Review:

Defn: Almost Sure Convergence

$X_n \xrightarrow{a.s.} X$ if $X_n(s) \to X(s)$ for all $s \in A$

where $\mathbb{P}(A) = 1$.

$(\mathbb{P}(\{s \mid X_n(s) \to X(s)\}) = 1$.

Defn: Convergence in Probability

We say $X_n \xrightarrow{P} X$ if

$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1$

$\to \left(\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0 \right)$

Notice: ① $X_n \xrightarrow{a.s.} X \iff \mathbb{P}(\lim_{n\to\infty} |X_n - X| < \varepsilon) = 1$

② $X_n \xrightarrow{P} X \iff \lim_{n\to\infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1$

switch lim and $\mathbb{P}$

Theorem: a.s. $\Rightarrow$ i.p.

If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{P} X$.

(generally, the converse is false)
(a.s. is stronger than i.p.)

Ex. Consider $S=[0,1]$ w/ uniform density

$X_1(s) = \cancel{s}+1$

$X_2(s) = \cancel{s}+\mathbb{1}_{[0,1/2]}(s)$

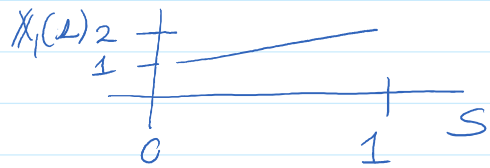
$X_3(s) = s+\mathbb{1}_{[1/2,1]}(s)$

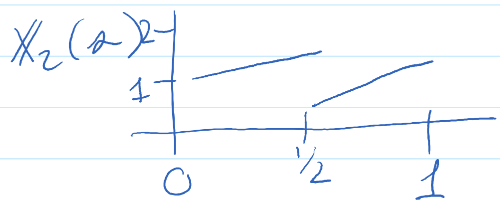
$X_4(s) = s+\mathbb{1}_{[0,1/3]}(s)$

$X_5(s) = s+\mathbb{1}_{[1/3,2/3]}(s)$

$X_6(s) = s+\mathbb{1}_{[2/3,1]}(s)$

$X_7(s) = s+\mathbb{1}_{[0,1/4]}(s)$





Let $X(s)=s$

Q: $X_n \xrightarrow{P} X$?

Need to show:

