Lecture 21 - Karlin-Rubin

Consider building the LRT in the traditional way

$$\lambda(\underset{\sim}{x}) = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{f_{\hat{\theta}_0}(\underset{\sim}{x})}{f_{\hat{\theta}}(\underset{\sim}{x})} \qquad [L(\theta) = f_\theta(\underset{\sim}{x})]$$

We have seen that often the LRT is based on some sufficient stat.

Alternative LRT: if T is a sufficient stat for θ let $g_\theta(t)$ be the PMF/PDF of T

might call

$$L^*(\theta) = g_\theta(t)$$

could define

$$\lambda^*(t) = \frac{L^*(\hat{\theta}_0)}{L^*(\hat{\theta})} = \frac{g_{\hat{\theta}_0}(t)}{g_{\hat{\theta}}(t)}$$

I could form a HT procedure that rejects when $\lambda^* \leq c$.

Punchline: this is equivalent to the LRT.

One way of building LRT is to get SS first and build such a test $\lambda^* \leq c$.

first and build such a test $\lambda^* \le c$.

<u>Why?</u> Recall that the MLE is a fn of the SS. So

$$\hat{\theta}_0 = \text{fn}(t) \quad \text{and} \quad \hat{\theta} = \text{fn}(t)$$

So

$$\lambda^*(t) = \frac{L^*(\hat{\theta}_0)}{L^*(\hat{\theta})} = \frac{g_{\hat{\theta}_0(t)}(t)}{g_{\hat{\theta}(t)}(t)}$$

a fn of t

<u>Theorem:</u>

$$\lambda^*(t) = \lambda^*(t(\underset{\sim}{x})) = \lambda(\underset{\sim}{x}) \quad \forall \underset{\sim}{x}$$

<u>Pf.</u>

$$\lambda(\underset{\sim}{x}) = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \frac{\max_{\theta \in \Theta_0} f_\theta(\underset{\sim}{x})}{\max_{\theta \in \Theta} f_\theta(\underset{\sim}{x})}$$

T sufficient

$$= \frac{\max_{\theta \in \Theta_0} g_\theta(t)\cancel{h(\underset{\sim}{x})}}{\max_{\theta \in \Theta} g_\theta(t)\cancel{h(\underset{\sim}{x})}}$$

$$= \frac{\max_{\theta \in \Theta_0} L^*(\theta)}{L^*(\theta)} = \lambda^*(t)$$

$$\frac{\quad}{\max_{\theta \in \Theta} L^*(\theta)} = \lambda^*(t)$$

Neyman - Pearson Lemma:

Consider testing

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta = \theta_a$$

w/ the LRT that rejects when

(*) $$\lambda = \frac{L(\theta_0)}{L(\theta_a)} \le c$$

where we choose c so that

(**) $$P_{\theta_0}(\lambda \le c) = \alpha$$ [Size α test]

(a) Sufficient: Any test satisfying (*) and (**) is a UMP level α test for this hypothesis.

(b) Necessity: Every UMP level α test for this hypothesis is (**) a size α test and has a rej. region equiv. to (*)

and has a rej. region equiv. to (*)

[up to a prob. zero set]

Corollary: If I test $H_0: \theta = \theta_0$ v. $H_a: \theta = \theta_a$

Let T be a suff. stat. for θ and $g_\theta(t)$ be its PMF/PDF then the test

that rejects iff

$$\lambda = \frac{g_{\theta_0}(t)}{g_{\theta_a}(t)} \leq c$$

when c is s.t. $P_{\theta_0}(\lambda \leq c) = \alpha$

is the UMP level α test.

Ex. Let $X_1, X_2 \overset{iid}{\sim} \text{Bernoulli}(\theta)$

test: $H_0: \theta = 1/2$ v. $H_a: \theta = 3/4$

Note: $T = X_1 + X_2$ is sufficient for θ

and $T \sim \text{Bin}(2, \theta)$

$g_\theta(t) = \binom{2}{t} \theta^t (1-\theta)^{2-t}$

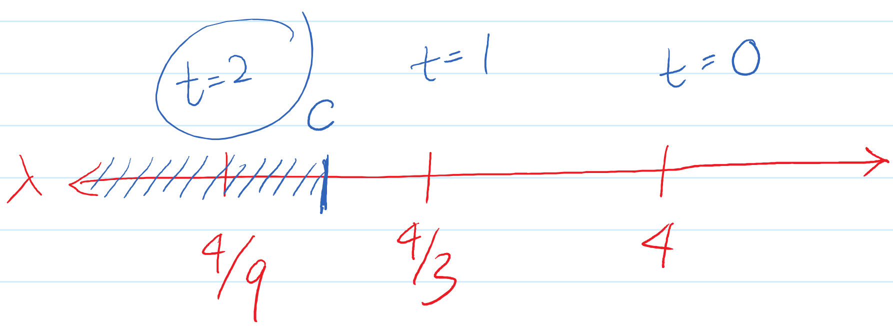
$$\cancel{\binom{2}{t}} \left(\frac{1}{\ }\right)^t \left(\frac{1}{\ }\right)^{2-t}$$

