# Lecture 23: p-values / Bayesian Inference

Often we report the outcome of a HT using a p-value.

**Defn: P-value**

A p-value is a test statistic $p(\underset{\sim}{x})$ where

$$0 \le p(\underset{\sim}{x}) \le 1 \quad \forall \underset{\sim}{x}$$

idea: small $p(\underset{\sim}{x})$ gives evidence of $H_a$

large $p(\underset{\sim}{x})$ gives evidence of $H_0$

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Recall, a HT is just a partition of $\mathcal{X}$ into $A$ and $R$, and we can use p-val's to defn $R$ as

$$R = \{ \underset{\sim}{x} : p(\underset{\sim}{x}) \le \alpha \}$$

i.e. rej. $H_0$ when p-val is suff. small.

We say a p-value is valid if

$\forall\ 0 \le \alpha \le 1$ and $\theta \in \Theta_0$

$$\mathbb{P}_\theta(p(\underset{\sim}{X}) \le \alpha) \le \alpha$$

stochastically bounded by a $U(0,1)$ RV

$$F_p(\alpha) \le \alpha$$

CDF of $U(0,1)$

Allowable (typical) $P(\underset{\sim}{X}) \sim U(0,1)$ for $\theta \in \Theta_0$

So $F_p(\alpha) = \alpha$.

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If $P$ is valid and I set up a test

$$R = \{ \underset{\sim}{x} : p(\underset{\sim}{x}) \le \alpha \}$$

then this test is level $\alpha$.

reason: under null $\theta \in \Theta_0$

$$\begin{aligned}
\mathbb{P}_\theta(\text{reject}) &= \mathbb{P}_\theta(P(\underset{\sim}{X}) \le \alpha) \\
&= F_p(\alpha) \\
&\le \alpha
\end{aligned}$$

by defn that $P$ is valid.

Ex. Consider $H_0: \theta = \theta_0$ v. $H_a: \theta \neq \theta_0$

and I get a test stat $T$ and

$$R = \{ \underset{\sim}{x} : T(\underset{\sim}{x}) \text{ is large} \}$$

Let

$$p(\underset{\sim}{x}) = \mathbb{P}_{\theta_0}\Big( \underbrace{T(\underset{\sim}{X})}_{\text{rad}} \geq \underbrace{T(\underset{\sim}{x})}_{\text{obs.}} \Big) = 1 - F_T(T(\underset{\sim}{x}))$$

= prob. under $H_0$ that I obs. a test stat $T$ as big or larger than my observed test stat $T(\underset{\sim}{x})$.

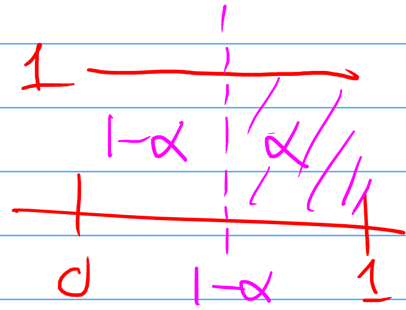

Claim: $P$ is a valid p-value

i.e. under $H_0$ $P \sim U(0,1)$

$$F_p(\alpha) = \mathbb{P}_{\theta_0}(P(\underset{\sim}{X}) \leq \alpha)$$

$$= \mathbb{P}_{\theta_0}(1 - F_T(T(\underset{\sim}{X})) \leq \alpha)$$

$$= \mathbb{P}_{\theta_0}\Big(\underbrace{F_T(T(\underset{\sim}{X}))}_{U \sim U(0,1)} \geq 1-\alpha\Big)$$




$$= \mathbb{P}_{\theta_0}(U \geq 1-\alpha)$$

$$= \alpha$$

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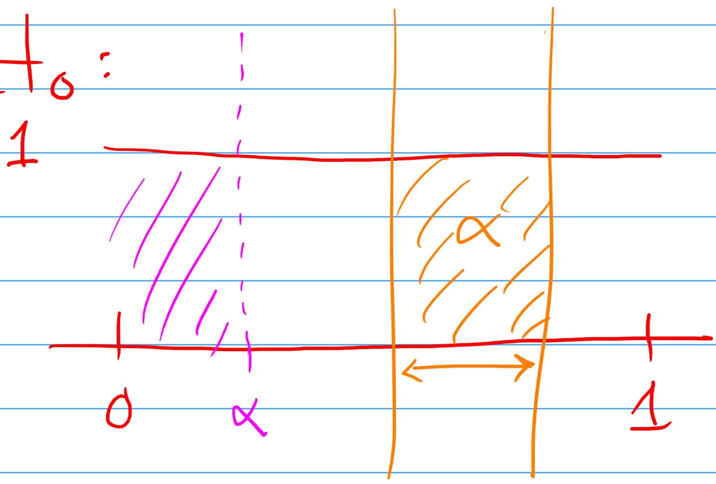
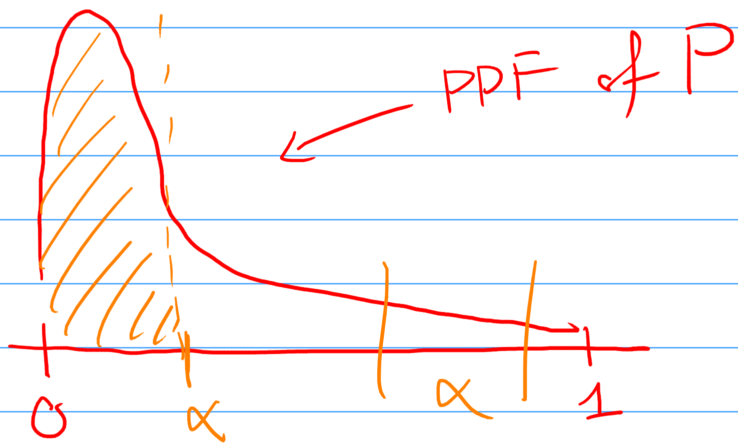
Punchline: under $H_0$ $P \sim U(0,1)$.

