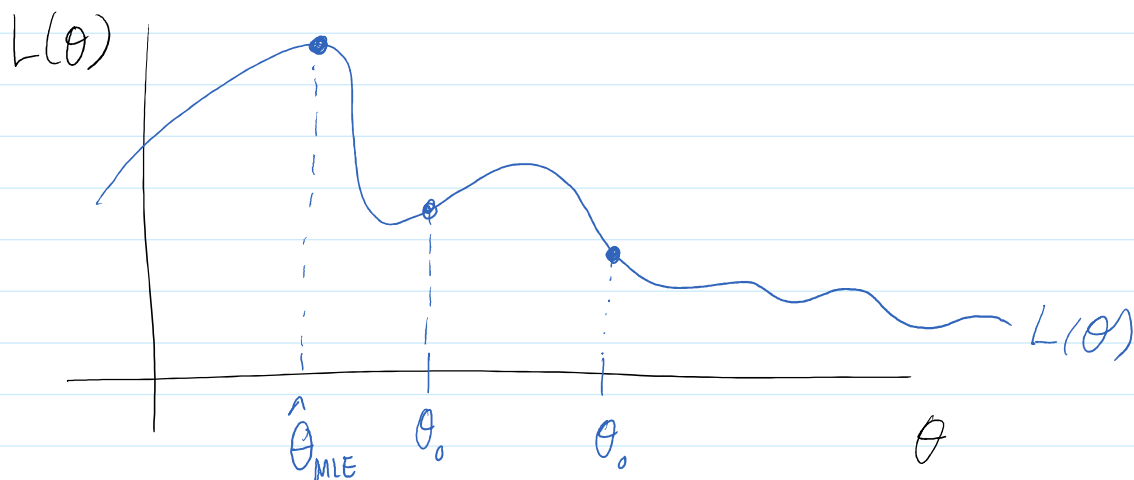


Recap:

$$L(\theta) = f_{\theta}(\underline{x}) \quad ; \quad \ell(\theta) = \log L(\theta)$$



Ex. $X_n \stackrel{iid}{\sim} \text{Pois}(\lambda)$

Let's get the MLE

$$\begin{aligned} \textcircled{1} L(\lambda) &= f_{\lambda}(\underline{x}) = \prod_{n=1}^N \frac{e^{-\lambda} \lambda^{x_n}}{x_n!} \\ &= \frac{e^{-N\lambda} \lambda^{\sum_n x_n}}{\prod_n x_n!} \end{aligned}$$

$$\begin{aligned} \ell(\lambda) &= \log L(\lambda) = \log(e^{-N\lambda}) + \log(\lambda^{\sum x_n}) - \log\left(\prod_n x_n!\right) \\ &= (-N\lambda) \log(e) + \left(\sum_n x_n\right) \log \lambda - \sum_n \log(x_n!) \end{aligned}$$

$$= (-N) \underbrace{\log(e)}_1 + \left(\sum_n x_n\right) \log \lambda - \sum_n \log(x_n!)$$

② find where $\frac{\partial \ell}{\partial \lambda} = 0$

$$\frac{\partial \ell}{\partial \lambda} = -N + \left(\sum_n x_n\right) \frac{1}{\lambda} = 0$$

$$\Rightarrow N = \left(\sum_n x_n\right) \frac{1}{\lambda}$$

so $\boxed{\hat{\lambda}_{MLE} = \frac{1}{N} \sum_n x_n = \bar{X}}$

Ex. $X_n \stackrel{iid}{\sim} \text{Exp}(\lambda)$ recall: $E[X_n] = \frac{1}{\lambda}$

Find the MLE for λ .

$$\begin{aligned} \textcircled{1} L(\lambda) &= f_{\lambda}(\underline{x}) = \prod_{n=1}^N \lambda e^{-\lambda x_n} \\ &= \lambda^N e^{-\lambda \sum_n x_n} \end{aligned}$$

$$\ell(\lambda) = \log L(\lambda) = \log(\lambda^N) + \log(e^{-\lambda \sum_n x_n})$$

$$= N \log \lambda - \lambda \sum_n x_n \log(e)^{\uparrow 1}$$

$$(2) \quad \frac{\partial \ell}{\partial \lambda} = 0$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{N}{\lambda} - \sum_n x_n = 0$$

$$\boxed{\hat{\lambda} = \frac{1}{\bar{x}}}$$

Ex. $X_n \stackrel{iid}{\sim} \text{Exp}(\beta)$; $\beta = 1/\lambda$

so that $E[X_n] = \beta$

What is MLE for β ?

