Lecture 4 - Minimal, Complete, Ancillary, Estimation, and Method of Moments (MOMs)

Useful Fact:

$\to \mathbb{1}(A)\mathbb{1}(B) = \mathbb{1}(A \text{ and } B)$

$\to \prod_n \mathbb{1}(A_n) = \mathbb{1}(\text{all } A_n)$

$$\to \prod_n \mathbb{1}(x_n > \theta) = \mathbb{1}(\text{all } x_n > \theta)$$
$$= \mathbb{1}(x_{(1)} > \theta)$$

$$\to \prod_n \mathbb{1}(x_n < \theta) = \cdots = \mathbb{1}(x_{(N)} < \theta)$$

Ex. Suff. Stats.

$$f_\theta(\underset{\sim}{x}) = g(t,\theta)\,h(\underset{\sim}{x})$$

then t is sufficient for θ.

$X_n \overset{iid}{\sim} U(a, 10)$, $0<a<10$

Find a SS for a:

$$f_\theta(\underset{\sim}{x}) = \prod_n \frac{1}{10-a}\mathbb{1}(x_n > a)\mathbb{1}(x_n < 10)$$

$$= (10-a)^{-N} \prod_n \mathbb{1}(x_n > a) \prod_n \mathbb{1}(x_n < 10)$$

$$= \underbrace{(10-a)^{-N} \mathbb{1}(\overset{t}{x_{(1)}} > a)}_{g(a,t)} \underbrace{\mathbb{1}(x_{(N)} < 10)}_{h(\underset{\sim}{x})}$$

$$g(a,t) = (10-a)^{-N} \mathbb{1}(t > a)$$

$$t = x_{(1)}$$

$$h(\underset{\sim}{x}) = \mathbb{1}(x_{(N)} < 10)$$

Note: any invertible function of a SS is also sufficient.

Defn: Statistic

If $X_n \overset{iid}{\sim} f_\theta$ then a Statistic T is a function of the $X_1, \ldots, X_n$ whose formula doesn't depend on θ.

Ex, $X_n \overset{iid}{\sim} N(\mu, 1)$

$\bar{X} \sim N(\mu, 1/N)$ the dist. of $\bar{X}$ depends on μ

then $T = \bar{X}$ is a statistic.

But, $T = \mu$ is not.

Nor is $T = \bar{X} - \mu$.

Defn: Ancillary Quantity

An ancillary quant. is a fn of the data $Q(\underline{X})$

whose dist. doesn't depend on θ.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ know this

then $\bar{X} \sim N(\mu, \sigma^2/N)$

not a stat.

so $Q = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{N}} \sim N(0,1)$

ancillary quantity

Defn: Ancillary Stat.

T is an ancillary stat if

1. it is ancillary to θ
2. it is a stat

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$

def a stat

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$

let $R = X_{(N)} - X_{(1)}$. ← def a stat (no μ)

$$X_n = \mu + Z_n \quad \text{where } Z_n \overset{iid}{\sim} N(0,1)$$

Similarly

$$X_{(N)} = \mu + Z_{(N)}$$
$$X_{(1)} = \mu + Z_{(1)}$$

So

$$R = X_{(N)} - X_{(1)} = \mu + Z_{(N)} - \mu - Z_{(1)}$$
$$= Z_{(N)} - Z_{(1)}$$

← neither have a dist. that depends on μ

So the dist of R doesn't depend on μ.

Concept: Minimal Sufficiency

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$ and $\bar{X}$ suff. for μ.

What about

$$(\bar{X}, \bar{X}^3)\,? \leftarrow \text{wasteful}$$

This is also suff for μ.

Minimal Sufficiency = sufficient but no larger than it needs to be.

Concept: Complete Sufficient

Typically same as minimal.

Defn: T is complete suff. if I can't make an ancillary stat from it (non-trivially).

Theorem: (don't need to worry too much)

If $X_n \overset{iid}{\sim} f_\theta$ and θ is 1-dim'l

and $f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\theta) \exp(t(\underset{\sim}{x})\, w(\theta))$

is an exp. fam.

Then t is

1. sufficient for θ
2. minimally suff.
3. complete sufficient.

Theorem: Basu's Theorem

→ Suff contains all info about θ

→ Ancillary contains no info about θ

If T is suff for θ and S is ancillary to θ (both stats) then

$$T \perp\!\!\!\perp S.$$

Theorem: $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then $\bar{X} \perp\!\!\!\perp S^2_{N-1}$

pf. $\bar{X}$ is sufficient for μ
S^2 is ancillary for μ

by Basu's theorem $\bar{X} \perp\!\!\!\perp S^2_{N-1}$.

Said: $\frac{N-1}{\sigma^2} S^2_{N-1} \sim \chi^2(N-1)$

alt. $\boxed{S^2_{N-1} \sim \frac{\sigma^2}{N-1} \chi^2(N-1)}$ no μ.

