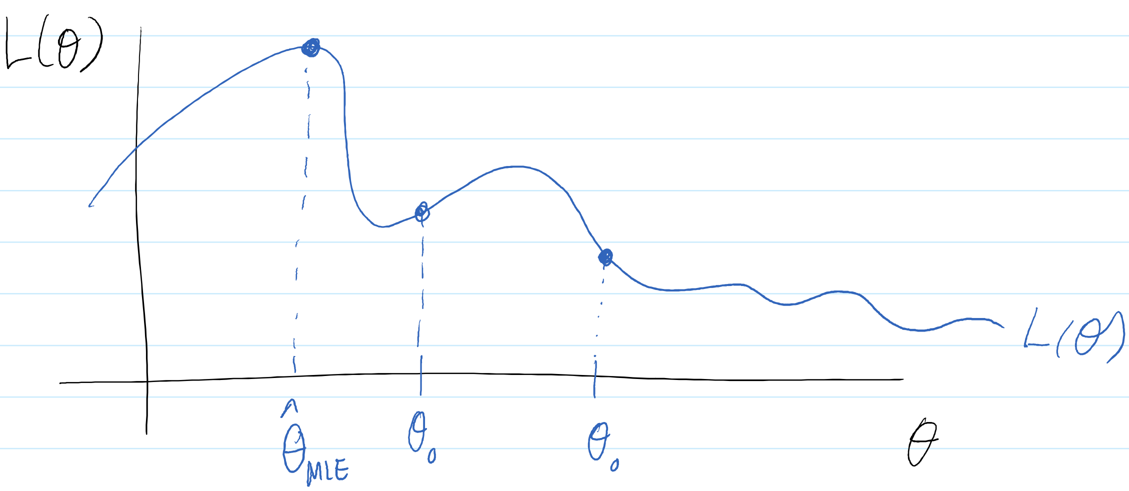
Recap:

$$L(\theta) = f_\theta(\underset{\sim}{x}) \quad ; \quad \ell(\theta) = \log L(\theta)$$



Ex.

$$X_n \overset{iid}{\sim} \text{Pois}(\lambda)$$

Let's get the MLE

(1) $$L(\lambda) = f_\lambda(\underset{\sim}{x}) = \prod_{n=1}^{N} \frac{e^{-\lambda} \lambda^{x_n}}{x_n!}$$

$$= \frac{e^{-N\lambda} \lambda^{\sum_n x_n}}{\prod_n x_n!}$$

$$\ell(\lambda) = \log L(\lambda) = \log(e^{-N\lambda}) + \log(\lambda^{\sum x_n}) - \log\left(\prod_n x_n!\right)$$

$$= (-N\lambda)\log(e) + \left(\sum_n x_n\right)\log\lambda - \sum_n \log(x_n!)$$

$$= (-N\lambda)\underbrace{\log(e)}_{1} + \left(\sum_n x_n\right)\log\lambda - \sum_n \log(x_n!)$$

② find where $\frac{\partial \ell}{\partial \lambda} = 0$

$$\frac{\partial \ell}{\partial \lambda} = -N + \left(\sum_n x_n\right)\frac{1}{\lambda} = 0$$

$$\Rightarrow \quad N = \left(\sum_n x_n\right)\frac{1}{\lambda}$$

so $\boxed{\hat{\lambda}_{MLE} = \frac{1}{N}\sum_n x_n = \overline{X}}$

Ex. $X_n \overset{iid}{\sim} Exp(\lambda)$ recall: $E[X_n] = 1/\lambda$

Find the MLE for λ.

