Lecture 9: CRLB

$X_n \overset{iid}{\sim} \text{Pois}(\lambda)$ then $I_N(\lambda) = \frac{N}{\lambda}$

let $\psi = \sqrt{\lambda} \iff \lambda = \psi^2$

Q: what's the $I_N(\psi)$?

Reparameterize Pois(λ) in terms of ψ

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(\psi^2)^x e^{-\psi^2}}{x!} = f_\psi(x)$$

$I(\psi) = -\mathbb{E}\left[\frac{\partial^2}{\partial \psi^2} \log f_\psi(x)\right]$ and $I_N(\psi) = N I(\psi)$

$$\rightarrow \log f_\psi(x) = x \log(\psi^2) - \psi^2 - \log(x!)$$
$$= 2x \log \psi - \psi^2 - \log(x!)$$

$$\rightarrow \frac{\partial}{\partial \psi} \cdots = \frac{2X}{\psi} - 2\psi$$

$$\rightarrow \frac{\partial^2}{\partial \psi^2} = -\frac{2X}{\psi^2} - 2$$

$$\rightarrow I(\psi) = -\mathbb{E}\left[\frac{-2X}{\psi^2} - 2\right] = \frac{2}{\psi^2}\mathbb{E}X + 2$$

$$= \frac{2}{\psi^2}\psi^2 + 2 = 4$$

$$\rightarrow \boxed{I_N(\psi) = 4N}$$

What's the general relation?

Recall: $y = f(x) \Leftrightarrow x = f^{-1}(y)$

$$\frac{dy}{dx} \ \underline{\text{rel?}} \ \frac{dx}{dy}.$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Theorem: Fisher Info Under Transf.

If $\theta = \tau(\psi)$ $[\Leftrightarrow \psi = \tau^{-1}(\theta)$ if τ invertible$]$

then

$$I(\theta) = \left(\frac{\partial \psi}{\partial \theta}\right)^2 I(\psi)$$

equiv.

$$I(\psi) = \left(\frac{\partial \theta}{\partial \psi}\right)^2 I(\theta)$$

Revisit our example

$$\frac{\partial \lambda}{\partial \psi} = 2\psi$$

$$I(\lambda) = 1/\lambda$$

and $\psi = \sqrt{\lambda} \Leftrightarrow \lambda = \psi^2$

$$I(\psi) = \left(\frac{\partial \lambda}{\partial \psi}\right)^2 I(\lambda) = (2\psi)^2 \left(\frac{1}{\lambda}\right) = \cancel{4\psi^2/\psi^2}$$

$$= 4$$

Why do we care?

Theorem: If $X_n \overset{iid}{\sim} f_\theta$ when $\theta \in \Theta$

and $\hat{\theta}$ is unbiased for $\tau(\theta)$

⊛ and if f_θ is nice enough

read: 1-dim'l exp. fam ⊛

then

$$\mathrm{Var}(\hat{\theta}) \geq \left(\frac{\partial \tau}{\partial \theta}\right)^2 \Big/ I_N(\theta) = B = \text{Cramér-Rao Lower Bound}$$

Note: ① If $\tau(\theta) = \theta$ then $\frac{\partial \tau}{\partial \theta} = 1$

and $B = 1/I_N(\theta)$.

