Ex. let $X_n \overset{iid}{\sim} N(\mu, 1)$ and let $N = 4$.

An interval est of μ

$$L = \bar{X} - 1$$

$$U = \bar{X} + 1$$

why use an interval est.? why not just $\bar{x}$?

Point est

notice: $P(\bar{X} = \mu) = 0$

so we could attach some uncertainty to $\bar{X}$

i.e. $sd(\bar{X}) = 1/\sqrt{4}$

Alt.: use an interval est b/c

$$P(\boxed{\bar{X} - 1 \le \mu \le \bar{X} + 1}) > 0$$

In this case:

$= P(\bar{X} - \mu \le 1 \text{ and } \bar{X} - \mu \ge -1)$

$= P(|\bar{X} - \mu| \le 1)$

$$= P\Big(\underbrace{\frac{|\bar{X} - \mu|}{\sqrt{1/4}}}_{N(0,1)} \le 2\Big) \approx .95$$

So $[\bar{X} - 1, \bar{X} + 1]$ is (approx.) a 95% CI.

E.x. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ known

Want: $1-\alpha$ CI for μ.

Do by: inverting a size α test for μ

(any statement about data that has prob α under H_0)

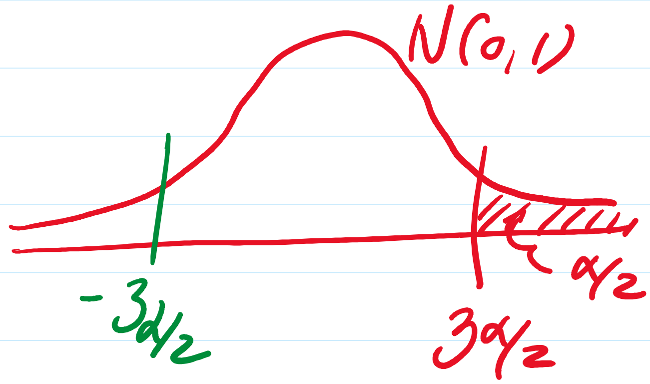
HT: $H_0: \mu = \mu_0$ v. $H_a: \mu \neq \mu_0$.

How abt:

$$\left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \right| > z_{\alpha/2}$$

under $H_0: \mu = \mu_0$ then

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \sim N(0,1)$$



So $$P\Big(\underbrace{\left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \right| > z_{\alpha/2}}_{\text{implicitly defines } R}\Big) = \alpha$$

So $A = \mathcal{X} \setminus R$

$$\to P\Big(\underbrace{\left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \right| \leq z_{\alpha/2}}_{\text{defines } A}\Big) = 1 - \alpha$$

defines A

"invert" = try to solve A for μ in middle...

manipulate

$$\boxed{P(L \le \mu_0 \le U) = 1 - \alpha}$$

in this case:

$$\left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \right| \le z_{\alpha/2}$$

$$\Leftrightarrow -z_{\alpha/2} \le \frac{\mu_0 - \bar{X}}{\sigma/\sqrt{N}} \le z_{\alpha/2}$$

$$\Leftrightarrow -\frac{\sigma}{\sqrt{N}} z_{\alpha/2} \le \mu_0 - \bar{X} \le \frac{\sigma}{\sqrt{N}} z_{\alpha/2}$$

$$\Leftrightarrow \underbrace{\bar{X} - \frac{\sigma}{\sqrt{N}} z_{\alpha/2}}_{} \le \mu_0 \le \underbrace{\bar{X} + \frac{\sigma}{\sqrt{N}} z_{\alpha/2}}_{}$$

$$\underbrace{\qquad\qquad}_{L} \qquad\qquad \underbrace{\qquad\qquad}_{U}$$

So $P(L \le \mu_0 \le U) = 1-\alpha$

So since this works for all μ_0 then

$$\min_{\mu} P(L \le \mu \le U) = 1-\alpha$$

i.e. $[L, U]$ is a $1-\alpha$ CI for μ.

Test Inversion

① Find a level (size) α test for

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta \ne \theta_0$$

i.e. a statement about $\underset{\sim}{X}$ s.t.

$$P_{\theta_0}(\text{statement}) = \alpha$$

this defines a test w/ rej. R

(2) Consider $A = \mathcal{X} \setminus R$ and try to "solve" for θ_0 in the middle:

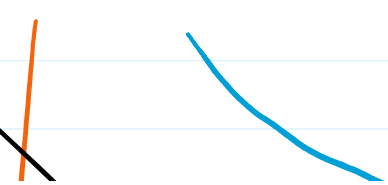
$$\underset{\sim}{X} \in A \rightsquigarrow L \le \theta_0 \le U$$

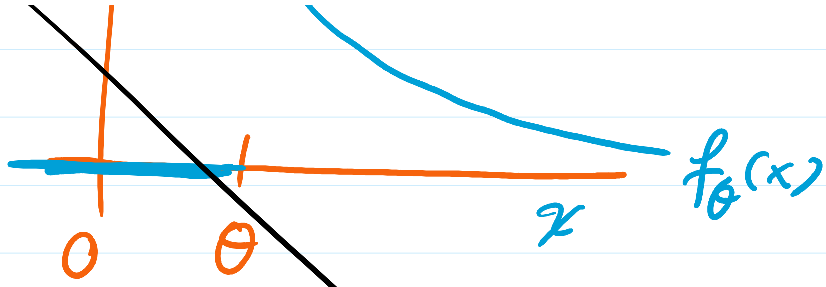
Then this $[L, U]$ is a $1-\alpha$ CI for θ.

