Lecture 22:

Often building CIs by inverting LRT is difficult.

Claim: HT $\leftrightsquigarrow$ CR

What is a level α HT?

Simply a region $R \subset \mathcal{X}$ so that

$$\max_{\theta \in \Theta_0} \mathbb{P}_\theta(\underset{\sim}{X} \in R) \leq \alpha$$

If $H_0: \theta = \theta_0$ v. $H_a: \dots$

simply a region $R \subset \mathcal{X}$ so that

$$\mathbb{P}_{\theta_0}(\underset{\sim}{X} \in R) \leq \alpha$$

or equiv. a region $A \subset \mathcal{X}$ so that

$$\mathbb{P}_{\theta_0}(\underset{\sim}{X} \in A) \geq 1 - \alpha$$

i.e. for $\boxed{H_0: \theta = \theta_0}$ a level α HT is just a statement about $\underset{\sim}{X}$ s.t.

$$\mathbb{P}_{\theta_0}(\text{statement}) \geq 1 - \alpha$$

defines A implicitly

Generalize i.e.

$$\mathbb{P}_{\theta_0}\left(\underbrace{Q(\underset{\sim}{X},\theta_0)\in A}_{\text{just some statement}}\right)\geq 1-\alpha$$

then Accept region is implicitly

$$\{\underset{\sim}{x} : Q(\underset{\sim}{x},\theta_0)\in A\}$$

So to build a CR I can (given dist X_n) come up w/ a statement

$$Q(\underset{\sim}{x},\theta_0)\in A$$

so that this statement has prob. $\geq 1-\alpha$ when $\theta=\theta_0$

[implicitly this defs a test
Accept $= \{\underset{\sim}{x} : Q(\underset{\sim}{x},\theta_0)\in A\}$]

then (maybe) a $1-\alpha$ CR is

$$C(\underset{\sim}{x}) = \{\theta : Q(\underset{\sim}{x},\theta)\in A\}$$

all θ where statement is true

Need: $\min_{\theta} \mathbb{P}_{\theta}(Q(\underset{\sim}{X},\theta)\in A)\geq 1-\alpha$

this is true if $\forall\theta\ \mathbb{P}(Q\in A)\geq 1-\alpha$

One way to ensure this is if

(1) dist. of Q doesn't depend on θ

② A doesn't depend on θ

↳ if both true then

$$\mathbb{P}_\theta(Q \in A) \geq 1-\alpha$$

doesn't depend on θ

This Q is called a pivot (pivotal quantity)

— basically what we called an ancillary quantity prev.

This is called creating a CR by pivoting:

① find some $Q = Q(\underline{X}, \theta)$ whose dist. doesn't depend on θ

② find some A that doesn't depend on θ so that $\mathbb{P}(Q \in A) \geq 1-\alpha$

③ then a $1-\alpha$ CR for θ is

$$C(\underline{X}) = \{\theta : Q(\underline{X}, \theta) \in A\}.$$

Ex: $X_n \overset{iid}{\sim} N(\mu, 1)$ want CI for μ.

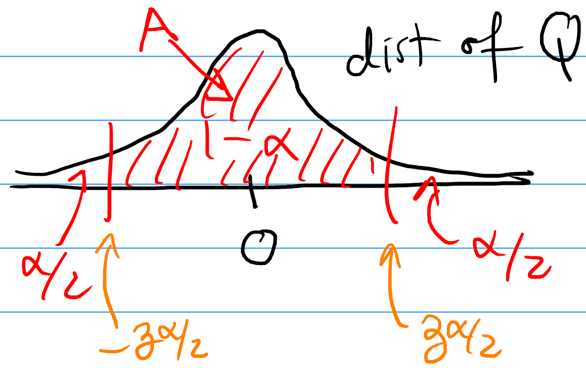
① get pivotal quantity:

