Sp 21 Mid 2 #5

CDF of X_n

$$F_n(x) = \left(1 - \exp(-(x + \log(n)))\right)^n$$

Claim: $X_n \xrightarrow{d} X$, what is F

CDF of X

Defn: $X_n \xrightarrow{d} X \iff F_n(x) \to F(x) \quad \forall x$

Fact: $\underbrace{\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c}$

$$\left[\begin{array}{l} e^a e^b = e^{a+b} \\ e^{\log a} = a \end{array}\right.$$

$$e^{-a} = 1/e^a$$

$$F_n(x) = \left(1 - e^{-(x + \log(n))}\right)^n$$

$$= \left(1 - e^{-x} e^{-\log(n)}\right)^n$$

$$= \left(1 - e^{-x}/e^{\log(n)}\right)^n$$

$$= \left(1 - \frac{e^{-x}}{n}\right)^n = e^{-e^{-x}}$$

by limit defn of e^c

↑ by limit degr

SP5 #2

Let $X_n \overset{iid}{\sim}$ Geometric (p)

Let $\boxed{\theta = \log(p)}$, Want: $I(\theta)$

Prev. problem: Get $I_N(p)$

$$L(p) = f_p(x) = (1-p)^{x-1} p$$

$$\ell(p) = (x-1)\log(1-p) + \log(p)$$

$$\frac{\partial \ell}{\partial p} = \frac{-(x-1)}{1-p} + \frac{1}{p}$$

$$\frac{\partial^2 \ell}{\partial p^2} = \frac{-(x-1)}{(1-p)^2} - \frac{1}{p^2}$$

$u = 1-p$

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

$$\frac{d}{dx} u(x) = -\frac{1}{u^2} u'$$

$$= -\frac{1}{(1-p)^2}(-1)$$

$$I(p) = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial p^2}\right] = \frac{\mathbb{E}[X] - 1}{(1-p)^2} + \frac{1}{p^2}$$

$$= \frac{\frac{1}{p} - 1}{(1-p)^2} + \frac{1}{p^2}$$

$$= \frac{\frac{\cancel{1-p}}{p}}{(1-p)^{\cancel{2}}} + \frac{1}{p^2}$$

$$\frac{\quad}{(1-p)^{\cancel{2}}} + \frac{\quad}{p^2}$$

$$= \frac{1}{p(1-p)} + \frac{1}{p^2}$$

$$= \frac{p + 1 - p}{p^2(1-p)} = \frac{1}{p^2(1-p)} = I(p)$$

So $I_N(p) = \dfrac{N}{p^2(1-p)}$

Theorem: $I_N(\theta) = \left(\dfrac{dp}{d\theta}\right)^2 I_N(p)$ $\qquad \theta = \log(p)$

$p = e^{\theta}$

$$= \left(e^{\theta}\right)^2 \left(\frac{N}{p^2(1-p)}\right) \qquad \frac{dp}{d\theta} = e^{\theta}$$

$$= e^{2\theta} \left(\frac{N}{e^{2\theta}(1-e^{\theta})}\right)$$

$$\boxed{I_N(\theta) = \frac{N}{1-e^{\theta}}}$$

Sp 8 # 9

$M_n(t) = \left(\dfrac{\lambda}{\lambda - t}\right)^n$ MGF of X_n

$$M_n(t) = \left(\frac{\lambda}{\lambda - t}\right)$$

$$Y_n = \frac{1}{n} X_n \qquad Y_n \xrightarrow{d} Z$$

$$M_{Y_n}(t) = M_n(t/n) = \left(\frac{\lambda}{\lambda - t/n}\right)^n$$

$$\lim_{n\to\infty}\left(1 + \frac{c}{n}\right)^{-n} = e^{-c} \qquad = \left(\frac{\lambda - t/n}{\lambda}\right)^{-n}$$

$$= \left(1 \underbrace{- \frac{t/\lambda}{n}}_{c}\right)^{-n}$$

$$\lim_{n\to\infty} M_{Y_n}(t) = \boxed{e^{\frac{1}{\lambda}t} = M(t)}$$

$$M(t) = \mathbb{E}e^{tX} = \sum_x \overbrace{P(X=x)}^{f(x)} e^{tx} = e^{\frac{1}{\lambda}t}$$

limiting dist is degenerate at $1/\lambda$

$$\boxed{P\left(Z = \frac{1}{\lambda}\right) = 1}$$

$$M_Z(t) = P(Z = 1/\lambda)e^{\frac{1}{\lambda}t} = e^{1/\lambda\, t}$$

$$\frac{X_n}{n} \xrightarrow{d} 1/\lambda \quad \text{so} \quad \frac{X_n}{n} \xrightarrow{P} 1/\lambda$$

n λ r

SP 8 #5

CDF of $Exp(\lambda)$ is $F(x) = 1 - e^{-\lambda x}$

$Exp(1)$ $\qquad = 1 - e^{-x}$

CDF of X

Given: $F_n(x) = (1 - (1 - \frac{1}{n})^{nx})$

CDF of X_n

Convergence in dis: $F_n(x) \to F(x) \ \forall x$

$$F_n(x) = 1 - (1 - \tfrac{1}{n})^{nx}$$

$$= 1 - \left[(1 - \tfrac{1}{n})^n\right]^x$$

$$\lim_{n \to \infty} F_n(x) = 1 - \left[\lim_{n \to \infty} (1 - \tfrac{1}{n})^n\right]^x$$

$$= 1 - (e^{-1})^x$$

$$= 1 - e^{-x} = F(x)$$

SP 7 #5

$X_n \sim Gamma(\alpha, \beta)$

UMVUE for $\frac{\alpha^2}{\beta^2} + \frac{\alpha}{N\beta^2} = \tau(\alpha, \beta)$

Want to use Lehmann-Scheffe

① find a SS for (α, β)

② Find fn of SS that has expectation τ...

(1) Factorization Theorem

$$f(\underset{\sim}{x}) = g(T, \theta)\, h(\underset{\sim}{x})$$

$$= g(\underbrace{T_1}_{\sum \lg X_n}, \underbrace{T_2}_{\bar{X}}, \alpha, \beta)\, \underbrace{h(\underset{\sim}{x})}_{1}$$

SP7 #2 $\quad X_n \overset{iid}{\sim} Exp(\lambda)$

$$\boxed{f_{X_{(1)}}(t) = N(1 - F(t))^{N-1} f(t)}$$

$$= N\left(1 - (\underbrace{1 - e^{-\lambda x}}_{F(x)})\right)^{N-1} \lambda e^{-\lambda x}$$

$$= N\left(e^{-\lambda x}\right)^{N-1} \lambda e^{-\lambda x} = N\, e^{-\lambda(N-1)x} e^{-\lambda x}$$

$= (N\lambda)\, e^{-(N\lambda)x}$ ← PDF of $\text{Exp}(N\lambda)$

$$\boxed{X_{(1)} \sim \text{Exp}(N\lambda)}$$

$\mathbb{E}X_n = \frac{1}{\lambda}$, $\boxed{\mathbb{E}X_{(1)} = \frac{1}{N\lambda}}$

So $\mathbb{E}[N X_{(1)}] = 1/\lambda$

i.e. $\boxed{T = N X_{(1)} \text{ is unbiased for } 1/\lambda.}$

(b) No, not based on sufficient stat.

① Show $\bar{X}$ is sufficient for λ using factorization theorem

② $\mathbb{E}\bar{X} = 1/\lambda$

so $\bar{X}$ is the UMVUE for $1/\lambda$.