Lecture 9: CRLB

Theorem: Cramér-Rao Lower Bound (CRLB)

If $X_n \overset{iid}{\sim} f_\theta$ for $\theta \in \Theta$ and if $\hat{\theta}$ is an unbiased est. of $\tau(\theta)$

(*) and if f_θ is nice enough

read: 1-dim'l exp. fams.

↳ θ is 1-dim'l e.g. $\theta \in \mathbb{R}$

then

$$\mathrm{Var}(\hat{\theta}) \geq \left(\frac{\partial \tau}{\partial \theta}\right)^2 \Big/ I_N(\theta) = B$$

Cramér-Rao lower bound

Notes:

(1) If $\tau(\theta) = \theta$ then $\frac{\partial \tau}{\partial \theta} = 1$

so $B = 1/I_N(\theta)$.

(2) If I can find some est. θ^* that is

(1) unbiased for $\tau(\theta)$

and (2) $\mathrm{Var}(\theta^*) = B$

then θ^* is the UMVUE.

(3) If I have an est. $\hat{\theta}$ that is unbiased for $\tau(\theta)$ but its variance is

$$\mathrm{Var}(\hat{\theta}) > B.$$

I don't know if $\hat{\theta}$ is the UMVUE or not.

Ex. $X_n \overset{iid}{\sim} \mathrm{Pois}(\lambda)$ ← exp. fam

Let $\hat{\lambda} = \bar{X}$ and $\tau(\lambda) = \lambda$

Know $E\hat{\lambda} = \lambda$, $\mathrm{Var}(\bar{X}) = \lambda / N$

↑ $\hat{\lambda}$ unbiased for λ

$I_N(\lambda) = N/\lambda$

so $\mathrm{Var}(\hat{\lambda}) = \lambda/N = 1/I_N(\lambda) = B$

So $\hat{\lambda}$ achieves the CRLB, and so it is the UMVUE.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ ← an exp. fam

$\hat{\mu} = \bar{X}$ then $E\hat{\mu} = \mu$ so its unbiased for $\tau(\mu) = \mu$

and $Var(\bar{X}) = \sigma^2/N$

$I_N(\mu) = N/\sigma^2$

So since $Var(\bar{x}) = CRLB$

then $\bar{X}$ is the UMVUE.

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$ ← is an exp. fam

Let $\tau(\lambda) = 1/\lambda$

let $T = \bar{X}$ then $ET = 1/\lambda$

So T is unbiased for $1/\lambda$

Let's get $I_N(\lambda)$

① get $\log f_\lambda(x) = \log(\lambda e^{-\lambda x})$

$= -\lambda x + \log(\lambda)$

② $\frac{\partial \log f_\lambda}{\partial \lambda} = -x + 1/\lambda$

$\frac{\partial^2 \log f_\lambda}{\partial \lambda^2} = -1/\lambda^2$

③ $I(\lambda) = -\mathbb{E}\left[\frac{\partial^2 \log f_\lambda(X)}{\partial\lambda^2}\right]$

$= -\mathbb{E}\left[-1/\lambda^2\right] = 1/\lambda^2$ $\quad \left\{ \begin{array}{l} I_N(\lambda) \\ = N/\lambda^2 \end{array} \right.$

$Var(T) = Var(\bar{X}) = (1/\lambda^2)/N = 1/\lambda^2 N$

$B = \frac{\left(\frac{\partial \tau}{\partial\lambda}\right)^2}{I_N(\lambda)} = \frac{\left(-1/\lambda^2\right)^2}{N/\lambda^2}$

$\tau(\lambda) = 1/\lambda$

$\frac{\partial\tau}{\partial\lambda} = -1/\lambda^2$

$= \frac{\lambda^2}{N\lambda^4} = \frac{1}{N\lambda^2} = Var(T)$

So $T = \bar{X}$ is the UMVUE for $\tau(\lambda) = 1/\lambda$.

Ex, Let $X_n \overset{iid}{\sim} U(0,\theta)$

Want UMVUE for $\tau(\theta) = \theta$.

① Unbiased est?

Can show that $\mathbb{E}[X_{(N)}] = \frac{N}{N+1}\theta$

So if $\hat{\theta} = \frac{N+1}{N} X_{(N)}$ then $\mathbb{E}[\hat{\theta}] = \theta$

So $\hat{\theta}$ is unbiased.

(2) Calc. var. $\mathrm{Var}(\hat{\theta}) = \frac{\theta^2}{N(N+2)}$

(3) Show $\mathrm{Var}(\hat{\theta}) = B$.

