Lecture 20: Karlin-Rubin

Consider alternative LRT: if T sufficient and let $g_\theta(T)$ to be the PMF/PDF of T

Traditional LRT $\quad \lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{f_{\hat{\theta}_0}(\underset{\sim}{x})}{f_{\hat{\theta}}(\underset{\sim}{x})}$

Let $L^*(\theta) = g_\theta(T)$ and then

$$\lambda^* = \frac{L^*(\hat{\theta}_0)}{L^*(\hat{\theta})} = \frac{g_{\hat{\theta}_0}(T)}{g_{\hat{\theta}}(T)}$$

If I reject when $\lambda^* \leq C$ then this equivalent to the standard LRT.

Reason this works is that MLE = function(T)

Pf. $\lambda^*(T) = \lambda(\underset{\sim}{x})$

$$\lambda(\underset{\sim}{x}) = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \frac{\max_{\theta \in \Theta_0} f_\theta(\underset{\sim}{x})}{\max_{\theta \in \Theta} f_\theta(\underset{\sim}{x})}$$

$$= \frac{\max_{\theta \in \Theta_0} \cancel{h(\underset{\sim}{x})}\, g(T,\theta)}{\max_{\theta \in \Theta} \cancel{h(\underset{\sim}{x})}\, g(T,\theta)}$$

$$= \frac{\max_{\theta \in \Theta_0} g(T,\theta)}{\max_{\theta \in \Theta} g(T,\theta)} \quad \propto g_\theta$$

$$= \lambda^*(T)$$

<u>Corollary to NP lemma</u>

If testing $H_0: \theta = \theta_0$ v. $H_a: \theta = \theta_a$ using the test that rejects if

$$\lambda = \frac{g_{\theta_0}(T)}{g_{\theta_a}(T)} \le c$$

where c is s.t. $P_{\theta_0}(\lambda \le c) = \alpha$

then this is the UMP level α test.

<u>Ex</u> $X_1, X_2 \overset{iid}{\sim} \text{Bern}(\theta)$

test $H_0: \theta = 1/2$ v. $H_a: \theta = 3/4$

<u>Note</u>: $T = X_1 + X_2$ is sufficient for θ and

$T \sim \text{Bin}(2, \theta)$

$$g_\theta(T) = \binom{2}{T} \theta^T (1-\theta)^{2-T}$$

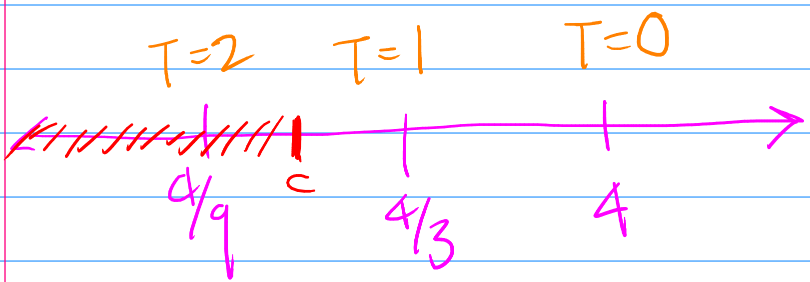
So

$$\lambda(T) = \frac{g_{\theta_0}(T)}{g_{\theta_a}(T)} = \frac{g_{1/2}(T)}{g_{3/4}(T)} = \frac{\binom{2}{T}\left(\frac{1}{2}\right)^T\left(\frac{1}{2}\right)^{2-T}}{\binom{2}{T}\left(\frac{3}{4}\right)^T\left(\frac{1}{4}\right)^{2-T}}$$

$$= \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{3}{4}\right)^T \left(\frac{1}{4}\right)^{2-T}}$$

LRT says reject when $\lambda \le c$

T	0	1	2
$\lambda(T)$	4	$4/3$	$4/9$



Ex. $4/9 < c < 4/3 \Rightarrow$ reject when $T=2$

and so $\alpha = \mathbb{P}_{\frac{1}{2}}(\lambda \le c) = \mathbb{P}_{\frac{1}{2}}(T=2)$

$$= \binom{2}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^0 = \frac{1}{4}$$

What about composite test?

Let's consider <u>one-sided tests</u>:

$$H_0: \theta \le \theta_0 \quad v. \quad H_a: \theta > \theta_0$$

Defn: Monotone Likelihood Ratio Property (MLR)

We say a family of PDFs/PMFs has the MLR property if $\forall\, \theta_1 < \theta_2$

f_θ (univar.)

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} \text{ is non-dec. as a fn of } x.$$

Theorem: If $\{f_\theta\}$ is an exp. fam. so that

$$f_\theta(x) = c(\theta)\, h(x) \exp(w(\theta)\, x)$$

and if $w(\theta)$ is non-decreasing in θ then this family has the MLR property.

