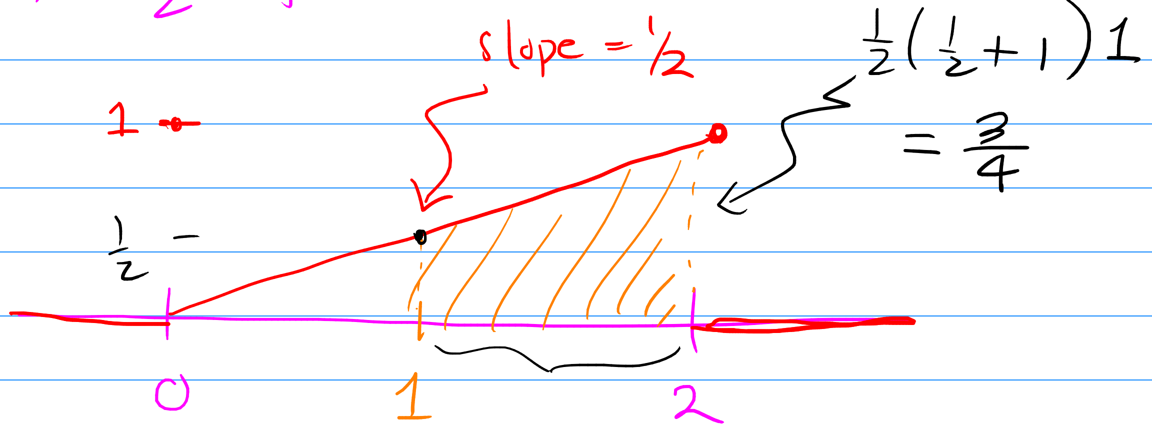
# Lecture 9

Ex.

$f(x) = \frac{x}{2}$ for $0 < x < 2$




$$P(X > 1) = \int_1^\infty f(x)\,dx$$

$$= \int_1^2 \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_1^2 = \frac{4}{4} - \frac{1}{4} = 3/4$$

$$\boxed{P(X \in A) = \int_A f(x)\,dx}$$

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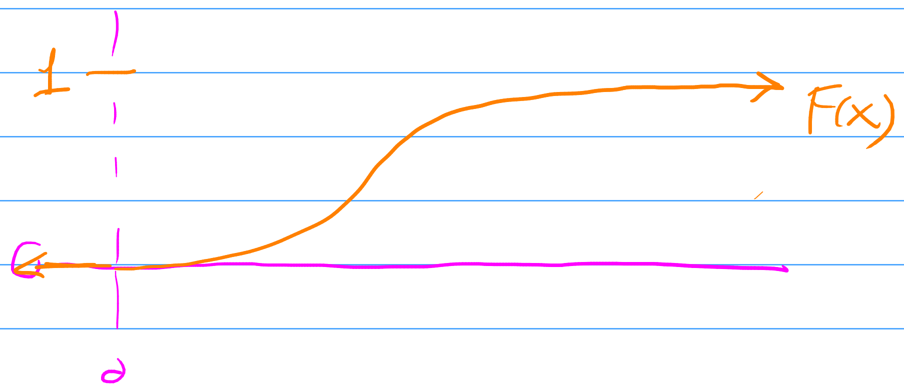
Ex. $F(x) = 1 - e^{-x}$ for $x > 0$




Q: $P(1 < X < 2)$

Way 1: $P(a < X \leq b) = F(b) - F(a)$

$$P(1 < X < 2) = F(2) - F(1)$$
$$= (1 - e^{-2}) - (1 - e^{-1})$$
$$= e^{-1} - e^{-2}$$

Way 2: $P(1 < X < 2) = \int_1^2 f(x)\,dx$

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}(1 - e^{-x}) = -(-e^{-x}) = e^{-x}$$

for $x > 0$

$$= \int_1^2 e^{-x}\,dx = -e^{-x}\Big|_1^2 = -e^{-2} - (-e^{-1})$$
$$= e^{-1} - e^{-2}$$

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**Theorem:** PMF/PDF characterization

A function $f$ is the PMF/PDF of some RV iff

1. $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
2. (discrete) $\sum_{x \in \mathbb{R}} f(x) = 1$

   (cts) $\int_{\mathbb{R}} f(x)\,dx = 1$

If $g(x) \geq 0$ and $\int_{\mathbb{R}} g(x)\,dx = c < \infty$

If I define $f(x) = \frac{1}{c} g(x)$

then $f$ is a PDF.

