Lecture 16 -- LRT

Tuesday, November 12, 2024 10:59 AM

Likelihood Ratio Test

Want to test

$$\boxed{H_0 : \theta \in \Theta_0} \quad v. \quad \boxed{H_a : \theta \in \Theta_a}$$

We define the likelihood ratio test (LRT) statistic as

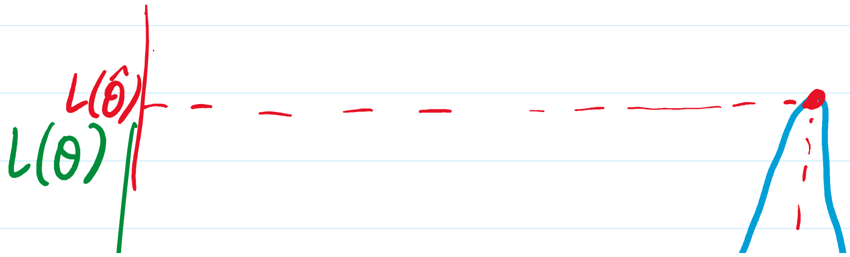
$$\lambda(\underset{\sim}{x}) = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \frac{\text{max of } L \text{ over } \Theta_0}{\text{max of } L \text{ overall}}$$

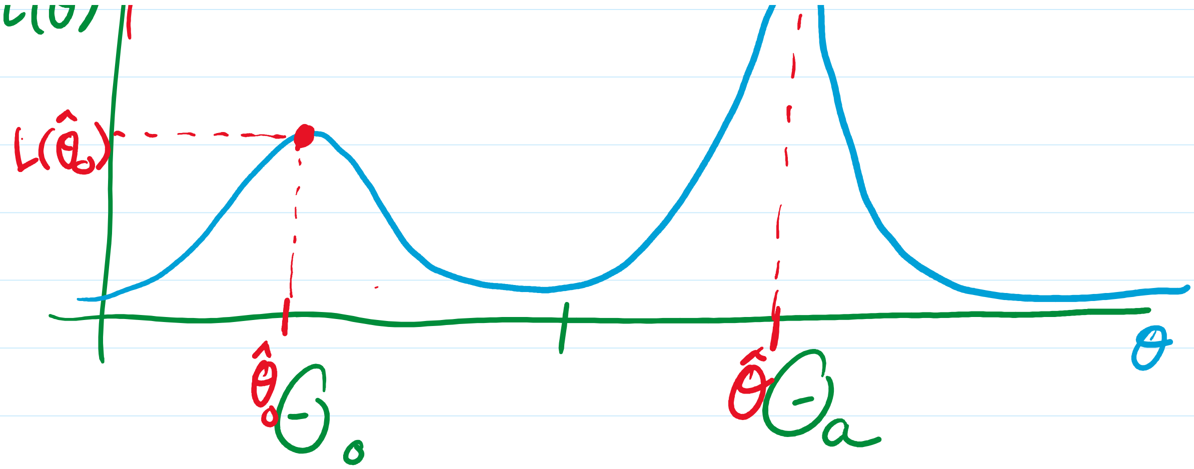
$$= \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} \leq 1$$

