Lecture 15:

$X_n \overset{iid}{\sim} Pois(\lambda)$

$Y_n = \mathbb{1}(X_n = 0) \overset{iid}{\sim} Bern(p)$

$p = P(X_n = 0) = e^{-\lambda}$

CLT: $\bar{Y} \sim AN(p, p(1-p)/N)$

$$\boxed{\sim AN(e^{-\lambda}, e^{-\lambda}(1-e^{-\lambda})/N)}$$

Which do we prefer?

$$ARE(\bar{Y}, e^{-\bar{X}}) = \frac{\text{asymp. var } e^{-\bar{X}}}{\text{asymp var } \bar{Y}}$$

$$= \frac{e^{-2\lambda}\lambda/\cancel{N}}{e^{-\lambda}(1-e^{-\lambda})/\cancel{N}} \quad \frac{e^{2\lambda}}{e^{2\lambda}}$$

$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots$

$$= \frac{\lambda}{e^{\lambda} - 1}$$

$$= \frac{\lambda}{\lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots} \quad > 0$$

$$= \frac{\lambda}{\lambda + \text{pos.}} < 1 \Rightarrow \text{so we prefer } e^{-\bar{X}}$$

Defn: Asymptotically Efficient

We say $\hat{\theta}$ is asymp. efficient for $\tau(\theta)$ if

$$\hat{\theta} \sim AN(\tau(\theta), B(\theta))$$

CRLB

$$B(\theta) = \left(\frac{\partial \tau}{\partial \theta}\right)^2 / I_N(\theta)$$

In prev. example,

$$e^{-\bar{X}} \sim AN(e^{-\lambda}, e^{-2\lambda} \lambda / N)$$

What's the CRLB?

$\rightarrow f(x) = \lambda^x e^{-\lambda} / x!$

$\rightarrow \log f(x) = x \log(x) - \lambda - \log(x!)$

$\rightarrow \frac{\partial}{\partial \lambda}[\cdots] = x/\lambda - 1$

$\rightarrow \frac{\partial^2}{\partial \lambda^2}[\cdots] = -x/\lambda^2$

$$I(x) = -E\left[\frac{\partial^2}{\partial \lambda^2} \log f(x)\right] = \frac{1}{\lambda^2} E[X] = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$I_N(\lambda) = N/\lambda \quad ; \quad \tau(\lambda) = e^{-\lambda}, \quad \frac{\partial \tau}{\partial \lambda} = -e^{-\lambda}$

$$B = \left(\frac{\partial \tau}{\partial \lambda}\right)^2 \Big/ I_N(\lambda) = \left(-e^{-\lambda}\right)^2 \Big/ \left(N/\lambda\right)$$

$$= \frac{\lambda e^{-2\lambda}}{N}$$

So $e^{-\bar{X}}$ is asymptotically efficient.

Theorem: MLEs are asymptotically efficient. (*)

$$\underbrace{\hat{\theta}_{MLE}}_{\text{MLE for } \tau(\theta)} \sim AN\left(\tau(\theta), \underbrace{\left(\partial \tau/\partial \theta\right)^2 / I_N(\theta)}_{\text{CRLB}}\right)$$

(*) ① likelihood fn is diffentiable

② $I(\theta) \neq 0$.

EXAM 2

Hypothesis Testing

Defn: Hypothesis

A hypothesis is a statement about a parameter

$$H_0: \theta \in \Theta_0 \quad \text{v.} \quad H_a: \theta \in \Theta_a$$

null hypothesis

alternative hypothesis

Constraint: ① $\Theta_0 \cap \Theta_a = \emptyset$

② $\Theta_0 \cup \Theta_a = \Theta$ all possible params

Ex. Let θ be the true pct. of defective items in a manufacturing procedure

$$\Theta = [0, 1]$$

$$H_0: \theta \leq 0.1 \quad \text{v.} \quad H_a: \theta > 0.1$$

$$\Theta_0 = [0, 0.1] \qquad \Theta_a = (0.1, 1]$$

Ex. Let θ denote the change in blood pressure after treatement w/ a new drug.

