Lecture 21 : Interval Estimation

Point estimation:

$$\hat{\theta} = \hat{\theta}(\underset{\sim}{x}) \in \Theta \qquad \text{idea } "\hat{\theta} \approx \theta"$$

Interval Estimation:

$$C = C(\underset{\sim}{X}) \subset \Theta \qquad \text{idea } "\theta \in C"$$

prefer if C is an interval.

Defn: Interval Estimator

An interval est. of $\theta \in \Theta \subset \mathbb{R}$ is a pair of functions

$$L = L(\underset{\sim}{x}) \quad \text{and} \quad U = U(\underset{\sim}{x})$$

that satisfy $L \leq U \ \forall \underset{\sim}{x}$.

<u>idea</u>: want to say "$L \leq \theta \leq U$" (approx.)

Sometimes want a one-sided interval e.g. $L = -\infty$ or $U = \infty$

we get $(-\infty, U]$ or $[L, \infty)$.

$\theta \leq U$ $\quad$ $L \leq \theta$

Ex. Let $X_1, \dots, X_4 \overset{iid}{\sim} N(\mu, 1)$ then an interval est. of μ is

$$[\underbrace{\bar{X}-1}_{L}, \underbrace{\bar{X}+1}_{U}]$$

might say $\bar{X}-1 \le \theta \le \bar{X}+1$.

Why use an interval est? Why not just use $\bar{X}$?

notice: $P(\bar{X}=\mu) = 0$

so we typically attach uncertainty to $\bar{X}$

i.e. $sd(\bar{X}) = 1/\sqrt{N}$

Alt. Use interval est. b/c

$$P(\underbrace{\bar{X}-1}_{\text{random}} \le \underbrace{\mu}_{\text{fixed}} \le \underbrace{\bar{X}+1}_{\text{random}}) > 0$$

$X_n \overset{iid}{\sim} N(\mu, 1)$

$\bar{X} \sim N(\mu, 1/4)$

$\bar{X}-\mu \sim N(0, 1/4)$

$$= P(\bar{X}-1 \le \mu, \bar{X}+1 \ge \mu)$$

$$= P(\bar{X}-\mu \le 1, \bar{X}-\mu \ge -1)$$

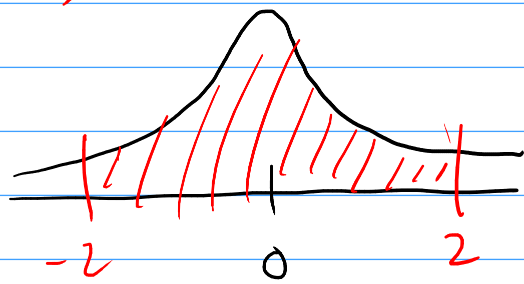
$$= P(-1 \le \bar{X}-\mu \le 1)$$

$$= P\Big(-2 \le \underbrace{\frac{\bar{X}-\mu}{\sqrt{1/4}}}_{N(0,1)} \le 2\Big)$$

$$= \mathbb{P}(|Z| \le 2)$$

where $Z \sim N(0,1)$

$$\approx .95$$



So I can use $[\bar{X}-1, \bar{X}+1]$ and say this contains μ about 95% of the time.

Defn: Coverage Prob.

For an int. est. $[L, U]$ of a param. θ, the coverage prob. is

$$\mathbb{P}_\theta(L \le \theta \le U).$$

depends on θ.

Defn: Confidence Coefficient

Worst case coverage prob. i.e.

$$1-\alpha = \min_{\theta \in \Theta} \mathbb{P}_\theta(L \le \theta \le U)$$

More generally if I have a set $C(\underset{\sim}{X}) \subset \Theta$

its assoc. conf. coef. is

$$1-\alpha = \min_{\theta \in \Theta} \mathbb{P}_\theta(\theta \in C).$$

Defn: Conf. Interval / Conf. Set.

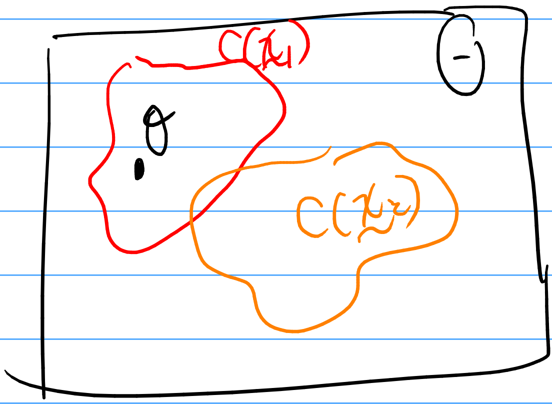
Conf. Interval = int. est. + conf. coef.

