Lecture 17 -- UMPs

Thursday, November 14, 2024 11:00 AM

Punchline: Sufficiency + LRT give UMP level α test.

Alternative LRT

Let T be a sufficient stat w/ PMF/PDF $g_\theta(t)$

Traditional LRT:

$$\lambda(\underset{\sim}{x}) = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{f_{\hat{\theta}_0}(\underset{\sim}{x})}{f_{\hat{\theta}}(\underset{\sim}{x})}$$

Alt. LRT: $L^*(\theta) = g_\theta(t)$

$$\lambda^*(t) = \frac{L^*(\hat{\theta}_0)}{L^*(\hat{\theta})} = \frac{g_{\hat{\theta}_0}(t)}{g_{\hat{\theta}}(t)}$$

Test where I reject when $\lambda^* \leq c$

Test where I reject when $\lambda^* \le c$ is equiv. to test where I reject when $\lambda \le c$.

Pf.

$$\lambda = \frac{\max_{\theta \in \Theta_0} f_\theta(\underset{\sim}{x})}{\max_{\theta \in \Theta} f_\theta(\underset{\sim}{x})} = \frac{\max_{\theta \in \Theta_0} g(\theta, t)\cancel{h(\underset{\sim}{x})}}{\max_{\theta \in \Theta} g(\theta, t)\cancel{h(\underset{\sim}{x})}} = \lambda^*$$

turns out $g(\theta, t) \propto g_\theta(t)$

Thm: Simple Hypotheses

Consider $H_0 : \theta = \theta_0$ v. $H_a : \theta = \theta_a$

and reject when $\lambda^* \le c$ where

c is chosen so that

$$P_{\theta_0}(\lambda^* \le c) = \alpha$$

Then this is the UMP size α test.

What about composite tests?

e.g.

$$H_0: \theta \leq \theta_0 \quad v. \quad H_a: \theta > \theta_0$$

Defn: Monotone Likelihood Ratio Property (MLR)

A fam of univariate dists $\{f_\theta\}_{\theta \in \Theta}$ has MLR prop if $\forall\ \theta_1 < \theta_2$

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} \text{ is non-decreasing in } x.$$

Thm: If f_θ is an exp. fam of the form

not $T(X)$

$$f_\theta(x) = c(\theta)\, h(x) \exp(w(\theta)\, x)$$

$$f_\theta(x) = c(\theta)\, h(x) \exp(w(\theta)\, x)$$

then this fam has MLR prop if

$w(\theta)$ is non-dec. in θ.

Pf.

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{c(\theta_2)\cancel{h(x)} \exp(w(\theta_2)x)}{c(\theta_1)\cancel{h(x)} \exp(w(\theta_1)x)}$$

$$\propto \exp((w(\theta_2) - w(\theta_1))x)$$

$$\approx e^{ax}$$

increasing in x if $a \geq 0$ i.e.

$w(\theta_2) \geq w(\theta_1)$.

Thm: If T has MLR prop. then a test of the form that rej. when

$$T > c$$

has a non-dec. power function.

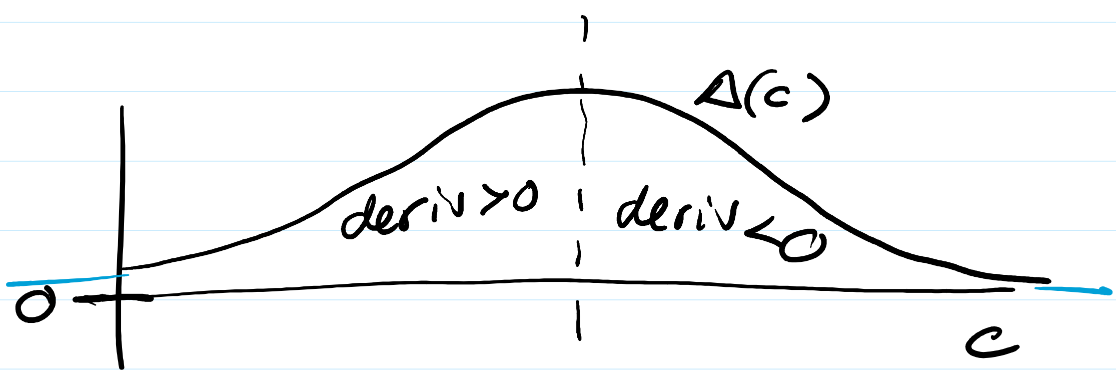
pf- show that $\theta_2 > \theta_1$ then $\beta(\theta_2) \geq \beta(\theta_1)$.

