<u>Claim:</u> Let $A(\theta_0)$ be the accept region of a level α test

$$H_0 : \theta = \theta_0 \quad v. \quad H_a : \ldots$$

then

$$C(\underset{\sim}{x}) = \{\theta \mid \underset{\sim}{x} \in A(\theta)\}$$

is a $1-\alpha$ confidence set.

<u>Converse:</u>

If $C(\underset{\sim}{x})$ is a $1-\alpha$ confidence set then for any $\theta_0 \in \Theta$

$$A(\theta_0) = \{\underset{\sim}{x} \mid \theta_0 \in C(\underset{\sim}{x})\}$$

is a α level test for

$$H_0 : \theta = \theta_0 \quad v. \quad H_a : \ldots$$

Two worlds:

HT: Fix some θ_0 want to test if

$$\theta \approx \theta_0 \quad \text{(or not)}$$

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \ldots$$

Determine some rule (R or A) to reject/accept based on data

set of reasonable $\underset{\sim}{x}$s if $\theta \approx \theta_0$

$$A(\theta_0) = \left\{ \frac{\text{set of } \underset{\sim}{x} \text{ where}}{\theta \approx \theta_0 \text{ is reasonable}} \right\} \subset \mathcal{X}$$

CI world: Fix some $\underset{\sim}{x}$, want to determine which $\theta \in \Theta$ are reasonable

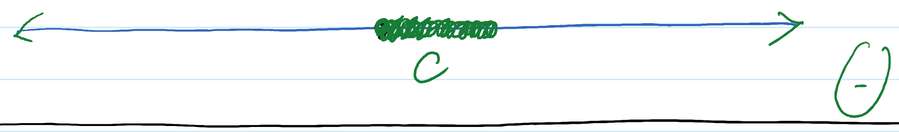
Determine some set C of reasonable θs

$$C(\underset{\sim}{x}) = \left\{ \begin{array}{c} \text{set of reasonable} \\ \theta - \text{given } \underset{\sim}{x} \end{array} \right\} \subset \Theta$$

Test inversion:

H_0 : truth is θ

$$C(\underset{\sim}{x}) = \{ \theta \mid \underset{\sim}{x} \in A(\theta) \}$$



Why does it produce a $1-\alpha$ CR for θ?

Typical steps for test inversion

① Defn a $\underline{\alpha\text{-level}}$ HT for

$$H_0: \theta = \theta_0 \quad v. \quad H_a: \ldots$$

i.e. make some statement about $\underset{\sim}{X}$ where

$$\mathbb{P}_{\theta_0}\big(\underbrace{\underset{\sim}{X} \in R(\theta_0)}_{\text{Some statement}}\big) \le \alpha$$

i.e. make some statement about $\underset{\sim}{X}$ where

$$\boxed{\mathbb{P}_{\theta_0}\big(\underset{\sim}{X} \in A(\theta_0)\big) \ge 1-\alpha}$$

nice if independent of θ_0

② Then

$$C(\underset{\sim}{x}) = \{\theta : \underline{\underset{\sim}{x} \in A(\theta)}\}$$

i.e. "invert" A by isolating θ

i.e. $\boxed{\theta \in C \iff \underset{\sim}{x} \in A(\theta)}$

i.e. $\boxed{\theta \in C \iff \underset{\sim}{x} \in A(\theta)}$

this works b/c

$$\mathbb{P}_\theta(\theta \in C) = \mathbb{P}_\theta(\underset{\sim}{X} \in A(\theta)) \geq 1-\alpha$$

So C is a $1-\alpha$ CR for θ.

$\underline{\text{Ex,}}$ $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ — $\underline{\alpha\text{-level for } H_0: \mu = \mu_0}$

$$A(\mu_0) = \left\{ \underset{\sim}{x} : \left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \right| \leq z_{\alpha/2} \right\}$$

$$= \left\{ \underset{\sim}{x} : \mu_0 - \frac{\sigma}{\sqrt{N}} z_{\alpha/2} \leq \underset{\sim}{\bar{X}} \leq \mu_0 + \frac{\sigma}{\sqrt{N}} z_{\alpha/2} \right\}$$

$\Updownarrow$

$$C(\underset{\sim}{x}) = \left\{ \mu \mid \mu - \frac{\sigma}{\sqrt{N}} z_{\alpha/2} \leq \bar{X} \leq \mu + \frac{\sigma}{\sqrt{N}} z_{\alpha/2} \right\}$$

$$= \left\{ \mu \mid \underbrace{\bar{X} - \frac{\sigma}{\sqrt{N}} z_{\alpha/2}}_{L} \leq \mu \leq \underbrace{\bar{X} + \frac{\sigma}{\sqrt{N}} z_{\alpha/2}}_{U} \right\}$$

