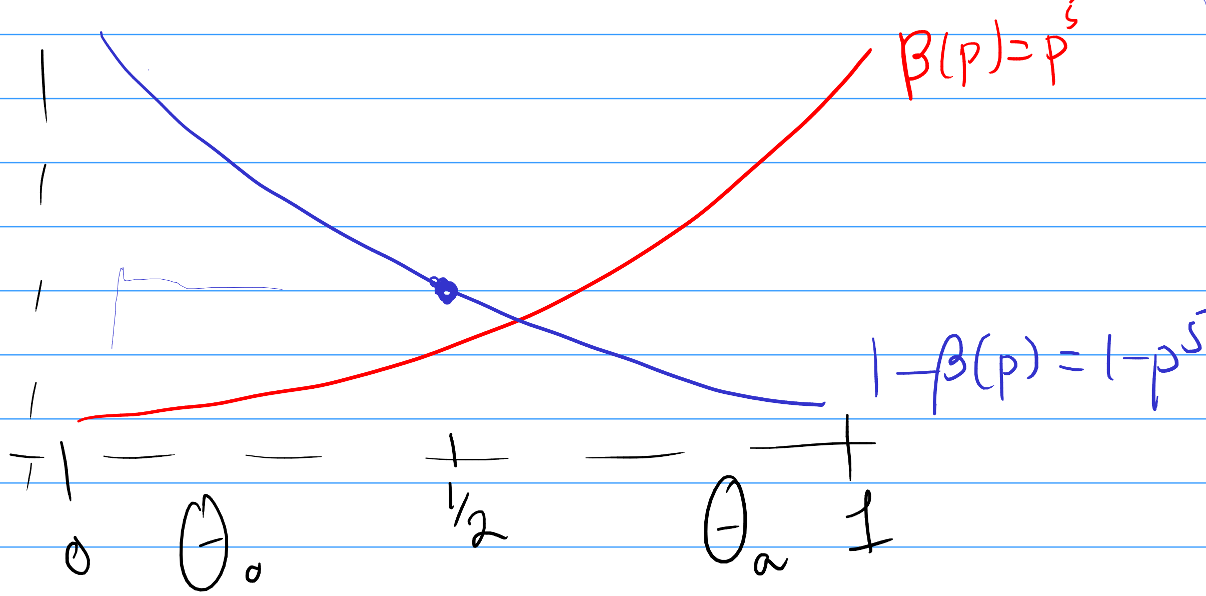
Lecture 16

What's max type II error prob?

when $\theta \in \Theta_a$ then $1-\beta(\theta) = $ prob. type II

so
$$\max_{\theta \in \Theta_a} 1-\beta(\theta) = \max_{p > 1/2} 1-p^5 = 1-\left(\tfrac{1}{2}\right)^5 = 31/32$$



Consider another test:

$$R = \{ \underset{\sim}{x} \mid T \geq 3 \}$$
$$= \{ \underset{\sim}{x} \mid T/5 = \bar{X} \geq 1/2 \}$$

What's the power function?