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$$

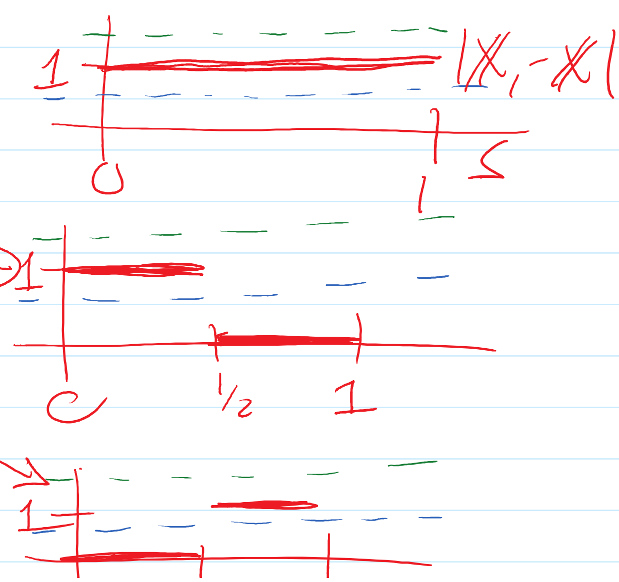
$|X_1 - X| = |1| = 1$

$|X_2 - X| = \mathbb{1}_{[0,1/2]}$

$|X_3 - X| = \mathbb{1}_{[1/2,1]}$

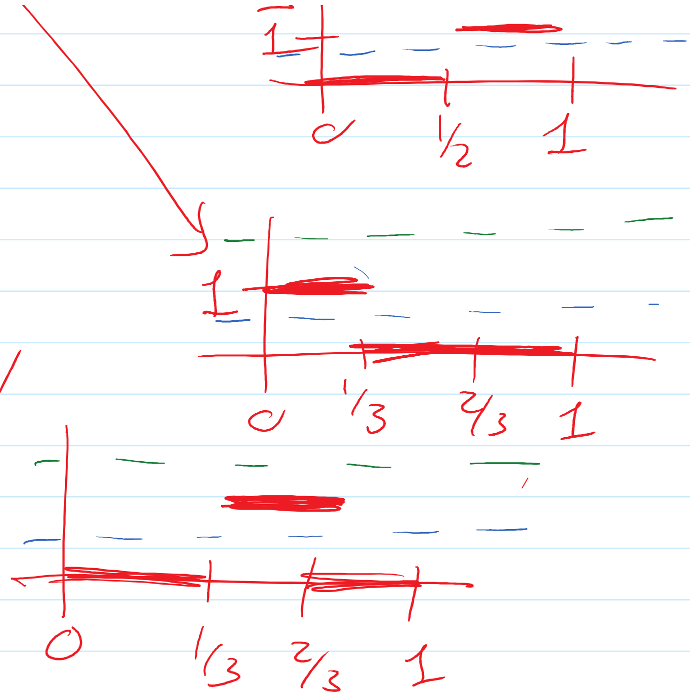
$|X_4 - X| = \mathbb{1}_{[0,1/3]}$

$|X_5 - X| = \mathbb{1}_{[1/3,2/3]}$



$$|X_5 - X| = \mathbb{1}_{[1/3,\, 2/3]}$$

$\vdots$



if $\boxed{\varepsilon > 1}$ then $P(|X_n - X| \geq \varepsilon) = 0 \quad \forall n$

so certainly

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

if $\boxed{0 < \varepsilon < 1}$

$P(|X_1 - X| \geq \varepsilon) = 1$

$P(|X_1 - X| \geq \varepsilon) = 1/2$

$P(|X_2 - X| \geq \varepsilon) = 1/2$

$\vdots \qquad = 1/3$ — this has a limit of zero.

$\vdots \quad 1/3$

$= 1/3 \qquad \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$

$\vdots \qquad = 1/4$

$\vdots$

$-\frac{1}{4}$

$\vdots$

So ... $\boxed{X_n \xrightarrow{P} X.}$

Does $X_n \xrightarrow{a.s.} X$?

$$A = \{ s \in S \mid X_n(s) \to X(s) \}$$

$$X_n \xrightarrow{a.s.} X \iff P(A) = 1.$$

pick any $s \in [0,1]$ consider $X_1(s), X_2(s), X_3(s), \ldots$

$X_n(s)$ will oscillate between s and $s+1$ forever

So it won't settle down, it has no limit

$$\forall s \quad X_n(s) \not\to X(s)$$

So $P(A) = 0$.

Hence $X_n \overset{a.s.}{\not\to} X$.

Defn: Convergence in distribution

We say (X_n) converges in distribution to X denoted

$$X_n \xrightarrow{d} X$$

if the CDFs converge pointwise.

i.e. if F_{X_n} is the CDF of X_n and F is the CDF of X then $\forall x \in \mathbb{R}$

