex. $X \sim U(\{1, 2, \dots, 7\})$

$P(X \in \{1, 3, 5\})$

$$= \sum_{i=1,3,5} f(i) = \sum_{i=1,3,5} \frac{1}{7} = 3/7.$$

Ex. Roll a die 60 times (independently)

X = # of 6s I roll

What's the PMF of X?

$$f(0) = P(X=0) = \underbrace{(5/6)(5/6)(5/6)\cdots(5/6)}_{60 \text{ times}}$$

$$= (5/6)^{60}$$

$$f(1) = \mathbb{P}(X=1) = \binom{60}{1}\left(\tfrac{1}{6}\right)\underbrace{\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)}_{59 \text{ times}}$$

$$f(2) = \mathbb{P}(X=2) = \binom{60}{2}\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)\underbrace{\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{8}\right)}_{58}$$

$$\boxed{f(x) = \mathbb{P}(X=x) = \binom{60}{x}\left(\tfrac{1}{6}\right)^{x}\left(\tfrac{5}{6}\right)^{60-x} \quad \text{for } x=0,1,2,\dots,60}$$

We call this type of RV a binomial RV.

I do a series of n independent tasks each w/ a binary outcome $\rightarrow$ 0/1, yes/no, success/fail

and the prob of a 1 is p $\in[0,1]$ for each task.

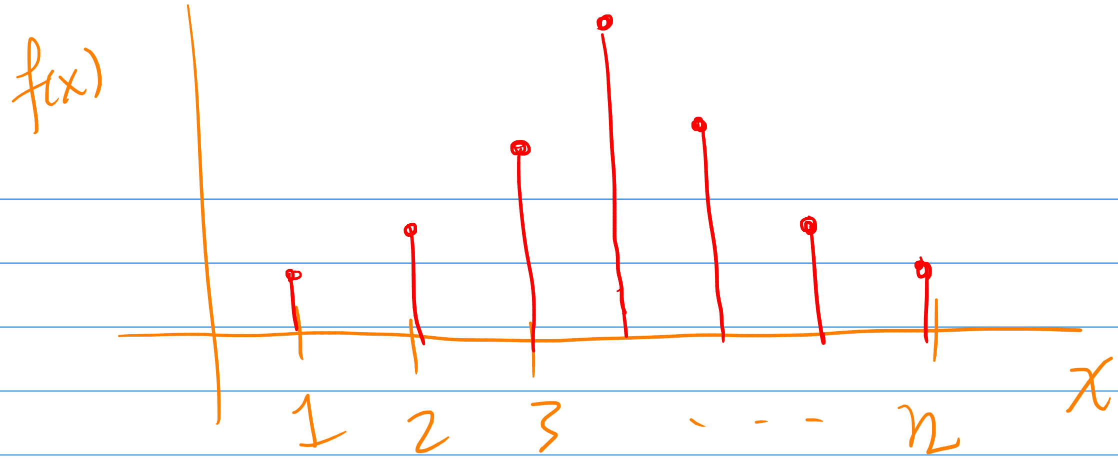
let X = # of 1s I observe.

