

Useful Fact:

$$\rightarrow \mathbb{1}(A) \mathbb{1}(B) = \mathbb{1}(A \text{ and } B)$$

$$\rightarrow \prod_n \mathbb{1}(A_n) = \mathbb{1}(\text{all } A_n)$$

$$\begin{aligned} \rightarrow \prod_n \mathbb{1}(x_n > \theta) &= \mathbb{1}(\text{all } x_n > \theta) \\ &= \mathbb{1}(x_{(1)} > \theta) \end{aligned}$$

$$\rightarrow \prod_n \mathbb{1}(x_n < \theta) = \dots = \mathbb{1}(x_{(n)} < \theta)$$

Ex. Suff. Stats.

$$f_{\theta}(\underline{x}) = g(t, \theta) h(\underline{x})$$

then t is sufficient for θ .

$$X_n \stackrel{\text{iid}}{\sim} U(a, 10), \quad \underline{0 < a < 10}$$

Find a SS for a :

$$f_{\theta}(\underline{x}) = \prod_n \frac{1}{10-a} \mathbb{1}(x_n > a) \mathbb{1}(x_n < 10)$$

$$\begin{aligned}
 &= (10-a)^{-N} \prod_n \mathbb{I}(x_n > a) \prod_n \mathbb{I}(x_n < 10) \\
 &= \underbrace{(10-a)^{-N} \mathbb{I}(x_{(n)}^t > a)}_{g(a, t)} \underbrace{\mathbb{I}(x_{(n)} < 10)}_{h(\underline{x})}
 \end{aligned}$$

$$g(a, t) = (10-a)^{-N} \mathbb{I}(t > a)$$

$$t = x_{(n)}$$

$$h(\underline{x}) = \mathbb{I}(x_{(n)} < 10)$$

Note: any invertible function of a SS is also sufficient.

Defn: Statistic

If $X_n \stackrel{iid}{\sim} f_\theta$ then a Statistic T is a function of the X_1, \dots, X_n whose formula doesn't depend on θ .

Ex. $X_n \stackrel{iid}{\sim} N(\mu, 1)$

then $T = \bar{X}$ is a statistic. $\bar{X} \sim N(\mu, 1/n)$ the dist. of \bar{X} depends on μ

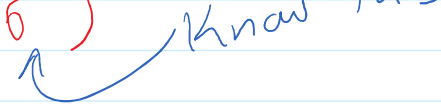
But, $T = \mu$ is not.

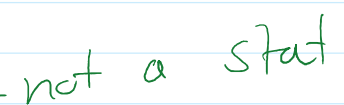
Nor is $T = X - \mu$.

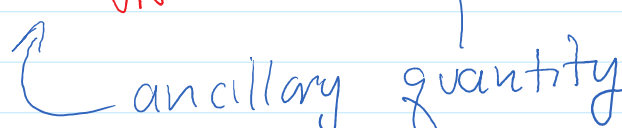
Defn: Ancillary Quantity

An ancillary quant. is a fn of the data $Q(\underline{X})$

whose dist. doesn't depend on θ .

Ex. $X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$  know this

then $\bar{X} \sim N(\mu, \sigma^2/N)$  not a stat.

so $Q = \frac{\bar{X} - \mu}{\sigma/\sqrt{N}} \sim N(0, 1)$  ancillary quantity

Defn: Ancillary Stat.

T is an ancillary stat if

① it is ancillary to θ

② it is a stat

Ex. $X_n \stackrel{\text{iid}}{\sim} N(\mu, 1)$

... def a stat

Ex. $X_n \sim N(\mu, 1)$

let $R = X_{(n)} - X_{(1)}$ ← def a stat (no μ)

$X_n = \mu + Z_n$ where $Z_n \stackrel{iid}{\sim} N(0, 1)$

Similarly

$$X_{(n)} = \mu + Z_{(n)}$$

$$X_{(1)} = \mu + Z_{(1)}$$

So

$$R = X_{(n)} - X_{(1)} = \mu + Z_{(n)} - \mu - Z_{(1)}$$

$$= Z_{(n)} - Z_{(1)}$$

neither have a dist. that depends on μ

So the dist of R doesn't depend on μ .

Concept : Minimal Sufficiency

Ex. $X_n \stackrel{iid}{\sim} N(\mu, 1)$ and \bar{X} suff. for μ .

What about

(\bar{X}, \bar{X}^3) ? ← wasteful

This is also suff for μ .

Minimal Sufficiency = sufficient but no larger than it needs to be.

Concept: Complete Sufficient

Typically same as minimal.

Defn: T is complete suff. if I can't make an ancillary stat from it (non-trivially).

Theorem: (don't need to worry too much)

If $X_n \stackrel{iid}{\sim} f_\theta$ and θ is 1-dim'l

and $f_\theta(x) = h(x) \ell(\theta) \exp(t(x) w(\theta))$

is an exp. fam.

Then t is

- (1) sufficient for θ
- (2) minimally suff.
- (3) complete sufficient.

