Lecture 17: More LRT

LRT says reject when $\bar{X} > a + \frac{\sigma}{\sqrt{N}} c^*$

Maybe want to make this a level α test.

i.e.

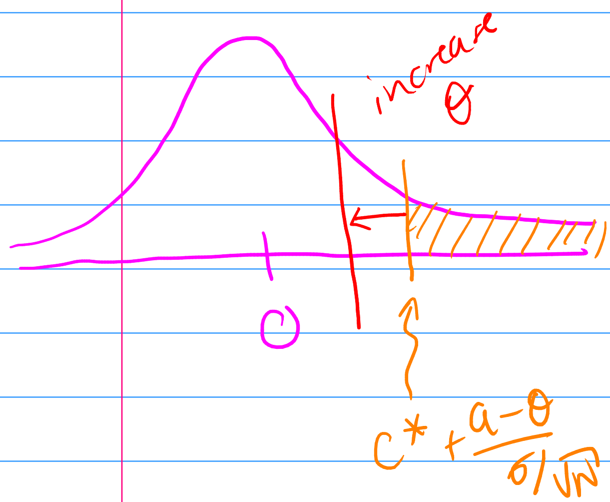
$$\max_{\underset{(-\infty, a]}{\theta \in \Theta_0}} \underbrace{P_\theta(\text{reject})}_{\beta(\theta)} \leq \alpha$$

$$\beta(\theta) = P_\theta(\lambda \leq c) = P_\theta\left(\frac{\bar{X} - a}{\sigma/\sqrt{N}} \geq c^*\right)$$

$$= P_\theta\left(\underbrace{\frac{\bar{X} - a}{\sigma/\sqrt{N}} + \frac{a - \theta}{\sigma/\sqrt{N}}}_{} \geq c^* + \frac{a-\theta}{\sigma/\sqrt{N}}\right)$$

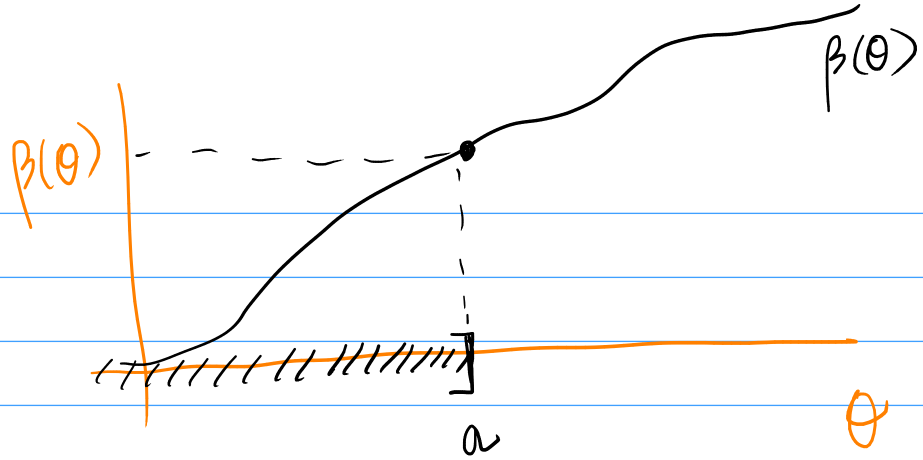
$$Z = \frac{\bar{X} - \theta}{\sigma/\sqrt{N}} \sim N(0,1)$$

$$= P_\theta\left(Z \geq c^* + \frac{a-\theta}{\sigma/\sqrt{N}}\right)$$



as $\theta \uparrow$, $P_\theta(Z \geq \cdots) \uparrow$

i.e. β monotonic in θ



$$\max_{\theta \in \Theta_0} \beta(\theta) = \max_{\theta \le a} \beta(\theta) = \beta(a)$$

So highest type I error rate at $\theta = a$

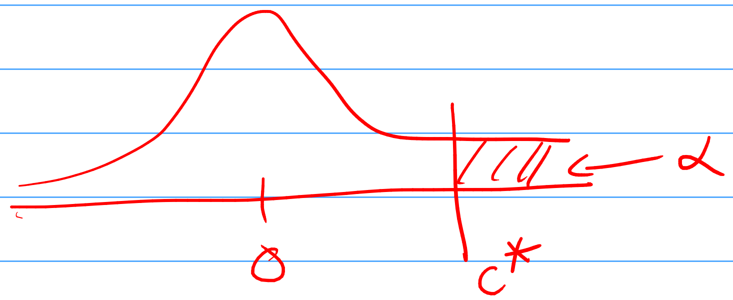
i.e. $\beta(a) = \mathbb{P}\left(Z \geq c^* + \frac{a-a}{\sigma/\sqrt{n}}\right) = \mathbb{P}(Z \geq c^*)$

