Rao - Blackwell

Let $\hat{\theta}$ is unbiased for $\tau(\theta)$ and W is a SS for θ then if

$$\varphi = \varphi(W) = \mathbb{E}[\hat{\theta} \mid W]$$

(Rao-Blackwellization of $\hat{\theta}$)

then

1. $\mathbb{E}\varphi = \tau(\theta)$
2. $Var\,\varphi \le Var\,\hat{\theta}$
3. φ is a statistic (no θ in formula)

Theorem: Lehmann - Scheffé

* complete sufficient

Basically: If I Rao-Blackwellize where W is a (complete) sufficient stat then φ is the UMVUE for $\tau(\theta)$.

If W is a sufficient stat* for θ and $\hat{\theta}$ is unbiased for $\tau(\theta)$ and a fn. of $\underset{\sim}{X}$ only through W — then $\hat{\theta}$ is the UMVUE.

Practically: (1) find SS for θ

(2) find some fn of SS that is unbiased for $\tau(\theta)$

that is the UMVUE.

Ex. $X_n \overset{iid}{\sim} U(0,\theta)$

what is the UMVUE for $\tau(\theta) = \theta$?

(1) Find a SS for θ : $X_{(N)}$

(2) Find some fn g of $X_{(N)}$ so that

$$\mathbb{E}[g(X_{(N)})] = \theta$$

In this case can show: $\mathbb{E}[X_{(N)}] = \frac{N}{N+1}\theta$

So if

$$g(X_{(N)}) = \frac{N+1}{N} X_{(N)}$$

then $\mathbb{E}\left[\frac{N+1}{N} X_{(N)}\right] = \frac{N+1}{N}\frac{N}{N+1}\theta = \theta$

So $\frac{N+1}{N} X_{(N)}$ is the UMVUE.

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pf of Lehmann - Scheffé $\hat{\theta} = \hat{\theta}(W)$

If V is another unbiased est. for $\tau(\theta)$

then
$$Var(\hat\theta) \le Var(V).$$

We do this by Rao-Blackwellizing V using W — my SS.

We'll show : $\hat\theta = E[\underline{V} \mid \underline{W}]$

Rao-Blackwell says :

if $\varphi(W) = E[V \mid W]$

then ① $E\varphi = \tau(\theta)$

② $Var(\varphi) \le Var(V)$

③ φ is a stat

Consider $g(W) = \hat\theta(W) - \varphi(W)$

We'll show $g \equiv 0 \ \forall\theta$

Completeness! $E[h(W)] = 0 \ \forall\theta \iff h \equiv 0$

If W is complete then since

$$E[g(W)] = E[\hat\theta(W) - \varphi(W)]$$
$$= E\hat\theta - E\varphi = \tau(\theta) - \tau(\theta) = 0$$

then this only happens if $g \equiv 0$

i.e. $g(w) = \hat{\theta} - \varphi = 0 \Rightarrow \hat{\theta} = \varphi$.

Theorem: UMVUEs are Unique

Let W_1 and W_2 be UMVUEs for $\tau(\theta)$ and $W_1 \neq W_2$.

Consider: $W_3 = \frac{1}{2}(W_1 + W_2)$

notice: $\mathbb{E}[W_3] = \frac{1}{2}\mathbb{E}W_1 + \frac{1}{2}\mathbb{E}W_2 = \tau(\theta)$