$$\lambda(t) = \lambda = \frac{g_{\theta_0}(t)}{g_{\theta_a}(t)} = \frac{g_{1/2}(t)}{g_{3/4}(t)} = \frac{\cancel{\binom{2}{t}}\left(\frac{1}{2}\right)^t\left(\frac{1}{2}\right)^{2-t}}{\cancel{\binom{2}{t}}\left(\frac{3}{4}\right)^t\left(1/4\right)^{2-t}}$$

$$= \frac{\left(1/2\right)^2}{\left(\frac{3}{4}\right)^t\left(\frac{1}{4}\right)^{2-t}}$$

LRT says rej. if $\lambda \le c$

t	0	1	2
$\lambda(t)$	4	4/3	4/9



For ex. if $\boxed{4/9 < c < 4/3}$ (i.e. rej. when $t=2$)

then $\alpha = P_{1/2}(\lambda \le c) = P_{1/2}(T=2) = \binom{2}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{2-2}$

$= \frac{1}{4}$

So this test is a UMP level .25 test

So this test is a UMP level .25 test for hypothesis.

What about composite hypotheses?

one-sided

Let's consider $H_0: \theta \le \theta_0$ v. $H_a: \theta > \theta_0$

Defn: Monotone Likelihood Ratio Property (MLR)

We say a family of PDFs/PMFs has the MLR property if $\theta_1 < \theta_2$

$$\boxed{\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)}}$$ is non-decreasing as a fn of x.

Theorem: If $\{f_\theta\}$ is an exp fam.

$t(x) = x$

$$f_\theta(x) = c(\theta)h(x)\exp(w(\theta)x)$$

and $w(\theta)$ is non-decreasing in θ — then this fam. has the MLR property.

Pf: $\theta_2 > \theta_1$

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{c(\theta_2)\cancel{h(x)}\exp(w(\theta_2)x)}{c(\theta_1)\cancel{h(x)}\exp(w(\theta_1)x)}$$

$$\propto \exp\big((w(\theta_2) - w(\theta_1))x\big)$$

If w non-dec. as a fn of θ then

$$w(\theta_2) - w(\theta_1) \geq 0$$

looks like e^{ax} where $a \geq 0$

is inc/non-dec. as a fn x.

Theorem: If T has the MLR property and we have a test that rej. when

$$T > c$$

then the power fn β of this test is non-decreasing.

Pf. $\theta_2 > \theta_1$ then $\beta(\theta_2) \geq \beta(\theta_1)$

i.e. $P(\underbrace{T > c}_{\text{I rej.}}) \geq P(T > c)$

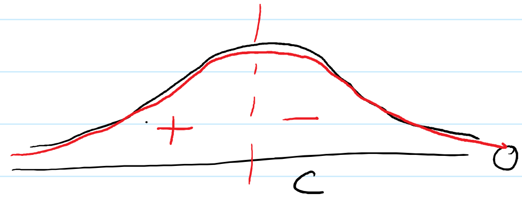
i.e.

$$\mathbb{P}_{\theta_2}(T>c) \geq \mathbb{P}_{\theta_1}(T>c)$$

CDF of T at θ

i.e. $$1-F_{\theta_2}(c) \geq 1-F_{\theta_1}(c)$$

i.e. $$\boxed{F_{\theta_1}(c) - F_{\theta_2}(c) \geq 0}$$



$$\frac{d[\quad]}{dc} = f_{\theta_1}(c) - f_{\theta_2}(c)$$

$$= f_{\theta_1}(c)\left(1 - \frac{f_{\theta_2}(c)}{f_{\theta_1}(c)}\right)$$

If MLR is non-dec. as fn c

as c goes $-\infty$ to ∞

→ deriv. goes + to − as c goes from $-\infty$ to ∞.