① $L(\lambda) = f_\lambda(\underset{\sim}{x}) = \prod_{n=1}^{N} \lambda e^{-\lambda x_n}$

$$= \lambda^N e^{-\lambda \sum_n x_n}$$

$$\ell(\lambda) = \log L(\lambda) = \log(\lambda^N) + \log\left(e^{-\lambda \sum_n x_n}\right)$$

$$= N\log\lambda - \lambda\sum_n x_n \cancel{\log(e)}^{1}$$

$= N \log \lambda - \lambda \sum_n x_n \cancel{\log(e)}$ –

② $\frac{\partial \ell}{\partial \lambda} = 0$

$\frac{\partial \ell}{\partial \lambda} = \frac{N}{\lambda} - \sum_n x_n = 0$

$$\boxed{\hat{\lambda} = \frac{1}{\bar{X}}}$$

Ex. $X_n \overset{iid}{\sim} \text{Exp}(\beta)$; $\beta = 1/\lambda$

so that $\mathbb{E}[X_n] = \beta$

What is MLE for β?

Know that MLE for λ is $\hat{\lambda} = 1/\bar{X}$

Want is MLE for $\beta = 1/\lambda$

so $\boxed{\hat{\beta}_{MLE} = \bar{X}}$

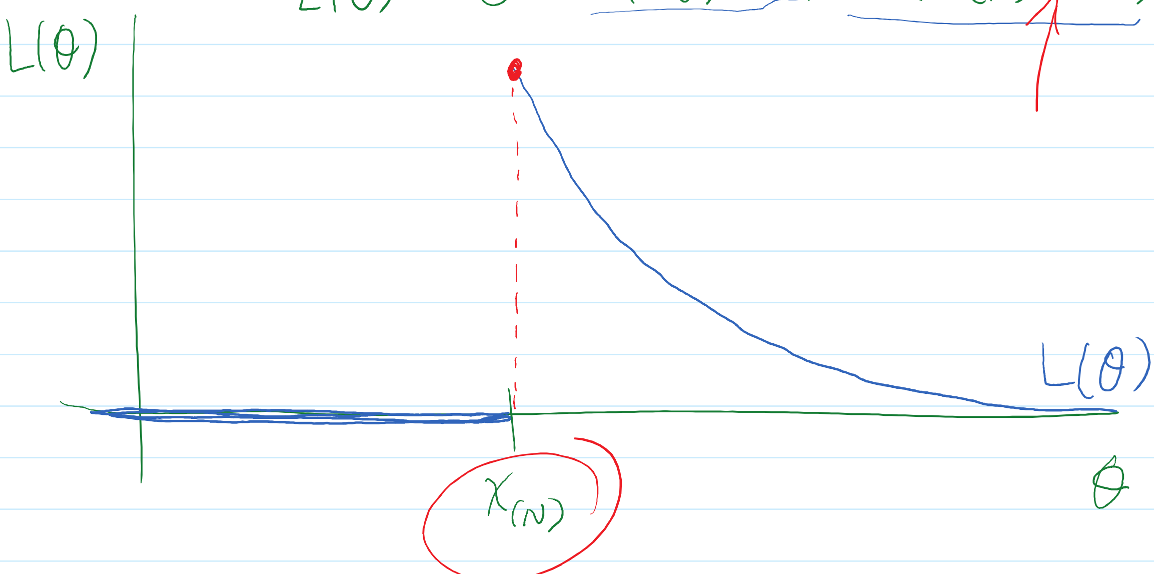
Ex. $X_n \overset{iid}{\sim} U(0, \theta)$ $\theta \geq 0$

Lets get MLE for θ.

(1) $L(\theta) = f_\theta(\underset{\sim}{x}) = \prod_{n=1}^{N} \frac{1}{\theta} \mathbb{1}(x_n \geq 0)\mathbb{1}(x_n \leq \theta)$

$$= \theta^{-N} \prod_n \mathbb{1}(x_n \geq 0) \prod_n \mathbb{1}(x_n \leq \theta)$$

$$L(\theta) = \theta^{-N} \underline{\mathbb{1}(x_{(1)} \geq 0)}\ \underline{\mathbb{1}(x_{(N)} \leq \theta)}$$



So $\hat{\theta} = x_{(N)}$.

