Defn: Likelihood Ratio Test (LRT)

Consider the hypothesis

$$H_0: \theta \in \Theta_0 \quad v. \quad H_a: \theta \in \Theta_a$$

Let

$$\lambda(\underset{\sim}{x}) = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}$$

$0 \le \lambda \le 1$

$\lambda \approx 0 \Rightarrow$ much more likely $\theta \in \Theta_a$ then $\theta \in \Theta_0$

(probably reject)

$\lambda \approx 1 \Rightarrow$ as likely that $\theta \in \Theta_0$ as $\theta \in \Theta_a$

$\boxed{\text{If } \hat{\theta} \in \Theta_0 \text{ then } \lambda = 1}$

Idea: reject if λ is small enough

LRT has rejection region

$$R = \{ \underline{x} \mid \lambda(\underline{x}) \le c \}$$

some threshold

Choose c to strike a balance between type I and II error.

Ex. Let $X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

Known

$$H_0 : \theta \le a \quad \text{V.} \quad \theta > a$$

Lets form our LRT

$$\Theta_0 = (-\infty, a] \quad \text{and} \quad \Theta_a = (a, \infty)$$

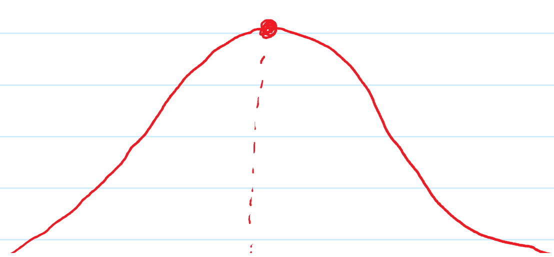
$$\Theta = \mathbb{R}$$

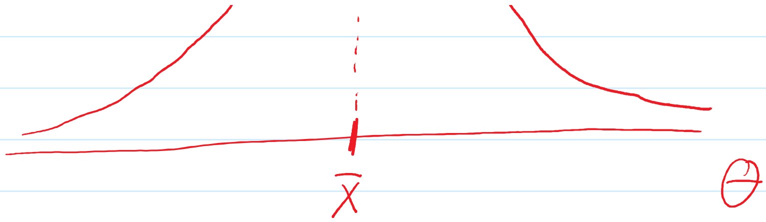
$$L(\theta) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_n - \theta)^2 \right)$$

looks quadratic in θ

$$= (2\pi)^{-N/2} (\sigma^2)^{-N/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=1}^{N} (x_n - \theta)^2 \right)$$

$L(\theta)$

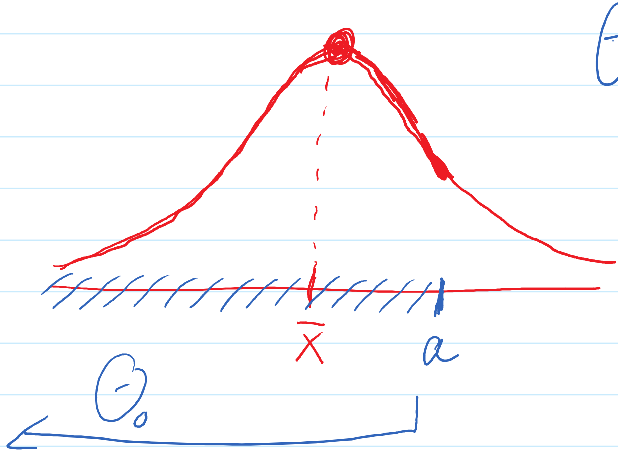




$\uparrow \hat{\theta}$

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{L(\hat{\theta}_0)}{L(\bar{X})}$$

