Lecture 19

LRT Example $\quad X_n \overset{iid}{\sim} \text{Shifted Exp}(1, \theta)$

$$f_\theta(x) = e^{-(x-\theta)} \mathbb{1}(x > \theta)$$

Consider $H_0: \theta \le 0$ v. $H_a: \theta > 0$

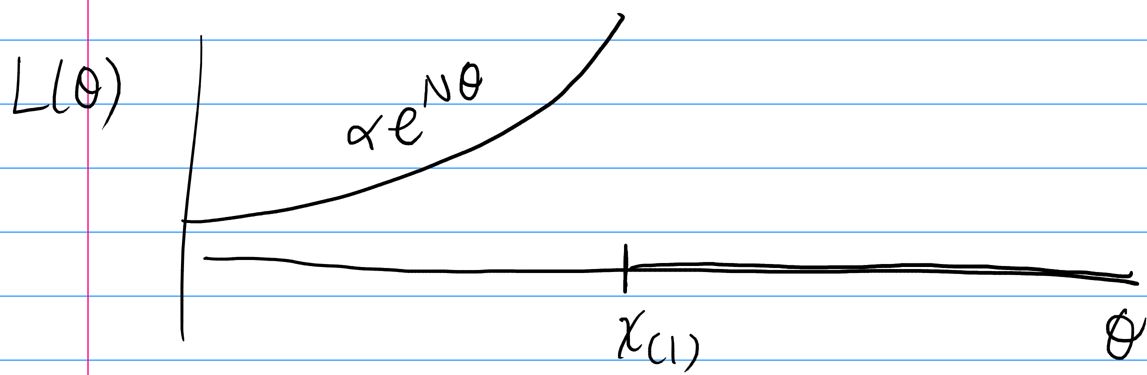
LRT

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} \quad ; \quad R = \{\lambda(\underset{\sim}{x}) \le c\}$$

$$L(\theta) = \prod_n e^{-(x_n - \theta)} \mathbb{1}(x_n > \theta)$$

$$= e^{-\sum_n (x_n - \theta)} \mathbb{1}(x_{(1)} > \theta)$$

$$= e^{-N\bar{x}} e^{N\theta} \mathbb{1}(x_{(1)} > \theta)$$

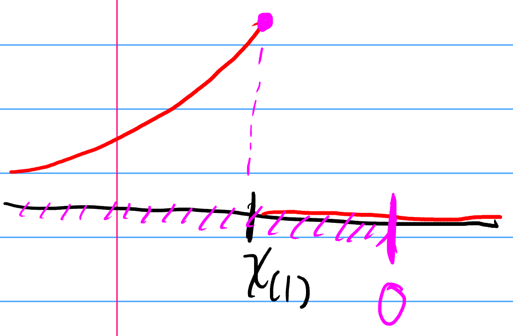


So $\boxed{\hat{\theta} = x_{(1)}}$

Now $H_0: \theta \le 0$ want to find $\hat{\theta}_0$

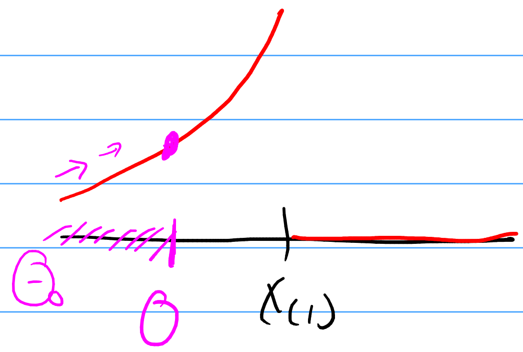
Two cases

<u>Case 1 : $X_{(1)} < 0$</u>



So $\hat{\theta}_0 = X_{(1)}$

<u>Case 2: $X_{(1)} \geq 0$</u>



So $\hat{\theta}_0 = \theta$

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \begin{cases} \dfrac{L(X_{(1)})}{L(X_{(1)})} = 1 & X_{(1)} < \theta \\ \dfrac{L(\theta)}{L(X_{(1)})} & X_{(1)} \geq \theta \end{cases}$$

So basically $\lambda = L(\theta)/L(X_{(1)})$.