Under $H_0$:




Typically higher power when we rej. when $P$ is small.

Ex.



# Bayesian Inference

this course

## Frequentist (classical) Inference

$\theta$ fixed but unknown.

## Bayesian Inference

$\Theta$ random

**Bayesian approach:**

1. prior dist.

$$\Theta \sim \pi$$ ← pmf or pdf of $\Theta$

$\pi(\theta)$

2. get data

$$\underset{\sim}{X} \mid \Theta = \theta \sim f(\underset{\sim}{x} \mid \theta)$$

sampling dist / likelihood

3. update/combine prior/likelihood to get posterior

$$\pi(\theta \mid \underset{\sim}{x}) = \frac{f(\underset{\sim}{x} \mid \theta)\pi(\theta)}{f(\underset{\sim}{x})} \propto f(\underset{\sim}{x} \mid \theta)\pi(\theta)$$

baye's' theorem

posterior $\propto$ likelihood $\times$ prior

$$f(\underset{\sim}{x}) = \int f(\underset{\sim}{x} \mid \theta)\pi(\theta)\,d\theta$$

Often estimate $\hat{\theta}$ as

$$\mathbb{E}[\Theta \mid \underset{\sim}{X}] \text{ or } \operatorname{mode}(\Theta \mid \underset{\sim}{X}).$$

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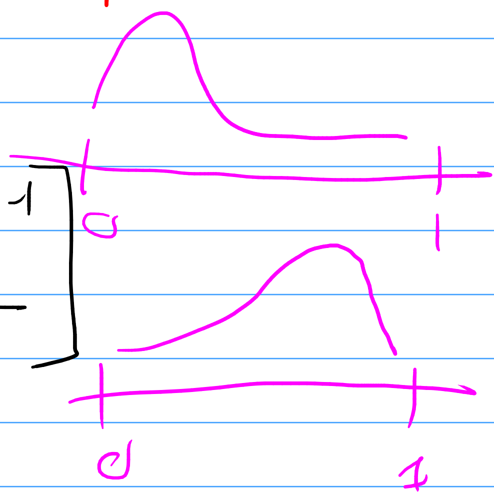
Ex. $P \sim \text{Beta}(\alpha, \beta)$ prior

$X_n \mid P = p \overset{iid}{\sim} \text{Bern}(p)$ likelihood

$$\pi(p \mid \underset{\sim}{x}) \propto f(\underset{\sim}{x} \mid p)\,\pi(p)$$

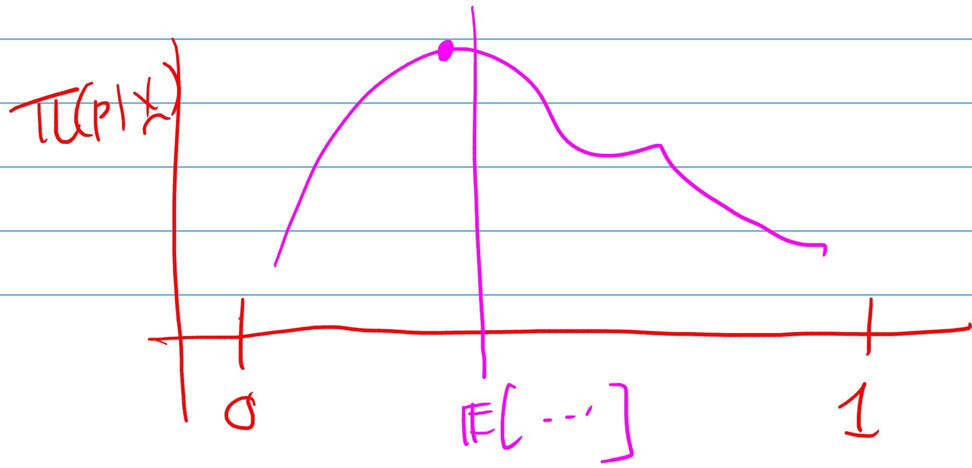
$$= \left[\prod_n p^{x_n}(1-p)^{1-x_n}\right]\left[\frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}\right]$$




$$\propto p^{N\bar{x}}(1-p)^{N-N\bar{x}}\, p^{\alpha-1}(1-p)^{\beta-1}$$

$$\propto \underbrace{p^{N\bar{x}+\alpha-1}(1-p)^{N-N\bar{x}+\beta-1}}_{\text{basically a Beta}(N\bar{x}+\alpha,\; N-N\bar{x}+\beta)}$$

i.e. $P \mid \underset{\sim}{X} \sim \text{Beta}(\qquad)$



$$\hat{p} = \mathbb{E}[p \mid \underset{\sim}{x}] = \frac{N\bar{x} + \alpha}{N\bar{x} + \alpha + N - N\bar{x} + \beta}$$

$$= w \underbrace{\bar{x}}_{\text{freq}} + (1-w) \underbrace{\frac{\alpha}{\alpha+\beta}}_{\mathbb{E}[p]}$$

$$w = \frac{N}{\alpha+\beta+N}$$

$w \to 1$ as $N \to \infty$