Lecture 22

W : 2:30 - 3:30
Thurs: 3:30 - 5:00
M: 10 am - 12:00 pm

} Extra office hours

Often report the outcome of a HT using a p-value

Defn: p-value

A p-value is a statistic $p(\underset{\sim}{x})$ where

$$0 \leq p(\underset{\sim}{x}) \leq 1$$

idea: small $p(\underset{\sim}{x})$ gives evidence of H_a

large $p(\underset{\sim}{x})$ gives evidence of H_0

Recall that a HT is just a partition of $\mathcal{X}$ into A and R

Can use $p(\underset{\sim}{x})$ to define R

$$R = \{ \underset{\sim}{x} : p(\underset{\sim}{x}) \leq \alpha \}$$

We say a p-value is <u>valid</u> if
$\forall \alpha \in [0,1]$ and $\theta \in \Theta_0$

$$\underbrace{\mathbb{P}_\theta(p(\underset{\sim}{X}) \le \alpha)}_{F_p(\alpha) \le \alpha} \le \alpha$$

$F_p(\alpha) \le \alpha \leftarrow$ CDF of $U(0,1)$

p is stochastically bounded by $U(0,1)$

Typically we actually have that

$$\underbrace{\mathbb{P}_\theta(P(\underset{\sim}{X}) \le \alpha)}_{F_p(\alpha)} = \alpha$$

i.e. $P(\underset{\sim}{X}) \sim U(0,1)$ for $\theta \in \Theta_0$

If P is valid then test that rejects when $P(\underset{\sim}{X}) \le \alpha$ is an α-level test.

<u>reason!</u>

$$\mathbb{P}_\theta(\text{reject}) = \mathbb{P}_\theta(P(\underset{\sim}{X}) \le \alpha) \le \alpha \quad \forall \theta \in \Theta_0$$

$\theta \in \Theta_0$

Ex. Consider

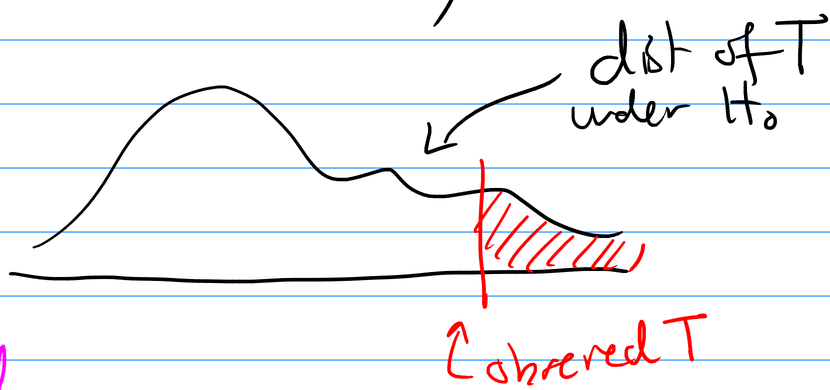
$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta \neq \theta_0$$

I get a test stat T and my test rejs when

$$R = \{ x : T(x) \text{ is large} \}$$

Let

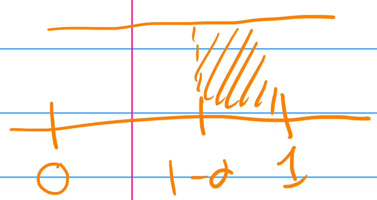
$$p(x) = \mathbb{P}_{\theta_0}\big(\overset{\text{random}}{\overline{T(X)}} \geq \overset{\text{observed}}{\overline{T(x)}} \big) = 1 - F_T(T(x))$$



this is a valid p-value

$$\mathbb{P}_{\theta_0}(p(X) \leq \alpha) = \mathbb{P}_{\theta_0}\big(1 - F_T(T(X)) \leq \alpha \big)$$

$$= \mathbb{P}_{\theta_0}\big(\underbrace{F_T(T(X))}_{U(0,1)} \geq 1 - \alpha \big) = \alpha$$

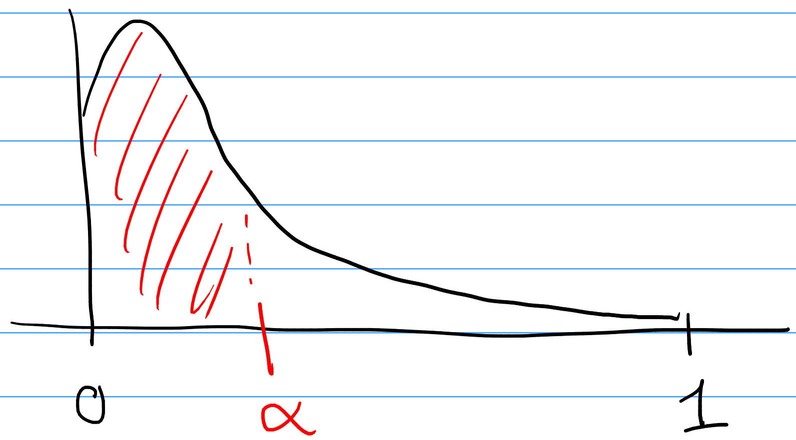


So this is valid.

Punchline: under H_0, $P \sim U(0,1)$.

Under H_0:

Under H_a: typically, p-value is small



Frequentists: prob = long run freq of events

Bayesians: prob = degree of belief / information

Sleeping Beauty

H → wake up monday Q → wake up weds

T → wake up monday Q → wake up Tuesday Q → wake up on weds

Practical!

Frequentist: θ fixed but unknown

Bayesian: Θ is random

Bayesian approach!

1. prior dist $\Theta \sim \pi$
2. get data

$$\underset{\sim}{X} \mid \Theta = \theta \sim \underbrace{f(\underset{\sim}{x} \mid \theta)}_{\text{likelihood}}$$

3. update / combine prior and likelihood to get posterior

$$\pi(\theta \mid \underset{\sim}{x}) = \frac{f(\underset{\sim}{x} \mid \theta)\, \pi(\theta)}{f(\underset{\sim}{x})} \propto f(\underset{\sim}{x} \mid \theta)\, \pi(\theta)$$

Then use $\pi(\theta|\underset{\sim}{X})$ for whatever, e.g. estimate $\hat{\theta}$ as

$$\mathbb{E}[\Theta|\underset{\sim}{X}].$$

Ex.

$$X_n \mid P=p \sim \text{Bern}(p)$$

$$P \sim \text{Beta}(\alpha,\beta)$$

prior

$$\pi(p|\underset{\sim}{X}) \propto f(\underset{\sim}{x}|p)\pi(p)$$

$$\text{Beta}(N\bar{X}+\alpha,\ N-N\bar{X}+\beta)$$

$$\hat{p} = \mathbb{E}[p|\underset{\sim}{X}] = \frac{N\bar{X}+\alpha}{N\bar{X}+\alpha+N-N\bar{X}+\beta}$$

$$= \omega\bar{X} + (1-\omega)\underbrace{\frac{\alpha}{\alpha+\beta}}_{\text{mean of prior}}$$

freq. est. of $\hat{p}$

$$\omega = \frac{N}{\alpha+\beta+N}$$