Lecture 14

$X \sim \text{Geom}(p)$

$$M(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{x=1}^{\infty} e^{tx} (1-p)^{x-1} p$$

$$= p \sum_{x=0}^{\infty} e^{t(x+1)} (1-p)^{x}$$

$$= pe^t \underbrace{\sum_{x=0}^{\infty} \left((1-p)e^t\right)^x}_{\text{geometric series}}$$

geometric series

$r = (1-p)e^t$

$|r| < 1$

$$= pe^t \left(\frac{1}{1-(1-p)e^t} \right)$$

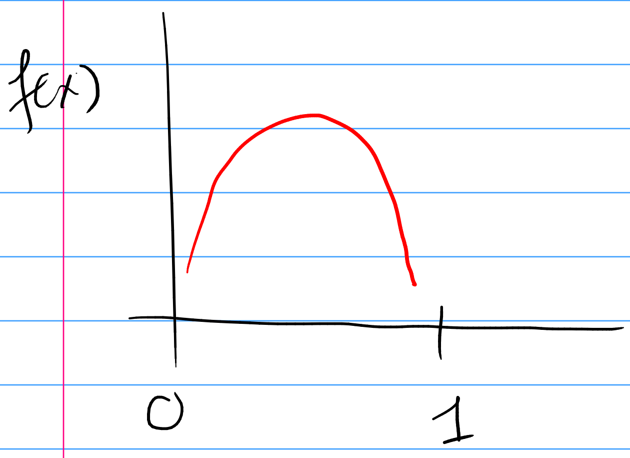
$$= \boxed{\frac{pe^t}{1-(1-p)e^t} \quad \text{for } (1-p)e^t < 1 \iff t < -\log(1-p)}$$

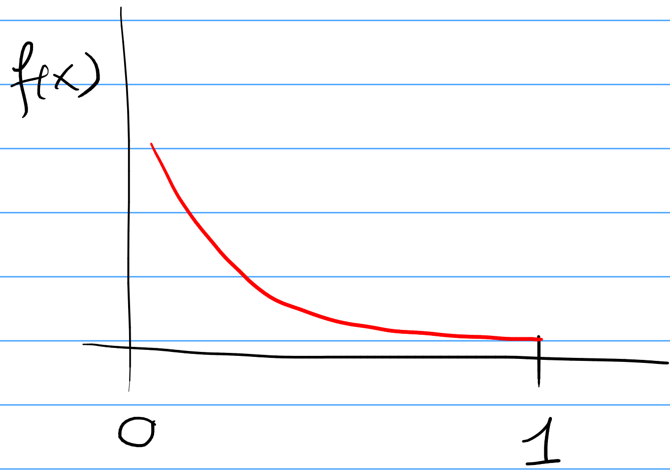
$$\mathbb{E}[X^2] = \left.\frac{d^2M}{dt^2}\right|_{t=0} = \frac{2-p}{p^2}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$$

Beta Distribution

- continuous dist w/ support on $[0,1]$





Beta Function : $B : \mathbb{R}^2 \to \mathbb{R}$

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

PDF: $X \sim \text{Beta}(a,b)$, $a,b > 0$

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } 0 < x < 1$$

Expectation:

looks like PDF of Beta(a+1, b)

$$E[X] = \int_0^1 x \, \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \, dx \qquad \Bigg| \qquad \frac{x^{a}(1-x)^{b-1}}{B(a+1,b)}$$

$$= \frac{B(a+1,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{a}(1-x)^{b-1}}{B(a+1,b)} \, dx}_{1}$$

$$= \frac{B(a+1,b)}{B(a,b)} = \frac{\dfrac{\overset{a\cancel{\Gamma(a)}}{\Gamma(a+1)}\cancel{\Gamma(b)}}{\Gamma(a+b+1)}}{\dfrac{\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{\Gamma(a+b)}}$$

$$= \frac{a}{\Gamma(a+b+1)} \, \Gamma(a+b)$$

$$= \frac{a \, \cancel{\Gamma(a+b)}}{(a+b)\,\cancel{\Gamma(a+b)}}$$

$$\boxed{E[X] = \frac{a}{a+b}}$$

Beta$(a+r, b)$

$$E[X^r] = \int_0^1 \frac{x^r x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx \qquad \Bigg| \quad \frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}$$

$$= \frac{B(a+r,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}\,dx}_{1}$$

$$\boxed{E[X^r] = \frac{B(a+r,b)}{B(a,b)}}$$

$$E[X^2] = \frac{B(a+2,b)}{B(a,b)}$$

$$= \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)} \Bigg/ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{(a+1)a\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{(a+b+1)(a+b)\cancel{\Gamma(a+b)}} \Bigg/ \frac{\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{\cancel{\Gamma(a+b)}}$$

$$E[X^2] = \frac{a(a+1)}{(a+b)(a+b+1)}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2$$

$$= \ldots$$

$$\boxed{= \frac{ab}{(a+b)^2(a+b+1)}}$$

EXAM 2

Transformations

If I know something about X

what do I know about $Y = g(X)$?

