# Lecture 18: Likelihood Ratio Test

**Recall**: $L(\theta) = f_\theta(\underset{\sim}{x})$

Want to test a hypothesis

$$H_0: \theta \in \Theta_0 \quad v. \quad H_a: \theta \in \Theta_a$$

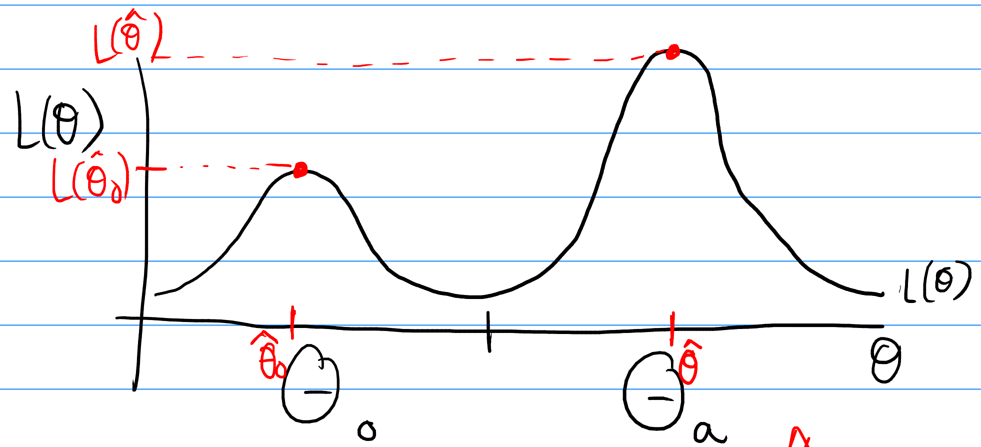
the Likelihood Ratio Test Statistic is defined as

$$\lambda(\underset{\sim}{x}) = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \frac{\text{max. val. of } L(\theta) \text{ in } \Theta_0}{\text{max val. of } L(\theta) \text{ in } \Theta} = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} \le 1$$

$\hat{\theta}$ = MLE

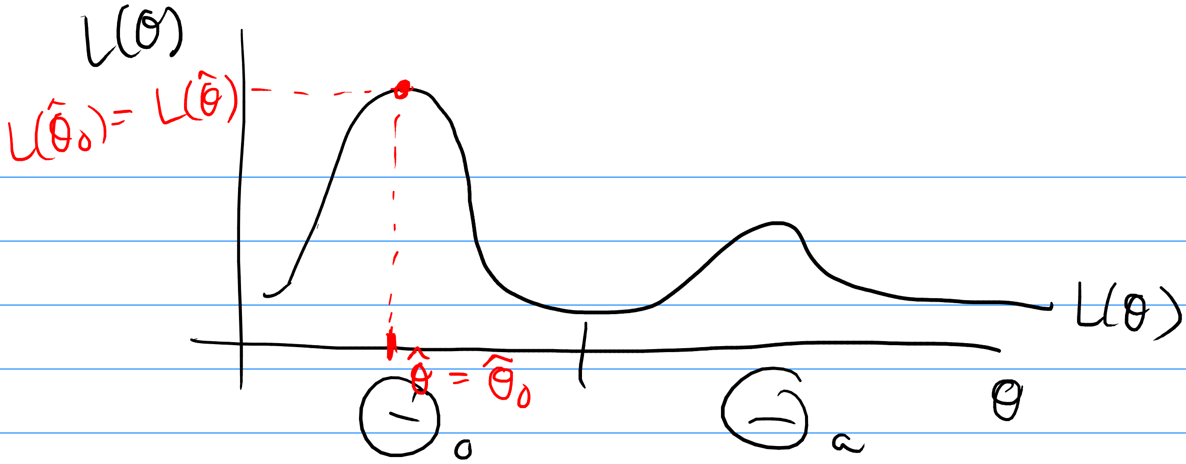
$\hat{\theta}_0$ = MLE restricted to $\Theta_0$

