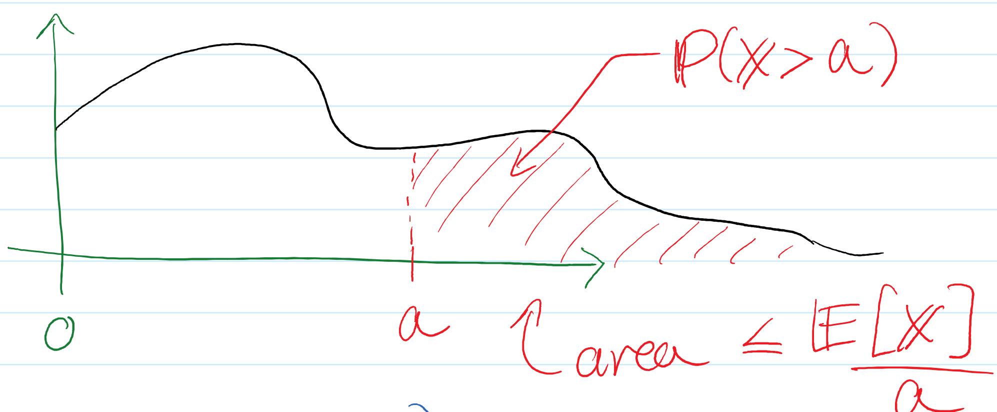
Inequalities

Theorem: Markov's Inequality

If $X \geq 0$ (X a non-neg. r.v. support$(X) \geq 0$)

then for $a \geq 0$ we have

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$



proof. (continuous case)

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^\infty x f(x)\,dx$$

$$\underbrace{\int_0^a x f(x)\,dx}_{A} + \underbrace{\int_a^\infty x f(x)\,dx}_{B}$$

$A \geq 0$ then $A + B \geq B$

$x \geq 0,\ f(x) \geq 0$

$\overbrace{\text{whole}}$ integral $A \geq 0$

notice: $x \geq a$

$$\geq \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty a f(x)\,dx = a \underbrace{\int_a^\infty f(x)\,dx}_{\mathbb{P}(X > a)}$$

$$= a\,\mathbb{P}(X > a)$$

all together: $\mathbb{E}[X] \geq a\,\mathbb{P}(X > a)$

or $\boxed{\mathbb{P}(X > a) \leq \frac{\mathbb{E}[X]}{a}}$.

tail prob.

Theorem: Chebyshev's Inequality

If X is a RV

where $\mu = \mathbb{E}[X]$ and $\sigma^2 = \mathrm{Var}(X)$.

then $\boxed{\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2}$.

$f(x)$

$$\mathbb{P}(|X-\mu| \geq \sigma k) \leq 1/k^2$$

<u>pf.</u> $Y = \dfrac{(X-\mu)^2}{\sigma^2}$ and $a = k^2$

<u>notice</u>: $Y \geq 0$ and so by Markov's Ineq. we have

$$\to \mathbb{P}(Y > a) \leq \frac{E[Y]}{a}$$

$$\to \mathbb{P}\left(\frac{(X-\mu)^2}{\sigma^2} > k^2\right) \leq \frac{1}{k^2}$$

$$E[Y] = E\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2} \underbrace{E[(X-\mu)^2]}_{Var(X)} = \frac{1}{\sigma^2}\sigma^2 = 1$$

$$\to \mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2$$

Various forms of Chebyshev:

① $\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2$

② $\mathbb{P}\left(\frac{|X-\mu|}{\sigma} < k\right) \geq 1 - 1/k^2$

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③ $\varepsilon = k\sigma \longleftrightarrow k = \varepsilon/\sigma$ and $1/k^2 = \sigma^2/\varepsilon^2$

$$\mathbb{P}(|X-\mu| \geq \varepsilon) \leq \sigma^2/\varepsilon^2$$

④ $\mathbb{P}(|X-\mu| < \varepsilon) \geq 1 - \sigma^2/\varepsilon^2$.

Ex. Let X = # of nails in a manufactured box by some factory

$$\mu = \mathbb{E}[X] = 1000$$

$$\sigma^2 = \mathrm{Var}(X) = 25 \quad (\sigma = 5)$$

what is the prob. that

$$994 \leq X \leq 1006$$

$$\mathbb{P}(\overset{1000-6}{994} \leq X \leq \overset{1000+6}{1006}) = \mathbb{P}(|X - 1000| \leq 6)$$

$$= \mathbb{P}\left(\frac{|X - \overset{\mu}{1000}|}{\sigma} \leq 1.2\right)$$

$$= \mathbb{P}\left(\frac{|X - 1000|}{\sigma_6} \le \frac{1.2}{k}\right)$$

$$\geq 1 - 1/k^2 \approx .3055$$

Convergence of RVs

Calc II: talked about convergence of seq. of numbers

$x_n \longrightarrow x$ where $x_n, x \in \mathbb{R}$

converges lim ...

