Lecture 11: Inequalities

Theorem: UMVUEs are Unique

If W_1 and W_2 are UMVUEs for $\tau(\theta)$ and $W_1 \neq W_2$.

Consider: $W_3 = \frac{1}{2}(W_1 + W_2)$

then

$$E[W_3] = \frac{1}{2} E W_1 + \frac{1}{2} E W_2$$
$$= \frac{1}{2}\tau(\theta) + \frac{1}{2}\tau(\theta) = \tau(\theta)$$

So W_3 is unbiased for $\tau(\theta)$

and $\mathrm{Var}(W_3) = \mathrm{Var}\left(\frac{1}{2}W_1 + \frac{1}{2}W_2\right)$

$$= \frac{1}{4}\mathrm{Var}(W_1) + \frac{1}{4}\mathrm{Var}(W_2) + \frac{1}{2}\mathrm{Cov}(W_1, W_2)$$

$\mathrm{Var}(aX) = a^2 \mathrm{Var} X$

$\mathrm{Var}(X+Y) = \mathrm{Var} X + \mathrm{Var} Y + 2\mathrm{Cov}(X,Y)$

$\mathrm{Cor}(W_1, W_2) \overset{=}{\leq} 1$

$\Downarrow$

$$\frac{\mathrm{Cov}(W_1, W_2)}{\sqrt{\mathrm{Var} W_1 \, \mathrm{Var} W_2}} \overset{=}{\leq} 1$$

$$\rightarrow \mathrm{Cov}(W_1, W_2) \overset{=}{\leq} \sqrt{\mathrm{Var}W_1 \, \mathrm{Var}W_2}$$

$$\mathrm{Var}(W_3) = \frac{1}{4}\mathrm{Var}(W_1) + \frac{1}{4}\mathrm{Var}(W_2) + \frac{1}{2}\mathrm{Cov}(W_1, W_2)$$

So

$$\mathrm{Var}(W_3) \overset{=}{\leq} \frac{1}{4}\mathrm{Var}(W_1) + \frac{1}{4}\mathrm{Var}(\cancelto{W_1}{W_2}) + \frac{1}{2}\sqrt{\mathrm{Var}W_1 \, \mathrm{Var}\cancelto{W_1}{W_2}}$$

$$\overset{=}{\leq} \frac{1}{4}\mathrm{Var}W_1 + \frac{1}{4}\mathrm{Var}W_1 + \frac{1}{2}\sqrt{(\mathrm{Var}W_1)^2}$$

$$\overset{=}{\leq} \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right)\mathrm{Var}(W_1)$$

$$\overset{=}{\leq} \mathrm{Var}(W_1)$$

We know that $\mathrm{Var}(W_3) \cancel{<} \mathrm{Var}(W_1)$

So all of those $\leq$ are really $=$

So $\mathrm{Cor}(W_1, W_2) = 1$

Another way: Showed $\mathrm{Cov}(W_1, W_2) = \sqrt{\mathrm{Var}W_1 \, \mathrm{Var}W_2}$

so $$\mathrm{cor}(W_1, W_2) = \frac{\mathrm{Cov}(W_1, W_2)}{\sqrt{\mathrm{Var}W_1 \, \mathrm{Var}W_2}} = 1$$

So W_1 and W_2 are perfectly correlated, i.e.!

$$W_1 = a W_2 + b$$

however

$$E W_1 = a \overbrace{E[W_2]}^{T(\theta)} + b = T(\theta)$$

So it must be that $a=1$ and $b=0$

thus $W_1 = W_2$.

