Review : Conditional Expectation

$$\underline{E[X \mid Y = y]} = \int x f(x|y)\,dx = g(y)$$

random version of this

Iterated Expectation

$$\rightarrow \quad \mathbb{E}[X] = \mathbb{E}\Big[\overbrace{\mathbb{E}[X \mid Y]}^{\text{a RV}}\Big]$$

Theorem: Law of Total Variance

$$\rightarrow \operatorname{Var}(X) = \mathbb{E}\big[\operatorname{Var}(X \mid Y)\big] + \operatorname{Var} \mathbb{E}[X \mid Y]$$

Ex. $X \mid Y = y \sim \operatorname{Bin}(y, p)$, $p \in [0,1]$

$Y \sim \operatorname{Pois}(\lambda)$, $\lambda > 0$

$\mathbb{E}[X]$

① $\mathbb{E}[X \mid Y = y] = yp$

② $\mathbb{E}[X \mid Y] = Yp$

③ $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[Yp] = p\,\mathbb{E}Y = p\lambda$

$\operatorname{Var}(X)$?

$\rightarrow \mathrm{Var}(X \mid Y = y) = y p(1-p)$

$\rightarrow \mathrm{Var}(X \mid Y) = \underline{Y p(1-p)}$

$$\mathrm{Var}(X) = \mathrm{Var}\, \mathbb{E}[X \mid Y] + \mathbb{E}\, \mathrm{Var}(X \mid Y)$$
$$= \mathrm{Var}(Y p) + \mathbb{E}[Y p(1-p)]$$
$$= p^2 \lambda + p(1-p)\lambda$$
$$= \underline{p\lambda}$$

Back to math. stats.

CRLB doesn't always allow us to find UMVUE.

Let's look at how to refine bad estimators into better estimators using conditioning

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Some facts

(1) Let $\hat{\theta}$ be unbiased for $\tau(\theta)$

so that $\mathbb{E}\hat{\theta} = \tau(\theta)$

Let $W = W(\underset{\sim}{X})$ to be some fn of $\underset{\sim}{X}$

(maybe a stat - maybe not could depend on θ)

Let $\varphi(W) = \varphi = \mathbb{E}[\hat{\theta} \mid W]$

a fn of $\underset{\sim}{X}$

could be a stat — or not

$\boxed{\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}X}$

In any case

$$\mathbb{E}\varphi = \mathbb{E}\big[\mathbb{E}[\hat{\theta}|W]\big] = \mathbb{E}\hat{\theta} = \tau(\theta)$$

If φ is a stat, then it is unbiased for $\tau(\theta)$.

2 $\text{Var}(\varphi) \le \text{Var}(\hat{\theta})$.

$\varphi = \mathbb{E}[\hat{\theta}|W]$

Law of total var says

$$\text{Var}\,\hat{\theta} = \underbrace{\text{Var}\,\mathbb{E}[\hat{\theta}|W]}_{\text{Var}(\varphi)} + \underbrace{\mathbb{E}\,\text{Var}(\hat{\theta}|W)}_{\ge 0}$$

So $\text{Var}(\hat{\theta}) \ge \text{Var}(\varphi)$.

Summary: Start w/ $\hat{\theta}$ where $\mathbb{E}\hat{\theta} = \tau(\theta)$

If I define $\varphi = \mathbb{E}[\theta | W]$

then ① $\mathbb{E}\varphi = \tau(\theta)$ ⟵

② $Var(\varphi) \leq Var(\hat{\theta})$ ⟵

Main issue: can't guarantee φ is a stat.

Ex. $X_n \overset{iid}{\sim} N(\theta, 1)$

define $\hat{\theta} = \frac{1}{2}(X_1 + X_2)$.

notice: $\mathbb{E}\hat{\theta} = \mathbb{E}\left[\frac{1}{2}(X_1 + X_2)\right] = \frac{1}{2}(\mathbb{E}X_1 + \mathbb{E}X_2)$

$= \frac{1}{2}(\theta + \theta) = \theta$

$Var\,\hat{\theta} = \frac{1}{4}\left(Var(X_1) + Var(X_2)\right) = \frac{1}{2}$

$W = X_1$

$\varphi = \mathbb{E}[\hat{\theta} | W] = \mathbb{E}\left[\frac{1}{2}(X_1 + X_2) \,|\, X_1\right]$

$= \frac{1}{2}\left(\mathbb{E}[X_1 | X_1] + \mathbb{E}[X_2 | X_1]\right)$

$= \frac{1}{2}\left(X_1 + \mathbb{E}[X_2]\right)$

Aside: $\mathbb{E}[Z | Z = 5] = 5$

$\mathbb{E}[Z | Z = 3] = 3$

Aside: $E[Z|Z=3]=3$ $\qquad = \frac{1}{2}(X_1 + E[X_2])$

$E[Z|Z=3]=3$

$\boxed{= \frac{1}{2}(X_1+\theta) = \varphi}$ $\otimes$

$E[Z|Z]=Z$

problem! not a stat

$$① \; E[\varphi] = E\left[\frac{1}{2}(X_1+\theta)\right] = \frac{1}{2}\left(E[X_1]+\theta\right)$$

$$= \frac{1}{2}(\theta+\theta) = \theta$$

$$② \; Var(\varphi) = \frac{1}{4}Var(X_1) = \frac{1}{4} < Var\,\hat{\theta} = \frac{1}{2}$$