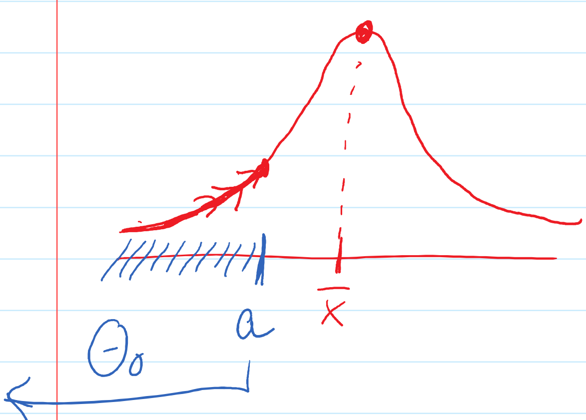
<u>two cases:</u> $\bar{X} \le a$



$$\hat{\theta}_0 = \max_{\theta \in \Theta_0} L(\theta) = \bar{X}$$

So
$$\lambda = \frac{L(\bar{X})}{L(\bar{X})} = 1$$

<u>Case 2:</u> $\bar{X} > a$



$$\hat{\theta}_0 = \max_{\theta \in \Theta_0} L(\theta) = a$$

$$\lambda = \frac{L(a)}{L(\bar{X})}$$

$$\hat{\theta}_0 = \begin{cases} \bar{X} & \bar{X} \le a \\ a & \bar{X} > a \end{cases}$$

$$\lambda = \begin{cases} 1 & \bar{x} \le a \leftarrow \text{never reject} \\ \dfrac{L(a)}{L(\bar{x})} & \boxed{\bar{x} > a} \leftarrow \text{sometimes reject} \end{cases}$$

LRT says reject $H_0 : \theta \le a$ if

$$\frac{L(a)}{L(\bar{x})} < c \quad \text{where} \quad c \in (0, 1)$$

$$\lambda = \frac{L(a)}{L(\bar{x})} = \frac{\cancel{(2\pi)^{-N/2}} \cancel{(\sigma^2)^{-N/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_n (x_n - a)^2\right)}{\cancel{(2\pi)^{-N/2}} \cancel{(\sigma^2)^{-N/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_n (x_n - \bar{x})^2\right)}$$

$$= \frac{\exp\left(-\frac{1}{2\sigma^2} \sum_n (x_n^2 - 2x_n a + a^2)\right)}{\exp\left(-\frac{1}{2\sigma} \sum_n (x_n^2 - 2x_n \bar{x} + \bar{x}^2)\right)}$$

$$= \frac{\exp\left(-\frac{1}{2\sigma^2}\left[\cancel{\sum_n x_n^2} - 2a \overbrace{\sum_n x_n}^{N\bar{x}} + Na^2\right]\right)}{\exp\left(-\frac{1}{2\sigma^2}\left[\cancel{\sum_n x_n^2} - 2\bar{x} \underbrace{\sum_n x_n}_{N\bar{x}} + N\bar{x}^2\right]\right)} \qquad \frac{1}{e^a} = e^{-a}$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(-2aN\bar{x} + Na^2 + \underline{2N\bar{x}^2} - \underline{N\bar{x}^2}\right)\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left[N\bar{x}^2 - 2aN\bar{x} + Na^2\right]\right)$$

$$\lambda = \exp\left(-\frac{N}{2\sigma^2}(\bar{X}-a)^2\right)$$

So LRT says reject when

$$\exp\left(-\frac{N}{2\sigma^2}(\bar{X}-a)^2\right) \le c$$

$$\iff -\frac{N}{2\sigma^2}(\bar{X}-a)^2 \le \log c$$

$$\iff (\bar{X}-a)^2 \ge \frac{-2\sigma^2 \log c}{N}$$

$\bar{X} > a$

$\bar{X} - a > 0$

$$\iff \bar{X} - a \ge \sqrt{\frac{-2\sigma^2 \log c}{N}}$$

$$\iff \bar{X} \ge a + \sqrt{\frac{-2\sigma^2 \log c}{N}}$$

$$\frac{\bar{X}-a}{\sigma/\sqrt{N}} \ge \overbrace{\sqrt{-2\log c}}^{c^*}$$

LRT says reject if $\bar{X}$ is more than c^* s.e.s. bigger than a

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How do we choose c^*?

Maybe want LRT to be a size α test

max $\mathbb{P}$(...

