Lecture 10 : Expectation

General trick : PMF/PDF trick

If I can recognize part of a calculation as either

$$\sum_x f(x) \quad \text{or} \quad \int_{\mathbb{R}} f(x)\,dx$$

I can replace these w/ 1.

Functions of RVs

Any function of a RV is itself a RV.

e.g. if X is a RV then so is

$$X^2, \ \log X, \ \sqrt{X}, \ \text{etc.}$$

Theorem: Law of the Unconscious Statistician

If $g: \mathbb{R} \to \mathbb{R}$ and X is a RV then

$$E[g(X)] = \begin{cases} \sum_x g(x) f(x) & \text{discrete} \\ \int_{\mathbb{R}} g(x) f(x)\,dx & \text{cts} \end{cases}$$

Ex. Let $X \sim \text{Exp}(\lambda)$ $\quad \lambda > 0$

$$f(x) = \lambda e^{-\lambda x}, \quad \text{for } x > 0$$

$$\mathbb{E}[X] = 1/\lambda$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^\infty \underbrace{x^2}_{u} \underbrace{\lambda e^{-\lambda x} dx}_{dv}$$

$u = x^2 \qquad v = -e^{-\lambda x}$

$du = 2x\,dx \qquad dv = \lambda e^{-\lambda x} dx$

$$= uv - \int v\,du = -x^2 e^{-\lambda x}\Big|_0^\infty + 2\int_0^\infty e^{-\lambda x} x\,dx$$

$$= 0 - 0 + \frac{2}{\lambda} \underbrace{\int_0^\infty x \underbrace{\lambda e^{-\lambda x}}_{f(x)} dx}_{\mathbb{E}[X]}$$

$$= \frac{2}{\lambda} \frac{1}{\lambda}$$

$$\boxed{\mathbb{E}[X^2] = 2/\lambda^2}$$

Ex. Cauchy Distribution

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$

$$E[X] = \int_{\mathbb{R}} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\, dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx$$

$$= \frac{2}{\pi} \lim_{T\to\infty} \int_0^T \frac{x}{1+x^2}\, dx = \infty$$

$$\frac{x}{1+x^2} \sim \frac{x}{x^2} = \frac{1}{x} \qquad \lim_{T\to\infty} \int_0^T \frac{1}{x}\, dx$$

$$\boxed{\begin{array}{ll} \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty & \int_0^{\infty} \frac{1}{x^2}\, dx < \infty \\ \sum_{i=1}^{\infty} \frac{1}{i} = \infty & \int_0^{\infty} 1/x\, dx = \infty \end{array}}$$

So $EX = \infty$.

Theorem: Properties of Expectation

1. Expectation is linear

$$\mathbb{E}[aX+b] = a\,\mathbb{E}[X] + b$$

pf (cts)

$$\mathbb{E}[\underbrace{aX+b}_{g(x)=ax+b}] = \int \overbrace{(ax+b)}^{g(x)} f(x)\,dx$$

$$= \int ax f(x)\,dx + \int b f(x)\,dx$$

$$= a\underbrace{\int x f(x)\,dx}_{\mathbb{E}X} + b\underbrace{\int f(x)\,dx}_{1}$$

$$= a\,\mathbb{E}X + b$$

2. If $\underbrace{X \geq 0}_{\text{support} \geq 0}$ then $\mathbb{E}X \geq 0$

pf (cts)

$$\mathbb{E}X = \underbrace{\int_0^\infty \underbrace{\underbrace{x}_{\geq 0} \underbrace{f(x)}_{\geq 0}}_{\geq 0} dx}_{\geq 0} \geq 0$$

③ If g_1 and g_2 are functions then

(i) $E[g_1(X)+g_2(X)] = E[g_1(X)] + E[g_2(X)]$

(ii) If $g_1(x) \le g_2(x)$ then

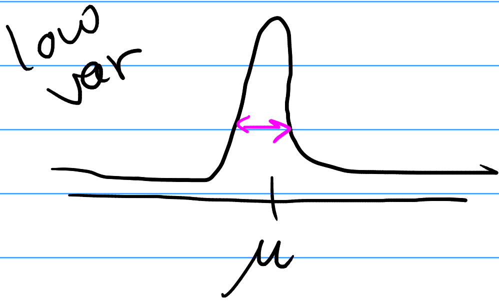
$$Eg_1(X) \le Eg_2(X)$$

④ If $a \le X \le b$ then $a \le EX \le b$.

<u>Defn:</u> Variance

$$\mu = EX = \text{location of dist}$$

$$\sigma^2 = Var(X) = \text{spread of dist}$$



$$Var(X) = E[(X-\mu)^2]$$

$$= E[(X - E[X])^2]$$

Standard deviation: $Sd(X) = \sqrt{Var(X)}$

Ex. $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$

$$\mu = EX = 1/\lambda \quad \text{and} \quad E[X^2] = 2/\lambda^2$$

$$Var(X) = E[\underbrace{(X-\mu)^2}_{g(x)=(x-\mu)^2}] = \int g(x) f(x)\,dx$$

$$= \int_0^\infty (x-\mu)^2 \lambda e^{-\lambda x}\,dx$$

correct, but I'm lazy

$$E[X^2 - 2\mu X + \overbrace{\mu^2}^{(EX)^2}]$$

$$= E[X^2] - 2\mu EX + \mu^2$$

$$= 2/\lambda^2 - 2\left(\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}\right) + \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - 2/\lambda^2 + 1/\lambda^2 = \boxed{1/\lambda^2 = Var(X)}$$

<u>Theorem</u>: Short-Cut formula for Variance

$$Var(X) = E[X^2] - E[X]^2.$$

<u>pf</u> $Var(X) = E[(X-\mu)^2]$

$$= E[X^2 - 2\mu X + \mu^2]$$

$$= \mathbb{E}[X^2] - 2\mu \mathbb{E}X + \mu^2$$

$$= \mathbb{E}[X^2] - 2(\mathbb{E}X)(\mathbb{E}X) + (\mathbb{E}X)^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Revisit ex, $\mathbb{E}X = 1/\lambda$, $\mathbb{E}X^2 = 2/\lambda^2$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

$$= 2/\lambda^2 - (1/\lambda)^2$$

$$= 1/\lambda^2.$$

Ex. $X \sim \mathrm{Bin}(n,p)$

$\mathbb{E}X = np$

$\mathbb{E}[X^2] = np(np - p + 1)$.

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

$$= np(np-p+1) - (np)^2$$

$$= \ldots$$

$$= np(1-p)$$

$$\mathrm{sd}(X) = \sqrt{np(1-p)}$$

Theorem:

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X).$$

two rules!

1. multiply by const $\rightarrow$ var goes like const^2

2. ignore additive consts

Pf

$$\mathrm{Var}(\underbrace{aX+b}_{Y}) = \underbrace{\mathbb{E}[(aX+b)^2]}_{\mathbb{E}[Y^2]} - \underbrace{\mathbb{E}[aX+b]^2}_{\mathbb{E}[Y]^2}$$

$$= \mathbb{E}[a^2X^2 + 2aXb + b^2] - (a\mathbb{E}X + b)^2$$

$$= a^2\mathbb{E}[X^2] + \cancel{2ab\mathbb{E}X} + \cancel{b^2} - (a^2(\mathbb{E}X)^2 + \cancel{2ab\mathbb{E}X} + \cancel{b^2})$$

$$= a^2(\mathbb{E}[X^2] - (\mathbb{E}X)^2)$$

$$= a^2\,\mathrm{Var}(X).$$