Theorem: Central Limit Theorem

If I have X_n that are iid w/ $\mathbb{E}X_n = \mu$

and $Var(X_n) = \sigma^2 < \infty$ then

$$\sqrt{N}\left(\frac{\bar{X} - \mu}{\sigma}\right) \xrightarrow{d} N(0,1).$$

Other forms: $\sqrt{N}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$

or $\bar{X} \sim AN(\mu, \sigma^2/N)$

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$

then $\mathbb{E}X_n = \frac{1}{\lambda} = \mu$, $Var(X_n) = \frac{1}{\lambda^2} = \sigma^2$

and so

$$\sqrt{N}\left(\frac{\bar{X} - \frac{1}{\lambda}}{\sqrt{\frac{1}{\lambda^2}}}\right) \xrightarrow{d} N(0,1)$$

Q: What about $g(\bar{X})$? $\frac{1}{\bar{X}}$, $\log \bar{X}$, ...

Theorem: First-Order Delta Method

If Y_n is a seq of RVs where

$$\sqrt{N}(Y_n - \theta) \xrightarrow{d} N(0, \psi^2) \quad \psi = \psi(\theta)$$

Obvious example is $Y_n = \bar{X}$, $\theta = \mu$, $\psi = \sigma^2$ then we have such a seq of Y_ns.

then if g is a differentiable function and $g'(\theta) \neq 0$, then

$$\sqrt{N}(g(Y_n) - g(\theta)) \xrightarrow{d} N(0, [g'(\theta)]^2 \psi^2)$$

Another way:

$$Y_n \sim AN(\theta, \psi^2/N)$$

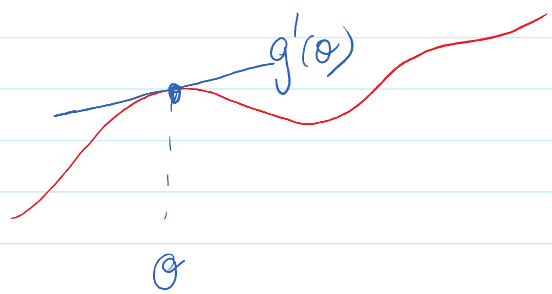
then $g(Y_n) \sim AN(g(\theta), [g'(\theta)]^2 \psi^2/N)$

Note: $\mathrm{Var}(cX) = c^2\,\mathrm{Var}(X)$

Pf. First-Order Taylor approx of g around θ is

$$\boxed{g(x) \approx g(\theta) + g'(\theta)(x-\theta)}$$

$$\frac{g(x)-g(\theta)}{x-\theta} \approx g'(\theta)$$



then

$$g(x) - g(\theta) \approx g'(\theta)(x-\theta)$$

so

$$\sqrt{n}\,(g(x) - g(\theta)) \approx g'(\theta)\,\sqrt{n}\,(x-\theta)$$

so

$$\sqrt{n}\,(g(Y_n) - g(\theta)) \approx \underbrace{g'(\theta)\,\underbrace{\sqrt{n}\,(Y_n - \theta)}_{\xrightarrow{d} N(0,\psi^2)}}_{\xrightarrow{d} N(0,\,[g'(\theta)]^2\psi^2)}$$

$$sd(g(X)) \approx |g'(\mu)|\, sd(X)$$

Ex. CLT says

$$\sqrt{n}\,(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$$

$$\sqrt{N}(\bar{X}-\mu) \xrightarrow{d} N(0,\sigma^2)$$

If $g(x) = \log(x)$ then $g'(x) = \frac{1}{x}$

$$\Rightarrow [g'(x)]^2 = \frac{1}{x^2}$$

and so the Δ-method says

$$\sqrt{N}(g(\bar{X})-g(\mu)) \xrightarrow{d} N(0, \underbrace{\tfrac{1}{\mu^2}}_{[g'(\mu)]^2}\sigma^2)$$

i.e. $\boxed{\bar{X} \sim AN(\mu, \sigma^2/N) \text{ then } \log\bar{X} \sim AN(\log\mu, \sigma^2/N\mu^2)}$

