WLLN:

If X_ns are uncorrelated and

① $EX_n = \mu$

② $\operatorname{Var} X_n = \sigma^2 < \infty$

then $\bar{X}_N \xrightarrow{p} \mu$.

Ex. $X_n \overset{iid}{\sim} \operatorname{Exp}(\lambda)$ then $EX_n = 1/\lambda$

and so $\bar{X}_N \xrightarrow{p} \frac{1}{\lambda}$

Consider $g(x) = 1/x$ is continuous so by CT mapping theorem

$$\frac{1}{\bar{X}_N} \xrightarrow{p} \lambda.$$

Ex. Consider:

$$S^2 = \frac{1}{N-1} \sum_{n=1}^{N} (X_n - \bar{X}_N)^2$$

If X_n are independent and $EX_n = \mu$, $\operatorname{Var} X_n = \sigma^2$

then $\boxed{E[S^2] = \sigma^2}$

then $\boxed{\mathbb{E}[S^2] = \sigma^2}$

Can show: $S^2 \xrightarrow{P} \sigma^2$ (consistency)

Want to show:

$$P(|S^2 - \sigma^2| \geq \varepsilon) \to 0$$

$$P(|Y - EY| \geq \varepsilon) \leq \frac{\operatorname{Var} Y}{\varepsilon^2}$$

S^2 σ^2

Chebyshev's

By Chebyshev's:

$$0 \leq P(|S^2 - \sigma^2| \geq \varepsilon) \leq \frac{\operatorname{Var} S^2}{\varepsilon^2}$$

If $\underline{\operatorname{Var}(S^2) \to 0}$ then $\underline{S^2 \xrightarrow{P} \sigma^2}$.

If $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then we can show:

$$\operatorname{Var}(S^2) = \frac{2\sigma^4}{N-1} \longrightarrow 0 \quad \text{as } N \to \infty$$

So S^2 consistent for σ^2.

By Cts mapping theorem:

$$\sqrt{S^2} \xrightarrow{P} \sqrt{\sigma^2}$$

i.e. $S \xrightarrow{P} \sigma$.

What about $\hat{\sigma}^2 = \frac{1}{N} \sum_{n=1}^{N} (X_n - \bar{X})^2$? Consistent?

Notice: $\hat{\sigma}^2 = \frac{N-1}{N} S^2$

Notice: $\hat{\sigma}^2 = \underbrace{\frac{N-1}{N}}_{c_N} S^2$

then $\hat{\sigma}^2 = c_N S^2$ and $c_n \to 1$

and so by our algebraic properties:

since $S^2 \xrightarrow{P} \sigma^2$ and $c_n \to 1$

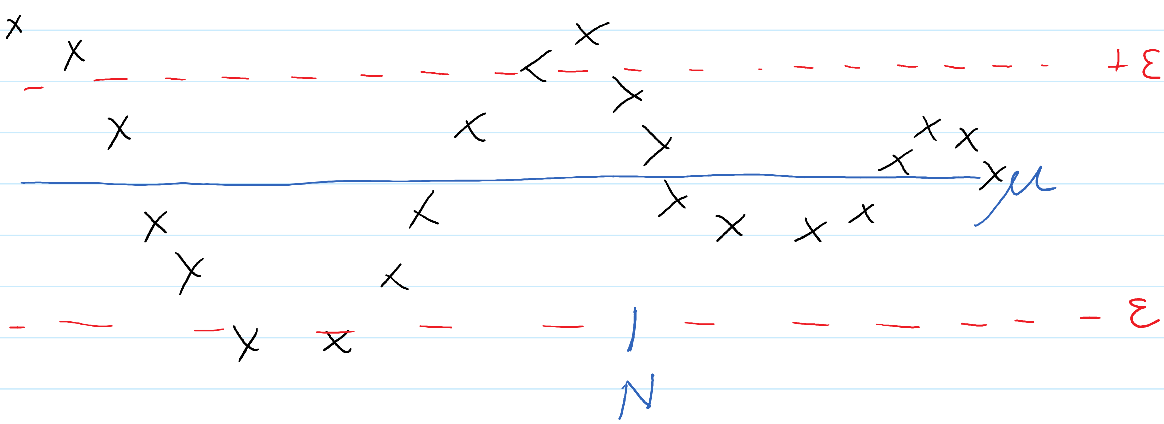
then $\hat{\sigma}^2 = c_N S^2 \xrightarrow{P} 1 \cdot \sigma^2 = \sigma^2$

Theorem: Strong Law of Large Numbers (SLLN)

If $X_n \overset{iid}{\sim}$ w/ $\mathbb{E}X_n = \mu$ and $Var(X_n) = \sigma^2 < \infty$

