Lecture 3: Sufficiency

Exp. families: $\underset{\sim}{X}_n \overset{iid}{\sim} f_\theta$ then $\theta \in \Theta \subset \mathbb{R}$

$$f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\theta) \exp\left(T(\underset{\sim}{x}) w(\theta)\right)$$

then X_ns are part of an Exponential Family

Can just check marginal

$$\rightarrow f_\theta(x) = h_0(x)\, c_0(\theta) \exp\left(T_0(x)\, w_0(\theta)\right)$$

$$f_\theta(\underset{\sim}{x}) = \prod_{n=1}^{N} f_\theta(x_n) = \prod_{n=1}^{N} h_0(x_n) c_0(\theta) \exp\left(T_0(x_n) w_0(\theta)\right)$$

$$= \underbrace{\prod_{n=1}^{N} h_0(x_n)}_{h(\underset{\sim}{x})}\ \underbrace{c_0(\theta)^N}_{c(\theta)} \exp\left\{\underbrace{\left(\sum_{n=1}^{N} T_0(x_n)\right)}_{T(\underset{\sim}{x})} \underbrace{w_0(\theta)}_{w(\theta)}\right\}$$

$$h(\underset{\sim}{x}) = \prod_n h_0(x_n) \qquad T(\underset{\sim}{x}) = \sum_n T_0(x_n)$$

$$c(\theta) = c_0(\theta)^N \qquad w(\theta) = w_0(\theta)$$

Ex. Pois $\quad X_n \overset{iid}{\sim} \mathrm{Pois}(\lambda)$ $\qquad \exp(\log(\lambda^{x_n})) = \exp(x_n \log \lambda)$

$$f_X(x_n) = \frac{\lambda^{x_n} e^{-\lambda}\, \mathbb{1}(x_n \in \mathbb{N}_0)}{x_n!} = \underbrace{\frac{\mathbb{1}(x_n \in \mathbb{N}_0)}{x_n!}}_{h_0(x_n)} \underbrace{(e^{-\lambda})}_{c_0(\lambda)} \exp(\underbrace{x_n}_{\substack{T_0(x_n)\\ \bar{x}_n}} \underbrace{\log \lambda}_{w_0(\lambda)})$$

So X_ns are jointly a Exp fam.

$$h(\underset{\sim}{x}) = \prod_n \mathbb{1}(x_n \in \mathbb{N}_0) \prod_n \frac{1}{x_n!} = \prod_n h_0(x_n)$$

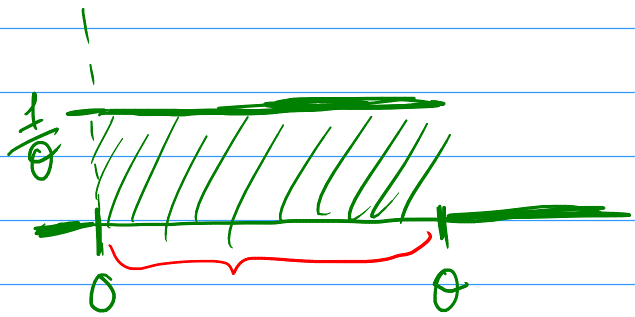
$$c(\lambda) = e^{-N\lambda} = c_0(\lambda)^N$$

$$T(\underset{\sim}{x}) = \sum_n x_n = \sum_n T_0(x_n)$$

$$w(\lambda) = \log \lambda = w_0(\lambda).$$

Ex. Poisson, Exp, Normal, Gamma, Beta are Exp. Fams

Ex. $X_n \overset{iid}{\sim} U(0,\theta)$



Exp fam?

$$f_\theta(x_n) = \frac{1}{\theta} \mathbb{1}(0 < x_n < \theta)$$

no way to write as $(\text{fn of } \underset{\sim}{x})(\text{fn of } \theta)$

So $U(0,\theta)$ is not an exp. fam.

More general fact: If the support of my dist depends on θ — then it isn't an exp. fam.

Setup: $X_n \overset{iid}{\sim} f_\theta$ where $\theta \in \Theta$

Defn: A statistic $T = T(\underset{\sim}{X})$ is sufficient for a parameter θ if

$$f_{\underset{\sim}{X} | T = t}(\underset{\sim}{x}) \text{ is "free" of } \theta$$

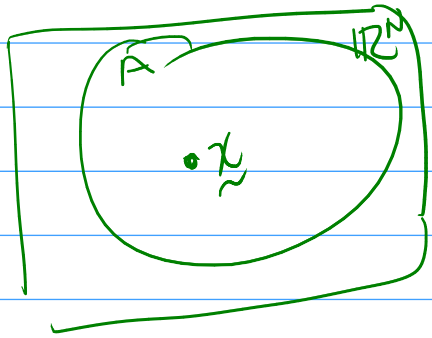
θ doesn't show up in formula...

