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Author(s): Stephen T. Goddard

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RANKING IN TOURNAMENTS AND GROUP DECISIONMAKING*

STEPHEN T. GODDARD†

This paper fully discusses methods for ranking a set of alternatives in the fairest possible way according to a minimum violations criterion. New methods, based on finding paths and circuits in graphs, are presented for ranking participants in round-robin and generalized tournaments, and for consensus and group decisionmaking problems. The objective of the paper is to review existing methods for tackling these problems, and compare them with the new methods, according to a "fairness" criterion, and the amount of computing required to reach a solution. It is shown that the new methods often exceed the existing methods in both fairness and reduced computing requirements. In particular, the new methods are generally more versatile than existing methods. This allows organizations to obtain the fairest ranking of a number of alternatives, according to their managers' or employees' wishes. Particular attention is given to incomplete rankings where insufficient exposure of an individual to some alternatives restricts that individual to ranking only the remaining alternatives.
(TOURNAMENT RANKINGS; DECISION MAKING)

1. Introduction

Everyday, each individual faces the problem of deciding which of a number of alternatives is most suitable for him. In business or other organizations, this problem is compounded immensely due to a number of individuals exerting influence over the same group of alternatives. This means that the choosing or ranking of alternatives in a way that is fairest to all is often a critical problem.

The problem of ranking a set of alternatives arises in a number of ways, most commonly in the cases of tournaments of competing players and democratic groups trying to reach a consensus opinion. The theory concerning ranking has developed along the lines of the method of paired comparisons, where only two alternatives are compared at a time. This situation is especially successful where an individual only needs to show a preference of one alternative over another rather than measuring each alternative numerically. These comparisons can be successfully modelled using a directed graph (digraph) where the vertices represent the alternatives and the edges are drawn between the vertices in the direction of the less preferred vertex.

In this paper the theoretical concepts originated in [1] for finding paths and circuits in graphs have been adapted to form new methods which give "fairer" rankings in many situations than alternative existing methods, according to experimentation. These new methods produce good results mainly due to an effective way of taking into account the quality of the opponents or alternatives, using the p-connectivity matrix discussed in §3.

2. Tournaments

A round-robin tournament is one in which a number of players play each other exactly once and no draws are allowed. The tournament can be represented by a complete asymmetric digraph, with n vertices $v_1, v_2, \ldots, v_n$ representing the players and edges (v_i, v_j) representing player i defeating player j.

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†General Electric Information Services Pty. Ltd., Melbourne, Australia.

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The adjacency matrix of the tournament $A = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge directed from vertex } i \text{ to vertex } j, \\ 0 & \text{otherwise.} \end{cases}$$

The score of a player is defined to be the number of opponents that player defeats, which is equal to the outdegree of the corresponding vertex in the digraph and also the sum of the corresponding row in the adjacency matrix.

Given the results of a completed tournament, the objective is to rank the participants as fairly as possible, without any ties. Many methods of ranking have been proposed, the most evident of them being ranking by scores, in which the players are ranked according to the number of players they have defeated. This method is the most commonly used one for actual tournaments, since it is easily understood and simple to use. Despite this, the method has some disadvantages. Firstly, it is common that ties occur, where two or more vertices have the same outdegree, which must be arbitrarily broken to achieve an ordered ranking. Secondly, and more importantly, this method does not take into account the quality of the opponents, thus allowing a win against a weak player to count as much as a win against a strong player.

Wei [10] and Kendall [5] developed a method which considered the strength of the opponents of each player. They defined the initial strength vector to be the row sums of the adjacency matrix of the tournament. An iterative procedure was then defined in which the $(n + 1)$th strength vector was given by the sum of the strengths of the players that each player defeated from the nth strength vector. Thus, the $(n + 1)$th strength vector is given by the row sums of the $(n + 1)$th power of the adjacency matrix of the tournament.

Use of the nth strength vector avoids the disadvantages discussed earlier to some extent, in that some ties can be broken by choosing n large enough, and the quality of the opponents is taken into account. In some cases, it is not possible to remove ties between vertices because of a type of "symmetry" between them. As shown in [1], if two vertices are similar (that is, if there is an automorphism which maps one vertex onto the other) no method can distinguish between the vertices. Since tournaments are represented by complete asymmetric digraphs, at least three vertices must be involved in an unbreakable tie. The best value of n to use for ranking will be discussed later, but Moon [6] has shown that under certain conditions, letting n tend to infinity produces a constant ranking.