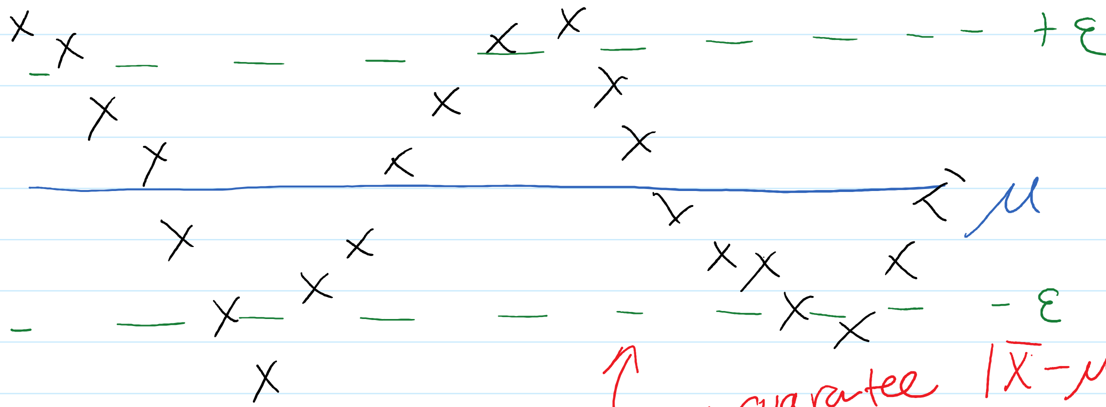
then $\bar{X}_N \xrightarrow{a.s.} \mu$.

Convergence of a set of numbers:



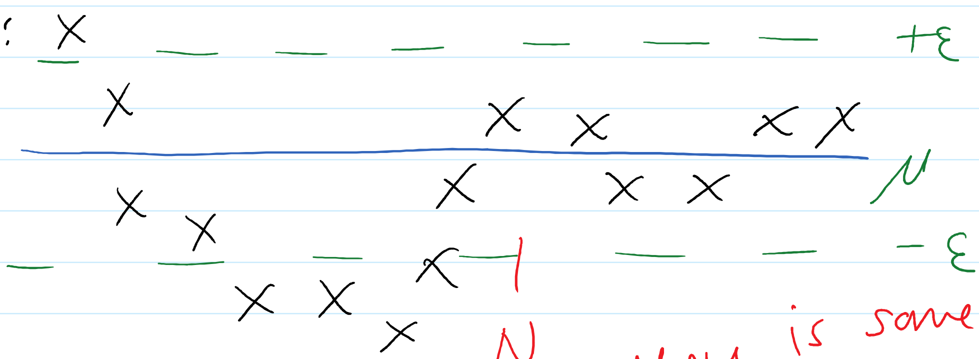
WLLN:

WLLN:



no guarantee $|\bar{X}-\mu| \le \varepsilon$
only that the prob. $|\bar{X}-\mu| \ge \varepsilon$ gets smaller as $n \to \infty$

SLLN:



there is some N so that beyond N $P(|\bar{X}-\mu| \ge \varepsilon) = 0$

Sums of RVs

① $\frac{1}{N} \sum_n X_n \to \mu$ (under some conditions) WLLN, SLLN

② $\sum_n X_n \to \infty$ (in general)

③ $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \xrightarrow{d}$ non-degenerate dist

↳ proper scaling

Theorem: Central Limit Theorem CLT

If I have X_n and they iid w/ $EX_n = \mu$ and $\text{Var}\, X_n = \sigma^2 < \infty$, then

$$\sqrt{N}\left(\frac{\bar{X}_N - \mu}{\sigma}\right) \xrightarrow{d} N(0,1).$$

If I ignore μ and σ then

$$\sqrt{N}\,\bar{X}_N = \sqrt{N}\,\frac{1}{N}\sum_{n=1}^{N} X_n = \frac{1}{\sqrt{N}}\sum_n X_n$$

Intuition:

$$\text{CLT}: \quad \bar{X} \approx N(\mu, \sigma^2/N)$$ ← intro stats version

Can't really write:

$$\cancel{\bar{X}_N \xrightarrow{d} N(\mu, \sigma^2/N)}$$

proper way:

1. $$\sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right) \xrightarrow{d} N(0,1)$$

2. $$\sqrt{N}(\bar{X}-\mu) \xrightarrow{d} N(0,\sigma^2)$$

3. $$\frac{\bar{X}-\mu}{\sigma/\sqrt{N}} \xrightarrow{d} N(0,1)$$

$$\mathbb{E}\bar{X} = \mu$$
$$\operatorname{Var}\bar{X} = \sigma^2/N \Rightarrow sd(\bar{X}) = \sigma/\sqrt{N}$$

Other notation: asymptotically normal

$$\bar{X} \sim AN(\mu, \sigma^2/N)$$

Ex. $X_n \overset{iid}{\sim} \text{Bernoulli}(p)$

$$\mu = \mathbb{E}X_n = p \quad \text{and} \quad \sigma^2 = \operatorname{Var}X_n = p(1-p)$$

