Lecture 17:

Hypothesis Test:

$$H_0 : \theta \in \Theta_0 \quad v. \quad H_a : \theta \in \Theta_a$$

where $\Theta_0 \cup \Theta_a = \Theta$ and $\Theta_0 \Theta_a = \phi$.

Hypothesis Testing Procedure

Idea: determine for which $\underset{\sim}{x}$ its more plausible that $\theta \in \Theta_0$ and for which $\underset{\sim}{x}$ its more plausible that $\theta \in \Theta_a$

If $\mathcal{X}$ is the support $\underset{\sim}{X} = (X_1, \ldots, X_N)$

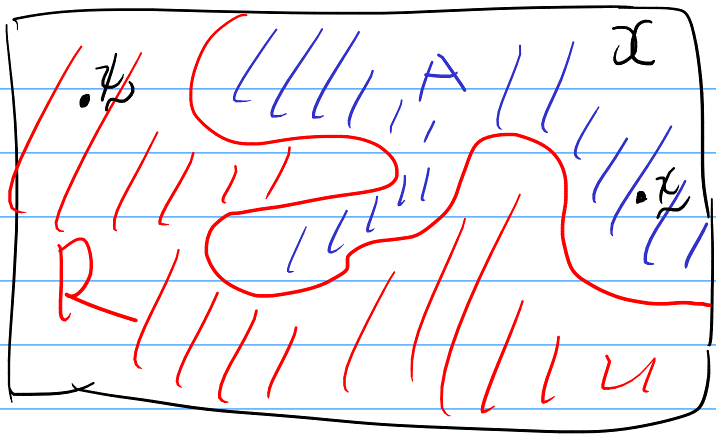
Typically $\mathcal{X} \subset \mathbb{R}^N$

A HT procedure is simply a rule that partitions $\mathcal{X}$ into

$$\mathcal{X} = A \cup R$$

accept region (accept H_0)

reject region (reject H_0)



We "reject H_0" if $\underset{\sim}{x} \in R$

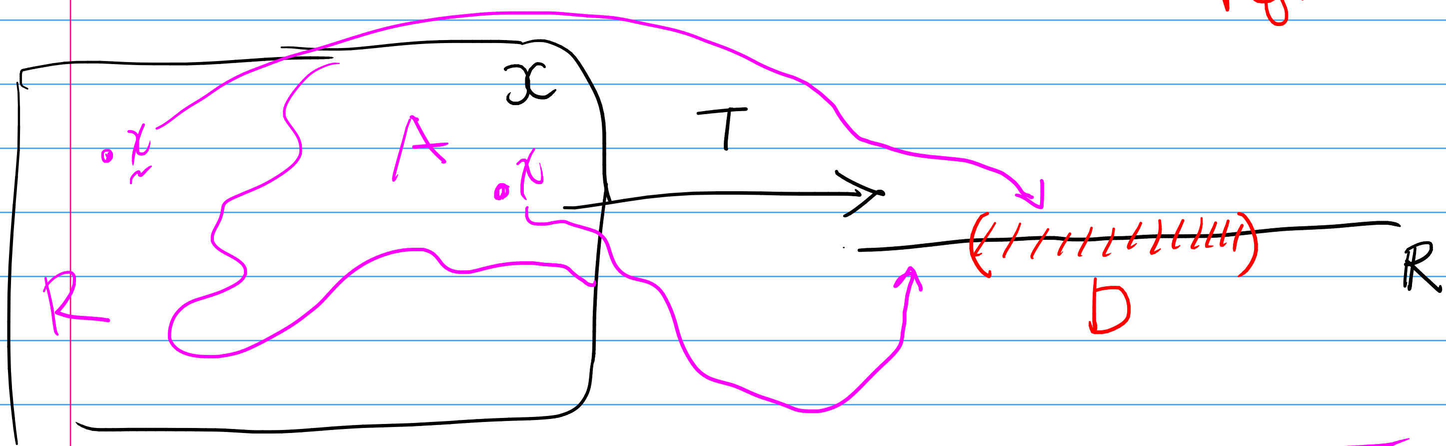
we "fail to reject H_0" if $\underset{\sim}{x} \in A$

Often we can define R (equiv. A) through a "test statistic" so that

T

$$R = \{ \underset{\sim}{x} \in \mathcal{X} \mid T(\underset{\sim}{x}) \in D \}$$

critical region



<u>ex.</u> $X_n \overset{iid}{\sim} f$ w/ mean θ

$$H_0: \theta > 5 \quad v. \quad H_a: \theta \le 5$$

$T = \overline{X}$ reject iff $\overline{X} \le 5$

E.g. if $\bar{X} = 100$ probably shouldn't reject

$\bar{X} = -10$ probably reject.

