Lecture 12 -- More Convergence

Thursday, October 17, 2024 11:00 AM

Theorem: Algebraic Props

Let $X_n \to X$ and $Y_n \to Y$ and $a, b \in \mathbb{R}$

conv. p or a.s.

then (1) $aX_n + bY_n \to aX + bY$

(2) $X_n Y_n \to XY$

$\left(X_n / Y_n \to X/Y \right)$

Constants are degenerate RVs so if

$$C_n \to C \quad \text{(as numbers)}$$

then

$$C_n \xrightarrow{a.s.} C \quad \text{(as RVs)}.$$

So if $C_n \to C$ (as numbers) and

$X_n \to X$ (p or a.s.) then

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(1) $aX_n + c_n \to aX + c$

(2) $c_n X_n \to cX$.

<u>What about convergence in dist?</u>

<u>Theorem</u>: Slutsky's Theorem

If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ then

(1) $X_n + Y_n \xrightarrow{d} X + c$

(2) $X_n Y_n \xrightarrow{d} Xc$

($X_n / Y_n \to X/c$)

<u>Theorem</u>: <u>Partial Converse</u>

<u>Know</u>: $p \Rightarrow d$

If $X_n \xrightarrow{d} c$ then $X_n \xrightarrow{p} c$.

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Theorem: Continuous Mapping Theorem (CMT)

If $g: \mathbb{R} \to \mathbb{R}$ is continuous and

$$X_n \to X \quad \text{(any type of convergence)}$$

then $g(X_n) \to g(X)$.

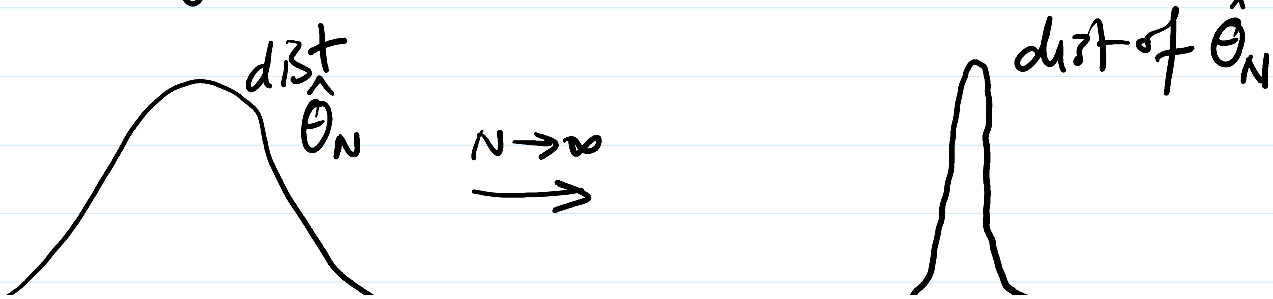
Defn: Consistent Estimator

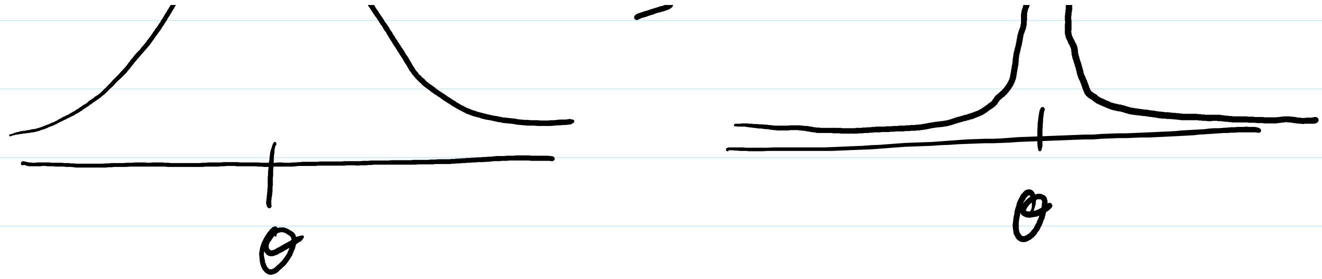
We say an estimator $\hat{\theta}_N$ is consistent for θ if

sample size on which est. is calc.