$$\bar{X} \sim N(\mu, 1/N)$$

$$\bar{X} - \mu \sim N(0, 1/N)$$

$$Q = \frac{\bar{X} - \mu}{1/\sqrt{N}} \sim N(0,1)$$

(2) Need some statement about Q w/ prob. $1-\alpha$



So

$$\mathbb{P}\left(-z_{\alpha/2} < Q < z_{\alpha/2}\right) = 1 - \alpha$$

(3) $$C(\underset{\sim}{x}) = \{\mu : Q \in A\}$$

$$= \{\mu : -z_{\alpha/2} < Q < z_{\alpha/2}\}$$

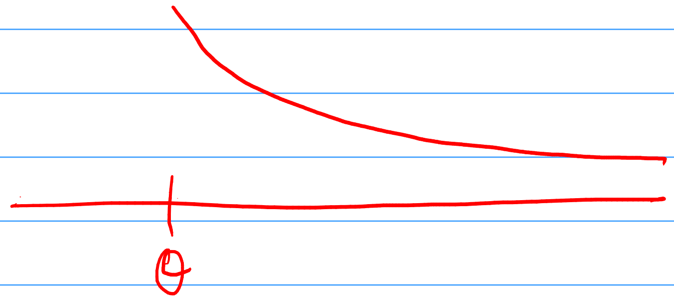
$$= \left\{\mu : -z_{\alpha/2} < \frac{\bar{X} - \mu}{1/\sqrt{N}} < z_{\alpha/2}\right\}$$

$$= \left\{\mu : \underbrace{\bar{X} - \frac{z_{\alpha/2}}{\sqrt{N}}}_{L} < \mu < \underbrace{\bar{X} + z_{\alpha/2}/\sqrt{N}}_{U}\right\}$$

Particularly easy to find pivots for Location/Scale Families

Loc. Fam:

Ex. Shifted Exponential



Location param μ so that

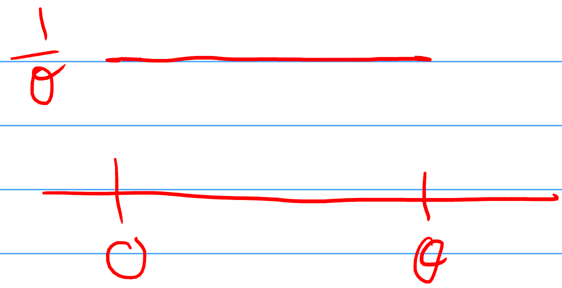
$f_\mu(x) = g(x-\mu)$

where g is free of μ

Pivot for Location families is $\bar{X} - \mu$.

Scale Family:

Ex. $U(0,\theta)$



Scale param σ so that

$f_\sigma(x) = g(x/\sigma)$

where g is free of σ

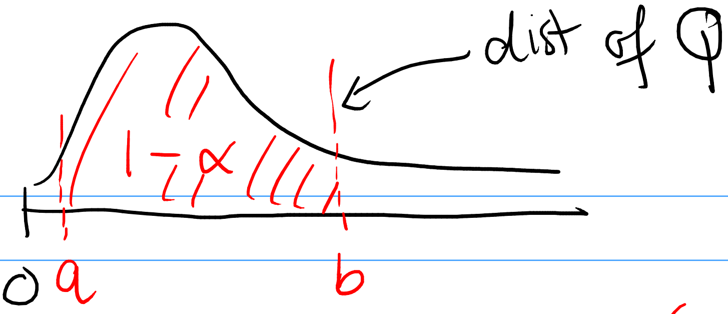
A pivot for scale family is $\bar{X}/\sigma$

Ex. Let $X_n \overset{iid}{\sim} Exp(\lambda)$

1 $T = \sum_n X_n \sim Gamma(N, \lambda)$

if $Q = \frac{2T}{\lambda} \sim Gamma(N, 2) \overset{d}{=} \chi^2(2N)$

2



choose a, b. s.t. $\mathbb{P}(a < \chi^2(2N) < b) = 1-\alpha$

3 $CI = \{\lambda : a < Q < b\}$

$$= \{\lambda : a < \frac{2T}{\lambda} < b\}$$

$$= \{\lambda : \frac{2T}{b} < \lambda < \frac{2T}{a}\}$$

So my CI is $\left[\frac{2\sum x_n}{b}, \frac{2\sum x_n}{a}\right]$

Practical Steps:

1. get $Q = Q(\underline{X}, \theta)$ whose dist free of θ

2. Find a, b s.t.
$$\mathbb{P}(a < Q < b) = 1-\alpha \quad (\text{or} \geq 1-\alpha)$$

3. Solve $a < Q < b$ for θ in middle
$$L < \theta < U$$

Very general way of finding a and b (in cts. case)

