

marginal dist \rightarrow
$$= f(x_1) f(x_2) \cdots f(x_N) \quad [\text{by independence}]$$

$$= \prod_{n=1}^N f(x_n)$$

[similar notation to $\sum_{i=1}^N x_i = x_1 + x_2 + \cdots$]

Ex. If $X_n \stackrel{iid}{\sim} \text{Exp}(\lambda)$

What's the joint?

$$f(\underline{x}) = \prod_{n=1}^N f(x_n)$$

$$= \prod_{n=1}^N (\lambda e^{-\lambda x_n} \mathbb{1}(x_n > 0))$$

$$= \lambda^N e^{-\lambda \sum_{n=1}^N x_n}$$

$\mathbb{1}(x_1 > 0) \mathbb{1}(x_2 > 0) \cdots \mathbb{1}(x_N > 0)$ Often see

If $Y \sim \text{Exp}(\lambda)$

$$f(y) = \lambda e^{-\lambda y} \text{ for } y > 0$$

$$= \lambda e^{-\lambda y} \mathbb{1}(y > 0)$$

indicator function

$$\mathbb{1}(\text{statement}) = \begin{cases} 0 & \text{statement} = \text{False} \\ 1 & \text{statement} = \text{True} \end{cases}$$

$$f(y) = \begin{cases} \infty & \mathbb{1}(y \in \text{Support}) \\ 0 & \text{else} \end{cases}$$

Defn: Statistic

Given (a random sample) $X_n \stackrel{iid}{\sim} f$ a

Statistic is a function $T: \mathbb{R}^N \rightarrow \mathbb{R}^d$

Statistic is a function $\bar{T}: \mathbb{R}^N \rightarrow \mathbb{R}^d$.
sample size

Typically $d \ll N$, often $d=1$.

input N numbers \xrightarrow{T} summary in lower dimension.

Ex. Arithmetic mean ($d=1$)

$$T(\underline{X}) = \frac{1}{N} \sum_{n=1}^N X_n = \bar{X}_N$$

Sample Variance

$$S_{N-1}^2 = \frac{1}{N-1} \sum_{n=1}^N (X_n - \bar{X}_N)^2$$

Sample SD:

$$S_{N-1} = \sqrt{S_{N-1}^2}$$

Minimum: $X_{(1)}$

Maximum: $X_{(N)}$

Range: $X_{(N)} - X_{(1)}$.

Defn: Sampling Distribution

A sampling distribution of a statistic T is the distribution of $T(\underline{X})$.

Theorem:

Sampling dist.

If $X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ then $\bar{X}_N \sim N(\mu, \sigma^2/N)$.

Theorem: Sums of RVs.

Let $g: \mathbb{R} \rightarrow \mathbb{R}$ and $X_n \stackrel{iid}{\sim} f$

$$\textcircled{1} \quad \mathbb{E}\left[\sum_{n=1}^N g(X_n)\right] = N \mathbb{E}[g(X_1)]$$

recall: $\mathbb{E}[A+B] = \mathbb{E}[A] + \mathbb{E}[B]$

then

expectation of any of them could use X_2, X_3, \dots

$$\mathbb{E}[g(X_1)] = \mathbb{E}[g(X_2)] = \dots$$

$$\begin{aligned} \mathbb{E}\left[\sum_{n=1}^N g(X_n)\right] &= \sum_{n=1}^N \mathbb{E}[g(X_n)] \leftarrow \\ &= N \mathbb{E}[g(X_1)]. \end{aligned}$$

$$\textcircled{2} \quad \text{Var}\left(\sum_{n=1}^N g(X_n)\right) = N \text{Var}(g(X_1)) \leftarrow$$

proof?

Similar to above using fact that $\text{Var}(A+B) = \text{Var}(A) +$

Var(B)

if $A \perp B$.
(works if uncorrelated)

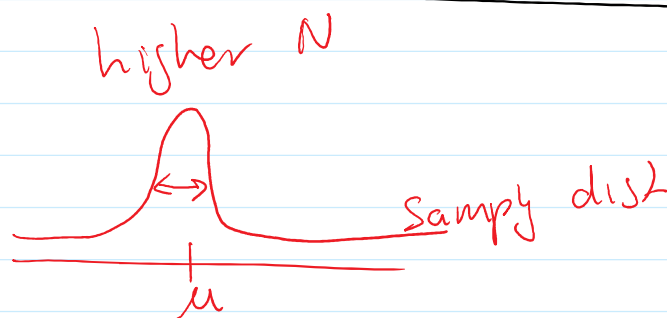
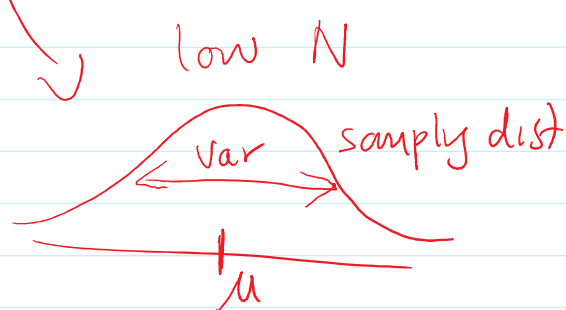
Theorem: If $X_n \stackrel{iid}{\sim} f$ and
 $E[X_n] = \mu$ and $Var(X_n) = \sigma^2$