$$H_0: \theta = 0 \quad v. \quad H_a: \theta \neq 0$$

$$\Theta = \mathbb{R}, \quad \Theta_0 = \{0\}, \quad \Theta_a = \mathbb{R} \setminus \{0\}$$

If θ is a 1-dim'l parameter,

(1) test of form

$$H_0: \theta \leq c \quad v. \quad H_a: \theta > c$$

or $\quad H_0: \theta \geq c \quad v. \quad H_a: \theta < c$

$\quad$ " $\quad >$ $\quad$ " $\quad \leq$

$\quad$ " $\quad <$ $\quad$ " $\quad \geq$

is called a <u>one-sided</u> test.

(2) a test of the form

$$H_0: \theta = c \quad v. \quad H_a: \theta \neq c$$

$\quad$ " $\quad \theta \neq c$ $\quad$ " $\quad \theta = c$

is called a <u>two-sided test</u>.

③ A test of the form

$$H_0: \theta = a \quad v. \quad H_a: \theta = b$$

is called a <u>simple test</u>.

<u>Idea</u>: want to collect data and use to determine which is more plausible, H_0 or H_a.

Need to determine for which $\underset{\sim}{x}$ its more plausible that $\theta \in \Theta_0$ v. for which more plausible that $\theta \in \Theta_a$.

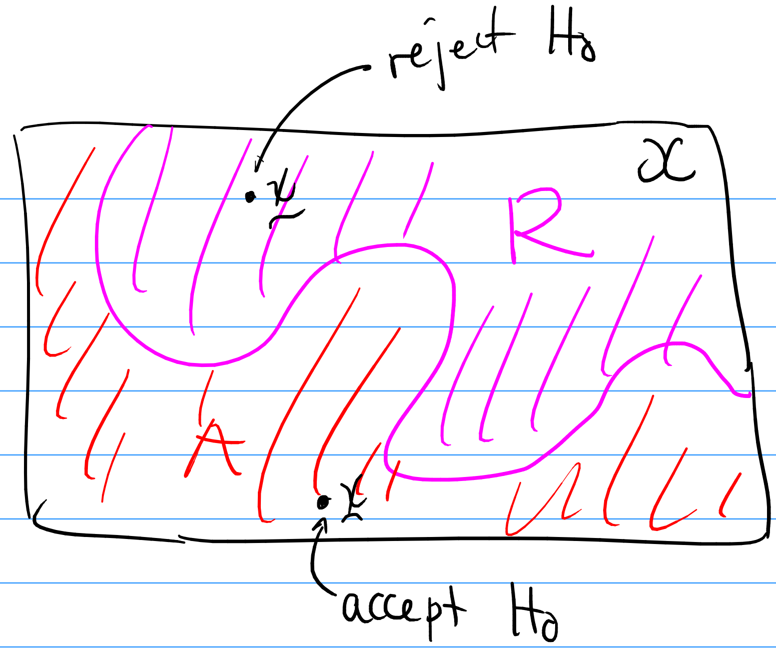
If $\mathcal{X}$ is the support for $\underset{\sim}{X} = (X_1, X_2, \dots,)$ (typically $\mathcal{X} = \mathbb{R}$)

A hypothesis testing procedure is simply a rule that partitions $\mathcal{X}$ into

$$\mathcal{X} = A \cup R$$

reject region (reject H_0)

accept region (accept H_0)