i.e. $P_{\theta_2}(T > c) \geq P_{\theta_1}(T > c)$

i.e. $1 - F_{\theta_2}(c) \geq 1 - F_{\theta_1}(c)$

F_θ is CDF of T w/ param θ

i.e. $\underbrace{F_{\theta_1}(c) - F_{\theta_2}(c)}_{\Delta(c)} \geq 0$



$$\lim_{c \to -\infty} \Delta(c) = 0 = \lim_{c \to \infty} \Delta(c)$$

$$\frac{d\Delta}{dc} = \frac{d}{dc}\left[F_{\theta_1}(c) - F_{\theta_2}(c)\right]$$

$$= f_{\theta_1}(c) - f_{\theta_2}(c)$$

$$= \underbrace{f_{\theta_1}(c)}_{\geq 0}\left(1 - \frac{f_{\theta_2}(c)}{f_{\theta_1}(c)}\right)$$

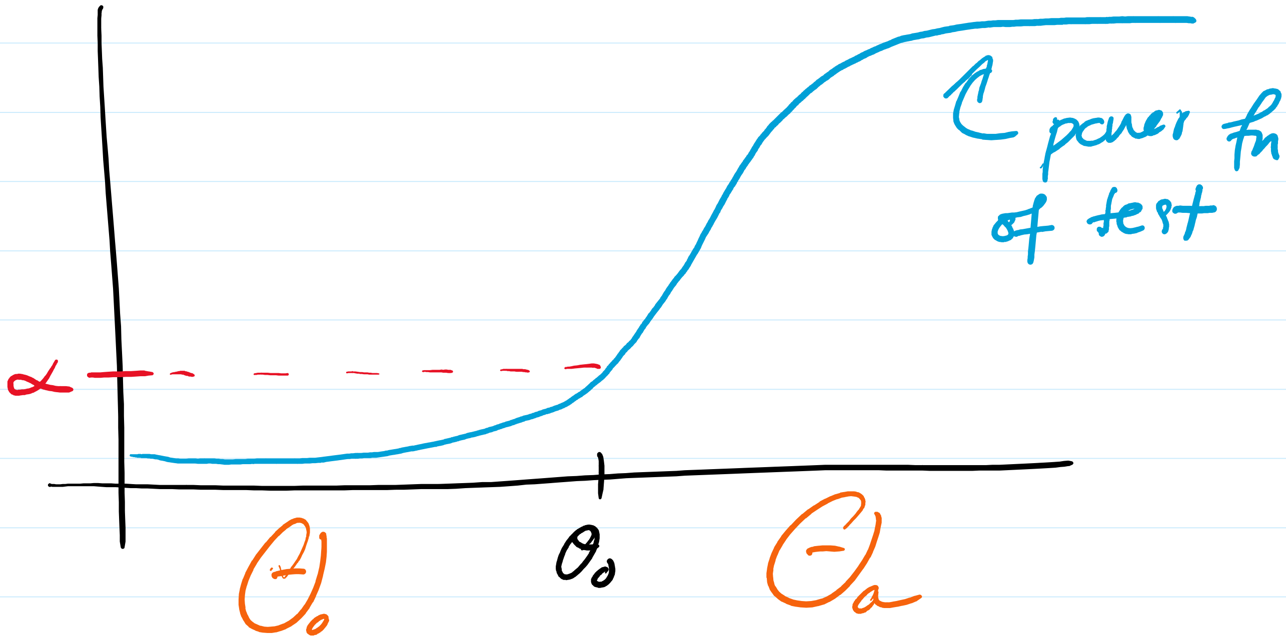
by MLR this is non-dec.

So overall $\frac{d\Delta}{dc}$ goes + to −.

<u>Why does this matter?</u>

$$H_0 : \theta \leq \theta_0 \quad v.\ H_a : \theta > \theta_0$$

and reject when $T > c$
and T has MLR



To ensure size α test just need to make

$$P_{\theta_0}(T > c) = \alpha.$$

[in general would need to ensure

$$\max_{\theta \in \Theta_0} P(T > c) = \alpha \text{]}$$

Thm: Karlin-Rubin

Consider testing

$$H_0 : \theta \le \theta_0 \quad v. \quad H_a : \theta > \theta_0$$

and let T be a sufficient stat for θ and have the MLR property.

Then the test that rejects when

$$T > c$$

where c chosen s.t.

$$P_{\theta_0}(T > c) = \alpha$$

is the UMP level α test.