where $\hat{\theta}_0$ = MLE restricted to Θ_0

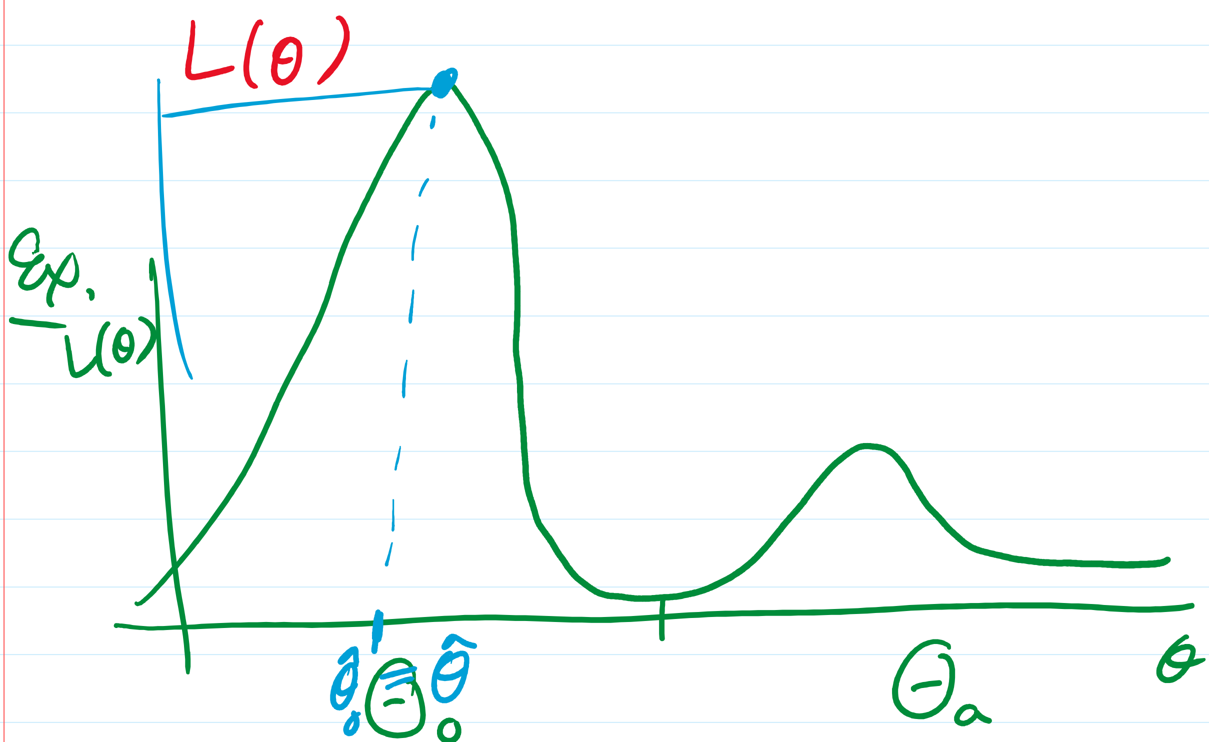
$\hat{\theta}$ = MLE

Ex.



