Convergence of RVs

Calc II : talked about convergen of a seq of numbers :

$$x_n \to x \quad \text{where } x_n, x \in \mathbb{R}$$

this class :

$$X_n \to X \quad \text{where } X_n, X \text{ are RVs}$$

Recall : $X_n : S \to \mathbb{R}$

for some $s \in S$ we have $X(s) \in \mathbb{R}$

We can talk about convergence of RVs as fns.

Defn : Pointwise Convergence of Functions

If (f_n) is a seq. of fns $f_n : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ then we say f_n converge pointwise to f if

$$f_n(x) \longrightarrow f(x) \quad \forall x \in \mathbb{R}$$

denoted $f \xrightarrow{\text{ptwse}} f.$

Fix x

$x=5$; $f_1(5), f_2(5), f_3(5), \ldots \quad f(5)$

Defn: Sure Convergence of R Vs

A seq of R Vs $X_1, X_2, \ldots$ Converges surley to X if

$$X_n \xrightarrow{\text{ptwse}} X.$$

i.e. $\forall s \in S$, $X_n(s) \longrightarrow X(s)$.

Defn: Almost Sure Convergence

We say a seq of RVs (X_n) converges almost surley to X if

$$X_n \xrightarrow{\text{ptwse}} X \text{ on some subset } A \subset S$$

where $\mathbb{P}(A) = 1$.

We denote this

$$X_n \xrightarrow{a.s.} X.$$

Basically: a.s. convergence is ptwse conv. everywhere in S except maybe some prob. zero event.

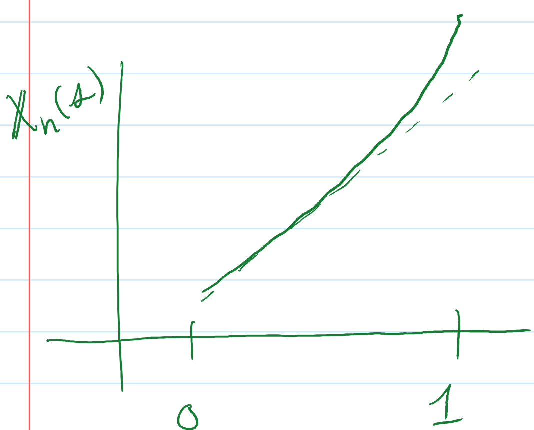
Other notation

$$\mathbb{P}(X_n \to X) = 1$$

means:
$$\mathbb{P}(\{s \mid X_n(s) \to X(s)\}) = 1.$$

Ex. Let $S = [0, 1]$ w/ uniform density

Let $X_n(s) = s + s^n$ and $X(s) = s$



Does: $X_n \xrightarrow{a.s.} X$?

Notice if $s \in [0,1)$ then

$$X_n(s) = s + s^n \xrightarrow{n} s = X(s)$$

however if $s = 1$

$$X_n(s) = 1 + 1^n = 2 \xrightarrow{n} 2 \neq X(s) = 1$$

don't converge ptwse here

$$\mathbb{P}(X_n \to X) = \mathbb{P}(\{s \mid X_n(s) \to X(s)\})$$
$$= \mathbb{P}([0,1)) = 1$$

So $X_n \xrightarrow{a.s.} X$ b/c this prob. is still 1.

Almost sure is a strong condition - can be difficult to establish.

Sometimes work w/ a slightly weaker condition called convergence in prob.

Defn: Convergence in Probability

We say a seq (X_n) converges in prob. to X denoted

$$X_n \xrightarrow{P} X$$

if

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1.$$

Pick $\varepsilon > 0$

$$\mathbb{P}(|X_1 - X| < \varepsilon), \mathbb{P}(|X_2 - X| < \varepsilon), \ldots$$

$\forall \varepsilon > 0$

$$\rightarrow \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1$$

$$\rightarrow \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$$

Theorem: a.s. $\Rightarrow$ i.p

If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{P} X$.

Ex. Consider $S = [0,1]$ w/ uniform prob. density

$$X_1(s) = \cancel{s} + 1$$
$$X_2(s) = \cancel{s} + \mathbb{1}_{[0,1/2]}(s)$$
$$X_3(s) = \cancel{s} + \mathbb{1}_{[1/2,1]}(s)$$
$$X_4(s) = \cancel{s} + \mathbb{1}_{[0,1/3]}(s)$$
$$X_5(s) = \cancel{s} + \mathbb{1}_{[1/3,2/3]}(s)$$
$$X_6(s) = \cancel{s} + \mathbb{1}_{[2/3,1]}(s)$$
$$\vdots$$

Let $X(s) = s$.

