## Defn: Size and level

We say a test is <u>size</u> $\alpha$ if

$$\underbrace{\max_{\theta \in \Theta_0} \beta(\theta)}_{\text{max type I error prob}} = \alpha$$

we say a test is <u>level</u> $\alpha$ if

$$\underbrace{\max_{\theta \in \Theta_0} \beta(\theta)}_{\text{max type I error prob.}} \leq \alpha$$

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## Defn: Likelihood Ratio Test (LRT)

Recall

$$L(\theta) = f_\theta(\underset{\sim}{x})$$

is called the likelihood function.

We want to test a hypothesis

$$H_0 : \theta \in \Theta_0 \quad v. \quad H_a : \theta \in \Theta_a$$

The LRT statistic is defined as

$$\lambda(\underline{X}) = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \frac{\text{max value of } L \text{ over } \Theta_0}{\text{max value of } L \text{ over entire space}}$$

$$= \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}$$

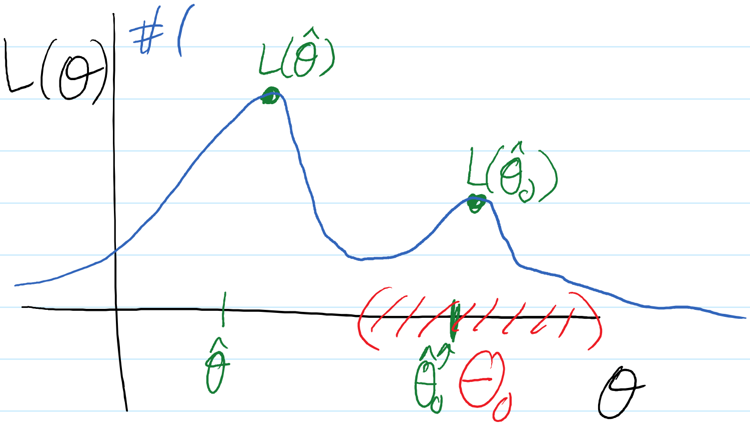
$\hat{\theta}_0$ = MLE over $\Theta_0$

$\hat{\theta}$: MLE over $\Theta$

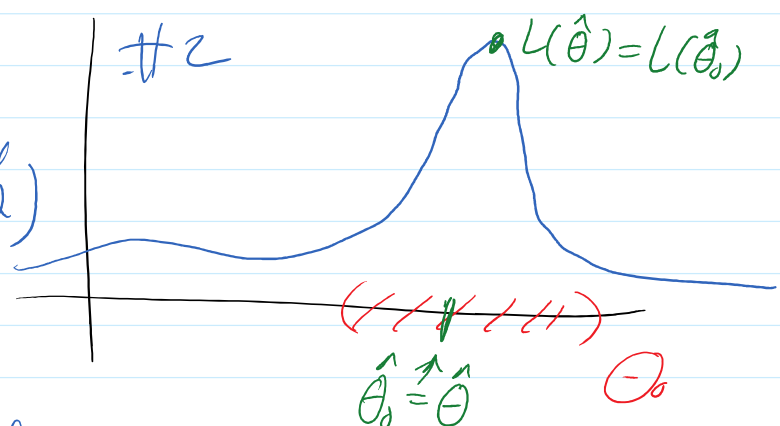
<u>note:</u> $L(\hat{\theta}) \geq L(\hat{\theta}_0)$

hence

$0 \leq \lambda \leq 1$




interpret this $\lambda$ as how likely data is if $\theta \in \Theta_0$ as compared to overall how likely my data is (unrestricted)




i.e. $\lambda \approx 1$ then

$H_0$ basically as likely as possible (i.e. don't reject $H_0$)

or $\lambda \approx 0$ then $H_0$ not as likely in comparison to other values of $\theta$

(i.e. maybe reject $H_0$)

The LRT says

$$R = \{ \underset{\sim}{X} \mid \lambda(\underset{\sim}{X}) \leq c \}$$

rejection region

for some $c \in [0,1]$

How to choose $c$?

