Extra OH: Thurs 2-3

Mon 2-3

Tues 3-4

P-values:

Back to HT,

often we report the results of a HT using a p-value

Defn: P-value

a p-value $p(\underset{\sim}{X})$ is a test stat where

$$0 \leq p(\underset{\sim}{X}) \leq 1$$

idea: small values give evidence of H_a and large values give evidence of H_0

Recall: a HT is just a partition of $\mathcal{X}$ into A and R - one way to define a test is to threshold p i.e.

$$R = \{ \underset{\sim}{x} : p(\underset{\sim}{x}) \leq \text{threshold} \}$$

We say a p-value is "valid" if
$\forall\ 0 \le \alpha \le 1$ and $\theta \in \Theta_0$

$$\mathbb{P}_\theta\left(p(\underset{\sim}{X}) \le \alpha\right) \le \alpha$$

$F_p(\alpha) \le \alpha$

stochastically bounded by $U(0,1)$

notice that if $F_p(\alpha) = \alpha$ then $p \sim U(0,1)$

If p is valid one I set up a test to reject when

$$R = \{ \underset{\sim}{x} \mid p(\underset{\sim}{x}) \le \alpha \}$$

this gives a level α test.

reason:

If p is valid then $\forall \alpha$, $\forall \theta \in \Theta_0$

$$\mathbb{P}_\theta(\text{reject } H_0)$$

$$= \mathbb{P}_\theta\left(p(\underset{\sim}{X}) \le \alpha\right)$$

$= F_p(\alpha) \le \alpha \Leftarrow$ level α test.

__Ex.__ Consider

$$H_0 : \theta = \theta_0 \quad \text{v.} \quad H_a : \theta \neq \theta_0$$

I define a test where

$$R = \{ \underset{\sim}{x} \mid T \text{ is large} \}$$

CDF of $T(\underset{\sim}{X})$ under $\theta = \theta_0$

let

$$p(\underset{\sim}{x}) \overset{\text{def}}{=} \underbrace{P_{\theta_0}\left(T(\underset{\sim}{X}) \geq T(\underset{\sim}{x})\right)}_{} = 1 - F_{\theta_0}(T(\underset{\sim}{x}))$$

observed test stat

under $H_0 : \theta = \theta_0$

$p(\underset{\sim}{x})$ = Prob. I observe a test test stat as or larger than $T(\underset{\sim}{x})$

If we do this then

$$F_p(\alpha) = P_{\theta_0}\left(p(\underset{\sim}{X}) \le \alpha\right)$$

$$= P_{\theta_0}\left(1 - F_{\theta_0}(T(\underset{\sim}{X})) \le \alpha\right)$$

$\rightarrow$ Probability Integral Trans

$F_X(X) \sim U(0,1)$

$$= \mathbb{P}_{\theta_0}\left(1 - F_{\theta_0}(T(\underset{\sim}{X})) \le \alpha\right) \qquad F_X(X) \sim U(0,1)$$

$$= \mathbb{P}_{\theta_0}\Big(\underbrace{F_{\theta_0}}_{\substack{\text{CDF of } T(\underset{\sim}{X}) \\ \text{under } \theta = \theta_0}}(T(\underset{\sim}{X})) \ge 1-\alpha\Big)$$

$U \sim U(0,1)$

$$= \mathbb{P}_{\theta_0}(U \ge 1-\alpha)$$

$$= 1 - F_U(1-\alpha)$$

$$F_p(\alpha) = 1 - (1-\alpha) = \alpha$$

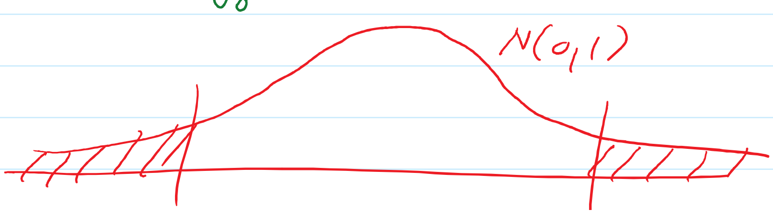
i.e. $p(\underset{\sim}{X}) \sim U(0,1)$.

