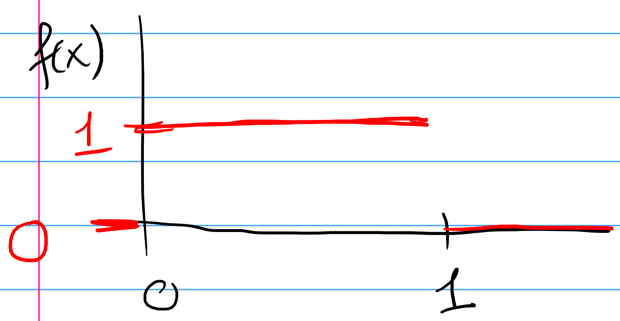
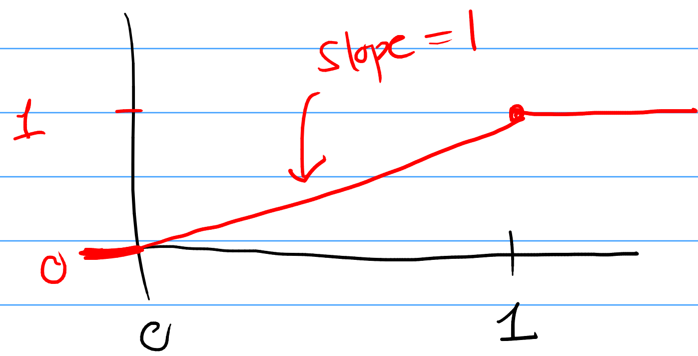
Lecture 15

OHs: Weds 1–2pm

Monday: 10–11am, 3–4pm

Ex. $X \sim U(0,1)$





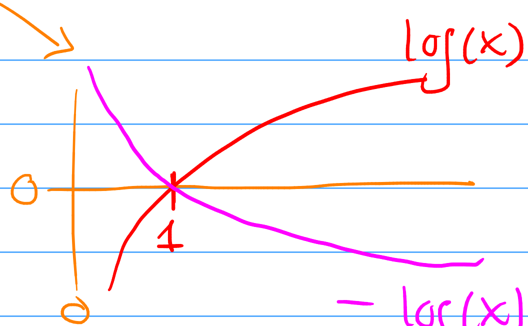
let $Y = -\log(X)$

$F_X(x) = x$ for $0 < x < 1$

$y = \boxed{g(x) = -\log(x)}$

$\Rightarrow -y = \log(x)$

$\Rightarrow \boxed{e^{-y} = x = g^{-1}(y)}$



$F_Y(y) = 1 - F_X(g^{-1}(y))$

$= 1 - F_X(e^{-y})$

$= 1 - e^{-y}$

$0 < x < 1 \Rightarrow \log(x) < 0$

$\Rightarrow y = -\log(x) > 0$

$0 < e^{-y} = \frac{1}{e^y} < 1$

If $Z \sim Exp(1)$ then $f(z) = e^{-z}$ for $z > 0$

$$F(z) = \int_0^z e^{-t}\,dt = 1 - e^{-z}$$

So CDF of Y is the CDF of an $Exp(1)$

So $Y \sim Exp(1)$

What about PDFs?

Theorem: If X is a cts RV and $Y = g(X)$

and if

(1) g is invertible

(2) g^{-1} is differentiable

then

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|}$$

pf. Case 1: g increasing $\to g^{-1}$ inc, $\frac{dg^{-1}}{dy} > 0$

Prev. thm said $F_Y(y) = F_X(g^{-1}(y))$

$$f_Y(y) = \frac{dF_Y}{dy} = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Case 2: g is decreasing $\to g^{-1}$ dec., $\frac{dg^{-1}}{dy} < 0$

