Lecture 14

$X \sim \text{Geom}(p)$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} (1-p)^{x-1} p$$

$$= p \sum_{x=0}^{\infty} e^{t(x+1)} (1-p)^{x}$$

$$= pe^{t} \sum_{x=0}^{\infty} \Big(\underbrace{(1-p)e^{t}}_{r}\Big)^{x}$$

$$= pe^{t} \frac{1}{1-(1-p)e^{t}}$$

$$\sum_{i=0}^{\infty} r^{i} = \frac{1}{1-r}$$

$$|r| < 1$$

$$\boxed{M(t) = \frac{pe^{t}}{1-(1-p)e^{t}} \quad \text{for } (1-p)e^{t} < 1 \iff t < -\log(1-p)}$$

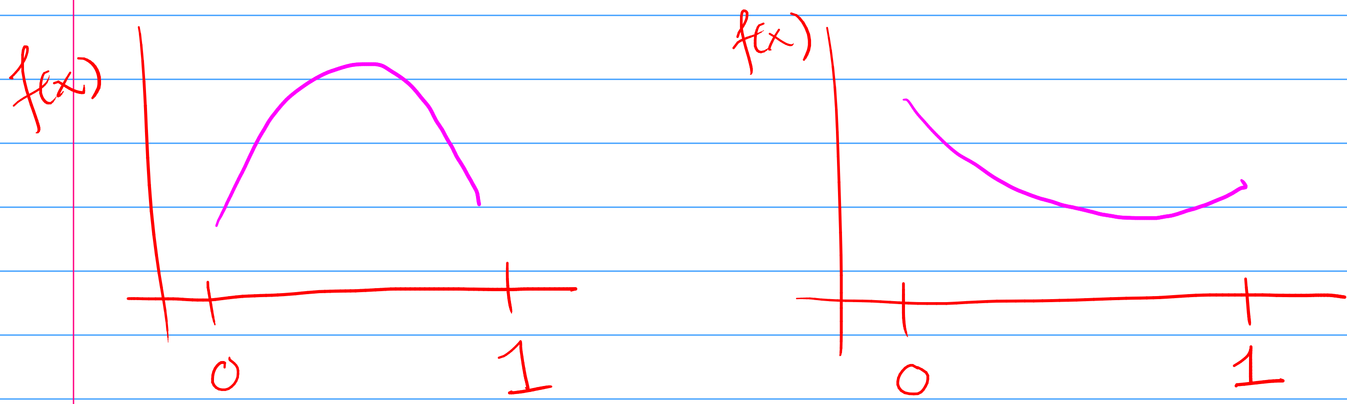
$$\mathbb{E}[X^2] = \left.\frac{d^2M}{dt^2}\right|_{t=0} = \cdots = \frac{2-p}{p^2}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \cdots = \frac{1-p}{p^2}.$$

Beta Distribution

- cts RV w/ support of [0, 1]



Beta Function: $a, b > 0$

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Beta dist: $X \sim \text{Beta}(a,b)$

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } 0 < x < 1.$$

Expectation

$$EX = \int_0^1 x \, \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

looks like PDF of Beta$(a+1, b)$.

$$\frac{x^{a}(1-x)^{b-1}}{B(a+1,b)}$$

$$= \frac{B(a+1,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{a}(1-x)^{b-1}}{B(a+1,b)}\,dx}_{\text{integrate to } 1}$$

$$= \frac{B(a+1,b)}{B(a,b)}.$$

$$= \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} \Big/ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{a\,\cancel{\Gamma(a)}\,\cancel{\Gamma(b)}}{(a+b)\Gamma\cancel{(a+b)}} \Big/ \frac{\cancel{\Gamma(a)}\,\cancel{\Gamma(b)}}{\Gamma\cancel{(a+b)}}$$

$$\boxed{E[X] = \frac{a}{a+b}}$$

$$\mathbb{E}[X^r] = \int_0^1 x^r \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx \quad \Big| \quad \frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}$$

Beta(a+r, b)

$$= \frac{B(a+r,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}\,dx}_{1}$$

$$\mathbb{E}[X^r] = \frac{B(a+r,b)}{B(a,b)}$$

$$\mathbb{E}[X^2] = \frac{B(a+2,b)}{B(a,b)}$$

$$= \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)} \Big/ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{(a+1)a\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{(a+b+1)(a+b)\cancel{\Gamma(a+b)}} \Big/ \frac{\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{\cancel{\Gamma(a+b)}}$$

$$\mathbb{E}[X^2] = \frac{a(a+1)}{(a+b+1)(a+b)}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= \frac{a(a+1)}{(a+b+1)(a+b)} - \left(\frac{a}{a+b}\right)^2$$

$$= \cdots$$

$$= \frac{ab}{(a+b)^2(a+b+1)}$$

EXAM 2

Transformations

If I know something about X,
what do I know about $Y = g(X)$?

