Review of last time:

Theorem: Rao - Blackwell

let $\hat{\theta}$ is unbiased for $\tau(\theta)$ and W is a sufficient stat for θ then if

$$\varphi = \varphi(W) = \mathbb{E}[\hat{\theta} \mid W]$$

(Rao-Blackwellization)

(1) $\mathbb{E}[\varphi] = \tau(\theta)$

(2) $Var(\varphi) \leq Var(\hat{\theta})$

(3) φ is a statistic (it doesn't depend on θ).

Theorem: UMVUEs are Unique

proof by contradiction

assume we have 2 UMVUEs W_1 and W_2 of $\tau(\theta)$. Assume that $W_1 \neq W_2$.

Consider: $W_3 = \frac{1}{2}(W_1 + W_2)$

notice: $\mathbb{E}[W_3] = \frac{1}{2}\mathbb{E}[W_1] + \frac{1}{2}\mathbb{E}[W_2]$

$$= \frac{1}{2}\tau(\theta) + \frac{1}{2}\tau(\theta) = \tau(\theta)$$

So W_3 is unbiased for $\tau(\theta)$

Also:

$$Var(W_3) = Var\left(\frac{1}{2}W_1 + \frac{1}{2}W_2\right)$$

$$= \frac{1}{4}Var(W_1) + \frac{1}{4}Var(W_2)$$

$$+ \frac{1}{2}\boxed{Cov(W_1, W_2)}$$

last time:

$$Cov(W_1, W_2)^2 \le Var(W_1)\,Var(W_2)$$

equiv.

$$Cov(W_1, W_2) \le \sqrt{Var(W_1)\,Var(W_2)}$$

(*) really =

Here,

$$Var(W_3) \le \frac{1}{4}Var(W_1) + \frac{1}{4}Var(W_2)$$

$$+ \frac{1}{2}\sqrt{Var(W_1)\,Var(W_2)}$$

Since W_1 and W_2 are both UMVUEs then

$$Var(W_1) = Var(W_2)$$

$Var(W_1)$

$$\underbrace{\tfrac{1}{4} Var(W_1) + \tfrac{1}{4} Var(W_1)}_{Var(W_1)} + \tfrac{1}{2} Var(W_1)$$

So $Var(W_3) \leq Var(W_1) = Var(W_2)$

$\circledast$ really $=$

Since W_1 and W_2 are UMVUES

it must be

$$Var(W_3) = Var(W_1) = Var(W_2)$$

So

$$Cov(W_1, W_2) = \sqrt{Var(W_1)\, Var(W_2)}$$

actually a $=$

i.e.

$$Cor(W_1, W_2) = \frac{Cov(W_1, W_2)}{\sqrt{Var(W_1)\, Var(W_2)}} = 1$$

If $Cor = 1$ then it must be $\exists\, a, b$ where

$$\boxed{W_1 = a + b W_2}$$

0 1

however:

$$\tau(\theta) = \mathbb{E}[W_1] = a + b\,\underbrace{\mathbb{E}[W_2]}_{\tau(\theta)}$$

So

$a = 0$ and $b = 1$

i.e. $\boxed{W_1 = W_2}$

Theorem: Lehmann-Scheffé

technical condition

Let $W = W(\underline{X})$ be a (complete) sufficient statistic for θ and let $\hat{\theta}$ be an unbiased estimator for $\tau(\theta)$ that depends on $\underline{X}$ only through W,

$$\hat{\theta} = \hat{\theta}(W) = \hat{\theta}(W(\underline{X}))$$

then $\hat{\theta}$ is the UMVUE for $\tau(\theta)$.

Basically: If I can form an unbiased est. for $\tau(\theta)$ out of a sufficient stat W - it is the UMVUE.

ex, $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

μ unknown, σ^2 known

Q: what is the UMVUE for μ? (hint: $\bar{X}$)

Use Lehmann-Scheffe

① Find SS for μ

$W = \bar{X}$ is sufficient for μ $= \tau(\mu)$

② Guess a function of W that is unbiased for μ

$\hat{\theta} = \bar{X} = W$ then $E[\hat{\theta}] = \mu$

③ hence $\bar{X}$ is the UMVUE.

Ex. Let $\tau(\mu) = \mu^2$.

① SS for μ: $W = \bar{X}$ suff. for μ

② Guess a fn of W whose expectation is $\mu^2 = \tau(\mu)$.

try: $\bar{X}^2$?

$$E[\bar{X}^2] = Var(\bar{X}) + E[\bar{X}]^2.$$

$$= \sigma^2/N + \mu^2$$

known

try: $\boxed{\hat{\theta} = \bar{X}^2 - \sigma^2/N} = W^2 - \sigma^2/N$

fn of X only through $\bar{X}$

$$E[\hat{\theta}] = E[\bar{X}^2] - \sigma^2/N$$

$$= \sigma^2/N + \mu^2 - \sigma^2/N = \mu^2$$

(3) hence, $\hat{\theta} = \bar{X}^2 - \sigma^2/N$ is the UMVUE for μ^2.