In [1] the p-connectivity matrix has been defined and found to be highly successful in generating Hamiltonian paths and circuits in arbitrary graphs. As will be seen, this matrix can be adapted for tie-breaking between players to produce fair rankings since the strengths of the player and the opponent are given equal weight when calculating the ranking.

Treating the tournament as a graph with adjacency matrix A, the p-connectivity matrix can be analogously defined. Assuming that the 1-connectivity matrix is equal to A and the p-connectivity matrix is (p_{ij}), the $(p + 1)$-connectivity matrix is $((p + 1)_{ij})$, where

$$(p+1)_{ij} = \left\{ \sum_{k=1}^{n} p_{ik} + \sum_{k=1}^{n} p_{jk} \right\} a_{ij}.$$

In [1], the interpretation of the p-connectivity matrix has been extensively discussed, and theory concerning its ability to break ties between vertices in a graph has been presented. In general throughout this paper, a_{ij} represents the degree by which i is preferred over j (in a win-lose situation, a_{ij} is one or zero), and the use of row sums in the definition of the p-connectivity matrix ensures that the players are ranked according to their total preferences.

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At each level, the p-connectivity matrix considers the total relationship of the players to a greater degree until at the $(n-1)$th level, no further connectivity matrices are required for tie-breaking (proved in [1]) since all the results of the tournament have successively been used. Thus, the p-connectivity matrix breaks ties between players depending upon the strengths of their opponents and their overall structure within the tournament, which implies that the row sums can be used to give a fair ranking of the players.

3. Minimum Violations

Previously, consideration has been given to methods which produce "fair" rankings. In this section, "fair" will be quantified in such a way that rankings can be numerically compared so as to choose the better ranking.

Given a ranking of the n players in the tournament, a violation occurs when a player is defeated by another player who is ranked below him. It is generally accepted that the ranking with the minimum number of violations is the fairest ranking since it represents the fewest possible upsets in the tournament.

Using this criterion, values of n and p in the nth strength vector and p-connectivity matrix may be used to produce rankings of the players. Rather than use the limiting case of n tending to infinity as suggested by Moon [6] many rankings should be obtained for all suitable values of n and p, and the ranking with minimum violations should then be chosen. This method is easily programmable and has been run on the CDC 130-730 computing system at the University of Melbourne. It can be shown that the method is efficient in the sense that the total number of operations is a polynomial function in the number of players, rather than an exponential or factorial.

THEOREM 1. *The algorithm terminates in at most $O(n^4)$ operations which means that the total number of operations is bounded above by a polynomial of degree* 4.

PROOF. The p-connectivity matrix may be calculated by $O(n^3)$ operations, since the row sums are found in $O(n)$ steps and then each element in the $n \times n$ matrix must be calculated separately. As mentioned above, this must be done at most $n-1$ times, and then the row sums must be sorted to give the ranking in $O(n^2)$ steps. This ranking must then be checked for violations in $O(n^2)$ operations by looking at half the elements in the adjacency matrix. Similarly, the strength vector may be found in $O(n^3)$ operations and the rest of the method is as above. Thus the algorithm requires at most $O(n^4)$ steps.

Instead of producing a ranking, a normalization procedure may be used to give a numerical measure of the performance of the players. The above algorithm for the p-connectivity matrix may be used replacing row sums with column sums to give an assessment of the players' relative weaknesses. As initially suggested by Ramanujacharyula [7], a ranking and numerical measure could then be determined from the quotients of the players' relative strengths and weaknesses, using normalization of the final vector of row sums.

4. Hamiltonian Rankings

Another method of ranking participants in a tournament is Hamiltonian ranking. If a tournament can be modelled by a completed asymmetric digraph, the participants may be ranked in the order given by a Hamiltonian path (a path passing through each vertex exactly once). Deo [3] has shown that all such tournaments have at least one directed Hamiltonian path. Thus, a Hamiltonian ranking is always available and the problem is to find and choose the most suitable path. The justification of this method is that each player has defeated the player ranked immediately below him.

However, Hamiltonian rankings can easily be misused to give wrong impressions of

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players' abilities. In many tournaments, a Hamiltonian circuit also exists in the digraph, and thus valid Hamiltonian rankings exist in which every player can be ranked either first or last! Possibly the best way of using this technique would be to enumerate all Hamiltonian paths in the tournament and then find the most acceptable according to a minimum violations criterion. Unfortunately, there is no efficient method known for total enumeration of Hamiltonian paths in digraphs, thus making this method unreasonable for large tournaments.