$$\beta(p) = \mathbb{P}_p(\underset{\sim}{X} \in R) = \mathbb{P}(T \geq 3)$$

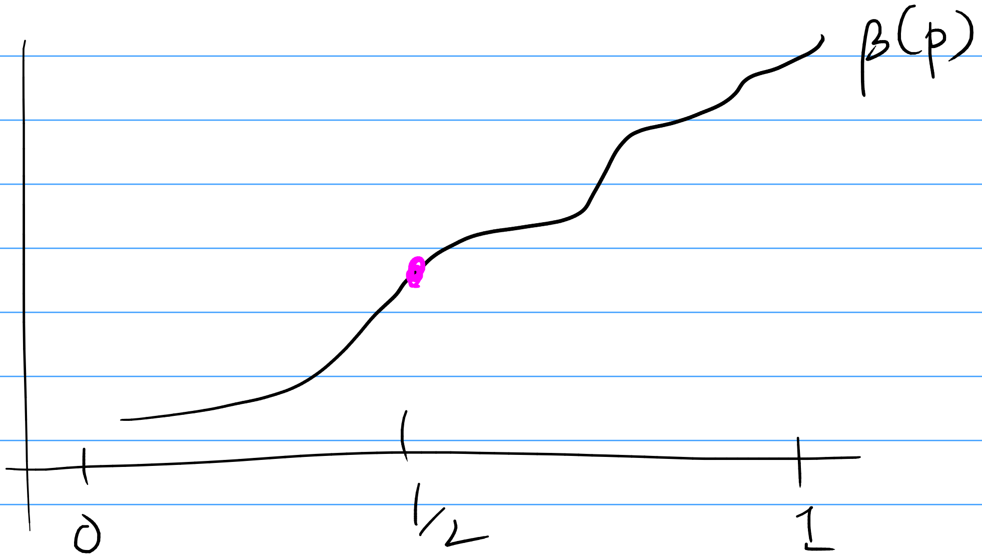
$$= \mathbb{P}(T=3) + \mathbb{P}(T=4) + \mathbb{P}(T=5)$$

$$= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5$$

$$= p^3 (6p^2 - 15p + 10)$$

Notice: $\frac{\partial \beta}{\partial p} = 30 p^2 (p-1)^2 > 0$

So β is increasing.



① max type I err prob. $= \beta(1/2)$

② max type II err. prob. $= 1 - \beta(1/2)$

Defn: Size and level α test

A test is size $\alpha \in [0,1]$ if

$$\alpha = \max_{\theta \in \Theta_0} \beta(\theta) = \text{max type I err prob.}$$

