Lecture 18 - Likelihood Ratio Test

$H_0: \theta \in \Theta_0$ v. $H_a: \theta \in \Theta_a$

Defn: HT procedure

Splits $\boxed{\mathcal{X} = A \cup R}$ ← partitions

If $\underset{\sim}{x} \in A \Rightarrow$ don't reject H_0

$\underset{\sim}{x} \in R \Rightarrow$ reject H_0

Defn: Test Function

The test function assoc. w/ a HT is a function φ

$$\varphi(\underset{\sim}{x}) = \mathbb{1}(\underset{\sim}{x} \in R) = \begin{cases} 1 & \underset{\sim}{x} \in R \\ 0 & \underset{\sim}{x} \notin R \end{cases}$$

Notice: $\mathbb{E}[\varphi(\underset{\sim}{X})] = \mathbb{E}[\mathbb{1}(\underset{\sim}{X} \in R)]$

$= P(\underset{\sim}{X} \in R)$

$\mathbb{E}[\mathbb{1}(A)] = P(A)$

↑ prob. I reject.

Defn: Type I and II errors

	Test Outcome: Fail reject $\underset{\sim}{x} \in A$	Reject H_0 $\underset{\sim}{x} \in R$
Null true $\theta \in \Theta$	Correct	Type I

Truth		
Null true $\theta \in \Theta_0$	Correct Decision	Type I error
Null false $\theta \in \Theta_a$	Type II error	Correct Decision

Goal: create a HT that minimizes type I and II errors

often: these goals are opposing

Defn: Power Function

For any $\theta \in \Theta$ the power function β is defined as

$$\rightarrow \beta(\theta) = \mathbb{E}_\theta[\varphi(\underset{\sim}{X})]$$

$$= \mathbb{P}_\theta(\underset{\sim}{X} \in R)$$

If truth is θ what is prob. I reject.

For $\theta \in \Theta_0$ then $\beta(\theta)$ is the prob. I make a type I error

make a type I error
(reject H_0 when $\theta \in \Theta_0$)

For $\theta \in \Theta_a$ then $\beta(\theta)$ is the prob. of correctly rejecting H_0

Conversely $1-\beta(\theta) = P(\underset{\sim}{X} \notin R)$
$=$ prob. of type II error.

Ex. $X_1, \dots, X_5 \overset{iid}{\sim} \text{Bernoulli}(p)$

$$H_0 : p \le \frac{1}{2} \quad \text{v.} \quad H_a : p > \frac{1}{2}$$

$$\left[\Theta = [0,1] \;;\; \Theta_0 = [0, \tfrac{1}{2}] \;;\; \Theta_a = (\tfrac{1}{2}, 1] \right]$$

Need a test:

