Lecture 4: Ancillary Statistics and Method of Moments

Factorization Theorem: $f(\underset{\sim}{x}) = g(\theta, T)\, h(\underset{\sim}{x})$

then T is sufficient for θ

Exp. Fam: $f(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\theta) \exp(T(\underset{\sim}{x})\, \omega(\theta))$

T is suff. for θ

$$\to f(x_n) = h_0(x_n)\, c_0(\theta) \exp(T_0(x_n)\, \omega(\theta))$$

$T(\underset{\sim}{x}) = \sum_n T_0(x_n)$ is sufficient for θ

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$ What's a SS for μ?

$$f(x_n) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_n - \mu)^2\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_n^2 - 2\mu x_n + \mu^2)\right)$$

$e^{a+b} = e^a e^b$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x_n^2\right)}_{h_0(x_n)} \exp(\underbrace{\mu}_{\omega(\mu)} \underbrace{x_n}_{T_0(x_n) = x_n}) \underbrace{\exp\left(-\frac{1}{2}\mu^2\right)}_{c_0(\mu)}$$

So this is an exp. fam w/

$$T(\underset{\sim}{x}) = \sum_{n=1}^{N} x_n$$

then T is sufficient for μ.

Theorem: any invertible fn of a SS is also Sufficient

So $\bar{X} = \frac{1}{N}\sum_{n=1}^{N} x_n = \frac{1}{N} T$

is also sufficient for μ.

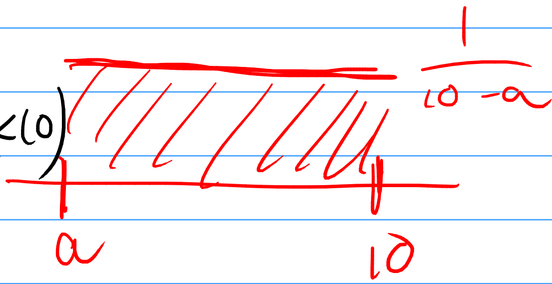
Ex. $X_n \overset{iid}{\sim} U(a, 10)$; $0 < a < 10$

Find a SS for a.

Since parameter a shows up in the support of the dist — its not an exp. fam.

$$f(\underset{\sim}{x}) = \prod_{n=1}^{N} f(x_n) = \prod_{n=1}^{N} \frac{1}{10-a} \mathbb{1}(a < x_n < 10)$$

$$= (10-a)^{-N} \prod_{n} \mathbb{1}(x_n > a)\mathbb{1}(x_n < 10)$$



Useful facts:

$$\prod_{n} \mathbb{1}(x_n > \text{value}) = \mathbb{1}(x_{(1)} > \text{value})$$

$$\prod_{n} \mathbb{1}(x_n < \text{value}) = \mathbb{1}(x_{(N)} < \text{value})$$

$$= (10-a)^{-N} \mathbb{1}(x_{(1)} > a)\mathbb{1}(x_{(N)} < 10)$$

$$= g(a, T)\, h(\underset{\sim}{x})$$

$T = X_{(1)}$

$g(a, T) = (10-a)^{-N}\mathbb{1}(T > a)$

$h(\underset{\sim}{x}) = \mathbb{1}(x_{(N)} < 10)$

so by Fact. Theorem, T is sufficient for a

Defn: Statistic

If $X_n \overset{iid}{\sim} f_\theta$ then a Statistic T is a function of the $X_1, \dots, X_N$ whose formula doesn't depend on θ.

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$

then $T = \bar{X}$ is a statistic

no μ in this formula

note: $\bar{X} \sim N(\mu, 1/N)$ ← dist. of $\bar{X}$ depends on μ

but $T = \mu$ is not a stat,

nor is $T = \bar{X} - \mu$

Defn: Ancillary Quantity

An ancillary quantity Q is a fn of the data $X_1, \dots, X_N$ whose dist doesn't depend on the unknown parameter.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ Known

$\bar{X} \sim N(\mu, \sigma^2/N)$ (not ancillary)

$$Q = \frac{\bar{X} - \mu}{\sigma/\sqrt{N}} \sim N(0,1) \quad \text{(is ancillary)}$$

Defn: Ancillary Statistic

T is an ancillary stat if
1. its fmla doesn't depend on θ
2. its dist. doesn't depend on θ.

Ex, $X_n \overset{iid}{\sim} N(\mu, 1)$

Let $R = X_{(N)} - X_{(1)}$

is a stat b/c no μ in fmla

note: $X_n = \underline{\mu} + \underline{Z_n}$ where $\underline{Z_n \overset{iid}{\sim} N(0,1)}$

and so

$$X_{(N)} = \mu + Z_{(N)}$$
$$X_{(1)} = \mu + Z_{(1)}$$

thus
$$R = X_{(N)} - X_{(1)} = \mu + Z_{(N)} - (\mu + Z_{(1)})$$
$$= Z_{(N)} - Z_{(1)}$$

no μ in dist.

So R is ancillary stat.

Theorem: Basu's Theorem

If T is a SS for θ and S is an ancillary stat. for θ then

$$T \perp\!\!\!\perp S.$$

Theorem: $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then $\bar{X} \perp\!\!\!\perp S^2$.

pf. $\bar{X}$ is sufficient for μ

S^2 is ancillary to μ

So by Basu's theorem: $\bar{X} \perp\!\!\!\perp S^2$.

