Lecture 21:

Pivoting

1. Find some quantity (pivot)

$$Q = Q(\underset{\sim}{X}, \theta)$$

whose dist doesn't depend on θ

(ancillary quantity)

2. Find some region A that doesn't depend on θ

where $P(Q \in A) \geq 1-\alpha$.

3. Then a $1-\alpha$ CR for θ is

$$C(\underset{\sim}{X}) = \{\theta : Q(\underset{\sim}{X}, \theta) \in A\}.$$

Reason this works is that

$$P_\theta(\theta \in C) = \underbrace{P_\theta(Q \in A)}_{\text{doesn't depend on } \theta} \geq 1-\alpha$$

dist of Q has no θ, and A has no θ

so $\min\limits_\theta \underbrace{P_\theta(Q \in A)}_{P_\theta(\theta \in C)} \geq 1-\alpha$

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$ want CI for μ.

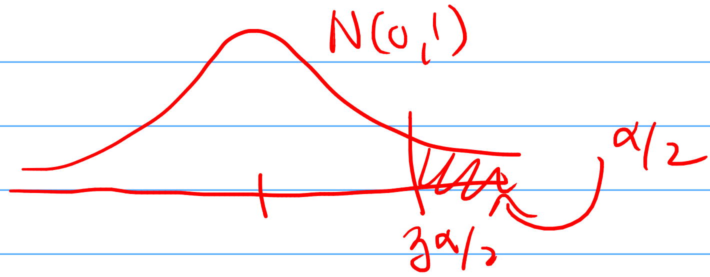
① get pivot: $\bar{X} \sim N(\mu, 1/N)$

$$\bar{X} - \mu \sim N(0, 1/N)$$

$$Q = \frac{\bar{X} - \mu}{1/\sqrt{N}} \sim N(0,1)$$

② $\mathbb{P}(-z_{\alpha/2} \le Q \le z_{\alpha/2}) = 1 - \alpha$

$A = [-z_{\alpha/2}, z_{\alpha/2}]$



③ $C = \{\mu : Q \in A\}$

$$= \{\mu : -z_{\alpha/2} \le Q \le z_{\alpha/2}\}$$

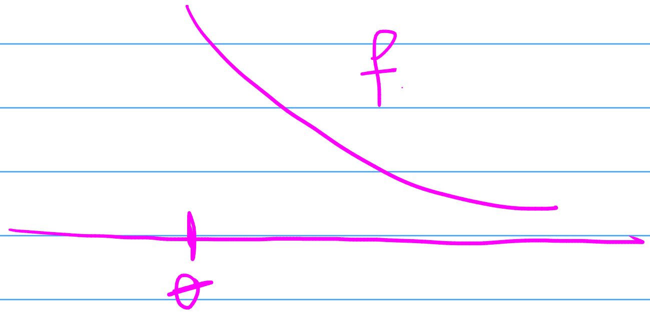
$$= \left\{\mu : -z_{\alpha/2} \le \frac{\bar{X} - \mu}{1/\sqrt{N}} \le z_{\alpha/2}\right\}$$

$$= \left\{\mu : \underbrace{\bar{X} - \frac{z_{\alpha/2}}{\sqrt{N}}}_{L} \le \mu \le \underbrace{\bar{X} + \frac{z_{\alpha/2}}{\sqrt{N}}}_{U}\right\}$$

Particularly easy to find pivots for "Location / Scale" families

Location Family

Ex. Shifted Exp.



Location parameter μ so that

$$f_\mu(x) = g(x-\mu)$$

where g is free of μ

$$f_\theta(x) = e^{-(x-\theta)} \mathbb{1}(x>\theta)$$

$$= g(x-\theta) \quad \text{when } g(x) = e^{-x} \mathbb{1}(x>0)$$

Scale Family

Ex. $U(0,\theta)$

Scale family:

Scale parameter σ so that

$$f_\sigma(x) = g(x/\sigma)\frac{1}{\sigma}$$

g free of σ

$$f_\theta(x) = \frac{1}{\theta}\mathbb{1}(0<x<\theta)$$

$$= \frac{1}{\theta} g(x/\theta) \quad \text{where } g(x) = \mathbb{1}(0<x<1)$$

Pivot for location family is $\bar{X} - \mu$

Pivot for scale family is $\bar{X}/\sigma$

<u>Ex.</u> Let $X_n \overset{iid}{\sim} \text{Exp}(\lambda)$

① <u>Get a pivot</u>

$$T = \sum_n X_n \sim \text{Gamma}(N, \lambda)$$

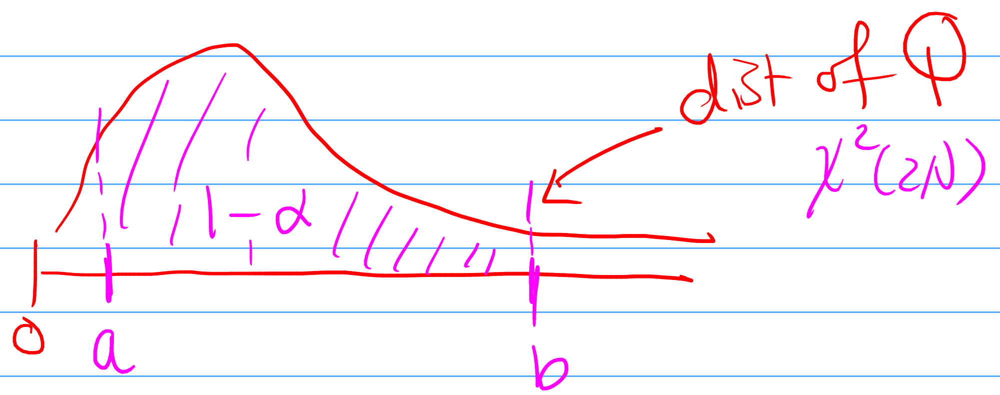
so if

$$Q = \frac{2T}{\lambda} \sim \text{Gamma}(N, 2) = \chi^2(2N)$$

a pivot.

② Need to choose some A s.t.

$$\mathbb{P}(Q \in A) \geq 1 - \alpha$$



③ then $\text{CI}: \{\lambda : a < Q < b\}$

$$= \left\{ \lambda : a \le \frac{2T}{\lambda} \le b \right\}$$

$$= \left\{ \lambda : \frac{1}{b} \le \frac{\lambda}{2T} \le 1/a \right\}$$

$$= \left\{ \lambda : \underbrace{\frac{2T}{b}}_{L} \le \lambda \le \underbrace{\frac{2T}{a}}_{U} \right\}$$

Practical Steps:

1. get some Q whose dist doesn't depend on θ
2. Find some a, b such that

$$\mathbb{P}(a < Q < b) = 1 - \alpha$$

3. Solve $a < Q < b$ for θ in middle

Very general way of doing this for cts dists

Recall that if $X \sim F_X$ then

$$Q = F_X(X) \sim U(0,1).$$

a pivot

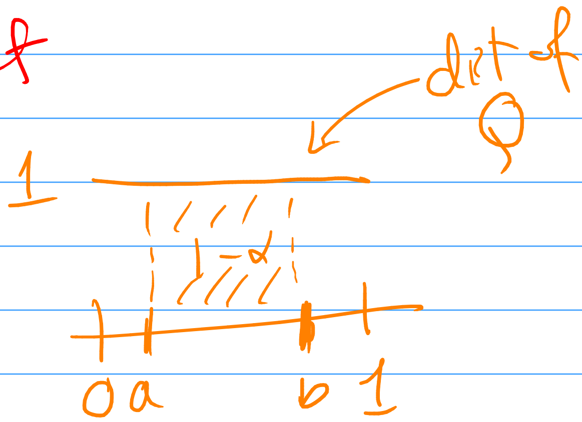
Let T be some statistic.

① $Q = F_T(T) \sim U(0,1)$

is a pivot

② Let $a = \alpha/2$

$b = 1-\alpha/2$



then $\mathbb{P}(a < Q < b) = 1-\alpha$

depends on θ

③ Solve $\frac{\alpha}{2} < F_T(T) < 1-\alpha/2$

for θ in middle

$\vdots$

$L < \theta < U$

Step ③ is easy if F_T is an invertible function of θ

Let $g(\theta) = F_T$ as a fn of θ

Then I need to solve

$$\frac{\alpha}{2} < g(\theta) < 1-\alpha/2$$

If g is increasing in θ then this gives

$$g^{-1}(\alpha/2) < \theta < g^{-1}(1-\alpha/2)$$

If g is dec. in θ then we get

$$g^{-1}(1-\alpha/2) < \theta < g^{-1}(\alpha/2).$$

Theorem: Universal Continuous Pivot

If T is a stat w/ CDF F_T and $g(\theta)$ is F_T as a fn of θ then

(a) If g is inc in θ a $1-\alpha$ CI is

$$L = g^{-1}(\alpha/2), \quad U = g^{-1}(1-\alpha/2)$$

(b) If g is dec in θ a $1-\alpha$ CI is

$$L = g^{-1}(1-\alpha/2), \quad U = g^{-1}(\alpha/2).$$

Ex. Let T be a stat w/ CDF given by

$$F_T(t) = \frac{1}{1+\exp(-(t-\mu))}$$

μ is my param

Let's create a $1-\alpha$ CI for μ.

$$g(\mu) = \frac{1}{1+\exp(-(t-\mu))}$$

this is decreasing in μ.

Let's get g^{-1}.

$$y = g(\mu) = \frac{1}{1+\exp(-(t-\mu))}$$

$$\Leftrightarrow \frac{1}{y} = 1+\exp(-(t-\mu))$$

$$\Leftrightarrow \frac{1}{y} - 1 = \exp(-(t-\mu))$$

$$\Leftrightarrow \log\left(\frac{1}{y}-1\right) = -(t-\mu)$$

$$\Longleftrightarrow t + \log\left(\frac{1}{y} - 1\right) = \mu = g^{-1}(\mu)$$

So our theorem says that

$$L = g^{-1}(1 - \alpha/2)$$
$$= t + \log\left(\frac{1}{1-\alpha/2} - 1\right)$$

$$U = g^{-1}(\alpha/2)$$
$$= t + \log\left(1/\alpha/2 - 1\right)$$