So W_3 unbiased for $\tau(\theta)$

$$Var(W_3) = Var\left(\frac{1}{2}W_1 + \frac{1}{2}W_2\right)$$

$$= \frac{1}{4}Var(W_1) + \frac{1}{4}Var(W_2) + \frac{1}{2}Cov(W_1, W_2)$$

Fact: $Cov(W_1, W_2) \overset{\text{really} =}{\leq} \sqrt{Var(W_1)Var(W_2)}$

(i.e. $cor \leq 1$)

$$Var(W_3) \overset{\text{really} =}{\leq} \frac{1}{4}Var(W_1) + \frac{1}{4}Var(\cancelto{W_1}{W_2}) + \frac{1}{2}\sqrt{Var(W_1)Var(\cancelto{W_1}{W_2})}$$

$$= \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right)Var(W_1)$$

$$= Var(W_1)$$

but $Var(W_3) \nleq Var(W_1)$

So $Cov(W_1, W_2) = \sqrt{Var(W_1) Var(W_2)}$

$$Cor(W_1, W_2) = \frac{Cov(W_1, W_2)}{\sqrt{Var(W_1) Var(W_2)}} = 1$$

So W_1 and W_2 are perfectly correlated.

So $W_1 = a W_2 + b$

However

$$\underbrace{E[W_1]}_{\tau(\theta)} = a \underbrace{E[W_2]}_{\tau(\theta)} + b$$

So it must be that $a = 1$ and $b = 0$

i.e. $W_1 = W_2$.

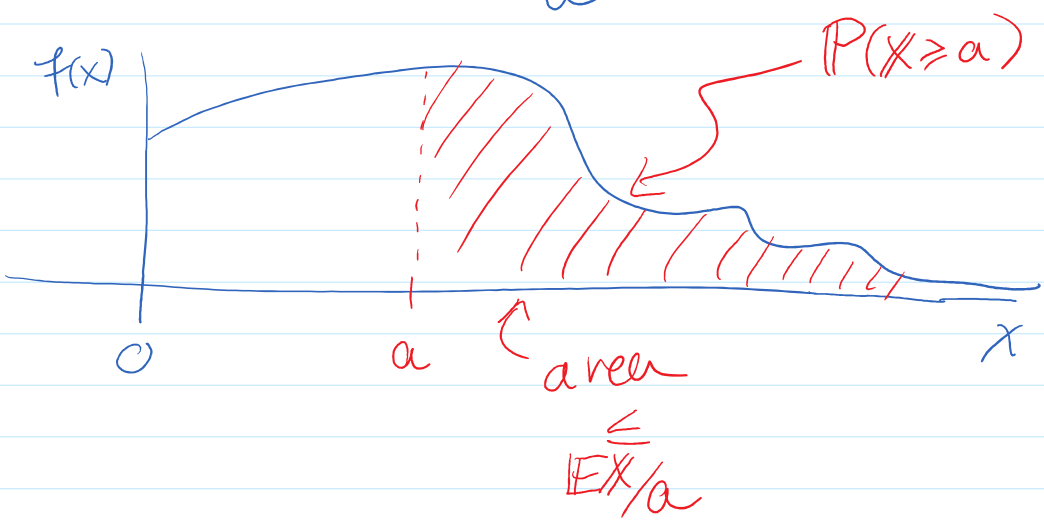
Inequalities

Theorem: Markov's Inequality

If $X \geq 0$ (support of $X \geq 0$)

then for any $a \geq 0$ we have

$$P(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$



(cts)

Pf. $$EX = \int_{\mathbb{R}} x f(x)\,dx = \int_0^\infty x f(x)\,dx$$

$$= \underbrace{\int_0^a \underbrace{x}_{\geq 0} \underbrace{f(x)}_{\geq 0}\,dx}_{\geq 0} + \int_a^\infty x f(x)\,dx$$

A B

$A \geq 0$

so

$A + B \geq B$

$$\geq \int_a^\infty x f(x)\,dx$$

over (a, ∞) $x \geq a$

$a \leq x$

$$\geq \int_a^\infty a f(x)\,dx$$

$$= a \int_a^\infty f(x)\,dx$$

$$= a\underbrace{\int_a^\infty f(x)\,dx}_{\mathbb{P}(X \geq a)}$$

So $\mathbb{E}X \geq a\,\mathbb{P}(X \geq a)$

or $\underbrace{\mathbb{P}(X \geq a)} \leq \dfrac{\mathbb{E}X}{a}$.

