Lecture 13: More Convergence

Ex. $X_i \overset{iid}{\sim} U(0,1)$

$Y_n = \max_{i=1,\dots,n} X_i$

$Z_n = n(1-Y_n)$

$Z_n \xrightarrow{d} Z.$

$F_n(z) = \mathbb{P}(Z_n \le z)$

$= \mathbb{P}(n(1-Y_n) \le z)$

$= \mathbb{P}(Y_n \ge 1 - z/n)$

$\max_{i=1,\dots,n} X_i$

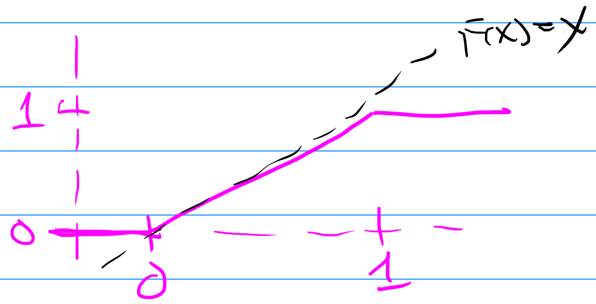
$= 1 - \mathbb{P}(Y_n \le 1 - z/n)$

$= 1 - \mathbb{P}(X_1 \le 1 - z/n, X_2 \le 1 - z/n, \dots, X_n \le 1 - z/n)$

$= 1 - \prod_{i=1}^{n} \mathbb{P}(X_i \le 1 - z/n)$

$X_i \overset{iid}{\sim} U(0,1)$

$= 1 - \mathbb{P}(X_i \le 1 - z/n)^n$



$= 1 - (1 - z/n)^n$

$= F_n(z)$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \qquad \lim_{n\to\infty}\left(1+\frac{c}{n}\right)^n = e^c}$$

$$F_n(z) = 1 - (1 - z/n)^n$$

$$= 1 - (1 + (-z)/n)^n$$

$$\to 1 - e^{-z} = F(z)$$ ← Exp(1) CDF

Punchline: $Z_n \to Z$ where $Z \sim \text{Exp}(1)$.

For a seq. of numbers $x_n, y_n \in \mathbb{R}$

If $x_n \to x$, $y_n \to y$

then ① $x_n + y_n \to x + y$

② $x_n y_n \to xy$

③ $x_n / y_n \to x/y$

④ $a x_n + b y_n \to ax + by$

Theorem: Algebraic Properties

Let $X_n \to X$ and $Y_n \to Y$ and $a, b \in \mathbb{R}$

and the convergence is either a.s. or i.p.
(not d)

then ① $aX_n + bY_n \to aX + bY$

② $X_n Y_n \to XY$ $(X_n/Y_n \to X/Y)$

Note: Constants are just degenerate RVs

So if $c_n \to c$ (as numbers)

then $c_n \xrightarrow{a.s.} c$ (as RVs)

So if $c_n \to c$ and $X_n \to X$ (a.s. or i.p)

then ① $aX_n + bc_n \to aX + bc$

② $c_n X_n \to cX$

What about convergence in dist?

Theorem: Slutsky's Theorem

If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ ← constant

then

① $X_n + Y_n \to X + c$

② $X_n Y_n \to Xc$

$\left(X_n / Y_n \to X / c \right)$

Theorem: Continuous Mapping Theorem

If $g: \mathbb{R} \to \mathbb{R}$ is a continuous function and

$X_n \to X$ (all types of convergence)

then $g(X_n) \to g(X)$

why?

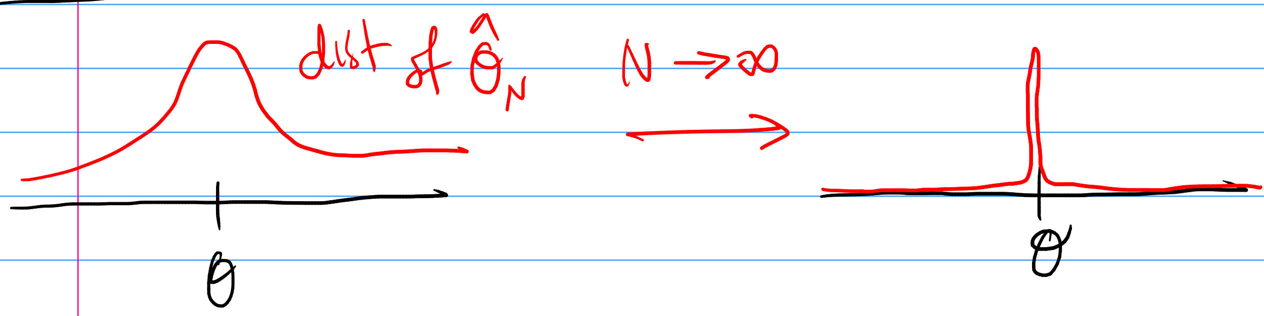
$$\lim_{x \to a} g(x) = g(a) = g\left(\lim_{x \to a} x\right)$$

If $x_n \to x$ then $g(x_n) \to g(x)$

Defn: Consistent Estimator

We say an estimator $\hat{\theta}_N$ is consistent for θ if

$$\hat{\theta}_N \xrightarrow{P} \theta.$$



(*) Consistency $\approx$ asymptotically unbiased

(*) for nice enough dist

Ex.

$$S^2 = \frac{1}{N-1} \sum_{n=1}^{N} (X_n - \bar{X})^2$$

(unbiased)

$$\mathbb{E} S^2 = \sigma^2$$

we also had

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{n=1}^{N} (X_n - \bar{X})^2$$

$$\mathbb{E}\hat{\sigma}^2 = \frac{N-1}{N} \sigma^2 \xrightarrow{n} \sigma^2$$

(asymptotically unbiased)

Theorem: MSE $\to 0$ then $\hat{\theta}_N$ is consistent.

