Quiz Problem 2

Let $X \in \mathbb{R}^{N \times P}$ be our design matrix and $Y = X\beta + \varepsilon$ where $\beta \in \mathbb{R}^P$. Let ε have a multivariate normal distribution so that $\varepsilon \sim N(0, \sigma^2 I)$ where $\sigma^2 > 0$ and I is the $N \times N$ identity matrix. Equivalently $Y \sim N(X\beta, \sigma^2 I)$. If our estimate of β is $\hat{\beta} = (X^T X)^{-1} X^T Y$ show that

$$\hat{\beta} \sim N(\beta, \sigma^2 (X^T X)^{-1}).$$

Hint: If $Z \sim N(\mu, \Sigma)$ where $\mu \in \mathbb{R}^N$ and $\Sigma \in \mathbb{R}^{N \times N}$ then if $B \in \mathbb{R}^{M \times N}$ we have $BZ \sim N(B\mu, B\Sigma B^T)$.