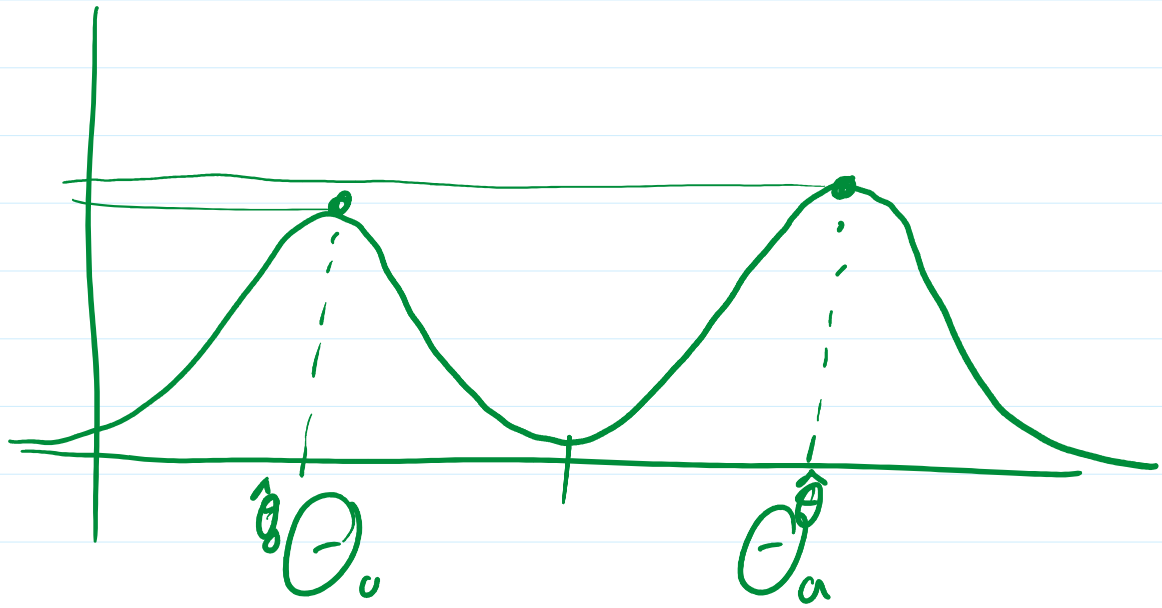


$$\lambda = \frac{L(\hat{\theta}_0)}{\hat{\ }} \ll 1 \rightsquigarrow \text{probably reject}$$



So $$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = 1 \rightsquigarrow \text{probably don't reject}$$

hard case:



$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} < 1 \quad \text{but} \quad \approx 1$$

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the LRT says I should reject when

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} \leq c$$

$0 \leq c \leq 1$

c is a thresh I choose to balance type I and II errors.

If c small, don't reject easily

If c small, don't reject easily,
less type I, more type II

If c is large, reject more easily,
more type I, less type II.

The rejection region for LRT is

$$R = \{\underset{\sim}{x} : \lambda(\underset{\sim}{x}) \leq c\}.$$

Ex. $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ known

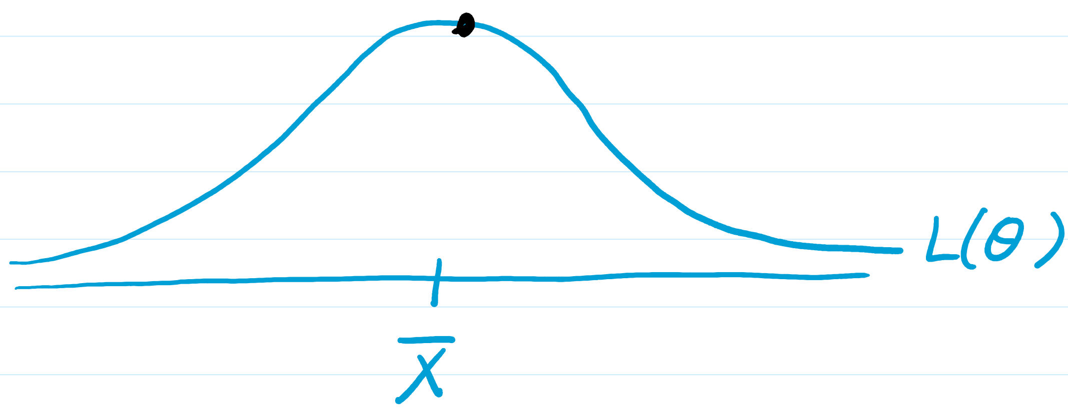
$$\boxed{H_0 : \theta \leq a} \quad v. \quad H_a : \theta > a$$

Let's derive the LRT

$$L(\theta) = \prod_n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_n - \theta)^2\right)$$

$$= \cdots (2\pi)^{-N/2} (\sigma^2)^{-N/2} \exp\left(-\frac{1}{2\sigma^2} \sum (x_n - \theta)^2\right)$$

$$= (2\pi)^{-N/2} (\sigma^2)^{-N/2} \exp\Big(\underbrace{-\frac{1}{2\sigma^2} \underbrace{\sum_n (x_n-\theta)^2}_{\text{quadratic in } \theta}}_{\text{looks like } e^{-(\theta-\bar{x})^2}}\Big)$$

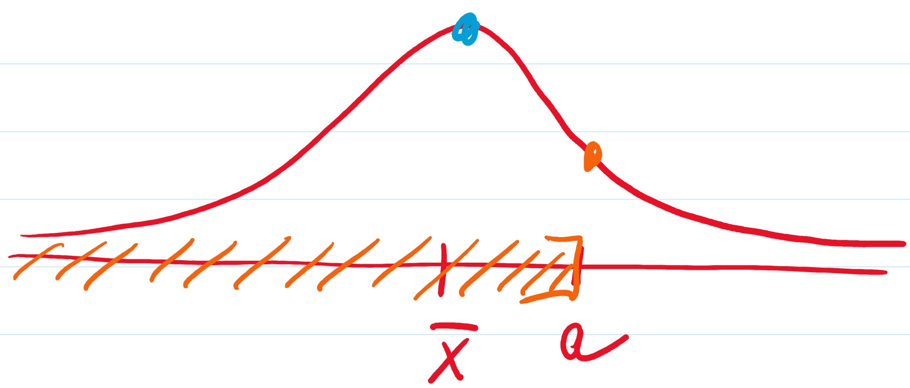


Clearly, $\hat{\theta} = \bar{X}$.