$$R = \{(1,1,1,1,1)\}$$

Could write in terms of a test stat

$$T = \sum_{n=1}^{5} X_n \sim \text{Bin}(5, p)$$

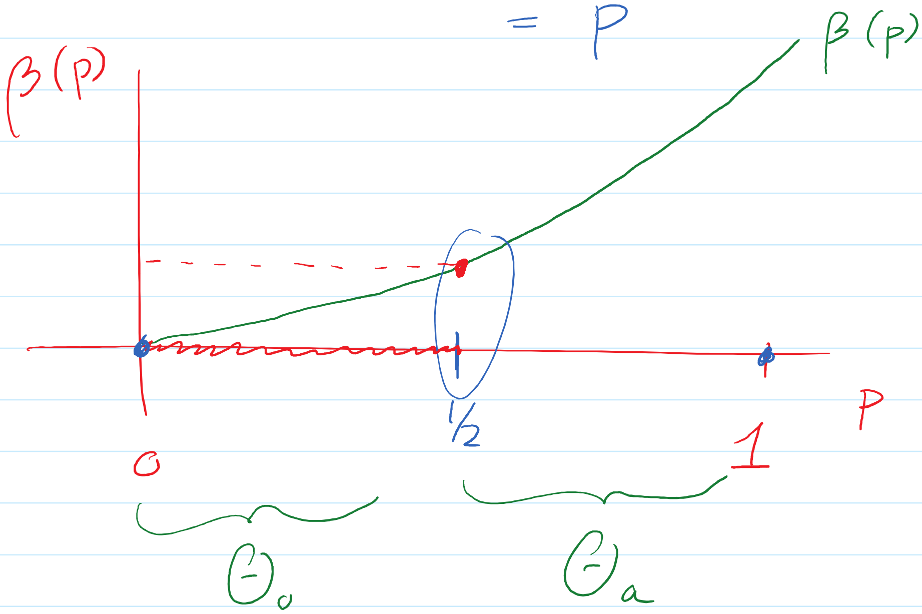
and let $R = \{ x \mid T = 5 \}$

and let $R = \{ \underset{\sim}{x} \mid T = 5 \}$

$D = \{5\}$

<u>What is β?</u>

$$\beta(p) = \mathbb{E}_p[\varphi(\underset{\sim}{X})] = \mathbb{P}(\underset{\sim}{X} \in R)$$
$$= \mathbb{P}(T \in D)$$
$$= \mathbb{P}(T = 5)$$
$$= \binom{5}{5} p^5 (1-p)^{5-5}$$
$$= p^5$$



① What is the max. prob. of a type I error?

For $\theta \in \Theta_0$ then $\beta(\theta) =$ prob. of type I error

so

max type I prob $= \max_{\theta} \beta(\theta)$

So

$$\text{max type I prob} = \max_{\theta \in \Theta_0} \beta(\theta)$$

$$= \max_{p \le 1/2} \beta(p)$$

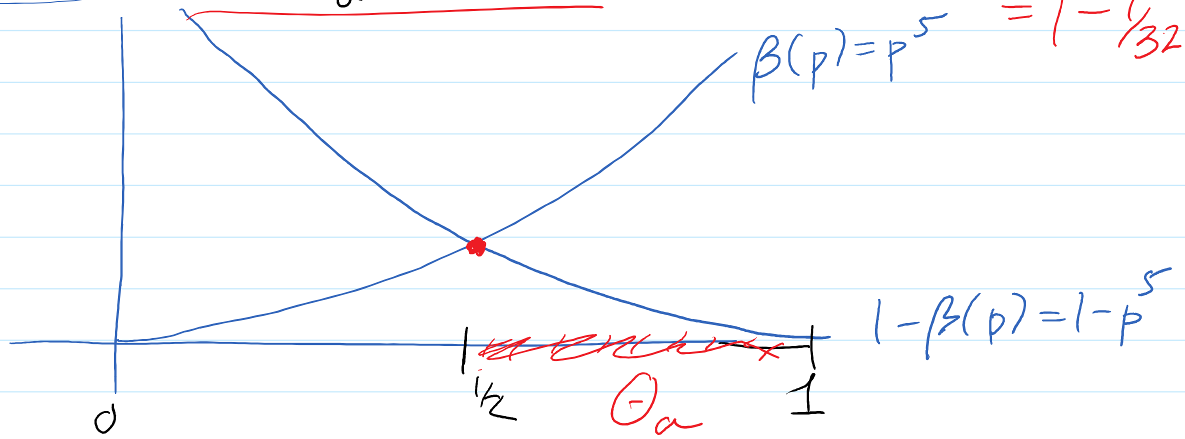
$$= \beta(1/2) = (1/2)^5 = 1/32$$

(2) what is max prob. of type II error.

$\theta_0 \in \Theta_a$ then $1 - \beta(\theta) =$ prob. of type II error

So

$$\text{max type II} = \max_{\theta \in \Theta_a} 1 - \beta(\theta) = \max_{p > 1/2} 1 - \beta(p) = 1 - \beta(\tfrac{1}{2}) = 1 - 1/32$$



Ex, $H_0 : p \le 1/2$ v. $H_a : p > 1/2$

Consider a different test

$$R = \{ x \mid T \ge 3 \}$$

$$R = \{ \underset{\sim}{x} \mid T \geq 3 \}$$

$$= \{ \underset{\sim}{x} \mid \text{pct of 1s} \geq 1/2 \}$$

In this case

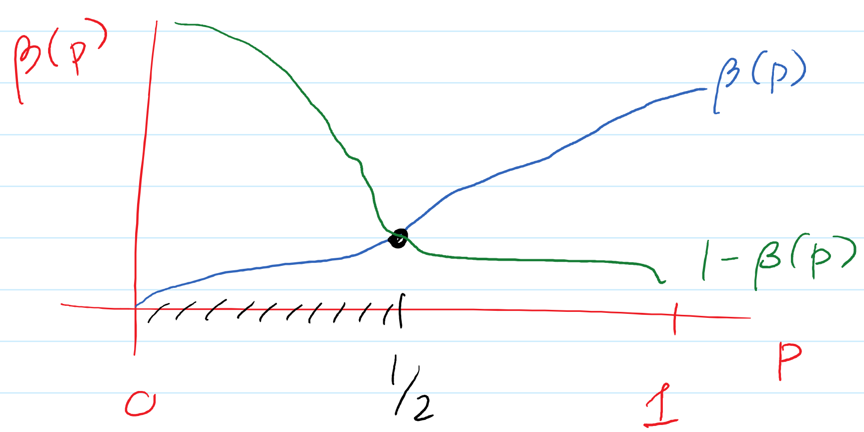
$$\beta(p) = \mathbb{P}(\underset{\sim}{X} \in R)$$
$$= \mathbb{P}(T \geq 3)$$
$$= \mathbb{P}(T=3) + \mathbb{P}(T=4) + \mathbb{P}(T=5)$$
$$= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p)^1 + \binom{5}{5} p^5 (1-p)^0$$