Then

① $E[\bar{X}_N] = \mu$ (avg. value of \bar{X}_N is μ)

② $Var(\bar{X}_N) = \sigma^2/N$ (more data is better)

③ $E[S_{N-1}^2] = \sigma^2$ (on average correctly estimates σ^2)



proof. ① $E[\bar{X}_N] = \mu$

$$E\left[\frac{1}{N} \sum_{n=1}^N X_n\right] = \frac{1}{N} \sum_{n=1}^N E[X_n] = \frac{1}{N} \sum_{n=1}^N \mu = \frac{1}{N} N\mu = \mu$$

$$(2) \text{Var}(\bar{X}_N) = \text{Var}\left(\frac{1}{N} \sum_{n=1}^N X_n\right) = \frac{1}{N^2} \text{Var}\left(\sum_{n=1}^N X_n\right)$$

= by prev. theorem

$$= \frac{1}{N^2} N \text{Var}(X_1)$$

$$= \frac{1}{N} \sigma^2$$

$$(3) \text{E}[S_{N-1}^2]$$

Fact: $\sum_{n=1}^N (X_n - \bar{X}_N)^2$
 $= \sum_{n=1}^N X_n^2 - N \bar{X}_N^2$

$$\frac{1}{N-1} \text{E}\left[\sum_{n=1}^N X_n^2 - N \bar{X}_N^2\right]$$

(recall: $\text{Var}(Y) = \text{E}[Y^2] - \text{E}[Y]^2$)

$$= \frac{1}{N-1} \text{E}\left[\sum_{n=1}^N X_n^2\right] - \frac{N}{N-1} \text{E}[\bar{X}_N^2]$$

Fact: $\text{E}[Y^2] = \text{Var}(Y) + \text{E}[Y]^2$

$$= \frac{1}{N-1} \sum_{n=1}^N \text{E}[X_n^2] - \frac{N}{N-1} \text{E}[\bar{X}_N^2]$$

recall $\text{E}[X_n] = \mu$ $\text{E}[\bar{X}_N] = \mu$
 $\text{Var}(X_n) = \sigma^2$ $\text{Var}(\bar{X}_N) = \sigma^2/N$

$$= \frac{1}{N-1} N(\sigma^2 + \mu^2) - \frac{N}{N-1} (\sigma^2/N + \mu^2)$$

$$= \text{algebra} \dots$$

$$= \sigma^2$$

Saw: $E[\bar{X}_N] = \mu$ \bar{X}_N is unbiased for μ
 $E[S_{N-1}^2] = \sigma^2$ S_{N-1}^2 is unbiased for σ^2

Defn: Unbiased

We say a statistic T is unbiased for a quantity θ if $E[T(X)] = \theta$.

Theorem: MGFs of \bar{X}_N .

Let $X_n \stackrel{iid}{\sim} f$ then

$$M_{\bar{X}_N}(t) = \left(M_{X_1}(t/N)\right)^N$$

proof.

$$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E\left[e^{t \frac{1}{N} \sum_{n=1}^N X_n}\right]$$

$$= E\left[\prod_{n=1}^N e^{t \frac{1}{N} X_n}\right] \quad (\text{property of exp } e^{a+b} = e^a e^b)$$

$$= \prod_{n=1}^N E\left[e^{t \frac{1}{N} X_n}\right] \quad (\text{by independence})$$

$$= \prod_{n=1}^N M_{X_n}(t/N) \quad (E[AB] = E[A]E[B])$$

$$= \underbrace{M_{X_1}(t/N) M_{X_1}(t/N) \cdots M_{X_1}(t/N)}_N$$

$$= (M_{X_1}(t/N))^N$$

Theorem
Let $X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ then $\bar{X}_N \sim N(\mu, \sigma^2/N)$.

proof

$$M_{\bar{X}}(t) = (M_{X_1}(t/N))^N$$

For $X_1 \sim N(\mu, \sigma^2)$

then

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

$$= \left(\exp\left(\mu t/N + \frac{\sigma^2 t^2}{2N^2}\right)\right)^N$$

$$\left[(e^a)^b = \overset{\text{algebra!}}{e^{ab}} \right]$$

$$= \exp\left(\mu t + \frac{\sigma^2}{N} t^2\right)$$

Notice that \uparrow is the MGF of a $N(\mu, \sigma^2/N)$

So $\bar{X} \sim N(\mu, \sigma^2/N)$.