Lecture 12 - Convergence

## Convergence of RVs

Calc II : talked about convergen of a seq of numbers :

$$x_n \to x \quad \text{where } x_n, x \in \mathbb{R}$$

this class :

$$X_n \to X \quad \text{where } X_n, X \text{ are RVs}$$

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Recall : $X_n : S \to \mathbb{R}$

for some $s \in S$ we have $X(s) \in \mathbb{R}$

We can talk about convergence of RVs as fns.

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**Defn: Pointwise Convergence of Functions**

If $(f_n)$ is a seq. of fns $f_n : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ then we say $f_n$ converge pointwise to $f$ if

$$f_n(x) \to f(x) \quad \forall x \in \mathbb{R}$$

$$f_n(x) \longrightarrow f(x) \quad \forall x \in \mathbb{R}$$

denoted $f \xrightarrow{\text{ptwse}} f$.

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Fix $x$

$x=5$ ; $f_1(5), f_2(5), f_3(5), \ldots \quad f(5)$

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Defn: Sure Convergence of RVs

A seq of RVs $X_1, X_2, \ldots$ converges surley to $X$ if

$$X_n \xrightarrow{\text{ptwse}} X.$$

i.e. $\forall s \in S$, $X_n(s) \longrightarrow X(s)$.

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Defn: Almost Sure Convergence

We say a seq of RVs $(X_n)$ converges almost surley to $X$ if

$$X_n \xrightarrow{\text{ptwse}} X \text{ on some subset } A \subset S$$

where $\mathbb{P}(A)=1$.

We denote this

We denote this

$$X_n \xrightarrow{a.s.} X.$$

Basically: a.s. convergence is pture conv. everywhere in $S$ except maybe some prob. zero event.

Other notation

$$\mathbb{P}(X_n \to X) = 1$$

means: $\mathbb{P}(\{s \mid X_n(s) \to X(s)\}) = 1.$

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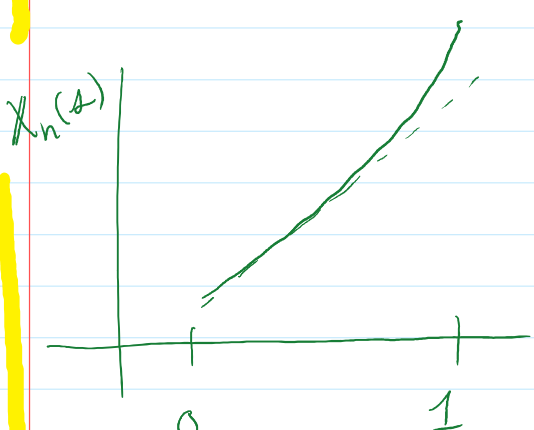
Ex. Let $S = [0, 1]$ w/ uniform density

Let $X_n(s) = s + s^n$ and $X(s) = s$




0 1 0 1

Does: $X_n \xrightarrow{a.s.} X$ ?

Notice if $s \in [0,1)$ then

$$X_n(s) = s + s^n \xrightarrow{n} s = X(s)$$

however if $s = 1$

$$X_n(s) = 1 + 1^n = 2 \xrightarrow{n} 2 \neq X(s) = 1$$

don't converge ptwse here

$$\mathbb{P}(X_n \to X) = \mathbb{P}(\{s \mid X_n(s) \to X(s)\})$$
$$= \mathbb{P}([0,1)) = 1$$

So $X_n \xrightarrow{a.s.} X$ b/c this prob. is still 1.

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Almost sure is a strong condition - can be difficult to establish.

Sometimes work w/ a slightly weaker condition called convergence in prob.

