MoM examples

Ex. $X_n \overset{iid}{\sim} U(0, \theta)$

Get MoM estimator:

$$\mathbb{E}[X_n] = \mu_1 = m_1 = \frac{1}{N}\sum_{n=1}^{N} x_n = \overline{X}$$

$$\mathbb{E}[X_n] = \frac{\theta + 0}{2} = \theta/2$$

Sys. of equ. $\frac{\theta}{2} = \overline{X}$

Solve for θ: $\boxed{\hat{\theta}_{MoM} = 2\overline{X}}$

Ex. $X_n \overset{iid}{\sim} \text{Beta}(\alpha, 1)$

1. $\mu_1 = \mathbb{E}X_n = \frac{\alpha}{\alpha+1}$

2. $m_1 = \overline{X}$

3. $\frac{\alpha}{\alpha+1} = \overline{X} \Rightarrow \alpha = \alpha\overline{X} + \overline{X}$

$$\Rightarrow \alpha - \alpha\overline{X} = \overline{X}$$

$$\Rightarrow \alpha(1-\overline{X}) = \overline{X}$$

$$\Rightarrow \hat{\alpha} = \frac{\overline{X}}{}$$

$$\Rightarrow \boxed{\hat{\alpha} = \frac{\bar{x}}{1-\bar{x}}}$$

Maximum Likelihood Estimation (MLE)

If $X_n \overset{iid}{\sim} f_\theta$ and $\theta \in \Theta$

Recall: joint dist of my data

$$f_\theta(\underset{\sim}{x}) = \prod_{n=1}^{N} f_\theta(x_n)$$

Typ. we think of this as a fn of $\underset{\sim}{x}$

$L : \Theta \to \mathbb{R}$

$L(\theta) = f_\theta(\underset{\sim}{x})$

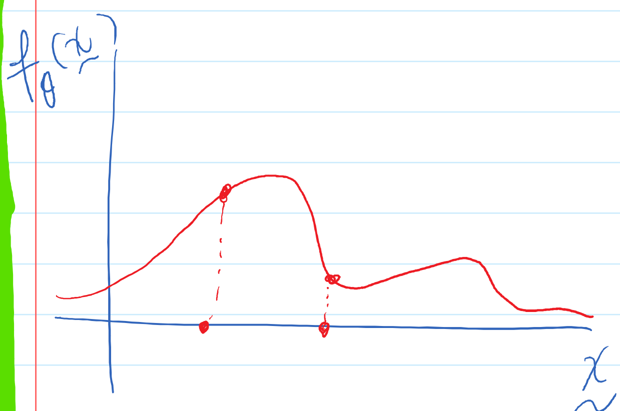
Way 1:

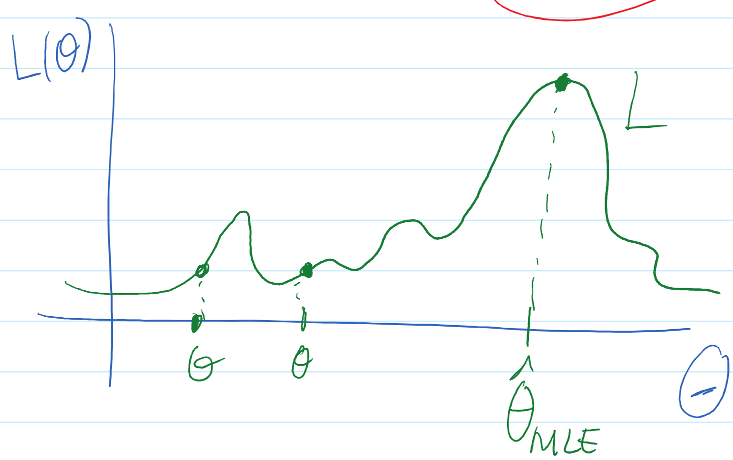
$f_\theta : \mathbb{R}^N \to \mathbb{R}$

Way 2:

think of as a fn of θ

Call it: the likelihood fn





Often it is useful to work with the

Often it is useful to work with the <u>log-likelihood function</u>

$$\ell(\theta) = \log L(\theta)$$

<u>Defn: Maximum Likelihood Estimator (MLE)</u>

<u>Idea</u>: want to estimate θ as value $\hat{\theta}$ with the largest likelihood

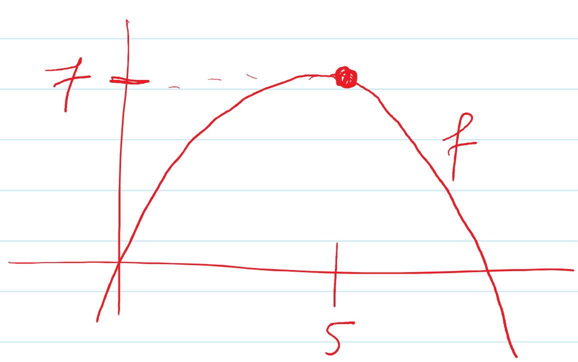
$$\hat{\theta}_{MLE} = \underset{\theta \in \Theta}{\arg\max}\ L(\theta)$$

the value of θ that makes $L(\theta)$ as large as possible.

