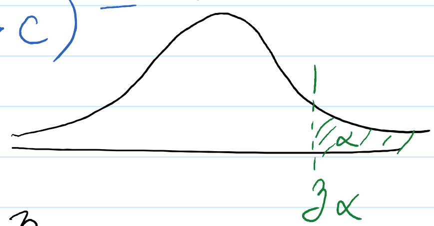
Ex. $H_0: \theta \le \theta_0$ v. $H_a: \theta > \theta_0$

$X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

Known

Let $T = \dfrac{\bar{X} - \theta_0}{\sigma/\sqrt{N}}$

So that $\mathbb{P}_{\theta_0}(T > c) = \alpha$



The test $T > c$ when $c = z_\alpha$

KR says this is the UMP level α test

So long as T has the MLR

Need to check that T has the MLR property.

MLR: if $\theta_1 < \theta_2$ then $\dfrac{f_{\theta_2}(t)}{f_{\theta_1}(t)}$ inc. fn of t.

$\bar{X} \sim N(\theta, \sigma^2/N)$

$T = \dfrac{\bar{X} - \theta_0}{\sigma/\sqrt{N}} \sim N(\underbrace{\theta - \theta_0}_{\mu}, 1)$

$T \sim N(\mu, 1)$

$f_\mu(t) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(t-\mu)^2\right)$

$\dfrac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2 - \frac{1}{2}\mu^2 + t\mu\right)$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}t^2 - \tfrac{1}{2}\mu^2 + t\mu\right)$$

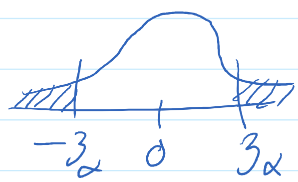
$$= \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}t^2\right)}_{h(t)} \underbrace{\exp\left(-\tfrac{1}{2}\mu^2\right)}_{c(\mu)} \exp\left(\underset{t}{t}\ \underset{w(\mu)=\mu}{\mu}\right)$$

Had a theorem said if w inc. as a fn of μ then T has the MLR property. ✓

Let's again consider $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

Test 1: rej. if $\dfrac{\bar{X} - \theta_0}{\sigma/\sqrt{N}} > z_\alpha$

Test 2: rej. if $\dfrac{\bar{X} - \theta_0}{\sigma/\sqrt{N}} < -z_\alpha$



Neyman–Pearson

$H_0: \theta = \theta_0$ v. $H_a: \theta = \theta_a$ ($\theta_a > \theta_0$) $\Rightarrow$ UMP α is T1

$H_0: \theta = \theta_0$ v. $H_a: \theta = \theta_a$ ($\theta_a < \theta_0$) $\Rightarrow$ UMP α is T2

Karlin–Rubin

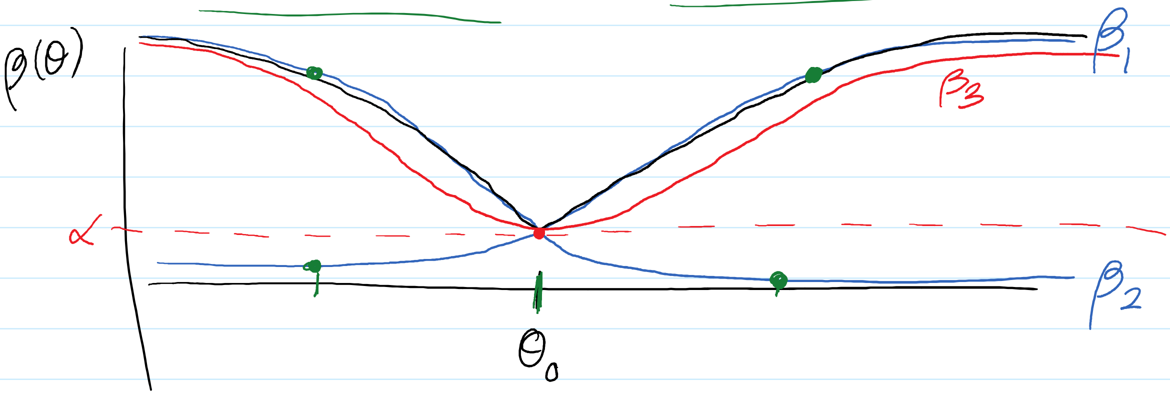
$H_0: \theta \le \theta_0$ v. $H_a: \theta > \theta_0$ $\Rightarrow$ UMP α is T1