$\underline{\text{Fact:}}$ cannot generally guarantee that test inversion gives an interval inversion gives an interval

Typically: two-sided test $\Leftrightarrow$ interval

one-sided $\Leftrightarrow$ one-sided interval

Ex. let $X_n \overset{iid}{\sim} \text{Exp}(\beta) \longrightarrow f(x) = \frac{1}{\beta} e^{-x/\beta}$

lets invert the LRT

$$H_0 : \beta = \beta_0 \quad v. \quad H_a : \beta \neq \beta_0$$

$$\lambda = \frac{L(\beta_0)}{L(\hat{\beta})} = \frac{\frac{1}{\beta_0^N} e^{-N\bar{X}/\beta_0}}{\frac{1}{\bar{X}^N} e^{-N}}$$

$\hat{\beta} = \bar{x}$

$$= \left(\frac{\bar{X}}{\beta_0}\right)^N e^N e^{-N\bar{X}/\beta_0}$$

$$A(\beta_0) = \left\{ \underset{\sim}{x} : \left(\frac{\bar{X}}{\beta_0}\right)^N e^N e^{-N\bar{X}/\beta_0} > c \right\}$$

choose c so that $\mathbb{P}(\underset{\sim}{X} \in A(\beta_0)) \geq 1-\alpha$

How do we make a CI?

$$\rightarrow C(x) = \left\{ \beta : \left(\frac{\bar{X}}{\beta}\right)^N e^N e^{-N\bar{X}/\beta} > c \right\}$$

$$\rightarrow \quad C(\underset{\sim}{x}) = \underbrace{\left\{ \beta : \left(\frac{\bar{x}}{\beta}\right)^N e^N e^{-N\bar{x}/\beta} > c \right\}}_{\boxed{L \le \beta \le U}}$$

Pivotal Quantities

→ Inverting LRT is difficult.

→ Alt. use pivotal quantities.

Defn: Pivotal Quantity

A RV $Q = Q(\underset{\sim}{X}, \theta)$ is called pivotal if the dist of Q doesn't depend on θ.

Idea:

I can create a CR

$$C(\underset{\sim}{x}) = \{\theta : Q \in A\}$$

doesn't depend on θ

$Q = Q(\underset{\sim}{X}, \theta)$

If I can find A so that

$$\underline{P_\theta(Q \in A)} \ge \underline{1-\alpha}$$

then C is a $1-\alpha$ CR for θ

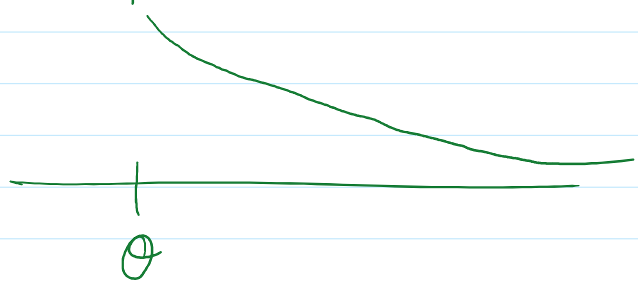
then C is a $1-\alpha$ CR for θ.

reason: $\min_{\theta} \mathbb{P}_{\theta}(\phi \in A) \geq 1-\alpha$

doesn't depend on θ
(same $\forall \theta$)

Particularly nice for Loc-Scale families

Ex. Loc. fam. Shifted Exp.

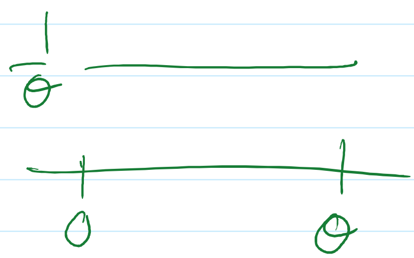


loc fam:

$f_{\mu}(x) = g(x-\mu)$

g free of μ

Ex. Scale Fam. $U(0,\theta)$



Scale fam:

$f_{\sigma}(x) = \frac{1}{\sigma} g(x/\sigma)$

g free of σ

Ex. Loc-Scale $N(\mu, \sigma^2)$

Pivots for I S

Pivots for LS

Type	Pivot
Loc.	$\bar{X}-\mu$
Scale.	$\bar{X}/\sigma$
Loc./Scale	$\frac{\bar{X}-\mu}{\sigma}$

<u>Ex.</u> Let $X_n \overset{iid}{\sim} \text{Exp}(\lambda)$ $\quad f(x)=\lambda e^{-\lambda x}$

$$T=\sum_n X_n \sim \text{Gamma}(N,\lambda)$$

$$Q=\frac{2T}{\lambda} \sim \text{Gamma}(N,2)=\chi^2(2N)$$

Lets find a, b
So that

$$\boxed{P(a \le Q \le b) \ge 1-\alpha}$$

then this defnes a $1-\alpha$ CI for λ.

$$P\left(a \le \frac{2T}{\lambda} \le b\right) \ge 1-\alpha$$

$$\Rightarrow \mathbb{P}\left(\frac{1}{b} \le \frac{\lambda}{2T} \le \frac{1}{a}\right) \ge 1-\alpha \quad \uparrow$$

$$\Rightarrow \mathbb{P}\left(\underbrace{\frac{2T}{b}}_{L} \le \lambda \le \underbrace{\frac{2T}{a}}_{U}\right) \ge 1-\alpha \quad \uparrow$$

So $\left[\frac{2\sum x_n}{b}, \frac{2\sum x_n}{a}\right]$ is a $1-\alpha$ CI for λ.

