Lecture 7: Random Variables

Informal Defn of Discrete/Cts RVs

① discrete RV : support RV is finite/cantable

Ex, X = sum of two dice

Ex. X = number of customers arriving in shop

② Continuous RV : support is not cantable

Ex, time/space

Defn: Cumulative Distribution Function (CDF)

If X is a RV then its CDF is a function

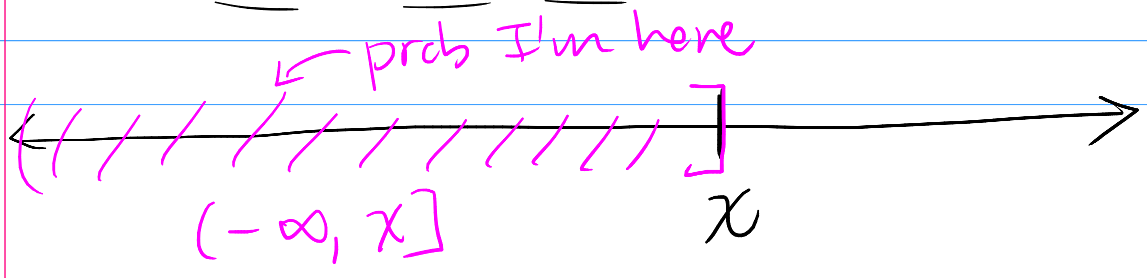
$$F : \mathbb{R} \longrightarrow \mathbb{R}$$

defined for $x \in \mathbb{R}$

$$F(x) = \mathbb{P}(X \leq x)$$

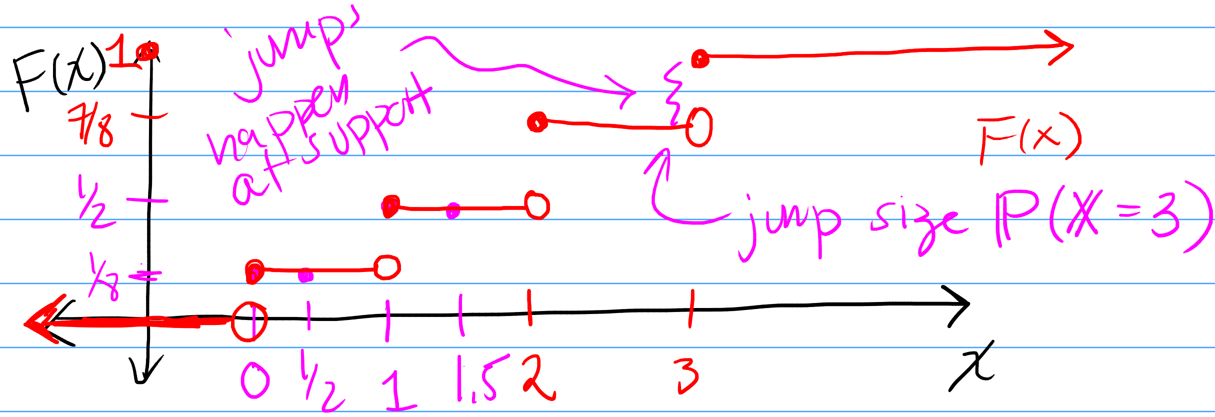
RV

a number



<u>Notation:</u> $P(X \le x) = P(X \in (-\infty, x])$
$= P(X^{-1}((-\infty, x]))$

<u>Ex.</u> Toss a coin 3 times, $X = \#\text{heads}$



$F(0) = P(X \le 0) = P(X = 0) = 1/8$

$F(1/2) = P(X \le 1/2) = P(X = 0) = 1/8$

$F(1) = P(X \le 1) = P(X=0) + P(X=1) = 1/2$

$F(1.5) = P(X \le 1.5) = P(X=1) = 1/2$

$F(2) = P(X \le 2) = 7/8$

$F(3) = P(X \le 3) = 1$

$F(4) = P(X \le 4) = 1$

$F(-1) = P(X \le -1) = 0$

Facts:

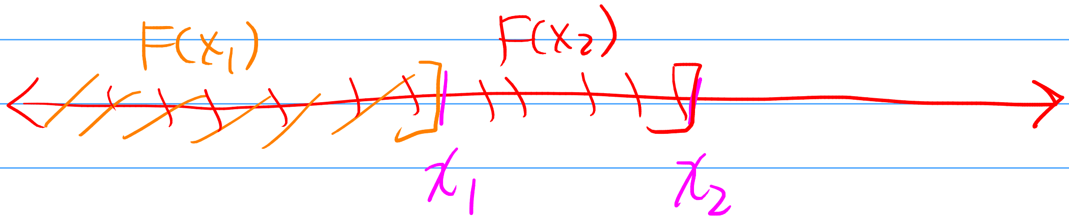
① $\boxed{0 \le F(x) \le 1}$

pf. $F(x) = \mathbb{P}(\cdots\cdots) \in [0,1]$

② $\boxed{\lim_{x \to -\infty} F(x) = 0 \quad , \quad \lim_{x \to \infty} F(x) = 1}$