$\underline{\text{Ex.}}$ $X_n \overset{iid}{\sim} N(\theta, 1)$ where $\theta \geq 0$.

The MLE must satisfy this constraint

$$\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmax}}\, L(\theta)$$

$\Theta = [0, \infty)$

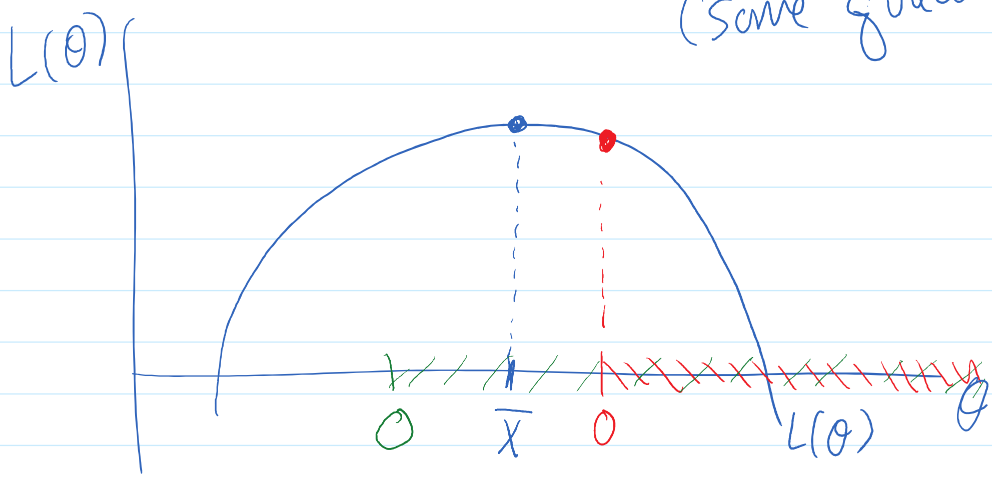
① $L(\theta) = \prod_n \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_n - \theta)^2\right)$

$= (2\pi)^{-N/2} \exp\left(\sum_n \left(-\frac{1}{2}(x_n - \theta)^2\right)\right)$

$= (2\pi)^{-N/2} \exp\left(-\frac{1}{2} \sum_n (x_n - \theta)^2\right)$

$\ell(\theta) = -\frac{N}{2} \log(2\pi) - \frac{1}{2} \sum_n (x_n - \theta)^2$

looks kinda like $-\theta^2$
(some quadratic in θ)



$\bar{X} > 0$ then $\hat{\theta} = \bar{X}$

$\bar{X} < 0$ then $\hat{\theta} = 0$

($\bar{X} < 0$ then $\theta = 0$

$\hat{\theta} = \max(0, \bar{X})$.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

both unknown
$\mu \in \mathbb{R}, \sigma^2 > 0$

lets get MLE for both μ and σ^2.

① $$L(\mu, \sigma^2) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_n - \mu)^2\right)$$

$$= (2\pi)^{-N/2} (\sigma^2)^{-N/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=1}^{N} (x_n - \mu)^2\right)$$

$$\ell(\mu, \sigma^2) = -\frac{N}{2}\log(2\pi) - \frac{N}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n - \mu)^2$$

$\tau = \sigma^2$

$$\ell(\mu, \tau) = \underbrace{-\frac{N}{2}\log(2\pi)}\ \underbrace{-\frac{N}{2}\log(\tau)} - \frac{1}{2\tau}\sum_{n=1}^{N}(x_n - \mu)^2$$

