Lecture 5:

Ex.

Let $X_n \overset{iid}{\sim} N(\theta, 1)$ and lets find the MLE

(1) Let's get $L(\theta)$ or $\ell(\theta)$

$$L(\theta) = f_\theta(\underset{\sim}{X}) = \prod_n f_\theta(X_n) = \prod_n \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(X_n-\theta)^2\right)$$

$\log(a^b) = b\log(a)$

$\log(ab) = \log(a) + \log(b)$

$\log(e^a) = a$

$$= (2\pi)^{-N/2} \exp\left(-\frac{1}{2}\sum_n (X_n-\theta)^2\right)$$

$$\ell(\theta) = -\frac{N}{2}\log(2\pi) - \frac{1}{2}\sum_n (X_n-\theta)^2$$

(2) Find $\underset{\theta}{\operatorname{argmax}}\ \ell(\theta)$

this is a calc I problem.

(i) Find $\frac{\partial \ell}{\partial \theta}$ and set (ii) equal to zero

then solve for θ.

(i)

$$\frac{\partial \ell}{\partial \theta} = \frac{\partial}{\partial \theta}\left[-\frac{N}{2}\log(2\pi) - \frac{1}{2}\sum_n (X_n-\theta)^2\right]$$

$$= 0 - \frac{1}{2}\sum_n 2(X_n-\theta)(-1)$$

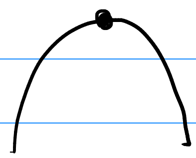
$$= \sum_n (X_n-\theta)$$

(ii) $$\frac{\partial \ell}{\partial \theta} = \sum_n (X_n - \theta) = 0$$

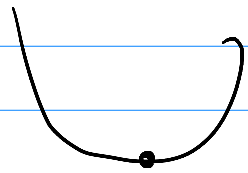
$$\Rightarrow \sum_n X_n - N\theta = 0$$

$$\Rightarrow \boxed{\hat{\theta}_{MLE} = \frac{1}{N} \sum_n X_n} = \overline{X}.$$

Technically, I need to check $\frac{\partial^2 \ell}{\partial \theta^2} < 0$



concave down

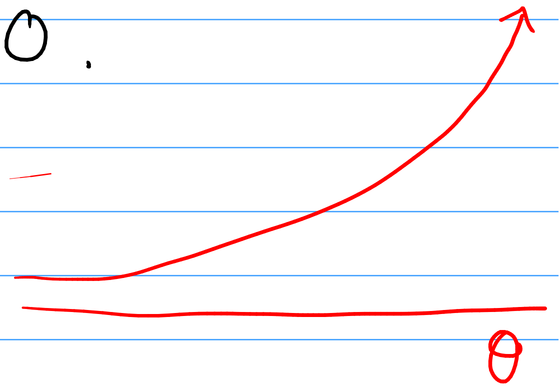


concave up

and check that $\lim_{\theta \to \pm\infty} L(\theta) = 0$.

For this example,

$$\frac{\partial^2 \ell}{\partial \theta^2} = -N < 0.$$



Theorem: MLEs are functions of SS

$$\hat{\theta}_{MLE} = \text{function}(T)$$

SS for θ.

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pf - Factorization Theorem

says if

$$L(\theta) = f_\theta(\underset{\sim}{X}) = g(T,\theta)\,h(\underset{\sim}{X})$$

then T is sufficient.

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, L(\theta) = \underset{\theta}{\operatorname{argmax}}\, g(T,\theta)\,h(\underset{\sim}{X})$$

$$= \underset{\theta}{\operatorname{argmax}}\, g(T,\theta)$$

$$= \text{function}(T)$$

$$\underset{x\in[0,1]}{\operatorname{argmax}}\, x^2 = \underset{x\in[0,1]}{\operatorname{argmax}}\, 5x^2$$

Ex. let $X_n \overset{iid}{\sim} \text{Bern}(p)$, $p \in [0,1]$.

Find the MLE.

1. Get $L(p)$ and $\ell(p)$

$$L(p) = f_p(\underset{\sim}{X}) = \prod_n f_p(x_n) = \prod_n p^{x_n}(1-p)^{1-x_n}\,\mathbb{1}(x_n = 0 \text{ or } 1)$$

$$= p^{\sum_n x_n}(1-p)^{N-\sum_n x_n} \prod_n \mathbb{1}(x_n = 0 \text{ or } 1)$$

$$\ell(p) = \log L(p) = \left(\sum_n x_n\right)\log(p) + \left(N - \sum_n x_n\right)\log(1-p) + \cancel{\log\left[\prod_n \mathbb{1}(x_n = 0 \text{ or } 1)\right]}$$

ignore
always 0

(i) $\frac{\partial \ell}{\partial p}$.

$$\frac{\partial \ell}{\partial p} = \left(\sum_n x_n\right)\frac{1}{p} + \left(N - \sum_n x_n\right)\frac{-1}{1-p}$$

$$= N\bar{x}/p - (N - N\bar{x})/(1-p)$$

(ii) Set eq. to zero, solve for p.

$$\frac{\cancel{N}\bar{x}}{p} - \frac{\cancel{N}(1-\bar{x})}{1-p} = 0$$

$$\Rightarrow (1-p)\bar{x} - p(1-\bar{x}) = 0$$

$$\Rightarrow \bar{x} - \cancel{p\bar{x}} - p + \cancel{p\bar{x}} = 0$$

$$\Rightarrow \bar{x} - p = 0$$

$$\Rightarrow \boxed{\hat{p}_{MLE} = \bar{x}}$$