Lecture 2: Axiomatic Probability

Defn: Sample Space

The set of possible outcomes.

Ex. Flip a coin.

$$S = \{H, T\}$$

Ex. Roll a six-sided die

$$S = \{1, 2, \dots, 6\}$$

Ex. Roll two dice

$$S = \{(1,1), (1,2), (2,1), \dots\}$$

Ex. Waiting time for bus

$$S = [0, \infty)$$

Ex. Number of customers

$$S = \{0, 1, 2, 3, \dots\}$$

Types of sample spaces:

(1) finite, $|S| < \infty$

(2) infinite, $|S| \geq \infty$

(i) Countable

(ii) un-countable

Defn: Outcome

We call elements of S "outcomes"

$$s \in S$$

outcome

sample space

Defn: Event

An event is a subset of the sample space.

$$E \subset S$$

event

Ex. $S = \{1, 2, 3, \ldots, 6\}$

$E = \{1, 2\}$

event that I roll a 1 or 2

We say an event "E happens" if the observed outcome of our experiment is in E.

Ex. $S \subset S$, so S is an event

the event that something happens

Ex. $\emptyset \subset S$, so $\emptyset$ is an event

event that nothing happens (?)

Axiomatic Probability

Given a sample space S

Want: for any event $E \subset S$ want to assign some measure of the prob. of E occuring

a prob. fn

Mathematically:

For each $E \subset S$ we assign $P(E)$

prob. of E

Rules for building $\mathbb{P}$ are

1. mathematically consistent
2. encode intuition about prob.

Defn: Probability Function

Given a sample space S a prob. fn $\mathbb{P}$ is a function

$$\mathbb{P} : 2^S \to \mathbb{R}$$

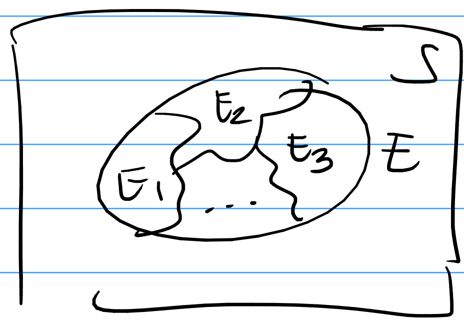
that satisfies the Kolmogorov Axioms

1. non-negativity

$$\mathbb{P}(E) \geq 0 \quad \forall E \subset S$$

2. Unit measure

$$\mathbb{P}(S) = 1$$



3. countable additivity

If $(E_i)_{i=1}^{\infty}$ is a partition of E $\left(E_i E_j = \phi, \bigcup_{i=1}^{\infty} E_i = E\right)$

then

$$\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

we will show

(1) This also holds for finite partitions

$$E = \bigcup_{i=1}^{n} E_i, \quad E_i \text{ disjoint}$$

$$P(E) = \sum_{i=1}^{n} P(E_i)$$

In particular, $n=2$

$$E = A \cup B, \quad AB = \emptyset,$$

$$P(E) = P(A) + P(B).$$

(2) Third axiom is like a distributive law

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$

if E_i disjoint.

Ex. Flip a coin:

$$S = \{H, T\}$$

What is a valid P on S?

$P(\{H\}) = 1/2$ $\quad P(\{H,T\}) = 1$

$P(\{T\}) = 1/2$ $\quad P(\emptyset) = 0$

Is this a valid $\mathbb{P}$?

① $\mathbb{P}(E) \geq 0 \quad \forall E \subset S$ ✓

② $\mathbb{P}(S) = 1$ ✓

③ $\mathbb{P}(\bigcup_i E_i) = \sum_i \mathbb{P}(E_i)$, E_i disjoint

interesting case

$E = S = \{H, T\}$, $E_1 = \{H\}$, $E_2 = \{T\}$

$1 = \mathbb{P}(S) = \mathbb{P}(E) = \mathbb{P}(E_1) + \mathbb{P}(E_2) = 1/2 + 1/2$

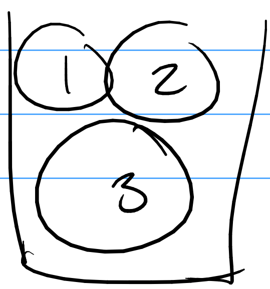
Ex $\quad S = \{H, T\}$

$\mathbb{P}(S) = 1$ and $\mathbb{P}(\{H\}) = \alpha$

$\mathbb{P}(\emptyset) = 0$ $\qquad \mathbb{P}(\{T\}) = 1 - \alpha$

where $0 \leq \alpha \leq 1$

Ex



$S = \{1, 2, 3\}$

$P_1 = 1/4$

$P_2 = 1/4$

$P_3 = 1/2$

[non-neg, sum to 1]

$\mathbb{P}(\{2,3\}) = p_2 + p_3 = 3/4$

$\mathbb{P}(\{1,2\}) = p_1 + p_2 = 1/2$

Theorem: Finite Sample Space

If $S = \{s_1, \ldots, s_n\}$, $|S| = n$,

and we choose some p_i where

(1) $p_i \geq 0$ and (2) $\sum_{i=1}^{n} p_i = 1$

and define $\mathbb{P}$ as

$$\mathbb{P}(E) = \text{sum } p_i \text{ for } i \text{ corresp to } s_i \in E$$

$$= \sum_{i : s_i \in E} p_i$$

then $\mathbb{P}$ is a valid prob. fn.