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ex Let $X_n \overset{iid}{\sim} Pois(\lambda)$ then the CLT says

$$\sqrt{N}(\bar{X}-\lambda) \xrightarrow{d} N(0,\lambda)$$

Consider $g(x) = 1/x$ then $g'(x) = -1/x^2$

so $[g'(x)]^2 = 1/x^4$

and thus the Δ-Method says

$$\sqrt{N}(g(\bar{X})-g(\mu)) \xrightarrow{d} N(0,[g'(\mu)]^2\sigma^2)$$

$$\sqrt{N}\,(g(\bar{X}) - g(\mu)) \to N(0, [g'(\mu)]^2 \sigma^2)$$

i.e.

$$\sqrt{N}\left(\frac{1}{\bar{X}} - \frac{1}{\lambda}\right) \xrightarrow{d} N\left(0, \left(\frac{1}{\lambda}\right)^4 \lambda\right)$$

$$= N\left(0, \frac{1}{\lambda^3}\right)$$

i.e. $\bar{X} \sim AN(\lambda, \lambda/N)$ then $\frac{1}{\bar{X}} \sim AN\left(\frac{1}{\lambda}, \frac{1}{N\lambda^3}\right)$

Ex. Variance-Stabilizing Transformation

Generically: $Y \sim AN(\theta, \psi^2/N)$ ψ^2 may depend on θ

Q: is there some transformation g so that

$$g(Y) \sim AN(g(\theta), \underbrace{\sim\sim}_{\text{doesn't depend on } \theta})$$

Soln: use Δ-method

b/c it says

$$g(Y) \sim AN(g(\theta), [g'(\theta)]^2 \psi^2)$$

Could do: choose g so that $[g'(\theta)]^2 \psi^2$ doesn't = constant

Could do: choose g so that $[g'(\theta)]\,\psi^2$ doesn't depend on θ.

So our condition is

$$\left.[g'(\theta)]^2\psi(\theta)^2 = c\;\right]\ \text{ODE}$$

Ex. $X_n \overset{iid}{\sim} Pois(\lambda)$

CLT: $\boxed{\sqrt{N}(\bar{X}-\lambda) \xrightarrow{d} N(0,\lambda)} \iff \underbrace{\bar{X}}_{\psi} \sim AN(\underbrace{\lambda}_{\theta}, \overbrace{\lambda/N}^{\psi(\lambda)^2})$

in this case our ODE

$$g'(\lambda)\frac{\lambda}{N} = c$$

$$\Rightarrow \left(\frac{dg}{d\lambda}\right)^2\frac{\lambda}{N} = c$$

$$\Rightarrow \frac{dg}{d\lambda} = \sqrt{\frac{cN}{\lambda}}$$

$$\Rightarrow dg = \sqrt{\frac{cN}{\lambda}}\,d\lambda$$

$$\Rightarrow g = \int dg = \int \frac{\sqrt{cN}}{\sqrt{\lambda}}\,d\lambda \;\propto \int \frac{1}{\sqrt{\lambda}}\,d\lambda$$

$$\propto \sqrt{\lambda}$$

$$\propto \sqrt{\lambda}$$

so $\boxed{g(x) = \sqrt{\lambda}}$

By Δ-method:

$$g(\bar{X}) \sim AN\left(g(\lambda), [g'(\lambda)]^2 \psi^2\right)$$

i.e.

$$\sqrt{\bar{X}} \sim AN\left(\sqrt{\lambda}, \underbrace{\left(\frac{1}{2\sqrt{\lambda}}\right)^2 \lambda}_{\frac{1}{4}\frac{1}{\lambda}\lambda = \frac{1}{4}}\right)$$

so $\sqrt{\bar{X}} \sim AN(\sqrt{\lambda}, \frac{1}{4})$.