Maybe want LRT to be a size α test

$$\boxed{\max_{\boxed{\theta\in\Theta_0}} \underbrace{P_\theta(\text{reject})}_{\beta(\theta)} = \alpha}$$

$$\underline{P_\theta(\text{reject } H_0)} = \beta(\theta)$$

$X_n \sim N(\theta, \sigma^2)$

$\bar{X} \sim N(\theta, \sigma^2/N)$

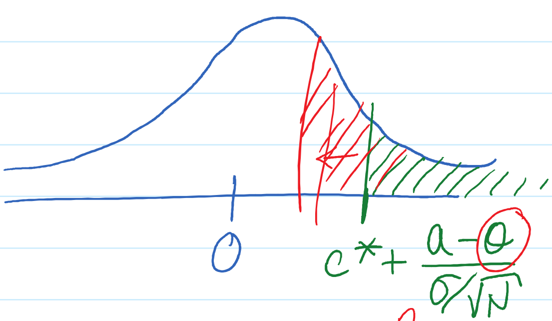
$\dfrac{\bar{X}-\theta}{\sigma/\sqrt{N}} \sim N(0,1)$

$$= P_\theta(\lambda \le c)$$

$$= P_\theta\left(\frac{\bar{X}-a}{\sigma/\sqrt{N}} \ge c^*\right)$$

$$= P_\theta\left(\frac{\bar{X}-a}{\sigma/\sqrt{N}} + \frac{a-\theta}{\sigma/\sqrt{N}} \ge c^* + \frac{a-\theta}{\sigma/\sqrt{N}}\right)$$

$$= P_\theta\Bigg(\underbrace{\frac{\bar{X}-\theta}{\sigma/\sqrt{N}}}_{Z\sim N(0,1)} \ge c^* + \frac{a-\theta}{\sigma/\sqrt{N}}\Bigg)$$



$H_0: \underbrace{\theta \le a}$ v. $H_a: \theta > a$

maximal when θ as large as possible

i.e. $\theta = a$

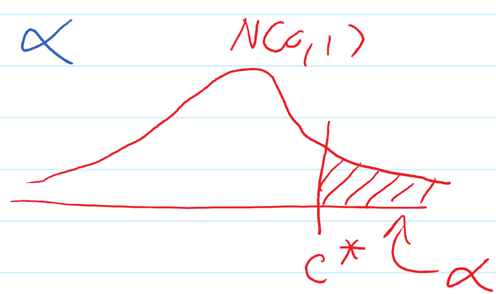
Then $\theta = a$

$$\beta(\theta) = \mathbb{P}_\theta\left(Z \geq c^* + \frac{a-a}{\sigma/\sqrt{N}}\right)$$

$$\boxed{= \mathbb{P}_\theta(Z \geq c^*)}$$

to make a size α test I need to choose c^* so that

$$\mathbb{P}(Z \geq c^*) = \alpha$$



$$F_Z(c^*) = \mathbb{P}(Z \leq c^*) = 1 - \alpha$$

$$c^* = F_Z^{-1}(F_Z(c^*)) = F_Z^{-1}(1-\alpha)$$

Ex.

$$H_0: \theta \leq 5 \qquad H_a: \theta > 5$$

$$x_1 = 1, \quad x_2 = 7, x_3 = 9, x_4 = 5, x_5 = 6$$

$N = 5$

$\sigma = 1.2$

$\alpha = .05$

$$\bar{x} = \frac{1+7+9+5+6}{-} = 5.6$$

5

$$\bar{X} - a = .6$$

$$z \; \frac{\bar{X} - a}{\sigma/\sqrt{N}} = \frac{.6}{1.2/\sqrt{5}} = \dots \; 1.11$$

$$c^* = F_Z^{-1}(1 - .05) = 1.64$$

$z < c^*$ so we don't have evidence to reject.

Defn: Uniformly Most-Powerful Tests (UMP)

Let $\mathcal{C}$ be a collection of tests testing the hypothesis

$$H_0 : \theta \in \Theta_0 \quad v. \quad H_a : \theta \in \Theta_a$$

A test w/ the power function $\beta^*(\theta) = \mathbb{P}_\theta(\underset{\sim}{X} \in R)$ is called the uniformly most powerful test (UMP) [for the collection $\mathcal{C}$] if

$$\beta^*(\theta) \geq \underset{\text{in } \mathcal{C}}{\beta(\theta)} \quad \forall \theta \in \Theta_a$$

for any other test in $\mathcal{C}$ w/ power fn β.