452: seqs of RVs

$X_n \longrightarrow X$ where X_n, X are RVs.

Recall: $X_n : S \longrightarrow \mathbb{R}$

for some $s \in S$ we calc $X(s) \in \mathbb{R}$

we can talk about convergence of RVs as convergence of functions.

Defn: Pointwise Convergence of Functions

Let (f_n) be a seq of functions $f_n : \mathbb{R} \to \mathbb{R}$

Let (f_n) be a seq of functions $f_n : \mathbb{R} \to \mathbb{R}$
and $f : \mathbb{R} \to \mathbb{R}$ is some other function,

we say the seq (f_n) converges pointwise to f
if for any $x \in \mathbb{R}$

means converges

$$f_n(x) \longrightarrow f(x).$$

denoted $f_n \xrightarrow{\text{pTwise}} f$

Fix x

Note $f_n(x) \in \mathbb{R}$ and $f(x) \in \mathbb{R}$,

$x=5$

$f_1(5), f_2(5), f_3(5), f_4(5), \ldots \longrightarrow f(5)$

as seq of numbers

$x=6$

$f_1(6), f_2(6), f_3(6), \ldots \longrightarrow f(6)$

Defn: Sure Convergence of RVs

A seq of RVs $X_1, X_2, X_3, \ldots$ converges surely to X if

$$X_n \xrightarrow{\text{pTwise}} X.$$

$\forall s \in S, \quad X_n(s) \longrightarrow X(s)$

Defn: Almost Sure Convergence

We say a seq (X_n) converge almost surley to X if X_n converge ptwse to X on some subset $A \subset S$ where $\mathbb{P}(A) = 1$.

Notation: $X_n \xrightarrow{a.s.} X$

basically: a.s. convergence is ptwse convergence everywhere in S except (maybe) some small set E w/ $\mathbb{P}(E) = 0$.

equiv. $X_n(s) \to X(s) \ \forall s \in A$ where $\mathbb{P}(A) = 1$.

Another way

For any $\varepsilon > 0$ and consider

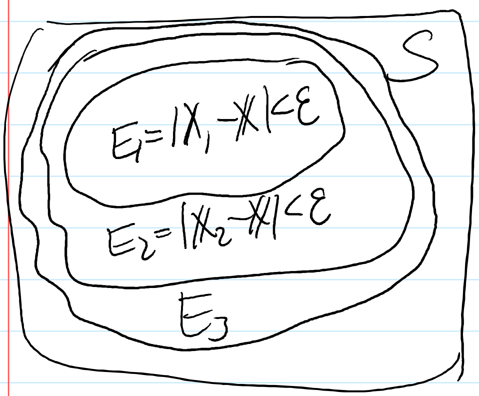
some event $E_n \subset S$

$$E_n = \text{"}|X_n - X| < \varepsilon\text{"}$$
$$= \{s \in S \text{ whe } |X_n(s) - X(s)| < \varepsilon\}$$