Know that MLE for λ is $\hat{\lambda} = 1/\bar{x}$

Want is MLE for $\beta = 1/\lambda$

so $\boxed{\hat{\beta}_{MLE} = \bar{x}}$

Ex. $X_n \stackrel{iid}{\sim} U(0, \theta)$ $\theta \geq 0$

Let's get MLE for θ .

$$\begin{aligned} \textcircled{1} \quad L(\theta) = f_{\theta}(\underline{x}) &= \prod_{n=1}^N \frac{1}{\theta} \mathbb{1}(x_n \geq 0) \mathbb{1}(x_n \leq \theta) \\ &= \theta^{-N} \prod_n \mathbb{1}(x_n \geq 0) \prod_n \mathbb{1}(x_n \leq \theta) \end{aligned}$$

$$L(\theta) = \theta^{-N} \underbrace{\mathbb{1}(x_{(1)} \geq 0)} \underbrace{\mathbb{1}(x_{(N)} \leq \theta)}$$



So $\hat{\theta} = x_{(N)}$.

Ex. $X_n \stackrel{iid}{\sim} N(\theta, 1)$ where $\theta \geq 0$.

The MLE must satisfy this constraint

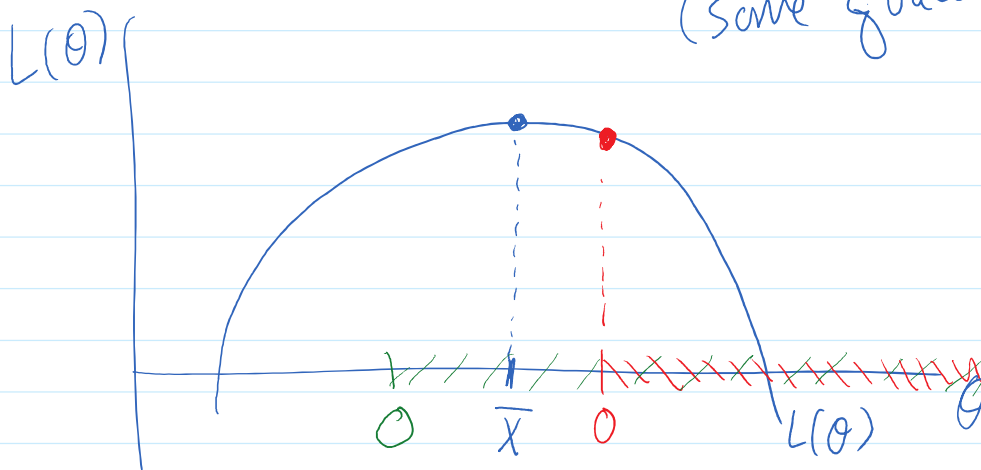
$$\hat{\theta} = \operatorname{argmax}_{\theta \in \Theta} L(\theta)$$

$$\Theta = [\sigma, \infty)$$

$$\begin{aligned} \textcircled{1} \quad L(\theta) &= \prod_n \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\chi_n - \theta)^2\right) \\ &= (2\pi)^{-N/2} \exp\left(\sum_n \left(-\frac{1}{2}(\chi_n - \theta)^2\right)\right) \\ &= (2\pi)^{-N/2} \exp\left(-\frac{1}{2} \sum_n (\chi_n - \theta)^2\right) \end{aligned}$$

$$\ell(\theta) = -\frac{N}{2} \log(2\pi) - \frac{1}{2} \sum_n (\chi_n - \theta)^2$$

↑ looks kinda like $-\theta^2$
(some quadratic in θ)



$$\begin{aligned} \bar{X} > 0 &\quad \text{then} \quad \hat{\theta} = \bar{X} \\ \bar{X} < 0 &\quad \text{then} \quad \hat{\theta} = 0 \end{aligned}$$

$$\hat{\theta} = \max(0, \bar{X}).$$

Ex. $X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$

both unknown
 $\mu \in \mathbb{R}, \sigma^2 > 0$

lets get MLE for both μ and σ^2 .