$$\hat{\theta}_N \xrightarrow{p} \theta.$$

Consistency $\overrightarrow{\approx}$ asymptotically unbiased





Ex: $S^2 = \frac{1}{N-1}\sum_{n=1}^{N}(X_n - \bar{X})^2$

Saw that $E[S^2] = \sigma^2$ (unbiased)

Also had

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(X_n - \bar{X})^2 = \frac{N-1}{N}S^2$$

Saw: $E[\hat{\sigma}^2] = \frac{N-1}{N}\sigma^2 < \sigma^2$

However, as $N \to \infty$ then $\frac{N-1}{N} \to 1$

and so

$$E[\hat{\sigma}^2] \to \sigma^2 \quad \text{(asymptotically unbiased)}$$

Theorem: $MSE(\hat{\theta}_N) \longrightarrow 0$ as $N \to \infty$

then $\hat{\theta}_N$ is consistent for θ.

Pf. Need to show:

$$\forall \varepsilon > 0 : \ P(|\hat{\theta}_N - \theta| \geq \varepsilon) \longrightarrow 0$$

as $N \to \infty$.

Markov's Ineq

$Y \geq 0$ then

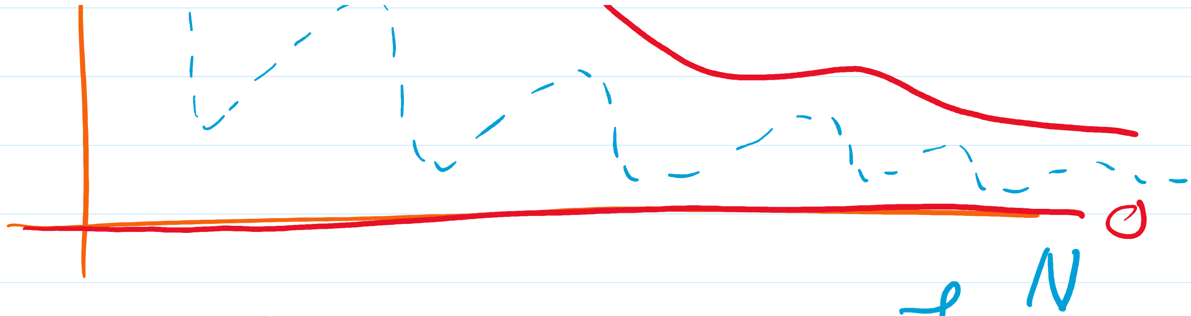
$$P(Y \geq a) \leq \frac{E[Y]}{a}.$$

$$0 \leq P(|\hat{\theta}_N - \theta| \geq \varepsilon)$$

$$= P\Big(\overbrace{\underbrace{(\hat{\theta}_N - \theta)}_{\geq 0}{}^2}^{Y} \geq \overbrace{\varepsilon^2}^{a}\Big)$$

$$\leq \frac{E[(\hat{\theta}_N - \theta)^2]}{\varepsilon^2}$$

$$= \frac{MSE(\hat{\theta}_N)}{\varepsilon^2}$$



By Squeeze theorem, if

$MSE(\hat{\theta}_N) \to 0$ then $P(|\hat{\theta}_N - \theta| \geq \varepsilon) \to 0$

i.e. $\hat{\theta}_N \xrightarrow{P} \theta$.

Ex, Showed prev. that if $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

then

$$MSE(S^2) = Bias(S^2)^2 + Var(S^2)$$

$$= 0 + \frac{2\sigma^4}{N-1}$$

$$= \frac{2\sigma^4}{N-1} \to 0$$

as $N \to \infty$

hence S^2 is consistent for σ^2.

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$$(S^2 \xrightarrow{p} \sigma^2)$$

What about $\hat{\sigma}^2$? $\hat{\sigma}^2 = \frac{N-1}{N} S^2$

$$\hat{\sigma}^2 = c_N S^2 \quad \text{where } c_N = \frac{N-1}{N}$$

and since $c_N \xrightarrow{N} 1$

and $S^2 \xrightarrow{p} \sigma^2$

then by my algebraic properties

$$\hat{\sigma}^2 = c_n S^2 \xrightarrow{p} 1 \cdot \sigma^2 = \sigma^2$$

So $\hat{\sigma}^2$ is also consistent.

