Theorem:

If $X_n \overset{iid}{\sim} f$ then

$M = $ mgf of each X_n

$N = $ sample size

$$M_{\bar{X}}(t) = \left(M(t/N)\right)^N$$

Pf. $\bar{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$

Generic:

$$M_{aZ+b}(t) = e^{tb} M(at)$$

$$M_{\bar{X}}(t) = \mathbb{E}\left[e^{t\bar{X}}\right]$$

$$= \mathbb{E}\left[e^{t\frac{1}{N}\sum_{n=1}^{N} X_i}\right]$$

$e^a e^b = e^{a+b}$

$e^{\sum a_i} = \prod e^{a_i}$

$$= \mathbb{E}\left[\prod_{n=1}^{N} e^{t/N X_n}\right]$$

$\mathbb{E}[AB] = (\mathbb{E}A)(\mathbb{E}B)$ iff $A \perp\!\!\!\perp B$

$$= \prod_{n=1}^{N} \mathbb{E}\left[e^{t/N X_n}\right]$$

$$= \prod_{n=1}^{N} M_{\cancel{X_n}}(t/N)$$

$$= M(t/N)^N$$

Ex, $X_n \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$
$\qquad\qquad\qquad\qquad a \quad \lambda$

$$f(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \mathbb{1}(x>0)$$

$$\frac{\ \ }{\Gamma(\alpha)} \quad \mathbb{1}(x>0)$$

$$M(t) = (1 - t/\beta)^{-\alpha}$$

$a \quad \lambda$

$$M_{\bar{X}}(t) = M(t/N)^N$$

$$= \left[(1 - t/N\beta)^{-\alpha}\right]^N$$

$$= (1 - t/(N\beta))^{-(N\alpha)}$$

replacing $\beta \to N\beta$, $\alpha \to N\alpha$

So $\bar{X} \sim \text{Gamma}(N\alpha, N\beta)$.

$\mathbb{E}\bar{X} = \alpha/\beta$?

Theorem: Normal Data

Let $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then

(sketch) ① $\bar{X} \sim N(\mu, \sigma^2/N)$

(later) ② $\bar{X} \perp\!\!\!\perp S^2_{N-1}$

(sketch) ③ $\frac{N-1}{\sigma^2} S^2_{N-1} \sim \chi^2(N-1)$

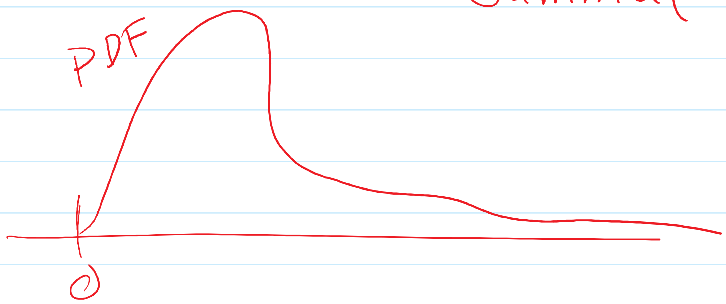
chi-sq. dist. w/ $N-1$ df.

Chi-Squared one parameter k - degrees of freedom

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2} \mathbb{1}(x>0)$$

$2^{k/2}\,\Gamma(k/2)$

$= \text{Gamma}(\alpha = k/2, \beta = 1/2)$



Facts:

1. $Z \sim N(0,1)$ then $Z^2 \sim \chi^2(1)$

2. $Z_1 \sim N(0,1)$, $Z_2 \sim N(0,1)$, $Z_1 \perp\!\!\!\perp Z_2$

   then $Z_1^2 + Z_2^2 \sim \chi^2(2)$

   Generically: $Z_i \overset{iid}{\sim} N(0,1)$

   then $\sum_{i=1}^{N} Z_i^2 \sim \chi^2(N)$

3. $Y_n \overset{indep}{\sim} \chi^2(k_n)$ then $\sum_{n=1}^{N} Y_n \sim \chi^2\left(\sum_{n=1}^{N} k_n\right)$

