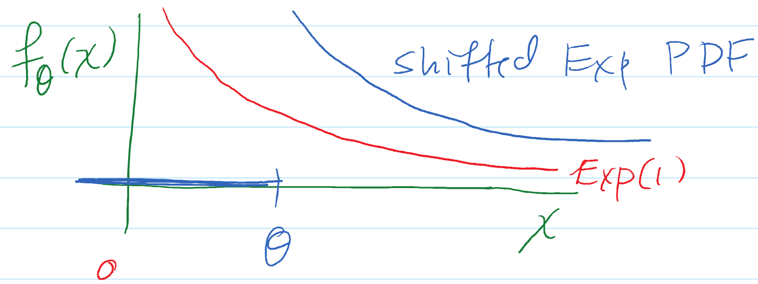
Lecture 20 - Neyman-Pearson Lemma

Ex. $X_n \overset{iid}{\sim}$ Shifted Exponential(θ)



$$f_\theta(x) = e^{-(x-\theta)} \text{ for } x > \theta$$

Consider $H_0: \theta \le 0$ v. $H_a: \theta > 0$

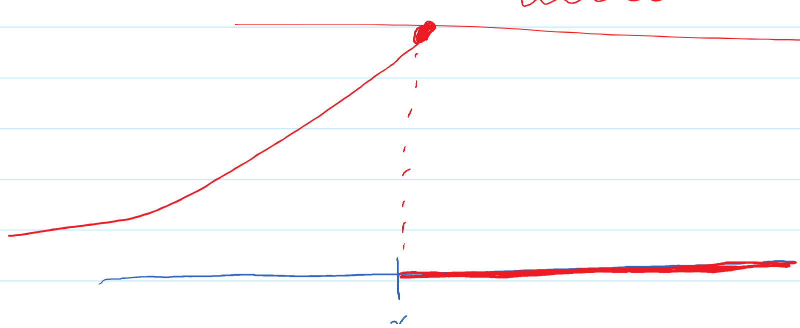
Let's derive the LRT : $\boxed{\lambda = \dfrac{L(\hat{\theta}_0)}{L(\hat{\theta})}}$

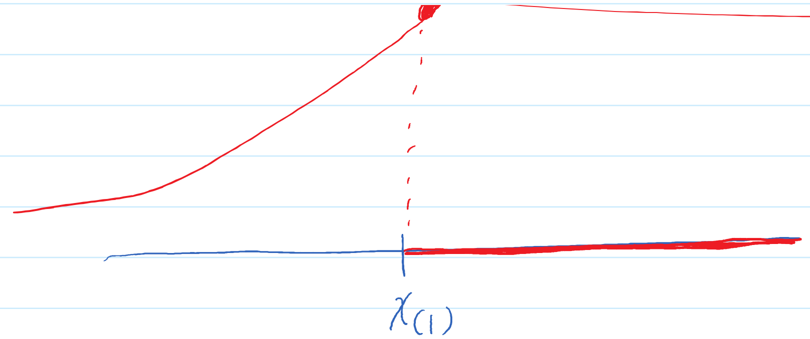
$$R = \{ \underset{\sim}{x} : \lambda(\underset{\sim}{x}) \le c \}$$

$$L(\theta) = \prod_{n=1}^{N} e^{-(x_n - \theta)} \mathbb{1}(x_n > \theta)$$

$$= e^{-\sum_n (x_n - \theta)} \prod_n \mathbb{1}(x_n > \theta)$$

$$\boxed{= e^{-N\bar{x}} e^{N\theta} \mathbb{1}(x_{(1)} > \theta) \propto e^{\theta}}$$

