Last lecture:

LLNs: Weak: X_n, uncorr, $E[X_n]=\mu$, $Var(X_n)=\sigma^2$
then $\bar{X}_n \xrightarrow{P} \mu$.

Strong: X_n iid etc
then $\bar{X}_n \xrightarrow{a.s.} \mu$.

Sums of RVs

① $\frac{1}{N}\sum_{n=1}^{N} X_n \longrightarrow \mu$ (under some conditions) LLNs

degenerate distribution

② $\sum_{n=1}^{N} X_n \longrightarrow \infty$ blow up / not converge

③ $\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n \xrightarrow{d}$ non-degenerate dist.

proper scaling

Theorem: Central Limit Theorem

If I have X_n that are iid w/ $E[X_n]=\mu$
and $Var(X_n)=\sigma^2<\infty$ then

$$\sqrt{N}\left(\frac{\bar{X}_N-\mu}{\sigma}\right) \xrightarrow{d} N(0,1).$$

Notice: $\sqrt{N}\,\bar{X} = \sqrt{N}\,\frac{1}{N}\sum_{n=1}^{N} X_n = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n.$

Intuition:

CLT: $\boxed{\bar{X} \approx N(\mu, \sigma^2/N).}$

Can't we rewrite CLT like

$$\bar{X}_N \xrightarrow{d} N(\mu, \sigma^2/N)\ ?$$

N in my limit! Not allowed!

Proper way:

① $\sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right) \xrightarrow{d} N(0,1)$

② $\sqrt{N}(\bar{X}-\mu) \xrightarrow{d} N(0,\sigma^2)$

③ $\frac{\bar{X}-\mu}{\sigma/\sqrt{N}} \xrightarrow{d} N(0,1)$

Other notation:

asymptotically normal

$$\bar{X} \sim AN(\mu, \sigma^2/N)$$

purely notational

$$\boxed{\sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right) \xrightarrow{d} N(0,1).}$$

<u>Ex.</u> $X_n \overset{iid}{\sim}$ Bernoulli(p)

recall that $\mu = E[X_n] = p$

and $\sigma^2 = Var(X_n) = p(1-p)$

<u>CLT says:</u>

$$\sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right) = \sqrt{N}\left(\frac{\bar{X}-p}{\sqrt{p(1-p)}}\right) \xrightarrow{d} N(0,1).$$

<u>In intro stats:</u> $\bar{X} = \hat{p}$ as estimate for "yes" responses to a poll.

$$\hat{p} \sim AN\left(p, \frac{p(1-p)}{N}\right)$$

maybe we form a 95% CI for $\hat{p}$ as

$$\hat{p} \pm 2\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}.$$

<u>Ex.</u> $X_n \overset{iid}{\sim}$ Pois(λ)

$$E[X_n] = \lambda = Var(X_n).$$

CLT:

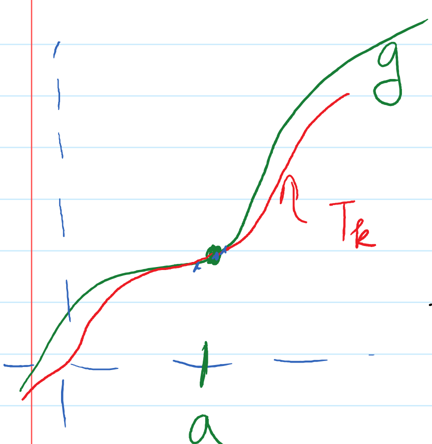
$$\sqrt{n}\left(\frac{\bar{X}-\lambda}{\sqrt{\lambda}}\right) \xrightarrow{d} N(0,1)$$

Recall: $\lim\limits_{n\to\infty}\left(1+\frac{c}{n}\right) = e^c.$

Theorem: Taylor's Theorem

I have a functn $g:\mathbb{R}\to\mathbb{R}$ that is k-times differentiable. The k^{th}-Order Taylor polynomial about $a\in\mathbb{R}$ is

$$T_k(x) = \sum_{r=0}^{k} \frac{g^{(r)}(a)}{r!}(x-a)^r$$



$$= \frac{g^{(0)}(a)}{0!}(x-a)^0 + \frac{g^{(1)}(a)}{1!}(x-a)^1 + \frac{g^{(2)}(a)}{2!}(x-a)^2 + \cdots$$

$$= g(a) + g^{(1)}(a)(x-a) \; \frac{g^{(2)}(a)}{2}(x-a)^2 + \cdots$$

Let $R = g(x) - T_k(x)$

then $R \to 0$ as $x \to a$

$\uparrow$ fast

Punchline: for $x \approx a$ we have $g(x) \approx T_k(x)$.

CLT: Statement: X_n iid, $\mathbb{E}[X_n] = \mu$, $Var(X_n) = \sigma^2 < \infty$,

$$\sqrt{N}\left(\frac{\overline{X} - \mu}{\sigma}\right) \xrightarrow{d} N(0,1).$$