Defn: Type I and II errors

Truth			
null is true	$\theta \in \Theta_0$	correct decision	Type I error
alt. is true	$\theta \in \Theta_a$	Type II error	correct decision
	procedure outcome	accept H_0	reject H_0

Goal: to create a procedure that minimizes prob. of Type I and II errors — equiv. max. prob. of correct decision.

often minimizing type I and II errors is opposing.

Defn: Power Function

For any $\theta \in \Theta$ the power function β is defined as

$$\beta(\theta) = P_\theta(\underset{\sim}{X} \in R)$$

prob. I reject if my param is θ

For $\theta \in \Theta_0$ then $\beta(\theta)$ is the prob. of a type I error.

For $\theta \in \Theta_a$ then $\beta(\theta)$ is the prob. of correctly rej. H_0

Equiv. $1-\beta(\theta)$ is the prob. of a type II error.

Ex. $X_1, \dots, X_5 \overset{iid}{\sim} \text{Bern}(p)$ $p \in [0,1]$

$$H_0 : p \le \tfrac{1}{2} \quad v. \quad H_a : p > 1/2$$

$$\left[\Theta = [0,1] \; ; \; \Theta_0 = [0, 1/2] \; ; \; \Theta_a = (1/2, 1] \right]$$

Need a HT:

$$R = \{(1,1,1,1,1)\}$$

Could write in terms of a test stat.

$T = \sum_{n=1}^{5} X_n$ then $R = \{\underset{\sim}{x} \mid T(\underset{\sim}{x}) = 5\}$

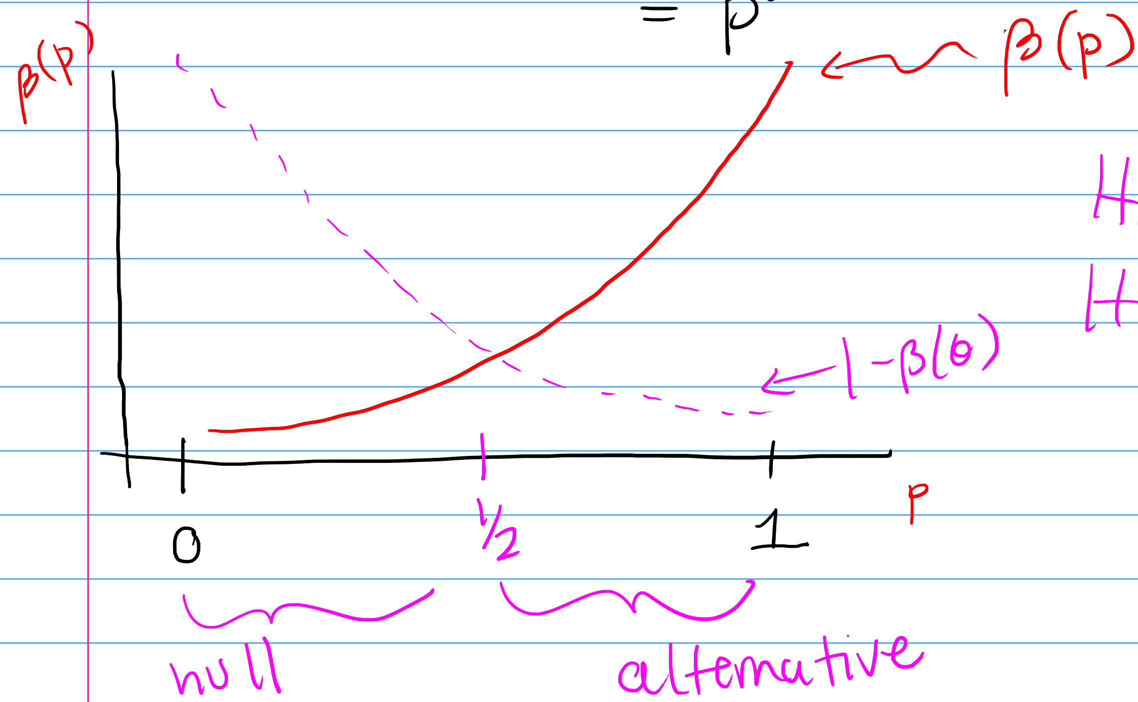
\# of 1s

$$T \sim \text{Bin}(5, p)$$

$D = \{5\}$ critical region

What is β?

$$\beta(p) = \mathbb{P}_p(\underset{\sim}{X} \in R) = \mathbb{P}_p(T=5)$$
$$= \binom{5}{5} p^5 (1-p)^{5-5}$$
$$= p^5$$



$H_0: p \le 1/2$
$H_a: p > 1/2$

① What is the max. type I err prob?