<u>Note:</u> $c = 0 \Rightarrow$ always accept

$c = 1 \Rightarrow$ always reject

$0 < c < 1$, inbetween, $\Rightarrow$ sometimes reject/accept

Choose $c$ to tradeoff type I and II errors.

e.g. $c \approx 1 \Rightarrow$ reject a lot

higher type I error

lower type II error

$c \approx 0 \Rightarrow$ not reject very much
lower type I error
higher type II error

Game: choose $c$ to make correct type I v. type II errors.

typically want to minimize type II error subject to a type I constraint.

e.g. build a level $\alpha$ test that minimizes type II error

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Ex, Let $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ known.

$$\boxed{H_0: \theta \leq a \quad v. \quad H_a: \theta > a.}$$

$\Theta_0 = (-\infty, a] \qquad \Theta_a = (a, \infty)$

Lets form a LRT $\qquad \Theta = \mathbb{R}$

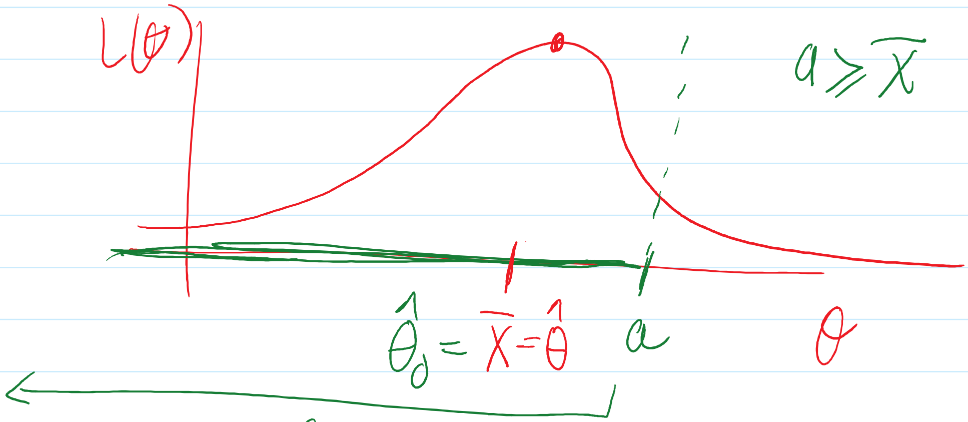
$$L(\theta) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_n - \theta)^2\right) = (2\pi)^{-N/2}(\sigma^2)^{-N/2}\exp(\cdots)$$

