# Lecture 8: UMVUEs

**Defn:** Uniformly Minimum Variance Unbiased Estimator (UMVUE)

**Note:** If $B(\hat{\theta}) = 0$ then $MSE(\hat{\theta}) = Var(\hat{\theta})$

We call $\theta^*$ the UMVUE of $\tau(\theta)$

e.g. $\theta^2$, $\log\theta$, $e^\theta$, ...

if it is

(1) unbiased for $\tau(\theta)$

$$\mathbb{E}[\theta^*] = \tau(\theta)$$

(2) minimum variance — uniformly

$$Var_\theta(\theta^*) \leq Var_\theta(\hat{\theta}) \quad \forall \text{ unbiased ests } \hat{\theta} \text{ of } \tau(\theta)$$

$$\forall \theta \in \Theta$$

How do find the UMVUE?

The likelihood function is important

↳ sufficiency

↳ MLE

Turns out deep connections among UMVUEs, sufficiency, MLEs, likelihood,

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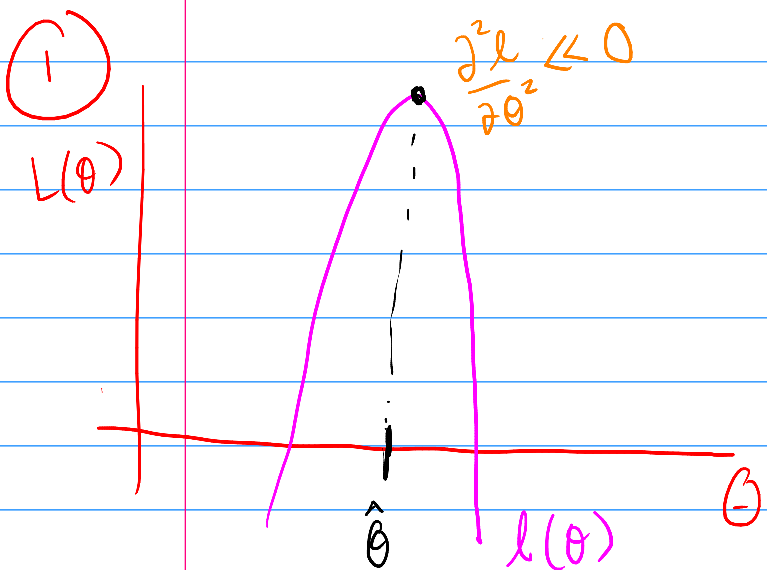
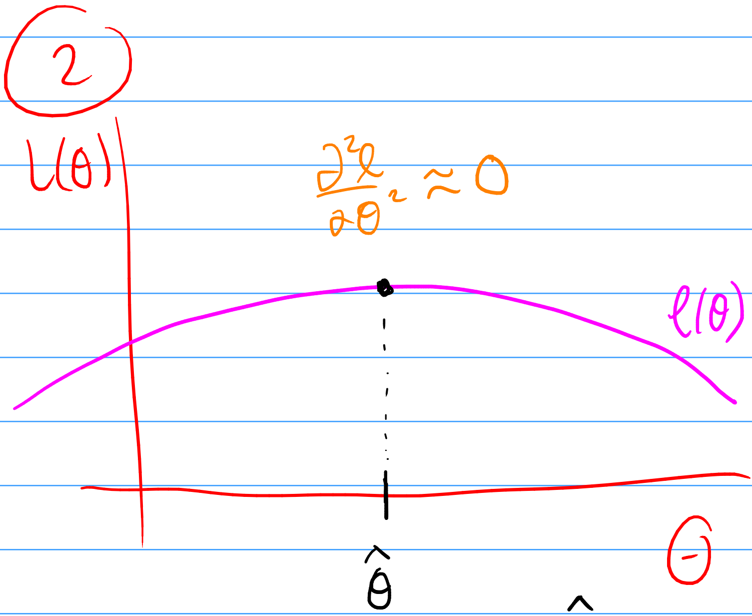
Consider estimating $\theta$ w/ a MLEs

two cases:



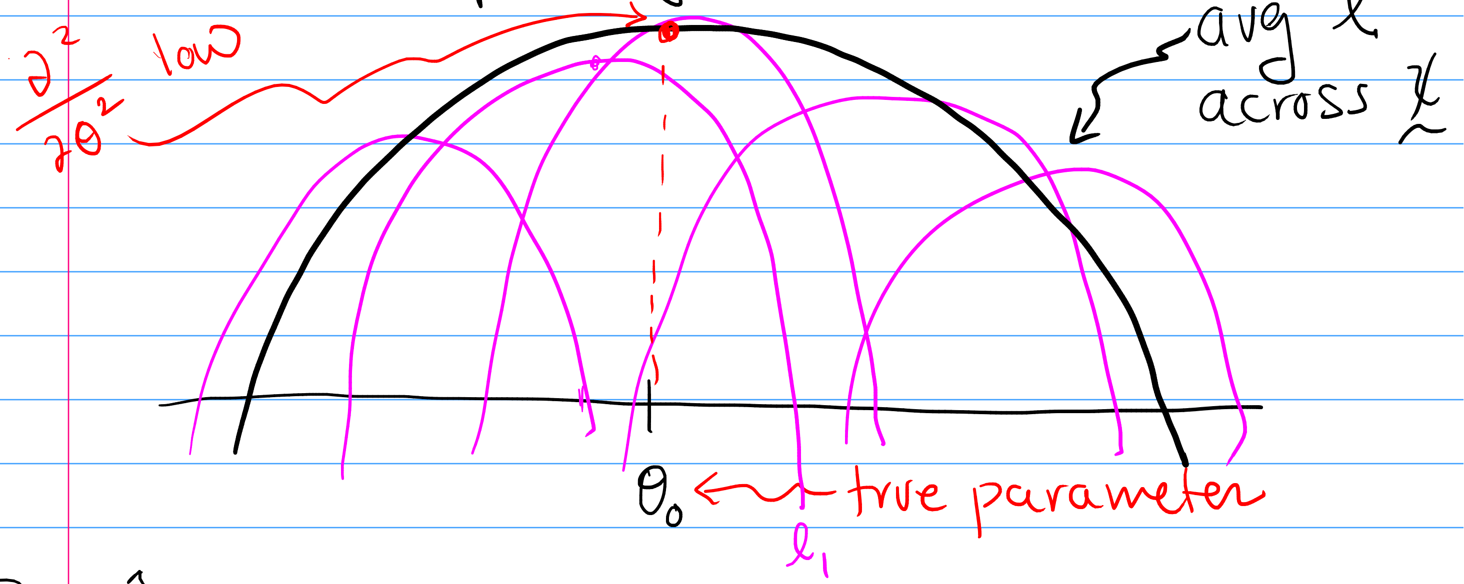

Strongly prefer $\hat{\theta}$ to other values of $\theta$



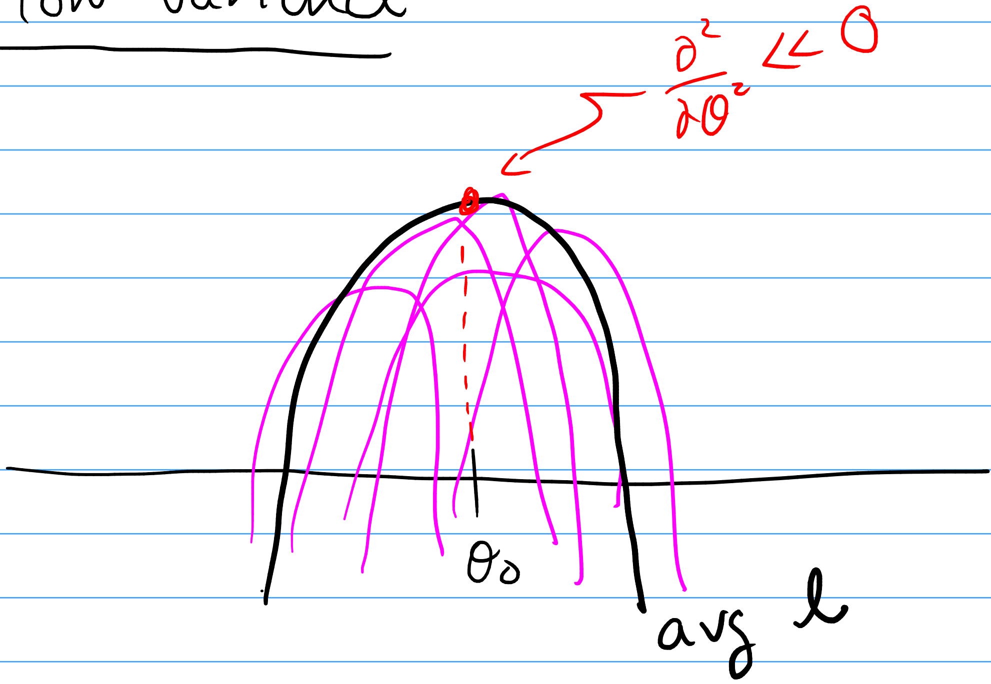

weakly prefer $\hat{\theta}$

## What about accuracy?

① $\hat{\theta}$ highly variable – so $\ell$ jumps around a lot depending on data $\underset{\sim}{x}$




② $\hat{\theta}$ low variance




Claim: concavity $\left(\frac{\partial^2}{\partial\theta^2}\right)$ of $\theta_0$ tells us something about possible accuracy of $\hat{\theta}$

(*) This only really works for "nice" distributions
Always makes sense for Exp. fams.