Need to find $\hat{\theta}_0 = \max_{\underbrace{\theta \in (-\infty, a]}_{\Theta_0}} L(\theta)$

Two cases:

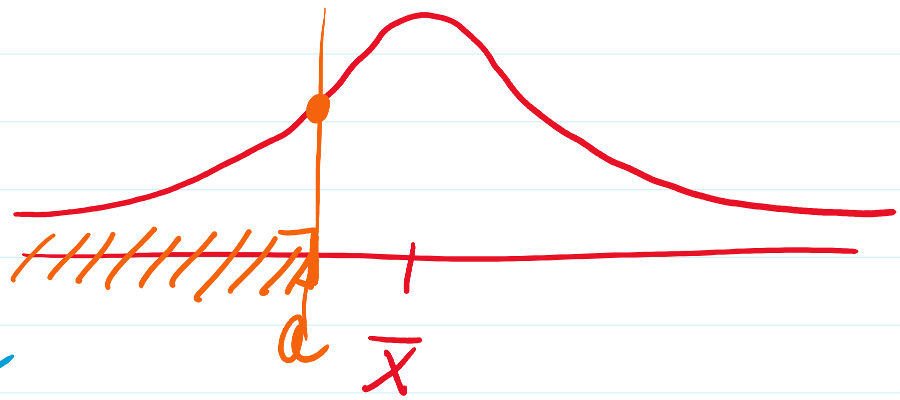
$\bar{X} \le a$



Since $\bar{X} \le a$ then $\hat{\theta}_0 = \bar{X}$

Since $\bar{x} = \theta$ then $\hat{\theta}_0 = \bar{x}$

$\underline{\bar{x} > a}$



So $\hat{\theta}_0 = a$

Overall then

$$\hat{\theta}_0 = \begin{cases} \bar{x} & \text{if } \bar{x} \le a \\ a & \text{if } \bar{x} > a \end{cases}$$

$H_0: \theta \le a$

$$\lambda(\underset{\sim}{x}) = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \begin{cases} \frac{L(\bar{x})}{L(\bar{x})} = 1 & \bar{x} \le a \\ \frac{L(a)}{L(\bar{x})} & \bar{x} > a \end{cases}$$

never reject

LRT says reject when $\lambda \le c < 1$

i.e. reject when

$$\frac{L(a)}{\ldots} \le c \qquad \left[L(a) \le c L(\bar{x})\right]$$

$$\frac{L(a)}{L(\bar{x})} \leq c \quad \left[L(a) \leq c\, L(\bar{x}) \right]$$

and $\boxed{\bar{x} > a.}$

$$\lambda = \frac{L(a)}{L(\bar{x})} = \frac{\cancel{(2\pi\sigma^2)^{-N/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_n (x_n - a)^2\right)}{\cancel{(2\pi\sigma^2)^{-N/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_n (x_n - \bar{x})^2\right)}$$

$$= \frac{\exp\left(-\frac{1}{2\sigma^2} \sum_n (\cancel{x_n^2} - 2a x_n + a^2)\right)}{\exp\left(-\frac{1}{2\sigma^2} \sum_n (\cancel{x_n^2} - 2 x_n \bar{x} + \bar{x}^2)\right)}$$

$$\boxed{\frac{e^{a+b}}{e^{a+c}} = \frac{\cancel{e^a} e^b}{\cancel{e^a} e^c}}$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(-2aN\bar{x} + Na^2 + 2N\bar{x}^2 - N\bar{x}^2\right)\right)$$