**Ex.**

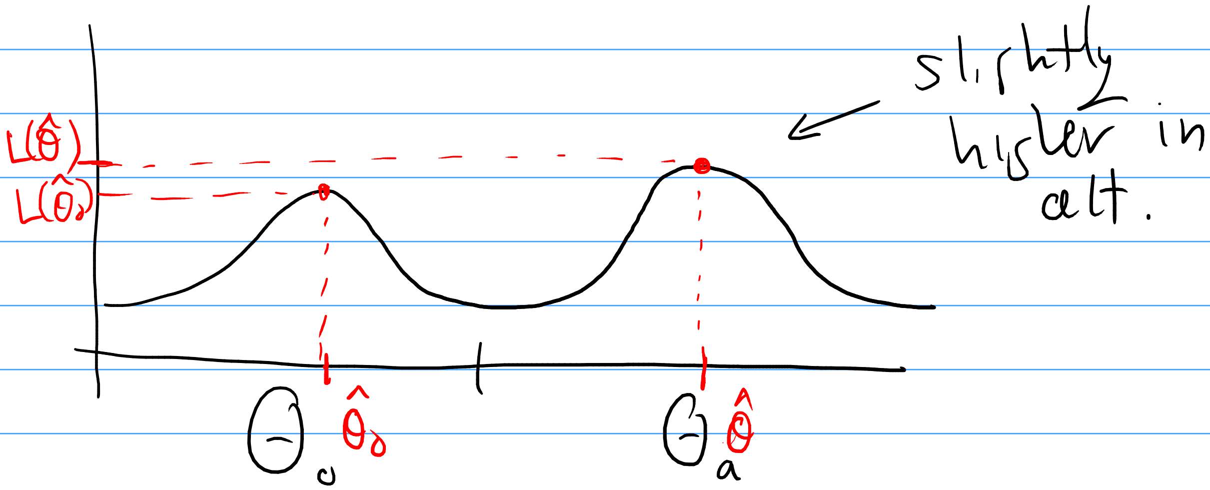



So $\lambda(\underset{\sim}{x}) = \dfrac{L(\hat{\theta}_0)}{L(\hat{\theta})} < 1$

↑ probably reject $H_0$




So $\lambda = \dfrac{L(\hat{\theta}_0)}{L(\hat{\theta})} = 1$ don't reject $H_0$




$\lambda \approx 1$ maybe reject $H_0$?

**Overall:** $\lambda \approx 1$ $H_a$ not much more likely than $H_0$

$\lambda \approx 0$ $H_a$ much more likely than $H_0$

The likelihood ratio test says reject when

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} \leq c$$

some threshold we can choose to balance Type I and II errors

Typically, $c$ small $\Rightarrow$ less type I
more type II

$c$ large $\Rightarrow$ reverse

Equiv. $R = \{ \underset{\sim}{x} \in \mathcal{X} : \lambda(\underset{\sim}{x}) \le c \}$

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Ex. $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

known

I can choose

$$H_0 : \theta \le a \quad v. \quad H_a : \theta > a$$

Let's form the LRT.

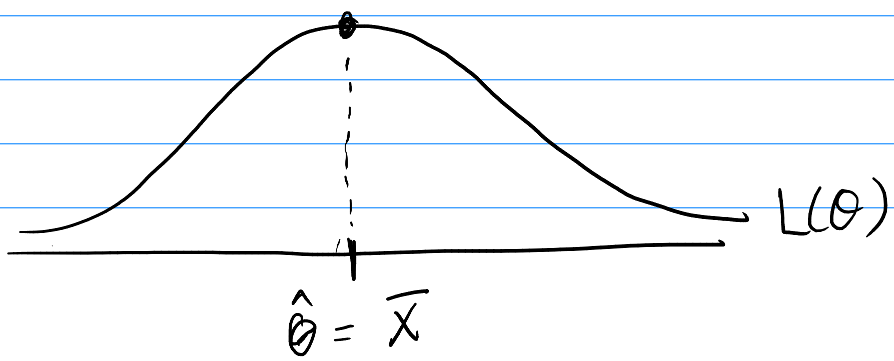
$$\Theta = \mathbb{R}, \quad \Theta_0 = (-\infty, a], \quad \Theta_a = (a, \infty)$$

$$L(\theta) = \prod_n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_n - \theta)^2\right)$$

$$= \dots = (2\pi)^{-N/2} (\sigma^2)^{-N/2} \exp\left(-\frac{1}{2\sigma^2} \sum_n (x_n - \theta)^2\right)$$

looks quadratic in $\theta$

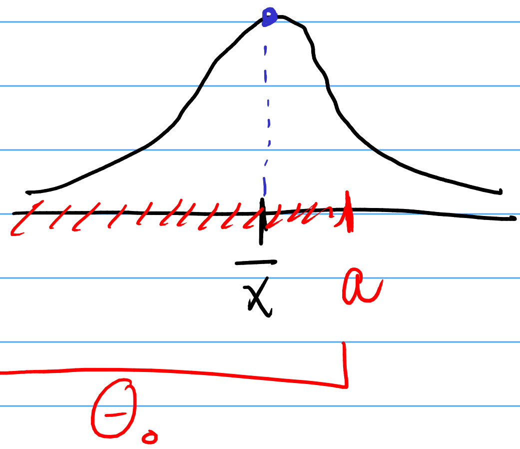
Kinda like $e^{-\theta^2}$



$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{L(\hat{\theta}_0)}{L(\bar{x})}$$

