Lecture 10: Lehmann-Scheffe

Last time:

If $\hat{\theta}$ is unbiased for $\tau(\theta)$, $E\hat{\theta} = \tau(\theta)$

and $\varphi = \varphi(W) = \mathbb{E}[\hat{\theta} | W]$ ← may or may not be a stat

(1) $E\varphi = \tau(\theta)$

(2) $\text{Var}\,\varphi \leq \text{Var}(\hat{\theta})$

Ex. $X_n \overset{iid}{\sim} N(\theta, 1)$

$\hat{\theta} = \frac{1}{2}(X_1 + X_2)$

Showed $E\hat{\theta} = \theta$ and $\text{Var}\,\hat{\theta} = \frac{1}{2}$

Let $W = \bar{X}$.

$$\varphi = \mathbb{E}[\hat{\theta} | W] = \mathbb{E}\left[\tfrac{1}{2}(X_1 + X_2) \,\middle|\, \bar{X}\right]$$

$$= \tfrac{1}{2}\mathbb{E}[X_1 | \bar{X}] + \tfrac{1}{2}\mathbb{E}[X_2 | \bar{X}]$$

same RV $\mathbb{E}[X_n | \bar{X}]$

$$= \tfrac{1}{2}\left(2\,\mathbb{E}[X_n | \bar{X}]\right)$$

$$= \mathbb{E}[X_n | \bar{X}]$$

$$= \frac{1}{N} N \mathbb{E}[X_n \mid \bar{X}]$$

$$= \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[X_n \mid \bar{X}]$$

$$= \mathbb{E}\left[\frac{1}{N} \sum_n X_n \mid \bar{X}\right]$$

$$= \mathbb{E}[\bar{X} \mid \bar{X}]$$

$$\varphi = \bar{X} \leftarrow \text{a stat.}$$

and by our theorem,

(1) $\mathbb{E}\varphi = \theta$

(2) $\underset{1/N}{\operatorname{Var}\varphi} \le \underset{\frac{1}{2}}{\operatorname{Var}(\hat{\theta})}$

Theorem: Rao-Blackwell Theorem

If $\hat{\theta}$ is unbiased stat for $\tau(\theta)$ and W is sufficient for θ then

if $\varphi = \mathbb{E}[\hat{\theta} \mid W]$

(1) $\mathbb{E}\varphi = \tau(\theta)$

(2) $\operatorname{Var}(\varphi) \le \operatorname{Var}(\hat{\theta})$

(3) φ is a stat (no θ in its formula)

$$\mathbb{E}g(X) = \int g(x) f(x) dx$$

Pf of 3

$$\varphi = \mathbb{E}[\hat{\theta} \mid W] = \mathbb{E}[\hat{\theta}(\underset{\sim}{X}) \mid W]$$

$$= \underbrace{\int \underbrace{\hat{\theta}(\underset{\sim}{x})}_{\text{no } \theta} \underbrace{f_{\underset{\sim}{X}|W}(\underset{\sim}{x})}_{\text{no } \theta} d\underset{\sim}{x}}_{\text{free of } \theta}$$

Theorem: Lehmann-Scheffé Theorem

If W is a (complete) sufficient statistic for θ and $\hat{\theta}$ is unbiased for $\tau(\theta)$ and $\hat{\theta}$ depends on $\underset{\sim}{X}$ only through W

tech. condition

$$\hat{\theta} = \hat{\theta}(W) = \hat{\theta}(W(\underset{\sim}{X}))$$

then $\hat{\theta}$ is the UMVUE for $\tau(\theta)$.

If I can find some function of a sufficient W that is unbiased for $\tau(\theta)$ it is the UMVUE.

Ex. Let $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ known

Want UMVUE for μ.

Use Lehman-Scheffe

1. find a SS for μ, $\bar{X}$
2. find a fn of $\bar{X}$ that is unbiased for μ

$\hat{\mu} = \bar{X}$ then $\mathbb{E}\hat{\mu} = \mu$.

Thus $\hat{\mu} = \bar{X}$ is the UMVUE.

Ex. Let $\tau(\mu) = \mu^2$.