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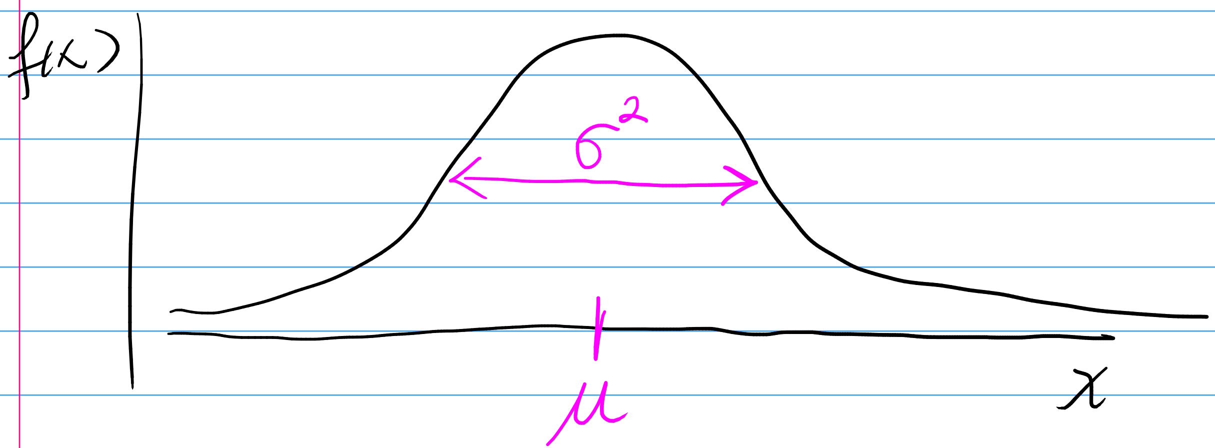
<u>Ex.</u> Normal Distribution (Gaussian)

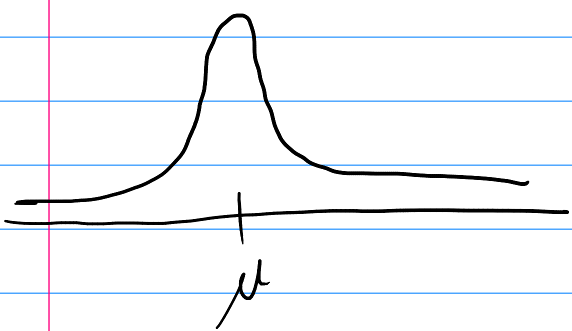
<u>notation:</u> $X \sim N(\mu, \sigma^2)$

$\sigma^2 > 0$, variance

$\mu \in \mathbb{R}$, mean




small $\sigma^2$




<u>large</u> $\sigma^2$

## Special Case:

Standard Normal: $\mu = 0$, $\sigma^2 = 1$.

$$X \sim N(0,1)$$

density

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right), \quad x \in \mathbb{R}$$

$\exp(a) = e^a$

Q: Is this a valid density?

(1) $f(x) \geq 0$ ✓

(2) $\int_{\mathbb{R}} f(x)\,dx = 1$

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$$\underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx}_{I > 0} = 1$$

Want: $I = 1 \iff I^2 = 1$

$$I^2 = I \cdot I = \left[\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)dx\right]\left[\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right)dy\right]$$

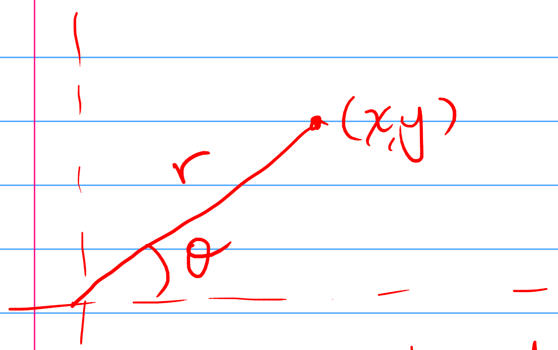
$$= \iint_{\mathbb{R}^2} \frac{1}{2\pi} \exp\left(-\frac{1}{2}x^2\right)\exp\left(-\frac{1}{2}y^2\right)dx\,dy$$