So to make this max $= \alpha$ then I should choose c^* so that

$$\mathbb{P}(Z \geq c^*) = \alpha.$$

If F_Z is the CDF of a $N(0,1)$ then

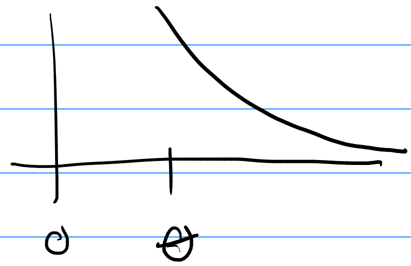
$$c^* = F_Z^{-1}(1-\alpha) = z_\alpha$$



Ex. $X_n \overset{iid}{\sim} \text{ShiftedExp}(\overset{\lambda}{1}, \theta)$

↳ add θ to $\text{Exp}(\lambda)$

$$f_\theta(x) = e^{-(x-\theta)} \mathbb{1}(x > \theta)$$



Consider $H_0: \theta \le 0$ v. $H_a: \theta > 0$

LRT:
$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}$$ and reject when $\lambda \le c$.

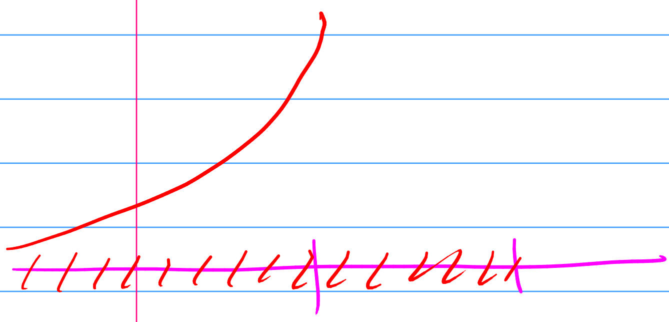
$$L(\theta) = \prod_n e^{-(x_n - \theta)} \mathbb{1}(x_n > \theta)$$
$$= e^{-N\bar{x}} e^{N\theta} \mathbb{1}(X_{(1)} > \theta)$$

So $\hat{\theta} = X_{(1)}$.

$H_0: \theta \le 0$, want $\hat{\theta}_0 =$ MLE over $(-\infty, 0]$

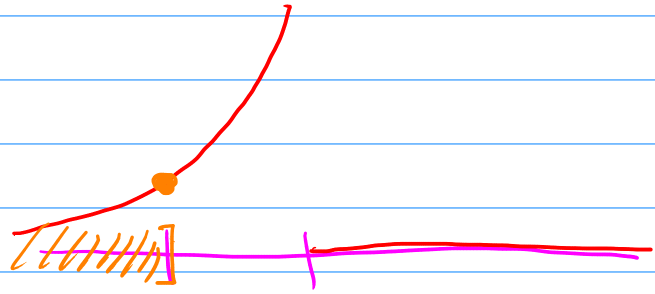
Two cases

Case 1: $X_{(1)} < \theta$



$\hat{\theta}_0 = X_{(1)}$

Case : $X_{(1)} \geq \theta$



$\hat{\theta}_0 = \theta$

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \begin{cases} \frac{L(X_{(1)})}{L(X_{(1)})} = 1 & X_{(1)} < \theta \\ L(\theta)/L(X_{(1)}) & X_{(1)} \geq \theta \end{cases}$$

So, basically $\lambda = L(\theta)/L(X_{(1)})$

$$= \frac{\cancel{e^{-N\bar{X}}}\, e^{N(\theta)}\, \cancel{1(X_{(1)} \geq \theta)}^{\,1}}{\cancel{e^{-N\bar{X}}}\, e^{NX_{(1)}}\, \cancel{1(X_{(1)} \geq X_{(1)})}}$$

$$= e^{-NX_{(1)}}$$

So LRT says reject when $\lambda \leq c$

i.e. $e^{-N X_{(1)}} \le c$

$\iff -N X_{(1)} \le \log c$

$\iff X_{(1)} \geq \underbrace{-\frac{1}{N}\log c}_{c^*} > 0$

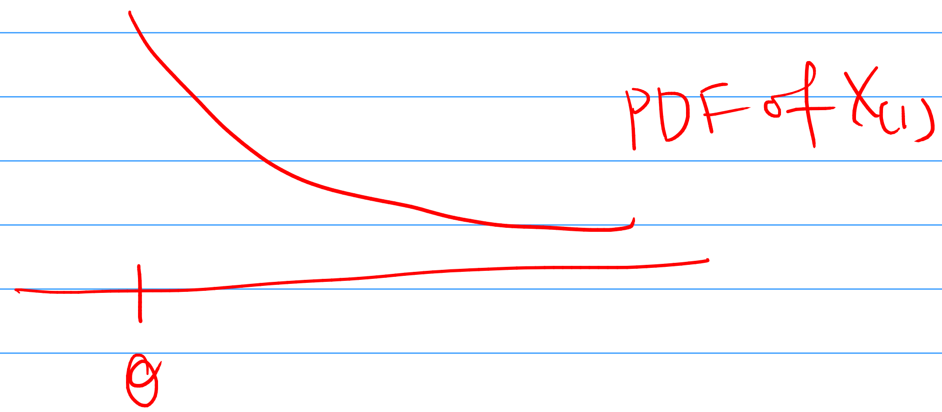
So LRT rejects when $X_{(1)}$ is sufficiently large.