Conf. Set. = Set. est. + conf. coef.

When we look at $\mathbb{P}_\theta(\theta \in C)$

θ fixed

$C = C(\underset{\sim}{X})$ = random



How do I build a conf. set/interval?

<u>Basically one way</u>: invert a hypothesis test.

$$HT \Longleftrightarrow \text{Conf. Set.}$$

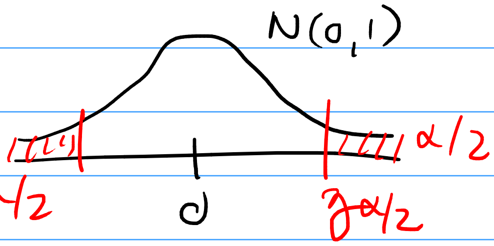
Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ ← known

Consider a HT for

$$H_0: \mu = \mu_0 \quad v. \quad H_a: \mu \neq \mu_0$$

I need a HT, last time saw that a α-level test for this was to reject when

$$\frac{|\bar{X} - \mu_0|}{\sigma/\sqrt{N}} > z_{\alpha/2} \longleftarrow z_{\alpha/2} = F_Z^{-1}(1 - \alpha/2)$$



$$R(\mu_0) = \left\{ \underset{\sim}{x} \in \mathcal{X} \;\middle|\; \frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{N}} > z_{\alpha/2} \right\}$$

↑ set of $\underset{\sim}{x}$ that don't agree w/ $H_0: \mu = \mu_0$

$$A(\mu_0) = \mathcal{X} \setminus R(\mu_0) = \left\{ \underset{\sim}{x} \in \mathcal{X} \;\middle|\; \frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{N}} \leq z_{\alpha/2} \right\}$$

↑ set of $\underset{\sim}{x}$ in agreement w/ $H_0: \mu = \mu_0$

$$-z_{\alpha/2} \le \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \le z_{\alpha/2}$$

$$\Leftrightarrow \mu_0 - z_{\alpha/2}\,\sigma/\sqrt{N} \le \bar{X} \le \mu_0 + z_{\alpha/2}\,\sigma/\sqrt{N}$$

$$\Leftrightarrow \underbrace{\bar{X} - \sigma/\sqrt{N}\, z_{\alpha/2}}_{L} \le \mu_0 \le \underbrace{\bar{X} + z_{\alpha/2}\,\sigma/\sqrt{N}}_{U}$$

Claim: $[L, U]$ is a $1-\alpha$ CI for μ.

<u>why?</u>

Test is level α i.e.

$$\mathbb{P}_{\mu_0}(\text{reject}) \le \alpha$$

$$\Leftrightarrow \mathbb{P}_{\mu_0}(\text{accept}) \ge 1-\alpha$$

$$\Leftrightarrow \mathbb{P}_{\mu_0}(\underset{\sim}{X} \in A(\mu_0)) \ge 1-\alpha$$

$$\Leftrightarrow \mathbb{P}_{\mu_0}(\bar{X} - \sigma/\sqrt{N}\, z_{\alpha/2} \le \mu_0 \le \bar{X} + \sigma/\sqrt{N}\, z_{\alpha/2}) \ge 1-\alpha$$

$$\Leftrightarrow \mathbb{P}_{\mu_0}(L \le \mu_0 \le U) \ge 1-\alpha$$

μ_0 arbitrary

so $\forall \mu : \mathbb{P}_\mu(L \le \mu \le U) \ge 1-\alpha$

so $\min_\mu \mathbb{P}_\mu(L \le \mu \le U) \ge 1-\alpha$

i.e. $[L, U]$ is a CI w/ conf. coef. $\ge 1-\alpha$.