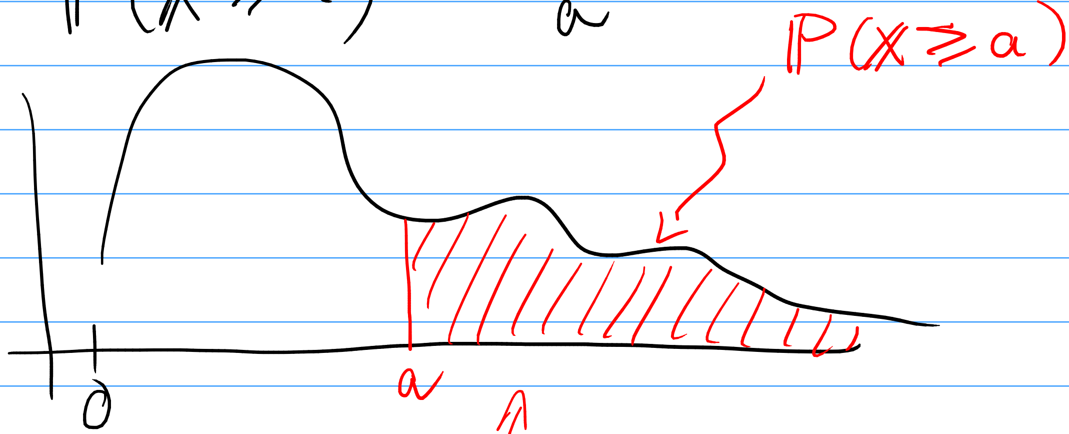
Inequalities

Theorem : Markov's Inequality

If $X \geq 0$ (support of $X \subset [0,\infty)$)

then for any $a \geq 0$ we have

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}X}{a}$$



$$\text{area} \leq \frac{\mathbb{E}X}{a}$$

Pf. (cts case)

$$\mathbb{E}[X] = \int_0^\infty x f(x)\,dx = \overbrace{\underbrace{\int_0^a \underbrace{x}_{\geq 0}\,\underbrace{f(x)}_{\geq 0}\,dx}_{\geq 0}}^{A} + \overbrace{\int_a^\infty x f(x)\,dx}^{B}$$

if $A \geq 0$ then $A + B \geq B$

$$\geq \int_a^\infty x f(x)\,dx \quad x \geq a$$

$$\geq \int_a^\infty a f(x)\,dx$$

$$= a \int_a^\infty f(x)\,dx$$

$$\Rightarrow \mathbb{E}X \geq a\,\mathbb{P}(X \geq a)$$

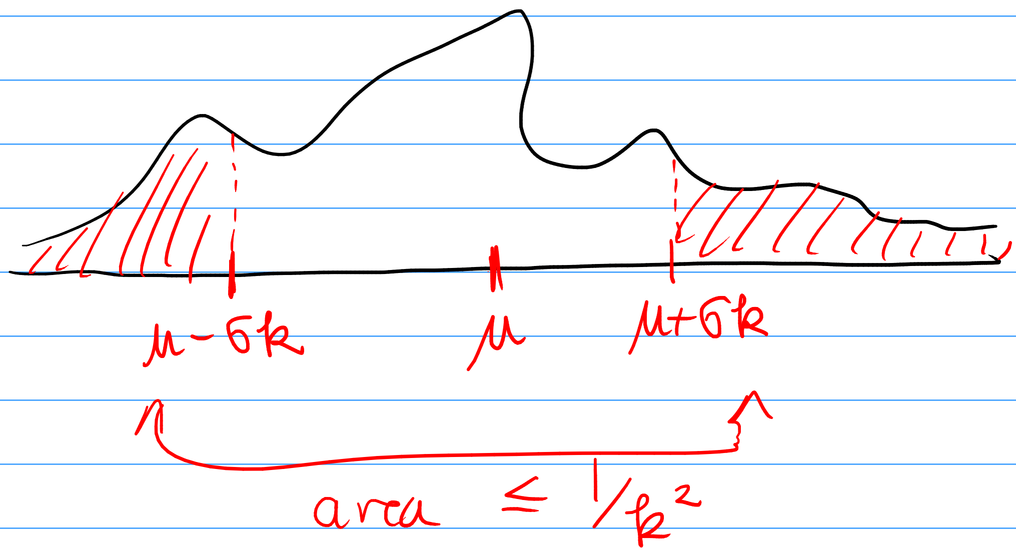
rearrange to get: $\mathbb{P}(X \geq a) \leq \dfrac{\mathbb{E}X}{a}$