② $\frac{\partial \ell}{\partial \mu} = 0$ and $\frac{\partial \ell}{\partial \tau} = 0$

$$\frac{\partial l}{\partial \mu} = -\frac{1}{2\tau}\sum_{n=1}^{N} 2(x_n-\mu)(-1) = \frac{1}{\tau}\sum_{n=1}^{N}(x_n-\mu)$$

$$= \frac{1}{\tau}\sum_n x_n - \frac{N}{\tau}\mu = 0$$

$$\Rightarrow \sum_n x_n - N\mu = 0$$

$$\Rightarrow \boxed{\hat{\mu} = \overline{X}}$$

$$\frac{\partial l}{\partial \tau} = -\frac{N}{2\tau} + \frac{1}{2\tau^2}\sum_{n=1}^{N}(x_n-\mu)^2 = 0$$

$$\Rightarrow -\frac{N\tau}{2} + \frac{1}{2}\sum_{n=1}^{N}(x_n-\mu)^2 = 0$$

$$\Rightarrow -\tau + \frac{1}{N}\sum_{n=1}^{N}(x_n-\mu)^2 = 0$$

$$\Rightarrow \tau = \frac{1}{N}\sum_{n=1}^{N}(x_n-\mu)^2$$

$$\Rightarrow \boxed{\hat{\tau} = \hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n-\overline{X})^2}$$

Ex. $X_n \overset{iid}{\sim} \text{Laplace}(\mu, \sigma)$

$$f(x) = \frac{1}{2\sigma}\exp\left(-\frac{1}{\sigma}|x-\mu|\right)$$

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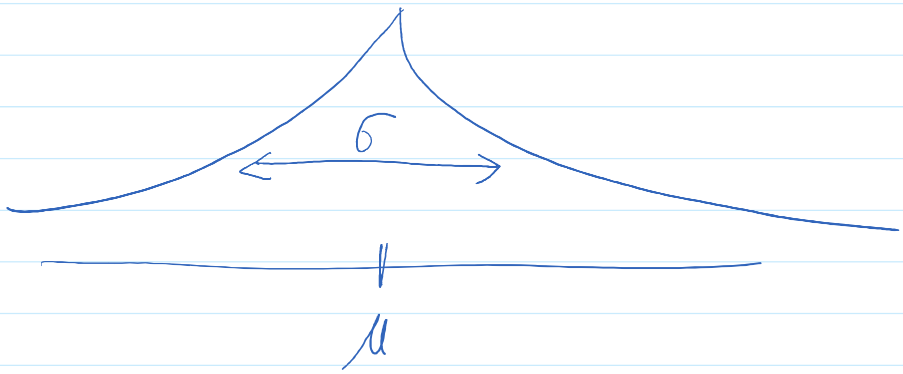

MLE?

① $$L(\mu,\sigma) = \prod_n \frac{1}{2\sigma} \exp\left(-\frac{1}{\sigma}|x_n-\mu|\right)$$

$$= 2^{-N}\sigma^{-N} \exp\left(-\frac{1}{\sigma}\sum_{n=1}^{N}|x_n-\mu|\right)$$

$$\ell(\mu,\sigma) = -N\log(2) - N\log(\sigma) - \frac{1}{\sigma}\sum_{n=1}^{N}|x_n-\mu|$$

② $$\underbrace{\frac{\partial \ell}{\partial \mu} = 0}_{\text{not diff'able}} \quad \text{and} \quad \underbrace{\frac{\partial \ell}{\partial \sigma} = 0}\ \checkmark$$

$$\frac{\partial \ell}{\partial \sigma} = \frac{-N}{\sigma} + \frac{1}{\sigma^2}\sum_{n=1}^{N}|x_n-\mu| = 0$$

So $\boxed{\hat{\sigma} = \frac{1}{N}\sum_{n=1}^{N} |x_n - \hat{\mu}|}$

How to get $\hat{\mu}$?

$$L(\mu, \sigma) \propto \exp\left(-\frac{1}{\sigma}\underbrace{\sum_n |x_n - \mu|}\right)$$

decreasing as this sum increases.

Want to make $\boxed{\sum_n |x_n - \mu|}$ as small as possible.

Turns out: $\hat{\mu} = \text{median}(\underline{x})$.