Lecture 14

Midterm on Nov. 10^{th}
QP/SP 5-8
Lectures 8-13

Continue example from last time:

Q: $\hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(X_n - \bar{X})^2$, is this consistent for σ^2?

Notice: $\hat{\sigma}^2 = \frac{N-1}{N} S^2 = C_n S^2$ when $C_n = \frac{N-1}{N}$

so by my algebraic properties

Since $C_n \to 1$ as $N \to \infty$

and $S^2 \xrightarrow{P} \sigma^2$

thus $\hat{\sigma}^2 = C_n S^2 \xrightarrow{P} 1 \cdot \sigma^2 = \sigma^2$

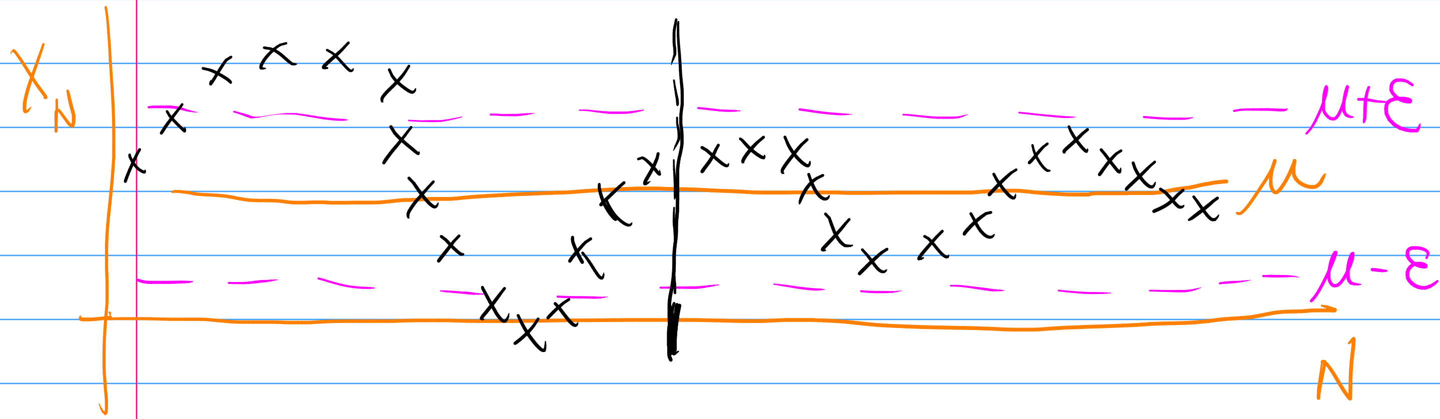
i.e. $\hat{\sigma}^2$ is consistent for σ^2.

Theorem: Strong Law of Large Numbers (SLLN)

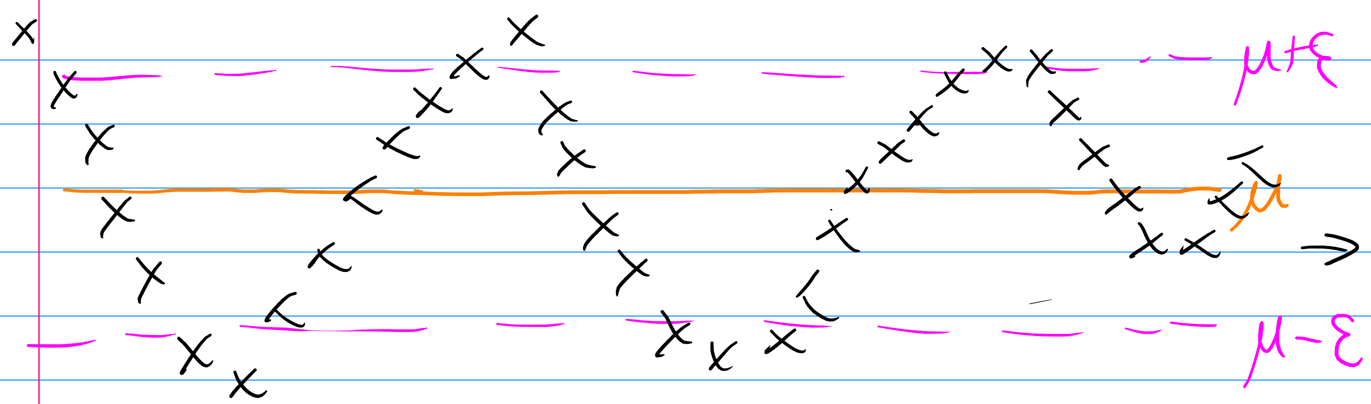
If $X_n \overset{iid}{\sim} f$ w/ $\mathbb{E}X_n = \mu$, $Var(X_n) = \sigma^2 < \infty$

then $\bar{X}_N = \frac{1}{N}\sum_{n=1}^{N} X_n \xrightarrow{a.s.} \mu$.