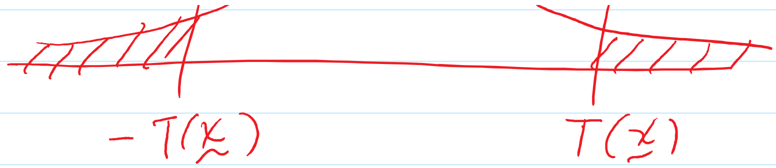
Ex. $H_0: \theta = \theta_0$ v. $H_a: \theta \ne \theta_0$

$$X_n \overset{iid}{\sim} N(\theta, \sigma^2)$$

$$T = \left| \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \right|$$

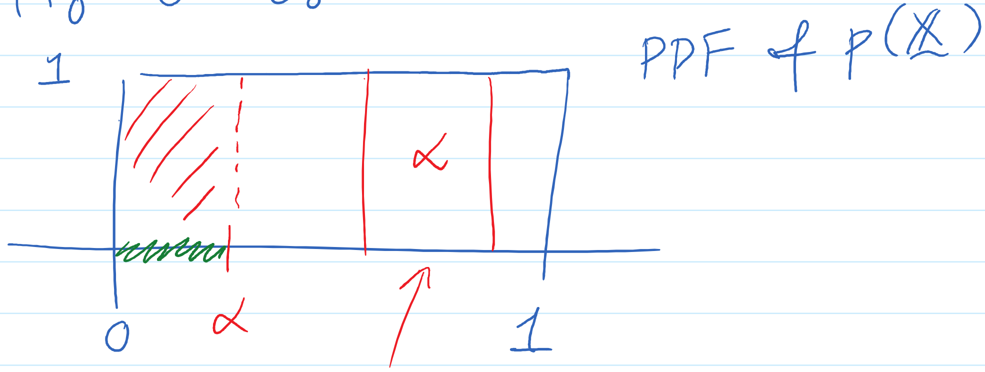
$$p(\underset{\sim}{x}) = \mathbb{P}_{\theta_0}\Big(T(\underset{\sim}{X}) \ge \overbrace{T(\underset{\sim}{x})}^{\text{obs. test stat}}\Big)$$





Claim: $p(\underset{\sim}{X}) \sim U(0,1)$ under H_0

Under $H_0: \theta = \theta_0$



Power and Sample Size

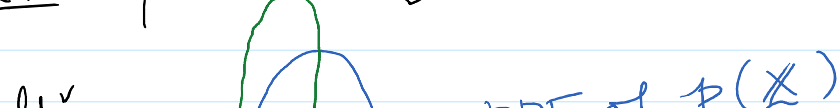
Power = ability to reject H_0

e.g. $\boxed{H_0: \theta = 0 \quad v. \quad H_a: \theta \neq 0}$

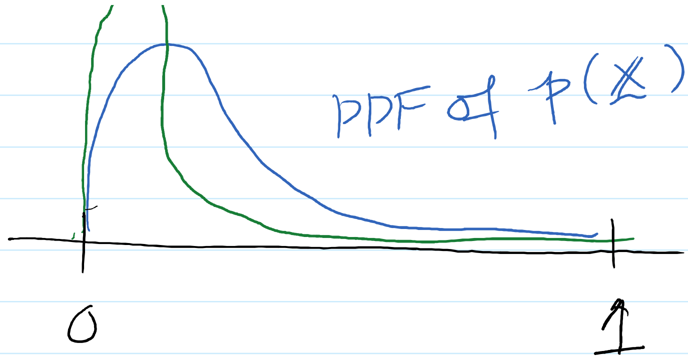
Power = ability to detect $\theta \neq 0$

Typically power $\uparrow$ as $N \uparrow$

alt: p-value $\downarrow$ as $N \uparrow$ $\Big\}$ under H_a



under H_a



<u>Instead:</u> $H_0: \theta \le \delta$ v. $H_a: \theta > \delta$

<u>Bayesian!</u>

<u>Frequentist:</u> θ fixed, unknown

<u>Bayesian:</u> θ random

<u>Bayesian approach:</u>

1. prior info/dist.