## Defn: Fisher Information (*)

For a single observation $X \sim f_\theta$ $(N=1)$

we define the Fisher Info about $\theta$

contained in $X$ as

$$I(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)\right] = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial\theta^2}\right]$$

If I have $N$ samples then the Info contained about $\theta$ is

$$I_N(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(\underset{\sim}{X})\right] = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial\theta^2}\right]$$

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**Theorem:** $I_N(\theta) = N\,I(\theta)$

Pf- $I_N(\theta) = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial\theta^2}\right]$

$$= -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(\underset{\sim}{X})\right]$$

$$= -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log\left[\prod_n f_\theta(X_n)\right]\right]$$

$$= -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\sum_n \log f_\theta(X_n)\right]$$

$$= -\sum_n \mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(X_n)\right]$$

$$= \sum_n \underbrace{-\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2} \log f_\theta(X_n)\right]}_{I(\theta)}$$

$$= N I(\theta)$$

$X_n \overset{iid}{\sim} f$

$\mathbb{E}X_1 = \mathbb{E}X_2 = \cdots$

$\mathbb{E}g(X_1) = \mathbb{E}g(X_2) = \cdots$

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Ex. $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$, $\lambda > 0$

Find $I_N(\lambda)$.

Let's find $I(\lambda)$ and multiply by $N$.

1 Find $\log f_\lambda(x)$

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!} \cancel{\mathbb{1}(x_n \in \mathbb{N}_0)}$$

can ignore

$$\log f_\lambda(x) = x \log(\lambda) - \lambda - \log(x!) + \cancel{\log \mathbb{1}(x_n \in \mathbb{N}_0)}$$

2 Take two derivs wrt $\lambda$

$$\frac{\partial}{\partial\lambda}[\cdots] = x/\lambda - 1$$

$$\frac{\partial^2}{\partial\lambda^2}[\cdots] = -\frac{x}{\lambda^2}$$

3 Promote $x$ to $X$

$$-\frac{X}{\lambda^2}$$

4) Calc $-\mathbb{E}[\cdots]$

we get: $-\mathbb{E}\left[-\frac{X}{\lambda^2}\right] = \frac{1}{\lambda^2}\mathbb{E}[X] = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

So $I(\lambda) = 1/\lambda$

and $I_N(\lambda) = N/\lambda$.

MLE for $\lambda$ is $\bar{X}$

$\mathbb{E}\bar{X} = \lambda$ ← unbiased for $\lambda$

$Var(\bar{x}) = \lambda/N = 1/I_N(\lambda)$

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<u>Ex.</u> Let $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\mu$: unknown, $\sigma^2$: known

<u>Let's get $I_N(\mu)$.</u>

1) $f_\mu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$

2) $\log f_\mu(x) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2$

3) $\frac{\partial \log f_\mu}{\partial \mu} = -\frac{1}{2\sigma^2} 2(x-\mu)(-1) = \frac{1}{\sigma^2}(x-\mu)$

$\frac{\partial^2 \log f_\mu}{\partial \mu^2} = -1/\sigma^2$

4) $I(\mu) = -\mathbb{E}\left[\frac{\partial^2 \log f_\mu}{\partial \mu^2}\right] = -\mathbb{E}[-1/\sigma^2] = 1/\sigma^2$

5) $I_N(\mu) = N/\sigma^2$

Suspiciously, $\bar{X}$ is unbiased for $\mu$

$$E[\bar{X}] = \mu$$

and $Var(\bar{X}) = \sigma^2/N = 1/I_N(\mu)$

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<u>Ex.</u> Revisit poisson, but consider $\psi = \sqrt{\lambda}$.

<u>Let's get $I_N(\psi)$.</u> $(\lambda = \psi^2)$

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(\psi^2)^x e^{(-\psi^2)}}{x!} = f_\psi(x)$$

1) $\log f_\psi(x) = 2x\log(\psi) - \psi^2 - \log(x!)$

2) $\frac{\partial}{\partial\psi}[\ldots] = \frac{2x}{\psi} - 2\psi$

$\log(a/b) = \log(a) - \log(b)$

$\log(ab) = \log(a) + \log(b)$

$\log(a^b) = b\log(a)$

$$\frac{\partial^2}{\partial\psi^2}[\ldots] = -\frac{2x}{\psi^2} - 2$$

3) $I(\psi) = -\mathbb{E}\left[\frac{\partial^2}{\partial\psi^2}\log f_\psi(X)\right] = -\mathbb{E}\left[-\frac{2X}{\psi^2} - 2\right]$

$$= \frac{2\mathbb{E}[X]}{\psi^2} + 2$$

$$= \frac{2\psi^2}{\psi^2} + 2 = 4$$

4) $I_N(\psi) = 4N$

## Theorem: Fisher Info For Transf.

If $\theta = \tau(\psi)$ [$\Leftrightarrow \psi = \tau^{-1}(\theta)$ if $\tau$ invertible]

then
$$I(\theta) = \left(\frac{\partial\psi}{\partial\theta}\right)^2 I(\psi)$$

equiv.
$$I(\psi) = \left(\frac{\partial\theta}{\partial\psi}\right)^2 I(\theta)$$

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Recall: $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

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Revisit $I_N(\lambda) = N/\lambda$

$$\psi = \sqrt{\lambda} \Leftrightarrow \boxed{\lambda = \psi^2}$$

$\dfrac{\partial\lambda}{\partial\psi} = 2\psi$

$$I(\psi) = \left(\frac{\partial\lambda}{\partial\psi}\right)^2 I(\lambda)$$

$$= (2\psi)^2 \, N/\lambda$$

$$= 4\psi^2 \cdot \frac{N}{\psi^2} = 4N$$

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