Lecture 15 : Delta Method

$\bar{X} \sim AN(\mu, \sigma^2/N)$

CLT: $X_n \overset{iid}{\sim} f$, $\mathbb{E}X_n = \mu$, $\text{Var}\, X_n = \sigma^2 < \infty$

then $\sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right) \xrightarrow{d} N(0,1)$

Theorem: First-Order Delta Method

If Y_N is a seq. of RVs where

$$\sqrt{N}(Y_n - \theta) \xrightarrow{d} N(0, \underbrace{\psi^2}_{\psi^2(\theta)})$$

then if g is a differentiable function and $g'(\theta) \neq 0$ then

$$\sqrt{N}(g(Y_n) - g(\theta)) \xrightarrow{d} N(0, [g'(\theta)]^2 \psi^2)$$

Another way:

$$Y_N \sim AN(\theta, \psi^2/N)$$

then $$g(Y_N) \sim AN(g(\theta), [g'(\theta)]^2 \psi^2/N)$$

The most obv. example is when $Y_N = \bar{X}_N$

then by CLT: $\bar{X}_N \sim AN(\mu, \sigma^2/N)$

and so by FO Δ-method:

$$\bar{X}_N \sim AN\left(g(\mu), [g'(\mu)]^2 \sigma^2/N\right)$$

Ex, CLT: $\sqrt{N}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$

If $g(x) = \log(x)$

then $g'(x) = 1/x$

and so (as long as $g'(\mu) \neq 0$)

then

$$\sqrt{N}\left(g(\bar{x}) - g(\mu)\right) = \sqrt{N}\left(\log(\bar{x}) - \log(\mu)\right)$$

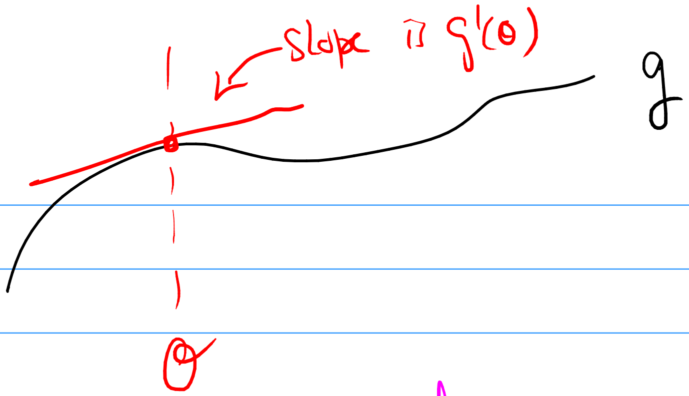
$$\xrightarrow{d} N\left(0, \frac{1}{\mu^2}\sigma^2\right)$$

i.e. $g(\bar{x}) = \log(\bar{x}) \sim AN\left(\log(\mu), \sigma^2/N\mu^2\right)$

Pf of Δ-method

Consider a taylor Approx. of g about θ

$$g(x) = g(\theta) + g'(\theta)(x - \theta)$$



Idea: $\mathbb{E}[cX+d] = c\,\mathbb{E}X + d$

and $\mathrm{Var}(cX+d) = c^2\,\mathrm{Var}X$

if near θ $g(x) \approx g(\theta) + g'(\theta)(x-\theta)$

then near θ $\mathbb{E}[g(X)] \approx g(\theta) + g'(\theta)\underbrace{[\mathbb{E}X - \theta]}_{0}$

$= g(\theta)$

and near θ $\mathrm{Var}(g(X)) \approx [g'(\theta)]^2\,\mathrm{Var}X$

more formally: $g(x) \approx g(\theta) + g'(\theta)(x-\theta)$

then $g(x) - g(\theta) \approx g'(\theta)(x-\theta)$

so $\sqrt{N}(g(x) - g(\theta)) \approx g'(\theta)[\sqrt{N}(x-\theta)]$

Plug in Y_N for x [recall: $\sqrt{N}(Y_N - \theta) \xrightarrow{d} N(0,\psi^2)$]

then

$$\underbrace{\sqrt{N}(g(Y_N) - g(\theta)) \approx g'(\theta)[\underbrace{\sqrt{N}(Y_N - \theta)}_{\xrightarrow{d} N(0,\psi^2)}]}_{\xrightarrow{d} N(0,[g'(\theta)]^2\psi^2).}$$

Ex. Let $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$ then CLT says

$$\sqrt{N}\left(\bar{X} - \lambda\right) \xrightarrow{d} N(0, \lambda)$$

consider $g(x) = 1/x$ then $g'(x) = -1/x^2$

and as long as $\lambda > 0$ $g'(\lambda) \neq 0$

and so $[g'(\lambda)]^2 = \left(-1/\lambda^2\right)^2 = 1/\lambda^4$

and the Δ-method says

$$\sqrt{N}\left(g(\bar{X}) - g(\lambda)\right) \xrightarrow{d} N\left(0, [g'(\lambda)]^2 \lambda\right)$$

i.e.

$$\sqrt{N}\left(1/\bar{X} - \frac{1}{\lambda}\right) \xrightarrow{d} N\left(0, 1/\lambda^3\right)$$