$$= \frac{\cancel{e^{-N\bar{X}}}\, e^{N\theta}}{\cancel{e^{-N\bar{X}}}\, e^{NX_{(1)}}} = e^{-NX_{(1)}}$$

Reject when $\lambda \leq c$

$$\Leftrightarrow e^{-NX_{(1)}} \leq c$$

$$\Leftrightarrow -NX_{(1)} \leq \log c$$

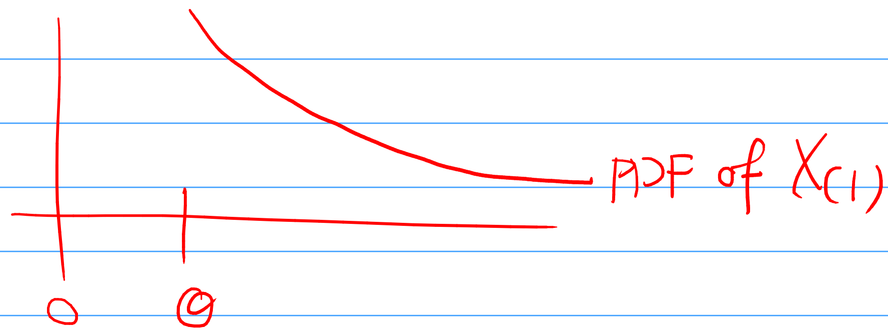
$$\Leftrightarrow X_{(1)} \geq \underbrace{-\frac{1}{N}\log c}_{c^*}$$

My LRT is to reject when $X_{(1)}$ is large enough ($\geq c^*$)

Choose c^* so that $\underbrace{\mathbb{P}_\theta(X_{(1)} > c^*)}= \beta(\theta)$

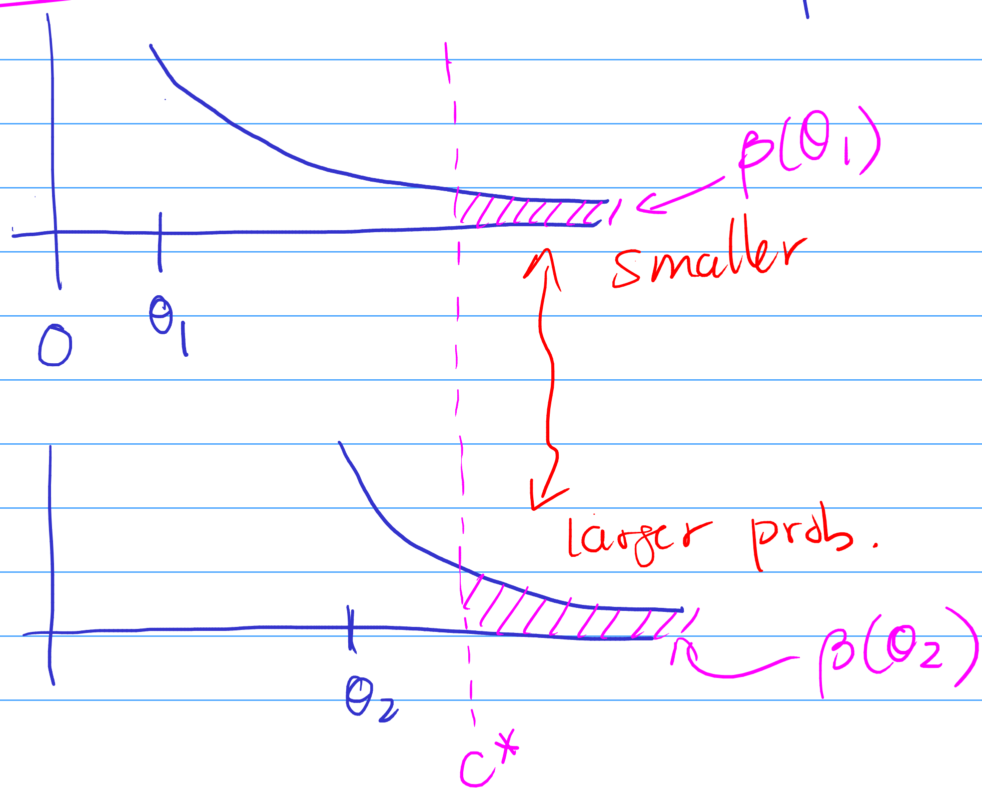
$$\max_{\theta \in \Theta_0} \mathbb{P}_\theta(\text{rej.}) = \alpha$$

