Lecture 2: Axiomatic Probability

Defn: Sample Space : S

The set of possible outcomes.

Ex. Flip a coin:

$$S = \{H, T\}$$

Ex. Roll a six sided die

$$S = \{1, 2, 3, \dots, 6\}$$

Ex. Roll two dice:

$$S = \{(1,1), (1,2), (2,1), (2,3), \dots\}$$

Ex. Waiting time for bus

$$S = [0, \infty)$$

Ex. Number of customers arriving at restaurant

$$S = \{0, 1, 2, 3, \dots\}$$

Types of sample spaces:

1. finite, $|S| < \infty$
2. infinite, $|S| \geq \infty$
   - (i) countable
   - (ii) uncountable

Defn: Outcome

We call elements of S "outcomes"

$$\underbrace{s}_{\text{outcome}} \in \underbrace{S}_{\text{sample space}}$$

Defn: Events

An event is a subset of S.

$$\underbrace{E}_{\text{an event}} \subset S$$

Ex. $S = \{1, \dots, 6\}$

$E = \{1,2\} \subset S$

↳ event that I roll a 1 or 2

Ex. $S = \{(i,j) : 1 \le i \le 6, 1 \le j \le 6\}$

$E = \{(1,2),(3,2)\} \subset S$

$F = \{(1,2),(2,3)\}$

We say an event E "happens" if the observed outcome of our experiment is in E

Ex. $S \subset S$

So S is an event

↳ event that something happens

Ex. $\emptyset \subset S$

So $\emptyset$ is an event

Axiomatic Probability

Given: a sample space S

Want: For any event E

assign some measure of the probability of E occuring.

prob. function

Mathematically:

For each $E \subset S$ we assign $\mathbb{P}(E)$

prob. of E

What are rules for $\mathbb{P}$?

1. mathematically consistent
2. encode some intuition about probability

Defn: Prob. Function

Given a sample space S a prob. fn $\mathbb{P}$ is a function

$$\mathbb{P} : 2^S \rightarrow \mathbb{R}$$

that satisfies the Kolmogorov Axioms

1. non-negative: $\mathbb{P}(E) \geq 0 \quad \forall E \subset S$

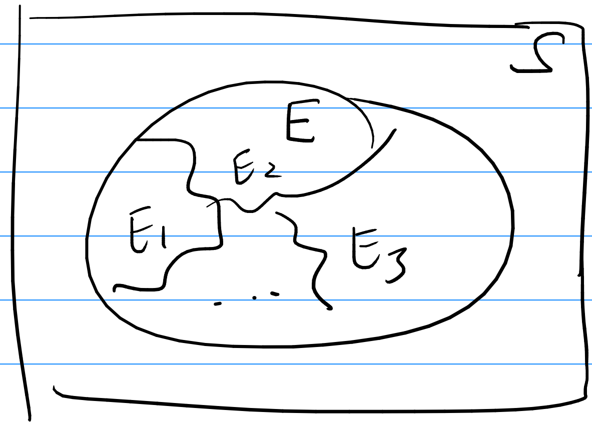
② **Unit-measure:**

$$\mathbb{P}(S) = 1$$

③ **Countable additivity**

If $(E_i)_{i=1}^{\infty}$ is a partition of E

$\left(E_iE_j = \emptyset,\ \bigcup_i E_i = E\right)$



then

$$P(E) = \sum_{i=1}^{\infty} P(E_i)$$

Comments on Axiom 3

① **distributive law**

$$\mathbb{P}\Big(\underbrace{\bigcup_i E_i}_{\text{disjoint}}\Big) = \sum_i \mathbb{P}(E_i)$$

② This also holds for finite unions

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{N} \mathbb{P}(E_i)$$

in particular two sets: $AB = \emptyset$

then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

<u>Ex.</u> Flip a coin

$$S = \{H, T\}$$

What's a valid $\mathbb{P}$ on S?

$$\mathbb{P}(\{H\}) = 1/2 \qquad \mathbb{P}(\underbrace{\{H, T\}}_{S}) = 1$$

$$\mathbb{P}(\{T\}) = 1/2 \qquad \mathbb{P}(\emptyset) = 0$$

Does this satisfy K-axioms?

1. $\mathbb{P}(E) \geq 0$ ✓
2. $\mathbb{P}(S) = 1$ ✓
3. $\mathbb{P}(\bigcup_i E_i) = \sum_i \mathbb{P}(E_i)$, for all disjoint E_i

<u>One example:</u>

$E = S$, $E_1 = \{H\}$, $E_2 = \{T\}$

$1 = P(S) = \mathbb{P}(E) = \mathbb{P}(E_1) + \mathbb{P}(E_2) = 1/2 + 1/2$

Ex. $S = \{H, T\}$

$\mathbb{P}(S) = 1 \qquad \mathbb{P}(\{H\}) = 3/4$

$\mathbb{P}(\emptyset) = 0 \qquad \mathbb{P}(\{T\}) = 1/4$

This also is a valid $\mathbb{P}$.

Ex.