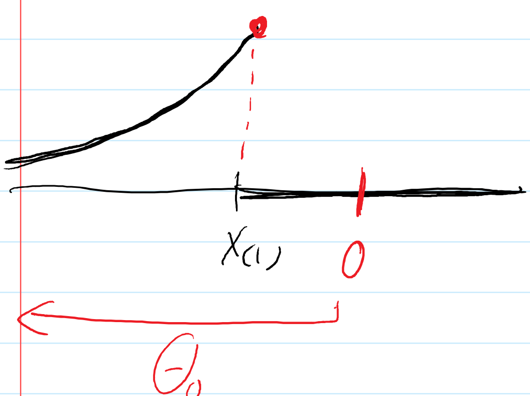




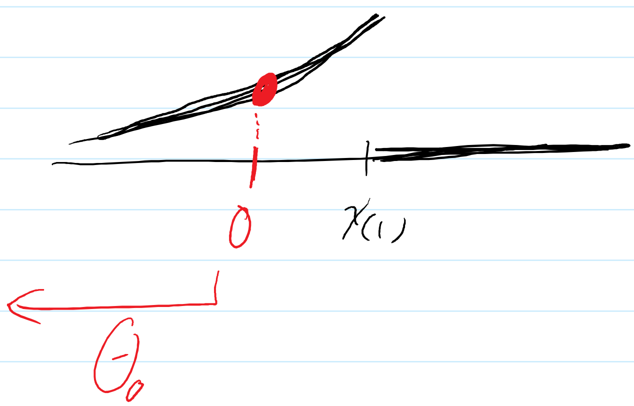
So $\hat{\theta} = x_{(1)}$

Find $\hat{\theta}_0$ = MLE restricted to $H_0 : \theta \leq 0$

Case 1 : $x_{(1)} < 0$ Case 2 : $x_{(1)} \geq 0$



$\hat{\theta}_0 = x_{(1)}$



$\hat{\theta}_0 = 0$

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \begin{cases} \frac{L(x_{(1)})}{L(x_{(1)})} = 1 & x_{(1)} < 0 \\ \boxed{\frac{L(0)}{L(x_{(1)})} \qquad x_{(1)} \geq 0} \end{cases}$$

$x_{(1)} \geq 0$

R

$x_{(1)} > c$

$$\lambda = \frac{L(0)}{L(x_{(1)})} = \frac{\cancel{e^{-N\bar{x}}}\, e^{N0}}{\cancel{e^{-N\bar{x}}}\, e^{N x_{(1)}}} = \boxed{e^{-N x_{(1)}} \leq c} \quad R$$

$$\Leftrightarrow -N x_{(1)} \leq \log c$$

$$\Leftrightarrow \boxed{x_{(1)} \geq \underbrace{-\frac{1}{N} \log c}_{c^*}}$$

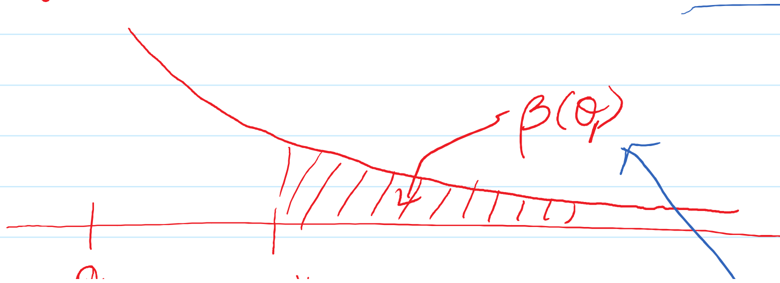
$$\boxed{\text{LRT says reject if } x_{(1)} \geq c^*}$$

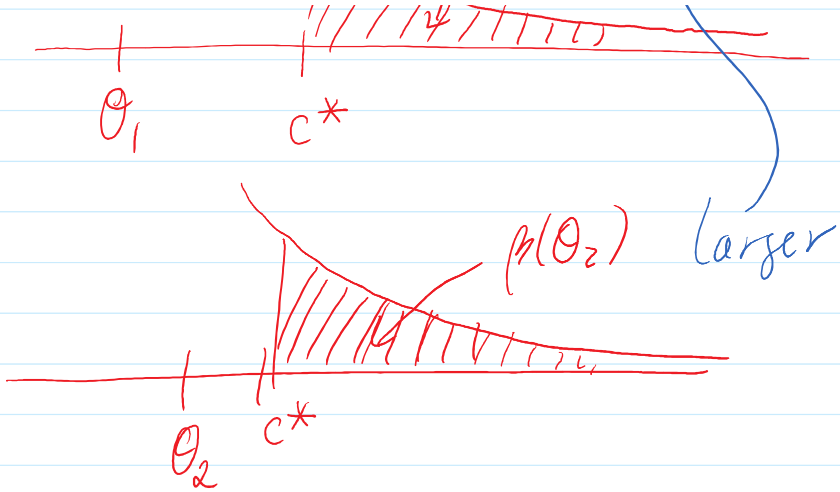
$$H_0: \theta < 0 \quad v. \quad H_a: \theta \geq 0$$

