Lecture 15 -- Hypothesis Testing

Tuesday, October 29, 2024 11:01 AM

Ex. Let θ be the percent. of defective items in a manufactering procedure

$$\theta \in \Theta = [0, 1]$$

$$H_0 : \theta \leq 0.1 \quad v. \quad H_a : \theta > 0.1$$

$$\Theta_0 = [0, 0.1] \qquad \Theta_a = (0.1, 1]$$

Ex. Let θ denote the change in blood pressure after some treatment

$$H_0 : \theta = 0 \quad v. \quad H_a : \theta \neq 0$$

$$\Theta = \mathbb{R}, \quad \Theta_0 = \{0\}, \quad \Theta_a = \mathbb{R} \setminus \{0\}$$

If θ is 1-dim'l :

① a test of the form

$$H_0 : \theta \leq c \quad v. \quad H_a : \theta > c$$

or $H_0 : \theta \geq c \quad v. \quad H_a : \theta < c$

$\quad : \quad < \quad : \quad \geq \quad :$

$\quad : \quad > \quad : \quad \leq$

is called a <u>one-sided test.</u>

② a test of the form

$$H_0 : \theta = c \quad v. \quad H_a : \theta \neq c$$

$\quad : \theta \neq c \quad : \quad \theta = c$

is called a <u>two-sided test</u>

③ a test of the form

$$H_0 : \theta = a \quad v. \quad H_a : \theta = b$$

is called a <u>simple test</u>.

<u>Idea</u>: want to collect data and use to determine which is more plausible H_0 or H_a.

Need to determine for which $\underset{\sim}{x}$ it is more plausible that $\theta \in \Theta_0$ v. for which $\underset{\sim}{x}$ more plausible $\theta \in \Theta_a$.

If $\mathcal{X}$ is the support of $\underset{\sim}{X}$

(typ. $\mathcal{X} = \mathbb{R}^N$)

then a hypothesis testing procedure is simply a partition of $\mathcal{X}$ into

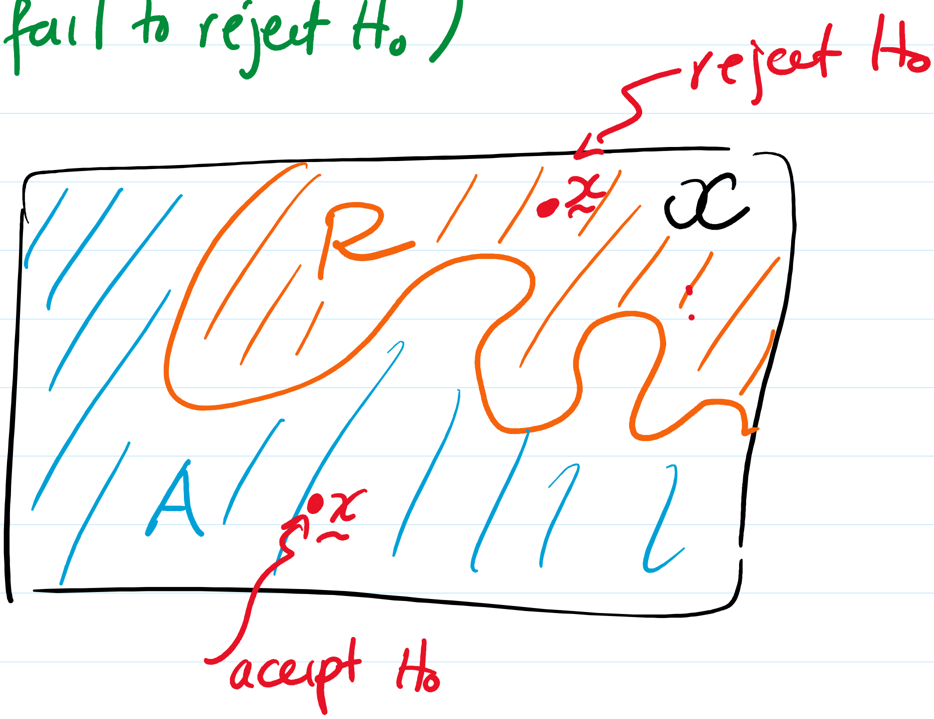
$$\mathcal{X} = A \cup R$$

disjoint

accept region

(accept H_0,
fail to reject H_0)

reject region
(rej. H_0)