Pf. $\theta_1 < \theta_2$

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{c(\theta_2)\, \cancel{h(x)} \exp(w(\theta_2)\, x)}{c(\theta_1)\, \cancel{h(x)} \exp(w(\theta_1)\, x)}$$

$$\propto \exp\big(x\, \underbrace{(w(\theta_2) - w(\theta_1))}_{>0}\big)$$

If w is non-dec. as a fn of θ then this looks like $\approx e^{ax}$ for $a > 0$

<u>Theorem:</u> If T has the MLR property, and we have a test that rejects when

$$T > c$$

then the power function of this test is non-decreasing.

pf. <u>Show:</u> $\theta_2 > \theta_1$ then $\beta(\theta_2) \geq \beta(\theta_1)$

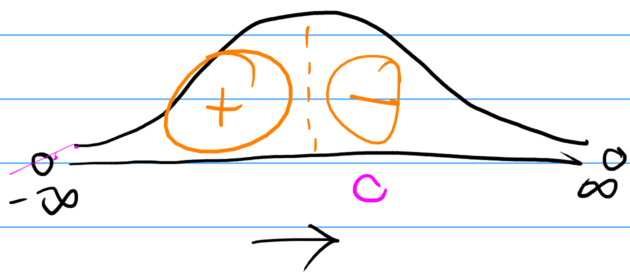
i.e. $P_{\theta_2}(T>c) \geq P_{\theta_1}(T>c)$ ← $F_\theta(c) =$ CDF of T at c

i.e. $1 - F_{\theta_2}(c) \geq 1 - F_{\theta_1}(c)$

i.e. $\Delta = F_{\theta_1}(c) - F_{\theta_2}(c) \geq 0 \quad \forall\ \theta_1 < \theta_2$

as $c \to -\infty$ $\Delta \to 0$

as $c \to \infty$ $\Delta \to 0$



$$\frac{d\Delta}{dc} = f_{\theta_1}(c) - f_{\theta_2}(c)$$

$$= \underbrace{f_{\theta_1}(c)}_{\geq 0}\left(1 - \frac{f_{\theta_2}(c)}{f_{\theta_1}(c)}\right)$$

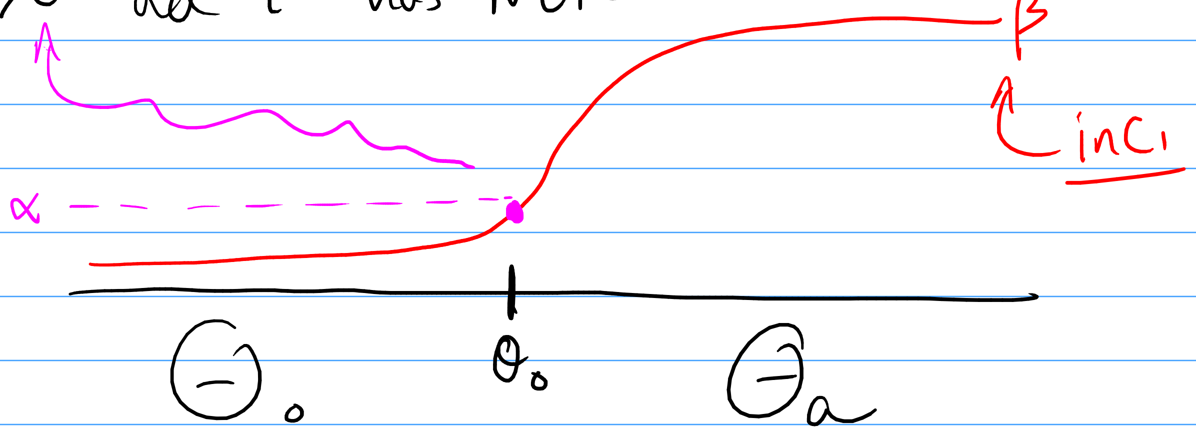
MLR says non-dec. as a fn of c

goes + to −

Why do we care?

$$H_0: \theta \le \theta_0 \quad \text{v.} \quad H_a: \theta > \theta_0$$

Rej. $T > c$ and T has MLR



Choose c s.t. $\mathbb{P}_{\theta_0}(T > c) = \alpha$

<u>Theorem:</u> <u>Karlin-Rubin Theorem</u>

Consider testing

$$H_0: \theta \le \theta_0 \quad \text{v.} \quad H_a: \theta > \theta_0$$

and let T be sufficient for θ and has the MLR property.

The test that rejects when $T > c$ where c is chosen so that

$$\alpha = \mathbb{P}_{\theta_0}(T > c)$$

is the UMP level α test.

Notes:

Alt test $\quad H_0: \theta \geq \theta_0 \quad v. \quad H_a: \theta < \theta_0$

by rej. when $T < c$

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $\quad \sigma^2$ known.

$$H_0: \mu \leq a \quad v. \quad H_a: \mu > a$$

Now, $\bar{X}$ sufficient for μ.