Theorem: Basu's Theorem

- Suff contains all info about θ
- Ancillary contains no info about θ

If T is suff for θ and S is ancillary to θ (both stats) then

$$T \perp\!\!\!\perp S.$$

Theorem: $X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$ then $\bar{X} \perp\!\!\!\perp S_{N-1}^2$

pf. \bar{X} is sufficient for μ } by Basu's
 S^2 is ancillary for μ } theorem
 $\bar{X} \perp\!\!\!\perp S_{N-1}^2$

Said: $\frac{N-1}{\sigma^2} S_{N-1}^2 \sim \chi^2(N-1)$

alt. $\boxed{S_{N-1}^2 \sim \frac{\sigma^2}{N-1} \chi^2(N-1)}$ no μ .

Point Estimation!

Setup: $X_n \stackrel{\text{iid}}{\sim} f_\theta$ when $\theta \in \Theta$

Defn: A point estimator for θ is just a statistic

$$\hat{\theta} = \hat{\theta}(\underline{X})$$

for pedants: \nwarrow random

$\hat{\theta}(\underline{X})$ is called an estimator

$\hat{\theta}(\underline{x})$ is called an estimate

\nearrow deterministic

Goal of this course:

- ① How do I form estimates?
 - ② How do I know they are good?
-

First approach: Method of Moments (MOMs)

Defn: the r^{th} moment of a RV X is

$$\mu_r = E[X^r]$$

Defn: the r^{th} sample moment

$$1, \frac{N}{r}, r$$

$$m_r = \frac{1}{N} \sum_{n=1}^N x_n^r$$

notice:

$$E[m_r] = E\left[\frac{1}{N} \sum_{n=1}^N x_n^r\right]$$

viewed as random

$$= \frac{1}{N} \sum_{n=1}^N E[x_n^r]$$

$$= \frac{1}{N} \sum_{n=1}^N \mu_r$$

$$= \mu_r$$

Ex $x_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$

both unknown

Want estimators for μ and σ^2

MoM: Calc first two pop. moments of $N(\mu, \sigma^2)$

① $\mu_1 = E[x_n] = \mu$

$$\mu_2 = E[x_n^2] = \text{Var}(x) + E[x_n]^2$$

$$= \sigma^2 + \mu^2$$

② Set these equal to sample moments

Set these equal to sample moments

$$\begin{cases} \mu_1 = \mu = \frac{1}{N} \sum_{n=1}^N x_n = \bar{X} \\ \mu_2 = \mu^2 + \sigma^2 = \frac{1}{N} \sum_{n=1}^N x_n^2 = \overline{X^2} \end{cases}$$

System of eqns

$$\mu = \bar{X}$$

$$\mu^2 + \sigma^2 = \overline{X^2}$$

Solve for μ and σ^2

$$\mu = \bar{X} \text{ so } \hat{\mu} = \bar{X}$$

Second eqn $\mu^2 + \sigma^2 = \overline{X^2}$

$$\Rightarrow \sigma^2 = \overline{X^2} - \mu^2$$

$$= \overline{X^2} - \bar{X}^2$$

show w/ algebra

$$\hat{\sigma}^2 = \overline{X^2} - \bar{X}^2 = \frac{1}{N} \sum_{n=1}^N (x_n - \bar{X})^2$$

Method of Moments

$X_n \stackrel{iid}{\sim} f_\theta$ where $\theta = (\theta_1, \dots, \theta_K)$

and let

μ_1, \dots, μ_K be the first K moments
of f_θ
and m_1, \dots, m_K be the first K sample moments.

Construct sys. of eqns

for θ s of $\left\{ \begin{array}{l} \mu_1 = m_1 \\ \mu_2 = m_2 \\ \vdots \\ \mu_K = m_K \end{array} \right.$

Solve for each θ_i in terms of X s.

Ex, $X_n \stackrel{iid}{\sim} \text{Bin}(k, p)$
 \uparrow both unknown

Find MOM for k and p .

① get pop. moments

$$\mu_1 = E[X_n] = kp$$

$$\begin{aligned} \mu_2 &= E[X_n^2] = \text{Var}(X_n) + E[X_n]^2 \\ &= kp(1-p) + k^2 p^2 \end{aligned}$$

② Form sys of eqns.

$$\left. \begin{array}{l} m_1 = \mu_1 \Rightarrow \textcircled{1} \bar{x} = kp \\ m_2 = \mu_2 \Rightarrow \textcircled{2} \bar{x}^2 = \underline{k}p(1-p) + \underline{k^2}p^2 \end{array} \right\}$$

Solve for k, p in terms of x s

$$\textcircled{2} \bar{x}^2 = \bar{x}(1-p) + \bar{x}^2$$

$$\Rightarrow \bar{x}^2 - \bar{x}^2 = \bar{x}(1-p)$$

$$\Rightarrow \frac{\bar{x}^2 - \bar{x}^2}{\bar{x}} = 1-p$$

$$= \boxed{\hat{p} = 1 - \frac{\bar{x}^2 - \bar{x}^2}{\bar{x}}}$$

$$\text{Use eqn } \textcircled{1} \bar{x} = kp \Rightarrow \boxed{\hat{k} = \frac{\bar{x}}{\hat{p}}}$$

ex. $X_n \stackrel{iid}{\sim} \text{Pois}(\lambda)$

Form MOM estimator.

① Calc. pop. moments.

$$\mu_1 = EX_n = \lambda$$

② set up sys of eqns

$$m_1 = \boxed{\bar{X} = \lambda} = \mu_1$$

$$\hat{\lambda} = \bar{X}_1$$