Goal: Find a UMVUE (Best estimator)

Deterministic: $\underset{\sim}{x} \in \mathbb{R}^N$ and $\theta \in \mathbb{R}$

Fns of $\underset{\sim}{x}$:	$f_\theta(\underset{\sim}{x})$	$\log f_\theta(\underset{\sim}{x})$	$\frac{\partial}{\partial\theta}\log f_\theta(\underset{\sim}{x})$ or $\frac{\partial^2}{\partial\theta^2}\log f_\theta(\underset{\sim}{x})$
Fn of θ :	$L(\theta) = f_\theta(\underset{\sim}{x})$	$\ell(\theta) = \log L(\theta)$	$\frac{\partial \ell}{\partial\theta}$ or $\frac{\partial^2 \ell}{\partial\theta^2}$

Random: Replace $\underset{\sim}{x}$ w/ $\underset{\sim}{X}$

$f_\theta(\underset{\sim}{X})$ or $\log f_\theta(\underset{\sim}{X})$ or $\frac{\partial}{\partial\theta}\log f_\theta(\underset{\sim}{X})$ or $\frac{\partial^2}{\partial\theta^2}\log f_\theta(\underset{\sim}{X})$

Score: $S_\theta = S_\theta(\underset{\sim}{X}) = \frac{\partial}{\partial\theta}\log f_\theta(\underset{\sim}{X}) = \frac{\partial \ell}{\partial\theta}$

Theorem: If we have nice enough f_θ
(Exponential families work)

then

1. $\mathbb{E}[S_\theta] = 0$ $\quad \left(\mathbb{E}\left[\frac{\partial}{\partial\theta}\log f_\theta(\underset{\sim}{X})\right] = 0\right)$

   $\left(\mathbb{E}\left[\frac{\partial \ell}{\partial\theta}\right] = 0\right)$

2. $Var(S_\theta) = \mathbb{E}[S_\theta^2] = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(\underset{\sim}{X})\right]$

② $Var(S_\theta) = E[S_\theta^2] = -E\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(\underline{X})\right]$

If $E[z]=0$
$Var(z) = E[z^2]$

$\left(E\left[\left(\frac{\partial \ell}{\partial\theta}\right)^2\right] = -E\left[\frac{\partial^2 \ell}{\partial\theta^2}\right]\right)$

my preference.

Defn: Fisher Info.

$N=1$ $\quad X \sim f_\theta$ then $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)\right]$

$N>1$ $\quad X_n \overset{iid}{\sim} f_\theta$ then $I_N(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(\underline{X})\right]$

$$\boxed{I_N(\theta) = -E\left[\frac{\partial^2 \ell}{\partial\theta^2}\right] = N\,I(\theta)}$$

Ex. $\quad X_n \overset{iid}{\sim} Pois(\lambda)$
what is $I_N(\lambda)$?

⓪ $I_N(\lambda) = N\,I(\lambda)$ so I only need to derive for $N=1$ sample

① $X \sim Pois(\lambda)$

② $f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

③ $\log f_\lambda(x) = x\log\lambda - \lambda - \log(x!)$

$= 1/\lambda$

Claim 1 (Redo): $\boxed{I_N(\lambda) = N\,I(\lambda)}$

by defn $I_N(\lambda) = -\mathbb{E}\left[\frac{\partial^2}{\partial\lambda^2} \log f_\lambda(\underset{\sim}{X})\right]$

$$\Rightarrow f_\lambda(\underset{\sim}{x}) = \prod_n \frac{e^{-\lambda}\lambda^{x_n}}{(x_n)!} = \frac{e^{-N\lambda}\lambda^{\Sigma x_n}}{\prod_n x_n!}$$

$$\Rightarrow \log f_\lambda(\underset{\sim}{x}) = -N\lambda + \Sigma x_n \log\lambda - \log\left(\prod x_n!\right)$$

$$\Rightarrow \frac{\partial}{\partial \lambda} \log f_\lambda(\underset{\sim}{x}) = -N + \frac{\sum_n x_n}{\lambda}$$

$$\Rightarrow \frac{\partial^2}{\partial \lambda^2} \log f_\lambda(\underset{\sim}{x}) = -\frac{\sum x_n}{\lambda^2}$$

$$\Rightarrow -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda^2}\right] = -\mathbb{E}\left[\frac{\partial^2}{\partial \lambda^2} \log f_\lambda(X)\right] = -\mathbb{E}\left[-\frac{\sum X_n}{\lambda^2}\right]$$

$$= \frac{\sum_n \mathbb{E}[X_n]}{\lambda^2} \quad \lambda$$

$$= \frac{N\lambda}{\lambda^2} = \frac{N}{\lambda}$$

Claim 2: $\mathbb{E}\left[\left(\frac{\partial \ell}{\partial \theta}\right)^2\right] = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \theta^2}\right]$

$X \sim \text{Pois}(\lambda)$

$$\Rightarrow \frac{\partial \ell}{\partial \lambda} = \frac{\partial}{\partial \lambda} \log f_\lambda(x) = \frac{x}{\lambda} - 1$$

$$\Rightarrow \mathbb{E}\left[\left(\frac{\partial \ell}{\partial \lambda}\right)^2\right] = \mathbb{E}\left[\left(\frac{X}{\lambda} - 1\right)^2\right]$$

$$= \mathbb{E}\left[\frac{X^2}{\lambda^2} - \frac{2X}{\lambda} + 1\right]$$

$$= \frac{\mathbb{E}[X^2]}{\lambda^2} - \frac{2\mathbb{E}[X]}{\lambda} + 1$$

$$= \frac{\lambda + \lambda^2}{\lambda^2} - \frac{2\lambda}{\lambda} + 1$$

$$\text{Var}(X) = \lambda = \mathbb{E}[X^2] - \underbrace{\mathbb{E}[X]^2}_{\lambda^2}$$