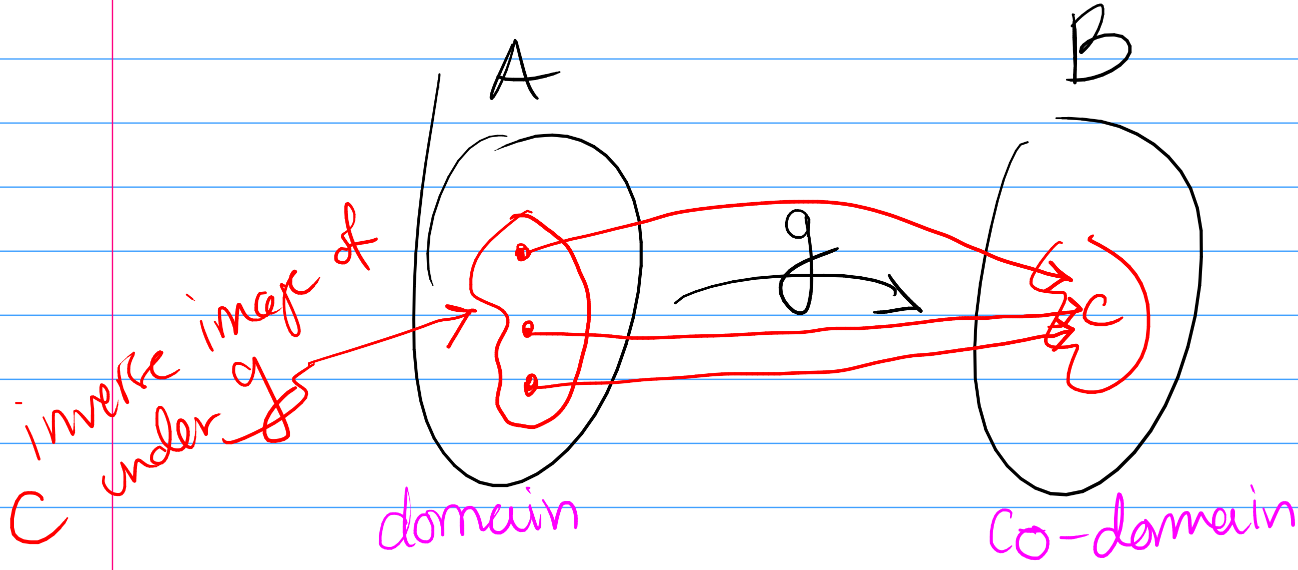
Discrete RVs:

Q: If I know f_X, what is f_Y?

PMF of X

PMF of Y

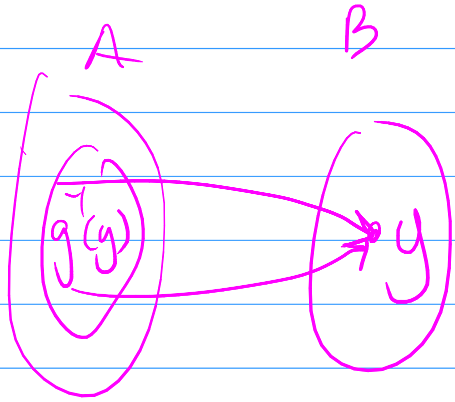
Inverse Images / Preimages



$$g^{-1}(C) = \{x \in A \mid g(x) \in C\}$$

<u>Single pt</u>: $y \in B$

$$g^{-1}(\{y\}) = g^{-1}(y)$$



If g is invertible, then g^{-1} is the inverse image

<u>Notice</u>:

$$f_Y(y) = P(Y=y)$$
$$= P(g(X)=y) \quad (*)$$

Case 1: g is invertible

$$(*) = \mathbb{P}(X = g^{-1}(y))$$

$$= f_X(g^{-1}(y))$$

Case 2: g isn't invertible

$$(*) = \mathbb{P}(X \in \underbrace{g^{-1}(y)}_{A})$$

$$= \sum_{x \in g^{-1}(y)} f_X(x)$$

$$= \sum_{x : g(x) = y} f_X(x)$$

$$\mathbb{P}(X \in A) = \sum_{x \in A} f_X(x)$$

Theorem: If X is discrete and $Y = g(X)$ then

$$f_Y(y) = \sum_{x : g(x) = y} f_X(x).$$

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ex. Let $X \sim Bin(n,p)$

n coin tosses

prob. p of H

$X = \#$ heads

Consider: $Y = n - X = \#$ tails

$$y = g(x) = n - x \iff x = g^{-1}(y) = n - y$$

$$f_Y(y) = \sum_{\substack{x:\, g(x)=y \\ x=g^{-1}(y)}} f_X(x) = \sum_{x=g^{-1}(y)} f_X(x)$$

$$= f_X(g^{-1}(y))$$

$$\boxed{f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}}$$

$$f_X(n-y)$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$

$$\binom{n}{n-y} = \binom{n}{y}$$

$$f_Y(y) = \binom{n}{y} q^y (1-q)^{n-y}$$

$$q = 1-p$$

pmf of $\underline{Bin(n,q)}$

$$Y \sim Bin(n, q = 1-p)$$

Cts RVs

Theorem: If X is cts and $Y = g(X)$ then

(1) if g is increasing then

$$F_Y(y) = F_X(g^{-1}(y))$$

(2) if g is decreasing then

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

true inverse

Pf Case 1: g is inc., then so is g^{-1}

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

$$= P(X \le g^{-1}(y))$$

$$= F_X(g^{-1}(y))$$

Case 2: g dec. then so is g^{-1}

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \ge g^{-1}(y))$$

$$= 1 - \mathbb{P}(X < g^{-1}(y))$$

cts RV

$$\xrightarrow{\text{cts RV}} = 1 - F_X(g^{-1}(y))$$