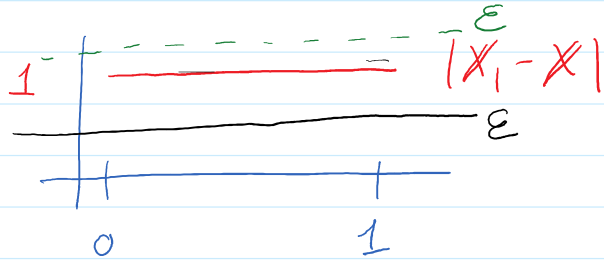
Q: $X_n \xrightarrow{P} X$?

need to establish

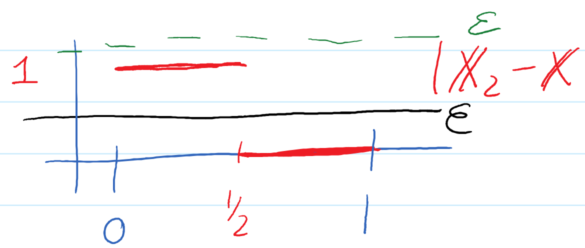
$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

note:

$$|X_1 - X| = 1$$

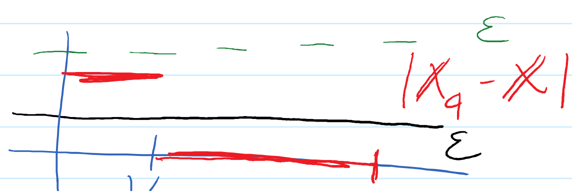


$$|X_2 - X| = \mathbb{1}(s \in [0, 1/2])$$

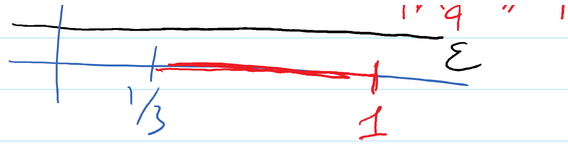


$$|X_3 - X| = \mathbb{1}(s \in [1/2, 1])$$

$$|X_4 - X| = \mathbb{1}(s \in [0, 1/3])$$



$|X_4 - X| = \mathbb{1}(\omega \in [0, 1/3])$



if $\underline{\varepsilon \geq 1}$ then $P(|X_n - X| \geq \varepsilon) = 0$

so certainly $\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$

if $0 < \varepsilon < 1$,

$$P(|X_1 - X| \geq \varepsilon) = 1$$
$$P(|X_2 - X| \geq \varepsilon) = 1/2$$
$$P(|X_3 - X| \geq \varepsilon) = 1/2$$
$$P(|X_4 - X| \geq \varepsilon) = 1/3$$
$$= 1/3$$
$$= 1/3$$
$$= 1/4$$
$$\vdots$$
$$= 1/5$$

in the limit this goes to zero

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

So $X_n \xrightarrow{P} X$.

Does $X_n \xrightarrow{a.s.} X$?

$$A = \{ s \mid X_n(s) \to X(s) \}$$

$$\mathbb{P}(A) = 1 \iff X_n \xrightarrow{a.s.} X.$$

pick any $s \in [0,1]$ and consider

$$X_1(s), X_2(s), \ldots \xrightarrow{?} X(s) = s$$

oscillates betwen s and $s+1$ forever

won't converge to s

So $\mathbb{P}(A) = 0$.

So $X_n \not\xrightarrow{a.s.} X$.

<u>Defn: Convergence in Distribution</u>

We say (X_n) converges in distribution to X

denoted

$$X_n \xrightarrow{d} X$$

if the CDFs converge pointwise.

i.e. if F_n is the CDF of X_n and

F is the CDF of X

then $F_n \xrightarrow{\text{ptwse}} F$

i.e. $\forall x \quad F_n(x) \to F(x)$.

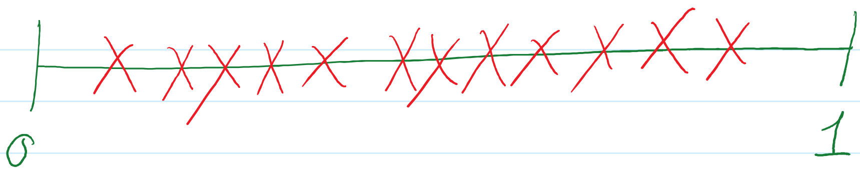
Theorem: i.p $\Rightarrow$ d

If $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{d} X$.

