Lecture 13:

$X_n \overset{iid}{\sim} Pois(\lambda)$; $\mathbb{E}X_n = Var\, X_n = \lambda$

So by WLLN we have $\bar{X} \xrightarrow{P} \lambda$

Consider $Y_N = 1/\bar{X}_N$

then by CMT we have that

$$Y_N = 1/\bar{X}_N \xrightarrow{P} 1/\lambda$$

Consider $Z_n = 1/(1+X_n^2)$

and what does $Z_n \xrightarrow{P}$?

WLLN that $\bar{Z} \xrightarrow{P} \mathbb{E}Z_n$

$$\sum_{x=0}^{\infty} \frac{1}{1+x^2} \lambda^x e^{-\lambda}/x! \approx .07$$

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$, $\mathbb{E}X_n = 1/\lambda$

So WLLN : $\bar{X} \xrightarrow{P} 1/\lambda$

By CMT we have $1/\bar{X} \xrightarrow{P} \lambda$.

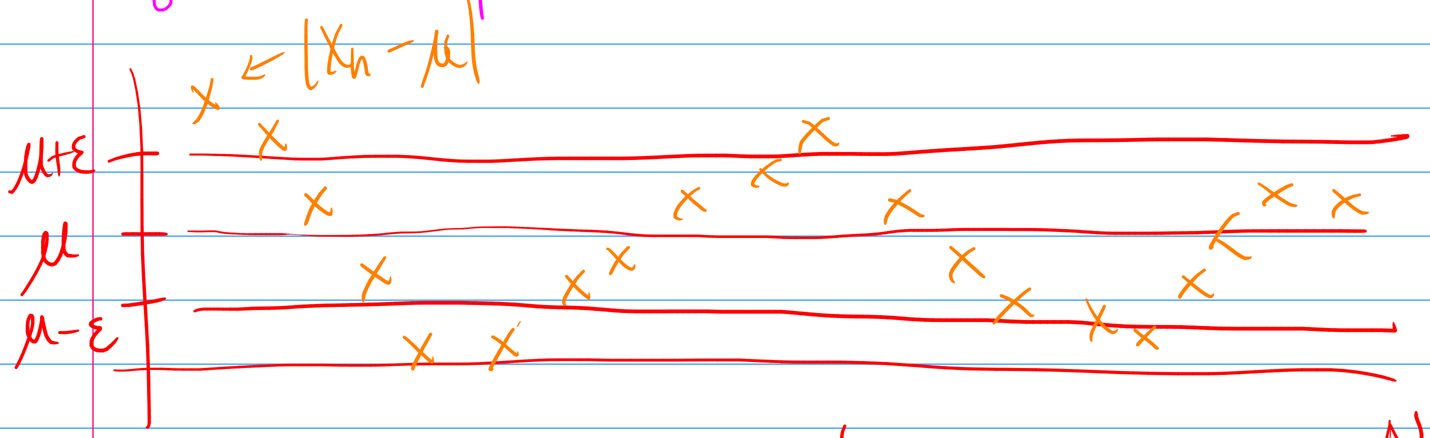
Theorem: Strong Law of Large Numbers (SLLN)

If $X_n \overset{iid}{\sim} f$ and $EX_n = \mu$, $\mathrm{Var}\, X_n = \sigma^2 < \infty$

then $\bar{X}_N = \frac{1}{N}\sum_{n=1}^{N} X_n \xrightarrow{a.s.} \mu$.

Convergence of Numbers: $x_n \to \mu$ ← more like convergence a.s.

Convergence in prob (WLLN):



as $N \to \infty$ prob I'm more than ε away from μ goes to zero

Sums of RVs

(1) $\sum_{n=1}^{N} X_n \longrightarrow \pm\infty$ (often)

(2) $\frac{1}{N}\sum_{n=1}^{N} X_n = \bar{X} \longrightarrow \mu$ (under some conditions)

constant

(3) $\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n \longrightarrow$ non-degenerate limit

proper scaling

Theorem: Central Limit Theorem

If $X_n \overset{iid}{\sim} f$ w/ $\mathbb{E}X_n = \mu$, $Var(X_n) = \sigma^2 < \infty$

then
$$\sqrt{N}\left(\frac{\bar{X}_N - \mu}{\sigma}\right) \xrightarrow{d} N(0,1).$$

Intuition: $\bar{X} \approx N(\mu, \sigma^2/N)$

Might like to say $\cancel{\bar{X} \xrightarrow{d} N(\mu, \sigma^2/N)}$

can't have N in my limit

Proper way to write CLT

① $\sqrt{N}\left(\dfrac{\overline{X}-\mu}{\sigma}\right) \xrightarrow{d} N(0,1)$

② $\sqrt{N}(\overline{X}-\mu) \xrightarrow{d} N(0,\sigma^2)$

③ $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{N}} \xrightarrow{d} N(0,1)$

