Lecture 4: Ancillary Statistics

Theorem: Sufficiency and Exp. Fams

If $X_n \overset{iid}{\sim} f_\theta$ and

$$f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})c(\theta)\exp(T(\underset{\sim}{x})w(\theta))$$

then $T = T(\underset{\sim}{X})$ is sufficient for θ.

Ex: $X_n \overset{iid}{\sim} \mathrm{Pois}(\lambda)$

Exp. Fam.

$$f(\underset{\sim}{x}) = \underbrace{\left(\prod_n \frac{1}{x_n!}\right)\prod_n \mathbb{1}(x_n \in \mathbb{N}_0)}_{h(\underset{\sim}{x})} \underbrace{e^{-N\lambda}}_{c(x)} \exp\Big(\underbrace{\left(\textstyle\sum_n x_n\right)}_{T(\underset{\sim}{x})}\underbrace{\lg\lambda}_{w(\lambda)}\Big)$$

So $T = \sum_n X_n$ is sufficient for λ

Pf. of theorem.

$$f_\theta(\underset{\sim}{x}) = \underbrace{h(\underset{\sim}{x})}_{h(\underset{\sim}{x})} \underbrace{c(\theta)\exp(\overbrace{T(\underset{\sim}{x})}^{T(\underset{\sim}{x})}w(\theta))}_{g(T,\theta)}$$

$$g(t,\theta) = c(\theta)\exp(t\,w(\theta))$$

then by Factorization theorem, T is a SS.

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$

$$f(x_n) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_n-\mu)^2\right) \qquad e^{a+b} = e^a e^b$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_n^2 - 2\mu x_n + \mu^2)\right)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x_n^2\right)}_{h_0(x_n)} \exp\Big(\overset{w(\mu)}{\mu} \underbrace{x_n}_{T_0(x_n)}\Big) \underbrace{\exp\left(-\frac{1}{2}\mu^2\right)}_{c_0(\mu)}$$

So

$$f(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\mu) \exp\left(T(\underset{\sim}{x})\, w(\mu)\right)$$

$$h(\underset{\sim}{x}) = \prod_n h_0(x_n)$$

$$c(\mu) = \prod_n c_0(\mu) = \exp\left(-N/2\, \mu^2\right)$$

$$T(\underset{\sim}{x}) = \sum_n T_0(x_n) = \sum_n x_n .$$

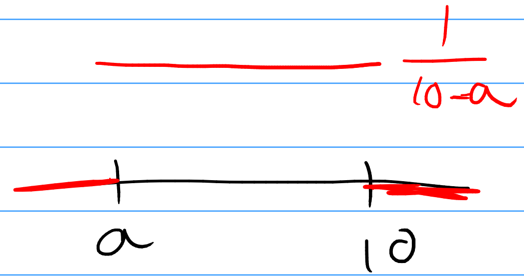
By prev. theorem, $T = \sum_n x_n$ is sufficient for μ.

Theorem: Any invertible function of a SS is also a SS.

Ex. $\bar{X}$ is sufficient for μ, above.

Ex. Let $X_n \overset{iid}{\sim} U(a, 10)$ where $0 < a < 10$

Find a SS for a.



Not an exp. fam.
since parameter affects support

$$f(\underset{\sim}{x}) = \prod_n f(x_n) = \prod_n \frac{1}{10-a} \mathbb{1}(a < x_n < 10)$$

$$= \left(\frac{1}{10-a}\right)^N \prod_n \mathbb{1}(X_n > a) \prod_n \mathbb{1}(X_n < 10)$$

$$= \overbrace{\left(\frac{1}{10-a}\right)^N \mathbb{1}(\underbrace{X_{(1)}}_{T} > a)}^{g(T,a)} \underbrace{\mathbb{1}(X_{(N)} < 10)}_{h(\underset{\sim}{x})}$$

$g(t,a) = \left(\frac{1}{10-a}\right)^N \mathbb{1}(t > a)$

$$= h(\underset{\sim}{x})\, g(T(\underset{\sim}{x}), a)$$

$$\prod_n \mathbb{1}(X_n > a) = \mathbb{1}(X_n > a \text{ for all } n)$$

$$= \mathbb{1}(X_{(1)} > a)$$

Defn: Statistic If $X_n \overset{iid}{\sim} f_\theta$ then a statistic T is a function of the X_ns that doesn't involve the unknown parameter θ in its formula.

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$

no μ in formula

then $T = \overline{X}$ is a statistic.

note! $\overline{X} \sim N(\mu, 1/N)$

not allowed

Ancillary Quantity

however $T = \overline{X} - \mu$ is not a statistic.

note! $T \sim N(0, 1/N)$

Defn! Ancillary Quantity $X_n \overset{iid}{\sim} f_\theta$.