1. SS for μ ! $\bar{X}$
2. find a fn of $\bar{X}$ unbiased for μ^2

$$\mathbb{E}[\bar{X}^2] = Var(\bar{X}) + \mathbb{E}[\bar{X}]^2$$
$$= \sigma^2/N + \mu^2$$

$$\mathbb{E}[\bar{X}^2 - \sigma^2/N] = \mu^2$$

So $\bar{X}^2 - \sigma^2/N$ — ① unbiased for μ^2 ② fn of $\bar{X}$

So Lehman-Scheffé say it is the UMVUE.

Ex. $X_n \overset{iid}{\sim} U(0,\theta)$

what's the UMVUE for $\tau(\theta) = \theta$?

① Find SS for θ : $X_{(N)}$

② Find fn of $X_{(N)}$ unbiased for θ.

Claim: $\mathbb{E}[X_{(N)}] = \frac{N}{N+1}\theta$

So $\hat{\theta} = \frac{N+1}{N} X_{(N)}$ then $\mathbb{E}\hat{\theta} = \theta$

and so Lehmann-Scheffé say it's the UMVUE.

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pf. of Lehmann-Scheffé

If $\hat{\theta}$ is unbiased, fn of complete sufficient stat W

then for any other unbiased est. V

$$\mathrm{Var}(\hat{\theta}) \leq \mathrm{Var}(V)$$

Rao-Blackwell says that if

$$\varphi = E[V|W]$$

then ① $E\varphi = \tau(\theta)$

② $\operatorname{Var}\varphi \le \operatorname{Var}(V)$

③ φ is a stat.

then $\operatorname{Var}\hat{\theta} = \operatorname{Var}\varphi \le \operatorname{Var}(V)$

We'll show that $\hat{\theta} = \varphi$.

Consider $g(W) = \hat{\theta}(W) - \varphi(W)$

will show that $g \equiv 0 \quad \forall \theta$

So $\hat{\theta}(W) - \varphi(W) = 0 \Rightarrow \hat{\theta} = \varphi$.

Completeness of W

Say W is "complete" if

$$E_\theta[h(W)] = 0 \ \forall \theta \Leftrightarrow h \equiv 0$$

h is the zero fn

$$E[g(W)] = E[\hat{\theta}(W) - \varphi(W)] = E[\hat{\theta}] - E[\varphi]$$

$$= \tau(\theta) - \tau(\theta)$$

$$= 0$$

If W is complete then $g \equiv 0$

Theorem: UMVUEs are Unique

Let W_1 and W_2 be UMVUEs and $W_1 \neq W_2$.

Consider: $W_3 = \frac{1}{2}(W_1 + W_2)$

1. $\mathbb{E}W_3 = \frac{1}{2}\mathbb{E}W_1 + \frac{1}{2}\mathbb{E}W_2 = \frac{1}{2}(\tau(\theta) + \tau(\theta))$
   $= \tau(\theta)$

   So W_3 unbiased for $\tau(\theta)$.

2. $Var(W_3) = Var\left(\frac{1}{2}W_1 + \frac{1}{2}W_2\right)$
   $= \frac{1}{4}Var(W_1) + \frac{1}{4}Var(W_2)$
   $+ \frac{1}{2}Cov(W_1, W_2)$

$\boxed{Cor(W_1, W_2) \overset{=}{\leq} 1} \Rightarrow \frac{Cov(W_1, W_2)}{\sqrt{Var(W_1)Var(W_2)}} \overset{=}{\leq} 1$

$\Rightarrow Cov(W_1, W_2) \overset{=}{\leq} \sqrt{Var(W_1)Var(W_2)}$

$Var(W_3) \overset{=}{\leq} \frac{1}{4}Var(W_1) + \frac{1}{4}Var(\underset{W_1}{\cancel{W_2}}) + \frac{1}{2}\sqrt{Var(W_1)Var(\underset{W_1}{\cancel{W_2}})}$

$$= \underbrace{\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{2}\right)}_{1} Var(W_1)$$

$$Var(W_3) = Var(W_1)$$

So $Cor(W_1, W_2) = 1$

$$W_1 = aW_2 + b$$

$$EW_1 = a\underbrace{EW_2}_{\tau(\theta)} + b = \tau(\theta)$$

must be that $a = 1$, $b = 0$

So $W_1 = W_2$.