What about the sample s.d. $S = \sqrt{S^2}$,

Is this consistant est. for σ?

By CMT $\sqrt{\cdot}$ is a cont fn and so

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Since $S^2 \xrightarrow{P} \sigma^2$

then $S = \sqrt{S^2} \xrightarrow{P} \sqrt{\sigma^2} = \sigma$.

So S is consistent for σ.

Intuition: $\overline{X}_N$ should be a good est. of $\mu = E[X_n]$.

Theorem: Weak Law of Large Numbers (WLLN)

If X_n are uncorrelated and

(1) $\mu = E[X_n]$

(2) $\sigma^2 = Var(X_n) < \infty$.

If $\overline{X}_N = \frac{1}{N}\sum_{n=1}^{N} X_n$ then

weak

$\overline{X} \xrightarrow{P} \mu$

weak

$$\bar{X}_N \xrightarrow{P} \mu.$$

pf. $MSE(\bar{x}) = Bias(\bar{x})^2 + Var(\bar{x})$

$E[\bar{x}] = \mu, \quad Var(\bar{x}) = \sigma^2/N$

$$= 0^2 + \sigma^2/N$$

$$= \sigma^2/N \to 0$$

as $N \to \infty$

So since $MSE \to 0$ then $\bar{X} \xrightarrow{P} \mu$.

Ex. $X_n \overset{iid}{\sim} Pois(\lambda)$

$$E[X_n] = \lambda = Var(X_n) < \infty$$

So WLLN says that

$$\bar{X}_N \xrightarrow{P} \lambda.$$

Ex. Consider $Y = \frac{1}{\quad} = g(X_n)$

Ex. Consider $Y_n = \frac{1}{1+X_n^2} = g(X_n)$

cts fn.

So by CMT I have that

$$g(\bar{X}_N) = \frac{1}{1+\bar{X}^2} \xrightarrow{P} g(\lambda) = \frac{1}{1+\lambda^2}.$$

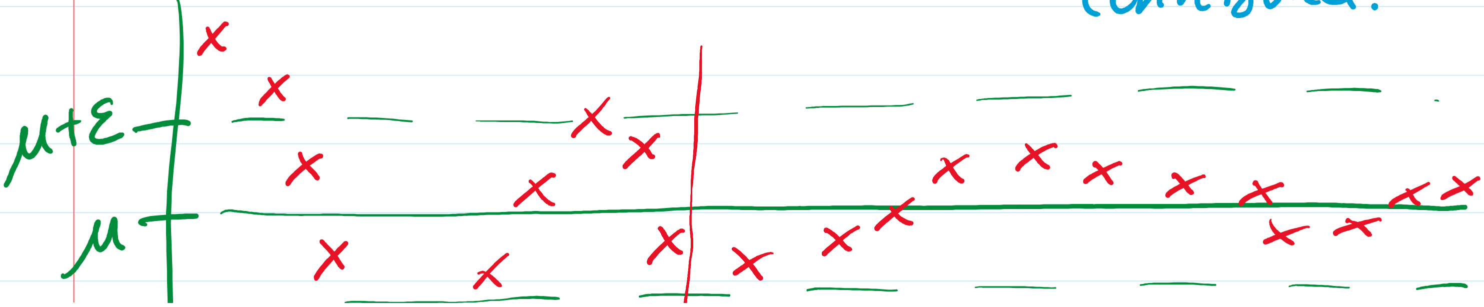
Theorem: Strong Law of Large Numbers (SLLN)