So $\mathbb{E}[X^2] = \lambda + \lambda^2$

$$\frac{\quad}{\lambda^2} - \frac{\quad}{\lambda} + 1$$

$$= \frac{1}{\lambda} + 1 - 2 + 1 = \frac{1}{\lambda} = I(\lambda)$$

Can I do this w/ $I_N(\lambda)$?
when working w/ $N > 1$

$$\boxed{\frac{\partial \ell}{\partial \lambda} = -N + \frac{\sum X_n}{\lambda}}$$

So calculate $\mathbb{E}\left[\left(\frac{\partial \ell}{\partial \lambda}\right)^2\right] = \mathbb{E}\left[\left(-N + \frac{\sum X_n}{\lambda}\right)^2\right]$

$$= \mathbb{E}\left[\left(\frac{\sum X_n}{\lambda}\right)^2\right] - \frac{2N}{\lambda}\mathbb{E}[\sum X_n] + N^2$$

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ex. Let $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

σ^2 known

μ unknown

What is $I_N(\mu)$?

① $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

② $\ell = \log f(x) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2$

③ $\frac{\partial \ell}{\partial \mu} = \frac{1}{\quad}(2)(x-\mu)(-1)$

$$\frac{\partial \ell}{\partial \mu} = -\frac{1}{2\sigma^2}(2)(x-\mu)(-1)$$

$$= \frac{1}{\sigma^2}(x-\mu)$$

④ $\frac{\partial^2 \ell}{\partial \mu^2} = \frac{-1}{\sigma^2}$

⑤ $-E\left[\frac{\partial^2 \ell}{\partial \mu^2}\right] = 1/\sigma^2 = I(\mu).$

⑥ $\boxed{I_N(\mu) = N/\sigma^2}$

$Var(\bar{X}) = \frac{\sigma^2}{N}$

Suspicious ???

Ex. $X_n \overset{iid}{\sim} Pois(\lambda)$

Recall $Var(X_n) = \lambda$ so $Sd(X_n) = \sqrt{\lambda} = \psi$

Reparameterize poisson in terms of $\psi = \sqrt{\lambda}$ $(\lambda = \psi^2)$

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(\psi^2)^x e^{-\psi^2}}{x!} = \frac{\psi^{2x} e^{-\psi^2}}{x!}$$

$$= f_\psi(x)$$

Use prev. procedure:

① $\ell = \log f_\psi = 2x \log \psi - \psi^2 - \log(x!)$

② $\frac{\partial \ell}{\partial \psi} = \frac{2x}{\psi} - 2\psi$

(3) $\frac{\partial^2 \ell}{\partial \psi^2} = \frac{-2x}{\psi^2} - 2$

(4) $I(\psi) = -E\left[\frac{\partial^2 \ell}{\partial \psi^2}\right] = -E\left[\frac{-2x}{\psi^2} - 2\right]$

$$= \frac{2E[x]}{\psi^2} + 2$$

$$= \frac{2\psi^2}{\psi^2} + 2 = 2 + 2 = 4.$$

(5) $I_N(\psi) = 4N.$

Derivative Review

$$y = f(x) \Leftrightarrow x = f^{-1}(y)$$

$$\frac{dy}{dx} \xleftrightarrow{\text{rel?}} \frac{dx}{dy}$$

Recall: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ or $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ $\leftarrow$

Theorem:

some fn

If $\theta = \tau(\psi)$ then

$$\boxed{I(\theta) = \left(\frac{\partial \psi}{\partial \theta}\right)^2 I(\psi)}$$

Transformation Theorem for Fisher Info.

Alt. $\boxed{I(\psi) = \left(\frac{\partial \theta}{\partial \psi}\right)^2 I(\theta)}$

Alt. $$\boxed{I(\psi) = \left(\frac{\partial\theta}{\partial\psi}\right)^2 I(\theta)}$$

b/c if $I(\theta) = \left(\frac{\partial\psi}{\partial\theta}\right)^2 I(\psi)$ then $I(\psi) = \left(\frac{1}{\frac{\partial\psi}{\partial\theta}}\right)^2 I(\theta)$

$$= \left(\frac{\partial\theta}{\partial\psi}\right)^2 I(\theta)$$

Revisit above example

$X_n \overset{iid}{\sim} Pois(\lambda)$ then $I(\lambda) = 1/\lambda$

then if $\psi = \sqrt{\lambda}$ then

$$I(\psi) = \left(\frac{d\lambda}{d\psi}\right)^2 I(\lambda)$$

$$= (2\psi)^2 \frac{1}{\lambda}$$

$$= (2\psi)^2 \frac{1}{\psi^2} = 4.$$

$\psi = \sqrt{\lambda}$

$\lambda = \psi^2$

so $\frac{d\lambda}{d\psi} = 2\psi$

Why we care?

Recall: $\hat{\theta}^*$ is the UMVUE for $\tau(\theta)$

(1) $E[\hat{\theta}^*] = \tau(\theta)$

(2) $Var(\hat{\theta}^*) \le Var(\hat{\theta})$ $\forall \theta$

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any other unbiased est.

Theorem: Cramer-Rao Lower Bound

If $X_n \sim f_\theta$ when $\theta \in \Theta$ and

$\hat{\theta}$ is unbiased for $\tau(\theta)$

(*) as long as f_θ is regular enough (i.e. exponential families) (*)

then

$$Var(\hat{\theta}) \geq \left(\frac{\partial \tau}{\partial \theta}\right)^2 \Big/ I_N(\theta).$$