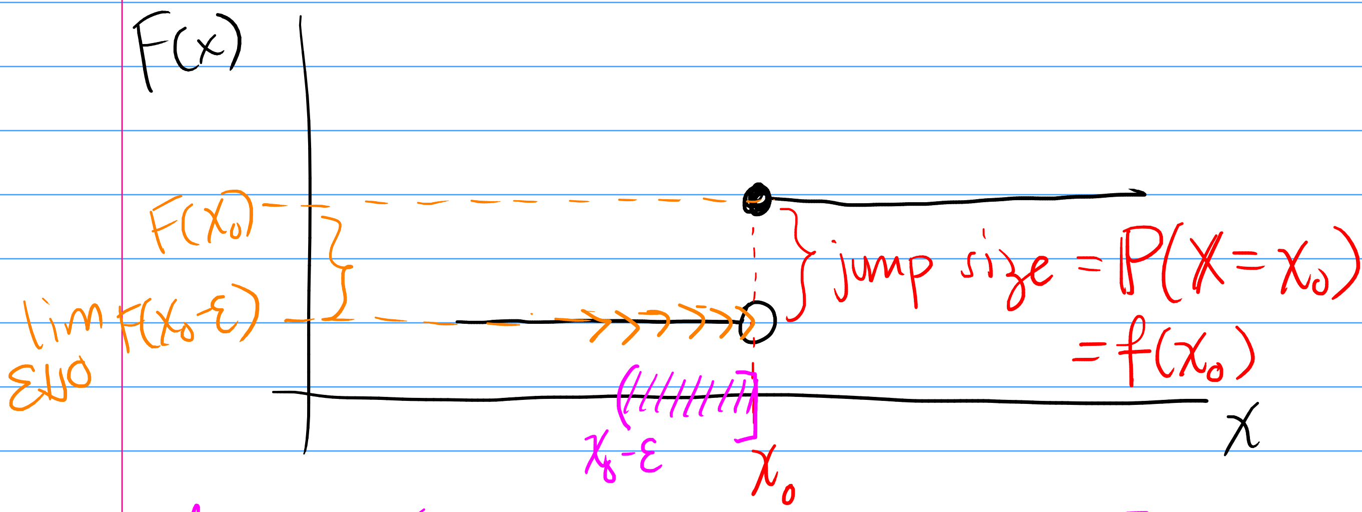
Then X has a binomial dist

notation: $X \sim \text{Bin}(n, p)$

$$f(x) = \binom{n}{x} p^{x}(1-p)^{n-x}$$

for $x=0,1,2,\dots,n$





Recall: $P(a < X \le b) = F(b) - F(a)$

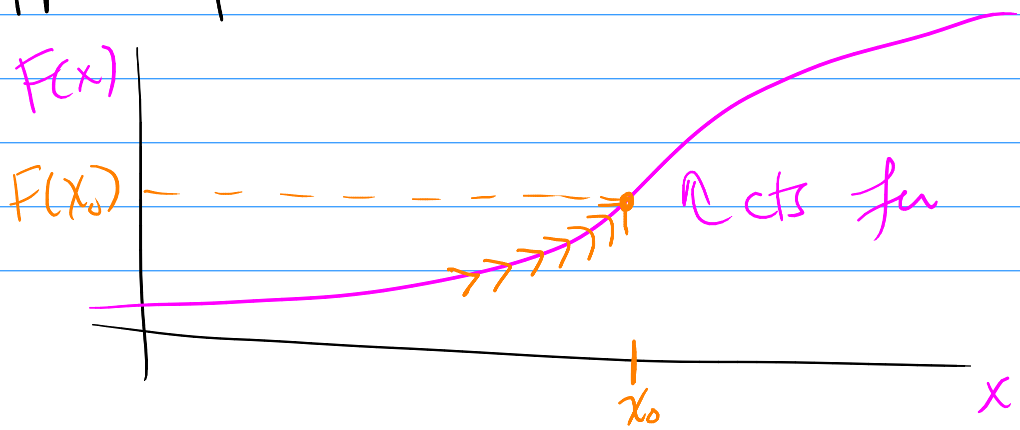
$$\lim_{\varepsilon \downarrow 0} P(x_0 - \varepsilon < X \le x_0) = \lim_{\varepsilon \downarrow 0} F(x_0) - F(x_0 - \varepsilon)$$

$$= F(x_0) - \lim_{\varepsilon \downarrow 0} F(x_0 - \varepsilon)$$

$$= \text{jump size}$$

$$\lim_{\varepsilon \downarrow 0} P(x_0 - \varepsilon < X \le x_0) = P(X = x_0) = f(x_0)$$

What happens for a cts RV?



$$P(X=x_0) = \cdots = F(x_0) - \underbrace{\lim_{\varepsilon \downarrow 0} F(x_0-\varepsilon)}_{F(x_0)}$$

$$= 0$$

Want: cts analog for PMF

$$F(x) = \sum_{i \le x} f(i)$$ ← discrete.

Defn: Probability Density Function (PDF)

(cts analog of PMF)

The PDF for a cts RV is a function

$$f : \mathbb{R} \to \mathbb{R}$$

defined for $x \in \mathbb{R}$ as the function so that

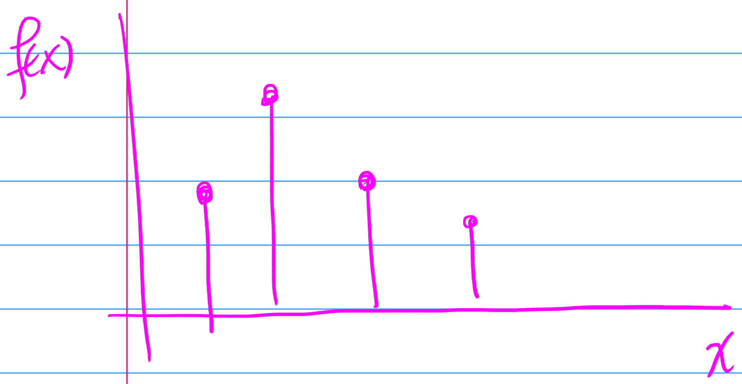
$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