Ex. $\underset{\sim}{x} \in \mathbb{R}^N$ and $A \subset \mathbb{R}^N$

$$\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x} \ \underline{\text{and}} \ \underset{\sim}{X} \in A)$$

two options: $\underset{\sim}{x} \in A$

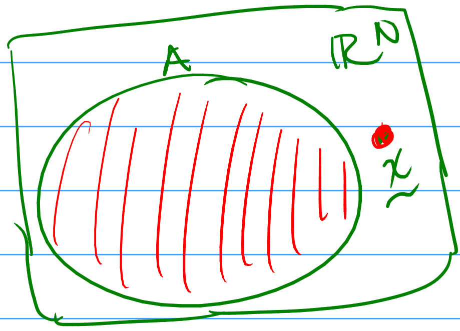
①



$$\mathbb{P}(\underbrace{\underset{\sim}{X} = \underset{\sim}{x}}_{\text{subset}} \ \overset{\cap}{\text{and}} \ \underset{\sim}{X} \in A)$$

$$= \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x})$$

② $\underset{\sim}{x} \notin A$



$$\overbrace{\mathbb{P}(\underline{\underset{\sim}{X} = \underset{\sim}{x}} \ \overset{\cap}{\text{and}} \ \underline{\underset{\sim}{X} \in A})}^{\emptyset}$$

$$= 0$$

$$\rightarrow \quad \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x} \text{ and } \underset{\sim}{X} \in A) = \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x}) \mathbb{1}(\underset{\sim}{x} \in A)$$

$$\rightarrow \quad \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x} \text{ and statement about } \underset{\sim}{X})$$
$$= \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x}) \mathbb{1}(\text{statement about } \underset{\sim}{x})$$

$$\rightarrow \quad \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x} \text{ and } T = t) = \boxed{\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x}) \mathbb{1}(T(\underset{\sim}{x}) = t)}$$

$T(\underset{\sim}{x}) = \sum_n x_n$; $T(\underset{\sim}{X}) = \sum_n X_n$

$$\rightarrow \quad f_{\underset{\sim}{X}, T}(\underset{\sim}{x}, t) = f_{\underset{\sim}{X}}(\underset{\sim}{x}) \mathbb{1}(T(\underset{\sim}{x}) = t)$$

$\underline{\text{Ex}}$ Let $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bernoulli}(\theta)$

$\theta \in [0,1]$

$f_T(t) = \mathbb{P}(T=t)$
$= \binom{3}{t} \theta^t (1-\theta)^{3-t}$

Bin PMF

Let $T = \underbrace{X_1 + X_2 + X_3}_{= \sum_{n=1}^{3} X_n} \sim \text{Bin}(3, \theta)$

$\underline{Q}$: Is T sufficient for θ?

$$f(\underset{\sim}{x} \mid T=t) = \frac{f_{\underset{\sim}{X}, T}(\underset{\sim}{x}, t)}{f_T(t)}$$

$$= \frac{\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x} \overset{\text{and}}{,} T=t)}{\mathbb{P}(T=t)}$$

$$= \frac{\underline{\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x})} \mathbb{1}(T(\underset{\sim}{x}) = t)}{\underline{\mathbb{P}(T=t)}}$$

$X_n \overset{iid}{\sim} \text{Bern}(\theta)$

$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x})$
$= \prod_{n=1}^{3} \theta^{x_n} (1-\theta)^{1-x_n}$
$= \theta^{\sum_n x_n} (1-\theta)^{3-\sum_n x_n}$

$$= \frac{\boxed{\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x})} \mathbb{1}(T(\underset{\sim}{x}) = t)}{\binom{3}{t} \theta^t (1-\theta)^{3-t}}$$

x_1, x_2, x_3

$T(\underset{\sim}{x}) = \sum_{n=1}^{3} x_n$

$$= \frac{\theta^{\sum_n x_n} (1-\theta)^{3-\sum_n x_n} \mathbb{1}(T(\underset{\sim}{x}) = t)}{\binom{3}{t} \theta^t (1-\theta)^{3-t}}$$

only need to consider $T(\underset{\sim}{x}) = \boxed{\sum_n x_n = t}$ (*)

$$= \frac{\cancel{\theta^t} \cancel{(1-\theta)^{3-t}}}{\binom{3}{t} \cancel{\theta^t} \cancel{(1-\theta)^{3-t}}} = \frac{1}{\binom{3}{t}} \text{ is free of } \theta$$

So $T = X_1 + X_2 + X_3$ is sufficient for θ.

Ex. Let $X_n \overset{iid}{\sim} f_\theta$

$$T = (X_1, X_2, \dots, X_N) = \underset{\sim}{X}$$

Q: is T suff. for θ?

$$f(\underset{\sim}{x} \mid T = t) = \frac{f_{\underset{\sim}{X}, T}(\underset{\sim}{x}, t)}{f_T(t)} = \frac{f_{\underset{\sim}{X}, \underset{\sim}{X}}(\underset{\sim}{x}, \underset{\sim}{x})}{f_{\underset{\sim}{X}}(\underset{\sim}{x})}$$

$$= f_{\underset{\sim}{X}}(\underset{\sim}{x}) \Big/ f_{\underset{\sim}{X}}(\underset{\sim}{x}) = 1$$

is free of θ $\Rightarrow T = \underset{\sim}{X}$ is sufficient for θ.

