Exp. fam.

joint.

Said:

$$f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\theta) \exp\left(t(\underset{\sim}{x})\, w(\theta)\right)$$

then this is an exp. fam.

Also said!

marginal

$$f_\theta(x_n) = h_0(x_n)\, c_0(\theta) \exp\left(t_0(x_n)\, w(\theta)\right)$$

then the joint is an exp.

This is true:

$$f_\theta(\underset{\sim}{x}) = \prod_n f_\theta(x_n) = \prod_n h_0(x_n)\, c_0(\theta) \exp\left(t_0(x_n)\, w(\theta)\right)$$

$$= \underbrace{\left(\prod_n h_0(x_n)\right)}_{h(\underset{\sim}{x})} \underbrace{c_0(\theta)^N}_{c(\theta)} \exp\left(\overbrace{\left(\sum_n t_0(x_n)\right)}^{t(\underset{\sim}{x})} w(\theta)\right)$$

$$\prod_n e^{a_n} = e^{\sum_n a_n}$$

$$h(\underset{\sim}{x}) = \prod_n h_0(x_n)$$

$$c(\theta) = c_0(\theta)^N \qquad w(\theta) \text{ same}$$

$$t(\underset{\sim}{x}) = \sum t_0(x_n)$$

$$t(\underset{\sim}{x}) = \sum_n t_0(x_n)$$

Last time:

$X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

① $e^a e^b = e^{a+b}$

② $e^{\log a} = a$

③ $\log a^b = b \log a$

Could do:

$$f_\lambda(x_n) = \frac{\lambda^{x_n} e^{-\lambda}}{x_n!} = \underbrace{\left(\frac{1}{x_n!}\mathbb{1}(x \in \mathbb{N}_0)\right)}_{h_0(x_n)} \underbrace{\left(e^{-\lambda}\right)}_{c_0(\lambda)} e^{\overbrace{x_n}^{t_0(x_n)} \overbrace{\log \lambda}^{w(\lambda)}}$$

$$t(\underset{\sim}{X}) = \sum_n X_n .$$

Notation:

dual life random and number

$$\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n \quad \text{v.} \quad \overline{x} = \frac{1}{N}\sum_{n=1}^{N} x_n$$

RV   number

e.g. $X_{(1)}$ v. $x_{(1)}$

$X_{(N)}$ v. $x_{(N)}$

S^2 v. s^2

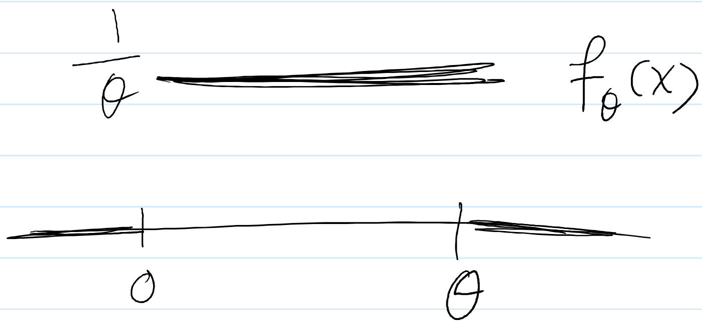
S^2 v. s^2

generically T v. t

Exp. fams

Examples: Poisson, Exp, Normal, Gama, Beta, ...

Ex. $X_n \overset{iid}{\sim} U(0, \theta)$



Exp fam?

$f_\theta(x) = \frac{1}{\theta} \mathbb{1}(0 < x < \theta)$

no way to separate into (fn x)(fn θ)

So $U(0,\theta)$ isn't an exp. fam.

General fact: If the support of my dist depends on θ, then it isn't an exp. fam.

