Lecture 9 - Cramer-Rao

Single Sample: $f_\theta(x)$, $\log f_\theta(x)$

Multiple samples:

$$L(\theta) = f_\theta(\underset{\sim}{x}) \quad , \quad \ell(\theta) = \log f_\theta(\underset{\sim}{x})$$

$$\frac{\partial \ell}{\partial \theta} \qquad \frac{\partial^2 \ell}{\partial \theta^2}$$

Score: basically treat $\frac{\partial \ell}{\partial \theta}$ as random (replace $\underset{\sim}{x}$, $\underset{\sim}{X}$)

$$S_\theta = \frac{\partial}{\partial \theta} \log f_\theta(\underset{\sim}{X})$$

Fisher Info:

one sample version

$$I(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial \theta} \log f_\theta(X)\right]$$

$$I_N(\theta) = N I(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial \theta^2} \log f_\theta(\underset{\sim}{X})\right]$$

$$= -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \theta^2}\right]$$

Ex. Let $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ assume known

What is the Fisher Info $I_N(\mu)$?

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① $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$

② $\log f(x) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2$

③ $\frac{\partial \log f(x)}{\partial \mu} = -\frac{1}{2\sigma^2} 2(x-\mu)(-1)$

$= \frac{1}{\sigma^2}(x-\mu)$

$\frac{\partial^2 \log f(x)}{\partial \mu^2} = -1/\sigma^2$

④ $I(\mu) = -\mathbb{E}\left[\frac{\partial^2 \log f}{\partial \mu^2}\right] = -\mathbb{E}\left[-1/\sigma^2\right]$

$= 1/\sigma^2$

$I_N(\mu) = N/\sigma^2$

$\left[\mathrm{Var}(\bar{X}) = \sigma^2/N \right]$

Ex. $X_n \overset{iid}{\sim} \mathrm{Pois}(\lambda)$

Recall $\mathbb{E}X = \lambda = \mathrm{Var}\, X$.

Recall $\mathbb{E} X_n = \lambda = \mathrm{Var}\, X_n$

$$Sd(X_n) = \sqrt{\lambda} = \psi \iff \psi^2 = \lambda$$

Reparameterize Pois in terms of ψ

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{\psi^{2x} e^{-\psi^2}}{x!} = f_\psi(x)$$

Can get $I_N(\psi)$ using prev. procedure,

① $\log f_\psi = 2x \log \psi - \psi^2 - \log(x!)$

② $\frac{\partial}{\partial \psi} \log f_\psi = \frac{2x}{\psi} - 2\psi$

$$\frac{\partial^2}{\partial \psi^2} \log f_\psi = \underbrace{-\frac{2x}{\psi^2} - 2}$$

$\lambda = \psi^2$

③ $I(\psi) = -\mathbb{E}\left[\quad\right] = \frac{2\mathbb{E}X}{\psi^2} + 2$

$$= \frac{2\psi^2}{\psi^2} + 2 = 4$$

$$\boxed{I(\psi) = 4N}$$

$$\boxed{I_N(\psi) = 4N}$$

Recall: $I_N(\lambda) = \dfrac{N}{\lambda}$

Derivative Review:

$$y = f(x) \iff x = f^{-1}(y)$$

$$\frac{dy}{dx} \overset{\text{rel.?}}{\longleftrightarrow} \frac{dx}{dy}$$

Recall: $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} \iff \dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$

Theorem: If $\theta = \tau(\psi)$

solve for

Then
$$\boxed{I(\theta) = \left(\frac{\partial \psi}{\partial \theta}\right)^2 I(\psi)}$$

$$I(\psi) = \left(\frac{\partial \theta}{\partial \psi}\right)^2 I(\theta)$$

Revisit Pois example,

$$I(\lambda) = 1/\lambda$$

$$\frac{d\lambda}{d\psi} = 2\psi$$

$$\psi = \sqrt{\lambda} \Leftrightarrow \psi^2 = \lambda$$

$$I(\psi) = \left(\frac{d\lambda}{d\psi}\right)^2 I(\lambda)$$

$$= (2\psi)^2 \frac{1}{\lambda}$$

$$= 4\psi^2 \frac{1}{\psi^2} = 4$$

Why do we care?

$\tau(\theta)$ could be θ

$\hat{\theta}^*$ is the UMVUE for $\tau(\theta)$,

if ① $E\hat{\theta}^* = \tau(\theta)$

② $Var(\hat{\theta}^*) \le Var(\hat{\theta})$

unbiased est of $\tau(\theta)$

Theorem: If $X_n \overset{iid}{\sim} f_\theta$ where $\theta \in \Theta$ and

$\hat{\theta}$ is unbiased for $\tau(\theta)$

(*) and f_θ is nice enough
read: 1-dim'l exp. family (*)

then

$$\mathrm{Var}(\hat{\theta}) \geq \frac{\left(\frac{\partial \tau}{\partial \theta}\right)^2}{I_N(\theta)}$$

called the Cramér-Rao lower bound (CRLB)

Note: if $\tau(\theta) = \theta$, $\left(\frac{\partial \tau}{\partial \theta}\right)^2 = 1^2 = 1$

So the CRLB is just $1/I_N(\theta)$

Comments:

① If I have an unbiased est. $\hat{\theta}^*$ for $\tau(\theta)$ and I can show that

$$\mathrm{Var}(\hat{\theta}^*) = \mathrm{CRLB}$$

then $\hat{\theta}^*$ is the UMVUE.

