Lecture 11: Inequalities and Convergence

Theorem: Markov's Inequality

If $X \geq 0$ (support $X \subset [0, \infty)$)

then for any $a \geq 0$ we have

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}X}{a}.$$

pf (cts case)

$$\mathbb{E}X = \int_0^\infty x f(x)\,dx = \underbrace{\int_0^a \underbrace{x}_{\geq 0} \underbrace{f(x)}_{\geq 0}\,dx}_{\geq 0} + \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty x f(x)\,dx \qquad \text{over } x \in [a, \infty) \quad x \geq a$$

$$\geq \int_a^\infty a f(x)\,dx$$

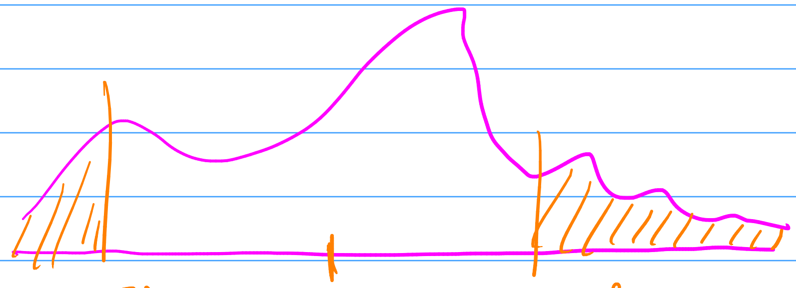
$$= a \int_a^\infty f(x)\,dx = a\,\mathbb{P}(X \geq a)$$

Theorem: Chebyshev's Inequality

If X is a RV w/ mean $\mu = \mathbb{E}X$, $\sigma^2 = \mathrm{Var}(X)$

then

$$\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2$$



Pf. If $Y = \frac{(X-\mu)^2}{\sigma^2}$ and $a = k^2$

by Markov's

$$(*) \quad \mathbb{P}(Y \geq a) \leq \frac{\mathbb{E}Y}{a}$$

notice: $\mathbb{E}Y = \mathbb{E}\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2}\mathbb{E}[(X-\mu)^2] = \frac{\sigma^2}{\sigma^2} = 1$

So $(*) = \mathbb{P}\left(\frac{(X-\mu)^2}{\sigma^2} \geq k^2\right) \leq 1/k^2$

$$\Rightarrow \mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2$$

Various Equiv. Vers. of Chebyshev.

(1) $P\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2$

(2) $P\left(\frac{|X-\mu|}{\sigma} < k\right) \geq 1 - 1/k^2$

let $\varepsilon = k\sigma \Leftrightarrow k = \varepsilon/\sigma \Leftrightarrow 1/k^2 = \sigma^2/\varepsilon^2$

(3) $P(|X-\mu| \geq \varepsilon) \leq \sigma^2/\varepsilon^2$

(4) $P(|X-\mu| < \varepsilon) \geq 1 - \sigma^2/\varepsilon^2$.

Ex. X = # of widgets produced by a factory

$\mu = EX = 1000$

$\sigma^2 = Var(X) = 25 \quad (\sigma = 5)$

What's the prob that

$994 \leq X \leq 1006$?

$P(994 \leq X \leq 1006)$

$= P(|X-1000| \leq 6)$

$= P\left(\frac{|X-\mu|}{\sigma} \leq \underbrace{1.2}_{k}\right) \geq 1 - 1/(1.2)^2$

$\approx 30\%$

Convergence

Calc II: convergence of numbers $x_n \in \mathbb{R}$

$$\lim_{n \to \infty} x_n = x$$

$$(x_n \to x)$$ notation

452: $X_n \longrightarrow X$

Recall: $X_n : S \longrightarrow \mathbb{R}$

for some $s \in S$ we have $X_n(s) \in \mathbb{R}$

Can define convergence of RVs as convergence of fns.

Defn: Pointwise Convergence of Functions

If $(f_n)_{n=1}^{\infty}$ is a seq of fns $f_n : \mathbb{R} \to \mathbb{R}$

and $f : \mathbb{R} \to \mathbb{R}$

We say the f_ns converge pointwise to f

denoted $f_n \xrightarrow{\text{ptwse}} f$

if

$$f_n(x) \to f(x) \quad \forall x \in \mathbb{R}.$$

Ex. $x = 5$

$f_1(5), f_2(5), f_3(5), \dots \quad \to f(5)$

Defn: Sure Convergence of RVs

A seq of RVs $(X_n)_{n=1}^{\infty}$ converges surely to X if $X_n \xrightarrow{\text{ptwse}} X$.

i.e. $\forall s \in S$ we have $X_n(s) \to X(s)$.