pf. Check K-axioms

(1) $\mathbb{P}(E) \geq 0$

$p_i \geq 0$

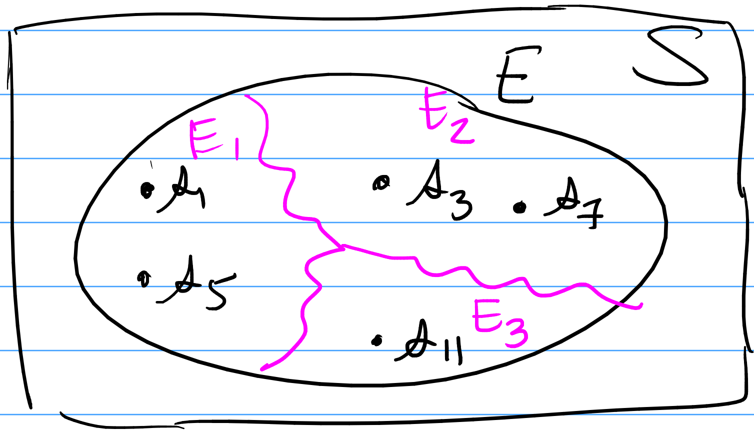
$$\mathbb{P}(E) = \sum_{\text{some } i} p_i \geq 0$$

② $\mathbb{P}(S) = 1$

$$\mathbb{P}(S) = \sum_{i:\,s_i \in S} p_i = \sum_{i=1}^{n} p_i = 1$$

③ If E_i partition E then $\mathbb{P}(\bigcup_i E_i) = \sum_i \mathbb{P}(E_i)$

Sketch!



Want:

$$\mathbb{P}(E) = \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3)$$

$$\mathbb{P}(E) = p_1 + p_3 + p_5 + p_7 + p_{11}$$

re-arrangement

$$\mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3) = (p_1 + p_5) + (p_3 + p_7) + (p_{11})$$

Basic Theorems

$\boxed{\text{Theorem: } \mathbb{P}(\emptyset) = 0}$

Pf.

$$S = \underbrace{S \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots}_{\text{partition of } S}$$

by axiom 3:

$$\mathbb{P}(S) = \mathbb{P}(S) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \cdots$$

So

$$P(S) = P(S) + \sum_{i=1}^{\infty} P(\phi)$$

So $\displaystyle\sum_{i=1}^{\infty} P(\phi) = 0$

the only way for this to work is if

$$P(\phi) = 0.$$

Finite Additivity:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i), \quad E_i \text{ disjoint}$$

Pf. $n = 2$

$$E = A \cup B, \quad AB = \phi$$

notice

$$E = \underbrace{A \cup B \cup \phi \cup \phi \cup \phi \cup \cdots}_{\text{partition of } E}$$

(Axiom 3)

so

$$P(E) = P(A) + P(B) + \underbrace{P(\phi) + P(\phi) + \cdots}_{0}$$

Hence

$$P(E) = P(A) + P(B).$$

For $n > 2$, use induction.

__Ex.__ E = "its raining"

$P(E) = 1/3$

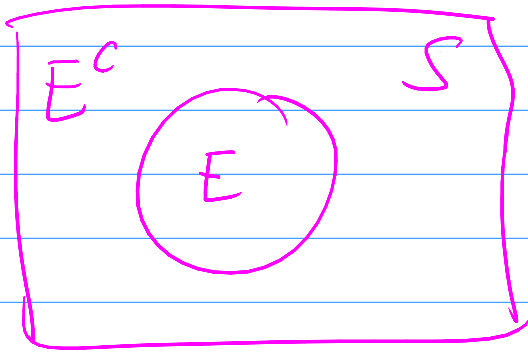
then $P(\text{"not raining"}) = P(E^c) = 2/3 = 1 - 1/3$

$\boxed{\text{Theorem: } P(E^c) = 1 - P(E)}$

pf

$S = E \cup E^c$

partition



thus

$1 = P(S) = P(E \cup E^c)$

$= P(E) + P(E^c)$

re-arrange to get $P(E^c) = 1 - P(E)$

$\boxed{\text{Theorem: } 0 \leq P(E) \leq 1}$

__Note:__ Axiom 1 says $P(E) \geq 0$

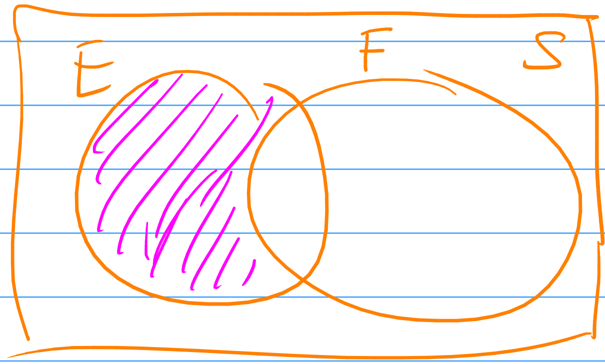
Similarly, $P(E^c) \geq 0$

So by prev. theorem $1 - P(E) \geq 0$

So, rearrange to get $P(E) \leq 1$.

Theorem: If $E, F \subset S$ then

$$\mathbb{P}(E \setminus F) = \mathbb{P}(EF^c) = \mathbb{P}(E) - \mathbb{P}(EF)$$



pf. $\underbrace{E = EF \cup EF^c}_{\text{partition of } E}$

by additivity

$$\mathbb{P}(E) = \mathbb{P}(EF) + \mathbb{P}(EF^c)$$

rearrange,

$$\mathbb{P}(EF^c) = \mathbb{P}(E) - \mathbb{P}(EF)$$