Intition for : $\frac{N-1}{\sigma^2} S_{N-1}^2 \sim \chi^2(N-1)$

$$S_{N-1}^2 = \frac{1}{N-1} \sum_{n=1}^{N} \underbrace{(X_n - \bar{X})^2}_{\text{kinda like } Z_n^2} \approx \chi^2(\ldots)$$

Sketch of this: Assume $\mu = 0$, $\sigma^2 = 1$

Sketch of this: Assume $\mu = 0,\ \sigma^2 = 1$

$$A = \begin{bmatrix} 1-\frac{1}{N} & & -\frac{1}{N} \\ & 1-\frac{1}{N} & \\ -\frac{1}{N} & & \ddots & \\ & & & 1-\frac{1}{N} \end{bmatrix}_{N \times N}$$

$\leftarrow$ residualizing mtx

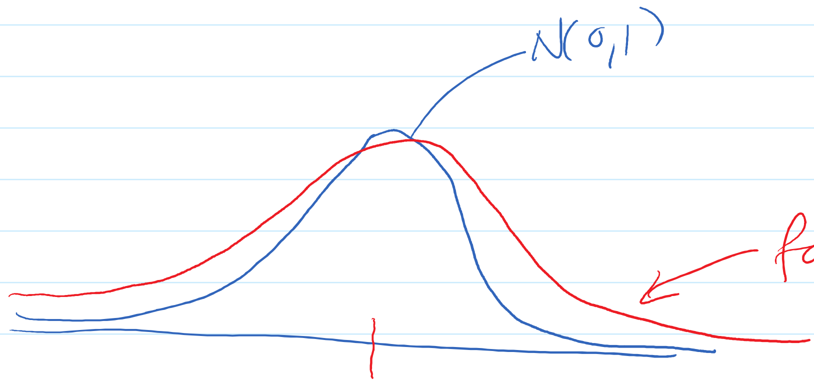
$\operatorname{rank}(A) = N-1$

$$(N-1) S_{N-1}^2 = \underset{\sim}{X}^T A \underset{\sim}{X} \sim \chi^2(\operatorname{rank}(A))$$

$\underset{\sim}{X} = (X_1, \dots, X_n)$

---

t-distribution: one parameter called degrees of freedom



$X \sim t(k)$

$$f(x) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma(k/2)} \frac{1}{\left(1 + x^2/k\right)^{\frac{k+1}{2}}}$$

Fact:

$U \sim N(0,1)$

$V \sim \chi^2(n)$ and $U \perp\!\!\!\perp V$

$U \sim N(0,1)$

$V \sim \chi^2(k)$ and $U \perp\!\!\!\perp V$

then $\boxed{\dfrac{U}{\sqrt{V/k}} \sim t(k)}$ prove: bivariate transformations

If $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then $\bar{X} \sim N(\mu, \sigma^2/N)$

What is the dist of

$$t = \frac{\bar{X} - \mu}{S_{N-1}/\sqrt{N}} \sim t(N-1)$$

Show: $\boxed{U = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{N}} \sim N(0,1)}$

$$E[U] = E\left[\frac{\bar{X}-\mu}{\sigma/\sqrt{N}}\right] = \frac{1}{\sigma/\sqrt{N}} E[\bar{X} - \mu] \quad 0$$

$$Var(U) = Var\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{N}}\right) = \frac{1}{\sigma^2/N} \underbrace{Var(\bar{X})}_{\sigma^2/N} = 1$$

$$\boxed{V = \frac{N-1}{\sigma^2} S^2 \sim \chi^2(N-1)}$$

and we know that $\boxed{U \perp\!\!\!\perp V.}$

$$\frac{U}{\sqrt{V/_{N-1}}} = \frac{\frac{\bar{X}-\mu}{\cancel{\sigma/\sqrt{N}}}}{\sqrt{\frac{\cancel{\frac{N-1}{\sigma^2}} S^2}{\cancel{N-1}}}} = \frac{\bar{X}-\mu}{S/\sqrt{N}} \sim t(N-1)$$