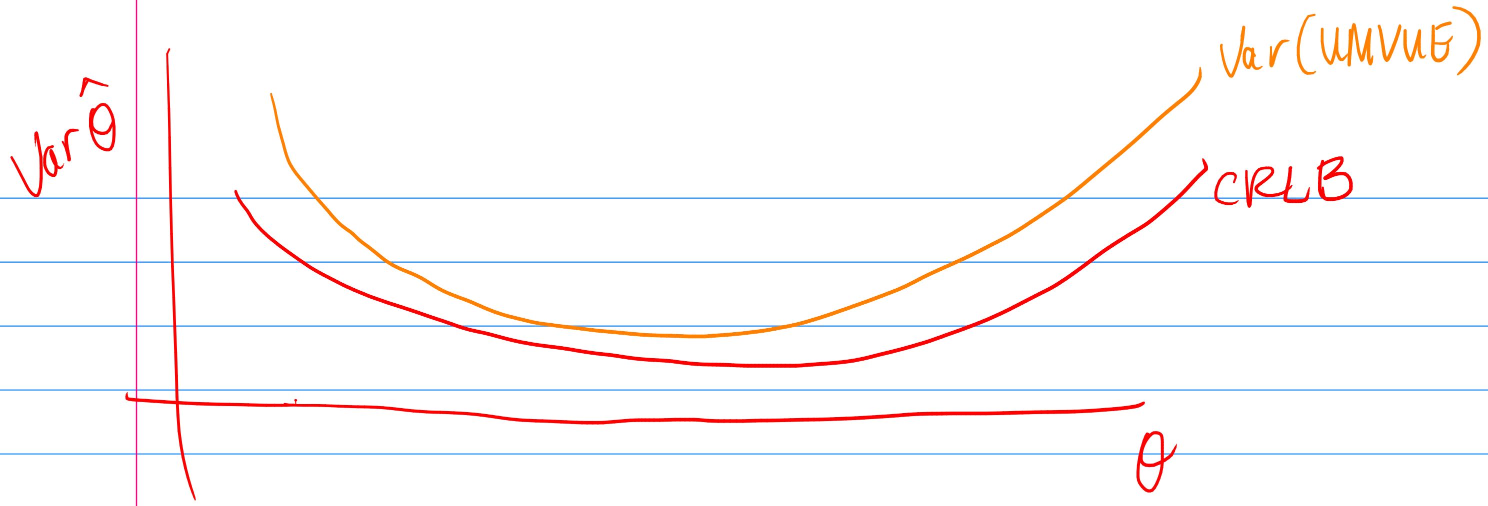
② If I can find an unbiased est θ^* for $\tau(\theta)$ and I can show that

$$\mathrm{Var}(\theta^*) = B = \mathrm{CRLB}$$

Cramer-Rao Lower Bound

so θ^* is the UMVUE.

③ If I have an estimator that is unbiased for $\tau(\theta)$ but its variance doesn't achieve the CRLB do we know if it is not the UMVUE? No,



Ex. $X_n \overset{iid}{\sim} Pois(\lambda)$

Let $\hat\lambda = \bar X$, $\tau(\lambda) = \lambda$

Since $E\hat\lambda = E\bar X = \lambda$ then $\hat\lambda$ is unbiased for $\tau(\lambda)$

$$Var(\hat\lambda) = Var(\bar X) = \frac{\lambda}{N}$$

$$CRLB = \frac{1}{I_N(\lambda)} = \frac{1}{N/\lambda} = \frac{\lambda}{N}$$

So since $Var(\hat\lambda) = \lambda/N = CRLB$

then $\hat\lambda$ is the UMVUE.

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$ Recall: $EX_n = 1/\lambda$

$Var X_n = 1/\lambda^2$

Goal: find UMVUE for $\tau(\lambda) = 1/\lambda$

(1) Propose an unbiased estimator: $\bar X$

$$\mathbb{E}\,\overline{X} = \frac{1}{\lambda} = \tau(\lambda) \quad \checkmark$$

② Calc $\mathrm{Var}(\overline{X})$.

$$\mathrm{Var}(\overline{X}) = \frac{1/\lambda^2}{N} = \frac{1}{N\lambda^2}$$

③ Calc. the CRLB

$$\tau(\lambda) = 1/\lambda$$

$$B = \left(\partial\tau/\partial\lambda\right)^2 / I_N(\lambda)$$

$$\frac{\partial\tau}{\partial\lambda} = -1/\lambda^2 \quad \text{so} \quad \left(\frac{\partial\tau}{\partial\lambda}\right)^2 = \frac{1}{\lambda^4}$$

$$I(\lambda) = -\mathbb{E}\left[\frac{\partial^2}{\partial\lambda^2}\log f_\lambda(x)\right]$$

$$\rightarrow f_\lambda(x) = \lambda e^{-\lambda x}\,\mathbb{1}(x>0)$$

$$\rightarrow \log f_\lambda = \log\lambda - \lambda x + \log \mathbb{1}(x>0)$$

$$\rightarrow \frac{\partial}{\partial\lambda}\log f_\lambda = \frac{1}{\lambda} - x$$