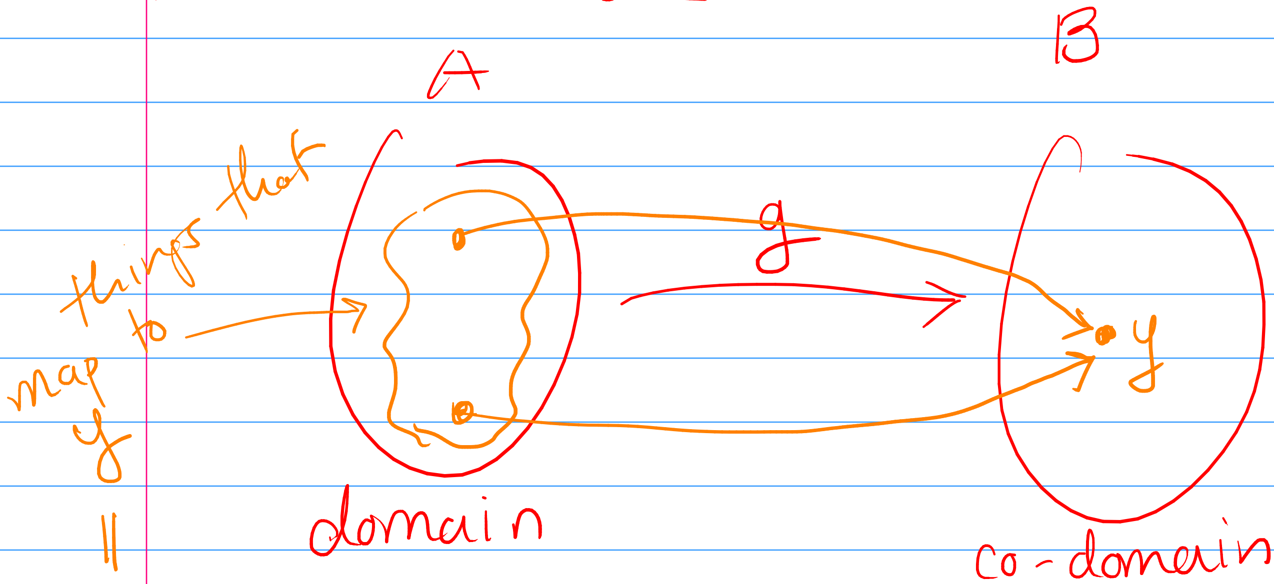
Discrete RVs

Q: If I know f_X can I get f_Y?

PMF of X

PMF of Y.

Inverse Images



= Inverse Image, Pre-image of $\{y\}$

<u>Notation:</u> $g^{-1}(\{y\})$

If g is invertible, then

$$g^{-1}(\{y\}) = g^{-1}(y)$$

inverse image ↑ ↑ inverse fn

<u>Notice:</u> $f_Y(y) = P(Y = y)$

$$= P(g(X) = y) \quad (*)$$

If g is invertible

$$(*) = P(X = g^{-1}(y))$$

$$= f_X(g^{-1}(y))$$

$$\boxed{P(X \in A) = \sum_{x \in A} f_X(x)}$$

If g isn't invertible then

$$(*) = P(X \in g^{-1}(y))$$

inverse image of y

$$= \sum_{x \in g^{-1}(y)} f_X(x)$$

$$= \sum_{x : g(x) = y} f_X(x)$$

<u>Theorem:</u> If X is discrete and $Y = g(X)$ then

$$f_Y(y) = \sum_{x : g(x) = y} f_X(x).$$

<u>Ex.</u> Let $X \sim Bin(n, p)$

Consider $Y = n - X$ # tails

$$y = g(x) = n - x \iff x = n - y = g^{-1}(y)$$

$$f_Y(y) = \sum_{x:g(x)=y} f_X(x)$$

$x : g(x) = y$

$n - x = y$

$x = n - y$

$$= \sum_{x=n-y} f_X(x)$$

$$= f_X(n-y)$$

Recall! $\quad f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{y}$$

$\binom{n}{n-y} = \binom{n}{y}$

define $q = 1 - p$

$$f_Y(y) = \binom{n}{y} q^y (1-q)^{n-y}$$

Bin $(n, q = 1-p)$

So $\quad Y \sim \text{Bin}(n, q)$

Continuous RVs : CDFs

Theorem: If X is cts and $Y = g(X)$ then

1. If g is increasing then

$$F_Y(y) = F_X(g^{-1}(y))$$

Inverse function

2. If g is decreasing then

$$F_Y(y) = 1 - F_X(g^{-1}(y)).$$

Pf Case 1 : g is increasing → g^{-1} increasing

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

Case 2: g dec. —— g^{-1} is dec.

$$F_Y(y) = \mathbb{P}(Y \leq y)$$
$$= \mathbb{P}(g(X) \leq y)$$
$$= \mathbb{P}(X \geq g^{-1}(y))$$
$$= 1 - \mathbb{P}(X < g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$

req. X is cts.