For $\theta \in \Theta$ then $\beta(\theta)$ = type I err prob

So

$$\text{max type I err prob} = \max_{\theta \in \Theta_0} \beta(\theta)$$
$$= \max_{p \le 1/2} \beta(p)$$
$$= \beta(1/2) = (1/2)^5 = 1/32$$

② what's max type II prob?

When $\theta \in \Theta_a$ then $1-\beta(\theta)$ = prob. of type II

So max type II $= \max_{\theta \in \Theta_a} 1-\beta(\theta)$

$$= \max_{p > 1/2} 1-\beta(p)$$

$$= \max_{p > 1/2} 1-p^5$$

$$= 1-(1/2)^5 = 1-1/32$$

Ex. Consider a different test

$$R = \{\underset{\sim}{x} \mid T \geq 3\}$$

$$= \{\underset{\sim}{x} \mid T/5 \geq 1/2\}$$

sample propn.

$$\beta(p) = \mathbb{P}_p(\underset{\sim}{X} \in R) = \mathbb{P}_p(T \geq 3)$$

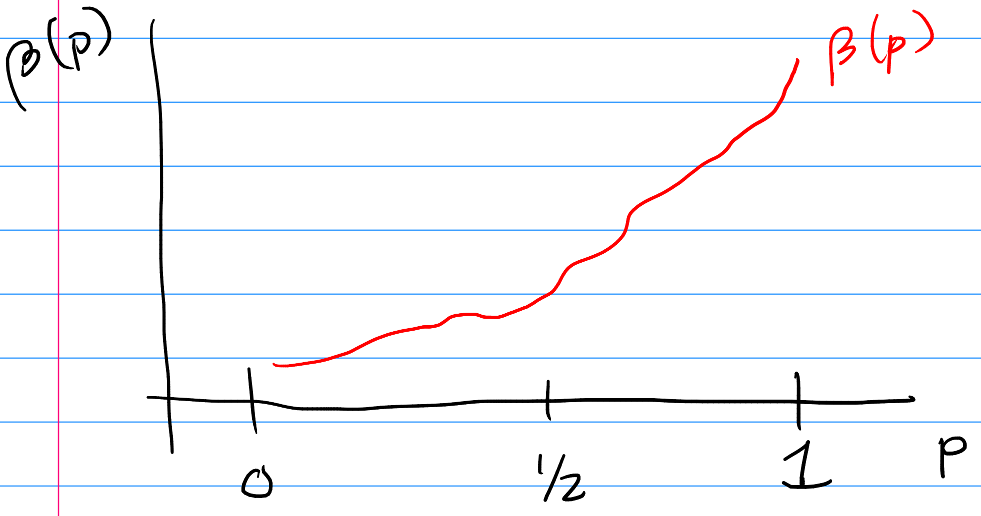
$$= \mathbb{P}(T=3) + \mathbb{P}(T=4) + \mathbb{P}(T=5)$$

$$= \binom{5}{3} p^3 (1-p)^{5-3} + \binom{5}{4} p^4 (1-p)^{5-4} + p^5$$

$$\vdots$$

$$= p^3 (6p^2 - 15p + 10)$$

$$\frac{\partial \beta}{\partial p} = 30p^2(p-1)^2 > 0$$



① max type I err prob. : $\max_{p \le 1/2} \beta(p) = \beta(1/2)$

② max type II err prob : $\max_{p > 1/2} 1-\beta(p) = 1-\beta(1/2)$

Defn! Size and Level α tests

A test is size $\alpha \in [0,1]$ if

$$\alpha = \max_{\theta \in \Theta_0} \beta(\theta) = \text{max. type I err.}$$

a level α test is if

$$\max_{\theta \in \Theta_0} \beta(\theta) \le \alpha.$$

Game: try to find tests that maximize $\beta(\theta)$ when $\theta \in \Theta_a$

subject to a constraint of being size or level α:

$$\max_{\theta \in \Theta_0} \beta(\theta) = \alpha \quad \text{or} \leq \alpha .$$

Ideal test: