Lecture 20 Interval Estimation

Point estimation:

$$\hat{\theta} = \hat{\theta}(\underset{\sim}{x}) \in \Theta \qquad \text{idea} \qquad \text{"}\hat{\theta} \approx \theta\text{"}$$

Interval Estimation:

$$C = C(\underset{\sim}{x}) \subset \Theta \qquad \text{idea} \qquad \text{"}\theta \in C\text{"}$$

prefer if C is an interval.

Defn: Interval Estimator

An interval est. of $\theta \in \Theta \subset \mathbb{R}$ is a pair of fns

$$L = L(\underset{\sim}{x}) \quad \text{and} \quad U = U(\underset{\sim}{x})$$

that satisfy $L \leq U \;\; \forall \underset{\sim}{x}$

idea: want to say "$L \leq \theta \leq U$"

Ex Let $X_n \overset{iid}{\sim} N(\mu, 1)$. Let $N = 4$.

Then an interval est. of μ is

$$[\underbrace{\bar{X} - 1}_{L}, \; \underbrace{\bar{X} + 1}_{U}]$$

might say "$\bar{X} - 1 \le \mu \le \bar{X} + 1$".

Why use an interval est? Just use $\bar{X}$?

Notice! $\mathbb{P}(\bar{X} = \mu) = 0$

So we typically attach some uncertainty to $\bar{X}$

i.e. $sd(\bar{X}) = 1/\sqrt{4}$

Alt: use an interval est. b/c

$$\mathbb{P}(\bar{X} - 1 \le \mu \le \bar{X} + 1) > 0$$

in this case

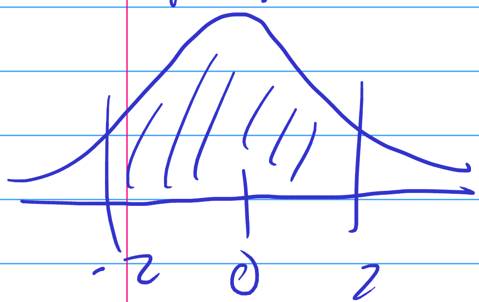
$$= \mathbb{P}(\bar{X} - \mu \le 1 \text{ and } \bar{X} - \mu \ge -1)$$

$$= \mathbb{P}(-1 \le \bar{X} - \mu \le 1)$$

$$= \mathbb{P}\left(-2 \le \frac{\bar{X} - \mu}{\sqrt{1/4}} \le 2\right) \approx 95\%$$

$N(0,1)$

N(0,1)



Defn: Coverage Prob.

For int. est. $[L, U]$ of a param θ the coverage prob is

$$\mathbb{P}_\theta (L \leq \theta \leq U)$$

depends on θ.

Defn: Confidence Coef.

Worst-case coverage prob.

$$1 - \alpha = \min_{\theta \in \Theta} \mathbb{P}_\theta (L \leq \theta \leq U)$$

more generally if I have a set $C(\underset{\sim}{X}) \subset \Theta$ its assoc. conf. coef is

$$1 - \alpha = \min_{\theta \in \Theta} \mathbb{P}_\theta (\theta \in C(\underset{\sim}{X}))$$

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conf interval = interval est. + conf. coef

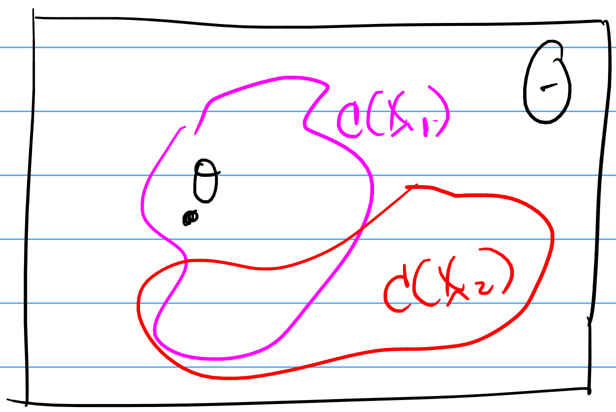
conf. set = set est. + conf. coef.

When we look at

$$\mathbb{P}_\theta(\theta \in C(\underline{X}))$$

random

fixed but unknown



How do I build a conf set/interval?

Basically one way ... invert a hypothesis test.

$$\text{HT} \longleftrightarrow \text{Conf. set.}$$

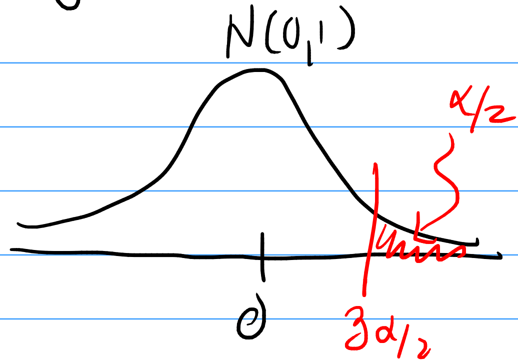
<u>Ex.</u> $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ known