If $MSE(\hat{\theta}_N) \xrightarrow{N} 0$ then $\hat{\theta}_N \xrightarrow{P} \theta$

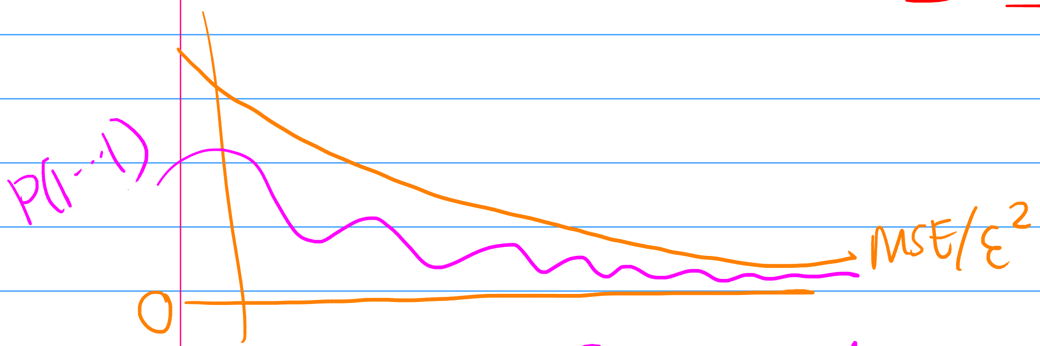
Pf: $\forall \varepsilon > 0$ $P(\underbrace{|\hat{\theta}_N - \theta|}_{\geq 0} \geq \varepsilon) \to 0$ as $N \to \infty$

Markov's $X \geq 0$ then $P(X \geq a) \leq \frac{EX}{a}$

$$0 \leq \boxed{P(|\hat{\theta}_N - \theta| \geq \varepsilon)} = P((\hat{\theta}_N - \theta)^2 \geq \varepsilon^2) \leq \frac{E[(\hat{\theta}_N - \theta)^2]}{\varepsilon^2}$$

$$= \frac{MSE(\hat{\theta}_N)}{\varepsilon^2} \to 0$$

as $n \to \infty$



So by Squeeze theorem

$$P(|\hat{\theta}_N - \theta| \geq \varepsilon) \to 0$$

and $\hat{\theta}_N \xrightarrow{P} \theta$.

Intuition: $\bar{X}_N$ should be a good estimator of $\mathbb{E}X_n$

Theorem: Weak Law of Large Numbers (WLLN)

If X_ns are uncorrelated and

① $\mathbb{E}X_n = \mu$

② $\operatorname{Var} X_n = \sigma^2 < \infty$

weak assumptions

then $\bar{X}_N = \frac{1}{N}\sum_{n=1}^{N} X_n \xrightarrow{P} \mu$.

weak form of convergence

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pf. $MSE(\bar{X}) = Bias(\bar{X})^2 + Var(\bar{X})$

$\mathbb{E}\bar{X} = \mu$ so $Bias(\bar{X})^2 = 0^2$

$Var\,\bar{X} = \sigma^2/N$

$MSE(\bar{X}) = \sigma^2/N \longrightarrow 0$ as $N \to \infty$

so $\bar{X} \xrightarrow{P} \mu$.

Ex. $X_n \overset{iid}{\sim} Pois(\lambda)$ then $\mathbb{E}X_n = \lambda = Var X_n$

then WLLN says $\bar{X}_N \xrightarrow{P} \lambda$

Can generalize WLLN

Assume $Var(X_n) = \sigma_n^2$ but

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \sigma_n^2 < \infty$$

then I can show that $\overline{X} \xrightarrow{P} \mu$.

$$MSE(\overline{X}) = Var(\overline{X}) = Var\left(\frac{1}{N}\sum_n X_n\right)$$

$$= \frac{1}{N^2}\sum_n Var(X_n)$$

$$= \frac{1}{N^2}\sum_n \sigma_n^2$$

$$= \underbrace{\frac{1}{N}}_{0}\underbrace{\left(\frac{1}{N}\sum_n \sigma_n^2\right)}_{<\infty}$$

$$\to 0$$

$X \sim N(0,1)$

$Cor(X, X^2) = 0$

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$; $\mathbb{E}X_n = 1/\lambda$

WLLN: $\overline{X} \xrightarrow{P} 1/\lambda$

$1/\overline{X} \xrightarrow{P} \lambda$? $\quad g(x) = 1/x$ for $x>0$ is continuous

So by CMT $g(\overline{X}) = 1/\overline{X} \xrightarrow{P} g(1/\lambda) = \lambda$

i.e. $1/\overline{X}$ is consistent for λ.

Ex. $S^2 = \frac{1}{N-1} \sum_n (X_n - \bar{X})^2$

Show that $ES^2 = \sigma^2$

want to show that $\boxed{S^2 \xrightarrow{P} \sigma^2}$.

If $MSE(S^2) \to 0$ then it is consistent.

$MSE(S^2) = Var(S^2)$

if $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

then $= \frac{2\sigma^4}{N-1} \longrightarrow 0$ as $N \to \infty$.

By our CMT that

$$S = \sqrt{S^2} \xrightarrow{P} \sqrt{\sigma^2} = \sigma$$

So S is consistent for σ.