Resut: If $X_n \overset{iid}{\sim} f$ then

sample size: $X_1, \dots, X_N$

$$M_{\bar{X}}(t) = \left(M_{X_1}(t/N)\right)^N$$

$M_{X_1} = M_{X_2} = M_{X_3} = \cdots$

Theorem:

If $X_n \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N(\mu, \sigma^2/N)$

Ex. $X_n \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$

$$f(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \text{ for } x > 0$$

$$= \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \mathbb{1}(x>0)$$

$$\mathbb{1}(x>0) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$

What is the dist of $\bar{X}$?

$$M_{\bar{X}}(t) = \left(M_{X_1}(t/N)\right)^N$$

$M(t) = (1 - t/\beta)^{-\alpha}$ for $t < \beta$

$$= \left[\left(1 - \frac{t}{N\beta}\right)^{-\alpha}\right]^N$$

$$= \left(1 - \frac{t}{N\beta}\right)^{-N\alpha}$$

recognize as the mgf of a Gamma$(N\alpha, N\beta)$

$= \left(1 - t/_{N\beta}\right)$ ← a Gamma$(N\alpha, N\beta)$

$$\bar{X} \sim \text{Gamma}(N\alpha, N\beta).$$

Theorem: Let $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

✓ (1) $\bar{X} \sim N(\mu, \sigma^2/N)$ $\qquad S^2_{N-1} = \frac{1}{N-1}\sum_{n=1}^{N}(X_n - \bar{X})^2$

(future) (2) $\bar{X} \perp\!\!\!\perp S^2_{N-1}$

(sketch) (3) $\frac{N-1}{\sigma^2} S^2_{N-1} \sim \chi^2(N-1)$

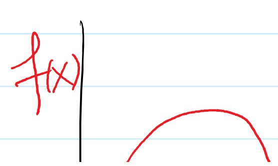
chi-squared distribution w/ N-1 deg. of freedom.

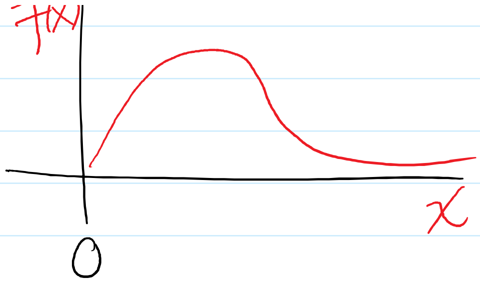
Fact: $\frac{N-1}{\sigma^2} S^2 \sim \chi^2(N-1)$

Chi-Squared dist. one parameter: k degrees of freedom

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2} \mathbb{1}(x>0)$$

$$= \text{Gamma}\left(\alpha = \frac{k}{2}, \beta = 1/2\right) \text{ PDF}$$





(*) Facts:

(1) $Z \sim N(0,1)$ then $Z^2 \sim \chi^2(1)$

(2) $Z_n \overset{iid}{\sim} N(0,1)$ then $\sum_{n=1}^{N} Z_n^2 \sim \chi^2(N)$

(3) $A_n \overset{indep}{\sim} \chi^2(k_n)$ then $\sum_{n=1}^{N} A_n \sim \chi^2\left(\sum_{n=1}^{N} k_n\right)$

Know these

Ex. $A_1 \sim \chi^2(3)$ and $A_2 \sim \chi^2(1.7)$, $A_1 \perp\!\!\!\perp A_2$

then $A_1 + A_2 \sim \chi^2(4.7)$

Sum of independent χ^2 is also χ^2
w/ a df = sum of dfs

Note: $S^2 = \sum_{n=1}^{N} (X_n - \bar{X})^2 \approx \sum Z^2$? (Kinda)

(For fun — not req. to know)