The method defined in [1] for finding Hamiltonian paths and circuits in arbitrary graphs builds a path sequentially by choosing that vertex which will minimize the remaining connections in the graph. In terms of a tournament, the objective is to start at the likely overall winner and build the path in one direction only towards the nodes that are "least connected", that is, the players that have lost the least number of times to the strongest players. By the nature of the method of choosing vertices, the Hamiltonian path that is obtained should give a fair ranking. If there is more than one likely overall winner, the method should be tried for all such vertices and the resultant paths compared to find the fairest ranking.

An advantage of this technique over the use of scores and the p-connectivity matrix is its applicability to tournaments other than classical round-robin tournaments. If a tournament is incomplete, any method which uses scores will be of little use. However, the tournament can still be modelled by a digraph and the heuristic method described above may be used to generate a fair ranking, although much care must be taken when choosing a starting vertex.

In the case of more than one (complete or incomplete) tournament, this method is still relevant. The set of n vertices may still represent the players, but the drawing of an edge from i to j if player i defeats player j may result in a multigraph. Although this in itself does not affect the existence of a Hamiltonian path, the adjacency matrix must be redefined where a_{ij} is equal to the number of edges directed from i to j, so that the p-connectivity matrix will consider the results of these extra encounters.

5. Generalized Tournaments

Until now, the tournaments have been represented graphically where the only outcome possible for an encounter between players i and j was either a win for i or a win for j. This is a limitation because in many tournaments, a draw is allowable between i and j, and indeed in some tournaments the result may reflect the magnitude of the defeat.

To allow for these extensions, a generalized tournament has been defined in which directed edges (i,j) and (j,i) join all vertices i and j, where $i \neq j$, and a weight α_{ij} is associated with each edge $e(i,j)$ such that $\alpha_{ij} + \alpha_{ji} = 1$. Thus a draw may be represented by $\alpha_{ij} = \alpha_{ji} = \frac{1}{2}$, or a strong victory for i over j may result in $\alpha_{ij} = 0.8$ and $\alpha_{ji} = 0.2$.

Although the Hamiltonian ranking methods of §4 fail (unless all α_{ij}'s are 1 or 0), the matrix methods may still be used.

6. Results

All the algorithms that have been discussed have been programmed in FORTRAN and run on the CDC 130-730 computing system at the University of Melbourne. All the methods were compared over approximately 30 tournaments using the criterion of minimum violations. From these, the new methods, the p-connectivity matrix and heuristic Hamiltonian ranking, produce better results than the older methods over 80% of the time. The results from the generalized tournaments considered showed that the p-connectivity matrix method was more effective than either ranking by scores or strength vectors, against at least 80% of the time.

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The following examples are actual chess tournaments in which the ability to consider the strength of the opponents gives the new methods a great advantage when considering minimum violations. It may be noted that these are generalized tournaments in which

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ defeats } j, \\ \frac{1}{2} & \text{if } i \text{ draws with } j, \\ 0 & \text{if } i \text{ is defeated by } j. \end{cases}$$

Players

	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
1	0	1	0	1	1	1	0	1	1	1	1	1	1	10
2	0	0	$\frac{1}{2}$	1	0	1	1	1	1	1	1	1	1	$9\frac{1}{2}$
3	1	$\frac{1}{2}$	0	1	1	1	0	0	1	$\frac{1}{2}$	1	1	1	9
4	0	0	0	0	1	0	0	1	1	1	1	1	1	7
5	0	1	0	0	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	1	0	7
6	0	0	0	1	$\frac{1}{2}$	0	1	0	1	1	$\frac{1}{2}$	1	1	7
7	1	0	1	1	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	6
8	0	0	1	0	0	1	1	0	$\frac{1}{2}$	0	1	0	1	$5\frac{1}{2}$
9	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	1	1	5
10	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	1	0	0	0	1	1	4
11	0	0	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	1	$3\frac{1}{2}$
12	0	0	0	0	0	0	1	1	0	0	$\frac{1}{2}$	0	1	$3\frac{1}{2}$
13	0	0	0	0	1	0	0	0	0	0	0	0	0	1

	Ranking	Violations
Scores	1 2 3 4 5 6 7 8 9 10 11 12 13 –	14
Strength Vector	1 3 2 5 7 6 4 8 9 10 12 11 13 –	14
p-connectivity matrix	1 2 3 5 6 7 4 8 9 10 12 11 13 –	13

