Lecture 10 : Rao - Blackwell Theorem

Some facts:

① If $\hat{\theta}$ is unbiased for $\tau(\theta)$

i.e. $E\hat{\theta} = \tau(\theta)$

let W be some function of $\underset{\sim}{X}$ i.e. $W = W(\underset{\sim}{X})$

(maybe a stat., maybe not)

$$\boxed{\begin{aligned} &\mathbb{E}[X|Y] = g(Y) \\ &\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}X \end{aligned}}$$

Consider

a RV

$$\varphi(W) = \varphi = \mathbb{E}[\hat{\theta} \mid W]$$

notice that

$$\mathbb{E}[\varphi] = \mathbb{E}[\mathbb{E}[\hat{\theta} \mid W]] = \mathbb{E}[\hat{\theta}] = \tau(\theta)$$

If φ is a statistic then it is unbiased for $\tau(\theta)$

② $Var(\varphi) \le Var(\hat{\theta})$.

Law of Total Variance

$$Var\,\hat{\theta} = Var\,\underbrace{\mathbb{E}[\hat{\theta} \mid W]}_{\varphi} + \underbrace{\mathbb{E}\,Var(\hat{\theta} \mid W)}_{\geq 0}$$

So $\operatorname{Var}\hat{\theta} = \operatorname{Var}\varphi + a$ where $a \geq 0$

i.e. $\operatorname{Var}(\hat{\theta}) \geq \operatorname{Var}(\varphi)$

Ex. $X_n \overset{iid}{\sim} N(\theta, 1)$

and let $\hat{\theta} = \frac{1}{2}(X_1 + X_2)$

$\hat{\theta}$ unbiased for θ: $\mathbb{E}\hat{\theta} = \frac{1}{2}\mathbb{E}X_1 + \frac{1}{2}\mathbb{E}X_2 = \frac{1}{2}\theta + \frac{1}{2}\theta$

$= \theta$

$$\operatorname{Var}\hat{\theta} = \frac{1}{2}\operatorname{Var}(X_1) = \frac{1}{2}$$

$W = X_1$

$$\varphi = \mathbb{E}[\hat{\theta} \mid W] = \mathbb{E}\left[\tfrac{1}{2}(X_1 + X_2) \mid X_1\right]$$

$$= \tfrac{1}{2}\mathbb{E}[X_1 \mid X_1] + \tfrac{1}{2}\mathbb{E}[X_2 \mid X_1]$$

$$= \tfrac{1}{2}X_1 + \tfrac{1}{2}\mathbb{E}[X_2]$$

$$= \tfrac{1}{2}X_1 + \tfrac{1}{2}\theta$$

$\mathbb{E}[z \mid z=5] = 5$

$\mathbb{E}[z \mid z=6] = 6$

$\mathbb{E}[z \mid z=y] = y$

$\mathbb{E}[z \mid z] = z$

$$\mathbb{E}[\varphi] = \mathbb{E}\left[\tfrac{1}{2}X_1 + \tfrac{1}{2}\theta\right] = \tfrac{1}{2}\mathbb{E}X_1 + \tfrac{1}{2}\theta = \tfrac{1}{2}\theta + \tfrac{1}{2}\theta = \theta$$

$$\mathrm{Var}(\varphi) = \mathrm{Var}\left(\tfrac{1}{2}X_1 + \tfrac{1}{2}\theta\right) = \tfrac{1}{4}\mathrm{Var}\,X_1 = \tfrac{1}{4}$$

Let $W = \bar{X}$

$$\varphi = \mathbb{E}[\hat{\theta} \mid W] = \mathbb{E}\left[\tfrac{1}{2}(X_1 + X_2) \mid \bar{X}\right]$$

$$= \tfrac{1}{2}\mathbb{E}[X_1 \mid \bar{X}] + \tfrac{1}{2}\mathbb{E}[X_2 \mid \bar{X}]$$

same

$$= \tfrac{1}{2} 2\, \mathbb{E}[X_n \mid \bar{X}]$$

$$= \mathbb{E}[X_n \mid \bar{X}]$$

$$= \tfrac{1}{N} N\, \mathbb{E}[X_n \mid \bar{X}]$$

$$= \tfrac{1}{N} \sum_{n=1}^{N} \mathbb{E}[X_n \mid \bar{X}]$$

$$= \mathbb{E}\left[\tfrac{1}{N}\sum_n X_n \mid \bar{X}\right]$$

$$= \mathbb{E}[\bar{X} \mid \bar{X}]$$

$$\varphi = \bar{X}$$

and by our theorem:

1. $\mathbb{E}\varphi = \mathbb{E}\bar{X} = \theta$
2. $\mathrm{Var}\,\varphi = \mathrm{Var}(\bar{X}) \leq \mathrm{Var}(\hat{\theta})$

Theorem: Rao-Blackwell Theorem

If $\hat{\theta}$ is unbiased for $\tau(\theta)$ and W is sufficient for θ, then if

$$\varphi = E[\hat{\theta} \mid W]$$

1. $E\varphi = \tau(\theta)$
2. $\operatorname{Var}\varphi \le \operatorname{Var}\hat{\theta}$
3. φ is a statistic (i.e. no θ in formula)

Pf of 3

$$\varphi = E[\hat{\theta}(\underset{\sim}{X}) \mid \overset{\text{sufficient}}{W}]$$

$$= \int \underbrace{\hat{\theta}(\underset{\sim}{x})}_{\text{no } \theta} \underbrace{f_{\underset{\sim}{X}|W}(\underset{\sim}{x})}_{\text{no } \theta} \, d\underset{\sim}{x}$$

doesn't depend on θ

$$Eg(X) = \int g(x) f(x) \, dx$$

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tech. conditions.

