Lecture 18: UMP Tests

Neyman-Pearson Lemma

Consider

$$H_0: \theta = \theta_0 \ \text{v.} \ H_a: \theta = \theta_a$$

and test w/ the LRT that rejects when

(I) $$\lambda = \frac{L(\theta_0)}{L(\theta_a)} \leq c$$

So that

(II) $$\mathbb{P}_{\theta_0}(\lambda \leq c) = \alpha$$ [size α test]

then

(a) Sufficiency Any test satisfying I and II is a UMP level α test for this hypothesis

(b) Necessity: Every UMP level α test for this hypothesis satisfies I and II.

Consider an alternate LRT: if T is sufficient for θ and let $g_\theta(T)$ be its PMF/PDF.

Traditional LRT: $\lambda = \dfrac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \dfrac{f_{\hat{\theta}_0}(\underset{\sim}{x})}{f_{\hat{\theta}}(\underset{\sim}{x})}$

Let $L^*(\theta) = g_\theta(T)$ and let

$$\lambda^* = \frac{L^*(\hat{\theta}_0)}{L^*(\hat{\theta})} = \frac{g_{\hat{\theta}_0}(T)}{g_{\hat{\theta}}(T)}$$

If I reject when $\lambda^* \leq c$ then this is equivalent to the standard LRT

Reason this works is that all MLEs are functions of the sufficient statistic.

$$\lambda(\underset{\sim}{x}) = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \frac{\max_{\theta \in \Theta_0} f_\theta(\underset{\sim}{x})}{\max_{\theta \in \Theta} f_\theta(\underset{\sim}{x})}$$

$f_\theta(\underset{\sim}{x}) = g(\theta, T)\,h(\underset{\sim}{x})$

$$= \frac{\max_{\theta \in \Theta_0} \cancel{h(\underset{\sim}{x})}\, g(\theta, T)}{\max_{\theta \in \Theta_0} \cancel{h(\underset{\sim}{x})}\, g(\theta, T)}$$

Believe : $g(\theta, T) \propto g_\theta(T)$ ← PDF/PMF of T

$$= \frac{\max_{\theta \in \Theta_0} g_\theta(T)}{\max_{\theta \in \Theta} g_\theta(T)} = \lambda^*(T)$$

Corollary to NP lemma

Test $H_0: \theta = \theta_0$ v. $H_a: \theta = \theta_a$

using a test that rejects when

$$\lambda^* = g_{\theta_0}(T) / g_{\theta_a}(T) \leq c$$

when c is chosen so that

$$\mathbb{P}_{\theta_0}(\lambda^* \leq c) = \alpha$$

then this is the UMP level α test.

Ex. $X_1, X_2 \overset{iid}{\sim} \text{Bern}(\theta)$

test $H_0: \theta = 1/2$ v. $H_a: \theta = 3/4$

Note: $T = X_1 + X_2$ is sufficient for θ.

and $T \sim \text{Bin}(2, \theta)$

So $g_\theta(T) = \binom{2}{T} \theta^T (1-\theta)^{2-T}$

$$\lambda = \frac{g_{1/2}(T)}{g_{3/4}(T)} = \frac{\cancel{\binom{2}{T}}\left(\frac{1}{2}\right)^T\left(1-\frac{1}{2}\right)^{2-T}}{\cancel{\binom{2}{T}}\left(\frac{3}{4}\right)^T\left(1-\frac{3}{4}\right)^{2-T}} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)^T\left(\frac{1}{4}\right)^{2-T}}$$

LRT says reject when $\lambda \leq c$

T	0	1	2
$\lambda(T)$	4	4/3	4/9

For example if $c \in (4/9, 4/3)$

$\lambda \leq c \iff$ rejecting when $T=2$

$\lambda \leq 1$

and so $\alpha = \mathbb{P}_{\frac{1}{2}}(\lambda \leq c) = \mathbb{P}_{\frac{1}{2}}(T=2) = \binom{2}{2}\left(\frac{1}{2}\right)^2$
$= 1/4$

So this test would be the UMP level 1/4 test.

Ex. $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ known

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta = \theta_a$$

and $\theta_a > \theta_0$

$T = \bar{X}$ is sufficient for θ

and $\bar{X} \sim N(\theta, \sigma^2/N)$

$$\lambda = \frac{g_{\theta_0}(\bar{X})}{g_{\theta_a}(\bar{X})} = \frac{\cancel{\frac{1}{\sqrt{2\pi\sigma^2/N}}} \exp\left(-\frac{1}{2\sigma^2/N}(\bar{X} - \theta_0)^2\right)}{\cancel{\frac{1}{\sqrt{2\pi\sigma^2/N}}} \exp\left(-\frac{1}{2\sigma^2/N}(\bar{X} - \theta_a)^2\right)}$$

$$= \exp\left(-\frac{1}{2\sigma^2/N}\left[(\bar{x} - \theta_0)^2 - (\bar{x} - \theta_a)^2\right]\right)$$

