Defn: Uniformly Most-Powerful Test (UMP)

Let $\mathcal{C}$ be a collection of tests, testing the hypothesis

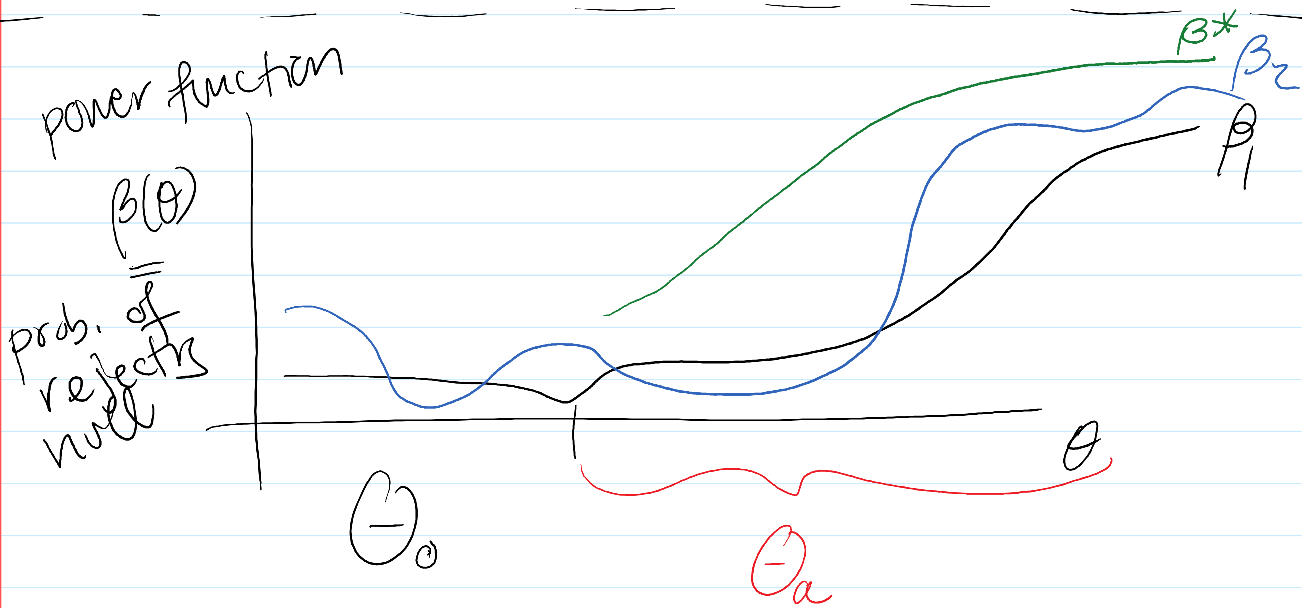
$$H_0 : \theta \in \Theta_0 \quad v. \quad H_a : \theta \in \Theta_a .$$

A test w/ power function $\beta^*(\theta) = \mathbb{P}_\theta(\underline{X} \in R)$ is called the uniformly most powerful test (UMP) [for this collection $\mathcal{C}$]

if

$$\beta^*(\theta) \geq \beta(\theta) \quad \forall \theta \in \Theta_a,$$

for any other test in $\mathcal{C}$ w/ power function β.



Defn: UMP level/size α test

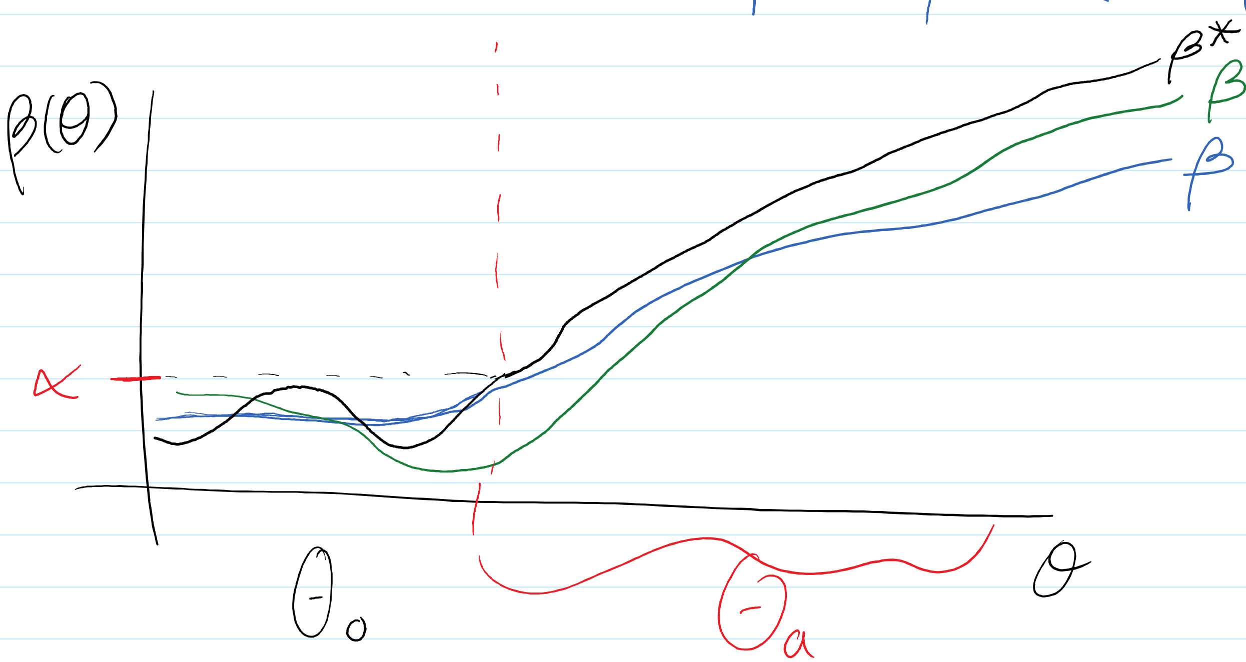
recall:

