Lecture 12 : Convergence

Defn: Almost Sure Convergence

$$X_n : S \to \mathbb{R}, \quad X : S \to \mathbb{R}$$

If I consider

$$A = \{ s \in S \mid X_n(s) \to X(s) \} \subset S$$

as $n \to \infty$

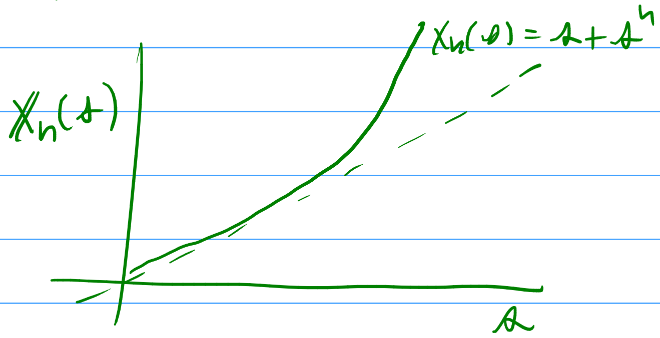
and $\mathbb{P}(A) = 1$

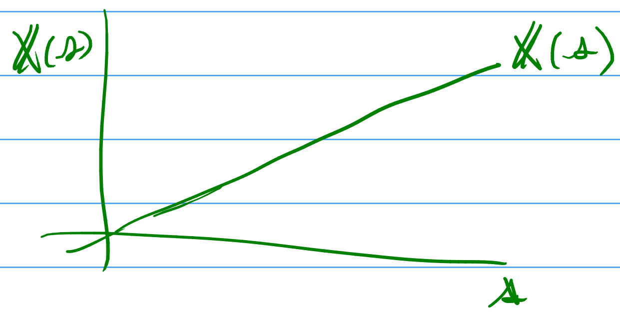
then X_n converge almost surely to X

$$X_n \xrightarrow{a.s.} X$$

Ex. $S = [0,1]$ w/ uniform density

let $X_n(s) = s + s^n$ and $X(s) = s$, for $s \in S$





Does $X_n \xrightarrow{a.s.} X$?

Notice that if $s \in [0,1)$

then

$X_n(s) = s + s^n \xrightarrow{n} s = X(s)$

however if $s = 1$

$X_n(s) = X_n(1) = 1 + 1^n = 2 \not\rightarrow 1 = X(s)$

$A = \{ s \mid X_n(s) \rightarrow X(s) \} = [0, 1)$

and $\mathbb{P}(A) = 1$

So $X_n \xrightarrow{a.s.} X$.

Almost sure convergence is a strong condition

can be difficult to establish.

Sometimes, work w/ weaker forms of convergence.

Defn: Convergence in Probability

We say a seq. (X_n) converges in prob. to X

denoted

$$X_n \xrightarrow{P} X$$

if $\forall \varepsilon > 0$ $\lim_{n \to \infty} P(|X_n - X| < \varepsilon) = 1$.

$\forall \varepsilon$ as $n \to \infty$ prob. X_n is less than ε away from X goes to 1.

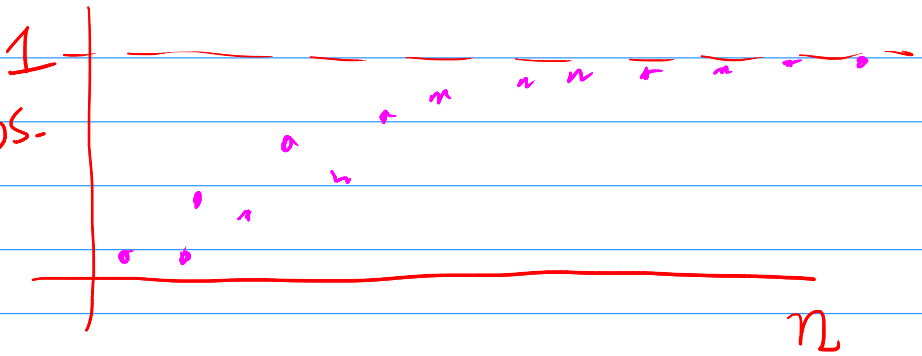
<u>equiv.</u>

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

Pick $\varepsilon > 0$

$$P(|X_1 - X| < \varepsilon), P(|X_2 - X| < \varepsilon), P(|X_3 - X| < \varepsilon), \ldots$$

$\forall \varepsilon > 0$



<u>Theorem:</u> a.s. $\Rightarrow$ i.p.