$$= \exp\left(-\frac{N}{2\sigma^2}\left(-2a\bar{x} + a^2 + \bar{x}^2\right)\right)$$

$$\lambda = \exp\left(-\frac{N}{2\sigma^2}(\bar{x} - a)^2\right)$$

and LRT says reject when $\lambda \leq C$

$$\Leftrightarrow \exp\left(-\frac{N}{2\sigma^2}(\bar{X}-a)^2\right) \leq C$$

$$\Leftrightarrow -\frac{N}{2\sigma^2}(\bar{X}-a)^2 \leq \log C$$

$$\Leftrightarrow \frac{N}{\sigma^2}(\bar{X}-a)^2 \geq -2\log C$$

when $\bar{X} > a$ then $\bar{X} - a > 0$

$$\downarrow$$

$$\Leftrightarrow \frac{\sqrt{N}}{\sigma}(\bar{X}-a) \geq \underbrace{\sqrt{-2\log C}}_{c^*}$$

$$\Leftrightarrow \boxed{\frac{\bar{X}-a}{\sigma/\sqrt{N}} \geq c^*}$$

$$\Leftrightarrow \boxed{\bar{X} \geq a + \frac{\sigma}{\sqrt{N}} c^*}$$

$H_0: \theta \leq a$

$\boxed{H_a: \theta > a}$

LRT $\Leftrightarrow$ reject when $\bar{X}$ is $a + c^* \, \mathrm{Std.}(\bar{X})$

$$a + c^* \, \text{s.d.}(\bar{X})$$

What's a reasonable value for c^*?

Maybe want to choose c^* so that this is a size α test

i.e.

$$\max \text{ type I error } = \alpha$$

i.e.

$$\max_{\theta \in \Theta_0} \underbrace{P_\theta(\text{reject})}_{\beta(\theta)} = \alpha$$

Θ_0: $(-\infty, a]$

$$\beta(\theta) = P_\theta(\lambda \le c) = P_\theta\left(\frac{\bar{X} - a}{\sigma/\sqrt{N}} \ge c^*\right)$$

$$= P_\theta\left(\underbrace{\frac{\bar{X} - a}{\sigma/\sqrt{N}} + \frac{a - \theta}{\sigma/\sqrt{N}}} \ge c^* + \frac{a - \theta}{\sigma/\sqrt{N}}\right)$$

$\bar{X} \sim N(\theta, \sigma^2/N)$ $\quad Z = \frac{\bar{X} - \theta}{\sigma/\sqrt{N}} \sim N(0,1)$