<u>Ex,</u>

$$\max_x f(x) = 7$$

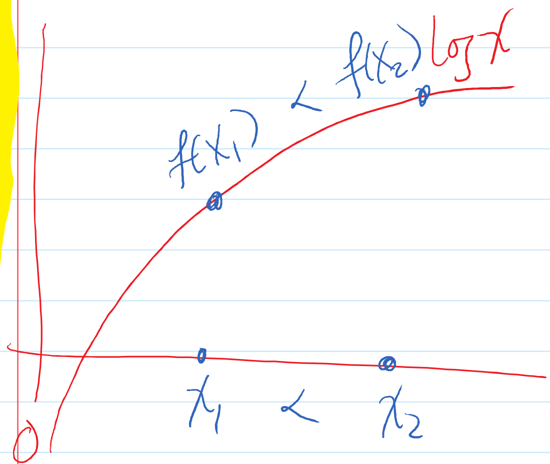
$$\underset{x}{\arg\max}\ f(x) = 5$$



Alt. defn

$$\hat{\theta}_{MLE} = \underset{\theta \in \Theta}{\arg\max}\, \ell(\theta) = \underset{\theta \in \Theta}{\arg\max}\, L(\theta)$$

equivalent b/c log is increasing



$x_1 < x_2$ then $\log(x_1) < \log(x_2)$

Ex. $X_n \overset{iid}{\sim} N(\theta, 1)$ where $\theta \in \mathbb{R}$

What's the MLE of θ?

1 Find $\ell(\theta)$

$$L(\theta) = f_\theta(\underline{x}) = \prod_{n=1}^{N} f_\theta(x_n) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_n - \theta)^2\right)$$

$$= (2\pi)^{-N/2} \exp\left(-\frac{1}{2}\sum_{n=1}^{N}(x_n - \theta)^2\right)$$

$$\ell(\theta) = \log L(\theta) = \log\left((2\pi)^{-N/2}\right) - \frac{1}{2}\sum_{n=1}^{N}(x_n - \theta)^2$$

$$= -\frac{N}{2}\log(2\pi) - \frac{1}{2}\sum_{n=1}^{N}(x_n - \theta)^2$$

$$= -\frac{N}{2}\log(2\pi) - \frac{1}{2}\sum_{n=1}(x_n - \theta)$$

(2) take a derivative

$$\frac{\partial \ell}{\partial \theta} = -\frac{1}{2}\sum_{n=1}^{N} 2(x_n-\theta)(-1) = \sum_{n=1}^{N}(x_n - \theta)$$

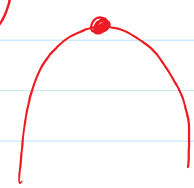
$$= \sum_{n=1}^{N} x_n - N\theta$$

critical pts where $\frac{\partial \ell}{\partial \theta} = 0$

so $\sum_{n=1}^{N} x_n - N\theta = 0 \Rightarrow \boxed{\hat{\theta}_{MLE} = \overline{X}}$

Technically: Need to check $\frac{\partial^2 \ell}{\partial \theta} < 0$

$-N$



and need to check

$$\lim_{\theta \to \pm\infty} L(\theta) = 0$$

Theorem: MLEs are based on Suff. Stats.

$$\hat{\theta}_{MLE} = \text{function}(T)$$

S.S. for θ.

Pf. Factorization theorem

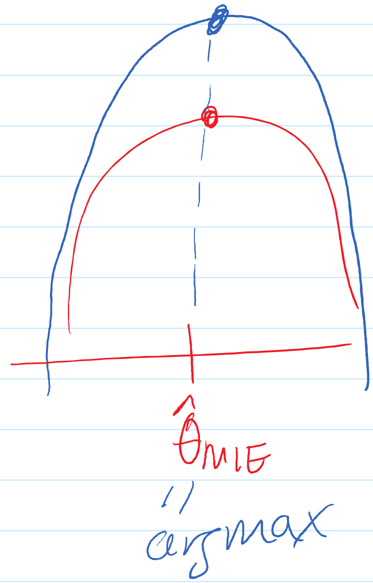
pf. Factorization theorem

$$L(\theta) = f_\theta(\underline{x}) = \underline{h(\underline{x})\, g(\theta, t)}$$

$$\hat{\theta}_{MLE} = \arg\max_\theta L(\theta)$$

$$= \arg\max_\theta h(\underline{x})\, g(\theta, t)$$

$$= \arg\max_\theta g(\theta, t)$$



Ex. Let $X_n \overset{iid}{\sim} \text{Bern}(p)$, $p \in [0, 1]$

What is $\hat{p}_{MLE}$?