Alt: $H_0: \theta \geq \theta_0$ v. $H_a: \theta < \theta_0$

then best test is to rej. when

$$T < c.$$

Ex. Let $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ σ^2 known.

Test $H_0: \mu > \mu_0$ v. $H_a: \mu < \mu_0$.

1. $T = \bar{X}$ is sufficient for μ.

2. MLR? $T = \bar{X} \sim N(\mu, \sigma^2/N)$

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2/N}} \exp\left(-\frac{N}{2\sigma^2}(t-\mu)^2\right)$$

$$= \frac{\sqrt{N}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{N}{2\sigma^2}(t^2 - 2\mu t + \mu^2)\right)$$

$$= \underbrace{\frac{\sqrt{N}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{Nt^2}{2\sigma^2}\right)}_{h(t)} \exp\left(\frac{N\mu t}{\sigma^2}\right) \cdot \underbrace{\exp\left(\frac{-N\mu^2}{2\sigma^2}\right)}_{c(\mu)}$$

$$\exp\left(\left(\frac{N\mu}{\sigma^2}\right)t\right)$$

$$w(\mu) = \frac{N\mu}{\sigma^2}$$

σ^2

Since $w(\mu)$ is non-dec in μ then this fam has MLR property

(3) So test is to reject when

$$\bar{X} > c$$

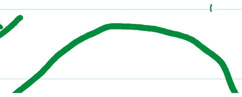
where c s.t.

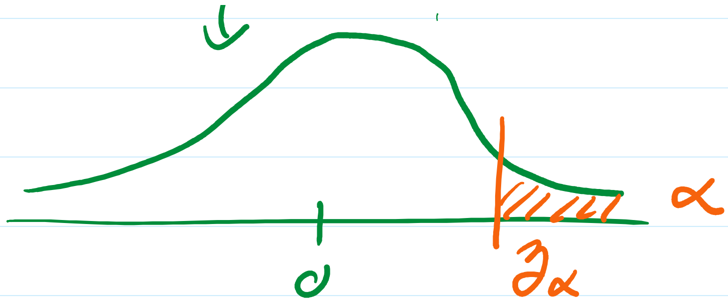
$$P_{\mu_0}(\bar{X} > c) = \alpha$$

When $\mu = \mu_0$, $\bar{X} \sim N(\mu_0, \sigma^2/N)$

so $$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \sim N(0,1)$$

$$P_{\mu_0}\left(\underbrace{\frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}}}_{N(0,1)} > \underbrace{\frac{c - \mu_0}{\sigma/\sqrt{N}}} \right) = \alpha$$





So $\dfrac{c-\mu_0}{\sigma/\sqrt{n}} = z_\alpha$

i.e. $\boxed{c = \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha}$

Confidence Intervals

point estimation: $\hat{\theta} = \hat{\theta}(\underset{\sim}{X}) \in \Theta$

idea: "$\hat{\theta} \approx \theta$"

interval/set estimation: $C = C(\underset{\sim}{X}) \subset \Theta$

idea: "$\theta \in C$"

Defn: Interval Estimator

An interval est of $\theta \in \Theta \subset \mathbb{R}$

An interval est of $\theta \in \Theta \subset \mathbb{R}$ is a pair of stats L and U s.t. $L(\underline{x}) \le U(\underline{x}) \ \forall \underline{x} \in \mathcal{X}$

<u>idea:</u> "$L \le \theta \le U$"

Defn: Coverage Prob

The coverage prob is

$$P_\theta(L \le \theta \le U)$$

deps on θ

<u>n.b.</u>: θ fixed, L, U random

Defn: Confidence Coef

Worst case coverage prob.

$$\underbrace{1-\alpha}_{} = \min_{\theta} P_\theta(L \le \theta \le U)$$

$$\underbrace{1-\alpha}_{\text{conf. coef.}} = \inf_\theta P_\theta(L \le \theta \le U)$$

Call an interval est + conf. coef a "confidence interval" (CI)

How do I build a CI?

Turns out that:

$$CI \iff HT$$

We can "invert" a size α HT to get a $1-\alpha$ CI.
(& vice-versa)

Aside: What is a level α HT?

Just a statement: "reject when $\underset{\sim}{x} \in R$"

s.t.

$$\max_{\theta \in \Theta_0} P(X \in R) \le \alpha.$$

$$\max_{\theta \in \Theta_0} P(\underset{\sim}{X} \in R) \leq \alpha.$$