Choose c^* so that

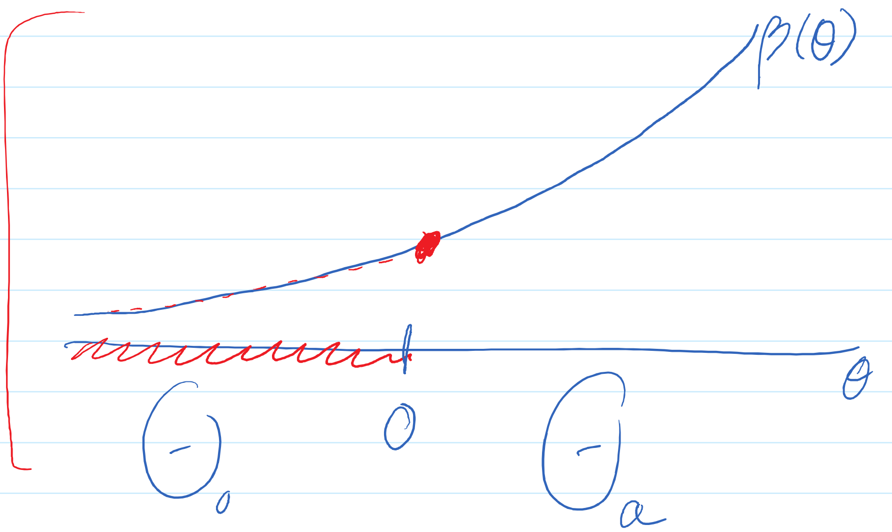
$$\max_{\text{null}\ \boxed{\theta \leq 0}} \underbrace{\mathbb{P}_\theta(\overbrace{X_{(1)} > c^*}^{\text{Prob. I reject}})}_{\beta(\theta)} = \alpha$$

$\beta(\theta) = \mathbb{P}_\theta(X_{(1)} > c^*)$, consider $\theta_1 < \theta_2$





So $\beta(\theta_2) > \beta(\theta_1)$ so β is increasing in θ



$$\max_{\theta \le 0} \beta(\theta) = \beta(0)$$

So I can choose c^* so that

$$\underbrace{\mathbb{P}_{\theta=0}(X_{(1)} > c^*)} = \alpha$$

If $\theta = 0$ $X_n \overset{iid}{\sim} Exp(1)$ so $X_{(1)} \sim Exp(N)$

So if F is the CDF of an $Exp(\lambda)$

then want

$$1 - F(c^*) = \alpha$$

i.e. $\boxed{c^* = F^{-1}(1-\alpha)}$

<u>Defn: Uniformly Most Powerful Test</u> (UMP)

Let $\mathcal{C}$ be a class of tests for the hypothesis

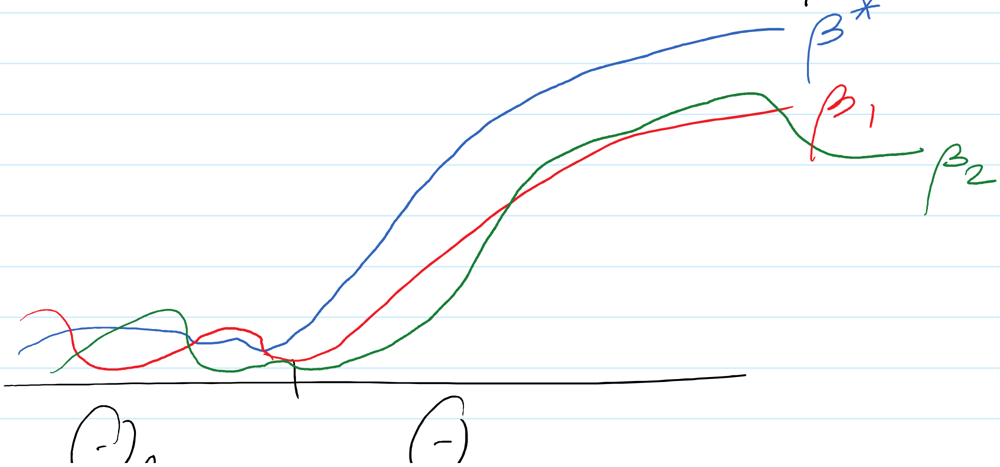
$$H_0 : \theta \in \Theta_0 \quad v. \quad H_a : \theta \in \Theta_a$$