① Write $L(p)$ and/or $\ell(p)$

$$\boxed{f_p(x) = p^{x}(1-p)^{1-x}\, \mathbb{1}(x = 0, 1)}$$

$$L(p) = \prod_n f_p(x_n)$$

$$= \prod_n p^{x_n}(1-p)^{1-x_n}\, \mathbb{1}(x_n = 0, 1)$$

$$= p^{\sum x_n}(1-p)^{N - \sum x_n} \prod_n \mathbb{1}(x_n = 0, 1)$$

$$\ell(p) = \log L(p) = \left(\sum_n x_n\right)\log(p) + \left(N - \sum_n x_n\right)\log(1-p)$$

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$$+ \log\left(\prod_n \mathbb{1}(x_n = 0,1)\right)$$

② Set derivative to zero

$$\frac{\partial \ell}{\partial p} = \left(\sum_n x_n\right)\frac{1}{p} + \left(N - \sum_n x_n\right)\frac{-1}{1-p} = 0$$

$$\Rightarrow \left(\sum_n x_n\right)(1-p) - \left(N - \sum_n x_n\right)p = 0$$

$$\Rightarrow \underbrace{\left(\sum_n x_n\right)}_{N\bar{x}} - \cancel{p\left(\sum_n x_n\right)} - Np + \cancel{p\left(\sum_n x_n\right)} = 0$$

$$\Rightarrow N\bar{x} - Np = 0$$

$$\boxed{\hat{p} = \bar{x}}$$

Contiue ex

$$\eta = \frac{p}{1-p} \leftarrow \text{odds}$$

$$\hookrightarrow \eta(1-p) = p$$

$$\Rightarrow \eta - \eta p = p$$

$$\Rightarrow \eta - \eta p = p$$

$$\Rightarrow \eta = (1+\eta)p$$

$$\Rightarrow \boxed{p = \eta/_{1+\eta}}$$

① Likelihood

$$L(p) = p^{N\bar{X}}(1-p)^{N-N\bar{X}}$$

$$\boxed{L(\eta) = \left(\eta/_{1+\eta}\right)^{N\bar{X}}\left(1-\eta/_{1+\eta}\right)^{N-N\bar{X}}}$$

$$\ell(\eta) = N\bar{X}\log\left(\eta/_{1+\eta}\right) + (N-N\bar{X})\log\left(\overbrace{1-\eta/_{1+\eta}}^{1/_{1+\eta}}\right)$$

recall: $\log(a/_b) = \log a - \log b$

$$= N\bar{X}\left[\log\eta - \log(1+\eta)\right] + (N-N\bar{X})\left[-\log(1+\eta)\right]$$

$$= N\bar{X}\log\eta - \cancel{N\bar{X}\log(1+\eta)} - N\log(1+\eta) + \cancel{N\bar{X}\log(1+\eta)}$$

$$\ell(\eta) = N\bar{X}\log\eta - N\log(1+\eta)$$

② $\partial\ell$

$$② \quad \boxed{\frac{\partial \ell}{\partial \eta} = 0}$$

$$\frac{\partial \ell}{\partial \eta} = \frac{N\bar{X}}{\eta} - \frac{N}{1+\eta} = 0$$

$$\Rightarrow (1+\eta)N\bar{X} - \eta N = 0$$

$$\Rightarrow \cancel{N}\bar{X} + \eta \cancel{N}\bar{X} - \eta \cancel{N} = 0$$

$$\eta = \boxed{\frac{p}{1-p}}$$

$$\Rightarrow \bar{X} = \eta(1-\bar{X})$$

$$\hat{p} = \bar{X}$$

$$\Rightarrow \boxed{\hat{\eta} = \frac{\bar{X}}{1-\bar{X}}} = \boxed{\frac{\hat{p}}{1-\hat{p}}}$$

Theorems Transformation for MLEs

If $\hat{\theta}$ is the MLE for θ then the MLE for $g(\theta)$ is $g(\hat{\theta})$.