Point Estimation:

Setup: $X_n \overset{iid}{\sim} f_\theta$ where $\theta \in \Theta$

Defn: A point estimator for θ is just a statistic

$$\hat{\theta} = \hat{\theta}(\underset{\sim}{X})$$

for pedants:

$\hat{\theta}(\underset{\sim}{X})$ is called an estimator ← random

$\hat{\theta}(\underset{\sim}{x})$ is called an estimate ← deterministic

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Goal of this course:

1. How do I form estimates?
2. How do I know they are good?

First approach: Method of Moments (MOMs)

Defn: the r^{th} moment of a RV X is

$$\mu_r = \mathbb{E}[X^r]$$

Defn: the r^{th} sample moment

$$\frac{1}{N}\sum^{N} x^r$$

$$m_r = \frac{1}{N}\sum_{n=1}^{N} x_n^r.$$

<u>notice:</u>

$$\mathbb{E}[m_r] = \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N} X_n^r\right]$$

viewed as random

$$= \frac{1}{N}\sum_{n=1}^{N} \mathbb{E}[X_n^r]$$

$$= \frac{1}{N}\sum_{n=1}^{N} \mu_r$$

$$= \mu_r$$

<u>Ex</u> $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ both unknown

Want estimators for μ and σ^2

<u>MoM</u>: Calc first two pop. moments of $N(\mu, \sigma^2)$

1. $\mu_1 = \mathbb{E}X_n = \mu$

$$\mu_2 = \mathbb{E}[X_n^2] = Var(X) + \mathbb{E}[X_n]^2$$

$$= \sigma^2 + \mu^2$$

2. Set these equal to sample moments

(2) Set these equal to sample moments

$$\boxed{\mu_1 = \mu = \mu^2 + \sigma^2} \quad \mu_1 = \mu = \frac{1}{N}\sum_{n=1}^{N} x_n = \overline{X}$$

$$\mu_2 = \mu^2 + \sigma^2 = \frac{1}{N}\sum_{n=1}^{N} x_n^2 = \overline{X^2}$$

→ system of eqns

$$\mu = \overline{X}$$

$$\mu^2 + \sigma^2 = \overline{X^2}$$

Solve for μ and σ^2

$$\mu = \overline{x} \quad \text{so} \quad \boxed{\hat{\mu} = \overline{x}}$$

Second eqn $\mu^2 + \sigma^2 = \overline{X^2}$

$$\Rightarrow \sigma^2 = \overline{X^2} - \mu^2$$

$$= \overline{x^2} - \overline{x}^2$$

show w/ algebra

$$\boxed{\hat{\sigma}^2 = \overline{x^2} - \overline{x}^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x})^2}$$

Method of Moments

$X_n \overset{iid}{\sim} f_\theta$ where $\theta = (\theta_1, \dots, \theta_K)$

and let

$\mu_1, \dots, \mu_K$ be the first K moments of f_θ

and $m_1, \dots, m_K$ be the first K sample moments.

Construct sys. of eqns

fn of θs

$$\begin{cases} \mu_1 = m_1 \\ \mu_2 = m_2 \\ \vdots \\ \mu_K = m_K \end{cases}$$

Solve for each θ_i in terms of Xs.

Ex, $X_n \overset{iid}{\sim} Bin(k, p)$

both unknown

Find MoM for k and p.

(1) get pop. moments.

$$\mu_1 = E[X_n] = kp$$

$$\mu_2 = E[X_n^2] = Var(X_n) + E[X_n]^2$$

$$= kp(1-p) + k^2p^2$$

(2) Form sys of eqns.

$$m_1 = \mu_1 \quad \Rightarrow \quad (1)\ \bar{x} = kp$$
$$m_2 = \mu_2 \qquad\quad (2)\ \overline{x^2} = kp(1-p) + k^2p^2$$

Solve for k, p in terms of xs

$$(2)\ \overline{x^2} = \bar{x}(1-p) + \bar{x}^2$$

$$\Rightarrow \overline{x^2} - \bar{x}^2 = \bar{x}(1-p)$$

$$\Rightarrow \frac{\overline{x^2} - \bar{x}^2}{\bar{x}} = 1-p$$

$$= \boxed{\hat{p} = 1 - \frac{\overline{x^2} - \bar{x}^2}{\bar{x}}}$$

Use eqn (1) $\bar{x} = kp \Rightarrow \boxed{\hat{k} = \frac{\bar{x}}{\hat{p}}}$

Ex. $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

Form MoM estimator.

(1) Calc. pop. moments

$$\mu_1 = \mathbb{E}X_n = \lambda$$

② Set up sys of eqns

$$m_1 = \boxed{\overline{X} = \lambda} = \mu_1$$

$$\hat{\lambda} = \overline{X},$$