Test Inversion

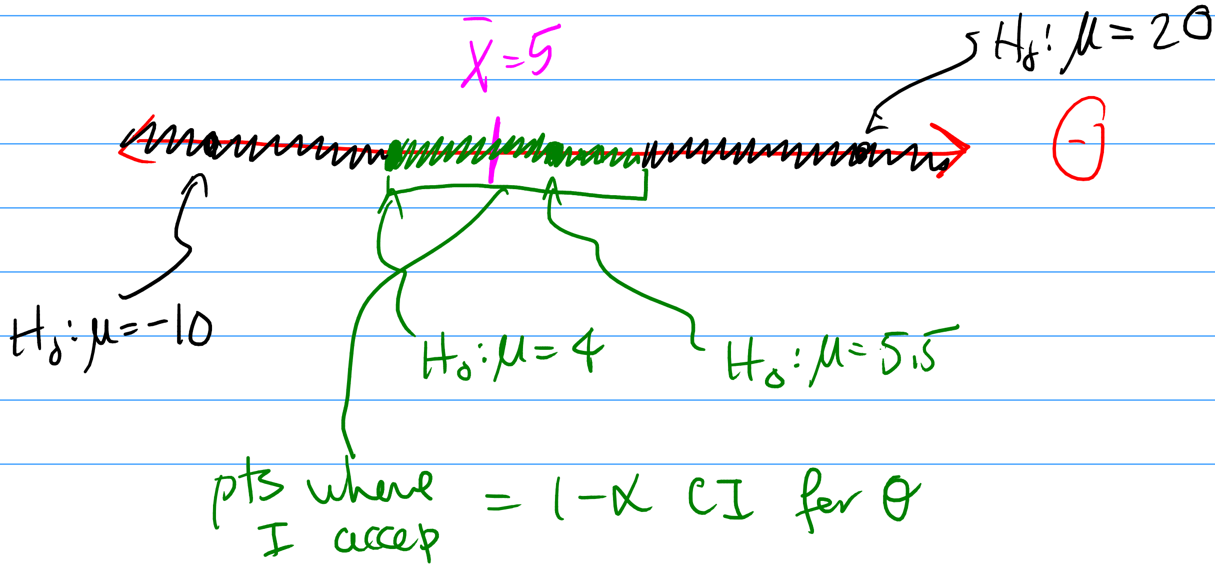
For $\theta_0 \in \Theta$ let $A(\theta_0)$ be the accept region of a α-level test for

$$H_0 : \theta = \theta_0 \quad v. \quad H_a : \theta \neq \theta_0$$

and then let

$$\boxed{C(\underset{\sim}{x}) = \{\theta \mid \underset{\sim}{x} \in A(\theta)\}}$$

this is a $1-\alpha$ conf. set.



Two worlds:

HT: fix θ_0 and test $H_0: \theta = \theta_0$

to do this I determine some rule

$$A(\theta_0) = \{\text{set of } \underset{\sim}{x} \text{ where } \theta = \theta_0 \text{ is reasonable}\}$$

$$\subset \mathcal{X}$$

CI: Fix $\underset{\sim}{x}$ want to determine which θ are consistent w/ $\underset{\sim}{x}$

$$C(\underset{\sim}{x}) = \{\text{set of } \theta \text{ consistent w/ } \underset{\sim}{x}\} \subset \Theta$$

Ex. Let $X_n \overset{iid}{\sim} \text{Exp}(\beta)$

$$\hookrightarrow \mathbb{E} X_n = \beta$$

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \text{ for } x > 0$$

Lets make a CI by inverting the LRT

$$H_0: \beta = \beta_0 \quad \text{v.} \quad H_a: \beta \neq \beta_0$$

$$\lambda = \frac{L(\hat{\beta}_0)}{L(\hat{\beta})} = \frac{L(\beta_0)}{L(\bar{X})} = \frac{\frac{1}{\beta_0^N} \exp(-N\bar{X}/\beta_0)}{\frac{1}{\bar{X}^N} \exp(-N)}$$

$$= \left(\frac{\bar{X}}{\beta_0}\right)^N e^N e^{-N\bar{X}/\beta_0}$$

$$R(\beta_0) = \{ \underset{\sim}{x} \mid \lambda(\underset{\sim}{x}) \le c \}$$

choose so that is a α level test

$$A(\beta_0) = \{ \underset{\sim}{x} \mid \lambda(\underset{\sim}{x}) > c \}$$

$$= \left\{ \underset{\sim}{x} \;\middle|\; \left(\frac{\bar{X}}{\beta_0}\right)^N e^N e^{-N\bar{X}/\beta_0} > c \right\}$$

$$C(\underset{\sim}{x}) = \left\{ \beta \;\middle|\; \left(\frac{\bar{X}}{\beta}\right)^N e^N e^{-N\bar{X}/\beta} > c \right\}$$