If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{P} X$.

<u>Ex.</u> Consider $S = [0,1]$ w/ uniform density

let $X(s) = s$

$X_1(s) = \cancel{s} + 1$ $\qquad |X_1(s) - X(s)| = 1$

$X_2(s) = \cancel{s} + \mathbb{1}(s \in [0, \tfrac{1}{2}])$ $\qquad |X_2(s) - X(\omega)| = \mathbb{1}(s \in [0, \tfrac{1}{2}])$

$X_3(s) = \cancel{s} + \mathbb{1}(s \in [\tfrac{1}{2}, 1])$ $\qquad |X_3(s) - X(s)| = \mathbb{1}(s \in [\tfrac{1}{2}, 1])$

$X_4(s) = \cancel{s} + \mathbb{1}(s \in [0, \tfrac{1}{3}])$ $\qquad \vdots$

$X_5(s) = \cancel{s} + \mathbb{1}(s \in [\tfrac{1}{3}, \tfrac{2}{3}])$

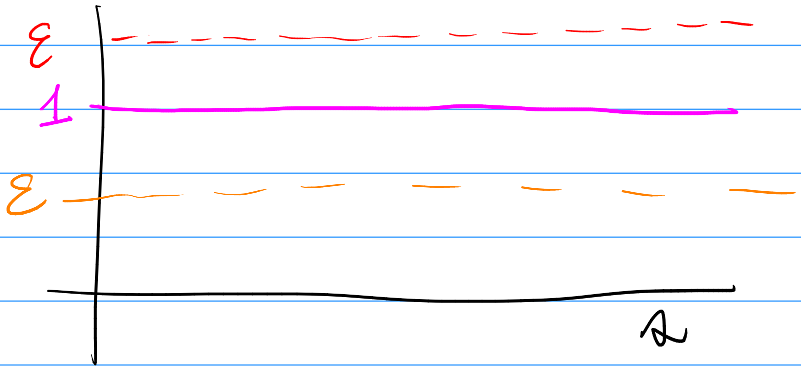
$X_6(s) = \cancel{s} + \mathbb{1}(s \in [\tfrac{2}{3}, 1])$

$\vdots$

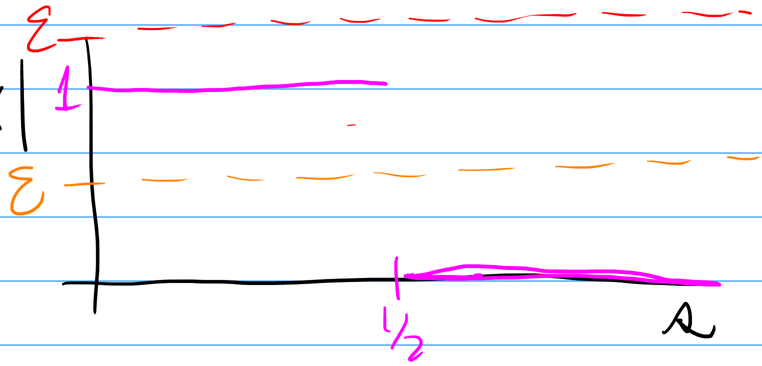
To show that $X_n \xrightarrow{P} X$ need to show