Prev. thm said $F_Y(y) = 1 - F_X(g^{-1}(y))$

$$f_Y(y) = \frac{dF_Y}{dy} = -f_X(g^{-1}(y))\frac{dg^{-1}}{dy}$$

$$= f_X(g^{-1}(y))\left(-\frac{dg^{-1}}{dy}\right)$$

$$= f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

Ex. Let $X \sim \text{Gamma}(k, \lambda)$

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}, \quad x > 0$$

Let $Y = 1/X$

$y = g(x) = 1/x \Rightarrow x = \frac{1}{y} = g^{-1}(y)$

So $\frac{dg^{-1}}{dy} = -\frac{1}{y^2}$

thus thrm says

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

$$= f_X\left(\frac{1}{y}\right)\left|-\frac{1}{y^2}\right|$$

$$\boxed{f_Y(y) = \frac{\lambda e^{-\lambda \frac{1}{y}}\left(\lambda \frac{1}{y}\right)^{k-1}}{\Gamma(k)} \frac{1}{y^2}, \quad y > 0}$$

called the Inverse Gamma dist

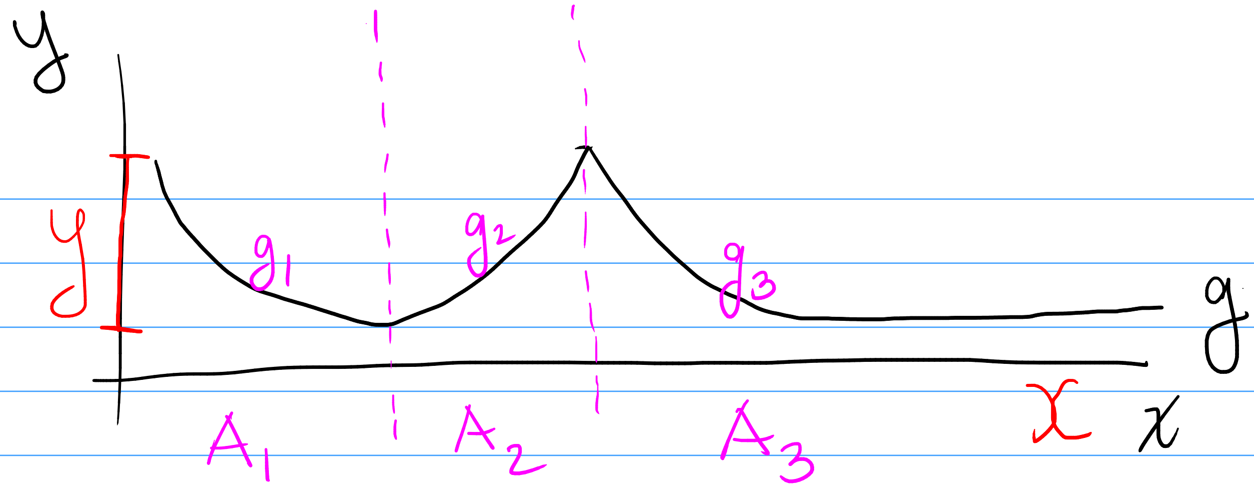
What about non-invertible g?

<u>Theorem</u>: let X is a cts RV w/ support $\mathcal{X}$ and assume that we can partition $\mathcal{X}$ into a collection of sets $A_1, \ldots, A_K$ so that if g_i is the function g restricted to A_i and

1. my prev thrm applies to each g_i

   [g_i is invertible on A_i
   g_i^{-1} is diff'able]

2. all of the g_i have the same range [$\mathcal{Y}$]



