Theorem: Neyman-Pearson Lemma

Consider a simple hypothesis test

$$H_0 : \theta = \theta_0 \quad v. \quad H_a : \theta = \theta_a$$

and use the LRT so that we reject H_0 if

$$\lambda = L(\theta_0) / L(\theta_a) \leq c$$

[where c is chosen so that our test is size α : $\mathbb{P}_{\theta_0}(\lambda \leq c) = \alpha$]

This is the UMP level α test.

Theorem: LRTs are functions of sufficient stats.

$$\lambda(\underset{\sim}{X}) = \lambda^*(T)$$

suff. stat for θ.

Goal: Find UMP level α test.

Simple hypothesis: Neyman-Pearson says use LRT (a suff. stat)

One-sided hypothesis: Karlin-Rubin says use LRT (a suff. stat)

need MLR property.

Two-sided hypotheses: :(

Beyond this course

Defn: Monotone Likelihood Ratio Property (MLR)

If $\{f_\theta\}$ is a family of dists. indexed by θ and if $\theta_2 > \theta_1$ we say this fam. of dists has the MLR property if

$$\frac{L(\theta_2)}{L(\theta_1)} = \frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} \leftarrow \text{a fn of } x$$

is monotone increasing as a fn of x.

Theorem: Karlin-Rubin

Consider testing

$$H_0: \theta \leq a \quad v. \quad H_a: \theta > a$$

and let T be a sufficient stat for θ and the dist. of T has the MLR property.

Consider the test that rejects when

$$T > c$$

when we choose c so that

$$\mathbb{P}_{\theta = a}(T > c) \leq \alpha$$

Then this is the UMP level α test.

Notes: ① alt. test $H_0: \theta \geq a$ v. $H_a: \theta < a$ by rejecting when $T < c$ ...

② This is basically the LRT. b/c the LRT is based on a suff stat.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where σ^2 is known

① Test $H_0: \mu \leq a$ v. $\boxed{H_a: \mu > a.}$

→ We know that $\bar{X}$ is suff. for μ.

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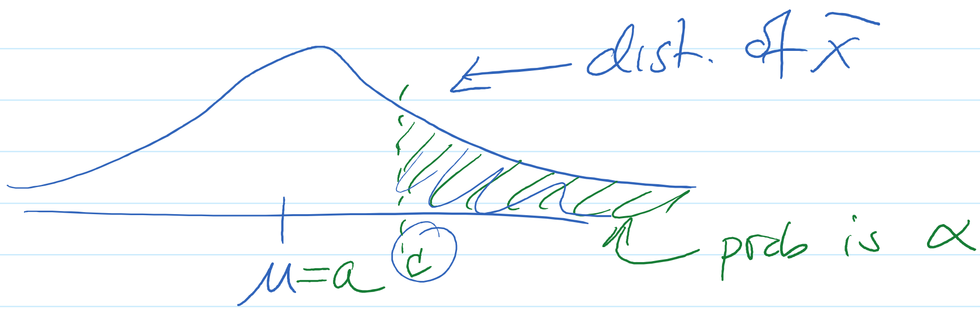
(*) → $\bar{X}$ has the MLR prop. b/c it follows an
→ exp. fam. [defer]

→ So the UMP level α test is to reject when

$$\boxed{\bar{X} > c.}$$

c chosen so that $\mathbb{P}_{\mu=a}(\bar{X} > c) = \alpha$

$$\bar{X} \sim N(\mu, \sigma^2/N)$$



$$\boxed{\bar{X} > c} \iff \boxed{\underbrace{\frac{\bar{X}-a}{\sigma/\sqrt{N}}}_{N(0,1)} > c^*}$$

$c^* = z_\alpha$

$$\boxed{c = a + z_\alpha \sigma/\sqrt{N}}$$

Theorem: Exp Fam and MLR

If $\{f_\theta\}$ is a fam of dists. that is an exp. fam. so that

$$\boxed{f_\theta(x) = c(\theta)\,h(x)\exp\big(w(\theta)\,x\big)}$$

then if $\boxed{w \text{ is inc. in } \theta}$, this fam has the MLR property.

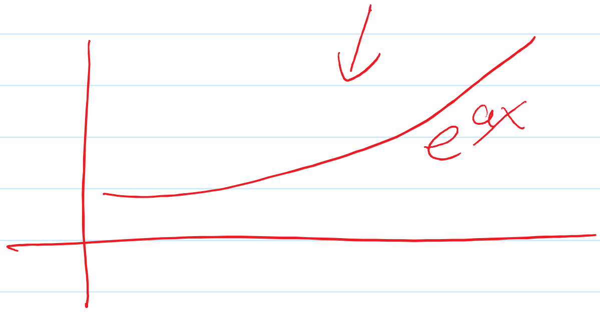
pf. $\theta_2 > \theta_1$

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{c(\theta_2)}{c(\theta_1)}\,\frac{\cancel{h(x)}}{\cancel{h(x)}}\,\frac{\exp\big(w(\theta_2)\,x\big)}{\exp\big(w(\theta_1)\,x\big)}$$

$$= \frac{c(\theta_2)}{c(\theta_1)}\exp\Big(\big(\underbrace{w(\theta_2) - w(\theta_1)}_{>0}\big)\,x\Big)$$

w is inc. then for $\theta_2 > \theta_1 \Rightarrow w(\theta_2) > w(\theta_1)$

looks like e^{ax} where $a > 0$



Revisit $\bar{X}$.

$\bar{X} \sim N(\mu, \sigma^2/N)$

PDF of $\bar{X}$

$$f(\bar{X}) = \frac{1}{\sqrt{2\pi\sigma^2/N}} \exp\left(-\frac{N}{\sigma^2}(\bar{X}-\mu)^2\right)$$

√ exp. fam.

$$= \frac{1}{\sqrt{2\pi\sigma^2/N}} \exp\left(-\frac{N}{\sigma^2}(\bar{X}^2 - 2\mu\bar{X} + \mu^2)\right)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma^2/N}} \exp\left(-\frac{N}{\sigma^2}\bar{X}^2\right)}_{h(\bar{x})} \underbrace{\exp\left(-\frac{N}{\sigma^2}\mu^2\right)}_{c(\mu)} \exp\left(\underbrace{2\frac{N}{\sigma^2}\mu}_{w(\mu)}\ \underbrace{\bar{X}}_{x}\right)$$

$w(\mu) = \frac{2N}{\sigma^2}\mu$ inc. in μ.