Extra OH: Thurs 2-3

Mon 2–3
Tues 3-4

P-values:

Back to HT,

often we report the results of a HT using a p-value

Defn: P-value

a p-value $p(\underset{\sim}{X})$ is a test stat where

$$0 \le p(\underset{\sim}{X}) \le 1$$

idea: small values give evidence of H_a and large values give evidence of H_0

Recall: a HT is just a partition of $\mathcal{X}$ into A and R – one way to define a test is to threshold p i.e.

$$R = \{\underset{\sim}{x} : p(\underset{\sim}{x}) \le \text{threshold}\}$$

$$R = \{ \underset{\sim}{x} : p(\underset{\sim}{x}) \le \text{threshold} \}$$

We say a p-value is "valid" if $\forall\ 0 \le \alpha \le 1$ and $\theta \in \Theta_0$

$$\mathbb{P}_\theta \left(p(\underset{\sim}{X}) \le \alpha \right) \le \alpha$$

$F_p(\alpha) \le \alpha$

stochastically bounded by $U(0,1)$

notice that if $F_p(\alpha) = \alpha$ then $p \sim U(0,1)$

If p is valid one I set up a test to reject when

$$R = \{ \underset{\sim}{x} \mid p(\underset{\sim}{x}) \le \alpha \}$$

this gives a level α test.

reason:

If p is valid then $\forall \alpha$, $\forall \theta \in \Theta_0$

$$\mathbb{P}_\theta(\text{reject } H_0)$$

$$= \mathbb{P}\left(p(X) \le \alpha \right)$$

$= \mathbb{P}_{\theta}\left(p(\underset{\sim}{X}) \le \alpha \right)$

$= F_p(\alpha) \le \alpha \longleftarrow$ level α test.

Ex. Consider

$$H_0 : \theta = \theta_0 \quad \text{v.} \quad H_a : \theta \neq \theta_0$$

I define a test where

$$R = \{ \underset{\sim}{x} \mid T \text{ is large} \}$$

let

$$p(\underset{\sim}{x}) \overset{\text{def}}{=} \underbrace{\mathbb{P}_{\theta_0}\left(T(\underset{\sim}{X}) \ge T(\underset{\sim}{x}) \right)}_{\text{under } H_0 : \theta = \theta_0} = 1 - F_{\theta_0}(T(\underset{\sim}{x}))$$

CDF of $T(\underset{\sim}{X})$ under $\theta = \theta_0$

observed test stat

$p(\underset{\sim}{x})$ = Prob. I observe a test test stat as or larger than $T(\underset{\sim}{x})$

If we do this then

$$F_p(\alpha) = \mathbb{P}_{\theta_0}\left(p(\underset{\sim}{X}) \le \alpha \right)$$

$\longrightarrow$ Probability Integral Trans

$= \mathbb{P}\left(1 - F(T(X)) \le \alpha \right)$

Integral trans

$F_X(X) \sim U(0,1)$

$$= \mathbb{P}_{\theta_0}\left(1 - F_{\theta_0}(T(\underset{\sim}{X})) \le \alpha\right)$$

$$= \mathbb{P}_{\theta_0}\left(\underbrace{F_{\theta_0}(T(\underset{\sim}{X}))}_{\text{CDF of } T(\underset{\sim}{X}) \text{ under } \theta=\theta_0} \ge 1-\alpha\right)$$

$U \sim U(0,1)$

$$= \mathbb{P}_{\theta_0}\left(U \ge 1-\alpha\right)$$

$$= 1 - F_U(1-\alpha)$$

$$F_p(\alpha) = 1 - (1-\alpha) = \alpha$$

i.e. $p(\underset{\sim}{X}) \sim U(0,1)$.

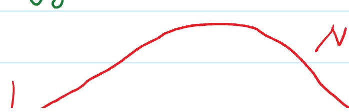
Ex. $H_0: \theta = \theta_0$ v. $H_a: \theta \neq \theta_0$

$$X_n \overset{iid}{\sim} N(\theta, \sigma^2)$$

$$T = \left| \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \right|$$

obs. test stat

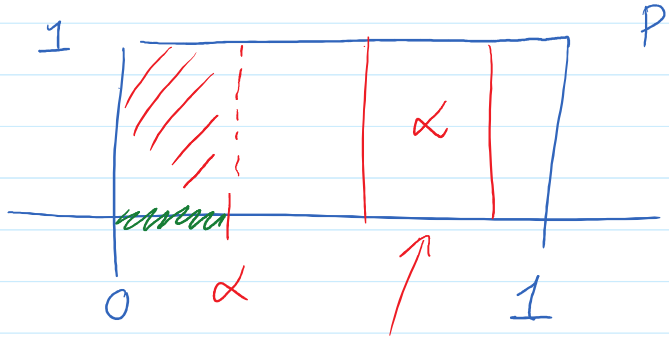
$$p(\underset{\sim}{x}) = \mathbb{P}_{\theta_0}\left(T(\underset{\sim}{X}) \ge T(\underset{\sim}{x})\right)$$