③ $\boxed{F \text{ is non-decreasing}}$

If $x_1 < x_2$ then $F(x_1) \le F(x_2)$



$$F(x_1) = \mathbb{P}(X \le x_1) \qquad\qquad F(x_2)$$

$$= \mathbb{P}(X \in (-\infty, x_1])$$

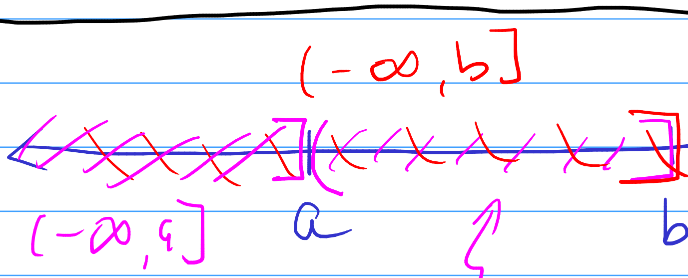
$$= \mathbb{P}(\underbrace{X^{-1}((-\infty, x_1])}_{E_1}) \le \mathbb{P}(\underbrace{X^{-1}((-\infty, x_2])}_{E_2})$$

Argue that $E_1 \subset E_2$

$(-\infty, x_1] \subset (-\infty, x_2]$

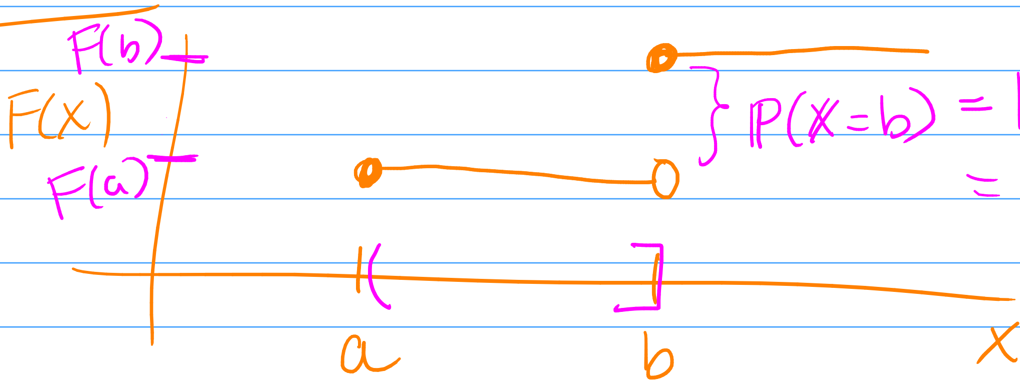
inverse images preserve subset rel.

④ $\boxed{\mathbb{P}(a < X \le b) = F(b) - F(a)}$ $a \le b$



$(a, b] = (-\infty, b] \setminus (-\infty, a]$

practical



$\mathbb{P}(X = b) = \mathbb{P}(a < X \le b)$

$= F(b) - F(a)$

⑤ $\boxed{F \text{ is right-continuous}}$

Recall: cts fn $\lim_{x \to a} F(x) = F(a)$

Right-continuous: $\boxed{\lim_{x \to a^+} F(x) = F(a)}$

Note: cts fn is right cts

Theorem: F is the CDF of some RV iff

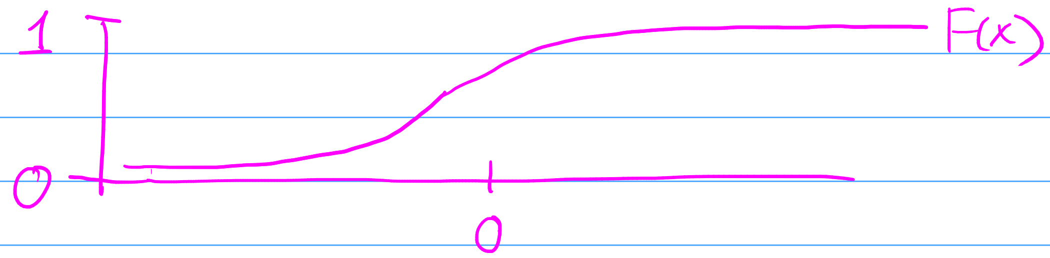
① $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

② F is non-decreasing

③ F is right cts

Ex. Let $F(x) = \frac{1}{1+e^{-x}}$ for $x \in \mathbb{R}$

Q: Is this a valid CDF?



Check 3 conditions

① $\lim_{x \to -\infty} F(x) = \frac{1}{1+e^{-(-\infty)}} = \frac{1}{\infty} = 0$

$\lim_{x \to \infty} F(x) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$

② non-decreasing: $\frac{dF}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} > 0$

3 Right cts?

Cts fn $\Rightarrow$ right cts

Defn: Identical Distribution

We say two RVs X and Y are equal in distribution if

$\forall A \subset \mathbb{R}$ we have

$$P(X \in A) = P(Y \in A).$$

We write: $X \stackrel{d}{=} Y$.