Said: $\frac{N-1}{\sigma^2} S^2 \sim \chi^2(N-1)$

so $S^2 \sim \underbrace{\frac{\sigma^2}{N-1} \chi^2(N-1)}_{\text{no } \mu}$.

Point Estimation:

Setup: $X_n \overset{iid}{\sim} f_\theta$ where $\theta \in \Theta$

Defn: A point estimator for θ is a statistic

$$\hat{\theta} = \hat{\theta}(\underset{\sim}{X})$$

that hopefully is close to θ i.e. $\hat{\theta} \approx \theta$.

terminology: $\hat{\theta}(\underset{\sim}{X})$ is called an estimator

$\hat{\theta}(\underset{\sim}{x})$ is called an estimate.

A goal of this course:

1. How do I form estimators?
2. How do I know if they are good?

First Approach: Method of Moments (MoM)

Defn: the r^{th} moment of a RV X is

$$\mu_r = \mathbb{E}[X^r]$$

Defn: the r^{th} sample moment is

$$m_r = \frac{1}{N}\sum_{n=1}^{N} X_n^r$$

notice:

$$\mathbb{E}[m_r] = \mathbb{E}\left[\frac{1}{N}\sum_n X_n^r\right]$$

$$= \frac{1}{N}\sum_n \underbrace{\mathbb{E}[X_n^r]}_{\mu_r}$$

$$= \frac{1}{N} N \mu_r = \mu_r$$

So maybe a good way of est. params is to let

$$\mu_r \approx m_r$$

$\underline{Ex.}$ $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ both unknown.

Want estimators $\hat{\mu}$ and $\hat{\sigma}^2$.

$\underline{MoM}$: 1 calc first two pop. moments of $N(\mu,\sigma^2)$
and 2 set equal to sample moments
and then solve for μ and σ^2.

1 $\mu_1 = \mathbb{E}X = \mu$

$\mu_2 = \mathbb{E}[X^2] = Var(X) + \mathbb{E}[X]^2 = \sigma^2 + \mu^2$

2 $\mu = \underline{\mu_1} \approx \underline{m_1} = \frac{1}{N}\sum_n x_n = \overline{x}$

$\sigma^2 + \mu^2 = \underline{\mu_2} \approx \underline{m_2} = \frac{1}{N}\sum_n x_n^2 = \overline{x^2}$

$\boxed{\mu = \overline{X}}$ and $\sigma^2 + \mu^2 = \overline{X^2}$

3 Solve these eqns for μ and σ^2 (in terms of $\underset{\sim}{x}$)

$\mu = \overline{X} \Rightarrow \boxed{\hat{\mu} = \overline{X}}$

$\sigma^2 + \mu^2 = \overline{X^2} \Rightarrow \sigma^2 = \overline{X^2} - \mu^2$

$\sigma^2 = \overline{X^2} - (\overline{x})^2$

So $\boxed{\begin{aligned}\hat{\sigma}^2 &= \overline{X^2} - \overline{X}^2 \\ &= \frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{X})^2\end{aligned}} = S_N^2$

Method of Moments

$X_n \overset{iid}{\sim} f_\theta$ where $\theta = (\theta_1, \dots, \theta_K)$

Let $\mu_1, \dots, \mu_K$ be the first K moments and $m_1, \dots, m_K$ are the first K sample moments

We form a system of eqns

$$\mu_1 = m_1$$
$$\mu_2 = m_2$$
$$\vdots$$
$$\underbrace{\mu_K}_{\text{depend on } \theta} = m_K$$

we then solve this system for $\theta_1, \dots, \theta_K$ in terms of $\underset{\sim}{X}$.

Ex. Let $X_n \overset{iid}{\sim} \mathrm{Bin}(k, p)$

let's find MoM ests for k and p.

1 Get pop. moments

$$\mu_1 = \mathbb{E}X_n = kp$$

$$\mu_2 = \mathbb{E}X_n^2 = \mathrm{Var}(X_n) + \mathbb{E}[X_n]^2$$
$$= kp(1-p) + k^2p^2$$

② Form sys. of eqns

$$\begin{cases} \mu_1 = kp = \overline{X} = m_1 \\ \mu_2 = kp(1-p) + k^2p^2 = \overline{X^2} = m_2 \end{cases}$$

③ Solve for k and p in terms of $\underset{\sim}{X}$.

$$\left\{ kp = \overline{X} \quad \text{and} \quad kp(1-p) + k^2p^2 = \overline{X^2} \right.$$

$$kp(1-p) + k^2p^2 = \overline{X^2}$$

$$\Downarrow \quad kp = \overline{X}$$

$$\overline{X}(1-p) + \overline{X}^2 = \overline{X^2}$$

$$\Downarrow \quad \overline{X}(1-p) = \overline{X^2} - \overline{X}^2$$

$$\Downarrow \quad 1-p = \frac{\overline{X^2} - \overline{X}^2}{\overline{X}}$$

$$\Downarrow \quad \boxed{\hat{p} = 1 - \frac{\overline{X^2} - \overline{X}^2}{\overline{X}}}$$

$$kp = \overline{X}$$

$$\Downarrow$$

$$\boxed{\hat{k} = \frac{\overline{X}}{\hat{p}}}$$