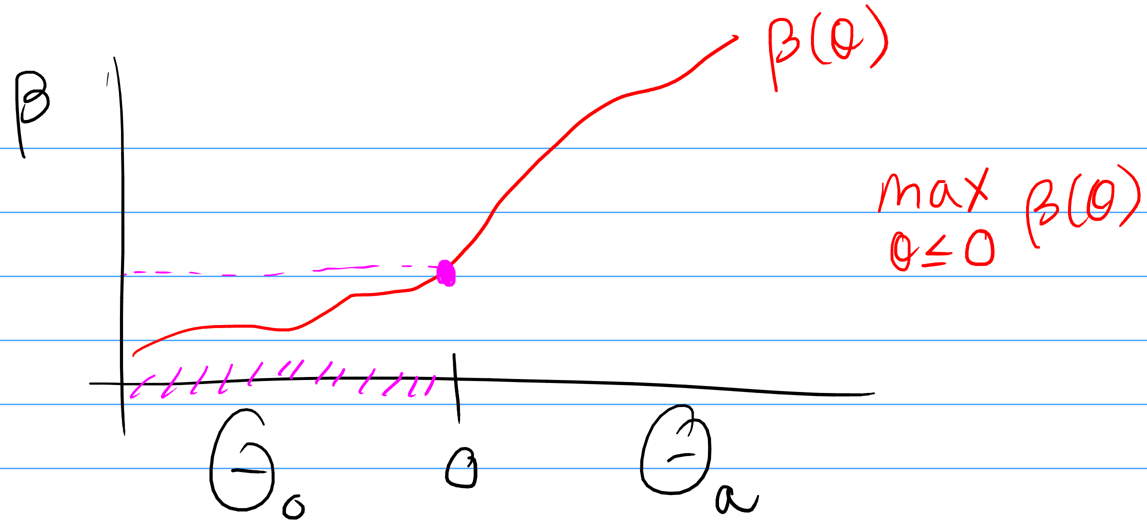
If $X_n \overset{iid}{\sim} \text{ShiftExp}(1, \theta)$ then $X_{(1)} \sim \text{ShiftExp}(N, \theta)$



Want to show that β is increasing

i.e. $\theta_2 > \theta_1$ want to show $\beta(\theta_2) > \beta(\theta_1)$





So I set c^* so that when $\theta = 0$

$$\beta(0) = P(\text{rej.}) = \mathbb{P}(X_{(1)} > c^*) = \alpha$$

So $c^* = F^{-1}(1-\alpha)$

F = CDF of Exp(N)

$F(t) = 1 - e^{-Nt}$

$F^{-1}(y) = -\frac{1}{N}\log(1-y)$

Defn: Uniformly Most Powerful Test (UMP)

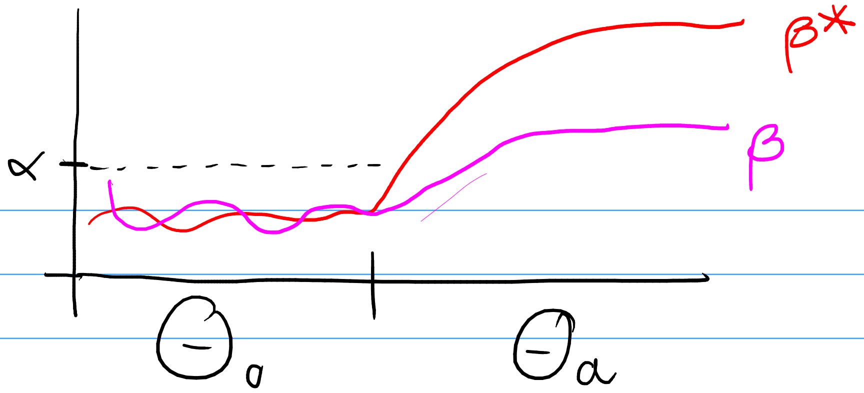
If $\mathcal{C}$ is a class of tests for

$$H_0: \theta \in \Theta_0 \quad \text{v.} \quad H_a: \theta \in \Theta_a$$

the test with the power function β^* is called the UMP test for this class if

$$\beta^*(\theta) \geq \beta(\theta) \quad \forall \theta \in \Theta_a$$

for any other test in $\mathcal{C}$ w/ power fn β.