Theorem: Chebyshev's Inequality

If X is a RV w/ $\mu = \mathbb{E}X$ and $\sigma^2 = \operatorname{Var} X$

then

$$\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq \frac{1}{k^2}.$$



Pf — let $Y = \left(\frac{X-\mu}{\sigma}\right)^2 = \frac{(X-\mu)^2}{\sigma^2}$

and $a = k^2$

Notice $Y \geq 0$ so apply Markov's

$$P(Y \geq a) \leq \frac{\mathbb{E}Y}{a}$$

$$\mathbb{E}Y = \mathbb{E}\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2}\mathbb{E}[(X-\mu)^2] = \frac{1}{\sigma^2}\operatorname{Var}X$$

$$= \frac{\sigma^2}{\sigma^2} = 1$$

$$\to P\left(\frac{(X-\mu)^2}{\sigma^2} \geq k^2\right) \leq 1/k^2$$

$\sqrt{x^2} = |x|$

$(\sqrt{x})^2 = x$

sqrt

$$= P\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2$$

↑ chebyshev.

Various Versions of Chebyshev's

1. $P\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2$
2. $P\left(\frac{|X-\mu|}{\sigma} < k\right) \geq 1 - 1/k^2$

$$\varepsilon = k\sigma \iff k = \varepsilon/\sigma \iff 1/k^2 = \sigma^2/\varepsilon^2$$

3. $P(|X-\mu| \geq \varepsilon) \leq \sigma^2/\varepsilon^2$

④ $P(|X-\mu| < \varepsilon) \geq 1 - \sigma^2/\varepsilon^2$

Ex. X = # nails produced in a box in some factory

$\mu = \mathbb{E}X = 1000$

$\sigma^2 = \mathrm{Var}\, X = 25 \quad (\sigma = 5)$

what is the prob that

$994 \leq X \leq 1006$

$P(994 \leq X \leq 1006)$

$= \mathbb{P}(|X - 1000| \leq 6)$

$= \mathbb{P}\left(\frac{|X-1000|}{\sigma} \leq \underbrace{1.2}_{k}\right)$

$\geq 1 - 1/(1.2)^2$

$\approx 30\%$

Convergence of RVs

Calc II : talk about convergence of seq of numbers, $x_n \in \mathbb{R}$, $x \in \mathbb{R}$ then

$$x_n \to x$$

$$\left(\lim_{n\to\infty} x_n = x \right)$$

this class: $X_n \longrightarrow X$

are RVs

Recall: $X_n : S \longrightarrow \mathbb{R}$

for some $s \in S$ we have $X_n(s)$

So we can talk about converge of RVs as convergence of functions.

Defn: Pointwise Convergence of Functions

If (f_n) is a seq of functions $f_n : \mathbb{R} \to \mathbb{R}$

and I have some $f : \mathbb{R} \to \mathbb{R}$

I say that f_n converge pointwise to f

if
$$f_n(x) \longrightarrow f(x) \quad \forall x \in \mathbb{R}.$$

We denote this $f_n \xrightarrow{\text{ptwse}} f$.

Ex, $x = 5$

$f_1(5), f_2(5), f_3(5), \ldots \longrightarrow f(5)$

Defn: Sure Convergence of RVs

A seq of RVs (X_n) converges surley to X if $X_n \xrightarrow{\text{ptwse}} X$,

i.e. $\forall s \in S$ then $X_n(s) \longrightarrow X(s)$.

Defn: Almost Sure Convergence of RVs

We say (X_n) converges almost surley to X if $X_n \xrightarrow{\text{ptwse}} X$ on some set $A \subset S$ where $\mathbb{P}(A) = 1$.

Denoted: $X_n \xrightarrow{\text{a.s.}} X$

Basically, a.s. convergence = pwise convergence everywhere in S except possibly some set w/ prob. zero.

<u>Other notation:</u>

$$\mathbb{P}(\{s \in S \mid X_n(s) \to X(s)\}) = 1$$

equiv.

$$\mathbb{P}(\{s \in S \mid X_n(s) \not\to X(s)\}) = 0.$$