$$= \ldots$$

$$= p^3 (6p^2 - 15p + 10)$$

Note: $\dfrac{\partial \beta}{\partial p} = 30 p^2 (p-1)^2 > 0$

so β is increasing in p



① max type I error :

$$\max_{p \le 1/2} \beta(p) = \beta(1/2)$$

② max type II error :

$$\max_{p > 1/2} 1-\beta(p) = 1-\beta(1/2)$$

Defn: Size and level of tests

We say a test is size $\alpha \in [0,1]$ if

$$\alpha = \max_{\theta \in \Theta_0} \beta(\theta) = \text{max. type I err. prob.}$$

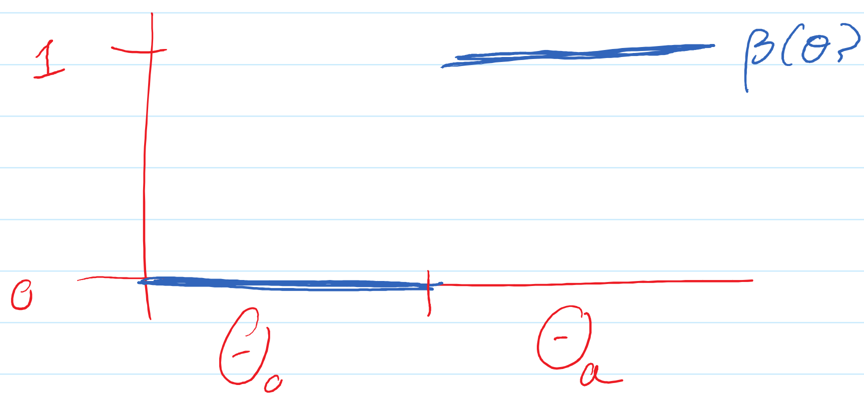
We say a test is level α if

$$\max_{\theta \in \Theta_0} \beta(\theta) \le \alpha$$

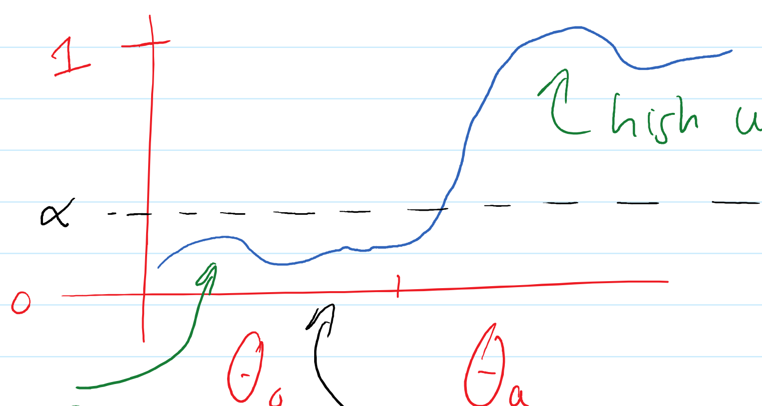
<u>Idea:</u> try to find tests that maximize power $\beta(\theta)$ when $\theta \in \Theta_a$

subject to constr. of being size/level α.

<u>Ideal test:</u>



Best we can hope for



high when $\theta \in \Theta_a$

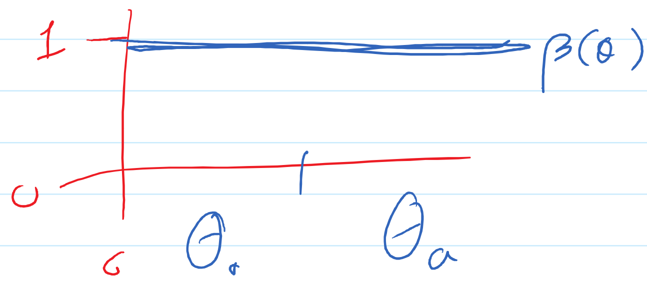
low $\theta \in \Theta_0$

if $\beta(\theta) \leq \alpha \;\; \forall \theta \in \Theta_0$
this is level α test

If we don't constrain I can come up w/ really dumb tests.

really dumb tests.