<u>Probability</u>: Given $X_n \overset{iid}{\sim} f_\theta$ parameter

We know $\theta = 5$

Calculate $\mathbb{P}(X_n = \cdots)$

<u>Statistics</u>! Given $X_n \overset{iid}{\sim} f_\theta$

I observe $x_1, \ldots, x_n$ but I don't know θ.

How can I estimate it?

What can we say about our estimator?

<u>Ex.</u> $X_n \overset{iid}{\sim} N(\mu, 1)$

How do I estimate μ? $\bar{X}$?

How good is this estimate?

How good is this estimate?

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$, $\lambda > 0$.

How do I estimate λ?

We work with parameterized families of distributions.

e.g.
- $N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}, \sigma^2 > 0$ Θ
- $Exp(\lambda)$ where $\lambda > 0$ Θ
- $Unif(0, \theta)$ where $\theta > 0$ Θ

Exponential family of distributions

Assume we have a parameter $\theta \in \Theta$ ← space of possible params.

and $X_n \overset{iid}{\sim} f_\theta$ and

joint

fns of θ not $\underset{\sim}{X}$

$$f_\theta(\underset{\sim}{X}) = h(\underset{\sim}{X})c(\theta)\exp\left(T(\underset{\sim}{X})w(\theta)\right)$$

we say it is an exponential family

fn of $\underset{\sim}{X}$ not θ

Ex. Let $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

$$f(\underset{\sim}{X}) = \prod_{n=1}^{N} f(x_n) = \prod_{n=1}^{N} \frac{1}{x_n!} \lambda^{x_n} e^{-\lambda} \mathbb{1}(x_n \in \mathbb{N}_0)$$

$$= \underbrace{\prod_{n=1}^{N}\left(\frac{1}{x_n!}\right) \prod_n \mathbb{1}(x_n \in \mathbb{N}_0)}_{h(\underset{\sim}{X})} \lambda^{\sum x_n} e^{-N\lambda}$$

$a = e^{\log a}$

$\log(a^b) = b\log(a)$

$$\lambda^{\sum x_n} = e^{\log(\lambda^{\sum x_n})}$$
$$= e^{(\sum x_n)\log\lambda}$$

$$= h(\underset{\sim}{X})\, e^{\overset{T(X)}{(\sum x_n)} \overset{w(x)}{\log\lambda}}$$

$$= \boxed{h(\underset{\sim}{X})\, d(\lambda) \exp(T(\underset{\sim}{X})\, w(\lambda))}$$

$d(\lambda) = e^{-N\lambda}$ → so $\text{Pois}(\lambda)$ is an Exp fam

Theorem: Only need to check the marginal of one observation.

If $X_n \overset{iid}{\sim} f_\theta$

(marginal of one obs) $f_\theta(x) = h(x)\, d(\theta) \exp(T(x)\, w(\theta))$

(... of one obs) $f_\theta(x) = h(x)\,c(\theta)\exp(\ldots)$

then jointly $f(\underset{\sim}{X})$ is an exp. fam.

i.e.

$$f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\,c(\theta)\exp\left(T(\underset{\sim}{X})\,w(\theta)\right).$$

Ex, revisit prev.

$X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!}\,\mathbb{I}(x \in \mathbb{N}_0)$$

$$= \underbrace{\frac{1}{x!}\,\mathbb{I}(x \in \mathbb{N}_0)}_{h(x)}\,\underbrace{e^{-\lambda}}_{c(x)}\,e^{\overset{T(x)}{x}\,\overset{w(\lambda)}{\log\lambda}}$$