$$\rightarrow \frac{\partial^2}{\partial\lambda^2}\log f_\lambda = -1/\lambda^2$$

$$\rightarrow I(\lambda) = -\mathbb{E}\left[-1/\lambda^2\right] = 1/\lambda^2$$

$$\rightarrow I_N(\lambda) = N/\lambda^2$$

$$B = \frac{\left(\frac{\partial \tau}{\partial \lambda}\right)^2}{I_N(\lambda)} = \frac{1/\lambda^4}{N/\lambda^2} = \frac{1}{N\lambda^2}$$

and so since $\operatorname{Var} \bar{X} = \frac{1}{N\lambda^2} = B$

then $\bar{X}$ is the UMVUE for $1/\lambda$.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ σ^2 is known

UMVUE for μ.

Saw earlier: $I_N(\mu) = N/\sigma^2$

1. $\mathbb{E}\bar{X} = \mu$ ($\bar{X}$ unbiased for μ)
2. $\operatorname{Var} \bar{X} = \sigma^2/N$
3. $B = 1/I_N(\mu) = \frac{1}{N/\sigma^2} = \sigma^2/N$

Since $\bar{X}$ is unbiased for μ, $\operatorname{Var} \bar{X} = \sigma^2/N = \text{CRLB}$

then $\bar{X}$ is the UMVUE for μ.

Ex. Let $X_n \overset{iid}{\sim} U(0, \theta)$

Want UMVUE for θ.

1. Propose an unbiased est.

Can show: $\mathbb{E}[X_{(N)}] = \frac{N}{N+1}\theta$

So $T = \frac{N+1}{N} X_{(N)}$ then $\mathbb{E}T = \theta$

So T is unbiased for θ.

(2) Calc Var. $Var(T) = \frac{\theta^2}{N(N+2)}$

(3) Show that $Var(T) = CRLB$

Need $I(\theta)$

$\rightarrow f_\theta(x) = \frac{1}{\theta} \mathbb{1}(0 < x < \theta)$

$\rightarrow \log f_\theta = -\log\theta + \log \mathbb{1}(0 < x < \theta)$

not diff'able

Not an exp. fam $\rightarrow$ CRLB does not apply.

Review: Iterated Expectation

a number

$\mathbb{E}[X \mid Y = y] = \int x f(x|y) dx = g(y)$

Could promote this to $g(Y) = \mathbb{E}[X \mid Y]$

a rand. var.

(1) Iterated Expectation

a RV

$\mathbb{E}[X] = \mathbb{E}_Y[\mathbb{E}[X \mid Y]]$

(2) Law of Total Variance

$$\mathrm{Var}(X) = \mathbb{E}[\underbrace{\mathrm{Var}(X|Y)}_{\text{RV}}] + \mathrm{Var}(\underbrace{\mathbb{E}[X|Y]}_{\text{RV}})$$

Ex. $\underbrace{X \mid Y = y}$ $\sim \mathrm{Bin}(y, p)$ where $p \in [0,1]$

$Y \sim \mathrm{Pois}(\lambda)$ for $\lambda > 0$

$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$

(1) $\mathbb{E}[X \mid Y = y] = yp$

(2) $\mathbb{E}[X|Y] = Yp$

(3) $\mathbb{E}[Yp] = p\,\mathbb{E}[Y] = p\lambda$

$\mathrm{Var}(X) = \mathbb{E}\,\underbrace{\mathrm{Var}(X|Y)} + \mathrm{Var}(\mathbb{E}[X|Y])$

$\mathrm{Var}(X \mid Y = y) = yp(1-p)$ $\qquad \mathbb{E}[X|Y] = Yp$

$\mathrm{Var}(X|Y) = Y p(1-p)$

$= \mathbb{E}[Yp(1-p)] + \mathrm{Var}(Yp)$

$= p(1-p)\,\mathbb{E}Y + p^2\,\mathrm{Var}(Y)$

$= p(1-p)\lambda + p^2\lambda = \cdots = p\lambda$