$\mu = \mathbb{E}X_n = p$ and $\sigma^2 = \mathrm{Var}X_n = p(1-p)$

CLT says:

$$\sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right) = \sqrt{N}\left(\frac{\bar{X}-p}{\sqrt{p(1-p)}}\right) \xrightarrow{d} N(0,1)$$

In intro stats: $\hat{p} = \bar{X}$ = sample proportion

$$\hat{p} \sim AN\left(p, \frac{p(1-p)}{N}\right)$$

maybe we form a 95% CI for p as

$$\hat{p} \pm 2\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}$$

Ex. $X_n \overset{iid}{\sim} \mathrm{Pois}(\lambda)$

$\mu = \mathbb{E}X_n = \lambda$ and $\sigma^2 = \mathrm{Var}\,X_n = \lambda$

CLT: $$\sqrt{N}\left(\frac{\bar{X}-\lambda}{\sqrt{\lambda}}\right) \xrightarrow{d} N(0,1)$$

Theorem:

Consider a seq of RVs X_n w/ MGFs M_{X_n}

Consider a seq of RVs X_n w/ MGFs M_{X_n} and a limiting RV X w/ MGF M then

if $M_{X_n} \to M$ (ptwse)

then $X_n \xrightarrow{d} X$.

<u>Fact:</u>

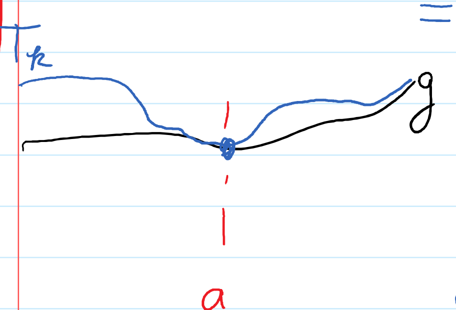
$$\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n \to e^c$$

<u>Theorem: Taylor's Theorem</u>

If I have a fn $g: \mathbb{R} \to \mathbb{R}$ that is k-times diff'able and we consider the k^{th} order Taylor polynomial about $a \in \mathbb{R}$

$$T_k(x) = \sum_{r=0}^{k} \frac{g^{(r)}(a)}{r!}(x-a)^k$$

$$= \frac{g^{(0)}(a)}{0!}(x-a)^0 + \frac{g^{(1)}(a)}{1!}(x-a)^1 + \frac{g^{(2)}(a)}{2!}(x-a)^2 + \cdots$$



$$= g(a) + g^{(1)}(a)(x-a) + \frac{g^{(2)}(a)}{2}(x-a)^2 + \cdots$$

$$\overline{\quad 2 \quad} (x-a)^{2} + \cdots$$

If $R = g(x) - T_k(x)$

then $R \to 0$ as $x \to a$

Punchline: if $x \approx a$ then $g(x) \approx T_k(x)$.

CLT: $\boxed{Y = \sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right)}$, $Y \xrightarrow{d} N(0,1)$.

$Y_n = \frac{X_n - \mu}{\sigma}$ Notice! $EY_n = 0$ and $\operatorname{Var} Y_n = 1$

$$Y = \sqrt{N}\left(\frac{\bar{X}-\mu}{\sigma}\right)$$

$$= \sqrt{N}\left(\frac{\frac{1}{N}\sum_{n=1}^{N} X_n - \frac{1}{N}\sum_n \mu}{\sigma}\right)$$

$$= \frac{\sqrt{N}}{N}\left(\frac{\sum_n X_n - \sum_n \mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{N}} \sum_n \left(\frac{X_n - \mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} Y_n$$

independent

Let $\underline{M}$ be the MGF of Y_n

then

$$\underline{M_Y(t)} = \prod_n M_{Y_n/\sqrt{N}}(t) = \prod_n M(t/\sqrt{N})$$

$$= M(t/\sqrt{N})^N$$

$k=2$ Taylor approx. of M at $a=0$

$$M(t) = \underbrace{M(0)}_{1} + \underbrace{\left.\frac{dM}{dt}\right|_{t=0}}_{0}(t-0) + \frac{\overbrace{\left.\frac{d^2M}{dt^2}\right|_{t=0}}^{1}(t-0)^2}{2}$$

$\mathbb{E}e^0$

$$= 1 + t^2/2$$

$$\underline{M_Y(t)} = M(t/\sqrt{N})^N$$

$$= \left(1 + \frac{t^2}{2N}\right)^N$$

$$\boxed{\lim_n \left(1 + \frac{t^2/2}{N}\right)^N}$$

$$\to e^{t^2/2}$$

MGF of $N(0,1)$.

So MGF of $Y \longrightarrow$ MGF of $N(0,1)$

So MGF of $Y \longrightarrow$ MGF of $N(0,1)$

So $Y \xrightarrow{d} N(0,1)$.