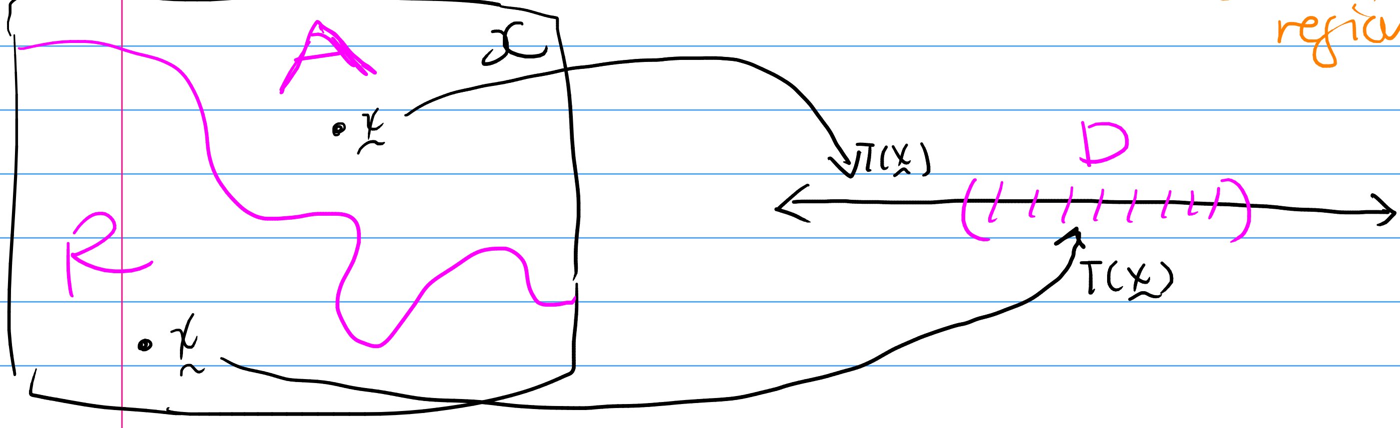
We "reject H_0" if $\underset{\sim}{x} \in R$

We "fail to reject H_0" if $\underset{\sim}{x} \in A$.

Often, define R (equiv. A) through a "test statistic" T, so that

$$R = \{ \underset{\sim}{x} \in \mathcal{X} : T(\underset{\sim}{x}) \in D \}$$

critical region



Ex. $X_n \overset{iid}{\sim} f_\theta$ w/ mean θ

$$H_0: \theta > 5 \quad v. \quad H_a: \theta \le 5$$

Let $T = \bar{X}$ and $D = (-\infty, 5]$

i.e. reject H_0 if $\bar{X} \le 5$

Defn: Type I and II errors

truth		accept H_0	reject H_0
Null True	$\theta \in \Theta_0$	Correct decision	Type I error
Alt. is true	$\theta \in \Theta_a$	Type II error	correct decision

} outcome of HT procedure

Goal: make a procedure that minimizes the prob. I make a Type I/II error.

Often, minimizing type I error prob increases my type II error prob.
and vice-versa

Defn: Power Function

For any $\theta \in \Theta$ the power function β is defined as

$$\beta(\theta) = \mathbb{P}_\theta(\underset{\sim}{X} \in R)$$

if true param is θ, prob. I reject H_0

For $\theta \in \Theta_0$ [null is true] then $\beta(\theta)$ is the prob ~~of~~ a type I error

For $\theta \in \Theta_a$ [alt. is true] then $\beta(\theta)$ is the prob. of correctly rejecting H_0

equiv.
$$1 - \beta(\theta) = \mathbb{P}(\underset{\sim}{X} \notin R)$$
$$= \text{prob. of type II error.}$$

$\underline{\text{Ex.}}$ $X_1, \dots, X_5 \overset{iid}{\sim} \text{Bern}(p)$ $0 \le p \le 1$

$$H_0: p \le 1/2 \quad v. \quad H_a: p > 1/2$$

$$\left[\Theta = [0,1], \quad \Theta_0 = [0, 1/2], \quad \Theta_a = (1/2, 1] \right]$$

$$R = \{(1,1,1,1,1)\}$$

Can write down in terms of a test stat:

$$T = \sum_{n=1}^{5} X_n \quad \text{then} \quad D = \{5\}$$

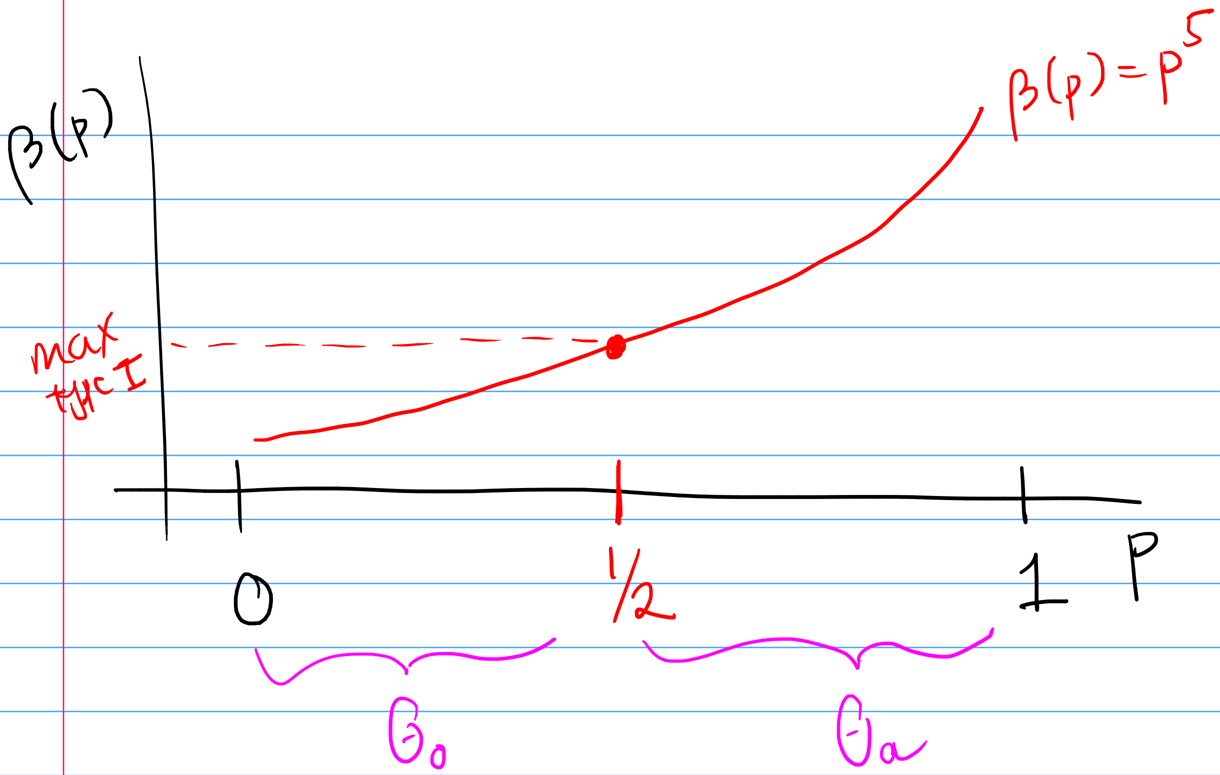
then $$R = \{ \underset{\sim}{x} \mid T(\underset{\sim}{x}) = 5 \}$$

<u>What is β?</u> $T \sim \text{Bin}(5, p)$

$$\beta(p) = \mathbb{P}_p(\underset{\sim}{X} \in R) = \mathbb{P}_p(T = 5)$$

$$= \binom{5}{5} p^5 (1-p)^{5-5}$$

$$= p^5$$



1) What is the max type I err. prob?

type I err. prob. is $\mathbb{P}(\underset{\sim}{X} \in R) = \beta(\theta)$ for $\theta \in \Theta_0$

So $\max_{\theta \in \Theta_0} \beta(\theta) = \max_{p \le 1/2} \beta(p)$

$= \beta(1/2) = \left(\frac{1}{2}\right)^5.$