Let's buid a HT for the hypothesis

$$H_0: \mu = \mu_0 \quad v. \quad H_a: \mu \neq \mu_0$$

A level α test for this hypothesis is to reject when

$$\left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \right| > z_{\alpha/2}$$



$$R(\mu_0) = \left\{ \underset{\sim}{x} : \frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{N}} > z_{\alpha/2} \right\}$$

set of $\underset{\sim}{x}$ that don't agree w/ $H_0 : \mu = \mu_0$

$$A(\mu_0) = \mathcal{X} \setminus R(\mu_0) = \left\{ \underset{\sim}{x} : \frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{N}} \le z_{\alpha/2} \right\}$$

set of $\underset{\sim}{x}$ that agree w/ H_0.

"invert" i.e. solve for μ_0 in the middle

$$-z_{\alpha/2} \le \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \le z_{\alpha/2}$$

$$\Leftrightarrow -z_{\alpha/2}\,\sigma/\sqrt{N} \le \bar{X} - \mu_0 \le \sigma/\sqrt{N}\, z_{\alpha/2}$$

$$\Leftrightarrow \underbrace{\bar{X} - \sigma/\sqrt{N}\, z_{\alpha/2}}_{L} \le \mu_0 \le \underbrace{\bar{X} + \sigma/\sqrt{N}\, z_{\alpha/2}}_{U}$$

Claim: $[L, U]$ is a $1-\alpha$ CI for μ.

Why? Need to show $\mathbb{P}_\mu(L \le \mu \le U) \geq 1-\alpha$

Test is level α so

$$\max_{\theta \in \Theta_0} \mathbb{P}_\theta(\text{reject}) \le \alpha$$

$$\iff \mathbb{P}_{\mu_0}(\text{reject}) \le \alpha$$

$$\iff \mathbb{P}_{\mu_0}(\underset{\sim}{X} \in R(\mu_0)) \le \alpha$$

$$\iff \mathbb{P}_{\mu_0}(\underset{\sim}{X} \in A(\mu_0)) \geq 1-\alpha$$

$$\iff \mathbb{P}_{\mu_0}\left(\overline{X} - \sigma/\sqrt{n}\, z_{\alpha/2} \le \mu_0 \le \overline{X} + \sigma/\sqrt{n}\, z_{\alpha/2}\right) \geq 1-\alpha$$

$$\iff \mathbb{P}_{\mu_0}(L \le \mu_0 \le U) \geq 1-\alpha$$

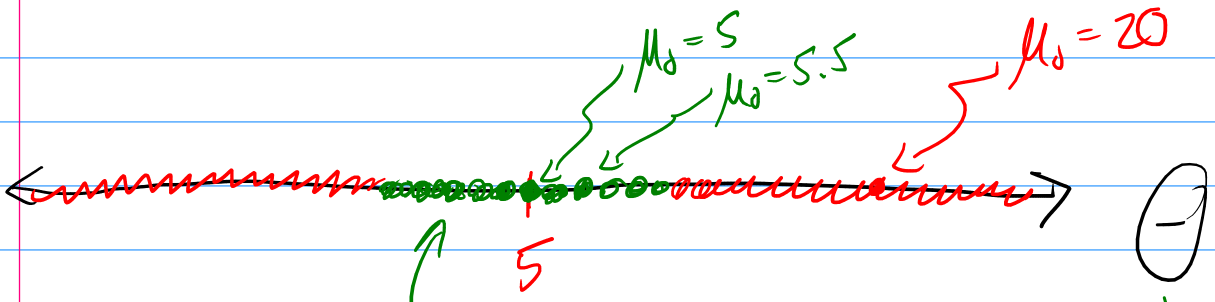
arbitrary

so $\forall \mu \quad \mathbb{P}_\mu(L \le \mu \le U) \geq 1-\alpha$

$$\min_{\mu} \mathbb{P}_\mu(L \le \mu \le U) \geq 1-\alpha$$

so L, U is a $1-\alpha$ CI for μ.

$H_0: \mu = \mu_0$, $\bar{X} = 5$



all μ_0 where I wouldn't reject
$H_0: \mu = \mu_0$
(i.e. I'd accept)
This is my CI.

Test Inversion

For $\theta_0 \in \Theta$ let $A(\theta_0)$ be the acceptence region of an α-level test for

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta \neq \theta_0$$

then if I let

$$C(\underset{\sim}{X}) = \{\theta : \underset{\sim}{x} \in A(\theta)\}$$

this is a $1-\alpha$ confidence set for θ.