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$$

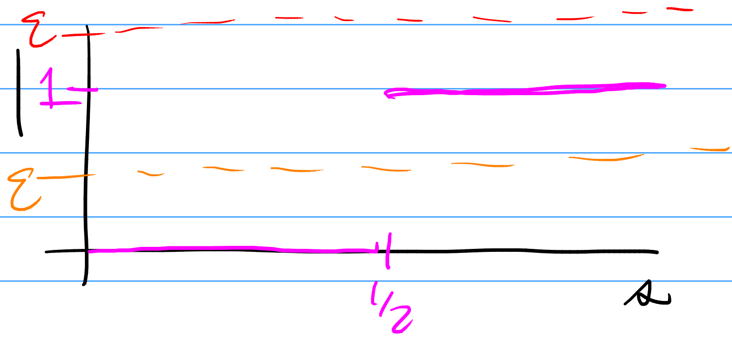
$|X_1 - X|$



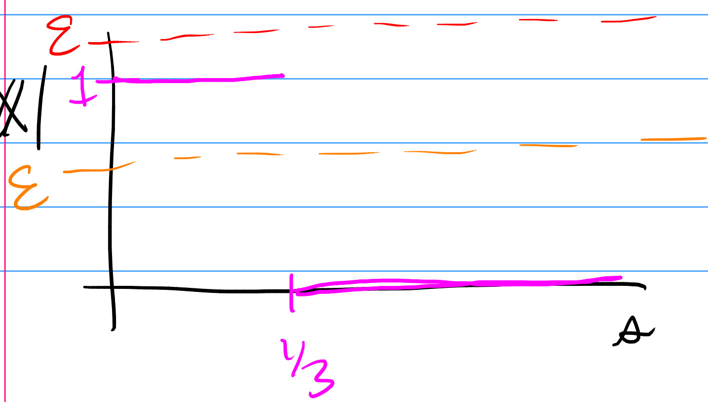
$|X_2 - X|$



$|X_3 - X|$



$|X_4 - X|$



If $\varepsilon > 1$ then $P(|X_n - X| \geq \varepsilon) = 0 \quad \forall n$

so $\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$

If $0 < \varepsilon \leq 1$

- $P(|X_1 - X| \geq \varepsilon) = 1$
- $P(|X_2 - X| \geq \varepsilon) = 1/2$
- $P(|X_3 - X| \geq \varepsilon) = 1/2$
- $P(|X_4 - X| \geq \varepsilon) = 1/3$

$= 1/3$

$= 1/3$ claim: limit is zero

$= 1/4$

$\vdots$

In all cases $\varepsilon > 0$

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

i.e. $X_n \xrightarrow{P} X$.

Does $X_n \xrightarrow{a.s.} X$?

$$A = \{ s : X_n(s) \to X(s) \}$$

and show $P(A) = 1$.

Pick any $s \in [0,1]$ and consider

$$X_1(s), X_2(s), X_3(s), \ldots \xrightarrow{?} X(s) ?$$

problem, this oscillates between s and $s+1$ forever. Doesn't settle down to a limit.

i.e. $P(A) = P(\emptyset) = 0$.

So $X_n \cancel{\xrightarrow{a.s.}} X$,

Defn: Convergence in Distribution

We say (X_n) converges in dist.

denoted $X_n \xrightarrow{d} X$

if the CDFs converge (pointwise)

i.e. if F_n is the CDF of X_n, F CDF of X

then

$$F_n(x) \to F(x) \quad \forall x$$

Theorem: i.p. $\Rightarrow$ d

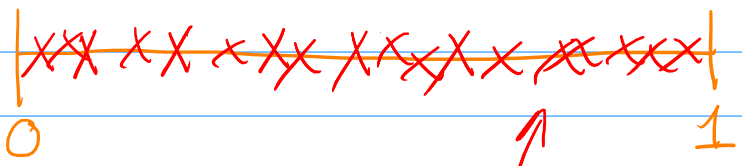
If $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{d} X$.

Chain: a.s. $\Rightarrow$ i.p. $\Rightarrow$ d

(gen, converses are false)

Ex. $X_i \overset{iid}{\sim} U(0,1)$

and $Y_n = \max_{i=1,\dots,n} X_i$ = max of first n X_i's

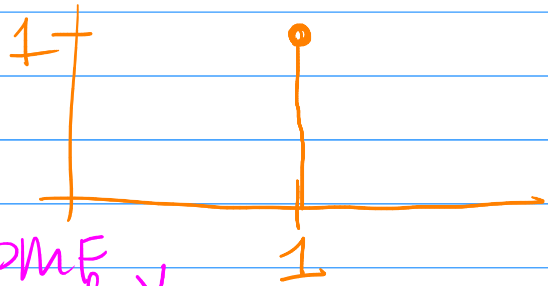


maximum Y_n gets ever closer to 1

intuition: $Y_n \to 1$ degenerate RV w/ all mass at 1

Could say

$Y_n \to Y$



PMF of Y

Show: $Y_n \xrightarrow{P} 1$

Need to show

$\forall \varepsilon > 0 \quad P(|Y_n - 1| \geq \varepsilon) \to 0$ as $n \to \infty$