This doesn't mean that $X = Y$. (as fns)

Ex, 3 coin flips.

$$X = \# \text{ heads}$$

$$Y = \# \text{ tails}$$

these are diffent RVs

$$X(HTT) = 1 \quad , \quad Y(HTT) = 2$$

However, $X \stackrel{d}{=} Y$.

$$P(X=0) = \frac{1}{8} = P(Y=0)$$

$$P(X=1) = 3/8 = P(Y=1)$$

Theorem: $X \overset{d}{=} Y$ iff $F_X = F_Y$ (as fns)

CDF X CDF Y

Ex: Toss a coin (indep) until a H appears.

$$S = \{H, TH, TTH, TTTH, \ldots\}$$

Let p be the prob I get a H on any flip.

X = # flips until I get a H

$s \in S$	$X(s)$
H	1
TH	2
TTH	3
TTTH	4
⋮	⋮

Support of X is $\{1, 2, 3, 4, \ldots\}$

Q: What's the CDF?

$$F(x) = \mathbb{P}(X \le x)$$

We'll look at $P(X = x)$

Let H_i = i^{th} toss is a H, $T_i = H_i^c$

$$\text{"}X = i\text{"} = T_1 T_2 T_3 \cdots T_{i-1} H_i$$

indep

So $P(X=i) = P(T_1 T_2 T_3 \cdots T_{i-1} H_i)$

$$= P(T_1)P(T_2)P(T_3) \cdots P(T_{i-1})P(H_i)$$

$$= (1-p)(1-p) \cdots (1-p)p$$

$$= (1-p)^{i-1} p$$

x integer, ≥ 0

$$\text{"}X \leq x\text{"} = \text{"}X=1\text{"} \cup \text{"}X=2\text{"} \cup \text{"}X=3\text{"} \cup \cdots \cup \text{"}X=x\text{"}$$

So

$F(x) = P(X \leq x) = P(X=1) + P(X=2) + \cdots + P(X=x)$

$$= \sum_{i=1}^{x} P(X=i)$$

$$= \sum_{i=1}^{x} (1-p)^{i-1} p$$

$$= p \sum_{i=0}^{x-1} (1-p)^{i}$$

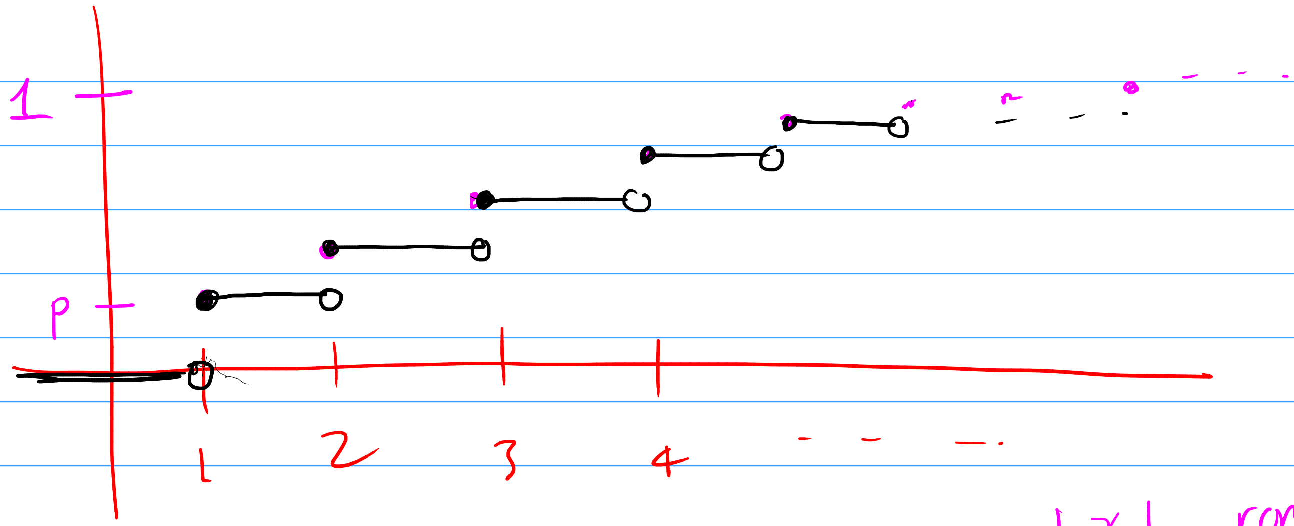
$$= \cancel{p} \frac{1-(1-p)^x}{1-\cancel{(1-p)}}$$

$$\boxed{\begin{array}{c}\text{Geometric Sum} \\ \displaystyle\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}\end{array}}$$

$r = 1-p$
$n = x$

$$F(x) = 1-(1-p)^x$$

$F(x) = 1-(1-p)^x$ for $x = 1,2,3,\ldots$



$\lfloor x \rfloor$ = round down

$$F(x) = \begin{cases} 0 & , \quad x < 1 \\ 1-(1-p)^{\lfloor x \rfloor} & , \quad x \geq 1 \end{cases}$$

<u>Defn</u>: Discrete/cts RVs

A <u>discrete</u> RV is one whose CDF is a step fn.

A <u>cts</u> RV is one whose CDF is a cts fn.
