If $X_n \overset{iid}{\sim} f_\theta$

① Score: $S_\theta = S_\theta(\underset{\sim}{X}) = \frac{\partial}{\partial\theta} \log f_\theta(\underset{\sim}{X}) = \frac{\partial \ell}{\partial \theta}$

Score = thinking of $\frac{\partial \ell}{\partial\theta}$ as random

(*) Theorem: $\mathbb{E}[S_\theta] = 0$

② Fisher Information

thinking about this as random

$$I_N(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2} \log f_\theta(\underset{\sim}{X})\right] = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial\theta^2}\right]$$

(*) Theorem:
$$= \mathbb{E}[S_\theta^2]$$
$$= \mathbb{E}\left[\left(\frac{\partial \ell}{\partial\theta}\right)^2\right]$$

(*) Required a certain amount of niceness in f_θ (not a concern for exp. fam.)

Facts: (i) $I_N(\theta) = N I(\theta)$

info from N samples

info from 1 sample

(ii) If θ is a fn of ψ

or ψ is a fn of θ

then $I(\theta) = \left(\frac{\partial \psi}{\partial \theta}\right)^2 I(\psi)$ ←

or $I(\psi) = \left(\frac{\partial \theta}{\partial \psi}\right)^2 I(\theta)$

Theorem: Cramer-Rao Lower Bound (CRLB) (*)

Ver1 If $X_n \overset{iid}{\sim} f_\theta$ and $\hat{\theta}$ is unbiased for θ

then

$$Var(\hat{\theta}) \geq \frac{1}{I_N(\theta)}$$

CRLB

Ver2 If $\hat{\theta}$ is unbiased for $\tau(\theta)$ then

$$Var(\hat{\theta}) \geq \left(\frac{\partial \tau}{\partial \theta}\right)^2 \Big/ I_N(\theta).$$

(*) This requies a nice/regular f_θ

(i.e. works for exp. families, not otherwise)

pf of Ver2 Notice $\tau = \tau(\theta)$ then

$$I_N(\theta) = \left(\frac{\partial \tau}{\partial \theta}\right)^2 I_N(\tau)$$

so $$I_N(\tau) = \frac{I_N(\theta)}{\left(\frac{\partial \tau}{\partial \theta}\right)^2}$$

$\left(\overline{\partial\theta}\right)$

So

$$Var(\hat{\theta}) \geq \frac{1}{I_N(\tau)} = \frac{1}{\frac{I_N(\theta)}{\left(\frac{\partial \tau}{\partial \theta}\right)^2}} = \frac{\left(\frac{\partial \tau}{\partial \theta}\right)^2}{I_N(\theta)}$$

ver 2

Comments:

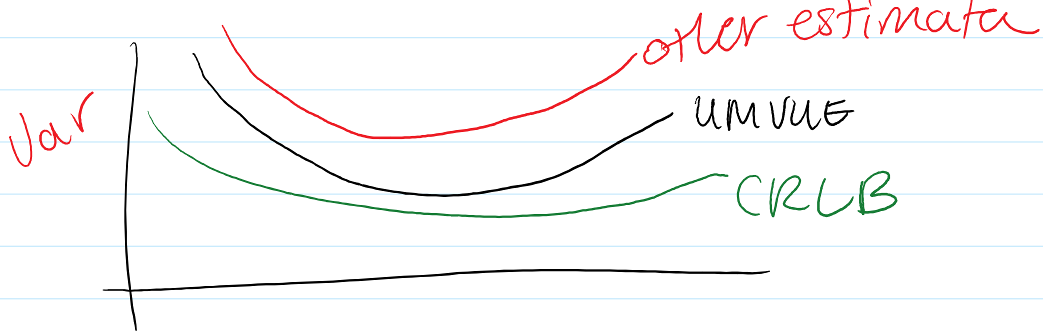
(1) If I have an unbiased estimator $\hat{\theta}^*$ for $\tau(\theta)$ $\left(\mathbb{E}[\hat{\theta}^*] = \tau(\theta)\right)$

and if I calculate that $Var(\hat{\theta}^*) = CRLB$

then $\hat{\theta}^*$ is the UMVUE.

(2) If an estimator doesn't achieve the CRLB I don't otherwise know it's not the UMVUE.

(More on attainment later)



Ex. $X_n \overset{iid}{\sim} Pois(\lambda)$ $\quad \mathbb{E}[X_n] = \lambda = Var(X_n)$

Let $\hat{\lambda} = \overline{X}$

what we know:

$$\mathbb{E}[\hat{\lambda}] = \lambda \quad (\hat{\lambda} \text{ unbiased for } \lambda)$$

$$\text{Var}(\hat{\lambda}) = \frac{\lambda}{N}$$

Lets calculate the CRLB.

$$B = \frac{1}{I_N(\lambda)} = \frac{1}{N/\lambda} = \frac{\lambda}{N}$$

last time

Notice that $\text{Var}(\hat{\lambda}) = B$

So $\hat{\lambda}$ achieves the CRLB hence it is the UMVUE for λ.

Steps:
1. propose unbiased estimator
2. calculate var of est.
3. show Var achieves CRLB
4. then est. is UMVUE

Consider $S^2 = \frac{1}{N-1} \sum_{n=1}^{N} (X_n - \bar{X})^2$

recall that $\mathbb{E}[S^2] = \text{Var}(X_n) = \lambda$

So S^2 is unbiased for λ.

However $\bar{X}$ is better at est. λ than S^2 since $\bar{X}$ is the UMVUE.

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$ Recall: $\mathbb{E}[X_n] = 1/\lambda$

$Var(X_n) = 1/\lambda^2$

Goal: find UMVUE for $1/\lambda$.