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**Defn: Convergence in Probability**

We say a seq $(X_n)$ converges in prob. to $X$ denoted

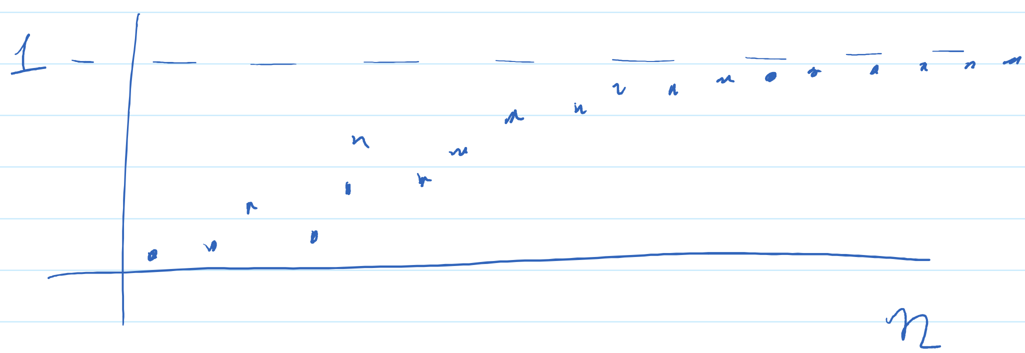
$$X_n \xrightarrow{P} X$$

if

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1.$$

Pick $\varepsilon > 0$

$$\mathbb{P}(|X_1 - X| < \varepsilon), \mathbb{P}(|X_2 - X| < \varepsilon), \ldots$$




$\forall \varepsilon > 0$

$$\to \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1$$

$$\to \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$$

**Theorem:** a.s. $\Rightarrow$ i.p

Theorem: a.s. $\to$ c.p

If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{P} X$.

Ex. Consider $S = [0,1]$ w/ uniform prob. density

$X_1(s) = \cancel{s} + 1$

$X_2(s) = \cancel{s} + \mathbb{1}_{[0,1/2]}(s)$

$X_3(s) = \cancel{s} + \mathbb{1}_{[1/2,1]}(s)$

$X_4(s) = \cancel{s} + \mathbb{1}_{[0,1/3]}(s)$

$X_5(s) = \cancel{s} + \mathbb{1}_{[1/3,2/3]}(s)$

$X_6(s) = \cancel{s} + \mathbb{1}_{[2/3,1]}(s)$

$\vdots$

let $X(s) = s$.

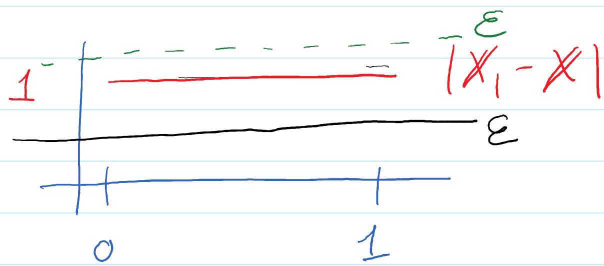
Q: $X_n \xrightarrow{P} X$?

need to establish

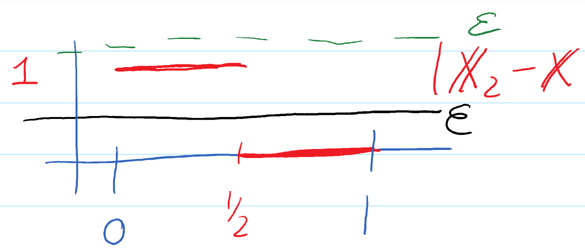
$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

note:

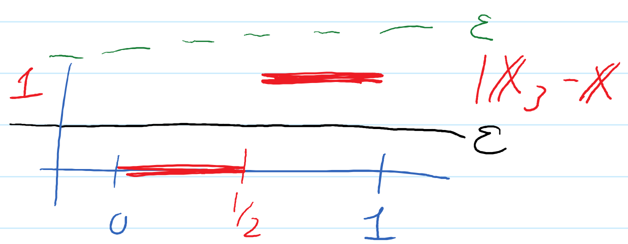
$|X_1 - X| = 1$



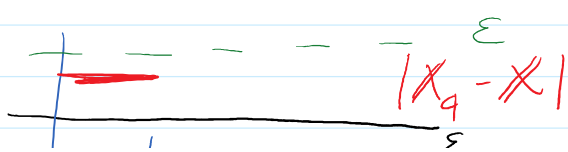

$|X_2 - X| = \mathbb{1}(s \in [0, 1/2])$



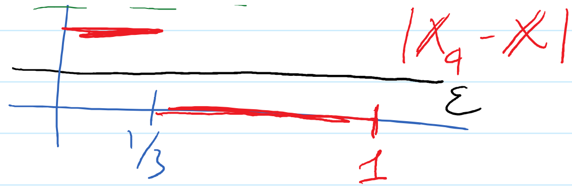

$|X_3 - X| = \mathbb{1}(s \in [1/2, 1])$




$|X_4 - X| = \mathbb{1}(s \in [0, 1/3])$



$|X_4 - X| = \mathbb{1}(\omega \in [0, \tfrac{1}{3}])$




if $\underline{\varepsilon \geq 1}$ then $P(|X_n - X| \geq \varepsilon) = 0$

so certainly $\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$

if $0 < \varepsilon < 1$,

$$P(|X_1 - X| \geq \varepsilon) = 1$$

$$P(|X_2 - X| \geq \varepsilon) = 1/2$$

$$P(|X_3 - X| \geq \varepsilon) = 1/2$$

$$\left.\begin{aligned} P(|X_4 - X| \geq \varepsilon) &= 1/3 \\ &= 1/3 \\ &= 1/3 \\ &= 1/4 \\ &\vdots \\ &= 1/5 \end{aligned}\right\} \text{ in the limit this goes to zero}$$

$$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$$

So $X_n \xrightarrow{P} X$.

Does $X \xrightarrow{a.s.} X$ ?

Does $X_n \xrightarrow{a.s.} X$?

$$A = \{ s \mid X_n(s) \to X(s) \}$$

$$\mathbb{P}(A) = 1 \iff X_n \xrightarrow{a.s.} X.$$

pick any $s \in [0,1]$ and consider

$$X_1(s), X_2(s), \ldots \xrightarrow{?} X(s) = s$$

oscillates between $s$ and $s+1$ forever

won't converge to $s$

So $\mathbb{P}(A) = 0$.

So $X_n \not\xrightarrow{a.s.} X$.

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Defn: Convergence in Distribution

We say $(X_n)$ converges in distribution to $X$

denoted

$$X_n \xrightarrow{d} X$$

if the CDFs converge pointwise.

i.e. if $F_n$ is the CDF of $X_n$ and

$F$ is the CDF of $X$

then $F_n \xrightarrow{ptwse} F$

then $F_n \longrightarrow F$

i.e. $\forall x \quad F_n(x) \longrightarrow F(x)$.

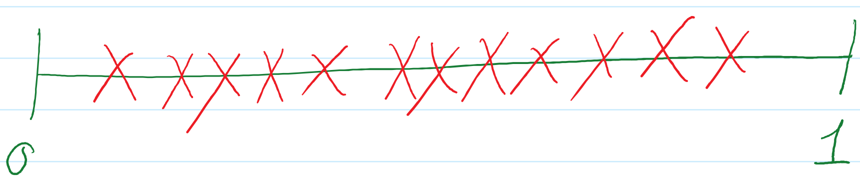
<u>Theorem:</u> i.p $\Rightarrow$ d

If $X_n \xrightarrow{p} X$ then $X_n \xrightarrow{d} X$.

<u>Chain:</u> a.s. $\Rightarrow$ i.p. $\Rightarrow$ d

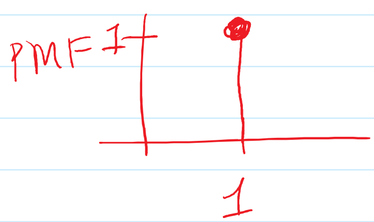
<u>Ex.</u> $X_i \overset{iid}{\sim} U(0,1)$

let $Y_n = \max\limits_{i=1,\dots,n} X_i$ = max of first $n$




Intuition: max gets close to 1.