Setup: $X_n \overset{iid}{\sim} f_\theta$, $\theta \in \Theta$

Def: Sufficiency

Defn: Sufficiency

A statistic $T = T(\underset{\sim}{X})$ is <u>sufficient</u> for a parameter θ if

$$f_{\underset{\sim}{X} \mid T=t}(\underset{\sim}{x}) \text{ is "free" of } \theta.$$

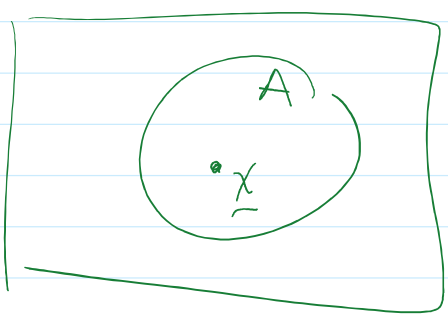
θ doesn't show up in the formula

Ex. $\underset{\sim}{x} \in \mathbb{R}^N$, $A \subset \mathbb{R}^N$

$$\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x} \text{ and } \underset{\sim}{X} \in A)$$

<u>two options:</u> $\underset{\sim}{x} \in A$

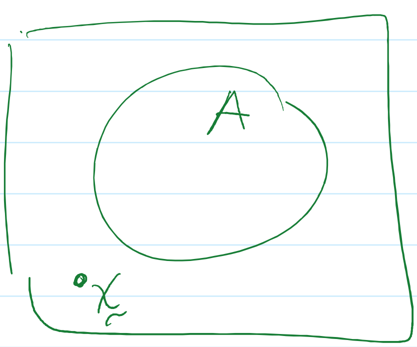
1



$$\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x} \text{ and } \underset{\sim}{X} \in A) = \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x})$$

$\underset{\sim}{x} \notin A$

2



$$\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x} \text{ and } \underset{\sim}{X} \in A) = \underline{\underline{0}}$$

<u>Clever</u>: $\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x} \text{ and Statement about } \underset{\sim}{X})$

$$= \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x})\, \mathbb{1}(\text{Statement about } \underset{\sim}{x})$$

$$= \mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x})\, \mathbb{1}(\text{Statement about } \underset{\sim}{x})$$

Joint dist: $f_{\underset{\sim}{X}, T}(\underset{\sim}{x}, t) = f_{\underset{\sim}{X}}(\underset{\sim}{x})\, \mathbb{1}(T(\underset{\sim}{X}) = t)$

Ex. Let $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bernalli}(\theta)$

$\theta \in [0, 1]$

Let $T = X_1 + X_2 + X_3 \sim \text{Bin}(3, \theta)$

is T sufficient for θ.

$T \sim \text{Bin}(3, \theta)$

$\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x}) = \prod_n \theta^{x_n}(1-\theta)^{1-x_n}\, \cancel{\mathbb{1}(x_n = 0 \text{ or } 1)}$

$= \theta^{\Sigma x_n}(1-\theta)^{3-\Sigma x_n}$

$$f(\underset{\sim}{x} \mid T = t) = \frac{f_{\underset{\sim}{X}, T}(\underset{\sim}{x}, t)}{f_T(t)}$$

$$= \frac{\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x}, T = t)}{\mathbb{P}(T = t)}$$

$$= \frac{\mathbb{P}(\underset{\sim}{X} = \underset{\sim}{x})\, \mathbb{1}(T(\underset{\sim}{X}) = t)}{\mathbb{P}(T = t)}$$

$$= \frac{\cancel{\theta^{\overset{t}{\Sigma x_n}}(1-\theta)^{3-\overset{t}{\Sigma x_n}}}\, \mathbb{1}(\Sigma x_n = t)}{\binom{3}{t}\cancel{\theta^{t}(1-\theta)^{3-t}}}$$

$$\binom{3}{t} \cancel{\theta} \cancel{(1-\theta)}$$

$$= \frac{\mathbb{1}(\sum X_n = t)}{\binom{3}{t}}$$ ← No θ!

So T is sufficient for θ.

<u>Ex.</u> Let $X_n \overset{iid}{\sim} f_\theta$

$$T = (X_1, X_2, \ldots, X_N)$$

Q: is this sufficient for θ?

$$f(\underset{\sim}{x} \mid T = t) = \frac{f_{\underset{\sim}{X}, T}(\underset{\sim}{x}, t)}{f_T(t)}$$

$$= \frac{f_{\underset{\sim}{X}, \underset{\sim}{X}}(\underset{\sim}{x}, \underset{\sim}{x})}{f_{\underset{\sim}{X}}(\underset{\sim}{x})}$$

$$= f_{\underset{\sim}{X}}(\underset{\sim}{x}) / f_{\underset{\sim}{X}}(\underset{\sim}{x}) = 1$$ ← free of θ.