looks quadratic in $\theta$

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}$$

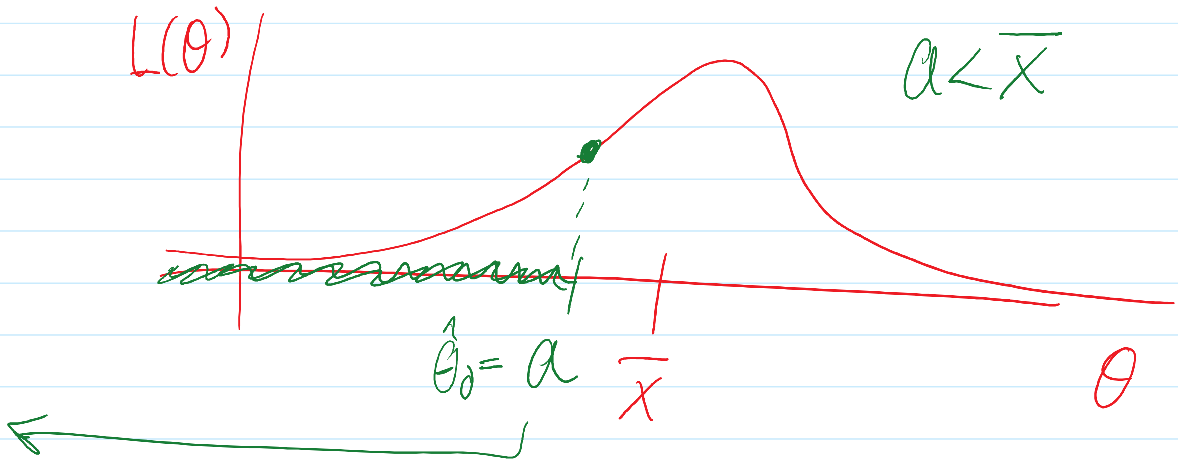
i.e. $\approx \exp(-\theta^2)$

we know: $\hat{\theta} = \bar{X}$




$\Theta_0 = (-\infty, a]$

$\boxed{\text{If } a \geq \bar{X} \text{ then } \hat{\theta}_0 = \bar{X}}$




$\Theta_0$

$\boxed{\text{If } a < \bar{X} \text{ then } \hat{\theta}_0 = a}$

$\boxed{\begin{array}{l} H_0: \theta \leq a \\ H_a: \theta > a \end{array}}$

Recap: $\hat{\theta} = \bar{X}$

Recap: $\hat{\theta} = \bar{X}$

$$\hat{\theta}_0 = \begin{cases} \bar{X}, & \text{if } \bar{X} \le a \\ a, & \text{if } \bar{X} > a \end{cases}$$

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \begin{cases} \frac{L(\bar{X})}{L(\bar{X})} = 1, & \text{if } \bar{X} \le a \\ \frac{L(a)}{L(\bar{X})}, & \text{if } \bar{X} > a \end{cases}$$

$$= \begin{cases} 1, & \text{if } \boxed{\bar{X} \le a} \leftarrow \text{never reject} \\ \underbrace{L(a)/L(\bar{X})} & \boxed{\text{if } \bar{X} > a} \leftarrow \text{sometimes reject} \end{cases}$$

LRT says reject if

$$\lambda \le \underbrace{c} \quad \text{when } c \in (0,1)$$

$$\lambda = \frac{L(a)}{L(\bar{X})} = \frac{\cancel{(2\pi)^{-N/2}}\cancel{(\sigma^2)^{-N/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_n (x_n - a)^2\right)}{\cancel{(2\pi)^{-N/2}}\cancel{(\sigma^2)^{-N/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_n (x_n - \bar{X})^2\right)}$$

$$e^{\cancel{a}+b} = \cancel{e^a} e^b \qquad = \frac{\exp\left(-\frac{1}{2\sigma^2}\left[\cancel{\sum_n x_n^2} - 2a\sum_n x_n + Na^2\right]\right)}{\cdots}$$

$$\frac{e^{\cancel{a}+b}}{\cancel{e^{a}}} = \frac{\cancel{e^{a}} e^{b}}{\cancel{e^{a}}}$$

$$\exp\left(-\frac{1}{2\sigma}\left[\cancel{\sum_n x_n^2} - 2\bar{X}\sum_n x_n + N\bar{X}^2\right]\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left[Na^2 - 2aN\bar{X} + 2N\bar{X}^2 - N\bar{X}^2\right]\right)$$

$$= \exp\left(-\frac{N}{2\sigma^2}\left[a^2 - 2a\bar{X} + \bar{X}^2\right]\right)$$

$$\lambda = \exp\left(-\frac{N}{2\sigma^2}\left[\bar{X} - a\right]^2\right)$$

So the LRT is to reject if

$$\lambda = \exp\left(-\frac{N}{2\sigma^2}(\bar{X}-a)^2\right) \le c$$

$$\iff \quad -\frac{N}{2\sigma^2}(\bar{X}-a)^2 \le \log c$$

$$\iff \quad (\bar{X}-a)^2 \ge -\frac{2\sigma^2}{N}\log c$$

recall: ignore case where $\bar{X} \le a$; assume $\bar{X} > a$

$$\iff \quad \bar{X} - a \ge \sqrt{\frac{-2\sigma^2 \log c}{N}}$$

$$\iff \quad \boxed{\frac{\bar{X}-a}{\sigma/\sqrt{N}} \ge \underbrace{\sqrt{-2\log c}}_{c^*}}$$