Recall that $Q = F_X(X) \sim U(0,1)$

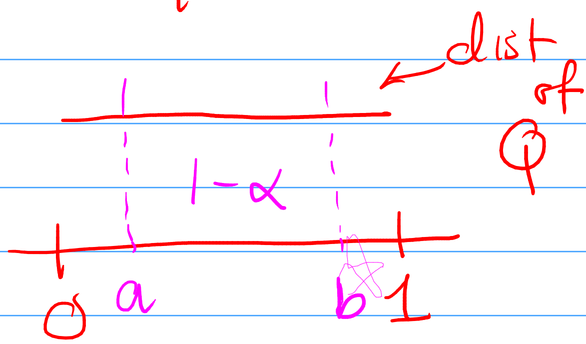
a pivot.

Let T be some statistic then

① Let $Q = F_T(T) \sim U(0,1)$

② $a = \alpha/2$

$b = 1 - \alpha/2$



③ Solve $\frac{\alpha}{2} < F_T(T) < 1 - \alpha/2$

for θ in middle

$L < \theta < U$

Step ③ is easy if F_T is invertible as a fn of θ

Let $g(\theta) = F_T$ as a fn of θ

want to solve

$\alpha/2 < g(\theta) < 1 - \alpha/2$

If g is inc. in θ then

$$\underbrace{g^{-1}(\alpha/2)}_{L} < \underbrace{g^{-1}(g(\theta))}_{\theta} < \underbrace{g^{-1}(1-\alpha/2)}_{U}$$

If g is dec. we get

$$\underbrace{g^{-1}(\alpha/2)}_{U} > \theta > \underbrace{g^{-1}(1-\alpha/2)}_{L}$$

Theorem: Continuous CDF pivot

If T is a stat w/ CDF F_T and

$g(\theta) = F_T$ as a fn of θ

then

(a) if g is inc. a $1-\alpha$ CI is

$L = g^{-1}(\alpha/2)$, $U = g^{-1}(1-\alpha/2)$

(b) if g is dec. a $1-\alpha$ CI is

$L = g^{-1}(1-\alpha/2)$, $U = g^{-1}(\alpha/2)$.

Ex. Let T be a stat w/ CDF given by

$$F_T(t) = \frac{1}{1+\exp(-(t-\mu))}$$

μ = param

Lets create a $1-\alpha$ CI for μ,

$$g(\mu) = \frac{1}{1+\exp(-(t-\mu))}$$

dec. fn of μ

If $y = g(\mu) \Rightarrow \frac{1}{y} = 1+\exp(-(t-\mu))$

$$\Leftrightarrow \frac{1}{y} - 1 = \exp(-t+\mu)$$

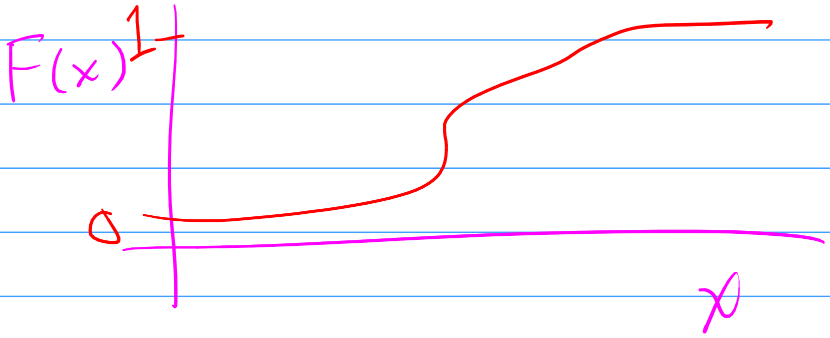
$$\Leftrightarrow \log\left(\frac{1}{y}-1\right) = -t+\mu$$

$$\Leftrightarrow \mu = t + \log\left(\frac{1}{y}-1\right) = g^{-1}(y)$$

then theorem says

$$L = g^{-1}(1-\alpha/2) = \boxed{T + \log\left(\frac{1}{1-\alpha/2}-1\right)}$$

$$U = g^{-1}(\alpha/2) = \boxed{T + \log\left(\frac{1}{\alpha/2}-1\right)}$$



$g(\theta) = F_\theta(x)$