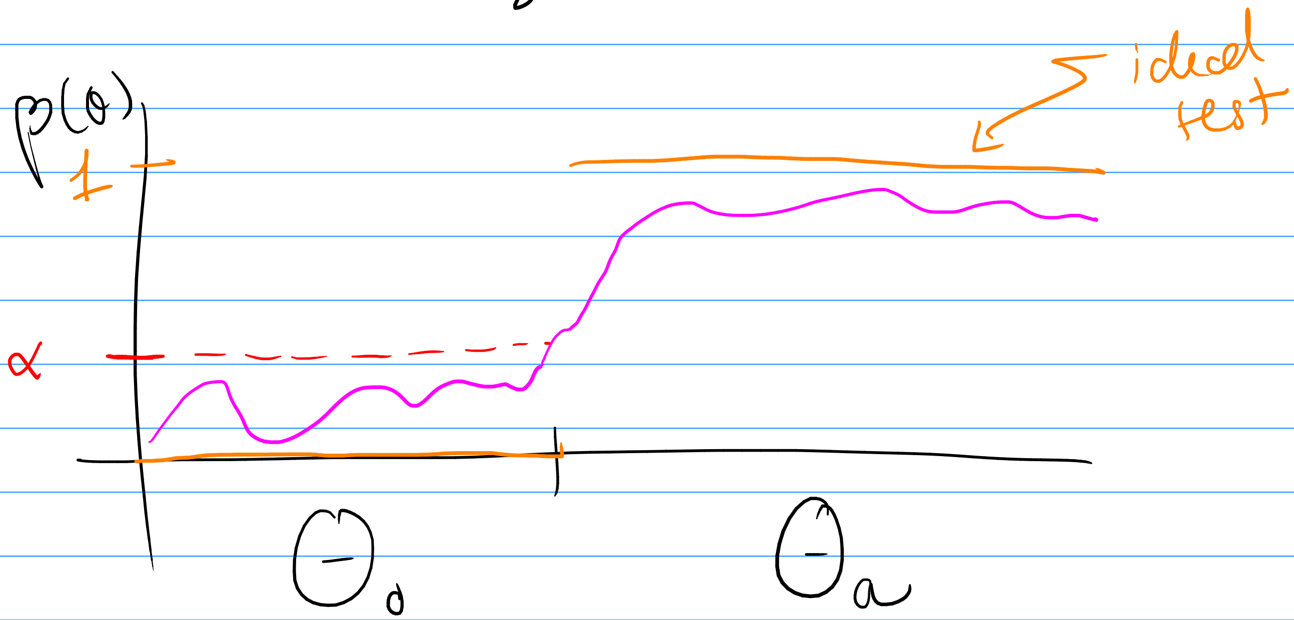
a <u>level</u> α test is one where

$$\max_{\theta \in \Theta_0} \beta(\theta) \le \alpha$$

<u>Game:</u> try to find test w/ maximum power

max $\beta(\theta)$ when $\theta \in \Theta_a$

s.t. it is size α or level α.



<u>Likelihood Ratio Test</u>

Want to test

$$H_0: \theta \in \Theta_0 \quad v. \quad H_a: \theta \in \Theta_a$$

We define the <u>likelihood ratio test statistic</u> (LRT)

as

$$\lambda(\underset{\sim}{x}) = \frac{\max\limits_{\theta\in\Theta_0} L(\theta)}{\max\limits_{\theta\in\Theta} L(\theta)} = \frac{\text{max val. of } L(\theta), \ \theta \text{ in null}}{\text{max val. of } L(\theta) \text{ overall}}$$

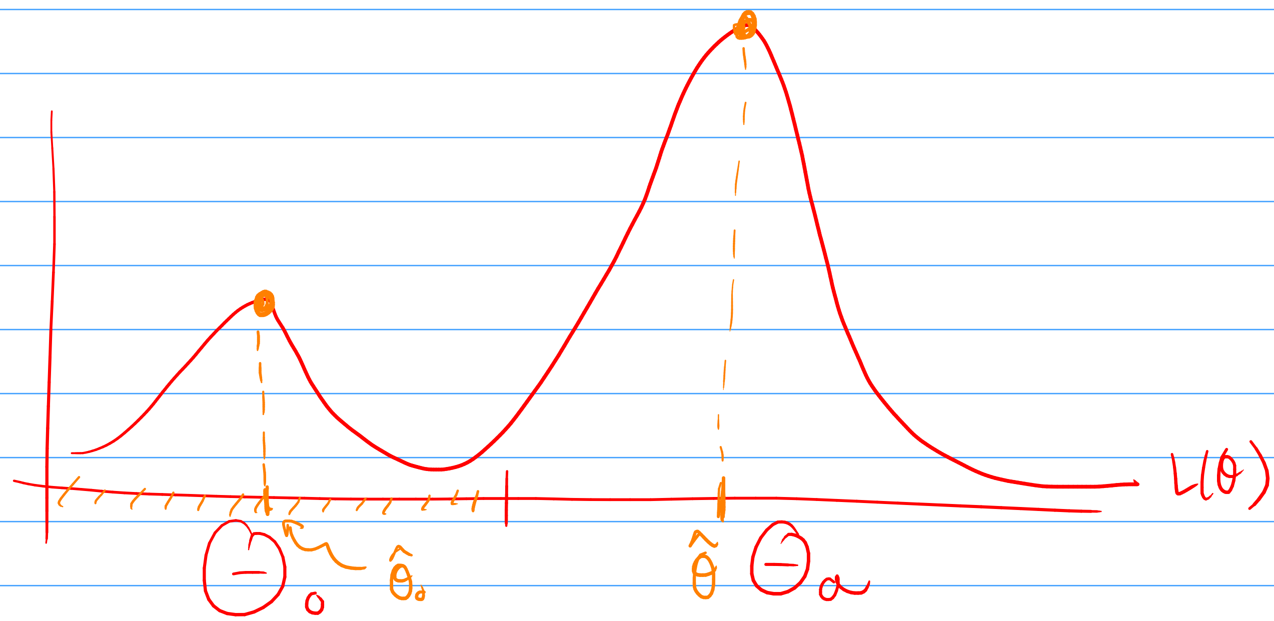
$$= \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}$$