$$F_{X_n}(x) \longrightarrow F(x)$$

$$\left(F_{X_n} \xrightarrow{\text{ptwise}} F \right)$$

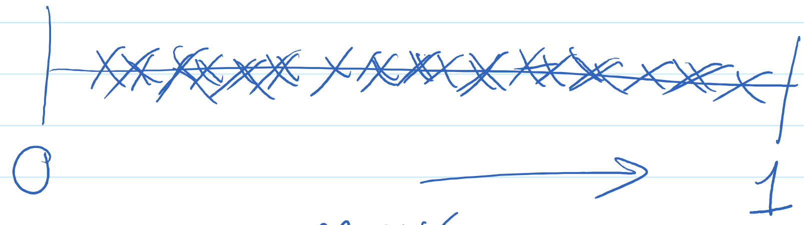
Convergence in dist

→ not that X_ns converge ptwise

→ the F_{X_n}s converge ptwise

Ex. $X_n \overset{iid}{\sim} U(0,1)$

and let $Y_n = \max\limits_{i=1,\dots,n} X_i$ = max of first n X_is

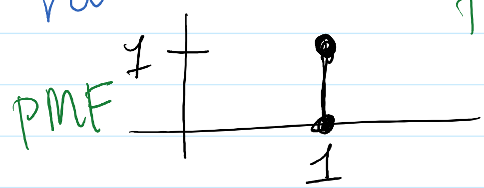


max heads towards 1

$\boxed{Y_n \xrightarrow{P} 1}$

degenerate random variable Y

$f(x) = \begin{cases} 1, & x = 1 \\ 0 & \text{else} \end{cases}$



$P(Y = 1) = 1$

want to show

$\boxed{P(|Y_n - Y| \geq \varepsilon) \xrightarrow{n} 0}$ notation for $\lim_{n \to \infty} \ldots$

$\downarrow$

$P(|Y_n - 1| \geq \varepsilon) \xrightarrow{n} 0$

$\parallel$

$P(|1 - Y_n| \geq \varepsilon)$ $\quad (Y_n \leq 1)$

$= P(1 - Y_n \geq \varepsilon)$

$= P(Y_n \leq 1 - \varepsilon)$ $\quad \max_{i=1,\ldots,n} X_i \leq 1 - \varepsilon$

$= P(\max_{i=1,\ldots,n} X_n \leq 1 - \varepsilon)$

$= P(X_1 \leq 1-\varepsilon, X_2 \leq 1-\varepsilon, X_3 \leq 1-\varepsilon, \ldots, X_n \leq 1-\varepsilon)$

$\prod_{i=1}^{n} P(X_i \leq 1 - \varepsilon)$ (indep)

$$= \prod_{i=1}^{n} P(X_i \le 1-\varepsilon) \quad \text{(indep)}$$

$$= \prod_{i=1}^{n} (1-\varepsilon) \quad \boxed{0 \le 1-\varepsilon \le 1} \quad \text{(identical dist)}$$

$0 \le \varepsilon \le 1$ $\qquad P(X_i \le a) = a$ for $0<a<1$

$$= \boxed{(1-\varepsilon)^n \xrightarrow{n} 0}$$

1 $\to$ for $0 \le \varepsilon \le 1$

2 $\boxed{\text{If } \varepsilon > 1 \quad P(|Y_n - 1| \ge \varepsilon) = 0 \quad \forall n}$

In both cases

$$P(|Y_n - 1| \ge \varepsilon) \xrightarrow{n} 0.$$

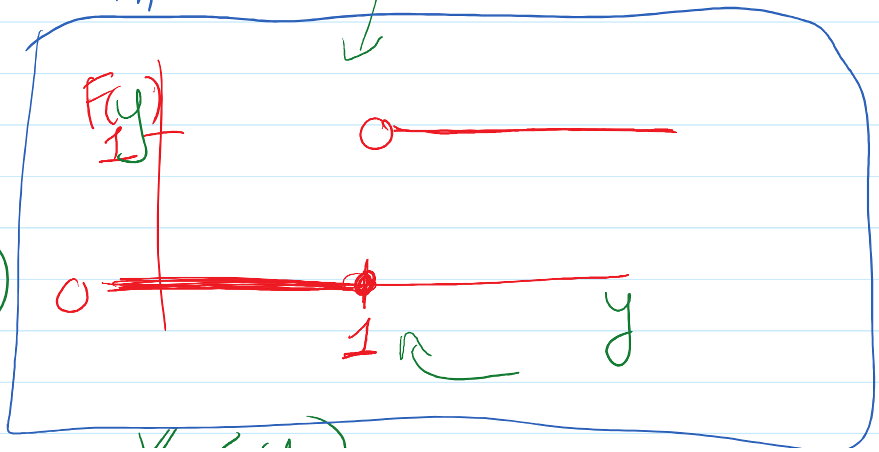
hence $Y_n \xrightarrow{P} 1$.