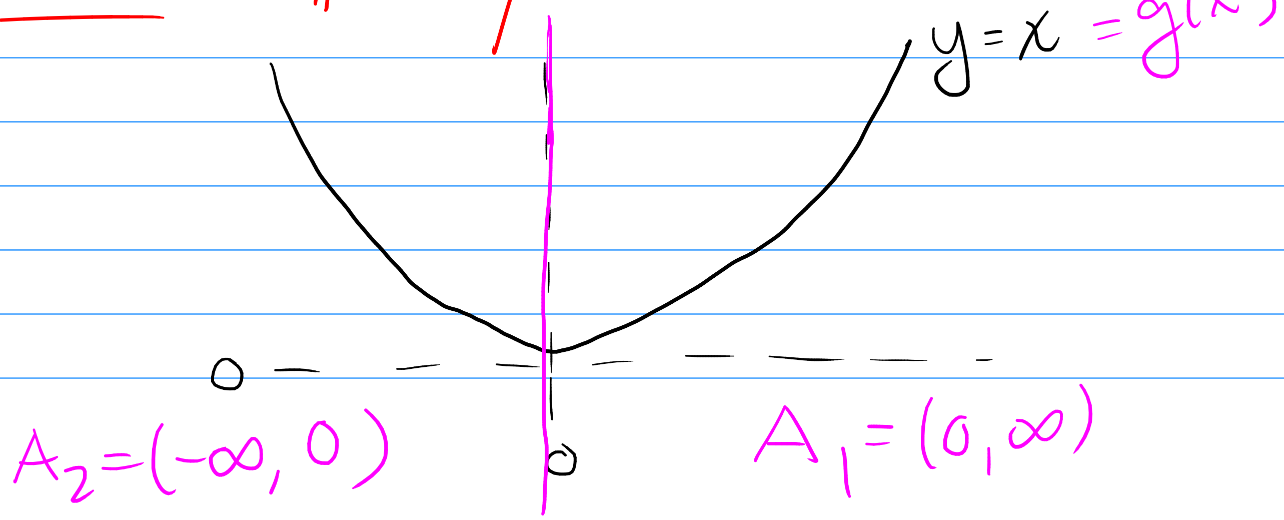
then if $Y = g(X)$ then

$$f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}}{dy} \right|$$

Ex. Chi-Squared Dist

If $X \sim N(0,1)$ and $Y = X^2$
then we say Y has a Chi-Sq. with
one degree of freedom.

Notation: $Y \sim \chi^2(1)$



$$A_1 = (0,\infty), \quad g_1(x) = x^2, \quad g_1^{-1}(y) = \sqrt{y}$$

$$\frac{dg_1^{-1}}{dy} = \frac{1}{2\sqrt{y}}$$

$$A_2 = (-\infty, 0), \quad g_2(x) = x^2, \quad g_2^{-1}(y) = -\sqrt{y}$$

$$\frac{dg_2^{-1}}{dy} = -\frac{1}{2\sqrt{y}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \quad \text{for all } x$$

$$f_Y(y) = f_X(g_1^{-1}(y))\left|\frac{dg_1^{-1}}{dy}\right| + f_X(g_2^{-1}(y))\left|\frac{dg_2^{-1}}{dy}\right|$$

$$= f_X(\sqrt{y})\left|\frac{1}{2\sqrt{y}}\right| + f_X(-\sqrt{y})\left|\frac{-1}{2\sqrt{y}}\right|$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\sqrt{y})^2\right)\frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(-\sqrt{y})^2\right)\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{y}}\left(\exp\left(-\frac{1}{2}y\right) + \exp\left(-\frac{1}{2}y\right)\right)$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}} e^{-\frac{1}{2}y}} \longleftarrow \text{PDF of } \chi^2(1)$$

(Type of Gamma)

Probability Integral Transformation

If X is a cts RV w/ CDF F_X

then

$$F_X(X) \sim U(0,1)$$

Pf. Assume that F_X is strictly increasing

then F_X^{-1} exists.

Let $Y = F_X(X) = g(X)$ where $g = F_X$

our CDF theorem says that

$$F_Y(y) = F_X(g^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

$0 < y < 1$

CDF of a $U(0,1)$

So $Y = F_X(X) \sim U(0,1)$

Know: how to generate $U \sim U(0,1)$

Want: generate some RV following CDF F

Let $Z = F^{-1}(U)$

then $F_Z(z) = \mathbb{P}(Z \le z)$

$$= \mathbb{P}(F^{-1}(U) \le z)$$

$$= \mathbb{P}(U \le F(z))$$

$\boxed{F_U(u) = u}$

$$= F_U(F(z))$$

$$= F(z)$$

So $Z \sim F$.

<u>Algo</u>: 1 generate $U \sim U(0,1)$

2 let $Z = F^{-1}(U)$

Want $\mathrm{Exp}(1)$

CDF is $F(x) = 1 - e^{-x} = y$

$$x = \boxed{F^{-1}(y) = -\log(1-y)}$$