Theorem: Lehmann - Scheffe Theorem

Let W be a (complete) sufficient statistic for θ and let $\hat{\theta}$ be an unbiased est. for $T(\theta)$ that depends on $\underset{\sim}{X}$ only through W:

$$\hat{\theta} = \hat{\theta}(W) = \hat{\theta}(W(\underset{\sim}{X}))$$

then $\hat{\theta}$ is the UMVUE for $T(\theta)$.

Basically: If I can find a unbiased est for $T(\theta)$ from a S.S. it is the UMVUE.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

known

unknown

Want: the UMVUE for μ.

Use Lehmann-Scheffe.

1. Find a SS for μ, $\bar{X}$.
2. Guess a fn of $\bar{X}$ unbiased for μ.

$\hat{\mu} = \bar{X}$

Since $E\hat{\mu} = E\bar{X} = \mu$

and is a fn of the SS, then it is the UMVUE.

Ex. Let $\tau(\mu) = \mu^2$.

1. get SS for μ, $\bar{X}$
2. Find a fn of $\bar{X}$ that is unbiased for $\tau(\mu) = \mu^2$.

$$E[\bar{X}^2] = Var(\bar{X}) + E[\bar{X}]^2$$

$$= \sigma^2/N + \mu^2$$

Consider $\widehat{\mu^2} = \bar{X}^2 - \sigma^2/N$

Now $E[\widehat{\mu^2}] = E[\bar{X}^2] - \sigma^2/N = \mu^2$.

So $\bar{X}^2 - \sigma^2/N$ is the UMVUE for μ^2.

Lehmann-Scheffe Takeaway

How to find the UMVUE

1. Find a SS for θ - call it W
2. Find a fn of W that is unbiased for $\tau(\theta)$
   - (i) Guess a fn $\hat{\theta}(W)$ so that
   $$\mathbb{E}[\hat{\theta}(W)] = \tau(\theta)$$
   - (ii) Use Rao-Blackwell, get some really simple unbiased est. V and form
   $$\hat{\theta} = \mathbb{E}[V \mid W]$$

Ex. $X_n \overset{iid}{\sim} U(0,\theta)$

What is the UMVUE for $\tau(\theta) = \theta$.

1. Find a SS for θ: $X_{(N)}$
2. Find some fn of $X_{(N)}$ that is unbiased for θ

Last time: $\mathbb{E}[X_{(N)}] = \frac{N}{N+1}\theta$

So if $\hat{\theta} = \frac{N+1}{N} X_{(N)}$

then $\mathbb{E}\hat{\theta} = \theta$ and $\hat{\theta}$ is a fn of the SS

so $\hat{\theta}$ is the UMVUE.

Pf. of Lehmann-Scheffe

If $\hat{\theta} = \hat{\theta}(W)$ where W is (complete) sufficient
and unbiased for $T(\theta)$
then for any other unbiased est V of
$T(\theta)$ want to show that

$$\mathrm{Var}(V) \geq \mathrm{Var}(\hat{\theta})$$

We'll do this by Rao-Blackwellizing V using W.

We'll show that $\mathbb{E}[V|W] = \hat{\theta}$.

Rao-Blackwell says:

if $\varphi = \varphi(W) = \mathbb{E}[V|W]$

then (1) $\mathbb{E}\varphi = T(\theta)$

(2) $\mathrm{Var}\,\varphi \leq \mathrm{Var}(V)$

(3) φ is a stat

If I can show that $\varphi = \hat{\theta}$ then

$$\underline{\operatorname{Var}\hat{\theta}} = \operatorname{Var}\varphi \leq \underline{\operatorname{Var}(V)}.$$

This is what I want.

Consider $g(W) = \hat{\theta}(W) - \varphi(W)$

we'll show that $g \equiv 0 \ \forall \theta$

<u>Completeness of W</u>

$$\mathbb{E}_\theta[h(W)] = 0 \ \forall \theta \iff h \equiv 0$$

Notice $\mathbb{E}_\theta g(W) = \mathbb{E}[\hat{\theta}] - \mathbb{E}[\varphi] = \tau(\theta) - \tau(\theta)$

$= 0 \quad \forall \theta$

If W is complete then $g \equiv 0$

So $g(W) = 0 = \hat{\theta} - \varphi$

So $\hat{\theta} = \varphi$.