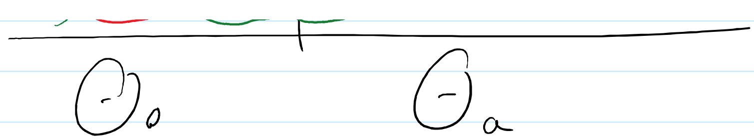
The test w/ the power function β^* is called the UMP test for this class $\mathcal{C}$

if

$$\beta^*(\theta) \geq \beta(\theta) \quad \forall \theta \in \Theta_a$$

for any other test w/ power fn β.





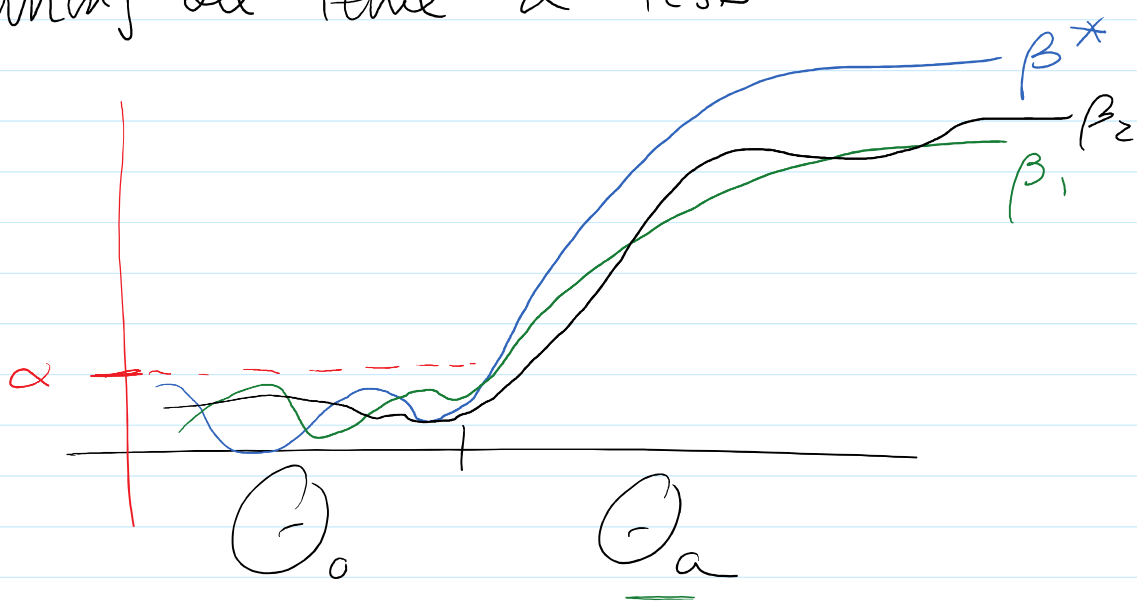
Defn: UMP level/size α test

Recall: size α test : $\max\limits_{\theta \in \Theta_0} \beta(\theta) = \alpha$

level α test : $\max\limits_{\theta \in \Theta_0} \beta(\theta) \le \alpha$

The UMP size α test is the UMP test among all size α tests

The UMP level α test is the UMP test among all level α tests



Consider a simple hypothesis

$$H_0 : \theta = \theta_0 \quad v. \quad H_a : \theta = \theta_a$$

$$\Theta = \{\theta_0, \theta_a\} \; ; \; \Theta_0 = \{\theta_0\}, \; \Theta_a = \{\theta_a\}$$

