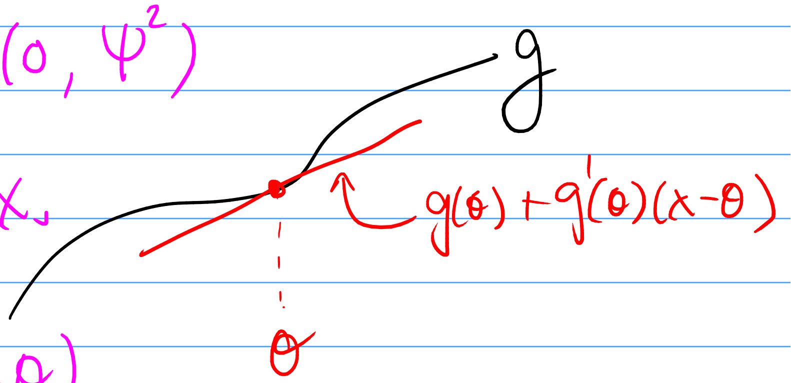
Lecture 14 : Delta Method

pf. of Δ-method

Consider a FO taylor approx. of g about θ

$$g(x) \approx g(\theta) + g'(\theta)(x-\theta)$$

Know: $\underbrace{\sqrt{N}(Y_N - \theta)}$ $\xrightarrow{d} N(0, \psi^2)$



plug Y_N into Taylor approx.

$$g(Y_N) \approx g(\theta) + g'(\theta)(Y_N - \theta)$$

$$\Leftrightarrow g(Y_N) - g(\theta) \approx g'(\theta)(Y_N - \theta)$$

$$\Leftrightarrow \sqrt{N}(g(Y_N) - g(\theta)) \approx \underbrace{g'(\theta)\underbrace{[\sqrt{N}(Y_N - \theta)]}_{\xrightarrow{d} N(0,\psi^2)}}$$

$$\sqrt{N}(g(Y_N) - g(\theta)) \xrightarrow{d} N(0, [g'(\theta)]^2 \psi^2)$$

Ex: $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, $\mu \neq 0$

What is asymptotic dist of $1/\bar{X}$?

CLT: $\sqrt{N}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$

Δ-method: $g(x) = 1/x$

$g'(x) = -1/x^2$, $g'(\mu) \neq 0$

$$\sqrt{N}\left(g(\bar{X}) - g(\mu)\right) \xrightarrow{d} N(0, \underbrace{[g'(\mu)]^2}_{} \sigma^2)$$

$$\left(-\frac{1}{\mu^2}\right)^2 \sigma^2 = \sigma^2/\mu^4$$

$$\sqrt{N}\left(\frac{1}{\bar{X}} - \frac{1}{\mu}\right) \xrightarrow{d} N(0, \sigma^2/\mu^4)$$

$$\frac{1}{\bar{X}} \sim AN\left(\frac{1}{\mu}, \frac{\sigma^2}{N\mu^4}\right)$$

<u>Ex.</u> <u>Variance Stabilizing Transformations</u>

Generically, $Y_N \sim AN(\theta, \varphi^2(\theta)/N)$

var could depend on θ

<u>Q</u>: is there some transformation g so that the asymptotic var <u>doesn't depend on θ</u>

Δ-method: $g(Y_N) \sim AN(g(\theta), \underbrace{\quad}_{\text{no } \theta})$

$[g'(\theta)]^2 \varphi^2(\theta) =$ const. wrt θ

differential equation

Ex. $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

CLT: $\bar{X} \sim AN(\lambda, \overbrace{\lambda/N}^{\varphi^2(\lambda)})$

ODE: $[g'(\lambda)]^2 \varphi^2(\lambda) = c$

$\Leftrightarrow [g'(\lambda)]^2 \lambda/N = c$

$\Leftrightarrow \left[\frac{dg}{d\lambda}\right]^2 = \frac{Nc}{\lambda}$

$\Leftrightarrow \frac{dg}{d\lambda} = \frac{\sqrt{N}\sqrt{c}}{\sqrt{\lambda}}$

$\Rightarrow dg = \frac{c}{\sqrt{\lambda}} d\lambda$

$g = \int dg = \int \frac{c}{\sqrt{\lambda}} d\lambda$

$\Rightarrow g \propto \sqrt{\lambda}$

let $g(x) = \sqrt{x}$, $g'(x) = \frac{1}{2\sqrt{x}}$, $g'(\lambda) \neq 0$

So Δ-method says

$$g(\bar{X}) = \sqrt{\bar{X}} \sim AN\Big(\underbrace{g(\lambda)}_{\sqrt{\lambda}}, \underbrace{\frac{\varphi^2(\lambda)[g'(\lambda)]^2}{N}}_{\underbrace{\frac{\lambda\left(\frac{1}{2\sqrt{\lambda}}\right)^2}{N}}_{\frac{1}{4N}}}\Big)$$

$$\sqrt{\bar{X}} \sim AN\left(\sqrt{\lambda}, \frac{1}{4N}\right)$$

Theorem: Second Order Delta Method

Assume
$$\sqrt{N}(Y_N - \theta) \xrightarrow{d} N(0, \psi^2)$$

let g be twice differentiable and $g'(\theta) = 0$

then
$$N\,(g(Y_N) - g(\theta)) \xrightarrow{d} \frac{\psi^2 g''(\theta)}{2} \chi^2(1).$$

so long as $g''(\theta) \neq 0$.