Practical Steps for using Pivoting

1. get some $Q(\underline{X},\theta)$ whose dist is free of θ
2. find a, b s.t.

$$\mathbb{P}(a \le Q \le b) \ge 1-\alpha$$

3. Solve statement $a \le Q(\underline{X},\theta) \le b$ for θ in middle. to get L, U.

Very general way of pivoting is [in cts case]

recall that $X \sim F_X$ then

$$Q = F(X) \sim U(0,1)$$

$$Q = F_X(X) \sim U(0,1)$$

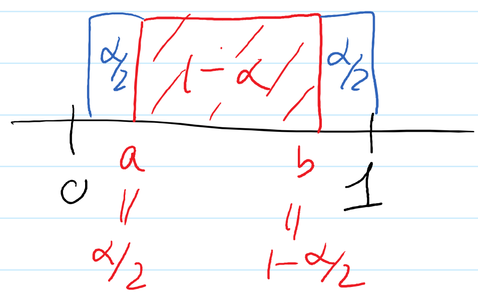
free of θ

a pivot!

① Let $Q = F_X(X)$ be the pivot.

② Let $a = \alpha/2$

$b = 1 - \alpha/2$



③ If I solve

$$\boxed{\alpha/2 \le F_X(X) \le 1 - \alpha/2}$$

for θ in the middle I can get a L and U defining a CI. [easy if F_X invertible as a fn of θ]

Let $g(\theta) = F_X(X)$ as a fn of θ

then I need to solve

$$\alpha/2 \le g(\theta) \le 1 - \alpha/2$$

if g is inc. then

$$g^{-1}(\alpha/2) \le \theta \le g^{-1}(1-\alpha/2)$$

$$\underbrace{g^{-1}(\alpha/2)}_{L} \le \theta \le \underbrace{g^{-1}(1-\alpha/2)}_{U}$$

If g dec. then

$$\underbrace{g^{-1}(1-\alpha/2)}_{L} \le \theta \le \underbrace{g^{-1}(\alpha/2)}_{U}.$$

Theorem: CDF pivot (for cts RVs)

Let T be a stat w/ CDF F_T. depends on θ

Let $g(\theta) = F_T$ as a fn of θ

1. if g inc. in θ then

$$L = g^{-1}(\alpha/2) \quad \text{and} \quad U = g^{-1}(1-\alpha/2)$$

2. g dec. in θ then

$$L = g^{-1}(1-\alpha/2) \quad \text{and} \quad U = g^{-1}(\alpha/2)$$

defines a $1-\alpha$ CI for θ.

Ex.

Assume we have a stat T w/ CDF

$$F_T(t) = \frac{1}{1+e^{-(t-\mu)}}$$ μ unknown param.

lets create a $1-\alpha$ CI for μ.

<u>Notice</u>: $g(\mu) = \dfrac{1}{1+e^{-t}e^{\mu}}$

is decreasing ~~on~~ μ.

If $y = g(\mu) = \dfrac{1}{1+e^{-(t-\mu)}}$

$$\Leftrightarrow \frac{1}{y} = 1 + e^{-(t-\mu)}$$

$$\Leftrightarrow \frac{1}{y} - 1 = e^{-(t-\mu)}$$

$$\Leftrightarrow \log\left(\frac{1}{y}-1\right) = -(t-\mu)$$

$$\Leftrightarrow \mu = \boxed{t + \log\left(\frac{1}{y}-1\right) = g^{-1}(y)}$$

Since g is decreasing then if we let

$$L = g^{-1}(1-\alpha/2)$$

$$L = g^{-1}(1-\alpha/2)$$
$$= t + \log\left(\frac{1}{1-\alpha/2} - 1\right)$$

and

$$U = g^{-1}(\alpha/2)$$
$$= t + \log\left(\frac{1}{\alpha/2} - 1\right)$$

These define a $1-\alpha$ CI for μ.

$$\left[t + \log\left(\frac{1}{1-\alpha/2} - 1\right),\ t + \log\left(\frac{1}{\alpha/2} - 1\right)\right]$$

is a $1-\alpha$ CI for μ.