Theorem: Factorization Theorem

T is sufficient for θ iff

there is a fn $g(\theta, T)$ and $h(\underset{\sim}{x})$ so that

$$\underbrace{f_\theta(\underset{\sim}{x})}_{\text{joint dist of } \underset{\sim}{X}} = \underbrace{g(\theta, T)}_{\text{fn of } \underset{\sim}{X} \text{ only thrgh } T(\underset{\sim}{X})} h(\underset{\sim}{x}).$$

Ex. Let $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(\theta)$

$\theta \in [0,1]$

and $T = X_1 + X_2 + X_3 = \sum_{n=1}^{3} X_n$

Is this sufficient?

$$f_\theta(\underset{\sim}{x}) = \prod_{n=1}^{3} f_\theta(x_n) = \prod_{n=1}^{3} \theta^{x_n}(1-\theta)^{1-x_n}\, \mathbb{1}(x_n = 0 \text{ or } 1)$$

$$= \theta^{\overset{T}{\sum_{n=1}^{3} x_n}}(1-\theta)^{3-\overset{T}{\sum_{n=1}^{3} x_n}} \prod_{n=1}^{3} \mathbb{1}(x_n = 0 \text{ or } 1)$$

$$= g(\theta, T)\, h(\underset{\sim}{x})$$

and so T is sufficient for θ.

$g(\theta, T) = \theta^T(1-\theta)^{3-T}$

$T = \sum_{n=1}^{N} x_n$

$h(\underset{\sim}{x}) = \prod_{n=1}^{3} \mathbb{1}(x_n = 0 \text{ or } 1)$

Let $X_n \overset{iid}{\sim} U(0,\theta)$ when $\theta \in \mathbb{R}$

Can I find a SS?

$f_\theta(x_n) = \frac{1}{\theta}\mathbb{1}(0 < x_n < \theta)$

$$f_\theta(\underset{\sim}{x}) = \prod_{n=1}^{N} f_\theta(x_n) = \prod_{n=1}^{N} \frac{1}{\theta}\mathbb{1}(0 < x_n < \theta)$$

$$= \theta^{-N} \prod_n \mathbb{1}(0 < x_n < \theta)$$

$\mathbb{1}(x_n > 0 \text{ and } x_n < \theta)$

$= \mathbb{1}(x_n < \theta)\mathbb{1}(x_n > 0)$

$\prod \mathbb{1}(A_i)$

$= \mathbb{1}(\text{all } A_i)$

$$= \theta^{-N} \prod_n \mathbb{1}(x_n < \theta) \prod_n \mathbb{1}(x_n > 0)$$

$$= \theta^{-N}\, \mathbb{1}(\text{all } x_n < \theta)\, \mathbb{1}(\text{all } x_n > 0)$$

$$= \underbrace{\theta^{-N} \mathbb{1}(x_{(N)} < \theta)}\, \mathbb{1}(x_{(1)} > 0)$$

$$= g(\theta, T)\, h(\underset{\sim}{x})$$

$g(\theta, T) = \theta^{-N} \mathbb{1}(T < \theta)$

$T = X_{(N)}$

$h(\underset{\sim}{x}) = \mathbb{1}(x_{(1)} > 0)$

So by the Factorization theorem $T = X_{(N)}$ is sufficient for θ.

Theorem:

<u>Sufficiency and Exp. Fams</u>

Let $X_n \overset{iid}{\sim} f_\theta$ and

JOINT

$$f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\theta) \exp\left(T(\underset{\sim}{x})\, \omega(\theta)\right)$$

$\hookrightarrow$ i.e. it is an exp. fam.

then $T(\underset{\sim}{X})$ is sufficient for θ.

<u>Ex.</u> $X_n \overset{iid}{\sim} \mathrm{Pois}(\lambda)$

$$f_\lambda(x_n) = \frac{\lambda^{x_n} e^{-\lambda}}{x_n!} \mathbb{1}(x_n \in \mathbb{N}_0)$$

$$= \underbrace{\left(\frac{1}{x_n!}\right) \mathbb{1}(x_n \in \mathbb{N}_0)}_{h_0(x_n)} \underbrace{e^{-\lambda}}_{c_0(\lambda)} \exp(\underbrace{x_n}_{T_0(x_n)} \underbrace{\log \lambda}_{w(\lambda)})$$

then

$h(\underset{\sim}{x}) = \prod_n h_0(x_n)$; $c(\lambda) = c_0(\lambda)^N$

$T(\underset{\sim}{x}) = \sum_n T_0(x_n) = \sum_n x_n$

Punchline: $T \overset{=\sum_n X_n}{}$ is sufficient for λ.

pf. $f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\theta)\, \exp(T(\underset{\sim}{x})\, w(\theta))$

let $h(\underset{\sim}{x}) = h(\underset{\sim}{x})$

$g(\theta, T) = c(\theta) \exp(T\, w(\theta))$

$T = T(\underset{\sim}{x})$

then $= g(\theta, T)\, h(\underset{\sim}{x})$

and so by Factorization Theorem, T is sufficient for θ.