FIGURE 1

Players

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	0	1	1	1	1	1	1	11
2	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	11
3	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	10
4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	10
5	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	1	0	$\frac{1}{2}$	1	1	1	1	1	1	1	$9\frac{1}{2}$
6	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$9\frac{1}{2}$
7	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	1	$8\frac{1}{2}$
8	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	8
9	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	7
10	1	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	1	$6\frac{1}{2}$
11	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	1	0	6
12	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	1	1	0	1	0	0	0	6
13	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	1	1	1	$5\frac{1}{2}$
14	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	1	1	$5\frac{1}{2}$
15	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	0	0	1	0	0	0	1	$3\frac{1}{2}$
16	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	1	1	0	0	0	0	$2\frac{1}{2}$

	Ranking	Violations
Scores	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 –	15
Strength Vector	2 1 3 4 5 6 7 8 12 9 10 11 14 13 15 16 –	13
p-connectivity matrix	2 1 3 4 6 5 7 8 9 12 10 11 14 13 15 16 –	12

FIGURE 2

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	Players 1	2	3	4	5	6	7	8	9	10	Total
1	0	0	1	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$6\frac{1}{2}$
2	1	0	0	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	1	1	6
3	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	6
4	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$	1	6
5	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	5
6	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	1	$4\frac{1}{2}$
7	0	1	0	1	0	$\frac{1}{2}$	0	1	1	0	$4\frac{1}{2}$
8	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	$\frac{1}{2}$	1	$2\frac{1}{2}$
9	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$2\frac{1}{2}$
10	0	0	0	0	0	0	1	0	$\frac{1}{2}$	0	$1\frac{1}{2}$

	Ranking											Violations
Scores	1	2	3	4	5	6	7	8	9	10	–	6
Strength Vector	1	2	4	3	7	5	6	9	8	10	–	6
p-connectivity matrix	1	4	2	3	5	7	6	9	8	10	–	5

FIGURE 3

7. Consensus and Group Decisionmaking

The general form of group decisionmaking is the ranking of a number of alternatives without ties so as to agree best with the wishes of individuals in a democracy. This situation has many practical applications such as elections, choosing job candidates, and so on. A number of methods exist which produce such a ranking, but in many cases, the ranking seems unfair.

The simplest of these methods is "majority rule" in which one alternative is ranked over another if and only if the majority of individuals agree with that particular ranking. This method fails when confronted with the voter's paradox, which makes ranking impossible according to the defined rules. An example of the voter's paradox with three alternatives A, B and C to be ranked is:

40%	rank	C	A	B
40%	rank	B	C	A
20%	rank	A	B	C.

In this case, majority rule implies that A ranks ahead of B, B ranks ahead of C and C ranks ahead of A, for which no valid ranking exists.

A more applicable method of ranking is known as the Borda count. Given the rankings of all individuals in the democracy, the Borda count of an alternative is the total number of alternatives ranked below it. The alternatives are then ranked in decreasing order of their Borda counts.

Roberts [8] has given an example in which he considers the ranking obtained from the Borda count to be unfair. Five individuals gave the following rankings for six alternatives:

P_1	P_2	P_3	P_4	P_5
x	x	x	x	y
y	y	y	y	z
z	z	z	z	u
u	u	u	u	v
v	v	v	v	w
w	w	w	w	x

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for which the Borda count ranking is $yxzuvw$, although four out of the five individuals ranked x first. This ranking also does not agree with the concept of minimum violations introduced in §3. Two new methods which appear to give fairer rankings are now presented.

The first method is an extension of the Borda count method which applies the p-connectivity matrix procedure to the matrix found by using the number of times one alternative is ranked over another. For example, in this case x is ranked over y four times and y is ranked over x once. Thus, in the matrix, the element (x, y) is 4, and (y, x) is 1. Thus the matrix is

$$\begin{array}{c@{\;}c@{\;}c}
 & \begin{matrix} x & y & z & u & v & w \end{matrix} & \\
\begin{matrix} x \\ y \\ z \\ u \\ v \\ w \end{matrix} & \begin{bmatrix} 0 & 4 & 4 & 4 & 4 & 4 \\ 1 & 0 & 5 & 5 & 5 & 5 \\ 1 & 0 & 0 & 5 & 5 & 5 \\ 1 & 0 & 0 & 0 & 5 & 5 \\ 1 & 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 20 \\ 21 \\ 16 \\ 11 \\ 6 \\ 1 \end{matrix}
\end{array}$$

The usual Borda count ranking is found using the row sums. Applying the p-connectivity procedure of §2 to this matrix yields the ranking $x\,y\,z\,u\,v\,w$, which is considered fairest according to both Roberts and the minimum violations criterion.

A possible disadvantage of this method is its inability to account for the magnitude of the individuals' preferences. For example, x ranked one position higher than y is treated the same as y ranked five places higher than x. The second method takes this "quality" into account.