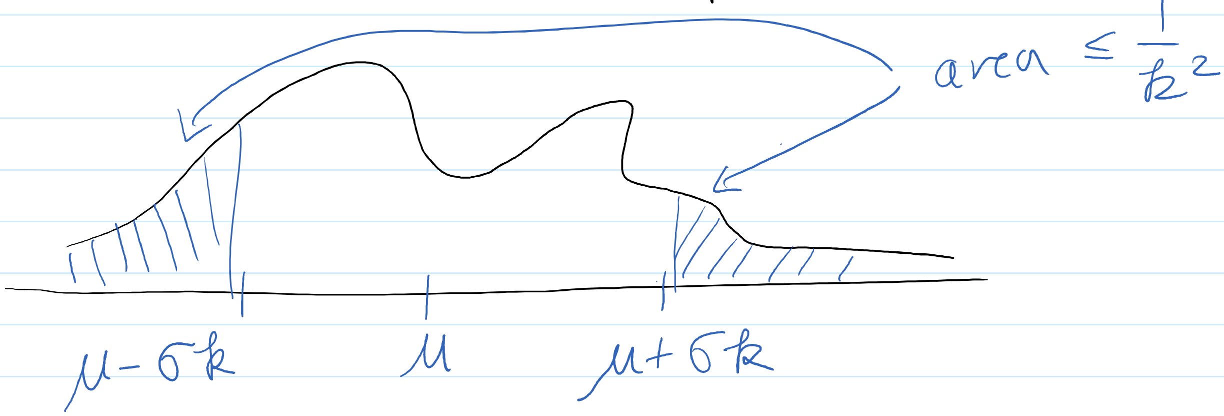
Theorem: Chebyshev's Inequality

If X is a RV and

$$\mu = \mathbb{E}X \quad \text{and} \quad \sigma^2 = \operatorname{Var} X$$

then

$$\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq \frac{1}{k^2}.$$



Pf. Let $Y = \dfrac{(X-\mu)^2}{\sigma^2}$ and $a = k^2$

notice: $Y \geq 0$ and so by Markov's ineq.

$$\mathbb{P}(Y \geq a) \leq \frac{\mathbb{E}Y}{a} \leftarrow 1$$

i.e.

$$\mathbb{P}\left(\frac{(X-\mu)^2}{\sigma^2} \geq k^2\right) \leq \frac{1}{k^2}$$

$$\mathbb{E}Y = \mathbb{E}\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2}\mathbb{E}[(X-\mu)^2]$$

$$= \frac{Var(X)}{\sigma^2} = 1$$

Sqrt

$$\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2.$$

Various Versions of Chebyshev's

① $\mathbb{P}\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2$

② $\mathbb{P}\left(\frac{|X-\mu|}{\sigma} < k\right) \geq 1 - 1/k^2 \leftarrow$

③ $\varepsilon = k\sigma \iff k = \varepsilon/\sigma$ and $\frac{1}{k^2} = \sigma^2/\varepsilon^2$

$$\mathbb{P}(|X-\mu| \geq \varepsilon) \leq \sigma^2/\varepsilon^2$$

④ $\mathbb{P}(|X-\mu| < \varepsilon) \geq 1 - \sigma^2/\varepsilon^2$.

Ex. X = # nails produced in a box in same factory

$\mu = \mathbb{E}X = 1000$

$\sigma^2 = \mathrm{Var}\,X = 25 \quad (\sigma = 5)$

what is the prob.

$994 \leq X \leq 1006$

$$\mathbb{P}(994 \leq X \leq 1006) = \mathbb{P}(|X - 1000| \leq 6)$$

$$= \mathbb{P}\Big(\frac{|X-1000|}{5} \leq \underbrace{1.2}_{k}\Big)$$

$$\geq 1 - \frac{1}{(1.2)^2}$$

$$\approx 30\%$$