$$\le 1$$

$\hat{\theta}_0$ = MLE restricted to Θ_0

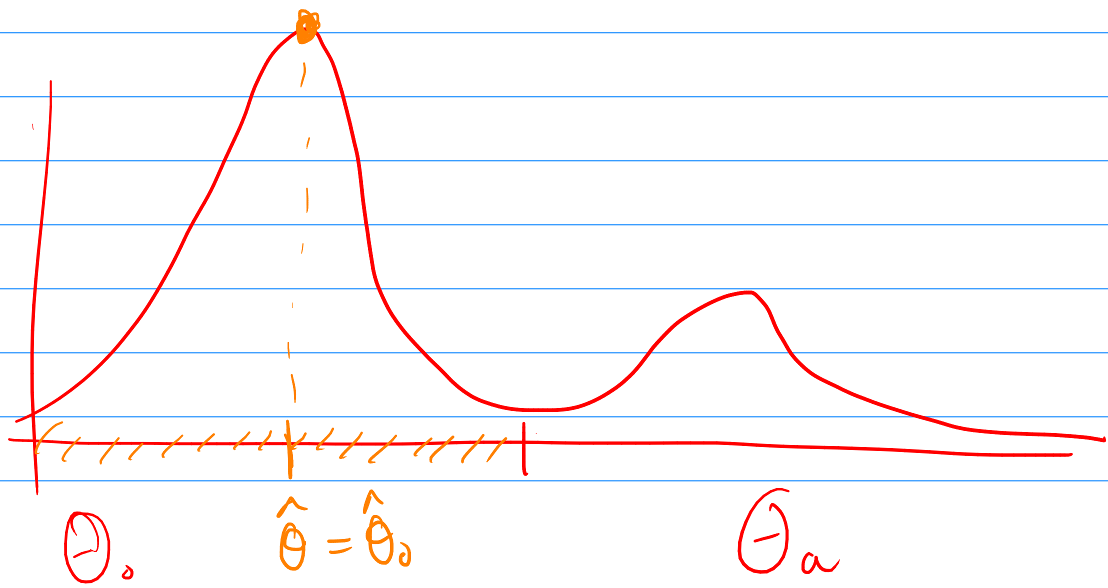
$\hat{\theta}$ = MLE

Ex.



$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} \ll 1 \rightsquigarrow \text{probably reject}$$

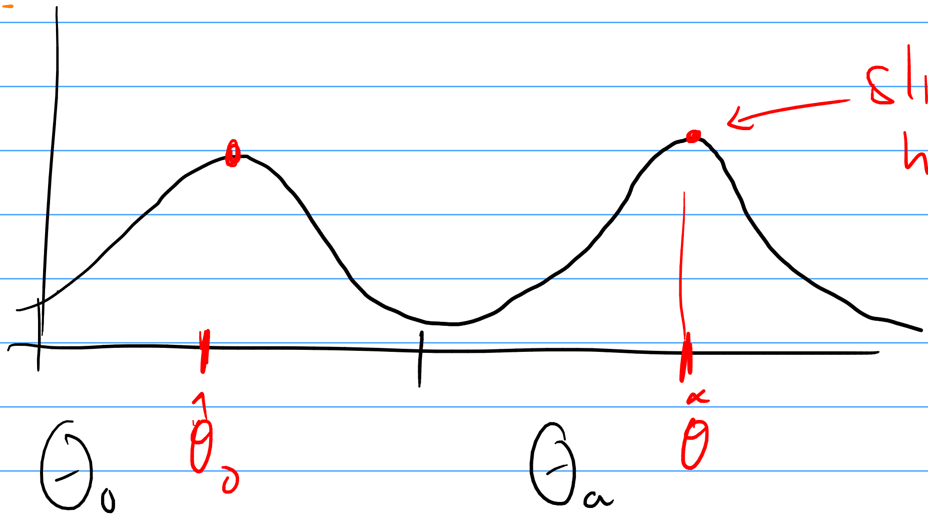
Ex.