$P(|Y_n - 1| \geq \varepsilon)$ $\quad Y_n \leq 1$

$= P(|1 - Y_n| \geq \varepsilon)$

$= P(1 - Y_n \geq \varepsilon)$ $\quad Y_n = \max\limits_{i=1,\dots,n} X_i$

$= P(Y_n \leq 1 - \varepsilon)$

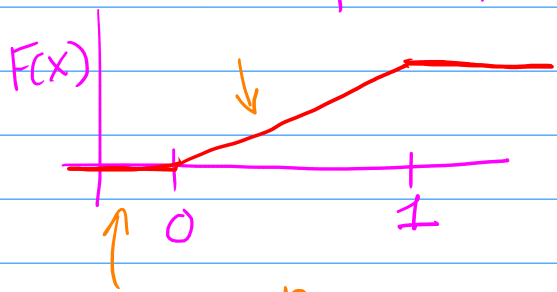
$= P(X_1 \leq 1-\varepsilon, X_2 \leq 1-\varepsilon, X_3 \leq 1-\varepsilon, \dots, X_n \leq 1-\varepsilon)$

independent

$= \underbrace{P(X_1 \leq 1-\varepsilon)\, P(X_2 \leq 1-\varepsilon) \cdots P(X_n \leq 1-\varepsilon)}_{\text{same dist — n of them}}$

$= P(X_n \leq 1-\varepsilon)^n$ $\quad U(0,1)$

CDF of $U(0,1)$



$= F_{X_n}(1-\varepsilon)^n$

If $\varepsilon \geq 1$ then $1 - \varepsilon < 0$ so $F_{X_n}(1-\varepsilon)^n = 0^n = 0$

If $0 < \varepsilon < 1$ then $F_{X_n}(1-\varepsilon)^n = (1-\varepsilon)^n$

All together

$$P(|Y_n - 1| \geq \varepsilon) = \begin{cases} 0, & \varepsilon > 1 \\ (1-\varepsilon)^n, & 0 < \varepsilon < 1 \end{cases}$$

and as $n \to \infty$

$$= \begin{cases} 0, & \varepsilon > 1 \\ 0, & 0 < \varepsilon < 1 \end{cases}$$

$$= 0$$

So $Y_n \xrightarrow{P} 1$.

Show $Y_n \xrightarrow{d} 1$.

Let $Y = 1$ CDF

$$F(y) = \begin{cases} 0, & y < 1 \\ 1, & y \geq 1 \end{cases}$$

$$F_n(y) = P(Y_n \leq y)$$

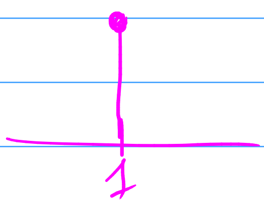
$$= P(\max_i X_i \leq y)$$

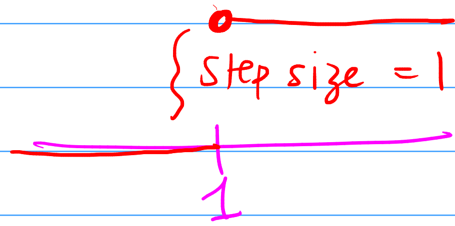
$$= P(X_1 \leq y, \ldots, X_n \leq y)$$

$$= P(X_n \leq y)^n$$

$$= \begin{cases} 0^n, & y < 0 \\ y^n, & 0 < y < 1 \\ 1^n, & y > 1 \end{cases}$$

PMF





$$= \begin{cases} 0, & y < 0 \\ y^n, & 0 < y < 1 \\ 1, & y > 1 \end{cases} \quad \text{as } n \to \infty \quad \to \begin{cases} 0, & y < 0 \\ 0, & 0 < y < 1 \\ 1, & y > 1 \end{cases} = F(y)$$

So $Y_n \xrightarrow{d} 1$.

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3 Types of Convergence

Almost Sure:

$$P(\{s \in S : X_n(s) \to X(s)\}) = 1$$

In Probability

$$\forall \varepsilon > 0 \quad \lim_{n \to \infty} P(|X_n - X| < \varepsilon) = 1$$

equiv. $$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

In Distribution

$$F_n \to F \quad \text{as} \quad n \to \infty$$

CDF of X_n CDF of X

Theorem: a.s. $\Rightarrow$ i.p $\Rightarrow$ d

generally, converses are false

Partial Converse: If $X_n \xrightarrow{d} c$ constant i.e. degenerate dist

then $X_n \xrightarrow{p} c$.