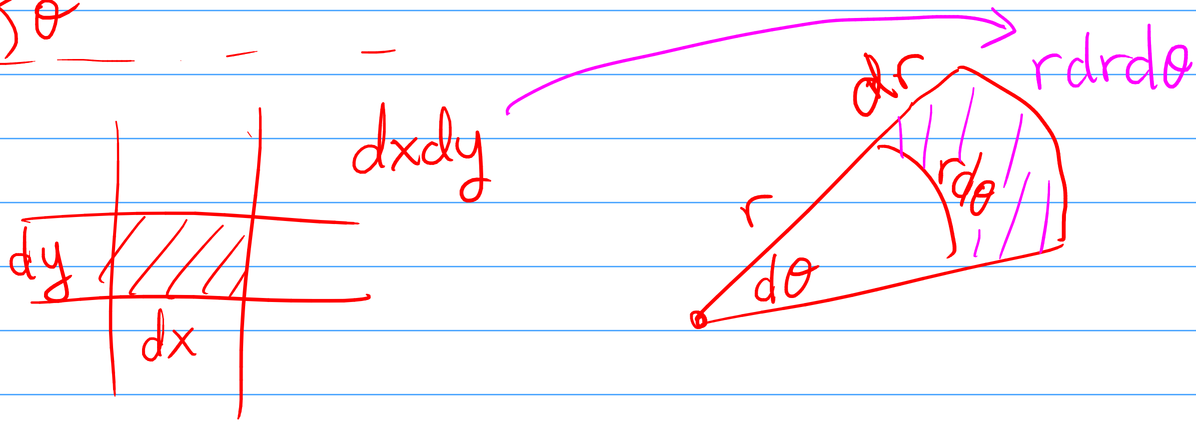
$$= \frac{1}{2\pi} \iint_{\mathbb{R}^2} \exp\left(-\frac{1}{2}(x^2+y^2)\right) dx\,dy$$

$$e^a e^b = e^{a+b}$$

## Polar Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$$




$$\frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-\frac{1}{2}(x^2+y^2)} dx\,dy = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-\frac{1}{2}r^2}\, r\,dr\,d\theta$$

u-sub

$u = \frac{1}{2}r^2 \qquad du = r\,dr$

$$\int_0^{\infty} e^{-u}\,du = -e^{-u}\Big|_0^{\infty}$$

$$= -0-(-1)$$

$$= 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 1 \, d\theta = \frac{1}{2\pi} 2\pi = 1.$$

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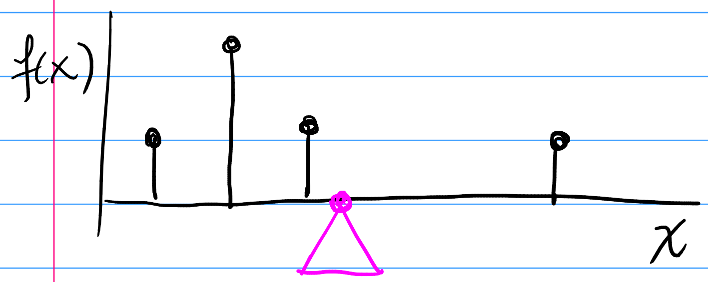
## Expected Value