So $\lambda = \dfrac{L(\hat\theta_0)}{L(\hat\theta)} = \dfrac{L(\hat\theta)}{L(\hat\theta)} = 1$

↑ don't reject

hard case:



$\lambda = L(\hat\theta_0)/L(\hat\theta) < 1$ but ≈ 1

The LRT says I should reject if

$$\lambda = \frac{L(\hat\theta_0)}{L(\hat\theta)} \le c$$

↑ $0 \le c \le 1$

c is some threshold I choose to trade off Type I and Type II err probs.

c small, don't reject easily, less type I
more type II

c large, reject easily, more type I
less type II

rejection region for LRT

$$R = \{ \underset{\sim}{x} : \lambda(\underset{\sim}{x}) \leq c \}$$

<u>Ex.</u> $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ Known

$$H_0 : \theta \leq a \quad v. \quad H_a : \theta > a$$

Let's derive the LRT

$$L(\theta) = \prod_n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_n - \theta)^2\right)$$

$$= \cdots = (2\pi)^{-N/2} (\sigma^2)^{-N/2} \exp\left(-\frac{1}{2\sigma^2} \sum_n (x_n - \theta)^2\right)$$

kinda quadratic in θ

$\approx e^{-\theta^2}$

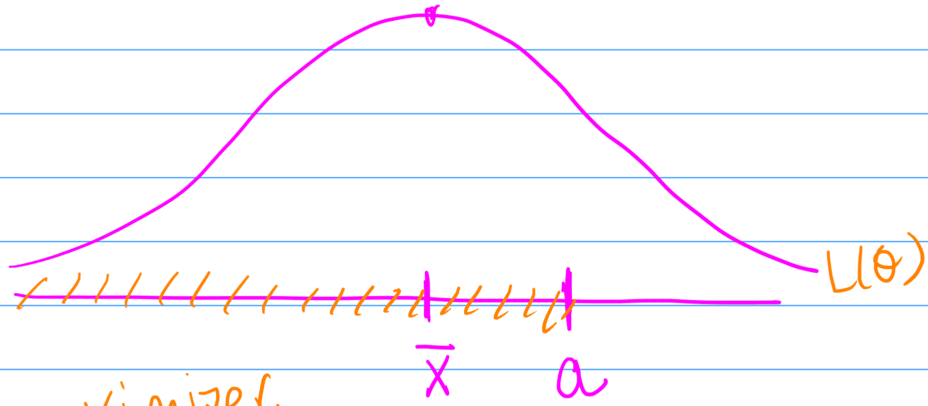
$$\boxed{\bar{X} = \hat{\theta}}$$

Need to find $\hat{\theta}_0$, $\Theta_0 = [-\infty, a]$

Two cases

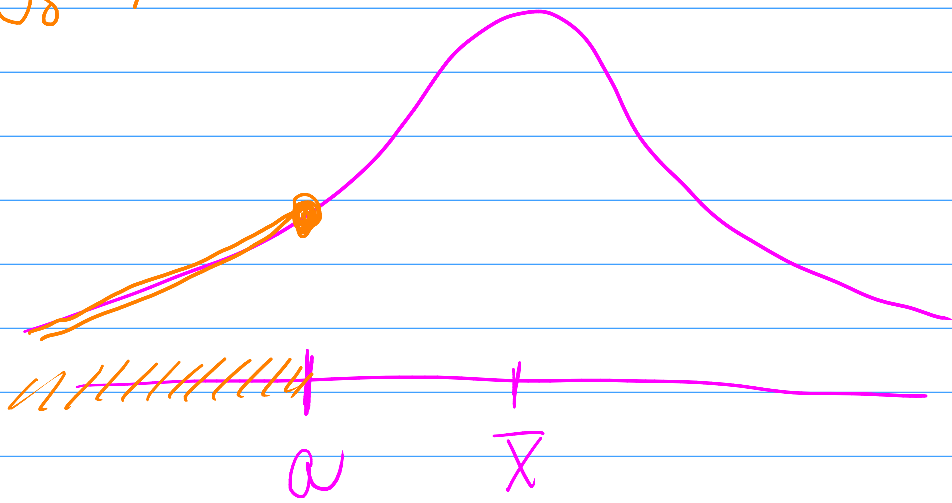
$\hat{\theta}_0 = \underset{\theta \le a}{\operatorname{argmax}}\, L(\theta)$