Ex. $X_n \overset{iid}{\sim}$ Shifted Exp$(1, \theta)$

take exp and add θ

$$f_\theta(x) = e^{-(x-\theta)} \mathbb{1}(x>0)$$





Consider: $H_0: \theta \leq 0$ v.

Ex. let $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

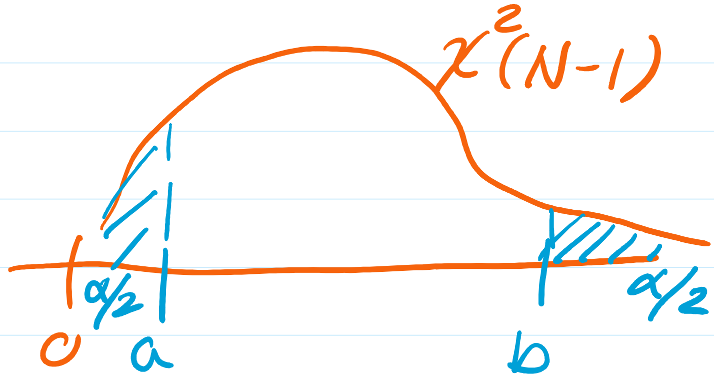
both unknown

Create a CI for σ^2.

Let's get size α test for

$$H_0: \sigma^2 = \sigma_0^2 \quad \text{v.} \quad H_a: \sigma^2 \neq \sigma_0^2.$$

Know: $\frac{N-1}{\sigma^2} S^2 \sim \chi^2(N-1)$



$\alpha/2$ $\alpha/2$

0 a b

choose a, b s.t.

$$P(\chi^2(N-1) \geq b) = \alpha/2$$

$$P(\chi^2(N-1) \leq a) = \alpha/2$$

then under $H_0: \sigma^2 \overset{?}{=} \sigma_0^2$

$$P\Big(\underbrace{\frac{N-1}{\sigma_0^2} S^2 \leq a \text{ or } \frac{N-1}{\sigma_0^2} S^2 \geq b}_{\text{defines } R}\Big)$$

$$= \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$$

defines R

$A = \mathcal{X} \setminus R$ is just complement of this

i.e. under $H_0: \sigma^2 = \sigma_0^2$

$$P\left(a \leq \frac{N-1}{\sigma_0^2} S^2 \leq b\right) = 1 - \alpha$$

try to solve this statement for

Try to solve this statement for

$$L \le \sigma_0^2 \le U$$

$$\iff \frac{1}{a} \ge \frac{\sigma_0^2}{(N-1)S^2} \ge \frac{1}{b}$$

$$\boxed{\begin{array}{c} c \le d \\ \frac{1}{c} \ge \frac{1}{d} \end{array}}$$

$$\iff \underbrace{\frac{(N-1)S^2}{b}}_{L} \le \sigma_0^2 \le \underbrace{\frac{(N-1)S^2}{a}}_{U}$$

So $[L, U]$ is a $1-\alpha$ CI for σ_0^2.

Pivoting : drop test pretence and jump to a statement w/ prob. $1-\alpha$

Easiest way to do:

1. Find some ancillary quant
$$Q = Q(X, \theta)$$

$$Q = Q(\underset{\sim}{X}, \theta)$$

[dist of Q doesn't depend on θ]

<u>Called</u>: pivot.

(2) Find some region A [doesn't dep on θ] so that

$$P(Q \in A) \geq 1-\alpha$$

(3) Solve $Q \in A$ for θ in middle:

$$L \leq \theta \leq U.$$

Then $[L, U]$ is a $1-\alpha$ CI for θ.

<u>reason this works:</u>

no θ in dist

no θ

$$P(L \leq \theta \leq U) = \underbrace{P(Q \in A)}_{\text{doesn't dep on } \theta} = 1-\alpha$$

doesn't dep on θ

So $$\min_{\theta} P(L \le \theta \le U) = 1-\alpha$$

Ex. $X_n \overset{iid}{\sim} \text{Exp}(\lambda)$

Want: $1-\alpha$ CI for λ.