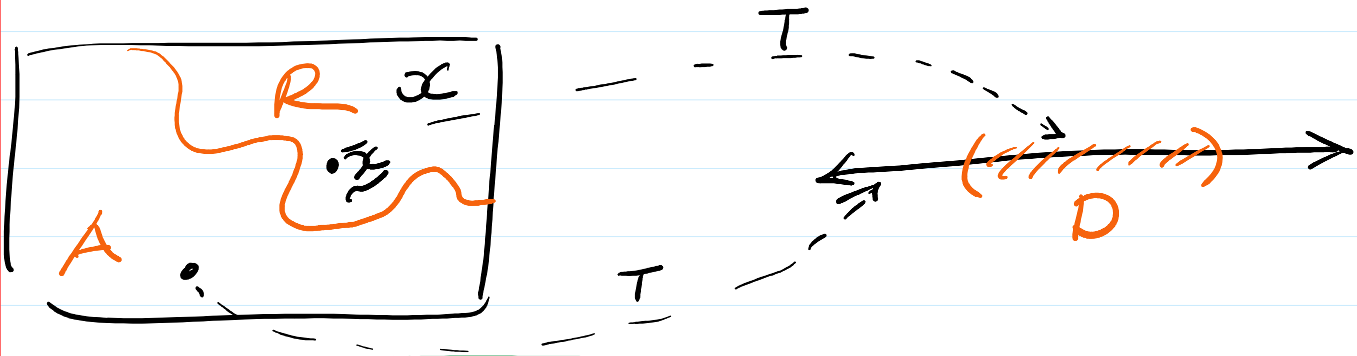
We "reject H_0" if $\underset{\sim}{x} \in R$

We "fail to reject H_0" if $\underset{\sim}{x} \in A$

Often, define R (equiv. A) through some "test statistic" T so that

$$R = \{ x \in \mathcal{X} : T(x) \in D \}$$

critical region (for D)



Ex. $X_n \overset{iid}{\sim} f_\theta$ with mean θ

$$H_0 : \theta > 5 \quad \text{v.} \quad H_a : \theta \leq 5$$

Let $T = \bar{X}$ and $D = (-\infty, 5)$

i.e. reject H_0 when $\bar{X} < 5$

Defn: Type I and II errors

Truth		
null true $\theta \in \Theta$...	Correct decision	Type I error

	accept H_0	reject H_0
null true $\theta \in \Theta_0$	Correct decision	Type I error
alt. true $\theta \in \Theta_a$	Type II error	Correct decision

Decision

Goal: come up w/ a HT procedure that minimizes type I and II errors

Often, minimizing type I error increases type II error, vice-versa.