Ex. Variance-Stabilizing Transformation

Generically $Y_N \sim AN\left(\theta, \psi^2(\theta)/N\right)$

variance depends on θ

Q: is there some trans g so that

$g(Y_N) \sim AN(g(\theta),$ doesn't depend on θ $)$

Soln: use Δ-method.

$$g(Y_N) \sim AN\left(g(\theta), [g'(\theta)]^2 \psi^2(\theta)/N\right)$$

could choose g to make

$$[g'(\theta)]^2 \psi^2(\theta)/N \text{ independent of } \theta$$

So our condition to satisfy is

$$[g'(\theta)]^2 \psi^2(\theta) = c$$

Ex. $X_n \overset{iid}{\sim} Pois(\lambda)$ $\qquad \psi^2(\lambda) = \lambda$

CLT: $\bar{X}_N \sim AN(\lambda, \lambda/N)$

ODE: $[g'(\lambda)]^2 \psi^2(\lambda) = c$

$\Rightarrow [g'(\lambda)]^2 \lambda = c$

$\Rightarrow \left(\frac{dg}{d\lambda}\right)^2 \lambda = c$

$\Rightarrow \frac{dg}{d\lambda} = \sqrt{c/\lambda}$

$\Rightarrow dg = \frac{\sqrt{c}}{\sqrt{\lambda}} d\lambda$

$\Rightarrow g = \int dg = \int \frac{\sqrt{c}}{\sqrt{\lambda}} d\lambda$

$\Rightarrow g \propto \sqrt{\lambda}$

<u>Claim:</u> $g(x) = \sqrt{x}$ (var. stab. trans for Pois(λ))

$\bar{X} \sim AN(\lambda, \lambda/N)$ $\qquad g'(x) = \frac{1}{2\sqrt{x}}$

$$\sqrt{\bar{x}} \sim AN\left(\sqrt{\lambda}, \underbrace{[g'(\lambda)]^2 \lambda/N}_{\left(\frac{1}{2\sqrt{\lambda}}\right)^2 \lambda/N \;=\; \frac{1}{4N}}\right)$$

<u>Theorem:</u> <u>Second Order Δ-method</u>

Assume $\sqrt{N}(Y_N - \theta) \xrightarrow{d} N(0, \psi^2(\theta))$

and $g'(\theta) = 0$ but g is twice diff'able

then

$$N(g(Y_N) - g(\theta)) \xrightarrow{d} \frac{\psi^2 g''(\theta)}{2} \chi^2(1)$$

$g''(\theta) \neq 0$

<u>Ex.</u> $X_n \overset{iid}{\sim} \text{Bern}(p)$ and let

$$g(t) = t \log(t/p) - (1-t)\log\left(\frac{1-t}{1-p}\right)$$

dist. metric btwn Bern(p) and a Bern(t)

KL divergence

Q: What can I say about $g(\bar{X})$?

CLT: $\sqrt{N}(\bar{X}-p) \xrightarrow{d} N(0, \underbrace{p(1-p)}_{\psi^2(p)})$

notice that

$$g'(t) = \log(t/1-t) - \log(p/1-p)$$

and so $g'(p) = 0$

Can't use FO Δ-method. Can use SO Δ-method.

$$g''(t) = \frac{1}{t} + \frac{1}{1-t} = \frac{1}{t(1-t)}$$

$$g''(p) = \frac{1}{p(1-p)} \neq 0 \quad (0<p<1)$$

So by the SO Δ-method,

$$N(g(\bar{X}) - g(p)) \xrightarrow{d} \frac{\psi^2(p)\, g''(p)}{2} \chi^2(1)$$

in our case

$$N(g(\bar{x}) - g(p)) \xrightarrow{d} \frac{p(1-p)\frac{1}{p(1-p)}\chi^2(1)}{2}$$

$$= \frac{1}{2}\chi^2(1)$$

pf. of SO Δ-method

Consider a SO Taylor expansion of $g(x)$

$$g(x) \approx g(\theta) + g'(\theta)(x-\theta) + \frac{g''(\theta)}{2}(x-\theta)^2$$

plug in Y_N into this

$$\left[\sqrt{N}(Y_N - \theta) \xrightarrow{d} N(0, \psi^2)\right]$$

$$g(Y_N) \approx g(\theta) + g'(\theta)(Y_N-\theta) + \frac{g''(\theta)}{2}(Y_N-\theta)^2$$

$g'(\theta)=0$

$$\approx g(\theta) + \frac{g''(\theta)}{2}(Y_N-\theta)^2$$

$$\Rightarrow g(Y_N) - g(\theta) \approx \frac{g''(\theta)}{2}(Y_N-\theta)^2$$

$$\Rightarrow N(g(Y_N)-g(\theta)) \approx \frac{g''(\theta)}{2}\Big[\underbrace{\sqrt{N}(Y_N-\theta)}_{\xrightarrow{d} N(0,\psi^2) = \psi N(0,1)}\Big]^2$$

$$\xrightarrow{d} \frac{g''(\theta)}{2}\left[\psi N(0,1)\right]^2$$

$$\xrightarrow{d} \frac{\psi^2 g''(\theta)}{2} \chi^2_{(1)}$$