$H_0: \theta \ge \theta_0$ v. $H_a: \theta < \theta_0$ $\Rightarrow$ UMP α is T2

$H_0: \theta \geq \theta_0$ v. $H_a: \theta < \theta_0 \Rightarrow$ UMP α is T2

Q: What about

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta \neq \theta_0 \quad ?$$



$\theta < \theta_0$ → T2

$\theta > \theta_0$ → T1

Test 3: $\dfrac{|\bar{X} - \theta_0|}{\sigma/\sqrt{N}} > z_{\alpha/2}$

when $\theta \approx \theta_0$

$\hat{\theta} < \theta_0$ → T_2

$\hat{\theta} > \theta_0$ → T_1

level α

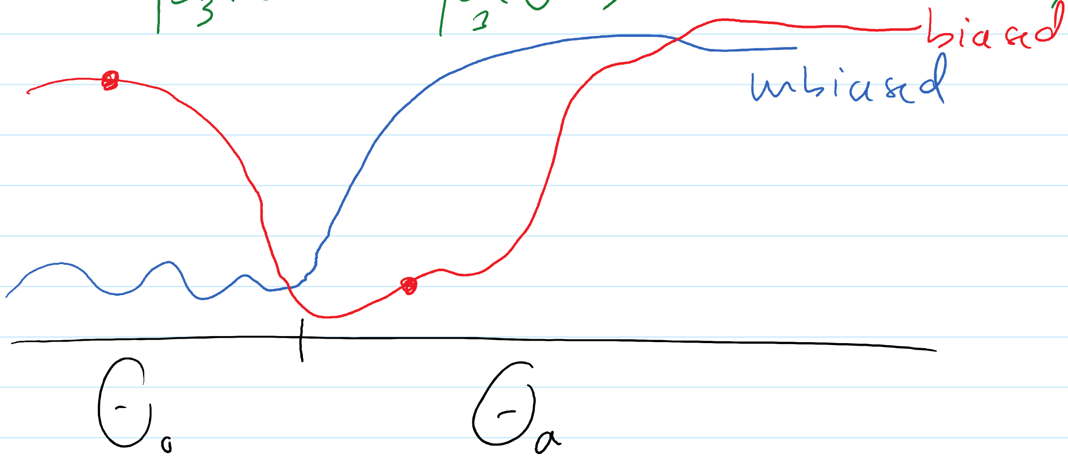
Test 3 the Uniformly Most Powerful Unbiased Test

Unbiased:

$\beta_3(\theta') \geq \beta_3(\theta'')$ the $\theta' \in \Theta_a$, $\theta'' \in \Theta_0$

biased

$\beta_3(\theta') \ge \beta_3(\theta'')$ whe $\theta' \in \Theta_a$, $\theta'' \in \Theta_0$



Interval Estimation

Point Estimation: $\hat{\theta} \approx \theta$

New: want to say $\theta \in C$ (approx.) when

$$C = C(\underset{\sim}{x}) \subset \Theta$$

preferably, C is some interval.

Defn: Interval Estimator

An interval est. of $\theta \in \Theta \subset \mathbb{R}$ is any pair of fns

$$L = L(\underset{\sim}{x}), \quad U = U(\underset{\sim}{x}) \text{ that}$$

satisfy $L \le U$.

satisfy $L = U$.

We say: $L \leq \theta \leq U$ (at least approx.)

Sometimes: might want a "one-sided" interval

i.e. $L = -\infty$ or $U = \infty$

so that our interval is

$(-\infty, U]$ or $[L, \infty)$.

Say: $\theta \leq U$ $\qquad$ $\theta \geq L$

Ex. Let $X_1, \ldots, X_4 \overset{iid}{\sim} N(\mu, 1)$ then an interval est. of μ is

$$[\underbrace{\bar{X} - 1}_{L}, \underbrace{\bar{X} + 1}_{U}].$$

might say: $\bar{X} - 1 \leq \theta \leq \bar{X} + 1$.

why? just use $\bar{X}$?

$$\mathbb{P}(\bar{X} = \mu) = 0.$$

Need to attach some meas. of error. ($sd(\bar{X})$)

fixed

Need to attach some meas. of error. (sd($\bar{X}$))

fixed

Alt. $\mathbb{P}(\bar{X} - 1 \le \mu \le \bar{X} + 1) > 0$

random

Ex. cont.

$$\boxed{\mathbb{P}(\mu \in [\bar{X}-1, \bar{X}+1])}$$

$$= \mathbb{P}(\bar{X}-1 \le \mu \le \bar{X}+1)$$

$$= \mathbb{P}(\bar{X}-1 \le \mu, \ \bar{X}+1 \ge \mu)$$

$X_1, \ldots, X_4 \overset{iid}{\sim} N(\mu, 1)$

$$= \mathbb{P}(\bar{X}-\mu \le 1, \ \bar{X}-\mu \ge -1)$$

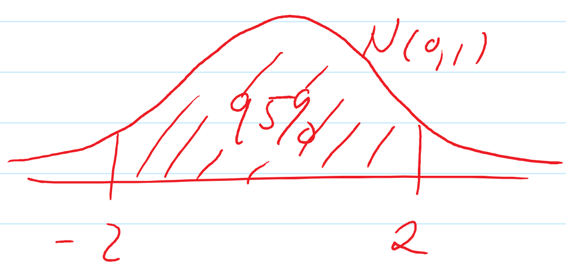
$\bar{X}-\mu \sim N(0, \tfrac{1}{4})$

$$= \mathbb{P}(-1 \le \bar{X}-\mu \le 1)$$

$$= \mathbb{P}\Big(-2 \le \underbrace{\frac{\bar{X}-\mu}{1/2}}_{N(0,1)} \le 2\Big)$$

$Z \sim N(0,1)$.

$$= \mathbb{P}(|Z| \le 2)$$



$$\approx .95$$

So the chance that $[\bar{X}-1, \bar{X}+1]$ covers μ is 95%.