Assume $\mu = 0$, $\sigma^2 = 1$ (claim 3 above)

$$A = \begin{bmatrix} 1-\frac{1}{N} & & -\frac{1}{N} \\ & \ddots & \\ -\frac{1}{N} & & 1-\frac{1}{N} \end{bmatrix} = \text{residualizing matrix}$$

$$\text{rank}(A) = N-1$$

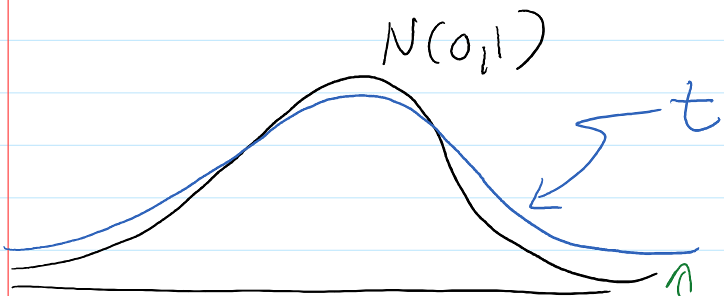
$$(N-1)S^2 = \underset{\sim}{X}^T A \underset{\sim}{X}$$

if Xs normal
quadratic form has
$\chi^2(\text{rank}(A))$

---

t-distribution

one parameter k
degrees of freedom



has fatter tails
(more prob. in tails)

$$f(x) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma(k/2)} \left(1 + \frac{x^2}{k}\right)^{-\frac{k+1}{2}}$$

Fact: If $U \sim N(0,1)$
$V \sim \chi^2(k)$

$U \perp\!\!\!\perp V$

then $\dfrac{U}{\sqrt{V/k}} \sim t(k)$

If $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then $\bar{X} \sim N(\mu, \sigma^2/N)$

$$U = \frac{\bar{X} - \mu}{\sigma/\sqrt{N}} \sim N(0,1)$$

$$E[U] = \frac{1}{\sigma/\sqrt{N}}\left(E[\bar{X}] - \mu\right) = 0$$

$$Var(U) = \frac{1}{\sigma^2/N} Var(\bar{X}) = 1$$

$$V = \frac{N-1}{\sigma^2} S^2 \sim \chi^2(\underline{N-1})$$

also $U \perp\!\!\!\perp V$. Hence

$$\frac{U}{\sqrt{V/N-1}} = \frac{\frac{\bar{X}-\mu}{\cancel{\sigma}/\sqrt{N}}}{\sqrt{\frac{N-1}{\cancel{\sigma^2}} S^2 / \cancel{N-1}}} = \boxed{\frac{\bar{X}-\mu}{S/\sqrt{N}} \sim t(N-1)}$$

Probability: Given $X_n \overset{iid}{\sim} f_\theta$ parameter

We know $\theta = 5$

Calalate $P(X_n = \dots)$

Statistics: Given $X_n \overset{iid}{\sim} f_\theta$

We don't know θ.

We want to use $X_1, \ldots, X_N$ to estimate θ.

How confident are we in our estimates.

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$

How estimate μ? $\bar{X}$?

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$ and $\lambda > 0$

How do I estimate λ?

We'll work w/ familes of parameterized distributions.

e.g. $N(\mu, \sigma^2)$ when $\mu \in \mathbb{R}$, $\sigma^2 > 0$

$Exp(\lambda)$ when $\lambda > 0$

$Unif(0, \theta)$ $\theta \in [10, 20]$

Exponential Family (one-dimension parameter)

Assume we have a parameter $\theta \in \Theta$ and the data $X_n \overset{iid}{\sim} f_\theta$ where

$$f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\theta) \exp\left(T(\underset{\sim}{x})\, w(\theta)\right)$$

depend on θ not $\underset{\sim}{x}$

depend on $\underset{\sim}{x}$ not θ

we call the family of dists. an $\boxed{\text{exp. family}}$.