(1) Propose unbiased est. for $1/\lambda$.

Let $T = \bar{X}$. Then $\mathbb{E}[T] = 1/\lambda$.

So T unbiased for $1/\lambda$.

(2) Calc. Var of T.

$$Var(T) = \frac{Var(X_n)}{N} = \frac{1/\lambda^2}{N} = \boxed{\frac{1}{N\lambda^2}}$$

(3) Calc CRLB for unbiased est. of $1/\lambda$.

Notice $\tau(\lambda) = 1/\lambda$.

Recall: $CRLB = \left(\frac{\partial \tau}{\partial \lambda}\right)^2 /$

$$\Big/ I_N(\lambda)$$

<u>part 1</u>: $\frac{\partial \tau}{\partial \lambda} = -1/\lambda^2$ so $\left(\frac{\partial \tau}{\partial \lambda}\right)^2 = 1/\lambda^4$

<u>part 2</u>: $I_N(\lambda) = NI(\lambda)$

$$I(\lambda) = -E\left[\frac{\partial^2}{\partial \lambda^2} \log f_\lambda(X_n)\right]$$

$f_\lambda(x) = \lambda e^{-\lambda x} \mathbb{1}(x>0)$

$\log f_\lambda(x) = \log \lambda - \lambda x$

$\frac{\partial}{\partial \lambda} \log f_\lambda(x) = \frac{1}{\lambda} - x$

$\frac{\partial^2}{\partial \lambda^2} \log f_\lambda(x) = -1/\lambda^2$

$-E\left[\frac{\partial^2}{\partial \lambda^2} \log f_\lambda(X)\right]$

$= -E[-1/\lambda^2] = 1/\lambda^2$

$$\boxed{I(\lambda) = 1/\lambda^2}$$

So $I_N(\lambda) = N/\lambda^2$

and $B = \left(\frac{\partial \tau}{\partial \lambda}\right)^2 \Big/ I_N(\lambda) = \frac{1/\lambda^4}{N/\lambda^2} = \boxed{\frac{1}{N\lambda^2}}$

(4) So $Var(T) = \frac{1}{N\lambda^2} = CRLB$

so T is the <u>UMVUE</u> for $1/\lambda$.

<u>Ex.</u> $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where σ^2 known.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where σ^2 known.

Want is the UMVUE for μ.

(1) propose an unbiased est. of μ.

Let $T = \bar{X}$. Then $E[T] = \mu$.

So $\bar{X}$ unbiased for μ.

(2) Calc. var of T.

$$Var(T) = Var(\bar{X}) = \sigma^2/N$$

(3) Calc. CRLB for μ. Note $\tau(\mu) = \mu$.

$$B = \frac{\left(\frac{\partial \tau}{\partial \mu}\right)^2}{I_N(\mu)} = \frac{1}{I_N(\mu)}$$

$$I(\mu) = -\mathbb{E}\left[\frac{\partial^2}{\partial \mu^2} \log f_\mu(x)\right]$$

$$f_\mu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(X-\mu)^2\right)$$

$$\log f_\mu(x) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}(X-\mu)^2$$

$$\frac{\partial}{\partial \mu} \log f_\mu(x) = -\frac{1}{2\sigma^2} 2(x-\mu)(-1) = \frac{1}{\sigma^2}(x-\mu)$$

$$\frac{\partial^2}{\partial \mu^2} \log f_\mu(x) = -1/\sigma^2$$

here $I(\mu) = -E[\;\;] = -(-1/\sigma^2) = 1/\sigma^2$

$$\rightarrow B = \frac{1}{I_N(\mu)} = \frac{1}{N\,1/\sigma^2} = \sigma^2/N$$

4 here $Var(T) = Var(\bar{X}) = \sigma^2/N = CRLB$

so $\bar{X}$ is the UMVUE for μ.

Ex. $X_n \overset{iid}{\sim} U(0,\theta)$, want UMVUE for θ.

1 Propose unbiased est. for θ.

let $T = \frac{N+1}{N} X_{(N)}$

Can show $E[X_{(N)}] = \frac{N}{N+1}\theta$

then $E[T] = E\left[\frac{N+1}{N} X_{(N)}\right]$

$$= \frac{N+1}{N} E[X_{(N)}]$$

$$= \frac{N+1}{N} \frac{N}{N+1} \theta = \theta$$

So T is unbiased for θ.

② Calc var of T.

$$\text{Var}(T) = \cdots = \frac{\theta^2}{N(N+2)}$$

451 calculation.

③ CRLB

$$f_\theta(x) = \frac{1}{\theta} \quad \text{for } \underbrace{0 < x < \theta}_{!!!}$$

$$\Rightarrow \log f_\theta = -\log \theta$$

$$\Rightarrow \frac{\partial}{\partial\theta} \log f_\theta = -1/\theta \quad \rightarrow \quad \frac{\partial^2}{\partial\theta^2} \log f_\theta = 1/\theta^2$$

$$-E\left[\frac{\partial^2}{\partial\theta^2}\right] = -1/\theta^2$$

$$I(\theta) = E[S_\theta^2] = E\left[\left(-\frac{1}{\theta}\right)^2\right] = 1/\theta^2$$

$$\Rightarrow I_N(\theta) = N/\theta^2$$

$$\text{So CRLB} \overset{?}{=} \frac{1}{I_N(\theta)}$$

$$= \theta^2/N$$

④ ... ? Notice $\mathrm{Var}(T) = \frac{\theta^2}{N(N+2)} < \frac{\theta^2}{N}$?

None of this applies b/c $U(0,\theta)$ is not regular enough (not exp. fam).