   $\Theta \sim \pi(\theta)$

   PMF/PDF of Θ

2. get some data

   $\underset{\sim}{X} \mid \Theta = \theta \sim f(\underset{\sim}{x} \mid \theta)$ ← sampling dist, likelihood

③ update/combine prior and likelihood to get posterior

$$\pi(\theta|\underset{\sim}{x}) = \frac{f(\underset{\sim}{x}|\theta)\pi(\theta)}{f(\underset{\sim}{x})} \propto f(\underset{\sim}{x}|\theta)\pi(\theta)$$

posterior

likelihood × prior

$$f(\underset{\sim}{x}) = \int f(\underset{\sim}{x}|\theta)\pi(\theta)\,d\theta$$

Ex. $P \sim \text{Beta}(\alpha,\beta)$ prior

$X_n | P=p \sim \text{Bern}(p)$ likelihood

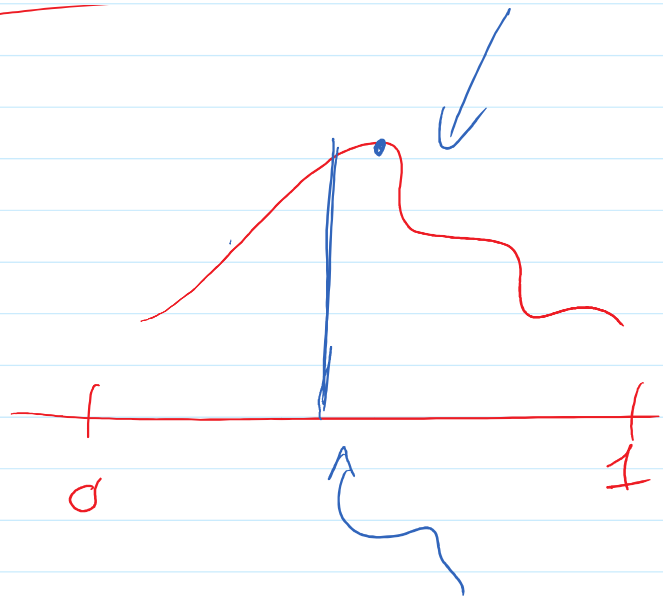
$$\pi(p|\underset{\sim}{x}) \propto \underline{f(\underset{\sim}{x}|p)\pi(p)}$$

$$\propto \left(\prod_n p^{x_n}(1-p)^{1-x_n}\right) \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}$$

$$\propto p^{N\bar{x}}(1-p)^{N-N\bar{x}}\, p^{\alpha-1}(1-p)^{\beta-1}$$

$$\propto p^{N\bar{x}+\alpha-1}(1-p)^{N-N\bar{x}+\beta-1}$$

$$\underline{P|\underset{\sim}{X} \sim \text{Beta}(\underline{N\bar{x}+\alpha},\ \underline{N-N\bar{x}+\beta})}$$



Could est. p as $\mathbb{E}[P \mid \underset{\sim}{X} = \underset{\sim}{x}] \approx \hat{p}$

$$\hat{p} = \mathbb{E}[P \mid \underset{\sim}{X} = \underset{\sim}{x}] = \frac{N\bar{x} + \alpha}{N\bar{x} + \alpha + N(1-\bar{x}) + \beta}$$

$$= \dots$$

$$= w\bar{x} + (1-w)\frac{\alpha}{\alpha+\beta}$$

weighted avg. of $\bar{x}$ and $\mathbb{E}P = \frac{\alpha}{\alpha+\beta}$

where $w = \frac{N}{\alpha+\beta+N}$, $0 < w < 1$

$w \to 1$ as $N \to \infty$