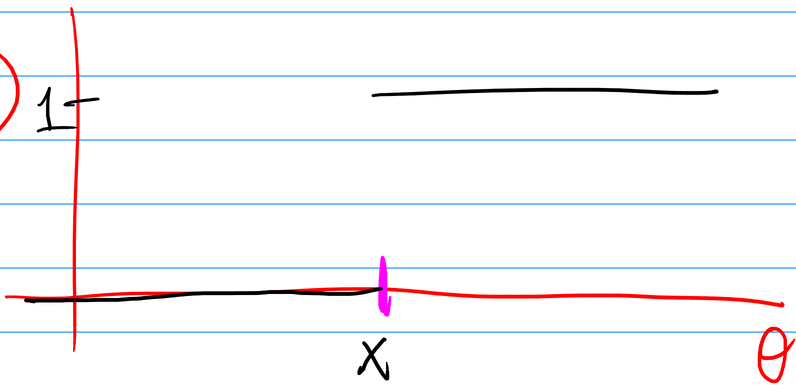
Need $I(\theta)$

$$f_\theta(x) = \frac{1}{\theta}\mathbb{1}(0<x<\theta)$$

$$\log f_\theta(x) = -\log(\theta) + \log \mathbb{1}(0<x<\theta)$$

$\frac{\partial}{\partial\theta}$ $U(0,\theta)$ NOT exp. fam.

$\mathbb{1}(0<x<\theta)$



Review:

Theorem: Iterated Expectation

a number

Background: $\underline{\mathbb{E}[X \mid Y = y]} = \int x f(x|y)\,dx = g(y)$

Can plug Y into g to get $g(Y)$

notation: $g(Y) = \underline{\mathbb{E}[X \mid Y]}$ a random variable

Iterated Expectation

$$E[X] = E_Y[E[X|Y]]$$

$$= E_Y[g(Y)]$$

Law of Total Variance

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y]).$$

Ex. $\quad X|Y=y \sim Bin(y,p) \qquad p \in [0,1]$

$Y \sim Pois(\lambda), \quad \lambda > 0$

EX? ① $E[X|Y=y] = yp$

② $E[X|Y] = Yp$

③ $E[E[X|Y]] = E[Yp] = pEY = p\lambda$

$Var(X)? = \underbrace{EVar(X|Y)} + \underbrace{Var\,E[X|Y]}$

① $Var(X|Y=y) = yp(1-p)$

② $Var(X|Y) = Yp(1-p)$

③ $EVar(X|Y) = p(1-p)EY = p(1-p)\lambda$

$Var(Yp) = p^2 Var(Y)$
$= p^2\lambda$

$$\text{Var}(X) = p(1-p)\lambda + p^2\lambda = p\lambda$$

Some facts:

① If $\hat{\theta}$ is unbiased for $\tau(\theta)$

$$E\hat{\theta} = \tau(\theta)$$

Let W be some function of $\underset{\sim}{X}$

(could be a stat - maybe not)

Consider

is a RV and is a fn of W

$$\varphi = \varphi(W) = E[\hat{\theta} \mid W]$$

notice that

iterated expectation

$$E_W\varphi = E_W[E[\hat{\theta} \mid W]] = E[\hat{\theta}] = \tau(\theta)$$

If φ is a stat (no θ in formula) then

$\Leftarrow$ φ is unbiased for $\tau(\theta)$.

② $\underline{\text{Var}(\varphi) \leq \text{Var}(\hat{\theta})}$.

$$\text{Var}(\hat{\theta}) = \text{Var}(\underbrace{E[\hat{\theta} \mid W]}_{\varphi}) + \underbrace{E(\text{Var}(\hat{\theta} \mid W))}_{\geq 0}$$

$$\text{Var}(\hat{\theta}) = \text{Var}(\varphi) + \text{something} \geq 0$$

So $\mathrm{Var}(\hat{\theta}) \geq \mathrm{Var}(\varphi)$

<u>Ex.</u> $X_n \overset{iid}{\sim} N(\theta, 1)$

Let $\hat{\theta} = \frac{1}{2}(X_1 + X_2)$

<u>Note:</u> $\mathbb{E}\hat{\theta} = \frac{1}{2}(\mathbb{E}X_1 + \mathbb{E}X_2) = \frac{1}{2}(\theta+\theta) = \theta$

so $\hat{\theta}$ is unbiased for $\tau(\theta) = \theta$.

$$\mathrm{Var}(\hat{\theta}) = \frac{1}{4}(\mathrm{Var}\,X_1 + \mathrm{Var}\,X_2) = \frac{1}{4}(1+1)$$
$$= 1/2$$

Let $W = X_1$

$$\varphi = \mathbb{E}[\hat{\theta} \mid W] = \mathbb{E}\left[\tfrac{1}{2}(X_1 + X_2) \mid X_1\right]$$

$$\boxed{\begin{array}{l} \mathbb{E}[z \mid z=5] = 5 \\ \mathbb{E}[z \mid z=3] = 3 \\ \mathbb{E}[z \mid z] = z \end{array}}$$

$$= \frac{1}{2}\mathbb{E}[X_1 \mid X_1] + \frac{1}{2}\mathbb{E}[X_2 \mid X_1]$$
$$= \frac{1}{2}X_1 + \frac{1}{2}\mathbb{E}[X_2]$$
$$= \frac{1}{2}X_1 + \theta/2$$

$$\mathbb{E}[\varphi] = \frac{1}{2}\mathbb{E}X_1 + \theta/2 = \theta/2 + \theta/2 = \theta = \mathbb{E}[\hat{\theta}]$$

$$\mathrm{Var}(\varphi) = \frac{1}{4}\mathrm{Var}(X_1) = \frac{1}{4}(1) = \frac{1}{4}$$

Consider instead $W = \overline{X}$.

$$\varphi = \mathbb{E}[\hat{\theta} \mid W] = \ ?$$