Could choose c^* so that

$$\max_{\theta \in \Theta_0} \beta(\theta) = \alpha$$

$P_\theta(\text{reject})$
$= P_\theta(X_{(1)} \geq c^*)$

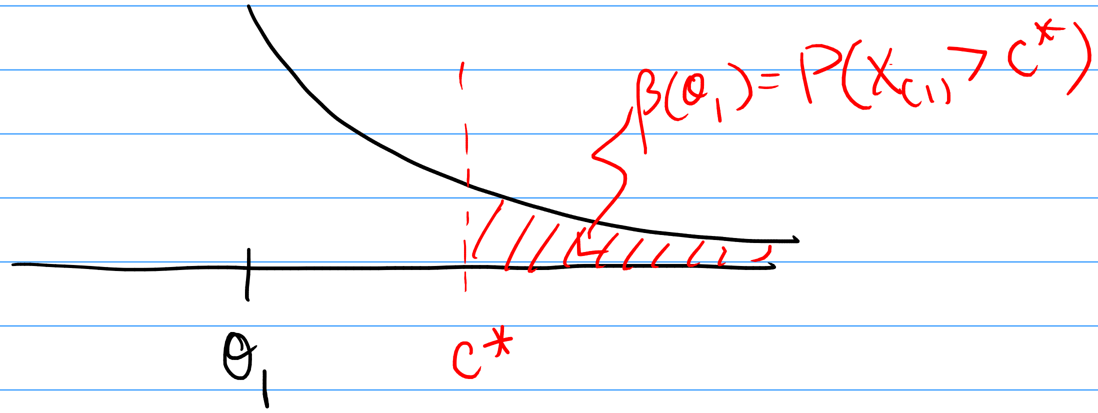
Turns out that if $X_n \overset{iid}{\sim} \text{Shifted Exp}(1, \theta)$
then $X_{(1)} \sim \text{shifted Exp}(N, \theta)$