$\boxed{Y_n \xrightarrow{p} 1}$ $\longleftarrow$ 1 = degenerate RV w/ all mass at 1




$P(Y=1) = 1$

Want to show $\forall \varepsilon > 0$

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$$\mathbb{P}(|Y_n - 1| \geq \varepsilon) \xrightarrow{n} 0$$

$Y_n \leq 1$

$$\mathbb{P}(|1 - Y_n| \geq \varepsilon)$$

$$= \mathbb{P}(1 - Y_n \geq \varepsilon)$$

$Y_n = \max_{i=1,\dots,n} X_i$

$$= \mathbb{P}(Y_n \leq 1-\varepsilon)$$

$$= \mathbb{P}(X_1 \leq 1-\varepsilon, X_2 \leq 1-\varepsilon, \dots, X_n \leq 1-\varepsilon)$$

independent

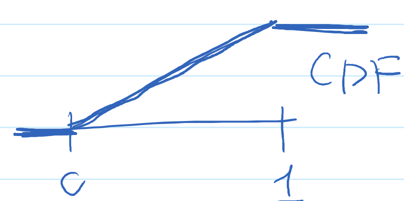
$$= \mathbb{P}(X_1 \leq 1-\varepsilon)\,\mathbb{P}(X_2 \leq 1-\varepsilon) \cdots \mathbb{P}(X_n \leq 1-\varepsilon)$$

$$= \mathbb{P}(X_i \leq 1-\varepsilon)^n$$

If $\varepsilon > 1$ then $1 - \varepsilon < 0$ so

$$\mathbb{P}(X_i \leq 1-\varepsilon) = 0 \quad \text{b/c } X_i \geq 0$$

If $0 < \varepsilon < 1$




$$\mathbb{P}(X_i \leq 1-\varepsilon) = F_{X_i}(1-\varepsilon)$$

$$= 1 - \varepsilon$$

$$= \begin{cases} 0 & \varepsilon \geq 1 \\ (1-\varepsilon)^n & 0 < \varepsilon < 1 \end{cases} = \mathbb{P}(|Y_n - 1| \geq \varepsilon)$$

$$\begin{cases} (1-\varepsilon)^n & 0<\varepsilon<1 \end{cases} = \mathbb{P}(|Y_n - 1| \geq \varepsilon)$$

$\to 0$ as $n \to \infty$ in both cases

So $\lim\limits_{n\to\infty} \mathbb{P}(|Y_n - 1| \geq \varepsilon) = 0$

ad so $Y_n \xrightarrow{P} 1$.

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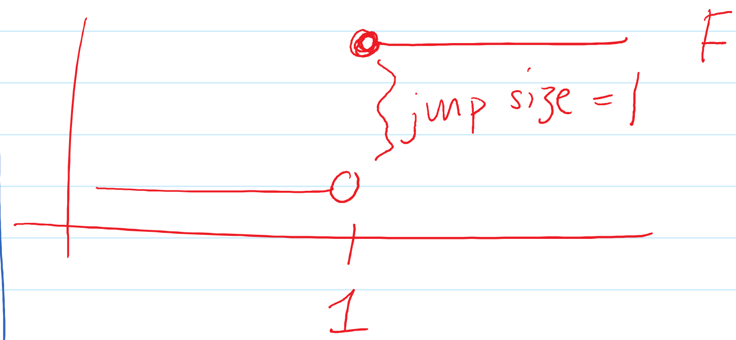
Show $\boxed{Y_n \xrightarrow{d} 1}$ $\quad F_n \xrightarrow{\text{ptwse}} F$




$$F_n(y) = \mathbb{P}(Y_n \leq y)$$

$$= \mathbb{P}(\max_i X_i \leq y)$$

$$= \mathbb{P}(X_1 \leq y, X_2 \leq y, \dots, X_n \leq y)$$

$$= \mathbb{P}(X_i \leq y)^n$$

$$= F_{X_n}(y)^n \qquad X_n \sim U(0,1)$$

$$F(y) = \begin{cases} 0 & y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$= \begin{cases} 0 & y \leq 0 \\ y^n & 0<y<1 \\ 1 & y \geq 1 \end{cases} \xrightarrow{n} \begin{cases} 0 & y \leq 0 \\ 0 & 0<y<1 \\ \end{cases}$$

$$\begin{cases} y & 0<y<1 \\ 1 & y\geq 1 \end{cases} \to \underbrace{\begin{cases} 0 & 0<y<1 \\ 1 & y\geq 1 \end{cases}}_{F(y)}$$

So $F_n \xrightarrow{\text{ptwise}} F$ so $Y_n \xrightarrow{d} 1$.