The **UMP size α test** is the UMP among all tests where

$$\max_{\theta \in \Theta_0} \beta(\theta) = \alpha$$

The **UMP level α test** is the UMP among all tests where

$$\max_{\theta \in \Theta_0} \beta(\theta) \le \alpha$$

Consider the simple hypothesis

$$H_0 : \theta = \theta_0 \quad v. \quad H_a : \theta = \theta_a$$

in this case

$$\Theta = \{\theta_0, \theta_a\} \; ; \; \Theta_0 = \{\theta_0\} \; ; \; \Theta_a = \{\theta_a\}$$

and the LRT is

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \begin{cases} \dfrac{L(\theta_0)}{L(\theta_0)} = 1 & L(\theta_0) > L(\theta_a) \\[2ex] \dfrac{L(\theta_0)}{L(\theta_a)} & L(\theta_a) > L(\theta_0) \end{cases}$$

and so I reject when $\lambda \le c$

i.e. $\dfrac{L(\theta_0)}{L(\theta_a)} \le c \iff L(\theta_0) \le c\,L(\theta_a)$

$\iff L(\theta_a) \ge k\,L(\theta_0)$

$k = 1/c$

we choose c/k so that

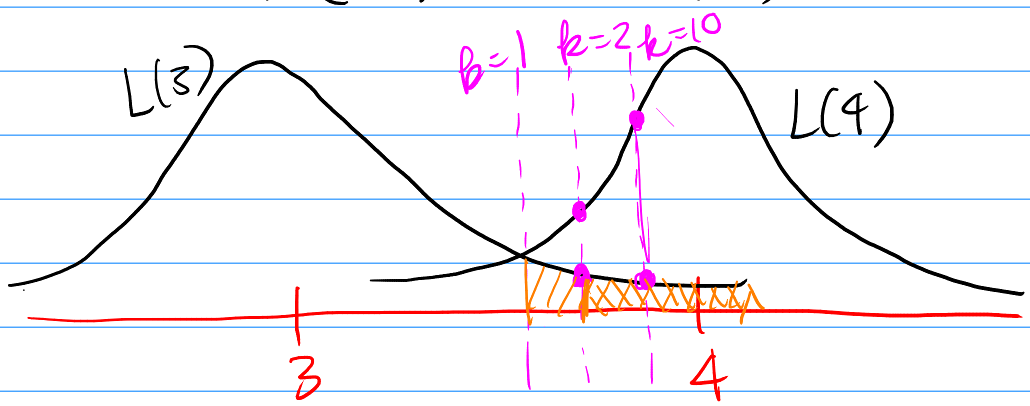
$$\mathbb{P}_{\theta_0}\left(L(\theta_0) \le c\,L(\theta_a)\right) \le \alpha$$

Ex. $X \sim N(\mu, 1)$

$H_0: \mu = 3$ v. $H_a: \mu = 4$

LRT says reject if

$$L(4) \ge k\,L(3)$$



Neyman–Pearson Lemma

Consider testing

$$H_0: \theta = \theta_0 \quad \text{v.} \quad H_a: \theta = \theta_a$$

w/ the LRT that rejects when

I $$\lambda = \frac{L(\theta_0)}{L(\theta_a)} \leq c$$

so that

II $$\mathbb{P}_{\theta_0}(\lambda \leq c) = \alpha \qquad [\text{size } \alpha \text{ test}]$$

a) **Sufficiency** Any test satisfying I and II is a UMP level α test for this hypothesis.

b) **Necessity** Every UMP level α test for this hypothesis satisfies I and II (up to some prob. zero set)