Defn: Almost Sure Convergence

We say (X_n) converges almost surely to X if $X_n(s) \to X(s)$ for $s \in A \subset S$ where $P(A) = 1$.

Denote this as $X_n \xrightarrow{a.s.} X$.

Ex. $S = [0,1]$ w/ uniform density

Let $X_n(s) = s + s^n$

$X(s) = s$

Does $X_n \xrightarrow{a.s.} X$?

For $s \in [0,1)$ then $s^n \to 0$ as $n \to \infty$

$$X_n(s) = s + s^n \xrightarrow{n} s = X(s)$$

Notice that if $s = 1$ then

$$X_n(s) = X_n(1) = 1 + 1^n = 2 \not\to X(1) = 1$$

So $A = [0,1)$ and $\mathbb{P}([0,1)) = 1$

hence $X_n \xrightarrow{a.s.} X$.

Defn: Convergence In Prob.

We say (X_n) converges in prob. to X

denote $X_n \xrightarrow{P} X$

if

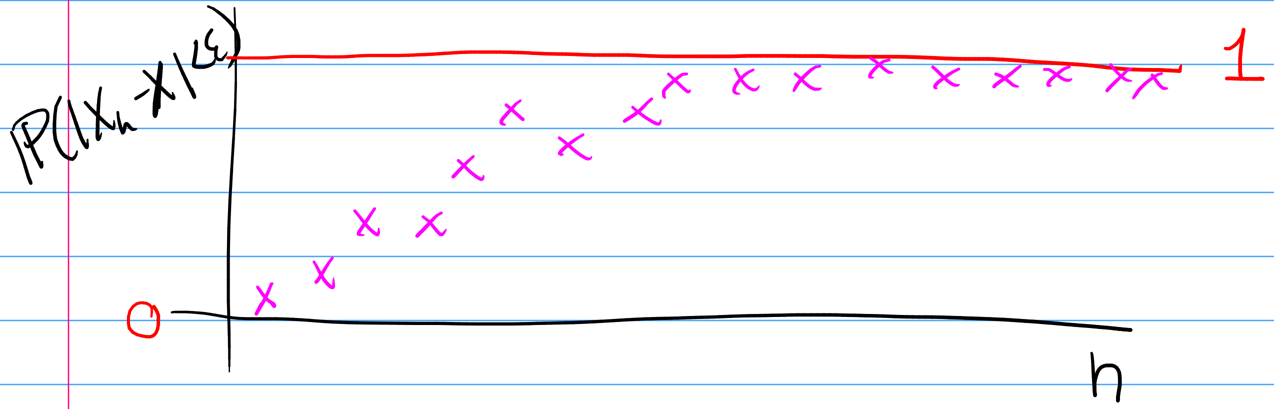
$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1,$$

equiv.

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$$

Pick $\varepsilon > 0$.

Calc. $\mathbb{P}(|X_1 - X| < \varepsilon), \mathbb{P}(|X_2 - X| < \varepsilon), \dots$



Defn: Convergence In Distribution

We say (X_n) converges in dist. to X if the CDFs converge pointwise i.e.

$$F_n \xrightarrow{\text{ptwse}} F$$

CDF of X_n → F_n; CDF of X → F

i.e. $\quad F_n(x) \to F(x) \quad \forall x.$

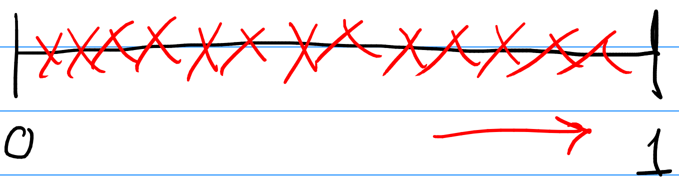
Theorem:

$$\text{a.s.} \Rightarrow \text{i.p.} \Rightarrow \text{d}$$

Converses are typically false.

Ex. Let $X_i \overset{iid}{\sim} U(0,1)$

and let $Y_n = \max_{i=1,\dots,n} X_i$ = max of first n X_is.