To find $\hat{\theta}_0$, two cases $\bar{x} \le a$ and $\bar{x} > a$

$\bar{x} \le a$

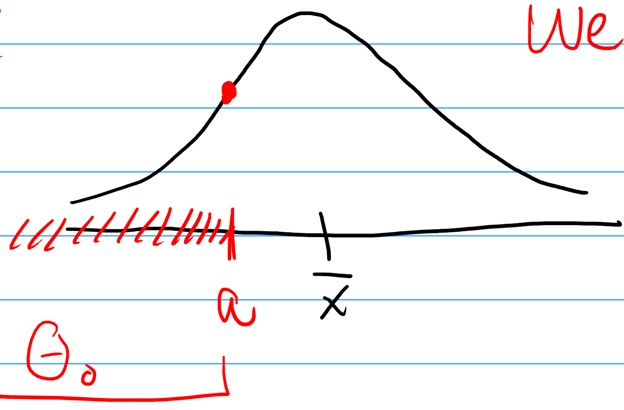



$$\hat{\theta}_0 = \underset{\theta \in \Theta_0}{\operatorname{argmax}} L(\theta)$$

$$\Theta_0 = (-\infty, a]$$

we get $\hat{\theta}_0 = \bar{x}$

$\bar{x} > a$




we find $\hat{\theta}_0 = a$

So
$$\hat{\theta}_0 = \begin{cases} \bar{x} & \text{if } \bar{x} \le a \\ a & \text{if } \bar{x} > a \end{cases}$$

So

$$\lambda(\underset{\sim}{x}) = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \begin{cases} \frac{L(\bar{x})}{L(\bar{x})} = 1 & \bar{x} \le a \\ \boxed{\frac{L(a)}{L(\bar{x})} \quad \bar{x} > a} \end{cases}$$

$H_0: \theta \le a$

$H_a: \theta > a$

So LRT says to reject when

$$\frac{L(a)}{L(\bar{x})} \leq c \quad \text{where } c \in (0,1)$$

$$\frac{L(a)}{L(\bar{x})} = \frac{\cancel{(2\pi)^{-N/2}}\cancel{(\sigma^2)^{-N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_n (x_n - a)^2\right)}{\cancel{(2\pi)^{-N/2}}\cancel{(\sigma^2)^{-N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_n (x_n - \bar{x})^2\right)}$$

$$= \frac{\exp\left(-\frac{1}{2\sigma^2}\sum_n(\cancel{x_n^2} - 2a x_n + a^2)\right)}{\exp\left(-\frac{1}{2\sigma^2}\sum_n(\cancel{x_n^2} - 2\bar{x} x_n + \bar{x}^2)\right)} \qquad \frac{e^{a+b}}{e^{a+c}}$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(-2aN\bar{x} + Na^2 + 2N\bar{x}^2 - N\bar{x}^2\right)\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(N\bar{x}^2 - 2aN\bar{x} + Na^2\right)\right)$$

$$= \exp\left(-\frac{N}{2\sigma^2}(\bar{x} - a)^2\right)$$

So the LRT says to reject when

$$\lambda = \exp\left(-N/2\sigma^2 (\bar{x} - a)^2\right) \leq c$$

$$\Leftrightarrow \quad -N/2\sigma^2 (\bar{x} - a)^2 \leq \log c$$

$$\Leftrightarrow \quad \frac{N}{\sigma^2}(\bar{x} - a)^2 \geq -2\log c$$

$$\sqrt{a^2} = |a|$$

$$\Leftrightarrow \quad \frac{\sqrt{N}}{\sigma}(\bar{X}-a) \geq \sqrt{-2\log c}$$

we assume $\bar{X} > a$

$\bar{X} - a > 0$

$$\Leftrightarrow \quad \frac{\bar{X}-a}{\sigma/\sqrt{N}} \geq \underbrace{\sqrt{-2\log c}}_{c^*}$$

some number $> 0$

So LRT says to reject when

$$\boxed{\frac{\bar{X}-a}{\sigma/\sqrt{N}} \geq c^*} \Leftrightarrow \bar{X} \geq a + c^* \sigma/\sqrt{N}$$

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How should we choose $c^*$?

Maybe want the LRT to be level $\alpha$ test? (size $\alpha$ test)

$$\text{i.e} \quad \max_{\theta \in \Theta_0} \underbrace{P(\text{reject})}_{\beta(\theta)} \leq \alpha \quad (\text{level})$$