Ex. Poisson. $\lambda > 0$

$X_n \overset{iid}{\sim} \mathrm{Pois}(\lambda)$

$$\lambda^{x_1}\lambda^{x_2}\lambda^{x_3}\cdots\lambda^{x_N} = \lambda^{x_1+x_2+\cdots x_N}$$

$$f(\underset{\sim}{x}) = \prod_{n=1}^{N} \frac{\lambda^{x_n} e^{-\lambda}}{x_n!} = \prod_{n=1}^{N}\left(\frac{1}{x_n!}\right) \lambda^{\sum_n x_n} e^{-N\lambda}$$

$$e^{\log a} = a$$

$$\log(a^b) = b\log a$$

$$= \prod_{n=1}^{N}\left(\frac{1}{x_n!}\right) \exp\left(\log\left(\lambda^{\sum x_n}\right)\right) e^{-N\lambda}$$

$$= \underbrace{\prod_{n=1}^{N}\left(\frac{1}{x_n!}\right)}_{h(\underset{\sim}{x})} \exp\Big(\underbrace{(\textstyle\sum x_n)}_{T(\underset{\sim}{x})}\underbrace{\log(\lambda)}_{w(\lambda)}\Big) \underbrace{e^{-N\lambda}}_{c(\lambda)}$$

$$\exp\left(T(\underset{\sim}{x})\, w(\lambda)\right)$$

So this is an exponential family.

Ex. $X_n \sim \mathrm{Exp}(\lambda)$

$$f_\lambda(\underset{\sim}{x}) = \prod_{n=1}^{N}\left[\lambda e^{-\lambda x_n} \mathbb{1}(x_n > 0)\right]$$

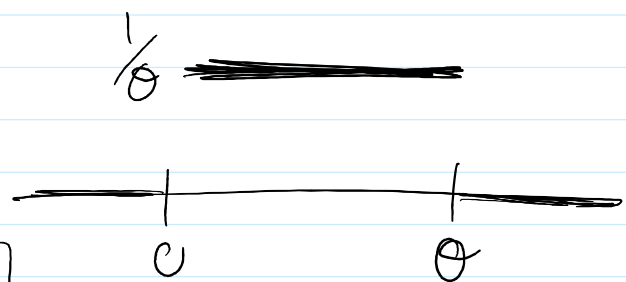
$$f_\lambda(\underset{\sim}{x}) = \prod_{n=1}^{N}\left[\cdots \mathbb{1}(x_n > 0)\right]$$

$$= \underbrace{\left[\prod_{n=1}^{N} \mathbb{1}(x_n > 0)\right]}_{h(\underset{\sim}{x})} \underbrace{\lambda^N}_{c(\lambda)} e^{\overbrace{-\lambda}^{w(\lambda)} \overbrace{\sum_{n=1}^{N} x_n}^{T(\underset{\sim}{x})}}$$

Other exponential families:

Poisson, Exp, Normal, Gamma, Beta, ...

<u>Ex</u> $X_n \overset{iid}{\sim} U(0, \theta)$



$$f(\underset{\sim}{x}) = \prod_n \left[\frac{1}{\theta} \mathbb{1}(0 < x_n < \theta)\right]$$

$$= \left(\frac{1}{\theta}\right)^N \prod_n \mathbb{1}(0 < x_n < \theta) \overset{??}{=} h(\underset{\sim}{x}) c(\theta) \exp\left(T(\underset{\sim}{x}) w(\theta)\right)$$

can't factor these as products

<u>General fact</u>: If my support depends on the unknown parameter it isn't an exp. family.

Theorem: If $X_n \overset{iid}{\sim} f_\theta$ and marginally f_θ is an exponential family (i.e.)

marginal of 1 obs. $\rightarrow$ $f_\theta(x) = c(\theta) h(x) \exp(T(x) w(\theta))$

then $f_\theta(\underset{\sim}{x})$ is an exponential family.

joint of all

Practical advice: just check marginal of one observation.

Ex. $X \sim Exp(\lambda)$

$$f(x) = \underbrace{\lambda}_{c(\lambda)} e^{-\lambda x} \underbrace{\mathbb{I}(x>0)}_{h(x)}$$

$T(x)$, $w(x)$