$$\cancel{\bar{x}^2} + \theta_0^2 - 2\bar{x}\theta_0 - (\cancel{\bar{x}^2} + \theta_a^2 - 2\bar{x}\theta_a)$$

$$= \theta_0^2 + \theta_a^2 - 2\bar{x}(\theta_0 - \theta_a)$$

Reject when $\lambda \le c$

$$-\frac{1}{2\sigma^2/N}\left[\theta_0^2 + \theta_a^2 - 2\bar{x}(\theta_0 - \theta_a)\right] \le \log c$$

$$\iff \theta_0^2 + \theta_a^2 - 2\bar{x}(\theta_0 - \theta_a) \ge -\frac{2\sigma^2}{N}\log c$$

$$\iff \bar{x}\left(-2(\theta_0 - \theta_a)\right) \ge -\frac{2\sigma^2}{N}\log c - \theta_a^2 - \theta_0^2$$

$$\iff \boxed{\bar{x} \ge c^*}$$

when I choose c^* so that

$$P_{\theta_0}(\lambda \le c) = P_{\theta_0}(\bar{X} \ge c^*) = \alpha$$

$$P_{\theta_0}\left(\underbrace{\frac{\bar{X}-\theta_0}{\sigma/\sqrt{N}}}_{Z\sim N(0,1)} \ge \frac{c^*-\theta_0}{\sigma/\sqrt{N}}\right) = \alpha$$

then $\dfrac{c^*-\theta_0}{\sigma/\sqrt{N}} = z_\alpha$

i.e. $c^* = \theta_0 + \frac{\sigma}{\sqrt{N}} z_\alpha$

So my UMP level α test is to reject when

$$\boxed{\bar{X} \ge \theta_0 + \frac{\sigma}{\sqrt{N}} z_\alpha}$$

What about composite tests?

Let's consider one-sided tests

$$H_0: \theta \le \theta_0 \quad \text{v.} \quad H_a: \theta > \theta_0$$

univariate dists

Defn: Monotone Likelihood Ratio Property (MLR)

We say a fam of dists f_θ, $\theta \in \Theta$ has the MLR property if $\forall \theta_1 < \theta_2$

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} \text{ is non-decreasing in } x$$

Corollary: If f_θ is an exp. fam of the form

$$f_\theta(x) = c(\theta)\, h(x) \exp(w(\theta)\, x)$$

then this fam has the MLR property if $w(\theta)$ is non-decreasing in θ.

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pf. $\theta_1 < \theta_2$

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{c(\theta_2)\cancel{h(x)} \exp(w(\theta_2)x)}{c(\theta_1)\cancel{h(x)} \exp(w(\theta_1)x)}$$

$$= \frac{c(\theta_1)}{c(\theta_2)} \exp\Big(\underbrace{(w(\theta_2) - w(\theta_1))}_{\geq 0} x\Big)$$

$\approx e^{ax}$ for $a \geq 0$

so this is non-dec. in x hence the MLR

Theorem: If T has the MLR property. and we construct a test that rejects when

$$T > c$$

then the power function of this test is non-decreasing.

Pf. Show if $\theta_2 > \theta_1$ then $\beta(\theta_2) \geq \beta(\theta_1)$

i.e. $\mathbb{P}_{\theta_2}(T > c) \geq \mathbb{P}_{\theta_1}(T > c)$

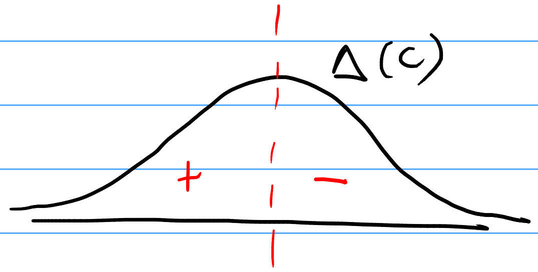
i.e. $1 - F_{\theta_2}(c) \geq 1 - F_{\theta_1}(c)$

CDF of T at θ_1

i.e. $\underbrace{F_{\theta_1}(c) - F_{\theta_2}(c)}_{\Delta} \geq 0$

as $c \to -\infty$ then $\Delta \to 0$

as $c \to \infty$ then $\Delta \to 0$



deriv goes + to −

$$\frac{d\Delta}{dc} = f_{\theta_1}(c) - f_{\theta_2}(c)$$

$$= \underbrace{f_{\theta_1}(c)}_{\geq 0}\underbrace{\left(1 - \frac{f_{\theta_2}(c)}{f_{\theta_1}(c)}\right)}$$