$S = \{1, 2, 3\}$

$p_1 = 1/4$
$p_2 = 1/4$
$p_3 = 1/2$

[non-neg and sum to 1]

$\mathbb{P}(\{1,2\}) = p_1 + p_2 = 1/4 + 1/4 = 1/2$

$\mathbb{P}(\{1,3\}) = p_1 + p_3 = 3/4$

Theorem: Finite Sample Spaces

If $S = \{s_1, \dots, s_n\}$, $|S| = n < \infty$

and we choose some p_i; $i = 1, \dots, n$

so that

① $p_i \geq 0$ and ② $\sum_{i=1}^{n} p_i = 1$

then a <u>valid</u> prob. fn is

$$\mathbb{P}(E) = \text{sum of } p_i \text{ corresp. to } s_i \in E$$

$$= \sum_{i:\, s_i \in E} p_i$$

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pf. ① $\mathbb{P}(E) \geq 0 \quad \forall E \subset S$

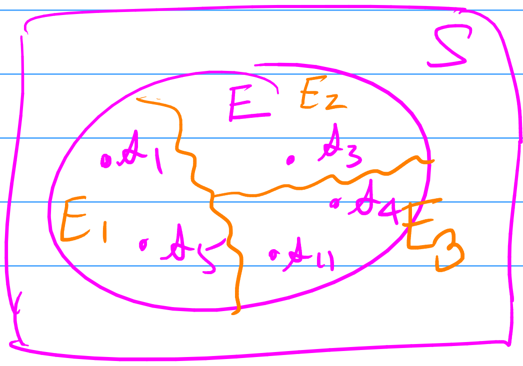
$$\mathbb{P}(E) = \sum_{i} \overset{\geq 0}{p_i} \geq 0$$

② $\mathbb{P}(S) = 1$

$$\mathbb{P}(S) = \sum_{i=1}^{n} p_i = 1$$

③ If E_i partition E then $\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$

sketch:



Show: $\mathbb{P}(E) = - - - - = \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3)$

$$P_1 + P_5 + P_3 + P_4 + P_{11} = P_1 + P_5 + P_4 + P_{11} + P_3$$

Basic Theorems

$\boxed{\text{Theorem: } \mathbb{P}(\emptyset) = 0}$

Pf. $S = \underbrace{S \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots}_{\text{partition of } S}$

by axiom 3,

$$\mathbb{P}(S) = \mathbb{P}(S) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \cdots$$
$$= \mathbb{P}(S) + \sum_{i=1}^{\infty} \mathbb{P}(\emptyset)$$

so $\sum_{i=1}^{\infty} \mathbb{P}(\emptyset) = 0$

only true if $\mathbb{P}(\emptyset) = 0$.

$\boxed{\text{Theorem: Finite Additivity}}$

Third Axiom: $\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$, E_i disj.

Finite Add: $\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i)$, E_i disj.

Pf. $E = A \cup B$, $AB = \emptyset$

notice: $E = A \cup B \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots$

$n=2$

and then by third Axiom,

$$\mathbb{P}(E) = \mathbb{P}(A) + \mathbb{P}(B) + \underbrace{\mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \cdots}_{0}$$

So

$$\mathbb{P}(E) = \mathbb{P}(A) + \mathbb{P}(B)$$

For $n > 2$, use induction.

Ex. $E =$ "its raining"

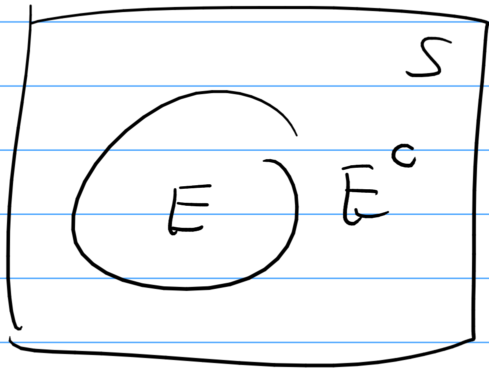
$$\mathbb{P}(E) = 1/3$$

$$\mathbb{P}(\text{"not raining"}) = 2/3$$

$$\mathbb{P}(E^c) = 1 - \mathbb{P}(E) = 1 - 2/3$$

$$\boxed{\underline{\text{Theorem}}: \ \mathbb{P}(E^c) = 1 - \mathbb{P}(E)}$$

<u>Pf</u>. $S = \underbrace{E \cup E^c}_{\substack{\text{disjoint} \\ \text{(partition)}}}$



$$\underset{\substack{\| \\ 1}}{P(S)} = \mathbb{P}(E \cup E^c) = \mathbb{P}(E) + \mathbb{P}(E^c)$$

So, rearranging, we get

$$\mathbb{P}(E^c) = 1 - \mathbb{P}(E).$$

$$\boxed{\text{Theorem: } \ 0 \leq \mathbb{P}(E) \leq 1}$$

<u>Pf</u>. $\mathbb{P}(E) \geq 0$ by Axiom 1

$\mathbb{P}(E^c) \geq 0$ 〃

and by prev theorem, $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$

so $1 - \mathbb{P}(E) \geq 0$ and so

rearranging $\mathbb{P}(E) \leq 1$

Theorem: If $E, F \subset S$, then

$$\mathbb{P}(E \setminus F) = \mathbb{P}(EF^c) = \mathbb{P}(E) - \mathbb{P}(EF)$$