Claim: $p(\underset{\sim}{X}) \sim U(0,1)$ under H_0

Under $H_0: \theta = \theta_0$

PPF of $p(\underset{\sim}{X})$



Power and Sample Size

Power = ability to reject H_0

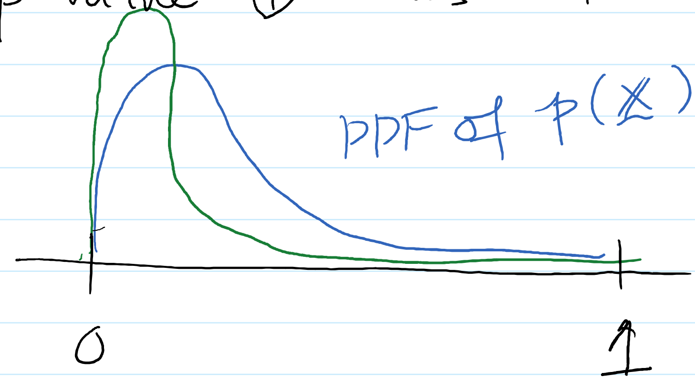
e.g. $\boxed{H_0: \theta = 0 \quad v. \quad H_a: \theta \neq 0}$

Power = ability to detect $\theta \neq 0$

$$\left.\begin{array}{l} \text{Typically power} \uparrow \text{ as } N \uparrow \\ \text{alt: p-value} \downarrow \text{ as } N \uparrow \end{array}\right\} \text{under } H_a$$

alt: p-value ↓ as N ↑) ...

under H_a



Instead: $H_0: \theta \le \delta$ v. $H_a: \theta > \delta$

Bayesian:

Frequentist: θ fixed, unknown

Bayesian: θ random

Bayesian approach:

1. prior info/dist.

   $\Theta \sim \pi(\theta)$
   PMF/PDF of Θ

2. get some data

   $\underset{\sim}{X} \mid \Theta = \theta \sim f(\underset{\sim}{x} \mid \theta) \leftarrow$ sampling dist, likelihood

$\underset{\sim}{X} \mid \Theta = \theta \sim f(\underset{\sim}{x} \mid \theta)$

③ update/combine prior and likelihood to get posterior

$$\pi(\theta \mid \underset{\sim}{x}) = \frac{f(\underset{\sim}{x} \mid \theta)\pi(\theta)}{f(\underset{\sim}{x})} \propto f(\underset{\sim}{x} \mid \theta)\pi(\theta)$$

posterior

likelihood × prior

$$f(\underset{\sim}{x}) = \int f(\underset{\sim}{x} \mid \theta)\pi(\theta)\,d\theta$$

Ex. $P \sim \text{Beta}(\alpha, \beta)$ prior

$X_n \mid P = p \sim \text{Bern}(p)$ likelihood

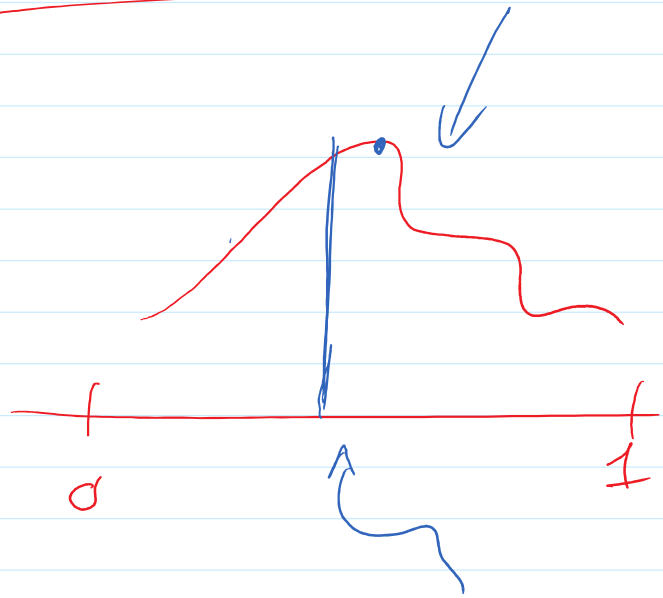
$$\pi(p \mid \underset{\sim}{x}) \propto f(\underset{\sim}{x} \mid p)\pi(p)$$

$$\propto \left(\prod_n p^{x_n}(1-p)^{1-x_n}\right) \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}$$

$$\propto p^{N\bar{x}}(1-p)^{N-N\bar{x}} \, p^{\alpha-1}(1-p)^{\beta-1}$$

$$\propto p^{N\bar{x}+\alpha-1}(1-p)^{N-N\bar{x}+\beta-1}$$

$P | \underset{\sim}{X} \sim \text{Beta}(N\bar{x}+\alpha, N-N\bar{x}+\beta)$



Could est. p as $\mathbb{E}[P | \underset{\sim}{X} = \underset{\sim}{x}] \approx \hat{p}$

$$\hat{p} = \mathbb{E}[P | \underset{\sim}{X} = \underset{\sim}{x}] = \frac{N\bar{x}+\alpha}{N\bar{x}+\alpha+N(1-\bar{x})+\beta}$$

$$= \dots$$

$$= w\bar{x} + (1-w)\frac{\alpha}{\alpha+\beta}$$

weighted avg. of $\bar{x}$ and $\mathbb{E}P = \frac{\alpha}{\alpha+\beta}$

where $w = \frac{N}{\alpha+\beta+N}$, $0 < w < 1$

$w \to 1$ as $N \to \infty$