Let $W = \bar{X}$ and re-do

$$\varphi = E[\hat{\theta}|w] = E\left[\frac{1}{2}(X_1+X_2)\,|\,\bar{X}\right]$$

$$= \frac{1}{2}\left(E[X_1|\bar{X}] + E[X_2|\bar{X}]\right)$$

$X_n \overset{iid}{\sim} N(\theta,1)$ so $E[X_1|\bar{X}] = E[X_2|\bar{X}] = \cdots$

$$= \frac{1}{2}\,2\,E[X_1|\bar{X}]$$

$$= E[X_1|\bar{X}]$$

$$= \frac{1}{N}N\,E[X_1|\bar{X}]$$

$$= \frac{1}{N}\sum^{N} E[X_i|\bar{X}]$$

$$= E\left[\frac{1}{N}\sum_{i=1}^{N} X_i \,\middle|\, \bar{X}\right]$$

$$= E[\bar{X} \mid \bar{X}]$$

$$\boxed{\varphi = \bar{X}} \quad (*)$$

this is a stat.

We also know:

① $E\varphi = \theta = E\bar{X}$

② $\underbrace{\mathrm{Var}\,\varphi}_{1/N} \le \underbrace{\mathrm{Var}\,\hat{\theta}}_{1/2}$

Theorem: Rao-Blackwell Theorem

Let $\hat{\theta}$ is unbiased for $\tau(\theta)$ and W is sufficient for θ. Then if

$$\varphi = E[\hat{\theta} \mid W]$$

① $E[\varphi] = \tau(\theta)$

② $\mathrm{Var}\,\varphi \le \mathrm{Var}\,\hat{\theta}$

③ φ is a statistic.

Pf. Fact ③

Pf. Fact 3

$$\varphi = \mathbb{E}[\underbrace{\hat{\theta}(\underset{\sim}{X})}_{g} \mid \overset{\text{sufficient}}{W}]$$

$$= \int \underbrace{\hat{\theta}(\underset{\sim}{x})}_{\substack{\text{a stat} \\ \text{no } \theta}} \underbrace{f_{\underset{\sim}{X}|W}(\underset{\sim}{x})}_{\substack{W \text{ sufficient,} \\ \text{no } \theta \text{ in this PDF}}} d\underset{\sim}{x}$$

$\mathbb{E}[g(X)] = \int g(x) f(x) dx$

$\mathbb{E}[g(X) \mid Y = y]$
$= \int g(x) f(x|y) dx$

Theorem: Lehmann - Scheffé

Let W be a (complete) sufficient statistic for θ and let $\hat{\theta}$ be an unbiased est. for $\tau(\theta)$ that depends on $\underset{\sim}{X}$s only through W,

tech. condition.

$$\hat{\theta} = \hat{\theta}(W) = \hat{\theta}(W(\underset{\sim}{X}))$$

then $\hat{\theta}$ is the UMVUE for $\tau(\theta)$.

Basically: If I can form an unbiased est. for $\tau(\theta)$ from a S.S. it is the UMVUE.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

μ: don't know

σ^2: Known

Q: what is the UMVUE for μ?

Use Lehmann - Scheffé

1. Find a SS for μ : $\bar{X}$

2. Guess a fn of $\bar{X}$ that is unbiased for μ

$\hat{\mu} = \bar{X}$ is unbiased for μ, $\mathbb{E}\bar{X} = \mu$.

3. So $\bar{X}$ is the UMVUE.

Ex. Let $\tau(\mu) = \mu^2$.

1. SS for μ : $\bar{X}$

2. Find / guess a fn of $\bar{X}$ that is unbiased for $\mu^2 = \tau(\mu)$.

Consider: $\bar{X}^2$, $\mathbb{E}[\bar{X}^2] = Var(\bar{X}) + (\mathbb{E}\bar{X})^2$

$\boxed{\bar{X} \sim N(\mu, \sigma^2/N)}$ $= \sigma^2/N + \mu^2$

Let $\hat{\mu^2} = \bar{X}^2 - \sigma^2/N$

$$\mathbb{E}[\hat{\mu^2}] = \mathbb{E}[\bar{X}^2] - \sigma^2/N = \mu^2$$

So $\bar{X}^2 - \sigma^2/N$ is the UMVUE for μ?

Takeaway:

How to find UMVUE for $\tau(\theta)$ w/ Lehman-Scheffé

1. Find SS for θ — call it W
2. Find a fn of W that is unbiased for $\tau(\theta)$
   - (i) Guess fn $\hat{\theta}(W)$ so that $\mathbb{E}[\hat{\theta}(W)] = \tau(\theta)$
   - (ii) Use Rao-Blackwell and condition using W,
     - $\rightarrow$ Find any unbiased est. of $\tau(\theta)$ call it V

$\Rightarrow \hat{\theta} = E[V|W]$ is the UMVUE.