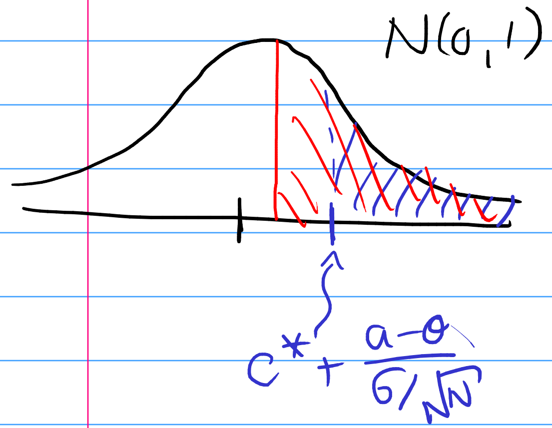
$$\beta(\theta) = P_\theta(\lambda \leq c) = P_\theta\left(\frac{\bar{X}-a}{\sigma/\sqrt{N}} \geq c^*\right)$$

$$= P_\theta\left(\frac{\bar{X}-a}{\sigma/\sqrt{N}} + \frac{a-\theta}{\sigma/\sqrt{N}} \geq c^* + \frac{a-\theta}{\sigma/\sqrt{N}}\right)$$

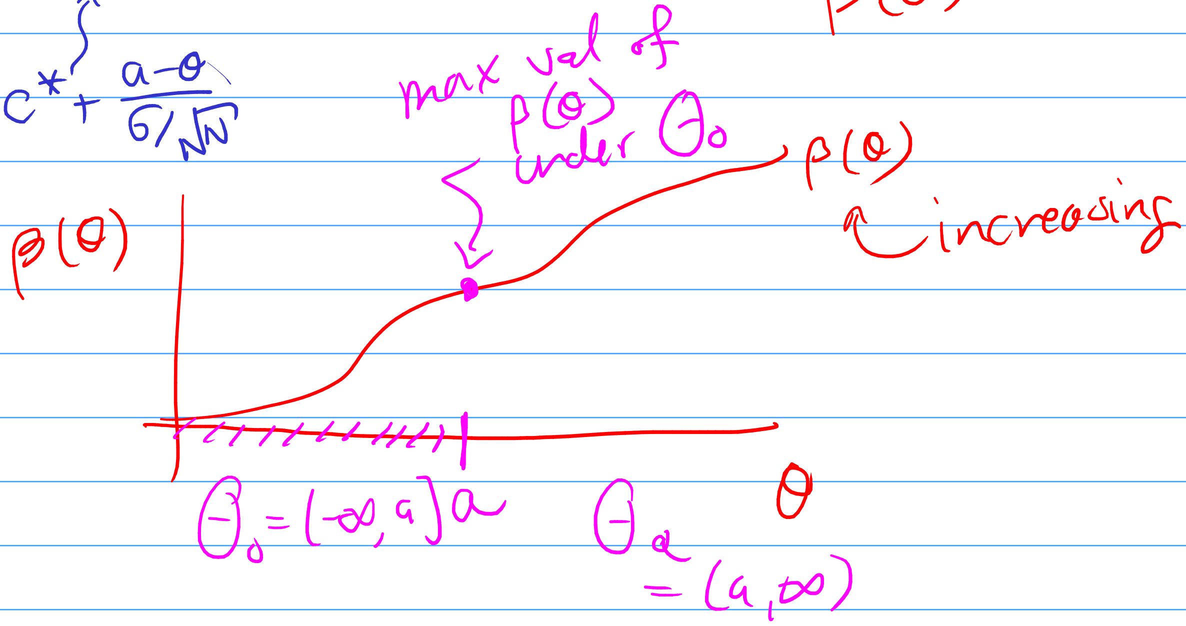
$X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

$\bar{X} \sim N(\theta, \sigma^2/N)$ $\quad = \mathbb{P}_\theta\left(\frac{\bar{X}-\theta}{\sigma/\sqrt{N}} \geq c^* + \frac{a-\theta}{\sigma/\sqrt{N}}\right)$

$z = \frac{\bar{X}-\theta}{\sigma/\sqrt{N}} \sim N(0,1)$ $\quad = \boxed{\mathbb{P}\left(\underbrace{Z}_{Z \sim N(0,1)} \geq c^* + \frac{a-\theta}{\sigma/\sqrt{N}}\right)}\ \beta(\theta)$




as $\theta \uparrow$, $\underbrace{\mathbb{P}(\dots)}_{=\beta(\theta)} \uparrow$




So max $\beta(\theta)$ under $H_0$ is at $\theta = a$

So if I want

$$\max_{\theta \in \Theta_0} \beta(\theta) = \beta(a) \leq \alpha$$

So choose $c^*$ so that $\beta(a) \leq \alpha$

$$\beta(a) = \mathbb{P}\left(Z \geq c^* + \frac{a-a}{\sigma/\sqrt{N}}\right) = \mathbb{P}(Z \geq c^*)$$

So choose $c^*$ so that $\overbrace{P(Z \geq c^*)}^{1-F_Z(c^*)} = \alpha$

If $F_Z$ is the CDF of a $N(0,1)$

then want

$$1 - F_Z(c^*) = \alpha$$

or

$$\boxed{c^* = F_Z^{-1}(1-\alpha)} = z_\alpha$$