Theorem: Second Order Δ-method

If

$$\sqrt{N}(Y_n - \theta) \xrightarrow{d} N(0, \psi^2)$$

and if g is twice-differentiable

but $g'(\theta) = 0$

multiplys $\chi^2(1)$ RV by ---

then

$$\boxed{N\left(g(Y_n) - g(\theta)\right) \xrightarrow{d} \frac{\psi^2 g''(\theta)}{2} \chi^2(1)}$$

$$\boxed{N\left(g(Y_n) - g(\theta)\right) \xrightarrow{d} \frac{\sigma^2 g''(\theta)}{2}\chi^2(1)}$$

Ex. Let $X_n \overset{iid}{\sim} \text{Bernoulli}(p)$ and let

$$g(t) = \underbrace{t\log(t/p) - (1-t)\log\left(\frac{1-t}{1-p}\right)}$$

$\hookrightarrow$ KL-divergence

(dist. measure between Bern(p) and Bern(t))

Q: what can I say about $g(\bar{X})$?

CLT: $\sqrt{N}(\bar{X} - p) \xrightarrow{d} N(0, p(1-p))$

$$\mathbb{E}X_n = p \quad \text{and} \quad Var(X_n) = p(1-p)$$

notice though that

$$g'(t) = \log\left(\frac{t}{1-t}\right) - \log\left(\frac{p}{1-p}\right)$$

and so $g'(p) = \log\left(\frac{p}{1-p}\right) - \log\left(\frac{p}{1-p}\right) = 0$

problem for First-Order Δ method.

Let's apply the second order Δ-method

Let's apply the second order Δ-method

$$g''(t) = \frac{1}{t} + \frac{1}{1-t} = \frac{1}{t(1-t)}$$

and so

$$N\left(g(\bar{x}) - g(p)\right) \xrightarrow{d} \frac{\psi^2 g''(p)}{2} \chi^2(1)$$

i.e.

$$N\left(g(\bar{x}) - g(p)\right) \xrightarrow{d} \underbrace{\frac{p(1-p)}{2} \frac{1}{p(1-p)} \chi^2(1)}_{\frac{1}{2}\chi^2(1)}$$

"proof" of Second Order Δ-method

Assumption: $\boxed{\sqrt{n}\left(Y_n - \theta\right) \xrightarrow{d} N(0, \psi^2)}$

and $g'(\theta) = 0$

Second order Taylor Expansion:

$$g(x) \approx g(\theta) + \cancelto{0}{g'(\theta)(x-\theta)} + \frac{g''(\theta)}{2}(x-\theta)^2$$

if $g'(\theta) = 0$ then this is

if $g'(\theta) = 0$ then this is

$$g(x) \approx g(\theta) + \frac{g''(\theta)}{2}(x-\theta)^2$$

So

$$g(Y/n) \approx g(\theta) + \frac{g''(\theta)}{2}(Y/n - \theta)^2$$

So

$$g(Y/n) - g(\theta) \approx \frac{g''(\theta)}{2}(Y/n - \theta)^2$$

multiply by N

$$N\left(g(Y/n) - g(\theta)\right) \approx \frac{g''(\theta)}{2} N (Y/n - \theta)^2$$

$$= \frac{g''(\theta)}{2}\Big(\underbrace{\sqrt{N}(Y/n - \theta)}_{\xrightarrow{d} N(0,\psi^2) \xrightarrow{d} \psi N(0,1)}\Big)^2$$

$$\xrightarrow{d} \underbrace{\frac{g''(\theta)}{2}\left(\psi N(0,1)\right)^2}_{\frac{g''(\theta)\psi^2}{2}\chi^2(1)}$$

Back to estimation:

For a finite sample we looked for estimators that are unbiased ...

For a finite sample we looked for estimators that are <u>unbiased</u> and have a <u>low variance</u>

Asymptotically, we want estimators that are

(1) asymptotically unbiased (consistency)

$\hat{\theta} \xrightarrow{P} \theta$

(2) asymptotic variance to be small.

<u>Theorem:</u> MLEs are consistent (*) Some conditions needed (works for Exp. fams)

If $\hat{\theta}$ is the MLE of $\tau(\theta)$ then

$$\hat{\theta} \xrightarrow{P} \tau(\theta)$$