Two worlds:

HT: fix θ_0 and test $H_0: \theta = \theta_0$

by determining some rule

$$A(\theta_0) = \{\text{set of } \underset{\sim}{x} \text{ consistent w/ } H_0\} \subset \mathcal{X}$$

CI: Fix $\underset{\sim}{x}$ and want to determine which θ are consistent w/ $\underset{\sim}{x}$

do this as

$$C(\underset{\sim}{x}) = \{\text{set of } \theta \text{ consistent w/ } \underset{\sim}{x}\} \subset \Theta$$

Ex. Let $X_n \overset{iid}{\sim} \text{Exp}(\beta)$

$$\mathbb{E} X_n = \beta$$

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \quad \text{for } x > 0$$

Let's make a CI for β by inverting the LRT.

$$H_0: \beta = \beta_0 \quad v. \quad H_a: \beta \neq \beta_0$$

$$\lambda = \frac{L(\hat{\beta}_0)}{L(\hat{\beta})} = \frac{L(\beta_0)}{L(\bar{X})} = \frac{\frac{1}{\beta_0^N} \exp(-N\bar{X}/\beta_0)}{\frac{1}{\bar{X}^N} \exp(-N)}$$

$$= \left(\frac{\bar{X}}{\beta_0}\right)^N e^N e^{-N\bar{X}/\beta_0}$$

LRT rejects when $\lambda \le c$

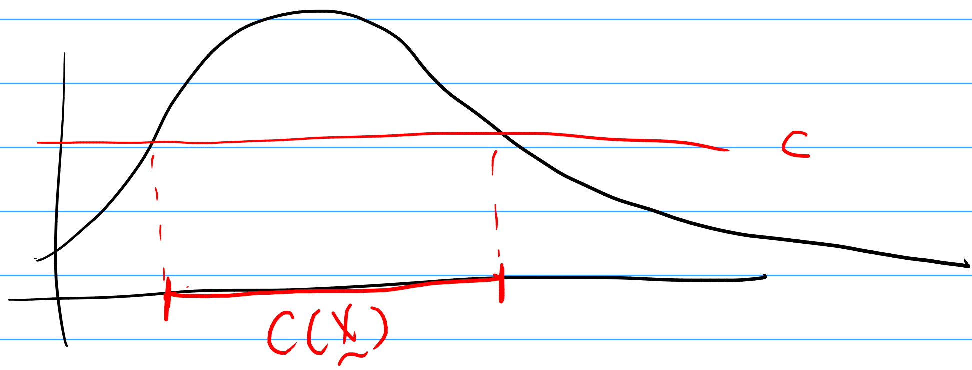
set to make some α level test

$$R(\beta_0) = \left\{ \underset{\sim}{x} : \left(\frac{\bar{X}}{\beta_0}\right)^N e^N e^{-N\bar{X}/\beta_0} \le c \right\}$$

$$A(\beta_0) = \left\{ \underset{\sim}{x} : \left(\frac{\bar{X}}{\beta_0}\right)^N e^N e^{-N\bar{X}/\beta_0} > c \right\}$$

$$C(\underset{\sim}{x}) = \left\{ \beta : \underset{\sim}{x} \in A(\beta) \right\}$$

$$= \left\{ \beta : \left(\frac{\bar{X}}{\beta}\right)^N e^N e^{-N\bar{X}/\beta} > c \right\}$$



Often trying to invert LRT is nasty.

What is a HT?

A level α HT is simply a region $R \subset \mathcal{X}$ so that

$$\max_{\theta \in \Theta_0} \mathbb{P}_\theta(\underset{\sim}{X} \in R) \le \alpha$$

So if I consider

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta \neq \theta_0$$

then a level α HT is simply a region R so that

$$\mathbb{P}_{\theta_0}(\underset{\sim}{X} \in R) \leq \alpha$$

or equiv. a region A where

$$\mathbb{P}_{\theta_0}(\underbrace{\underset{\sim}{X} \in A}_{\text{same statement about } \underset{\sim}{X}}) \geq 1-\alpha$$

So a level α HT is just a statement about $\underset{\sim}{X}$ so that

$$\mathbb{P}_{\theta_0}(\text{statement}) \geq 1-\alpha$$

implicitly defines A

Generalize!

$$\mathbb{P}_{\theta_0}\Big(\underbrace{Q(\underset{\sim}{X}, \theta_0) \in A}_{\text{"statement"}}\Big) \geq 1-\alpha$$

then this implicitly defines accept region of a HT as

$$\{ \underset{\sim}{Y} : Q(\underset{\sim}{X}, \theta_0) \in A \}$$

So to build a conf. set I can invert this:

$$C(\underset{\sim}{X}) = \{ \theta : Q(\underset{\sim}{X}, \theta) \in A \}.$$

This will work so long as

$$\min_{\theta} \mathbb{P}_{\theta} (Q(\underset{\sim}{X}, \theta) \in A) \geq 1 - \alpha$$

i.e. this statement has to have prob $\geq 1-\alpha$ for all possible θ.

One way to ensure this is if

① dist $Q(\underset{\sim}{X}, \theta)$ doesn't depend on θ

② A doesn't depend on θ

b/c then

$$\mathbb{P}_{\theta} (Q(\underset{\sim}{X}, \theta) \in A) \geq 1 - \alpha$$

dist doesn't depend on θ

no θ