④ $\overline{X} \sim AN(\mu, \sigma^2/N)$

asymptotically normal

Ex. $X_n \overset{iid}{\sim} \text{Bern}(p)$

$\mu = \mathbb{E}X_n = p$

$\sigma^2 = \text{Var}\, X_n = p(1-p)$

$\sigma = \sqrt{p(1-p)}$

CLT says something about $\overline{X} = \hat{p} =$ pct of 1s

CLT: $\sqrt{N}(\hat{p}-p) \xrightarrow{d} N(0, p(1-p))$

So for large N

$$\sqrt{N}(\hat{p}-p) \approx N(0, p(1-p))$$

or $\hat{p} \approx N\left(p, \frac{p(1-p)}{N}\right)$

95% of vals fall w/in $\hat{p} \pm 2\sqrt{\frac{p(1-p)}{N}}$

Intro stats: $\hat{p} \pm \underbrace{2\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}}_{\text{MOE}}$

Ex. $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

$$\mu = EX_n = \lambda = \text{Var}(X_n) = \sigma^2$$

so

CLT: $\sqrt{N}\left(\frac{\bar{x} - \lambda}{\sqrt{\lambda}}\right) \xrightarrow{d} N(0,1)$

i.e. $\bar{x} \approx N(\lambda, \lambda/N)$ for large N.

Theorem: MGFs and Convergence in Dist

Let X_n be a seq of RVs w/ MGFs M_n and X a RV w/ MGF M,

then if $M_n(t) \rightarrow M(t)$

we have $X_n \xrightarrow{d} X$.

Taylor Series

If g is k-times diff'able function then the <u>k^{th} order Taylor poly. about a</u>

$$T_k(x) = \sum_{r=0}^{k} \frac{g^{(r)}(a)}{r!} (x-a)^k$$

e.g. second order about zero is

$$T_2(x) = g(0) + g'(0)\,x + \frac{g''(0)\,x^2}{2!}$$

under some conditions

$$g(x) \approx T_k(x) \text{ when } x \approx a.$$

Pf. of CLT

$$Y_N = \sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right) \qquad \underline{\text{want}}:\ Y_N \xrightarrow{d} N(0,1)$$

$Z_n = \dfrac{X_n-\mu}{\sigma}$ then $\mathbb{E}Z_n = 0$ and $\operatorname{Var} Z_n = 1$

$$Y_N = \sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right)$$

$$= \sqrt{N}\left(\frac{\frac{1}{N}\sum_n X_n - \frac{1}{N}\sum_n \mu}{\sigma}\right)$$

$$= \frac{\sqrt{N}}{N}\left(\frac{\sum_n X_n - \sum_n \mu}{\sigma}\right)$$

$$= \frac{\sqrt{N}}{N}\sum_n\left(\frac{X_n-\mu}{\sigma}\right)$$

$$Y_N = \frac{1}{\sqrt{N}}\sum_n z_n \quad \text{want} \xrightarrow{d} N(0,1)$$

Get MGF of Y_N $\quad \left(M_{ax+b}(t) = e^{tb}M(at)\right)$

$$M_{Y_N}(t) = \prod_n M_{z_n}(t/\sqrt{N})$$

all MGFs same

$$= \underbrace{M(t/\sqrt{N})^N}$$

M = MGF of any z_n

Taylor approx of $M(t)$ [Second order around zero]

$$M(t) \approx \underbrace{M(0)}_{1 = \mathbb{E}e^{0\cdot z_n}} + \underbrace{M'(0)t}_{\mathbb{E}z_n = 0} + \frac{M''(0)t^2}{2}$$

Var $z_n = 1$

$$\approx 1 + t^2/2$$

$$\boxed{\left(1+\frac{c}{n}\right)^n \to e^c}$$

$$M_{Y_N}(t) = M(t/\sqrt{N})^N \approx \left(1 + t^2/2N\right)^N$$

$$\text{as } N\to\infty \quad \to e^{t^2/2}$$

MGF of $N(0,1)$

$$Y_N \xrightarrow{d} N(0,1).$$

Delta Method

Theorem: First Order Delta Methods

If Y_N is a seq of RVs where

$$\sqrt{N}\,(Y_N - \theta) \xrightarrow{d} N(0, \varphi^2(\theta))$$

Think of $Y_N = \bar{X}$, $\theta = \mu$, $\varphi^2 = \sigma^2$.

then if g is a diff'able function and $g'(\theta) \neq 0$ then

$$\sqrt{N}\,(g(Y_N) - g(\theta)) \xrightarrow{d} N(0, [g'(\theta)]^2 \varphi^2(\theta)).$$

Intuition: $Y_N \sim AN(\theta, \varphi^2/N)$

$$g(Y_N) \sim AN(g(\theta), [g'(\theta)]^2 \varphi^2/N)$$

Ex. $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$, $\lambda > 0$

CLT: $\sqrt{n}\left(\frac{\bar{X} - \lambda}{\sqrt{\lambda}}\right) \xrightarrow{d} N(0,1)$

or $\sqrt{n}(\bar{X} - \lambda) \xrightarrow{d} N(0, \lambda)$

Let $g(x) = \log(x)$; $g'(x) = 1/x$

$$\sqrt{n}\left(g(\bar{x}) - g(\lambda)\right) \xrightarrow{d} N(0, [g'(\lambda)]^2 \lambda)$$

FO Δ-method

$$\sqrt{n}\left(\log(\bar{x}) - \log(\lambda)\right) \xrightarrow{d} N(0, \underbrace{\tfrac{1}{\lambda^2}\lambda}_{1/\lambda})$$