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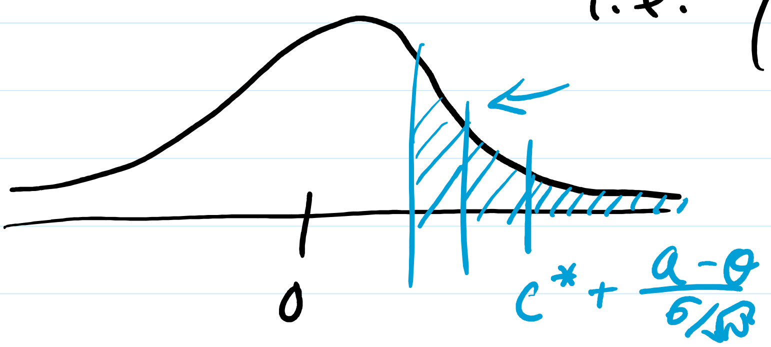
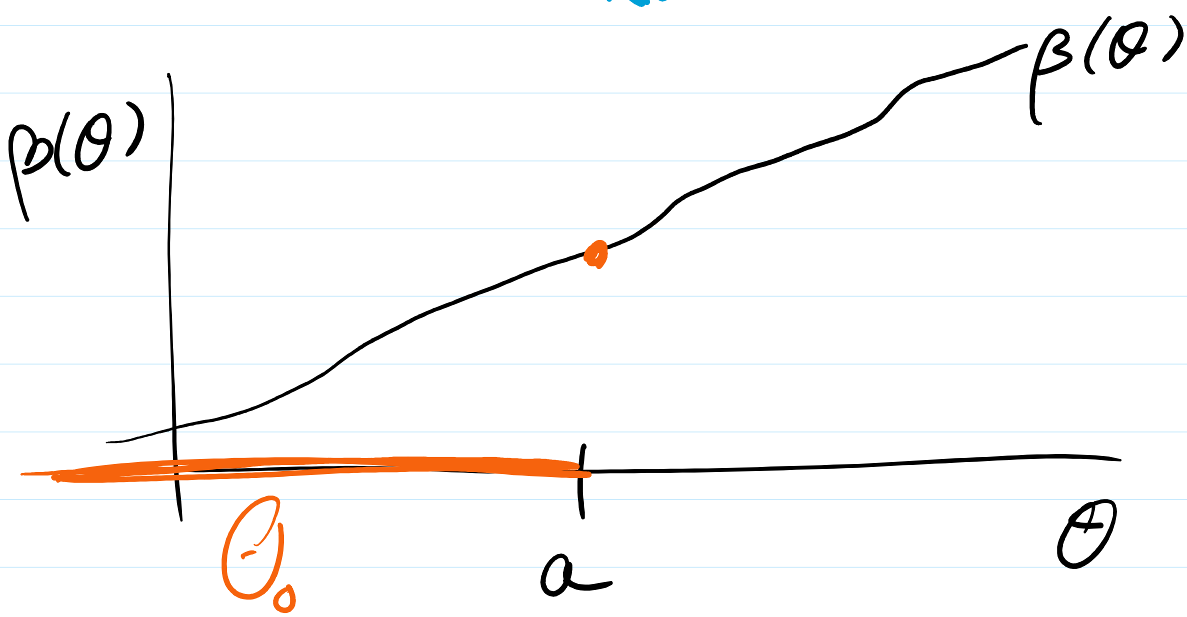
$$\boxed{\beta(\theta) = P_\theta\left(Z \geq c^* + \frac{a-\theta}{\sigma/\sqrt{N}}\right)}$$

as $\theta \uparrow$ then $\beta(\theta) \uparrow$

i.e. β is inc. in θ





$$\max_{\theta \in \Theta_0} \beta(\theta) = \beta(a) = P_a\left(Z \geq c^* + \cancel{\frac{a-a}{\sigma/\sqrt{N}}}^{\,0}\right)$$

$$= P_a(Z \geq c^*)$$

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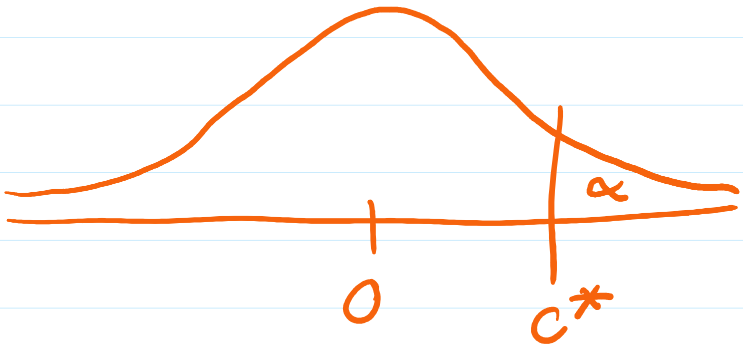
So should choose c^* so that

$$P(Z \geq c^*) = \alpha$$



If F_Z is the CDF of a $N(0,1)$ then this condition is

$$1 - F_Z(c^*) = \alpha$$

i.e. $\boxed{c^* = F_Z^{-1}(1-\alpha)}$

Defn: Uniformly Most Powerful (UMP) test

If $\mathcal{C}$ is a class of tests for the

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$$H_0: \theta \in \Theta_0 \quad v. \quad H_a: \theta \in \Theta_a$$