Defn: Power Function

For any $\theta \in \Theta$ the power function β is defined as

$$\beta(\theta) = P_\theta(\underset{\sim}{X} \in R)$$

θ

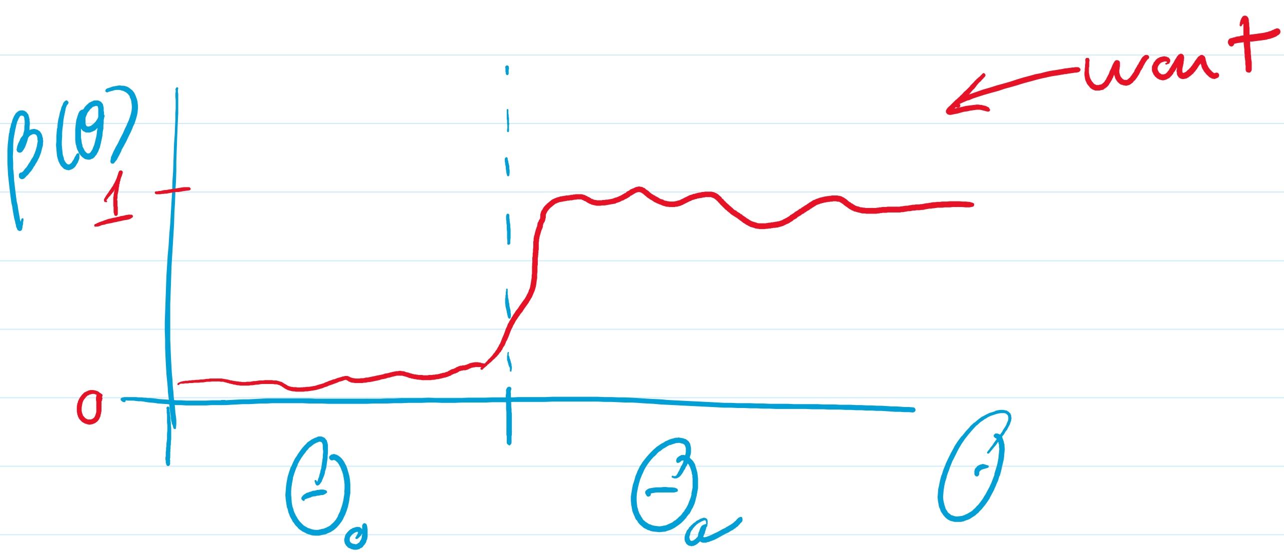
$\rightsquigarrow$ if true param is θ, prob I reject H_0

For $\theta \in \Theta_0$ [null true] then

$\beta(\theta)$ is the prob. of a type I error

For $\theta \in \Theta_a$ [alt. true] then

$\beta(\theta)$ is the prob. of correctly rej. H_0



equiv. when $\theta \in \Theta_a$ then

$$1 - \beta(\theta) = P(X \notin R)$$

$$1-\beta(\theta) = P(\underset{\sim}{X} \notin R)$$

is the prob. of a type II error.

Ex. $X_1, \dots, X_5 \overset{iid}{\sim} \text{Bern}(p)$, $0<p<1$

$$H_0 : p \le \tfrac{1}{2} \quad \text{v.} \quad H_a : p > 1/2$$

$$\Theta = [0,1], \ \Theta_0 = [0, 1/2], \ \Theta_a = (1/2, 1]$$

$$R = \{(1,1,1,1,1)\}$$

Can write in terms of a test stat:

$$T = \sum_{n=1}^{5} X_n \quad \text{and} \quad D = \{5\}$$

So that

$$R = \{ \underset{\sim}{x} \mid T(\underset{\sim}{x}) = 5 \}$$

What is β?