size α test : $\max_{\theta \in \Theta_0} \beta(\theta) = \alpha$

level α test : $\max_{\theta \in \Theta_0} \beta(\theta) \leq \alpha$

The UMP size α test is the UMP among the collection of size α tests for a hypothesis

The UMP level α test is the UMP among all level α tests of a particular hypothesis



Consider a simple hypothesis:

$$H_0: \theta = \theta_0 \quad \text{v.} \quad H_0: \theta = \theta_a$$

$$\Theta = \{\theta_0, \theta_a\} \; ; \; \Theta_0 = \{\theta_0\} \; ; \; \Theta_a = \{\theta_a\}$$

Consider the LRT:

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \begin{cases} \dfrac{L(\theta_0)}{L(\theta_0)}, & L(\theta_0) > L(\theta_a) \\[2ex] \dfrac{L(\theta_0)}{L(\theta_a)}, & L(\theta_0) < L(\theta_a) \end{cases}$$

$$\Rightarrow \lambda = \begin{cases} \boxed{1, \quad L(\theta_0) > L(\theta_a)} & \text{won't reject} \\[2ex] \dfrac{L(\theta_0)}{L(\theta_a)}, & L(\theta_a) > L(\theta_0) \end{cases}$$

recall for LRT I reject H_0 if $\lambda \le c$

(1) don't reject if $L(\theta_0) > L(\theta_a)$

(2) will reject if $\dfrac{L(\theta_0)}{L(\theta_a)} \le c$

i.e.

$$\boxed{L(\theta_0) \le c\, L(\theta_a)}$$

size α test

$$\max_{\theta \in \Theta_0} P(\lambda \le c) = \alpha$$

$=$

or

$\phi(\theta_0$ $\to$ $\boxed{L(\theta_0) = c\, L(\theta_a)}$

$=$ or

$P_{\theta_0}(\lambda \le c) = \alpha \to$ $\boxed{L(\theta_a) \ge k\, L(\theta_0)}$ where $k = 1/c$

punchline: for simple hypotheses the UMP level α test is the LRT.

Theorem: Neyman-Pearson Lemma

Consider testing

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta = \theta_a$$

with a LRT, so that I reject H_0 if

$$\lambda = L(\theta_0)/L(\theta_a) \le c$$

[where c is chosen so that $\underbrace{P_{\theta_0}(\lambda \le c) = \alpha}_{\text{our test is size } \alpha}$]

This is the UMP size α test.

Ex. Let $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ known

Test $H_0: \theta = a \quad v. \quad H_a: \theta = b$

Test

$$\boxed{H_0: \theta = a \quad v. \quad H_a: \theta = b}$$

$[b > a]$

Using LRT,

$$\lambda = \frac{L(a)}{L(b)} = \cdots \propto \exp\left\{\frac{N(b^2-a^2)+2(a-b)N\bar{X}}{2\sigma^2}\right\}$$

The LRT says reject if $\lambda \le c$

or $\log(\lambda) \le \log(c)$

$$\iff N(b^2-a^2)+2(a-b)N\bar{X} \le 2\sigma^2 \log(c)$$

$$\iff \bar{X} \ge \frac{2\sigma^2 \log c - N(b^2-a^2)}{2(a-b)N}$$

$$\iff \boxed{\bar{X} \ge c^*}$$

how do we choose c^*?
Want a size α test.