② If an estimator doesn't achieve the CRLB I don't otherwise know its not the UMVUE.

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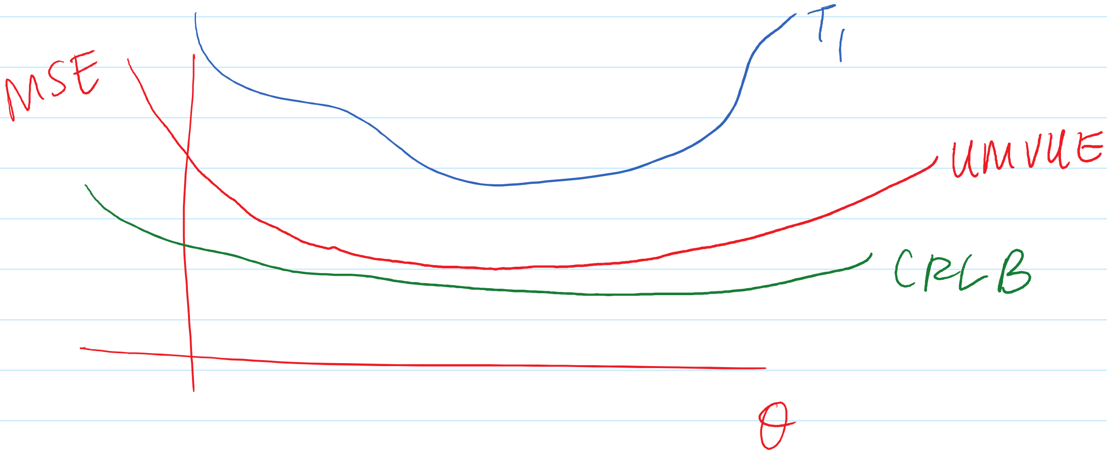

Ex. $X_n \overset{iid}{\sim} Pois(\lambda)$

let $\hat{\lambda} = \bar{X}$ Est λ, $\tau(\lambda) = \lambda$

$\mathbb{E}\bar{X} = \lambda$ (its unbiased for λ)

$Var(\bar{x}) = \frac{Var X_1}{N} = \frac{\lambda}{N}$

CRLB

$B = \frac{1}{I_N(\lambda)} = \frac{1}{N/\lambda} = \frac{\lambda}{N}$

General Steps:

1. propose unbiased est.
2. calc its variance
3. calc. CRLB
4. show 2 & 3 are equal $\Rightarrow$ this is UMVUE

(4) Show (2) & (3) are equal $\Rightarrow$ this is UMVUE

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$

Recall: $EX_n = 1/\lambda$

$Var\, X_n = 1/\lambda^2$

Goal: Find UMVUE for $1/\lambda$. $[\tau(\lambda) = 1/\lambda]$

(1) Propose unbiased est. ; $\bar{X}$

$E\bar{X} = 1/\lambda$;

(2) Calc. var

$$Var(\bar{X}) = \frac{1}{\lambda^2 N}$$

(3) Calc. CRLB for estimating $1/\lambda$.

$$\tau(\lambda) = 1/\lambda, \quad \frac{\partial \tau}{\partial \lambda} = -1/\lambda^2 ; \quad \boxed{\left(\frac{\partial \tau}{\partial \lambda}\right)^2 = \frac{1}{\lambda^4}}$$

$I(\lambda)$

$\rightarrow f_\lambda(x) = \lambda e^{-\lambda x}$

$\rightarrow f_X(x) = \lambda e$

$\rightarrow \log f_X = \log \lambda - \lambda x$

$\rightarrow \frac{\partial \log f_X}{\partial \lambda} = \frac{1}{\lambda} - x$

$\rightarrow \frac{\partial^2 \log f_X}{\partial \lambda^2} = -\frac{1}{\lambda^2}$

$\rightarrow I(\lambda) = -\mathbb{E}[-1/\lambda^2]$

$= 1/\lambda^2$

So $I_N(\lambda) = N/\lambda^2$

$$\frac{}{N/\lambda^2} = \frac{1}{N\lambda^2}$$

Since $\bar{X}$ unbiased for $1/\lambda$ and

$\mathrm{Var}(\bar{X}) = \frac{1}{N\lambda^2} = B$

$\bar{X}$ is the UMVUE.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ σ^2 is known

Saw earlier: $I_N(\mu) = N/\sigma^2$

If we want to estimate μ,

① $\mathbb{E}\bar{X} = \mu$ ($\bar{X}$ is unbiased for μ)

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② $\text{Var}\,\bar{X} = \sigma^2/N$

③ $B = 1/I_N(\mu) = \sigma^2/N$

④ $\bar{X}$ is the UMVUE.

Ex. Let $X_n \overset{iid}{\sim} U(0,\theta)$

Want the UMVUE for θ.

① Propose unbiased Est. for θ.

$$T = \frac{N+1}{N} X_{(N)}$$

Can show:

$$\mathbb{E}[X_{(N)}] = \frac{N}{N+1}\theta$$

then $ET = \theta$

② Calc. Variance

$$\text{Var}(T) = \frac{\theta^2}{N(N+1)}$$

③ CRLB

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$$f_\theta(x) = \frac{1}{\theta}\underbrace{\mathbb{1}(0<x<\theta)}$$

$$\log f_\theta(x) = -\log\theta + \log(\mathbb{1}(0<x<\theta)) \cdots$$