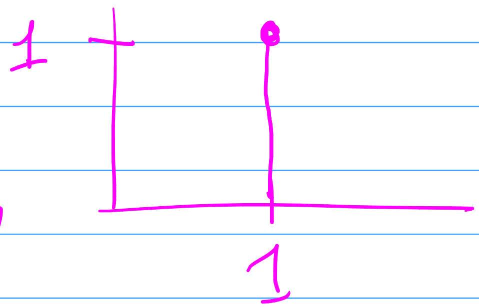


max creeps up towards 1 as $n \to \infty$

Intuition: $Y_n \to 1$ degenerate RV w/ all prob. mass at $x=1$

Could say $Y_n \to Y$

PMF of Y



Show $Y_n \xrightarrow{P} 1$.

Need to show $\forall \varepsilon > 0$ $\mathbb{P}(|Y_n - 1| \geq \varepsilon) \to 0$ as $n \to \infty$

$\mathbb{P}(|Y_n - 1| \geq \varepsilon)$

$= \mathbb{P}(|1 - Y_n| \geq \varepsilon)$

$= \mathbb{P}(1 - Y_n \geq \varepsilon)$ $\quad Y_n \leq 1$

$$= \mathbb{P}(Y_n \le 1-\varepsilon)$$

$$= \mathbb{P}(X_1 \le 1-\varepsilon, X_2 \le 1-\varepsilon, \dots, X_n \le 1-\varepsilon)$$

$$= \mathbb{P}(X_1 \le 1-\varepsilon)\,\mathbb{P}(X_2 \le 1-\varepsilon)\cdots\mathbb{P}(X_n \le 1-\varepsilon)$$

independence

$$= \mathbb{P}(X_i \le 1-\varepsilon)^n$$

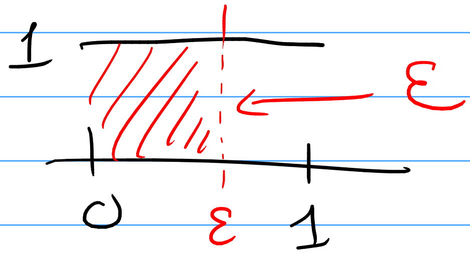
$X_i \overset{iid}{\sim} U(0,1)$

Case 1

If $\varepsilon > 1$ then $1-\varepsilon < 0$

So $\mathbb{P}(X_i \le 1-\varepsilon) = 0$

If $0 < \varepsilon \le 1$



then $\mathbb{P}(X_i \le 1-\varepsilon)^n = \varepsilon^n$

all together

$$\mathbb{P}(|Y_n - 1| \ge \varepsilon) = \begin{cases} 0, & \varepsilon > 1 \\ \varepsilon^n, & 0 < \varepsilon \le 1 \end{cases}$$

$$\to \begin{cases} 0, & \varepsilon > 0 \\ 0, & 0 < \varepsilon \le 1 \end{cases} = 0$$

as $n \to \infty$

So $Y_n \xrightarrow{P} 1$.

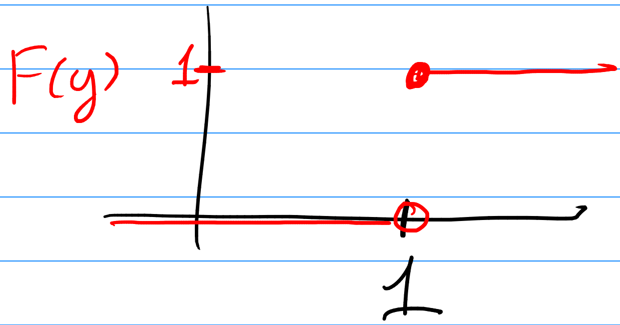
Show that $Y_n \xrightarrow{d} 1$

Need to show $F_n(y) \to F(y) \quad \forall y$

CDF of Y_n CDF of 1

Let's get CDF of Y_n.

$$F_n(y) = \mathbb{P}(Y_n \leq y)$$
$$= \mathbb{P}(\max_{i=1,\dots,n} X_i \leq y)$$
$$= \mathbb{P}(X_1 \leq y, X_2 \leq y, X_3 \leq y, \dots, X_n \leq y)$$
$$= \mathbb{P}(X_1 \leq y) \cdots \mathbb{P}(X_n \leq y)$$
$$= \mathbb{P}(X_i \leq y)^n$$



$$F_{X_i}(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 1 \\ x, & 0 \leq x \leq 1 \end{cases}$$