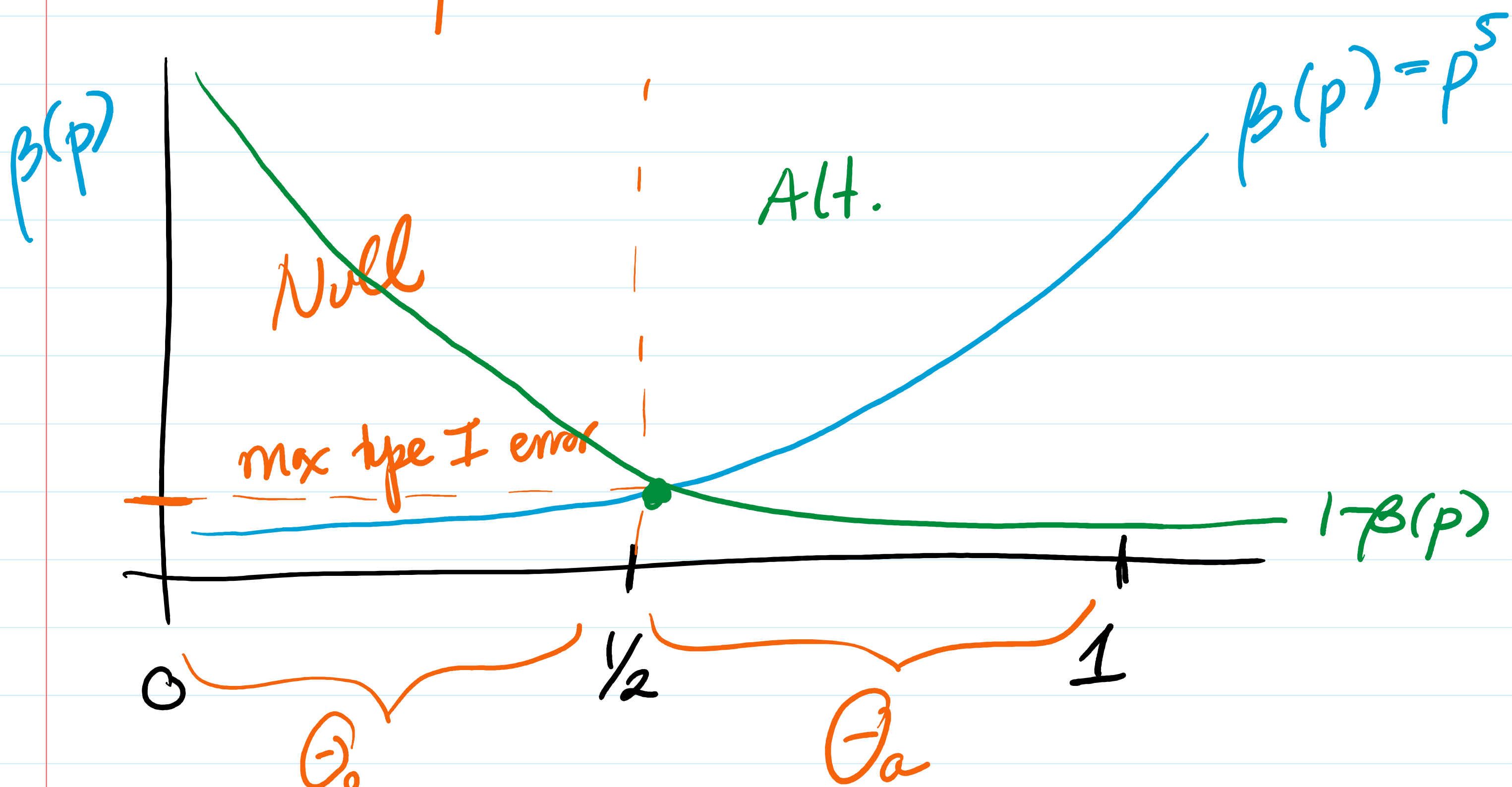
$T \sim Bin(5, p)$

$$\beta(p) = P_p(\underset{\sim}{X} \in R)$$

$$= P(T=5)$$

$$= \binom{5}{5} p^5 (1-p)^{5-5}$$

$$= p^5$$



① what is the max type I error prob?

type I error prob is $\beta(\theta) = P(\underset{\sim}{X} \in R)$ for $\theta \in \Theta_0$

So

$$\max_{\theta \in \Theta_0} \beta(\theta) = \max_{p \le 1/2} \beta(p)$$

$$= \beta(1/2) = (1/2)^5 = 1/32$$

(2) What's the max type II error prob?