E.g. Reject all the time



E.g. reject never

Defn: Likelihood Ratio Test

Recall: $L(\theta) = f_\theta(\underset{\sim}{x})$ ← likelihood fn.

We want to test a hypothesis:

$$H_0 : \theta \in \Theta_0 \quad v. \quad H_a : \theta \in \Theta_a$$

The Likelihood Ratio Test statistic (LRT) is defined as

$$\lambda(\underset{\sim}{x}) = \frac{\max\limits_{\theta \in \Theta_0} L(\theta)}{\max\limits_{\theta \in \Theta} L(\theta)} = \frac{\text{max. val. of } L \text{ over } \Theta_0}{\text{max val. of } L \text{ over } \Theta}$$

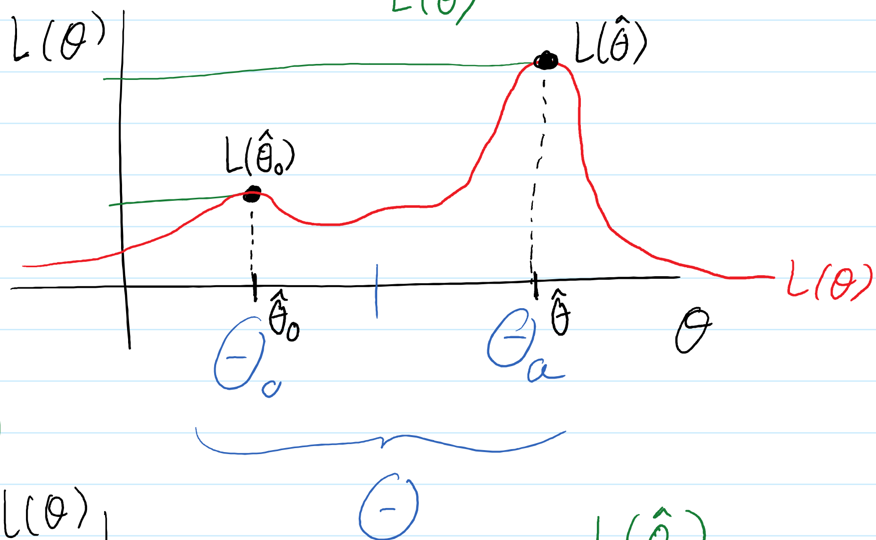
$\theta \in \Theta$ ... L over Θ

$$= \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}$$

MLE restricted to Θ_0

MLE

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} < 1$$



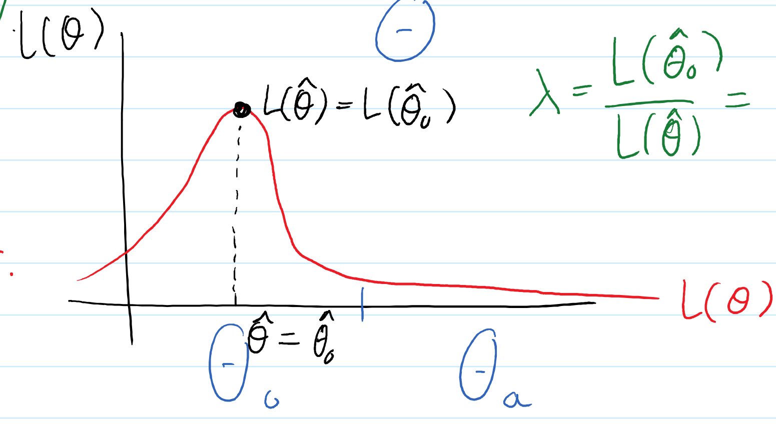
LRT: is to reject when $\lambda \ll 1$

i.e.

$$\boxed{R = \{\underset{\sim}{x} : \lambda(\underset{\sim}{x}) \le c\}}$$

some threshold chosen to balance Type I and II errors.

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = 1$$



Broad idea: $\lambda \simeq 0$ says Θ_a more likely than Θ_0

$\lambda \simeq 1$ Θ_a not more likely than Θ_0

Notice: $0 \le \lambda \le 1$

How to choose c? Balance α and power.

$c = 0 \Rightarrow$ never reject

$c = 1 \Rightarrow$ always reject