$$\boxed{Y_n \xrightarrow{d} 1}$$

$F_{Y_n} \longrightarrow F$

$$F_{Y_n}(y) = P(Y_n \le y)$$

$$= P(\max_{i=1,\ldots,n} X_n \le y)$$

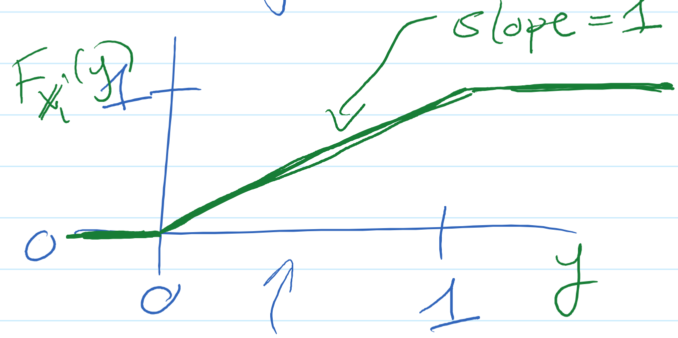


$$= \mathbb{P}(X_1 \le y, X_2 \le y, \dots, X_n \le y)$$

$$= \prod_{i=1}^{n} \mathbb{P}(X_i \le y) \qquad \text{(indep)}$$

$$= \begin{cases} y^n \xrightarrow{n} 0 & 0 \le y \le 1 \\ 1 & y \ge 1 \\ 0 & y \le 0 \end{cases}$$

CDF of X_i



limit as $n \to \infty$

$$= \begin{cases} 0 & 0 \le y \le 1 \\ 1 & y \ge 1 \\ 0 & y \le 0 \end{cases} = F(y)$$

Show $F_{Y/n} \xrightarrow{\text{ptwse}} F$

hence $Y/n \xrightarrow{d} 1.$

Theorem: i.p. $\Rightarrow$ d

If $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{d} X$.

Corollary: chain

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$$

$$\boxed{a.s. \Rightarrow p \Rightarrow d.}$$

Revisit this example

$$\boxed{Z_n = n(1 - Y_n)}$$ ← distributional limit?

$Y_n = \max_{i=1,\dots,n} X_i$ and $X_i \overset{iid}{\sim} U(0,1)$

$$\boxed{F_{Z_n}(z)} = P(Z_n \le z)$$

$$= P(n(1-Y_n) \le z)$$

$$= P(1 - Y_n \le z/n)$$

$$= P(Y_n \ge 1 - z/n)$$

$$= 1 - P(Y_n \le 1 - z/n)$$

$$= 1 - \prod_{i=1}^{n} P(X_i \le 1 - z/n) \quad [X_i \sim U(0,1)]$$

$$= 1 - \prod_{i=1}^{n} F_{X_i}(1 - z/n)$$

$$F_{X_i}(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & t \le 0 \\ 1, & t \ge 1 \end{cases}$$

$$= 1 - F_{\cdot}(1 - z/\cdot)^n$$

$$(1, \ t \geq 1$$

$$= 1 - F_{X_i}(1 - z/n)^n$$

$$= \begin{cases} \boxed{1 - (1 - z/n)^n} & 0 \leq 1 - z/n \leq 1 \quad [0 \leq z/n \leq 1] \\ 1 & 1 - z/n \leq 0 \quad [z/n \geq 1] \\ 0 & 1 - z/n \geq 1 \quad [z/n \leq 0] \end{cases}$$

what is the limit of this?

<u>Recall</u>:

① $$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

② $$\lim_{n \to \infty} (1 + c/n)^n = e^c$$

③ $$\lim_{n \to \infty} 1 - (1 + c/n)^n = 1 - e^c$$

let $c = -z$

then

$$\underbrace{F_{Z_n}(z)} = 1 - (1 + -z/n)^n \longrightarrow \underbrace{1 - e^{-z}}_{Exp(1)}$$

$$\boxed{Z_n = n(1 - Y/n) \xrightarrow{d} Exp(1).}$$

Know: a.s. $\Rightarrow$ p $\Rightarrow$ d

partial converse

If $X_n \xrightarrow{d} c$ const./degenerate dist concentrated at c

then $X_n \xrightarrow{P} c$.

[Says: if limit is const. i.p. = i.d.]