Ex. $X_n \overset{iid}{\sim} \text{Bern}(p)$

let $g(x) = x \log(x/p) - (1-x)\log\left(\frac{1-x}{1-p}\right)$

KL divergance between two Bernalli

What can I say about $g(\bar{X})$?

CLT: $\sqrt{N}(\bar{X} - p) \xrightarrow{d} N(0, \underbrace{p(1-p)}_{\psi^2})$

Notice: $g'(x) = \log(x/1-x) - \log(p/1-p)$

so $g'(p) = 0$

Can't use FO Δ-method.

However,

$$g''(x) = \frac{1}{x(1-x)}$$

and $0<p<1$ we have $g''(p) \neq 0$

By SO Δ- method we have

$$N\left(g(\bar{x}) - g(p)\right) \xrightarrow{d} \underbrace{\frac{\psi^2 g''(p)}{2}}_{\frac{p(1-p)\frac{1}{p(1-p)}}{2} = \frac{1}{2}} \chi^2(1)$$

all together,

$$N\left(g(\bar{x}) - g(p)\right) \xrightarrow{d} \frac{1}{2}\chi^2(1).$$

<u>pf. of SO Δ-method</u>

Second order Taylor approx. of g is

$$g(x) \approx g(\theta) + g'(\theta)(x-\theta) + \frac{1}{2}g''(\theta)(x-\theta)^2$$

if $g'(\theta)=0$ then this becomes

$$g(x) \approx g(\theta) + \frac{1}{2}g''(\theta)(x-\theta)^2$$

<u>Know:</u> $\sqrt{N}(Y_N - \theta) \xrightarrow{d} N(0,\psi^2)$

plug in to SO Taylor approx.

$$g(Y_N) - g(\theta) \approx \frac{1}{2} g''(\theta) (Y_N - \theta)^2$$

$$\sqrt{N}\left(\frac{Y_N - \theta}{\psi}\right) \xrightarrow{d} N(0,1)$$

so $$N\frac{(Y_N - \theta)^2}{\psi^2} \xrightarrow{d} \chi^2(1)$$

$$N\left(g(Y_N) - g(\theta)\right) \approx \frac{\psi^2}{2} g''(\theta) \underbrace{\left[\frac{\sqrt{N}(Y_N - \theta)}{\psi}\right]^2}_{\xrightarrow{d} \chi^2(1)}$$

Back to estimation

For finite samples, looked at making estimators

1. unbiased
2. low variance

For large samples (asymptotically) we want estimators that are

1. asymptotically unbiased [consistency]
2. asymptotically has low variance

Theorem: MLEs are Consistent (*)

(*) need some technical conditions... works for exp. fams.

If $\hat{\theta}_{MLE}$ is the MLE for θ

then $\hat{\theta}_{MLE} \xrightarrow{P} \theta$.

Note: τ is continuous,

then $\underbrace{\tau(\hat{\theta}_{MLE})}_{\text{this is the MLE for } \tau(\theta)} \xrightarrow{P} \tau(\theta)$

Defn: Asymptotically Normal

We say $\hat{\theta}$ is asymptotically normal w/

(1) asymptotic mean $\tau(\theta)$

(2) asymptotic variance $v(\theta)$

if

$$\sqrt{N}\left(\hat{\theta} - \tau(\theta)\right) \xrightarrow{d} N(0, v(\theta))$$

Notation:

$$\hat{\theta} \sim AN\left(\tau(\theta), v(\theta)/N\right)$$

Defn: Asymptotic Relative Efficiency (ARE)

If T_N and W_N are est. for $\tau(\theta)$ and

$$T_N \sim AN(\tau(\theta), \sigma_T^2)$$

$$W_N \sim AN(\tau(\theta), \sigma_W^2)$$

then the ARE of W_N w.r.t. T_N is

$$ARE(W_N, T_N) = \sigma_T^2 / \sigma_W^2$$

If $ARE < 1$ we prefer T_N

$ARE > 1$ we prefer W_N.

Ex. Let $X_n \overset{iid}{\sim} Pois(\lambda)$ and let $\tau(\lambda) = e^{-\lambda}$

$P(X_n = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda}$

Way 1: $\bar{X}$ is the MLE for λ

So $e^{-\bar{X}}$ is the MLE for $e^{-\lambda}$

Way 2: Let $Y_n = \mathbb{1}(X_n = 0) \sim Bern(p)$

$$p = P(X_n = 0) = e^{-\lambda}$$

$$Y_n \sim Ber(e^{-\lambda})$$

Then $E\bar{Y} = e^{-\lambda}$

$\bar{Y}$ = % of data that is zero.

Q: Which is better?

(1) $e^{-\bar{X}}$ or (2) $\bar{Y}$

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(1) CLT says $\bar{X} \sim AN(\lambda, \lambda/N)$

What about $e^{-\bar{X}}$? Let $g(x) = e^{-x}$ use Δ-method.

$g'(x) = -e^{-x}$; $g'(\lambda) = -e^{-\lambda} \neq 0$

Use FO Δ-method:

$$g(\bar{x}) \sim AN(\underbrace{g(\lambda)}_{e^{-\lambda}}, \underbrace{[g'(\lambda)]^2 \lambda/N}_{(e^{-2\lambda})\lambda/N})$$

So $$\boxed{e^{-\bar{X}} \sim AN(e^{-\lambda}, e^{-2\lambda}\lambda/N)}$$