$$\begin{aligned} \textcircled{1} L(\mu, \sigma^2) &= \prod_{n=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_n - \mu)^2\right) \\ &= (2\pi)^{-N/2} (\sigma^2)^{-N/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=1}^N (x_n - \mu)^2\right) \end{aligned}$$

$$\ell(\mu, \sigma^2) = -\frac{N}{2} \log(2\pi) - \frac{N}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{n=1}^N (x_n - \mu)^2$$

$$\tau = \sigma^2$$

$$\ell(\mu, \tau) = \underbrace{-\frac{N}{2} \log(2\pi)} - \underbrace{\frac{N}{2} \log(\tau)} - \frac{1}{2\tau} \sum_{n=1}^N (x_n - \mu)^2$$

$$\textcircled{2} \frac{\partial \ell}{\partial \mu} = 0 \quad \text{and} \quad \frac{\partial \ell}{\partial \tau} = 0$$

$$\frac{\partial \ell}{\partial \mu} = -\frac{1}{2\tau} \sum_{n=1}^N 2(x_n - \mu)(-1) = \frac{1}{\tau} \sum_{n=1}^N (x_n - \mu)$$

$$= \frac{1}{L} \sum_n x_n - \frac{N}{L} \mu = 0$$

$$\Rightarrow \sum_n x_n - N\mu = 0$$

$$\Rightarrow \boxed{\hat{\mu} = \bar{x}} \leftarrow$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = -\frac{N}{2L} + \frac{1}{2L^2} \sum_{n=1}^N (x_n - \mu)^2 = 0$$

$$\Rightarrow -\frac{NL}{2} + \frac{1}{2} \sum_{n=1}^N (x_n - \mu)^2 = 0$$

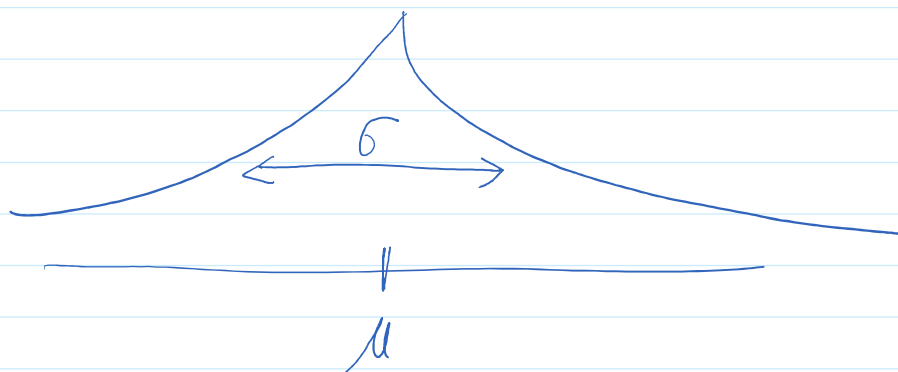
$$\Rightarrow -L + \frac{1}{N} \sum_{n=1}^N (x_n - \mu)^2 = 0$$

$$\Rightarrow L = \frac{1}{N} \sum_{n=1}^N (x_n - \mu)^2$$

$$\Rightarrow \boxed{\frac{\hat{L}}{L} = \hat{\sigma}^2 = \frac{1}{N} \sum_{n=1}^N (x_n - \bar{x})^2}$$

Ex. $X_n \stackrel{iid}{\sim} \text{Laplace}(\mu, \sigma)$

$$f(x) = \frac{1}{2\sigma} \exp\left(-\frac{1}{\sigma} |x - \mu|\right)$$



MLE?

$$\textcircled{1} \quad L(\mu, \sigma) = \prod_n \frac{1}{2\sigma} \exp\left(-\frac{1}{\sigma} |x_n - \mu|\right)$$

$$= 2^{-N} \sigma^{-N} \exp\left(-\frac{1}{\sigma} \sum_{n=1}^N |x_n - \mu|\right)$$

$$\ell(\mu, \sigma) = -N \log(2) - N \log(\sigma) - \frac{1}{\sigma} \sum_{n=1}^N |x_n - \mu|$$

$$\textcircled{2} \quad \underbrace{\frac{\partial \ell}{\partial \mu}}_{\text{not differentiable}} = 0 \quad \text{and} \quad \underbrace{\frac{\partial \ell}{\partial \sigma}} = 0 \quad \checkmark$$

$$\frac{\partial \ell}{\partial \sigma} = -\frac{N}{\sigma} + \frac{1}{\sigma^2} \sum_{n=1}^N |x_n - \mu| = 0$$

$$\text{So} \quad \boxed{\hat{\sigma} = \frac{1}{N} \sum_{n=1}^N |x_n - \hat{\mu}|}$$

$$\sum_{n=1}^N |x_n - \mu|$$

How to get $\hat{\mu}$?

$$L(\mu, \sigma) \propto \exp\left(-\frac{1}{\sigma} \sum_n |x_n - \mu|\right)$$

decreasing as this sum increases.

Want to make

$\sum_n |x_n - \mu|$ as small as possible.

Turns out: $\hat{\mu} = \text{median}(\underline{x})$.