prob. of type II err $= 1 - \beta(\theta)$ when $\theta \in \Theta_a$

So

$$\max_{\theta \in \Theta_a} 1 - \beta(\theta) = \max_{p > 1/2} 1 - p^5$$

$$= 1 - \beta(1/2)$$

$$= 1 - (1/2)^5 = 31/32$$

$= 1 - (1/2)^5 = 31/32$

Consider another test

$$R = \{ \underset{\sim}{x} \mid T \geq 3 \}$$

$$= \{ \underset{\sim}{x} \mid \bar{X} \geq 3/5 \}$$

$T \sim Bin(5, p)$

What's the power function

$$\beta(p) = P(\underset{\sim}{X} \in R) = P(T \geq 3)$$

$$= P(T=3) + P(T=4) + P(T=5)$$

$$= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + p^5$$

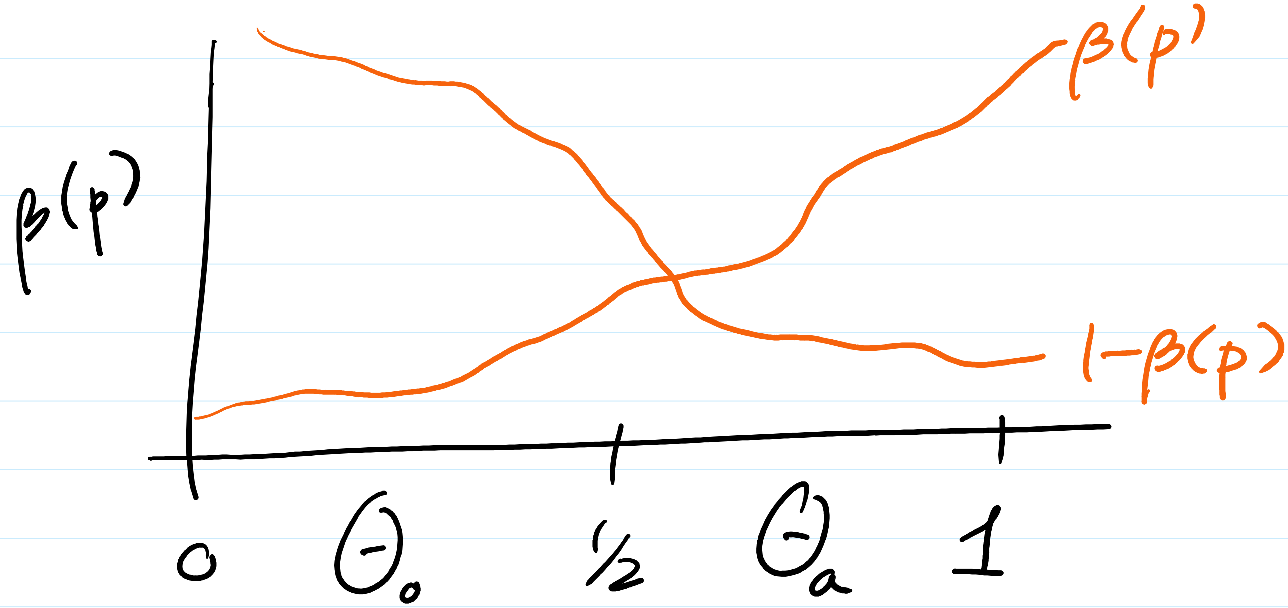
$$\vdots$$

$$= p^3 (6p^2 - 15p + 10)$$

notice: $\frac{\partial \beta}{\partial p} = 30p^2 (p-1)^2 > 0$

$\Rightarrow \beta$ is increasing

so β is increasing



(1) max type I error prob $= \beta(1/2)$

(2) max type II error prob $= 1-\beta(1/2)$.

Defn: Size and Level tests

A test is <u>size</u> $\alpha \in [0,1]$ if

$$\alpha = \max_{\theta \in \Theta_0} \beta(\theta) = \text{max. type I error}$$

a <u>level</u> α test is where

a <u>level</u> α test is where

$$\max_{\theta \in \Theta_0} \beta(\theta) \le \alpha .$$

<u>Game</u>: find test w/ <u>max. power</u> when $\theta \in \Theta_a$ s.t. it being either size α or level α.

ideal test