define $\lim_{n\to\infty} E_n = \lim_{n\to\infty} |X_n - X| < \varepsilon$

$_{n\to\infty}$ E_n $_{n\to\infty}$

$$= \bigcup_n E_n = \bigcup_n \text{"}|X_n - X| < \varepsilon\text{"}$$



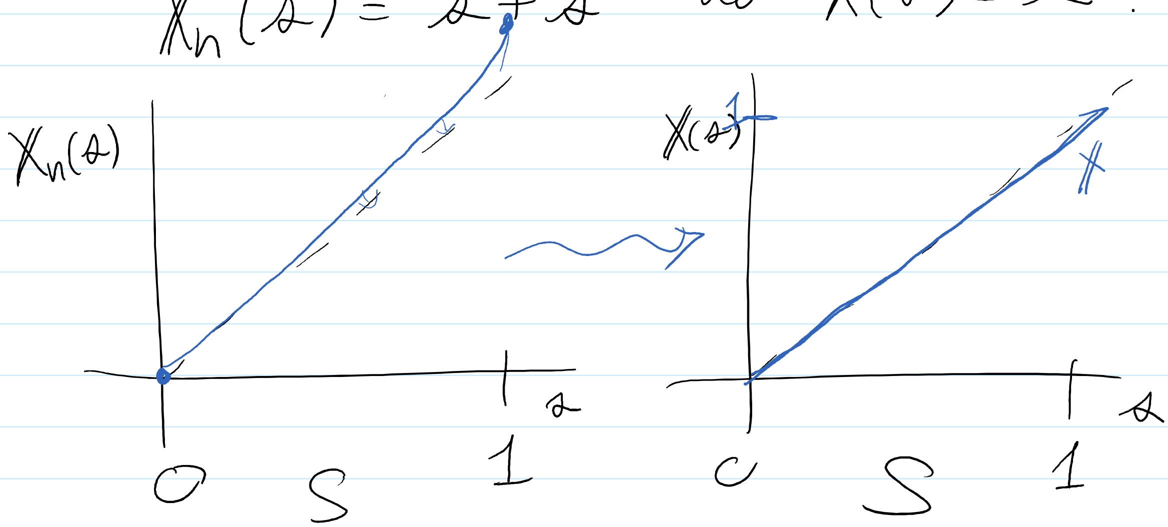
$$\boxed{\begin{gathered} X_n \xrightarrow{a.s.} X \\ \Updownarrow \\ P\left(\lim_{n\to\infty} |X_n - X| < \varepsilon\right) = 1 \end{gathered}}$$

Alt. $P\left(\lim_{n\to\infty} |X_n - X| \geq \varepsilon\right) = 0$

Ex. Let $S = [0,1]$ w/ uniform density

Let

$$X_n(s) = s + s^n \quad \text{and} \quad X(s) = s.$$



Does $X_n \xrightarrow{a.s.} X$?

Notice that $s \in [0,1)$ then

$$X_n(s) = s + \underbrace{s^n}_{\downarrow n \atop 0} \longrightarrow s = X(s) \quad \text{(ptwise)}$$

however if $s=1$

$$X_n(1) = 2 = 1 + 1^n \not\longrightarrow 1 = X(1)$$

who cares?

$$\mathbb{P}(X_n(s) \to X(s)) = 1$$

$$A = \{s \in S \mid X_n(s) \to X(s)\} = [0,1)$$

and $\mathbb{P}(A) = 1$

we're not bothered that no convergence when $s=1$ b/c $\mathbb{P}(\{1\}) = 0$.

Almost Sure convergence is a strong condition. Difficult to establish.

Sometimes, we can establish a weaker form of convergence

Defn: Convergence in Probability

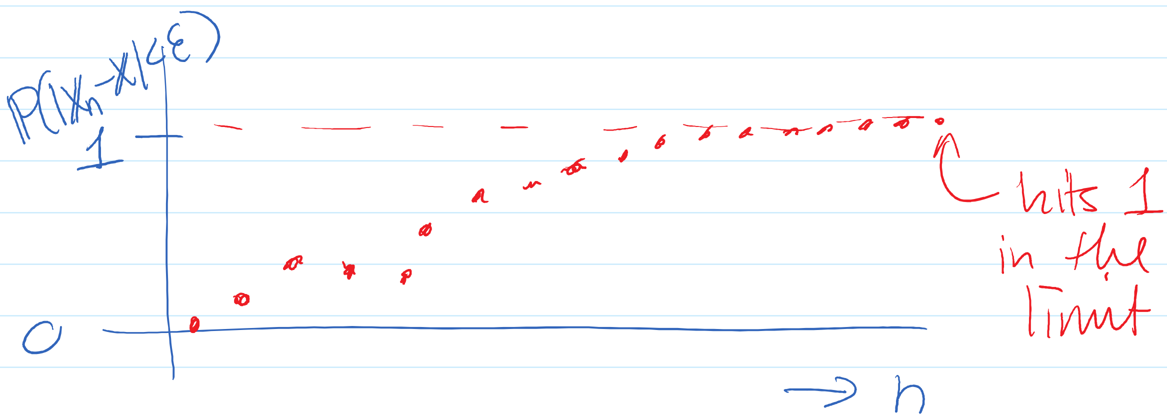
We say (X_n) converges in prob. to X denoted

$$X_n \xrightarrow{P} X$$

If $\forall \varepsilon > 0$, $\lim_{n \to \infty} P(|X_n - X| < \varepsilon) = 1$.

Pick $\varepsilon > 0$,

$$P(|X_1 - X| < \varepsilon), P(|X_2 - X| < \varepsilon), P(|X_3 - X| < \varepsilon), \ldots$$



equiv. $\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$.