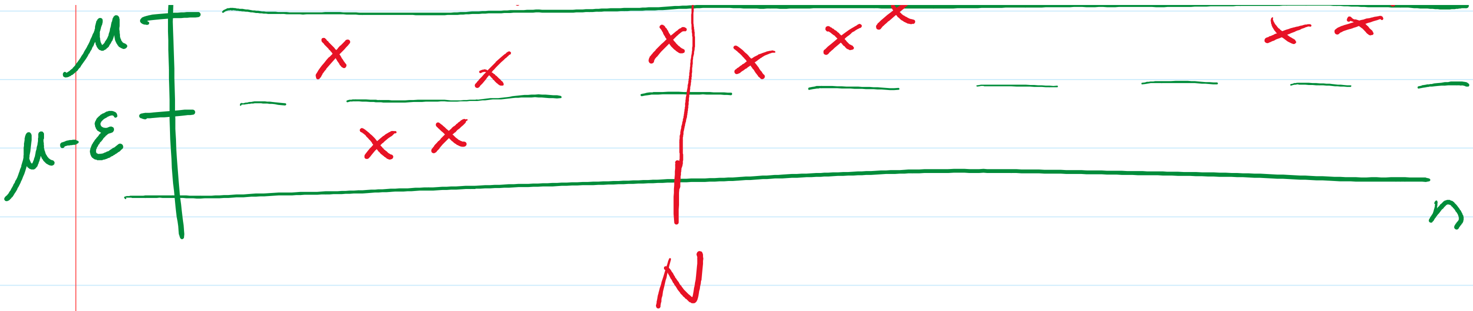
If $X_n \overset{iid}{\sim} f$ when $\mu = EX_n$, $\sigma^2 = Var(X_n) < \infty$

then $\bar{X}_N \xrightarrow{a.s.} \mu$.

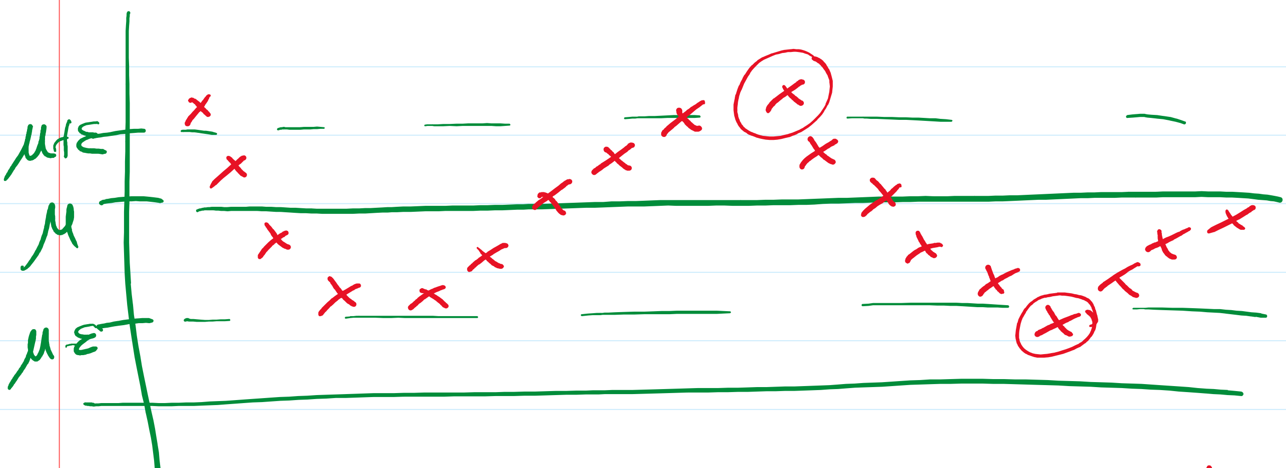
Convergence of Numbers $x_n \to \mu$

More like almost sure convergence.





Convergence in prob. $X_n \xrightarrow{P} \mu$



as $N \to \infty$ the prob I am outside of $\mu \pm \varepsilon$ goes to zero asymptotically

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Sums of R.Vs

① $\sum_{n=1}^{N} X_n \longrightarrow \pm\infty$ often,

② $\frac{1}{N}\sum_n X_n = \bar{X} \longrightarrow \mu$ (under some conditions)

(2) $\frac{1}{N}\sum_n X_n = \bar{X} \longrightarrow \mu$ (under some conditions)

constant

(3) $\underbrace{\frac{1}{\sqrt{N}}}_{\text{proper scaling.}} \sum_{n=1}^{N} X_n \longrightarrow$ some non-degenerate dist (often)