Convergence of a seq of numbers $x_n \to \mu$

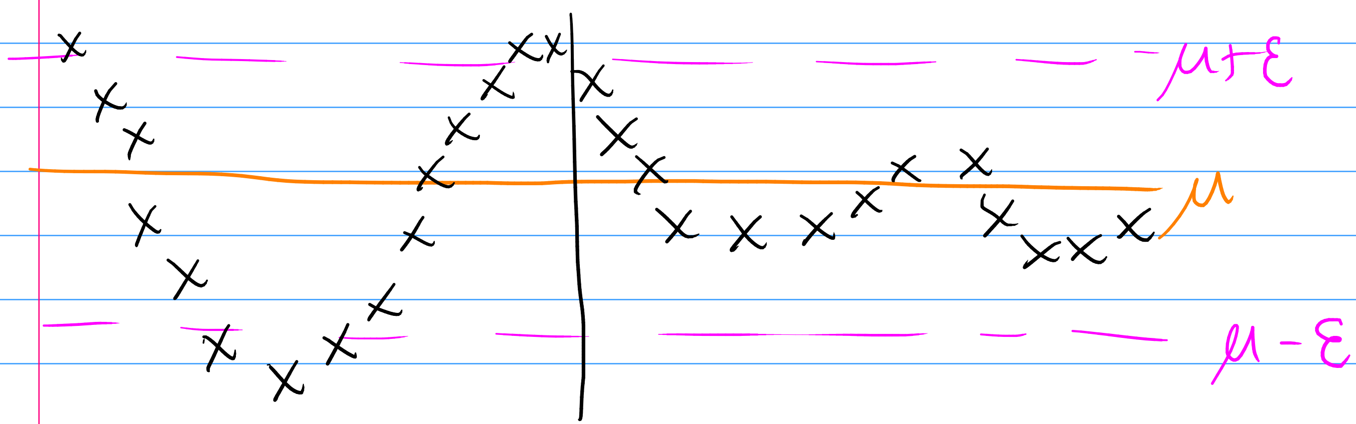


WLLN: $\bar{X} \xrightarrow{P} \mu$



So as $N \to \infty$ Prob $\bar{X}_N$ is more than ε away from μ goes to zero.

SLLN: $\bar{X} \xrightarrow{a.s.} \mu$



Sums of RVs

1. $\sum_n X_n \to \infty$

2. $\frac{1}{N} \sum_n X_n \to \mu$ (under same condition WLLN, SLLN)

   constant

3. $\frac{1}{\sqrt{N}} \sum_n X_n \to$ non degenerate dist

   proper scaling

Theorem: Central Limit Theorem (CLT)

If $X_n \overset{iid}{\sim} f$ w/ $\mathbb{E} X_n = \mu$, $\text{Var} X_n = \sigma^2 < \infty$

$$\sqrt{N} \left(\frac{\bar{X}_N - \mu}{\sigma} \right) \xrightarrow{d} N(0,1).$$

If $\mu = 0$ and $\sigma = 1$ then

$$\sqrt{N} \bar{X} = \sqrt{N} \frac{1}{N} \sum_n X_n = \frac{1}{\sqrt{N}} \sum_n X_n \to N(0,1)$$

Intuition CLT: $\bar{X} \approx N(\mu, \sigma^2/N)$

would like to say: $\cancel{\bar{X}_N \xrightarrow{d} N(\mu, \sigma^2/N)}$

no N in limit

Proper way to write CLT

(1) $\sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right) \xrightarrow{d} N(0,1)$

(2) $\sqrt{N}\left(\bar{X}-\mu\right) \xrightarrow{d} N(0,\sigma^2)$

(3) $\frac{\bar{X}-\mu}{\sigma/\sqrt{N}} \xrightarrow{d} N(0,1)$

z-score

$E\bar{X}=\mu$

$\operatorname{Var}\bar{X}=\sigma^2/N \to \operatorname{sd}\bar{X} = \sigma/\sqrt{N}$