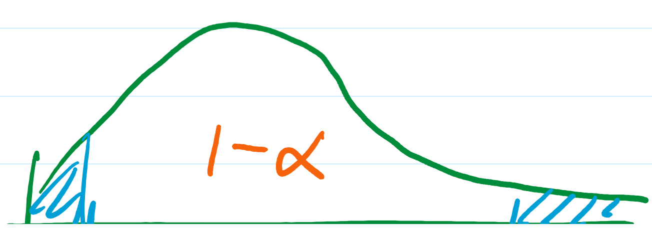
① Get pivot:

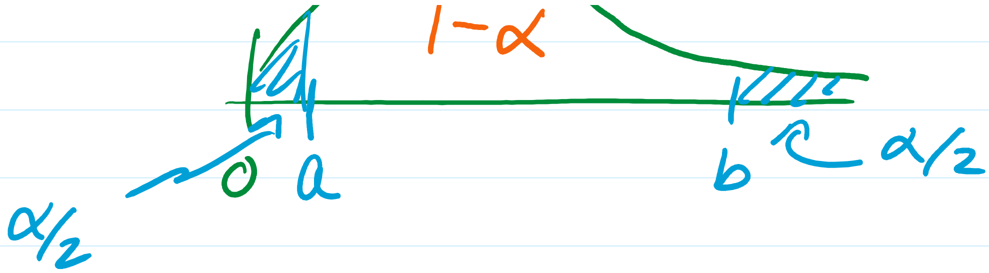
$$T = \sum_n X_n \sim \text{Gama}(N, \lambda)$$

$$Q = \frac{2T}{\lambda} \sim \text{Gamma}(N, 2) = \chi^2(2N)$$

is a pivot!

② Need A s.t. $P(Q \in A) = 1-\alpha$





let $A = [a, b]$ then $P(Q \in A) = 1-\alpha$

(3) $Q \in A$

$$\iff a \le \frac{2T}{\lambda} \le b$$

$$\iff \frac{1}{a} \ge \frac{\lambda}{2T} \ge 1/b$$

$$\iff \underbrace{\frac{2T}{b}}_{L} \le \lambda \le \underbrace{\frac{2T}{a}}_{U}$$

Continuous Pivots

Recall that if $X \sim F_X$ then

$$Q = F_X(X) \sim U(0,1)$$

always a pivot

← always a pivot

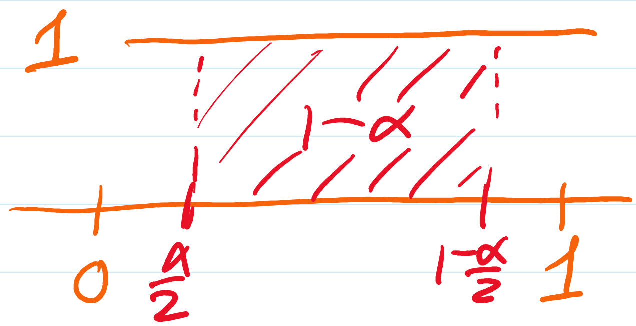
Let T be a stat (dist of T may dep on θ)

(1) $Q = F_T(T) \sim U(0,1)$
is a pivot

(2) Let $A = [a, b]$ where $a = \frac{\alpha}{2}$, $b = 1 - \frac{\alpha}{2}$

then

$$P(Q \in A) = P(a \le Q \le b)$$
$$= P\left(\frac{\alpha}{2} \le Q \le 1 - \frac{\alpha}{2}\right) = 1 - \alpha$$



(3) Solve $\frac{\alpha}{2} \le Q \le 1 - \alpha/2$ for

(3) Solve $\frac{\alpha}{2} \le Q \le 1 - \alpha/2$ for my param θ in middle:

$$\frac{\alpha}{2} \le F_T(T) \le 1 - \alpha/2$$

F_T deps on θ

let $g(\theta) = F_T$ as a fn of θ

$$\frac{\alpha}{2} \le g(\theta) \le 1 - \frac{\alpha}{2}$$

If g is increasing fn (invertible)

so

$$\underbrace{g^{-1}\left(\tfrac{\alpha}{2}\right)}_{L} \le \theta \le \underbrace{g^{-1}(1-\alpha/2)}_{U}$$

if g is dec. then

$$\underbrace{g^{-1}(1-\alpha/2)}_{L} \le \theta \le \underbrace{g^{-1}(\alpha/2)}_{U}$$

$$\underbrace{g(\ldots)}_{L} \quad \underbrace{g(\ldots)}_{U}$$

Theorem: Universal Continuous Pivot

If T is a stat w/ CDF F_T and

g is F_T as a fn θ

then a $1-\alpha$ CI for θ is

(a) $L = g^{-1}(\alpha/2)$, $U = g^{-1}(1-\alpha/2)$

if g is increasing (in θ)

(b) $L = g^{-1}(1-\alpha/2)$, $U = g^{-1}(\alpha/2)$

if g is dec.

Ex. Let T be a stat w/ CDF

$$F_T(t) = \frac{1}{1 + \exp(-(t-\mu))}$$

(μ is my param)

$$g(\mu) = \frac{1}{1+\exp(-(t-\mu))}$$

my claim: dec. in μ.

$$g^{-1}(y) = t + \log\left(\frac{1}{y} - 1\right)$$

then a $1-\alpha$ CI for μ is

$$L = g^{-1}(1-\alpha/2)$$

$$= \boxed{t + \log\left(\frac{1}{1-\alpha/2} - 1\right)}$$

$$U = g^{-1}(\alpha/2)$$

$$= \boxed{t + \log\left(\frac{1}{\alpha/2} - 1\right)}$$