If $X$ is a RV then the mean or expected value of $X$

denote $\mathbb{E}[X]$

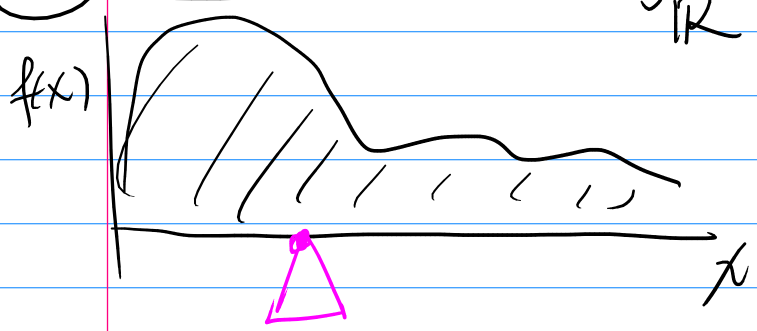
is defined as

① discrete $\mathbb{E}[X] = \sum_{x \in \text{Support}} x f(x)$ PMF



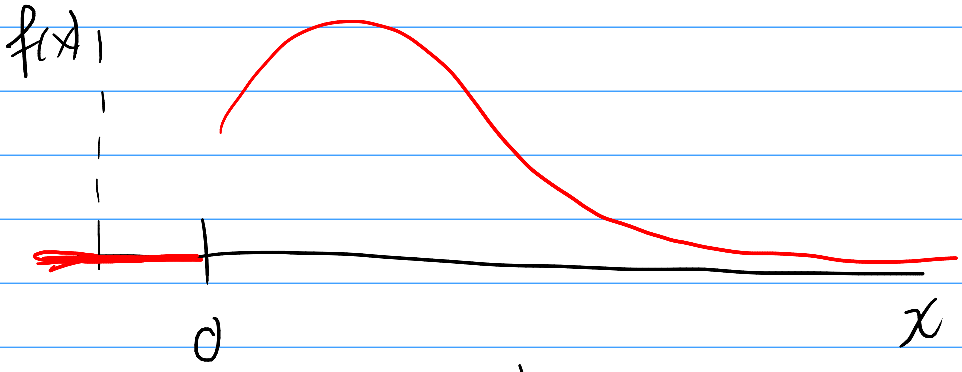

PDF

② cts: $\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) \, dx$



Ex. Let $X \sim \mathrm{Exp}(\lambda)$ $\quad \lambda > 0$, rate

"X dist as exponential dist"




$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$

Q: $E[X]$?

$$EX = \int_{\mathbb{R}} x f(x)\, dx = \int_0^\infty x \lambda e^{-\lambda x}\, dx$$

By parts integration! $\int u\, dv = uv - \int v\, du$

$u = x \qquad v = -e^{-\lambda x}$

$du = dx \qquad dv = \lambda e^{-\lambda x}\, dx$

$$\int u\, dv = uv - \int v\, du = -x e^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\, dx$$

$$\frac{x}{e^x} = \frac{x}{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots}$$

$$(0 - 0) + \int_0^\infty e^{-\lambda x}\, dx$$

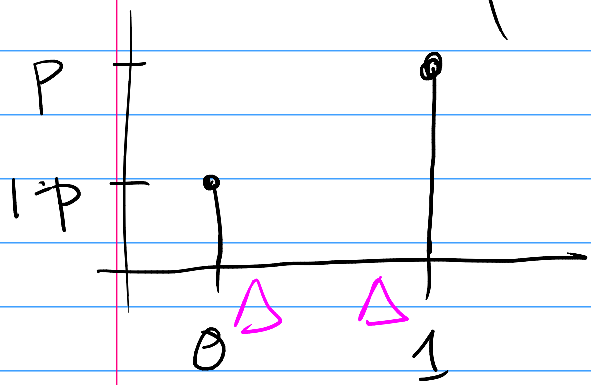
$$= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = -\frac{1}{\lambda}(0 - 1) = \frac{1}{\lambda}$$

Ex. $X \sim \text{Bern}(p)$ $\quad p \in [0,1]$

bernoulli dist.

$X$ = any binary experiment w/ outcome 0/1 and a prob. $p$ of getting a 1

$$f(x) = \begin{cases} 1-p & , x = 0 \\ p & , x = 1 \end{cases}$$




$$E[X] = \sum_{x=0,1} x f(x)$$
$$= (0) f(0) + (1) f(1)$$
$$= 0(1-p) + (1)p$$
$$= p$$

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Ex. $X \sim \text{Bin}(n, p)$ $\quad p \in [0,1]$

integer $\geq 0$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x = 0, 1, \ldots, n$$

# Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

$x = p$
$y = 1-p$

$$(x+y)^2 = \underset{\binom{2}{2}}{1} x^2 y^0 + \underset{\binom{2}{1}}{2} xy + \underset{\binom{2}{0}}{1} x^0 y^2$$

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = 1 \; ?$$

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$$E[X] = \sum_{x=\cancel{0}1}^{n} x f(x) = \sum_{x=1}^{n} \boxed{x \binom{n}{x}} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$\to x \binom{n}{x}$$

$$= x \frac{n!}{\cancel{x}!(n-x)!} \quad (x-1)!$$

$$= \frac{n(n-1)!}{(x-1)!((n-1)-(x-1))!}$$

$$= n \binom{n-1}{x-1}$$

$y = x-1 \iff x = y+1$

$$= n \sum_{y=0}^{n-1} \binom{n-1}{y} p^{y+1} (1-p)^{n-(y+1)}$$

$$= np \sum_{y=0}^{n-1} \underbrace{\binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{\text{PMF of } Bin(n-1,p)}$$

Sum of PMF of Bin over its support $= 1$

$$= np(1)$$