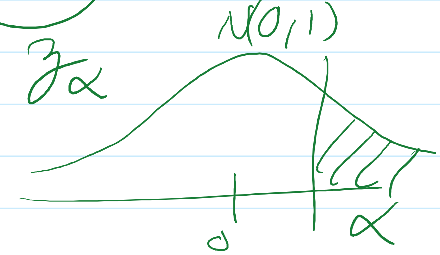
$$\iff \frac{\bar{X}-a}{\sigma/\sqrt{N}} \ge \frac{c^*-a}{\sigma/\sqrt{N}}$$

under H_0 $\sim N(0,1)$

so If I want $P\left(\frac{\bar{X}-a}{-} \ge \frac{c^*-a}{-}\right) = \alpha$

so If I want $P_{\theta=a}\left(\frac{\bar{X}-a}{\sigma/\sqrt{N}} \geq \frac{c^*-a}{\sigma/\sqrt{N}}\right) = \alpha$

z_α



so $\boxed{c^* = a + \sigma/\sqrt{N}\, z_\alpha}$

My test to reject when

$\rightarrow \boxed{\bar{X} \geq a + \sigma/\sqrt{N}\, z_\alpha}$

$\Updownarrow$

$\boxed{\frac{\bar{X}-a}{\sigma/\sqrt{N}} \geq z_\alpha}$

This is a size α test and by Neyman-Pearson it is the UMP size α-test.

Ex. Let $X \sim \text{Bin}(2, \theta)$ unknown.

flipping 2 coins w/ unknown prob. of H, θ

Test hypothesis

$$H_0: \theta = 1/2 \quad v. \quad H_a: \theta = 3/4$$

Using a LRT

$$\lambda = \frac{L(1/2)}{L(3/4)} = \frac{f_{1/2}(x)}{f_{3/4}(x)}$$

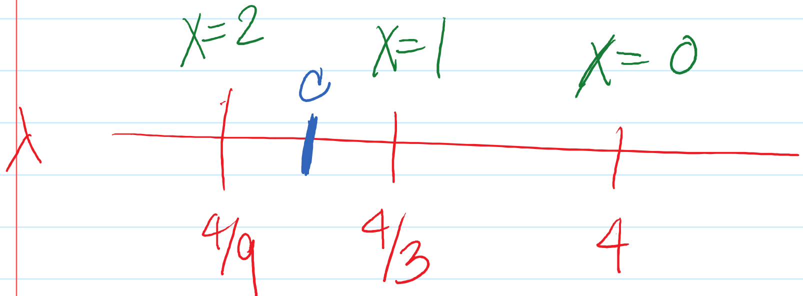
reject if $\boxed{\lambda \le c.}$

X can take on 3 values: 0, 1, 2

X	0	1	2
λ	4	4/3	4/9

$$\lambda(0) = \frac{P_{\theta=1/2}(X=0)}{P_{\theta=3/4}(X=0)} = \frac{\binom{2}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^2}{\binom{2}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^2}$$

$$= 4$$



If choose any $4/9 < c < 4/3$ (reject only when $X=2$)

then

$$\alpha = P_{\theta=1/2}(\lambda \le c) = P_{\theta=1/2}(X=2) = 1/4$$

So this test by Neyman-Pearson lemma is the UMP size $\alpha = 1/4$ test.

Theorem: LRTs are a function of a sufficient stat.

some function not involving $\underset{\sim}{X}$

$$\lambda(\underset{\sim}{X}) = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \lambda^*(T)$$

sufficient stat

proof: Recall that MLEs $\hat{\theta}_0$ and $\hat{\theta}$ are functions of suff. stats.

Defn: Monotone Likelihood Ratio Property (MLR)

We say a family of PDFs $\{f_\theta\}$ has the MLR property if

$$\frac{L(\theta_2)}{L(\theta_1)} = \frac{f_{\theta_2}(x)}{f_{\theta_1}(x)}$$ is non-decreasing as a function of x

where $\theta_2 > \theta_1$.

<u>Theorem:</u> If $\{f_\theta\}$ is an exp. family,

$$f_\theta(x) = c(\theta)h(x)\exp(w(\theta)x)$$

then if $w(\theta)$ is non-decreasing in θ, this family has the MLR property.

<u>pf.</u>

$$\frac{L(\theta_2)}{L(\theta_1)} = \frac{c(\theta_2)\cancel{h(x)}\exp(w(\theta_2)x)}{c(\theta_1)\cancel{h(x)}\exp(w(\theta_1)x)}$$

$$\propto \exp\Big(\underbrace{(w(\theta_2) - w(\theta_1))}_{\geq 0} x\Big) \approx \underbrace{e^{ax}}_{\text{inc. in } x}$$

w non-decreasing means: $\theta_2 > \theta_1$ then $w(\theta_2) \geq w(\theta_1)$