Note: by the Fundamental theorem of Calc

$$\frac{dF}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)$$

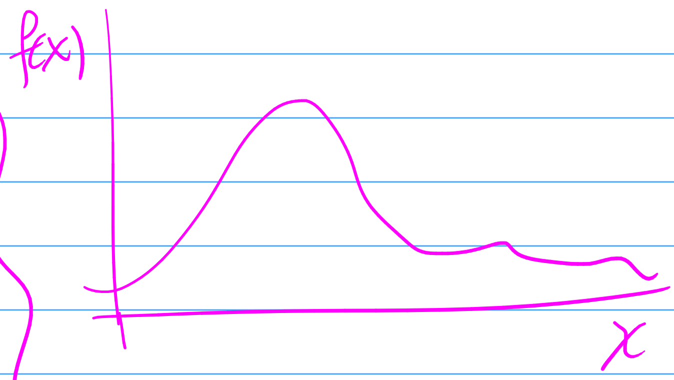
[PDF = deriv. of CDF]

discrete: PMF



$$f(x) = P(X = x)$$

Continuous: PDF

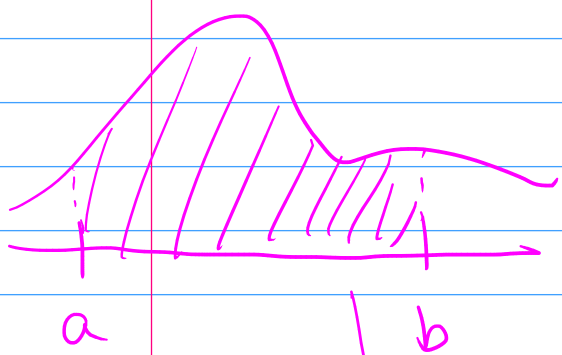


$$f(x) \neq P(X = x)$$

Properties

$$P(a < X \le b) = F(b) - F(a)$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$



$$= \int_{a}^{b} f(t)\,dt$$

In cts case, $P(X=a) = P(X=b) = 0$

$$P(a < X \le b) = P(a \le X \le b)$$

$$= P(a \le X < b)$$

$$= P(a < X < b)$$

Generally

(discrete) $P(X \in A) = \sum_{i \in A} f(i)$

(cts) $P(X \in A) = \int_A f(t)\,dt$

Ex.

$$F(x) = \frac{1}{1+e^{-x}}$$

What's the corresp. PDF?

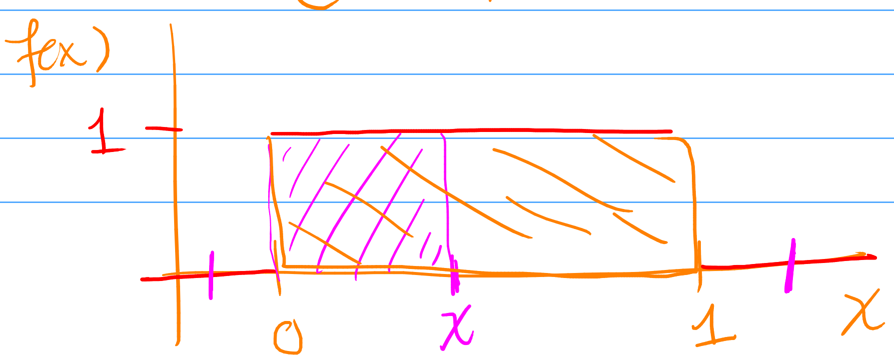
$$f(x) = \frac{dF}{dx} = \cdots = \frac{e^{-x}}{(1+e^{-x})^2}$$

Ex. Continuous Uniform Dist. (on $(0,1)$)

$$X \sim U(0,1)$$

means

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{else} \end{cases}$$



What's the CDF?

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$x<0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

$0 < x < 1$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 1\,dt = x$$

$x > 1$ $$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{1} 1\,dt = 1$$

All together,

$$F(x) = \begin{cases} 0, & x<0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$