then the test w/ power function β^* is called the UMP test for this class if

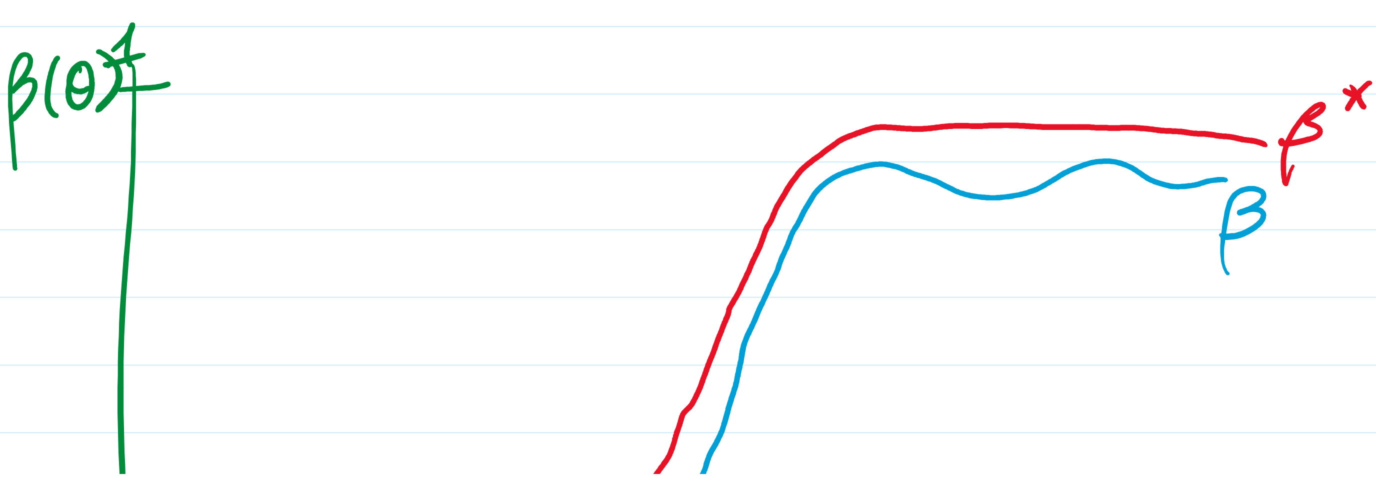
$$\beta^*(\theta) \geq \beta(\theta) \quad \forall \theta \in \Theta_a$$

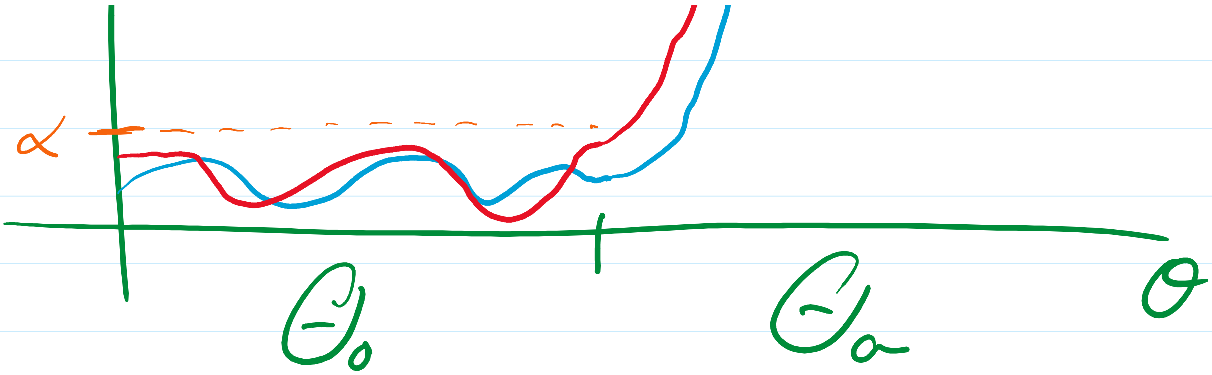
for any other test in $\mathcal{C}$ w/ power β.

UMP level α test

the UMP among all tests where

$$\max_{\theta \in \Theta_0} \beta(\theta) \leq \alpha. \quad \left[\text{level } \alpha \text{ test}\right]$$





α
Θ₀
Θₐ
θ