(notation that means above (1)-(3))

(4) $\bar{X} \sim AN(\mu, \sigma^2/N)$

asymptotically normal

Ex. $X_n \overset{iid}{\sim} \text{Bern}(p)$

$\mu = EX_n = p \quad , \quad \sigma^2 = \operatorname{Var}X_n = p(1-p)$

$\sigma = \sqrt{p(1-p)}$

CLT says

% of 1s

$$\sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right) = \sqrt{N}\left(\frac{\bar{X}-p}{\sqrt{p(1-p)}}\right) \xrightarrow{d} N(0,1)$$

Intro stats: $\hat{p} = \bar{X}$ = sample proportion

CI: $\hat{p} \pm \underbrace{2\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}}_{\text{MOE}} = \hat{p} \pm 2\,\operatorname{sd}(\hat{p})$

$$\hat{p} = \bar{X} \sim AN\left(p, \frac{p(1-p)}{N}\right)$$

Ex. $X_n \overset{iid}{\sim} Pois(\lambda)$

$$\mu = EX_n = \lambda = Var X_n = \sigma^2$$

$$\text{so } \sigma = \sqrt{\lambda}$$

CLT: $\bar{X} \sim AN(\lambda, \lambda/N)$

Theorem: MGFs and Convergence in Dist.

If I have a seq of RVs X_n w/ MGFs M_n and a limit X w/ MGF M

then if $M_n \to M$ (pointwise)

then $X_n \xrightarrow{d} X$.

Theorem: Taylor's Theorem

If g is k-times diff'able then the k^{th} order Taylor poly about a is

$$T_k(x) = \sum_{r=0}^{k} \frac{g^{(r)}(a)}{r!}(x-a)^k$$

then $T_k(x) \to g(x)$ as $x \to a$

i.e $T_k \approx g$ when $x \approx a$

e.g. $g(x) \approx g(a) + \frac{g'(a)}{1}(x-a) + \frac{g''(a)}{2}(x-a)^2$
$+ \frac{g'''(a)}{3!}(x-a)^3 + \cdots$

when $x \approx a$

Pf. of CLT

$Y_N = \sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right)$ want: $Y_N \xrightarrow{d} N(0,1)$

$Z_n = \frac{X_n - \mu}{\sigma}$ = standarized X_n

notice: $\mathbb{E} Z_n = 0$ $\text{Var} Z_n = 1$

$$Y_N = \sqrt{N}\left(\frac{\bar{X}_N - \mu}{\sigma}\right)$$

$$= \sqrt{N}\left(\frac{\frac{1}{N}\sum_n X_n - \frac{1}{N}\sum_n \mu}{\sigma}\right)$$

$$= \frac{\sqrt{N}}{N}\left(\frac{\sum_n X_n - \sum_n \mu}{\sigma}\right)$$

$$= \frac{\sqrt{N}}{N}\sum_n\left(\frac{X_n - \mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{N}}\sum_n Z_n$$

$$Y_N = \frac{1}{\sqrt{N}} \sum_n Z_n$$ ← indep w/ mean 0, Var 1

Get MGF of Y_N

independence of Z_n

$M_{aX+b}(t) = e^{tb} M(at)$

$$M_{Y_N}(t) = \prod_n M_{Z_n}(t/\sqrt{N})$$

$$= M(t/\sqrt{N})^N$$

MGF of any Z_n

2^{nd} order Taylor approx. of $M(t)$ about 0

$$M(t) \approx \underbrace{\overset{1}{M(0)}}_{\mathbb{E}e^{0X}=1} + \underbrace{\frac{\overset{0}{M'(0)}}{1}}_{\mathbb{E}Z_n = 0}(t-0) + \underbrace{\frac{M''(0)}{2}}_{\frac{Var(Z_n)}{2}}(t-0)^2$$

$\frac{1}{2}t^2$

$$= 1 + \frac{t^2}{2}$$

$$M_{Y_N}(t) = M(t/\sqrt{N})^N \approx \left(1 + \frac{t^2/2}{N}\right)^N$$

$$\to e^{t^2/2} \quad \text{as } N \to \infty$$

MGF of $N(0,1)$

So $Y_N \xrightarrow{d} N(0,1)$.