Defer: check that $\bar{X}$ has MLR property

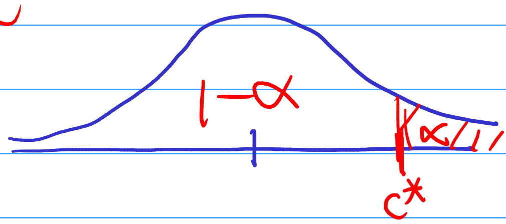
So the UMP level α test is to reject when

$$\bar{X} > c$$

where c s.t. $P_a(\bar{X} > c) = \alpha$

have prev. shown that $c = a + \frac{\sigma}{\sqrt{N}} F_Z^{-1}(1-\alpha)$

equiv. $\quad \dfrac{\bar{X} - a}{\sigma/\sqrt{N}} > F_Z^{-1}(1-\alpha) = c^*$



Need to check that $\bar{X}$ has MLR

$T = \bar{X} \sim N(\mu, \sigma^2/N)$

$$f_\mu(T) = \frac{1}{\sqrt{2\pi\sigma^2/N}} \exp\left(-\frac{N}{2\sigma^2}(T-\mu)^2\right)$$

$$\propto \exp\left(-\frac{N}{2\sigma^2}(T^2 - 2T\mu + \mu^2)\right)$$

$$= \underbrace{\exp\left(\frac{-N}{2\sigma^2}T^2\right)}_{h(T)} \exp\left(\frac{NT\mu}{\sigma^2}\right) \underbrace{\exp\left(-\frac{N\mu^2}{2\sigma^2}\right)}_{c(\mu)}$$

$\exp\left(\left(\frac{N\mu}{\sigma^2}\right) T\right)$

$\hookrightarrow w(\mu) = \frac{N\mu}{\sigma^2}$

is non-dec. in μ.

So T has MLR property

Consider again $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ known.

T1: rej. if $\frac{\bar{X} - \theta_0}{\sigma/\sqrt{N}} > z_\alpha = F_Z^{-1}(1-\alpha)$

T2: rej. if $\frac{\bar{X} - \theta_0}{\sigma/\sqrt{N}} < -z_\alpha$

Neyman Pearson

$\theta_a > \theta_0$

$H_0: \theta = \theta_0$ v. $H_a: \theta = \theta_a \Rightarrow$ T1 UMP α

$H_0: \theta = \theta_0$ v. $H_a: \theta = \theta_a \Rightarrow$ T2 UMP α

$\theta_0 > \theta_a$

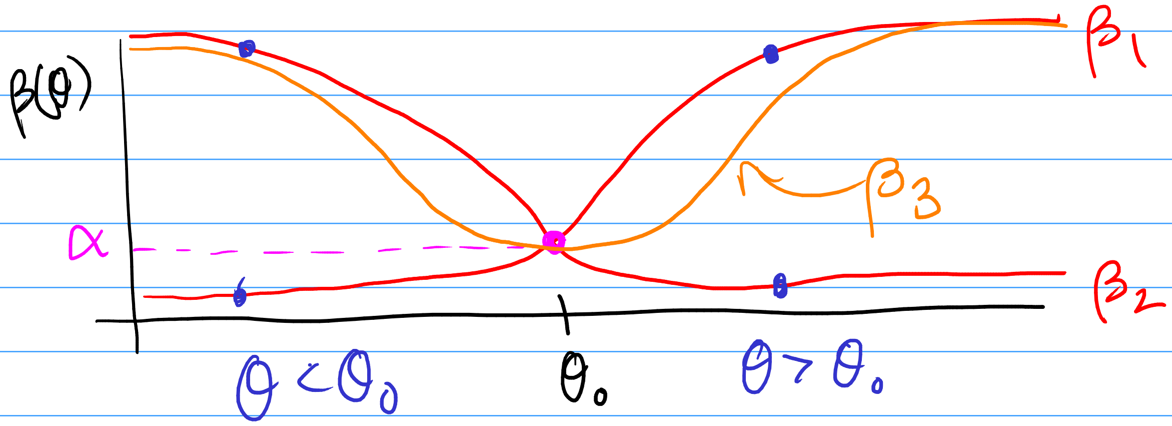
Karlin Rubin

$H_0: \theta \le \theta_0$ v. $H_a: \theta > \theta_0 \Rightarrow$ T1 UMP α

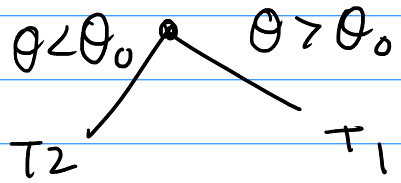
$H_0: \theta \geq \theta_0$ v. $H_a: \theta < \theta_0 \Rightarrow$ T2 UMP α

Q: What about $H_0: \theta = \theta_0$ v. $H_a: \theta \neq \theta_0$

No UMP level α test.



Like to do



Test 3: $\dfrac{|\bar{X}-\theta_0|}{\sigma/\sqrt{N}} > z_{\alpha/2}$

$\hat{\theta} < \theta_0$ $\quad$ $\hat{\theta} > \theta_0$

T2 $\quad$ T1

Test 3 : UMP level α unbiased test (UMPU α)

Unbiased test means β higher in Θ_a than Θ_0