Defn: For an interval estimator $[L, U]$ of a parameter θ we define the coverage probability to be coverage probability to be

$$\mathbb{P}_\theta(L \le \theta \le U)$$

Defn: Confidence Coefficient

Worst-case coverage prob.

For an interval est. $[L, U]$ the conf. coef. is

$$1-\alpha = \min_{\theta \in \Theta} \mathbb{P}_\theta(L \le \theta \le U)$$

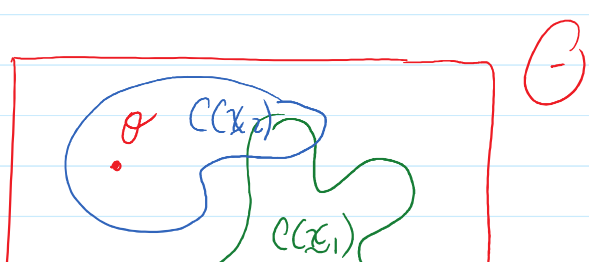
Defn: Confidence Interval

An interval est. along w/ its conf coef

Defn: Confidence Set

A set $C(\underset{\sim}{X}) \subset \Theta$ and its assoc. conf. coef

$$1-\alpha = \min_{\theta \in \Theta} \mathbb{P}(\theta \in C)$$



How do we build a confidence set / interval.

Basically one way: invert a hypothesis test

$$HT \Leftrightarrow \text{Conf. Set.}$$

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ known

Consider HT for

$$H_0: \mu = \mu_0 \quad v. \quad H_a: \mu \neq \mu_0$$

For some fixed sig. level α the UMPU level α test is to reject when

$$\frac{|\bar{X} - \mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2}$$

$$R(\mu_0) = \left\{ \underset{\sim}{x} \in \mathcal{X} \;\middle|\; \frac{|\bar{X} - \mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2} \right\}$$

$$R(\mu_0) = \left\{ \underset{\sim}{x} \in \mathcal{X} \;\middle|\; \frac{|\bar{X} - \mu_0|}{\sigma/\sqrt{N}} > z_{\alpha/2} \right\}$$

reject when data not in agreement w/ μ_0

$$\underline{A(\mu_0)} = \mathcal{X} \setminus R(\mu_0)$$

$$= \left\{ \underset{\sim}{x} \in \mathcal{X} \;\middle|\; \frac{|\bar{X} - \mu_0|}{\sigma/\sqrt{N}} \le z_{\alpha/2} \right\}$$

data is in good agreement w/ μ_0

$$\Leftrightarrow \quad -z_{\alpha/2} \le \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \le z_{\alpha/2}$$

$$\Leftrightarrow \mu_0 - \frac{\sigma}{\sqrt{N}} z_{\alpha/2} \le \bar{X} \le \mu_0 + \frac{\sigma}{\sqrt{N}} z_{\alpha/2}$$

$$\Leftrightarrow \underbrace{\bar{X} - \frac{\sigma}{\sqrt{N}} z_{\alpha/2}}_{L} \le \mu_0 \le \underbrace{\bar{X} + \frac{\sigma}{\sqrt{N}} z_{\alpha/2}}_{U}$$

Since my test is size α then

$$\mathbb{P}_{\mu_0}(\text{rej}) = \alpha$$

$$\mathbb{P}_{\mu_0}(\text{rej}) = \alpha$$

$$\iff \mathbb{P}_{\mu_0}(\text{accept}) = 1 - \alpha$$

and so

$$(*) \quad \mathbb{P}_{\mu_{\cancel{0}}}\left(\bar{X} - \frac{\sigma}{\sqrt{N}} z_{\alpha/2} \le \mu_{\cancel{0}} \le \bar{X} + \frac{\sigma}{\sqrt{N}} z_{\alpha/2} \right) = 1 - \alpha$$

true $\forall \mu$

So if
$$L = \bar{X} - \sigma/\sqrt{N}\, z_{\alpha/2}$$
$$U = \bar{X} + \sigma/\sqrt{N}\, z_{\alpha/2}$$

then
$$\min_{\mu \in \mathbb{R}} \mathbb{P}(L \le \mu \le U) = 1 - \alpha$$

and so ... $[L, U]$ is a $1-\alpha$ conf. interval.

Test Inversion

For any $\theta_0 \in \Theta$ let $A(\theta_0)$ be the accept. region for the α-level test

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta \neq \theta_0$$

$H_0: \theta = \theta_0$ v. $H_a: \theta \neq \theta_0$

and let

$$C(\underset{\sim}{x}) = \{\theta \mid \underset{\sim}{x} \in A(\theta)\} \subset \Theta$$

Then C is a $1-\alpha$ confidence set.