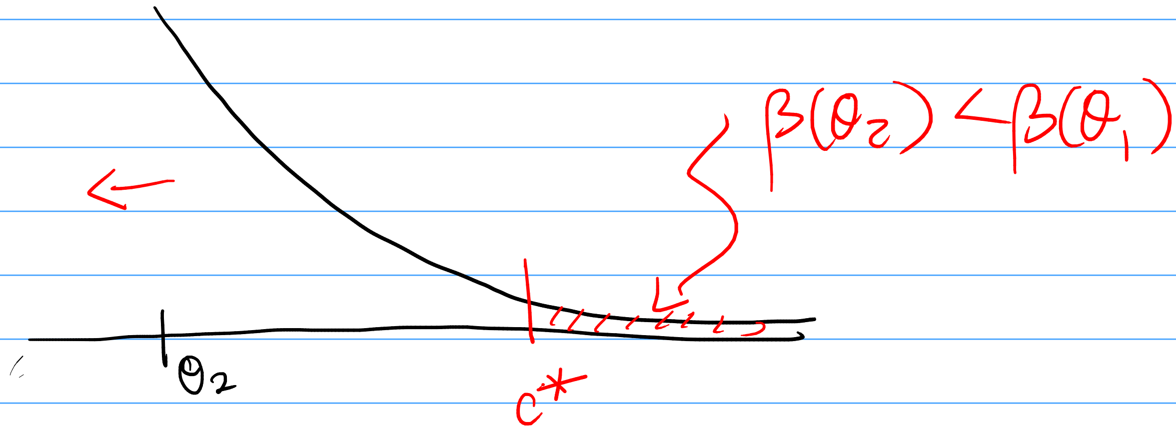
Want to show that β is increasing in θ

If $\theta_1 > \theta_2$ then $\beta(\theta_1) > \beta(\theta_2)$

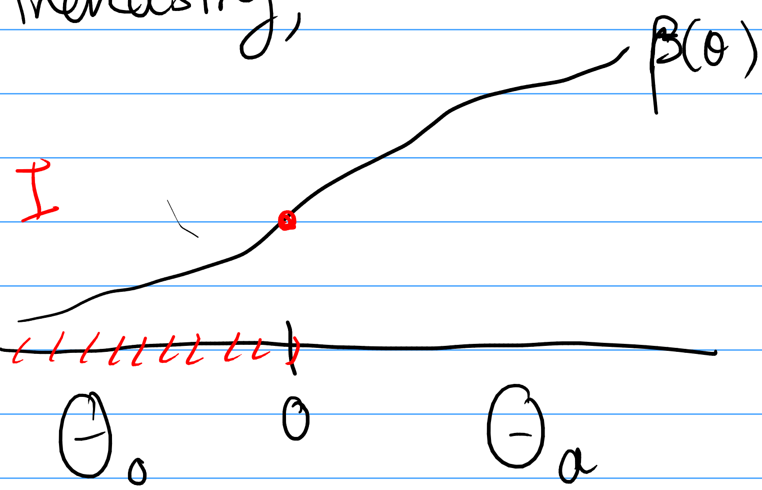
$\underline{\theta_1}$:



$\underline{\theta_2}$



So β is increasing,



So max type I err prob is at $\theta = 0$

and is

$\beta(0) = \mathbb{P}(X_{(1)} > c^*)$ when $X_{(1)} \sim Exp(N)$

So want $\beta(0) = \alpha$

$$\iff P(X_{(1)} > c^*) = \alpha$$

$$\iff P(X_{(1)} \leq c^*) = 1-\alpha$$

$$\iff F_{X_{(1)}}(c^*) = 1-\alpha$$

$$\iff 1-e^{-Nc^*} = 1-\alpha$$

$$\iff e^{-Nc^*} = \alpha$$

$$\iff \boxed{c^* = -\tfrac{1}{N}\log\alpha}$$

CDF of $Exp(\lambda)$ is $1-e^{-\lambda x}$

<u>Defn:</u> Uniformly Most Powerful (UMP) test

If $\mathcal{C}$ is a class of tests for

$$H_0: \theta \in \Theta_0 \quad v. \quad H_a: \theta \in \Theta_a$$

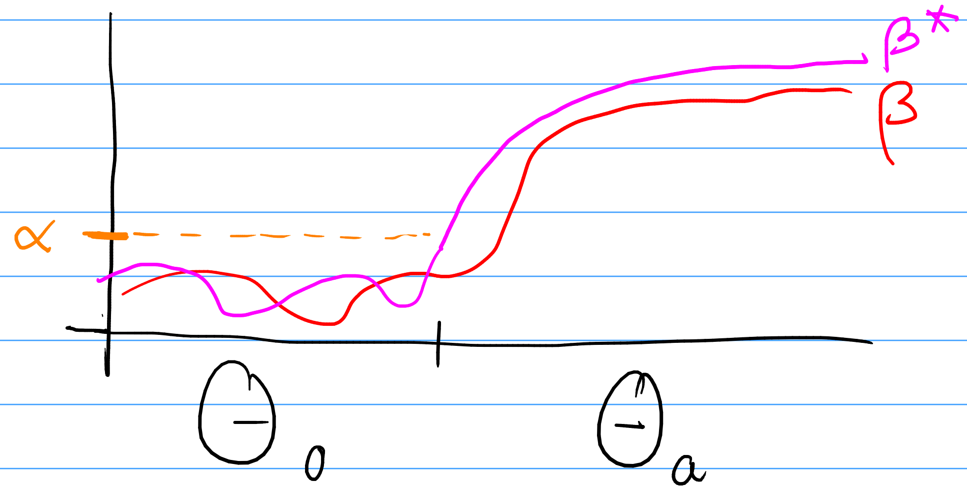
the test with a power function β^* is called the UMP test for this class if

$$\beta^*(\theta) \geq \beta(\theta) \quad \forall \theta \in \Theta_a$$

for any other test in $\mathcal{C}$ w/ power fn β.

UMP level-α test: the UMP test among all tests where

$$\max_{\theta \in \Theta_0} \beta(\theta) \le \alpha$$



Consider the simple hypothesis

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta = \theta_a$$

So $\Theta = \{\theta_0, \theta_a\}$, $\Theta_0 = \{\theta_0\}$, $\Theta_a = \{\theta_a\}$

In this case the LRT is

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \begin{cases} 1 = L(\theta_0)/L(\theta_0) & L(\theta_0) > L(\theta_a) \\ L(\theta_0)/L(\theta_a) & L(\theta_0) < L(\theta_a) \end{cases}$$

Really only care about second case, so basically LRT

$$\lambda = L(\theta_0)/L(\theta_a)$$

and I reject when $\lambda \leq C$

$$\lambda \leq C \iff L(\theta_0)/L(\theta_a) \leq C \iff L(\theta_0) \leq C L(\theta_a)$$

alt if $k = 1/c > 1$

then the LRT rejects if $L(\theta_a) \geq k L(\theta_0)$