$H_0: \theta \le a$
$H_a: \theta > a$

$\Rightarrow$ LRT says reject if

$$\boxed{\frac{\bar{X}-a}{\sigma/\sqrt{N}} \ge c^*}$$

$\bar{X}$ is more than $c^*$ s.e.s. above $a$

How do we determine $c^*$

control for type I error

$$\boxed{\underbrace{\max_{\theta \in \Theta_0} P(\text{reject})}_{\text{max type I error prob}} = \alpha}$$

$$P_\theta(\text{reject } H_0) = P_\theta(\lambda \le c)$$

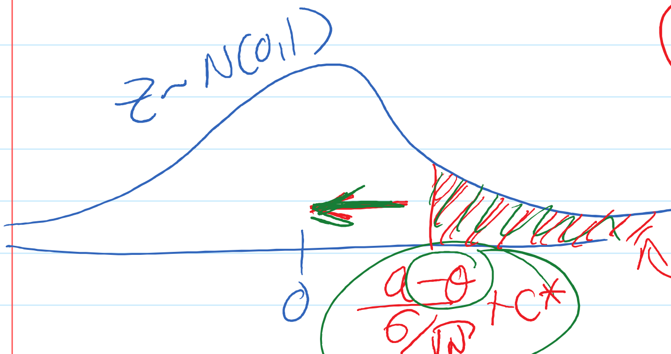
$$= P_\theta\left(\frac{\bar{X}-a}{\sigma/\sqrt{N}} \ge c^*\right)$$

$$= P\left(\underbrace{\frac{\bar{X}-a}{\sigma/\sqrt{N}} + \frac{a-\theta}{\sigma/\sqrt{N}}}_{} \ge \frac{a-\theta}{\sigma/\sqrt{N}} + c^*\right)$$

$$\frac{\bar{X}-\theta}{\sigma/\sqrt{N}} \sim N(0,1)$$

$$= P\left(Z \ge \frac{a-\theta}{\sigma/\sqrt{N}} + c^*\right)$$

Q: max value over $\theta \in \Theta_0$?




Q: max value over $\theta \in \Theta_0$?

$$\max_{\theta \le a} \mathbb{P}\left(z \ge \frac{a-\theta}{\sigma/\sqrt{n}} + c^*\right)$$

$$= \mathbb{P}\left(z \ge \frac{\theta-\theta}{\sigma/\sqrt{n}} + c^*\right)$$

$$= \boxed{\mathbb{P}(z \ge c^*)}$$

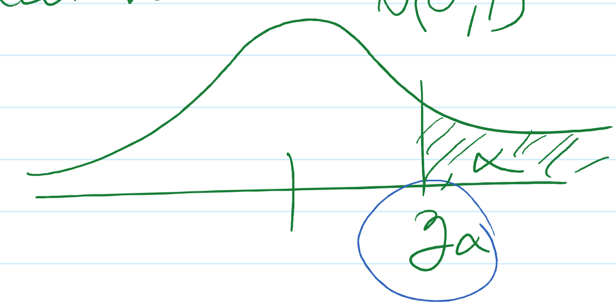
max at $\theta = a$

How do I chose $c^*$ to keep max type I err prob. at $\alpha$?

choose $c^*$ so that

$$\mathbb{P}(z \ge c^*) = \alpha$$

i.e. $\boxed{c^* = z_\alpha}$ = the point at which $\alpha$ prob to right on a standard normal $N(0,1)$




look up in table

Takeaway:

my size $\alpha$ LRT is just to reject

when

$$\frac{\bar{X} - a}{\sigma / \sqrt{n}} \geq z_\alpha$$