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proof of Lehmann-Scheffé

statement: $\hat{\theta} = \hat{\theta}(W)$ and W sufficient (complete) then if $E[\hat{\theta}] = \tau(\theta)$, $\hat{\theta}$ is the UMVUE for $\tau(\theta)$

Let V be another unbiased est. of $\tau(\theta)$

(*) We'll show: $Var(\hat{\theta}) \leq Var(V)$ i.e. $\hat{\theta}$ is the UMVUE.

Rao-Blackwell says

If $\varphi(W) = \mathbb{E}[V|W]$

then $\boxed{① \ Var(\varphi) \leq Var(V)}$

② $\mathbb{E}\varphi = \tau(\theta)$

Let $g(W) = \hat{\theta}(W) - \varphi(W)$

then

$$\mathbb{E}[g(W)] = \mathbb{E}[\underbrace{\hat{\theta}}_{\hat{\theta}(W)}] - \mathbb{E}[\underbrace{\varphi}_{\varphi(W)}]$$

$$= \tau(\theta) - \tau(\theta)$$

$$= 0$$

this is true $\forall \theta$ so $g(W) = 0$

(using completeness of W)

i.e. $0 = g(W) = \hat{\theta}(W) - \varphi(W)$

so $\hat{\theta}(W) = \varphi(W)$.

So $Var(\hat{\theta}) = Var(\varphi) \leq Var(V)$.

Takeaway Message , for $\tau(\theta)$

Takeaway Message

How to find UMVUE for $\tau(\theta)$ w/ Lehman-Scheffé

1. Find a sufficient stat W for θ
2. Find a fn $\hat{\theta}(W)$ so that
$$\mathbb{E}[\hat{\theta}(W)] = \tau(\theta)$$
   - → (i) Guess $\hat{\theta}(w)$ so that $\mathbb{E}[\hat{\theta}] = \tau(\theta)$
   - → (ii) Use Rao-Blackwell
     - → Find any unbiased est for $\tau(\theta)$ (call it V)
     - → Let $\hat{\theta} = \mathbb{E}[V|W]$ ← the UMVUE

Revisit $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 known

want the UMVUE for μ^2.

Lets use Rao-Blackwell:

(1) Find any unbiased est. of μ^2:

For $m \neq n$

$$\mathbb{E}[X_n X_m] = \mathbb{E}[X_n]\,\mathbb{E}[X_m]$$
$$= \mu \cdot \mu = \mu^2$$

So let $V = X_n X_m$

(2) Rao-Blackwellize using $W = \bar{X}$

$$\hat{\theta} = \mathbb{E}[V \mid W]$$
$$= \mathbb{E}[X_n X_m \mid \bar{X}]$$
$$= \frac{1}{N(N-1)} \sum_{n \neq m} \mathbb{E}[X_n X_m \mid \bar{X}]$$
$$= \frac{1}{N(N-1)} \mathbb{E}\Big[\sum_{n \neq m} X_n X_m \mid \bar{X}\Big]$$

$$\left(\sum X_n\right)^2 = \sum X_n^2 + \underbrace{\sum_{n \neq m} X_n X_m}_{\text{rearrange}}$$

$$= \frac{1}{N(N-1)} \mathbb{E}\Big[\left(\sum X_n\right)^2 - \sum X_n^2 \mid \bar{X}\Big]$$

(short-cut variance formula)

$$(N-1)S^2 + N\bar{X}^2$$

$$= \frac{1}{N(N-1)} \mathbb{E}\Big[(N\bar{X})^2 - (N-1)S^2 - N\bar{X}^2 \mid \bar{X}\Big]$$

$$= \frac{1}{N(N-1)} \left(N^2\bar{X}^2 - (N-1)\sigma^2 - N\bar{X}^2\right)$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$
$$\mathbb{E}[X^2] = Var(X) + \mathbb{E}(X)^2$$
$$\frac{1}{N}\sum X_n^2 = S^2 + \left(\frac{\sum X_n}{N}\right)^2$$

$N(N-1)($

$$= \frac{1}{N(N-1)}\left(N(N-1)\bar{X}^2 - (N-1)\sigma^2\right)$$

$$\frac{1}{N}\sum X_n^2 = S^2 + \left(\frac{\sum X_n}{N}\right)^2$$

$$\mathbb{E}(A|A) = A$$

$$= \boxed{\bar{X}^2 - \sigma^2/N}$$

my UMVUE.

Ex. $X_n \overset{iid}{\sim} U(0,\theta)$. ← UMVUE for θ?

(1) Find ss for θ

Claim: Show $X_{(N)}$ is sufficient for θ.

(2) Want an unbiased est. of θ made from $X_{(N)}$.

Claim: $\mathbb{E}[X_{(N)}] = \frac{N}{N+1}\theta$

Let $\hat{\theta} = \frac{N+1}{N} X_{(N)}$

then $\mathbb{E}[\hat{\theta}] = \frac{N+1}{N}\frac{N}{N+1}\theta = \theta$

and $\hat{\theta}$ is a fn of $X_{(N)}$

So $\hat{\theta}$ is the UMVUE for θ.