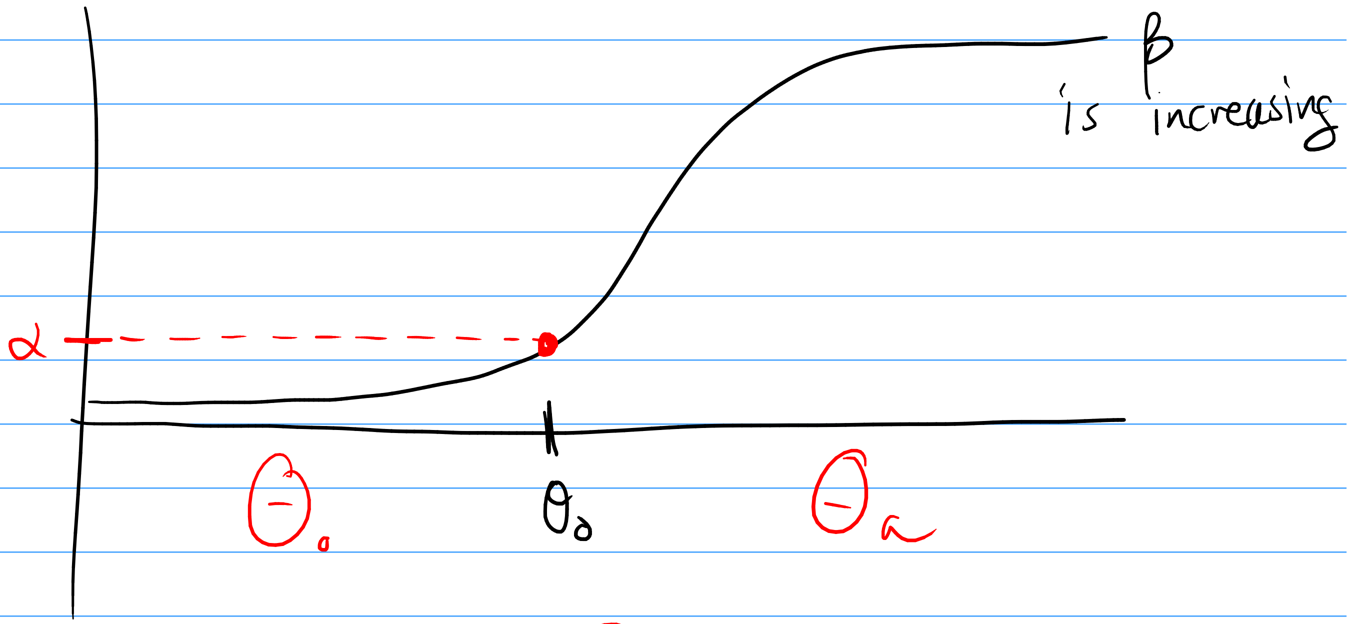
If f_θ has MLR then $f_{\theta_2}(c)/f_{\theta_1}(c)$ is increasing in c

So $1 - f_{\theta_2}(c)/f_{\theta_1}(c)$ is decreasing

Why do we care?

$$H_0: \theta \le \theta_0 \quad v. \quad H_a: \theta \geq \theta_0$$

reject when $T > c$ (T has MLR ....)



If choose c s.t. $\mathbb{P}_{\theta_0}(T > c) = \alpha$ then I have a level α test.

Theorem: Karlin-Rubin Theorem

Consider testing

$$H_0: \theta \le \theta_0 \quad v. \quad H_a: \theta > \theta_0$$

and let T be sufficient for θ and have the MLR property

Then the test that rejects when $T > c$ when c is chosen so that

$$\mathbb{P}_{\theta_0}(T > c) = \alpha$$

is the UMP level α test.

Alt. $H_0: \theta \geq \theta_0$ v. $H_a: \theta < \theta$

UMP test rej. when $T < c$

Ex. $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

$$H_0: \theta \leq \theta_0 \quad v. \quad H_a: \theta > \theta_0$$

Again, $\bar{X}$ is sufficient and so reject when

$$\bar{X} > c$$

where $\mathbb{P}_{\theta_0}(\bar{X} > c) = \alpha$

saw $c = \theta_0 + \sigma/\sqrt{n}\, z_\alpha$

To prove optimality, show $\bar{X}$ has MLR.

To do this note $\bar{X} \sim N(\theta, \sigma^2)$

so $f_\theta(\bar{x}) = \frac{1}{\sqrt{2\pi\sigma^2/N}} \exp\left(-\frac{1}{2\sigma^2/N}(\bar{x}-\theta)^2\right)$

$$-\frac{1}{2\sigma^2/N}\left(\bar{x}^2 - 2\bar{x}\theta + \theta^2\right)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma^2/N}} e^{\left(-\frac{\theta}{2\sigma^2/N}\right)}}_{c(\theta)} \underbrace{e^{-\frac{\bar{x}^2}{2\sigma^2/N}}}_{h(x)} \exp\left(\overset{w(\theta)}{+\frac{2\bar{x}}{2\sigma^2/N}\theta}\right)$$

inc. fn of θ

So $T = \bar{X}$ has MLR.