Chain: a.s. $\Rightarrow$ i.p. $\Rightarrow$ d

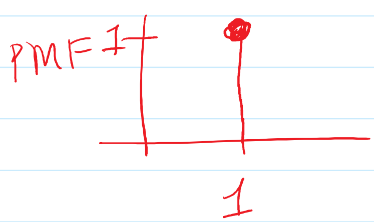
Ex. $X_i \overset{iid}{\sim} U(0,1)$

Let $Y_n = \max_{i=1,\dots,n} X_i$ = max of first n



Intuition: max gets close to 1.

$\boxed{Y_n \xrightarrow{P} 1}$ $\leftarrow$ 1 = degenerate RV w/ all mass at 1



$P(Y=1) = 1$

Want to show $\forall \varepsilon > 0$

$$P(|Y_n - 1| \geq \varepsilon) \xrightarrow{n} 0$$

$$\mathbb{P}(|1 - Y_n| \geq \varepsilon) \qquad Y_n \leq 1$$

$$= \mathbb{P}(1 - Y_n \geq \varepsilon)$$

$$= \mathbb{P}(Y_n \leq 1 - \varepsilon) \qquad Y_n = \max_{i=1,\dots,n} X_i$$

$$= \mathbb{P}(X_1 \leq 1-\varepsilon, X_2 \leq 1-\varepsilon, \dots, X_n \leq 1-\varepsilon)$$

independent

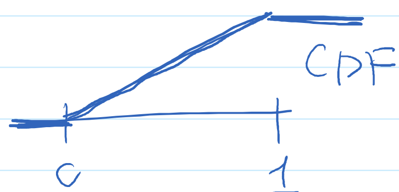
$$= \mathbb{P}(X_1 \leq 1-\varepsilon)\,\mathbb{P}(X_2 \leq 1-\varepsilon) \cdots \mathbb{P}(X_n \leq 1-\varepsilon)$$

$$= \mathbb{P}(X_i \leq 1-\varepsilon)^n$$

If $\varepsilon > 1$ then $1 - \varepsilon < 0$ so

$$\mathbb{P}(X_i \leq 1-\varepsilon) = 0 \quad \text{b/c } X_i \geq 0$$

If $0 < \varepsilon < 1$



$$\mathbb{P}(X_i \leq 1-\varepsilon) = F_{X_i}(1-\varepsilon)$$
$$= 1 - \varepsilon$$

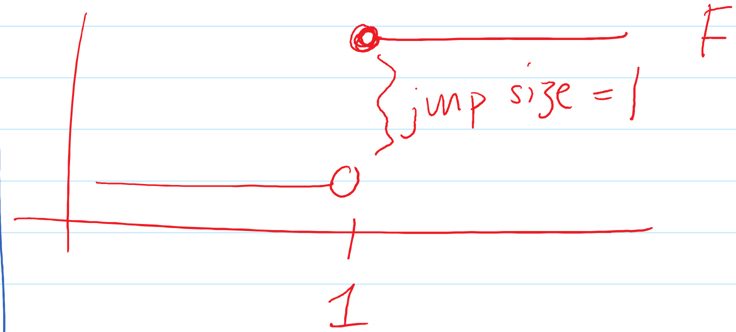
$$= \begin{cases} 0 & \varepsilon \geq 1 \\ (1-\varepsilon)^n & 0 < \varepsilon < 1 \end{cases} = \mathbb{P}(|Y_n - 1| \geq \varepsilon)$$

$\to 0$ as $n \to \infty$ in both cases

So $\lim_{n\to\infty} \mathbb{P}(|Y_n - 1| \geq \varepsilon) = 0$

and so $Y_n \xrightarrow{P} 1$.

Show $\boxed{Y_n \xrightarrow{d} 1}$ $\quad F_n \xrightarrow{\text{ptwse}} F$



$$F_n(y) = \mathbb{P}(Y_n \leq y)$$

$$= \mathbb{P}(\max_i X_i \leq y)$$

$$= \mathbb{P}(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y)$$

$$F(y) = \begin{cases} 0 & y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$= \mathbb{P}(X_i \leq y)^n$$

$$= F_{X_n}(y)^n$$

$X_n \sim U(0,1)$

$$= \begin{cases} 0 & y \leq 0 \\ y^n & 0 < y < 1 \\ 1 & y \geq 1 \end{cases} \xrightarrow{n} \underbrace{\begin{cases} 0 & y \leq 0 \\ 0 & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}}_{F(y)}$$

So $F_n \xrightarrow{\text{ptwse}} F$ so $Y_n \xrightarrow{d} 1$

So $F_n \xrightarrow{\text{pt. wise}} F$ so $Y_n \xrightarrow{d} 1$.