$\bar{X} \le a$



Since $\bar{X} \le a$ and it is the over-all maximizer, then $\hat{\theta}_0 = \bar{X}$

$\bar{X} > a$



$\hat{\theta}_0 = a$

Overall,

$$\boxed{\hat{\theta}_0 = \begin{cases} \bar{x} & \bar{x} \le a \\ a & \bar{x} > a \end{cases}}$$

$H_0: \theta \le a$

So.

$$\lambda(\underset{\sim}{x}) = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \begin{cases} \dfrac{L(\bar{x})}{L(\bar{x})} = 1 \ \text{(never reject)} & \bar{x} \le a \\[2ex] \dfrac{L(a)}{L(\bar{x})} & \bar{x} > a \end{cases}$$

LRT says, reject if $\lambda \le c$

Case $\bar{X} > a$

$$\lambda = \frac{L(a)}{L(\bar{X})} = \frac{\cancel{(2\pi\sigma^2)^{-N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_n (X_n - a)^2\right)}{\cancel{(2\pi\sigma^2)^{-N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_n (X_n - \bar{X})^2\right)}$$

$$= \frac{\exp\left(-\frac{1}{2\sigma^2}\sum_n (X_n^2 - 2aX_n + a^2)\right)}{\exp\left(-\frac{1}{2\sigma^2}\sum_n (X_n^2 - 2\bar{X}X_n + \bar{X}^2)\right)}$$

$$= \frac{\exp\left(-\frac{1}{2\sigma^2}\left(\cancel{N\overline{X^2}} - 2aN\bar{X} + Na^2\right)\right)}{\exp\left(-\frac{1}{2\sigma^2}\left(\cancel{N\overline{X^2}} - 2N\bar{X}^2 + N\bar{X}^2\right)\right)}$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(-2aN\bar{X} + Na^2 + 2N\bar{X}^2 - N\bar{X}^2\right)\right)$$

$$= \exp\left(-\frac{N}{2\sigma^2}\left(\underbrace{-2a\bar{X} + a^2 + \bar{X}^2}\right)\right)$$

$$\lambda = \exp\left(-\frac{N}{2\sigma^2}(\bar{X} - a)^2\right)$$

LRT says reject when $\lambda \leq c$

$$\Leftrightarrow \quad \exp\left(-\frac{N}{2\sigma^2}(\bar{X} - a)^2\right) \leq c$$

$$\Leftrightarrow \quad -\frac{N}{2\sigma^2}(\bar{X} - a)^2 \leq \log(c)$$

$$\Leftrightarrow \quad \frac{N}{\sigma^2}(\bar{X} - a)^2 \geq -2\log c$$

assuming $\bar{X} > a \iff \bar{X} - a > 0$

$$\iff \frac{\sqrt{N}}{\sigma}(\bar{X} - a) \geq \sqrt{-2\log c}$$

$$\iff \frac{\bar{X} - a}{\sigma/\sqrt{N}} \geq \sqrt{-2\log c} \quad c^*$$

$$\iff \boxed{\bar{X} \geq a + \frac{\sigma}{\sqrt{N}} c^*}$$

$H_0: \theta \leq a$

$H_a: \theta > a$

reject if $\bar{X} \leq a + c^* \, s.d.s(\bar{X})$