An ancillary quantity Q is a fn of the data whose dist. doesn't involve θ.

Defn! Ancillary Statistic

is a stat. T that is also ancillary.

[No θ in formul for T, no θ in dist. of T.]

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$

$R = X_{(N)} - X_{(1)}$.

is a statistic, no μ in the formula

note! $X_n \overset{d}{=} \mu + Z_n$ where $Z_n \overset{iid}{\sim} N(0, 1)$

and $X_{(N)} = \mu + Z_{(N)}$

$X_{(1)} = \mu + Z_{(1)}$

So $R = X_{(N)} - X_{(1)}$

$= (\mu + Z_{(N)}) - (\mu + Z_{(1)})$

$= \cancel{\mu} + Z_{(N)} - \cancel{\mu} - Z_{(1)}$

$= Z_{(N)} - Z_{(1)} \leftarrow$ dist. doesn't depend on μ

So it's ancillary

Hence R is an ancillary statistic.

<u>Theorem</u>: Basu's Theorem

If T is a SS for θ and S is a ancillary stat for θ then

$$T \perp\!\!\!\perp S$$

<u>Theorem</u>: $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then $\bar{X} \perp\!\!\!\perp S^2$.

<u>pf.</u> $\bar{X}$ is sufficient for μ

S^2 is ancillary to μ

then Basu's theorem says $\bar{X} \perp\!\!\!\perp S^2$.

$\rightarrow$ <u>said</u>: $\frac{N-1}{\sigma^2} S^2 \sim \chi^2(N-1)$

no μ, ancillary $S^2 \sim \frac{\sigma^2}{N-1} \chi^2(N-1)$

Point Estimation

Setup: $X_n \overset{iid}{\sim} f_\theta$ where $\theta \in \Theta$

Defn: A point estimator of θ is a statistic
$$\hat{\theta} = \hat{\theta}(\underset{\sim}{X})$$
Hopefully $\hat{\theta} \approx \theta$.

Goals: ① How do I build $\hat{\theta}$?
② How do I know if its good?

First approach: Method of Moments (MoM)

Defn: the r^{th} moment of a RV X is
$$\mu_r = \mathbb{E}[X^r].$$

Defn: the r^{th} sample moment is
$$m_r = \frac{1}{N}\sum_{n=1}^{N} x_n^r$$

notice: $\mathbb{E}[m_r] = \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N} X^r\right]$

$$= \frac{1}{N}\sum_{n=1}^{N} \mathbb{E}[X^r]$$

on avg. m_r gives us μ_r

$$= \frac{1}{N}\sum_{n=1}^{N} \mu_r$$

$$= \mu_r$$

A reasonable strategy is to assume

$$m_r \approx \mu_r .$$

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ both unknown

Want ests $\hat{\mu}$ and $\hat{\sigma}^2$.

Look at moments:

$\mu_1 = \mathbb{E}[X_n] = \mu$

$\mu_2 = \mathbb{E}[X_n^2] = Var(X_n) + \mathbb{E}[X_n]^2$

$= \sigma^2 + \mu^2$

Set

$$\left.\begin{aligned} \frac{1}{N}\sum_n X_n &= \boxed{m_1 = \mu_1} = \mu \\ \overline{X^2} = \frac{1}{N}\sum_n X_n^2 &= \boxed{m_2 = \mu_2} = \sigma^2 + \mu^2 \end{aligned}\right\} \text{system of eqns}$$

So we can solve this system of eqns for μ and σ^2.

$$\overline{X} = \mu$$

$$\overline{X^2} = \sigma^2 + \mu^2$$

Then solve for μ and σ^2

$$\boxed{\hat{\mu} = \overline{X}}$$

$$\boxed{\hat{\sigma}^2 = \overline{X^2} - \hat{\mu}^2 = \overline{X^2} - \overline{X}^2}$$

$$= \frac{1}{N} \sum_{n=1}^{N} (X_n - \overline{X})^2$$

denom of $1/N$

MoM: (1) Find pop. moments $E[X^r]$

(2) Set equal to sample moments

(3) Solve sys of eqns for unknown params.

Ex. Let $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

Lets get MoM est. $\hat{\lambda}$.

(1) Pop. moments: $\mathbb{E}[X_n] = \lambda$

(2) Set equal to sample moments: $m_1 = \overline{X} = \lambda$

(3) Solve for λ: $\boxed{\hat{\lambda} = \overline{X}}$