Consider the LRT:

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \begin{cases} \frac{L(\theta_0)}{L(\theta_0)} = 1 & L(\theta_0) > L(\theta_a) \\ \frac{L(\theta_0)}{L(\theta_a)} & L(\theta_0) < L(\theta_a) \end{cases}$$

So the LRT says reject if

$$\boxed{\begin{aligned} &\lambda = \frac{L(\theta_0)}{L(\theta_a)} \le c \\ \text{i.e. } \; &L(\theta_0) \le c\, L(\theta_a) \\ \text{or } \; &L(\theta_a) \ge k\, L(\theta_0) \quad \text{when } k = 1/c \end{aligned}}$$

We choose c/k so that ← size α LRT

$$\mathbb{P}_{\theta_0}\left(\frac{L(\theta_0)}{L(\theta_a)} \le c\right) = \alpha$$

Punchline: for such simple hypotheses, this LRT size α test is the UMP size α test.

Theorem: Neyman – Pearson Lemma

Consider testing

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \theta = \theta_a$$

with a LRT so that I reject H_0 if

$$\lambda = L(\theta_0) / L(\theta_0) \le C$$

when C is chosen so that our test is size/level α $[\mathbb{P}_{\theta_0}(\lambda \le c) = \alpha]$

Then this is the UMP size α test

Ex. Let $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ ← known

lets test

$$H_0: \theta = a \quad v. \quad H_a: \theta = b \qquad a < b$$

using the LRT

$$\lambda = \frac{L(a)}{L(b)} \propto \exp\left\{ \frac{N(b^2 - a^2) + 2(a-b)N\bar{X}}{2\sigma^2} \right\}$$

So the LRT says to reject if $\lambda \le c$

$$\iff \frac{N(b^2-a^2)+2(a-b)N\bar{X}}{2\sigma^2} \le \log c$$

$$\iff N(b^2-a^2)+2(a-b)N\bar{X} \le 2\sigma^2 \log c$$

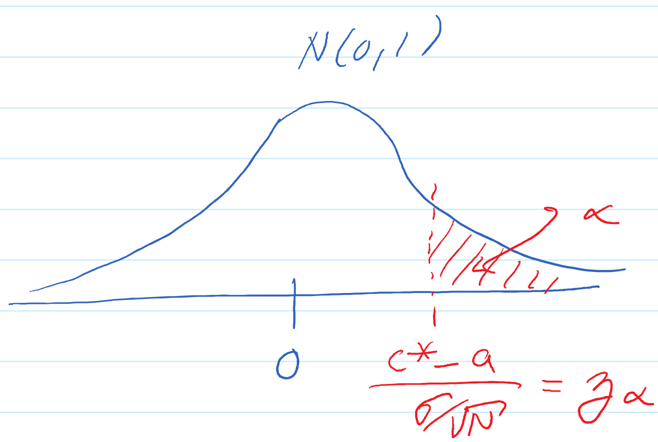
$$\iff \bar{X} \geq \underbrace{\frac{2\sigma^2 \log c - N(b^2-a^2)}{2(a-b)N}}_{c^*}$$

$$\iff \boxed{\bar{X} \geq c^*}$$ choose to make test size α

$$P_{\theta=a}(\bar{X} \geq c^*) = \alpha$$

$$\parallel$$

$$P_{\theta=a}\left(\underbrace{\frac{\bar{X}-a}{\sigma/\sqrt{N}}}_{N(0,1)} \geq \frac{c^*-a}{\sigma/\sqrt{N}} \right)$$



then $\boxed{c^* = a + z_\alpha \sigma/\sqrt{N}}$

This is is the UMP level α test.

Ex. $X \sim Bin(2, \theta)$

flip 2 coins w/ unknown prob of H - θ

test $H_0: \theta = 1/2$ v. $H_a: \theta = 3/4$

Using the LRT

$$\lambda = \frac{L(1/2)}{L(3/4)}$$

reject if $\lambda \le C$.