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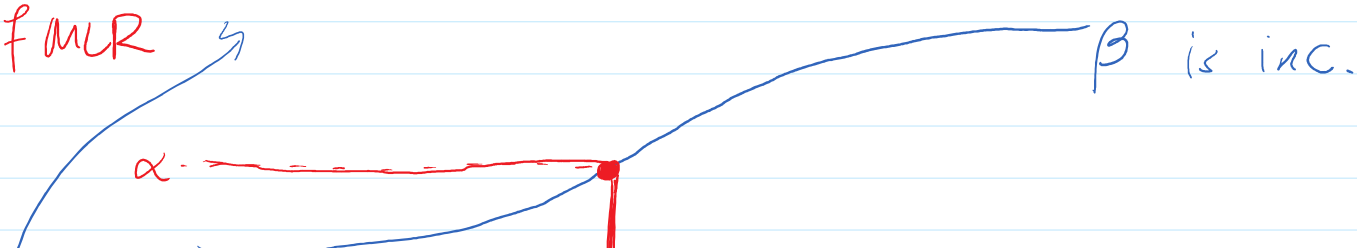
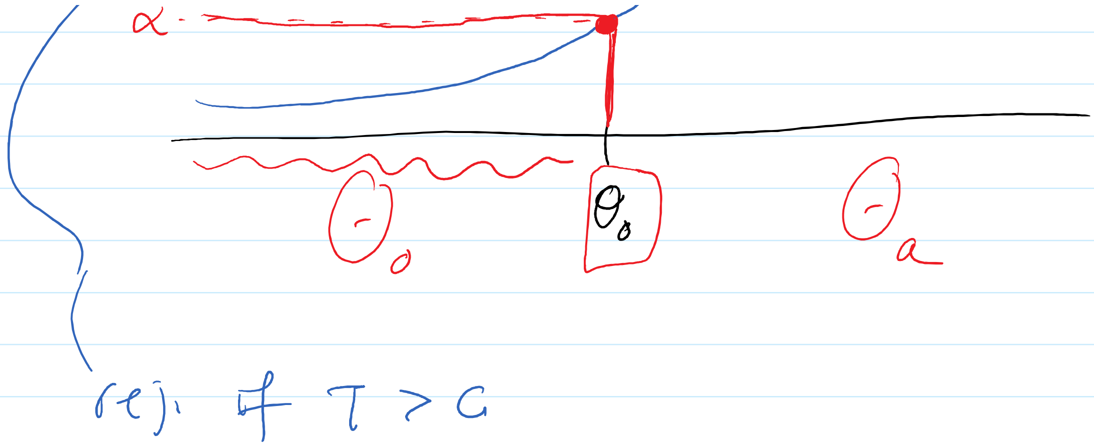
why do we care?

Test: $H_0: \theta \leq \theta_0$ v. $H_a: \theta > \theta_0$

If MLR ↗





Punchline: this is the UMP level α-test.

Theorem: Karlin-Rubin

Consider testing

$$H_0 : \theta \le \theta_0 \quad \text{v.} \quad H_a : \theta > \theta_0$$

and let T be sufficient for θ and have the MLR property.

The test that rejects when $T > c$

when c is chosen so that

$$P_{\theta=\theta_0}(T > c) = \alpha$$

is the UMP level α test.

Notes: (1) Alt. test

$$H_0: \theta \geq \theta_0 \quad v. \quad \underline{H_a: \theta < \theta_0}$$

and rej. if $\underline{T < c}$ ... this is the UMP level α test.

(2) This is basically the LRT b/c we are rejecting based on a S.S.

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ σ^2 known.

Test: $\underline{H_0: \mu \leq a \quad v. \quad H_a: \mu > a}$

Note: $\overline{X}$ is sufficient for μ,

$$\bar{X} \sim N(\mu, \sigma^2/N)$$

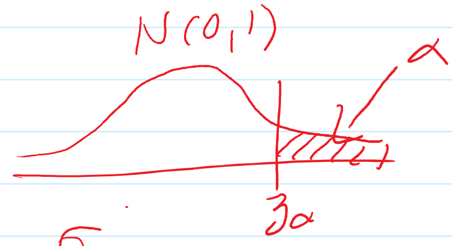
$\longrightarrow$ check $\overline{X}$ has MLR property [defer] ✓

Then the UMP level α test is to reject when $\overline{X} > c$

where c is chosen so that

$P(\bar{X} \quad)$ $N(0,1)$ α

$$\mathbb{P}_{\mu=a}(\bar{X} > c) = \alpha$$



We have shown that $c = a + \frac{\sigma}{\sqrt{N}} z_\alpha$