Factorization Theorem

T is sufficient for θ iff

there is a fn $g(\theta, t)$ and $h(\underset{\sim}{x})$ so that

$$f_\theta(\underset{\sim}{x}) = g(\theta, t)\, h(\underset{\sim}{x}).$$

only has $\underset{\sim}{X}$s through t

Ex. $X_n \overset{iid}{\sim} \text{Bern}(\theta)$

$$T = \sum_{n=1}^{N} X_n \quad , \quad t = \sum_n x_n$$

is T sufficient for θ?

$$f_\theta(\underset{\sim}{x}) = \prod_n \theta^{x_n}(1-\theta)^{1-x_n}\, \mathbb{1}(x_n = 0 \text{ or } 1)$$

$$= \underbrace{\theta^{\overbrace{\sum_n x_n}^{t}}(1-\theta)^{N-\overbrace{\sum_n x_n}^{t}}}_{g(\theta,t)\, =\, \theta^t(1-\theta)^{N-t}} \underbrace{\prod_n \mathbb{1}(x_n = 0 \text{ or } 1)}_{h(\underset{\sim}{x})}$$

Then $f_\theta(\underset{\sim}{x}) = g(\theta, t)\, h(\underset{\sim}{x})$ so T is suff. for θ.

Ex. Let $X_n \overset{iid}{\sim} U(0,\theta)$

Can I find a suff. stat. for θ?

$$f(\underset{\sim}{x}) = \prod_n \frac{1}{\theta} \mathbb{1}(0 < x_n < \theta)$$

$$= \theta^{-N} \underbrace{\prod_n \mathbb{1}(0 < x_n < \theta)}$$

all x_n btwn. 0 and θ

min > 0 and max < θ

$$= \theta^{-N} \mathbb{1}(x_{(1)} > 0) \mathbb{1}(x_{(N)} < \theta)$$

$$= g(\theta, t)\, h(\underset{\sim}{x})$$

Let $\boxed{t = x_{(N)}}$ so $x_{(N)}$ is sufficient for θ

and $g(\theta,t) = \theta^{-N} \mathbb{1}(x_{(N)} < \theta)$

$h(\underset{\sim}{x}) = \mathbb{1}(x_{(1)} > 0)$

Theorem: Exp. and Sufficiency

Let $X_n \overset{iid}{\sim} f_\theta$ and

joint $\rightarrow$ $f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x}) c(\theta) \exp(t(\underset{\sim}{x})\, w(\theta))$

so that it is an exp. fam.

so that it is an exp. fam.

Then $\underline{t(\underset{\sim}{X})}$ is sufficient for θ.

Let $X_n \overset{iid}{\sim} N(\mu, 1)$

$e^{a+b} = e^a e^b$

$$f(x_n) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_n-\mu)^2\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x_n^2 + x_n\mu - \frac{1}{2}\mu^2\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_n^2} e^{x_n\mu} e^{-\frac{1}{2}\mu^2}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_n^2}}_{h_0(x_n)} \underbrace{e^{-\frac{1}{2}\mu^2}}_{c_0(\mu)} \exp\left(\underbrace{x_n}_{t_0(x_n)} \underbrace{\mu}_{w(\mu)}\right)$$

$$t(\underset{\sim}{X}) = \sum_n t_0(x_n) = \sum_{n=1}^{N} x_n$$

So by our theorem $\sum_n x_n$ is suff. for θ.

$\underline{\text{Ex,}}$ Let $X_n \overset{iid}{\sim} \text{Exp}(\lambda)$

$$f(x_n) = \underbrace{\lambda}_{c_0(\lambda)} e^{\overbrace{-\lambda}^{w(\lambda)} \overbrace{x_n}^{t_0(x_n)}} \underbrace{\mathbb{1}(x_n > 0)}_{h_0(x_n)}$$

So $\mathrm{Exp}(\lambda)$ is a exp. fam.

and $t(\underset{\sim}{x}) = \sum_n t_0(x_n) = \sum_n x_n$.

So by our theorem $\sum_n x_n$ is suff. for λ.