Let $Y_n = \dfrac{X_n - \mu}{\sigma}$ then $\mathbb{E}[Y_n] = \frac{1}{\sigma}(\underbrace{\mathbb{E}[X_n]}_{\mu} - \mu) = 0$

$$Var(Y_n) = \frac{1}{\sigma^2} Var(X_n) = \frac{\sigma^2}{\sigma^2} = 1$$

$$\boxed{Y = \sqrt{N}\left(\frac{\overline{X} - \mu}{\sigma}\right)}$$

$$= \sqrt{N}\left(\frac{\frac{1}{N}\sum_{n=1}^{N} X_n - \mu}{\sigma}\right)$$

$$= \frac{\frac{\sqrt{N}}{N}\left(\sum_{n=1}^{N} X_n - N\mu\right)}{\sigma}$$

$\Sigma(X_n - \mu) = \Sigma X_n - N\mu$

$$= \frac{\sqrt{N}}{N} \sum_{n=1}^{N}\left(\frac{X_n - \mu}{\sigma}\right)$$

$$\boxed{= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} Y_n}$$

recall that my Ys are independent

Consider $\qquad M(t) = \text{mgf of } Y_n$

$$M_Y(t) = M_{\frac{1}{\sqrt{N}}\sum_{n=1}^{N} Y_n}(t) = M\left(t/\sqrt{N}\right)^N$$

two rules:

$$\begin{cases} M_{\sum X_i}(t) = M(t)^N \\ M_{aX}(t) = M_X(at) \end{cases}$$

Consider a second order Taylor Approx of $M(t)$ at $t=0$.

$$\begin{array}{c} \quad 1 \quad + \mathbb{E}[Y_n]\,t \quad + \mathbb{E}[Y_n^2]\,t^2 + R \\ M(t) = M(0) + M'(0)(t-0) + \frac{M''(0)}{2}(t-0)^2 + R \end{array}$$

0 $\qquad 1$

$M(0) = \mathbb{E}[e^{0 Y_n}] = \mathbb{E}[1] = 1$

$$= 1 + \frac{t^2}{2} + R$$

hence

$$M_Y(t) = M\left(t/\sqrt{N}\right)^N$$

$$= \left(1 + \frac{t^2}{2N} + R\left(t/\sqrt{N}\right)\right)^N$$

as $N \to \infty$ then $t/\sqrt{N} \to 0$ so

$R \to 0$ (very fast)

So

$$M_Y(t) \sim \left(1 + \frac{t^2}{2N}\right)^N \sim \left(1 + \frac{c}{N}\right)^N$$

(c written above $\frac{t^2}{2}$)

i.e.

$$\lim_{n\to\infty} M_Y(t) = \underbrace{e^{t^2/2}}_{\text{MGF of } N(0,1)}$$

Have Shown:

$Y = \sqrt{N}\left(\frac{\overline{X} - \mu}{\sigma}\right)$ has an MGF $\xrightarrow{n}$ MGF of a $N(0,1)$.

So since the MFG converges to the MGF of a $N(0,1)$ so w/ the CDF (*)

then by defn:

$$\sqrt{N}\left(\frac{\overline{X} - \mu}{\sigma}\right) \xrightarrow{d} N(0,1).$$

Ex. If X_n that are iid w/ $\mu = 2$ ad $\sigma^2 = 9$ then CLT says

$$\sqrt{n}\left(\frac{\bar{X}-2}{3}\right) \xrightarrow{d} N(0,1).$$

Q: what about $\log \bar{X}$?

Theorem: First Order Delta Method
Let Y_n be a seq of RVs where

$$\sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0, \tau^2)$$

some constant

Ex. $Y_n = \bar{X}$ and $\theta = \mu$, $\tau^2 = \sigma^2$ the CLT

$$\sqrt{n}(\bar{X}-\mu) \xrightarrow{d} N(0, \sigma^2).$$

then if g is differentable and $g'(\theta) \neq 0$ we have

$$\sqrt{n}\left(g(Y_n) - g(\theta)\right) \xrightarrow{d} N\left(0, (g'(\theta))^2 \tau^2\right).$$

Ex. CLT says that

$$\sqrt{n}(\bar{X}-\mu) \xrightarrow{d} N(0, \sigma^2).$$

Let $g(x) = \log(x)$.

$$g'(x) = 1/x \Rightarrow (g'(\mu))^2 = 1/\mu^2$$

then Δ- method

$$\sqrt{n}\left(g(\bar{X}) - g(\mu)\right) \xrightarrow{d} N\left(0, (g'(\mu))^2\sigma^2\right)$$

i.e.

$$\boxed{\sqrt{n}\left(\log(\bar{X}) - \log(\mu)\right) \xrightarrow{d} N\left(0, \sigma^2/\mu^2\right)}$$

i.e.

$$\boxed{\bar{X} \sim AN\left(\mu, \sigma^2/n\right) \quad \text{and} \quad \log(\bar{X}) \sim AN\left(\log(\mu), \frac{\sigma^2}{n\mu^2}\right)}$$

Proof.

Taylor approx of g around θ

$$g(x) \simeq g(\theta) + g'(\theta)(x-\theta)$$

So

$$g(x) - g(\theta) \simeq g'(\theta)(x-\theta)$$

So if $\sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0, \tau^2)$

then

$$\sqrt{n}\left(g(Y_n) - g(\theta)\right) \simeq \sqrt{n}\, g'(\theta)(Y_n - \theta)$$

$$= g'(\theta)\sqrt{n}(Y_n - \theta)$$

So . . . $\xrightarrow{d} N(0, \tau^2)$

$$\longrightarrow N(0, (g'(\theta))^2 \tau^2)$$

Ex. Let $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

then CLT says that

$$\sqrt{N}\left(\bar{X} - \lambda\right) \xrightarrow{d} N(0, \lambda).$$

Consider $g(x) = 1/x$

then $g'(x) = -1/x^2$

and so

$$\sqrt{N}\left(\frac{1}{\bar{X}} - \frac{1}{\lambda}\right) \xrightarrow{d} N\left(0, \left(-1/\lambda\right)^2 \lambda\right)$$

$$= N(0, 1/\lambda)$$

i.e. $\boxed{\begin{array}{l} \bar{X} \sim AN(\lambda, \lambda/N) \\ 1/\bar{X} \sim AN(1/\lambda, 1/N\lambda). \end{array}}$