Another matrix is formed where position (x, y) is the total number of ranking positions that x ranks above y, and (y, x) is the total number of places that y ranks above x (again avoiding the use of negatives). In the above example, this matrix is

$$\begin{array}{c@{\;}c@{\;}c}
 & \begin{matrix} x & y & z & u & v & w \end{matrix} & \\
\begin{matrix} x \\ y \\ z \\ u \\ v \\ w \end{matrix} & \begin{bmatrix} 0 & 4 & 8 & 12 & 16 & 20 \\ 5 & 0 & 5 & 10 & 15 & 20 \\ 4 & 0 & 0 & 5 & 10 & 15 \\ 3 & 0 & 0 & 0 & 5 & 10 \\ 2 & 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 60 \\ 55 \\ 34 \\ 18 \\ 7 \\ 1 \end{matrix}
\end{array}$$

Again, the p-connectivity matrix procedure may be applied and the best ranking according to the criterion of minimum violations can be found. For the above example, the ranking $x\,y\,z\,u\,v\,w$ is again obtained.

Criteria other than minimum violations have also been discussed in the literature. The most popular of these was initiated by Kemeny and Snell [4] and was based on a measure of the distance between two rankings. The best ranking according to this criterion, given a particular distance measure, is the one which minimizes the total absolute distance. However this does not necessarily agree with the minimum violations criterion.

Cook and Seiford [2] presented the following example of ten committee members ranking five alternatives a, b, c, d and e as:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
a	c	e	a	d	a	b	a	b	d
e	e	b	c	a	d	e	d	e	c
c	b	a	e	e	c	d	b	a	a
b	a	d	b	c	b	a	c	d	b
d	d	c	d	b	e	c	e	c	e

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where their distance measure $d(P_i, P_j) = \sum_{k=1}^{5} |r_{ik} - r_{jk}|$, where r_{ik} is the rank of the kth alternative ranked by P_i, produced the ranking $[a\,e\,b\,c\,d]$ with 44 violations whereas both methods using the p-connectivity matrix produced $[a\,e\,b\,d\,c]$ with 40 violations.

Until now, the only ranking problems that have been considered are ordinal problems, in which each individual simply ranks all the alternatives in order of preference. In cardinal problems, not only a ranking but a degree of preference of one alternative over another must also be specified.

A slight generalization of the latter method using the p-connectivity matrix can easily account for this type of problem. If the degrees of preference were given numerical values, the method could count the sums of the degrees rather than the total number of ranking positions. Many other similar extensions exist for this method.

8. Decisionmaking Using Incomplete Rankings

In many cases, individuals are not able or allowed to rank all the alternatives that are available. Instead, they may rank only those alternatives that are particularly relevant to them. Although this is a realistic situation, the methods of §7 are unable to cope with the incomplete matrices that result.

Rogers [9] has considered this problem through the use of an "agreement matrix" A, where the (i, j) element is defined as the number of individuals who have ranked alternative i ahead of alternative j. Although Rogers' method is heuristic, he makes no attempt to show that it is efficient (that is, not exponential) with respect to the number of alternatives to be ranked.

A method was developed using Rogers' agreement matrix and the p-connectivity matrix which is efficient in the sense defined in §3, and takes into account the incomplete nature of the matrix.

Initially, the agreement matrix was subjected to the p-connectivity technique as in previous algorithms. However, instead of purely using the row sums for ranking, they were each divided by the number of times that alternative was ranked. This allowed for the incompleteness of the agreement matrix. The alternative with greatest adjusted row sum was ranked first. This alternative was then removed by deleting its row and column in the agreement matrix. By suitably adjusting the remaining row sums and number of rankings, the method could be used until all alternatives were ranked.

This method has been programmed and used on a number of examples, including the following example devised by Rogers [9]:

	Individual				
Alternative	1	2	3	4	5
A	1				7
B	5	1		6	5
C	4	2	5	5	
D		3	4	4	3
E	2	4		3	4
F			3		6
G			2	2	2
H	3		1	1	1

The ranking obtained using the above method was $H\,G\,D\,E\,F\,A\,C\,B$ which agreed with Rogers' optimal (minimum violations) ranking found by his method.

As for the similar algorithms presented in the earlier sections, this method is very flexible and can also be used with a weighted agreement matrix to account for the magnitude of an individual ranking.

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9. Applications to Organizational Decisionmaking

The previous sections have shown, in some detail, methods for obtaining fair rankings in a number of situations which may be of particular interest in the field of organizational decisionmaking.

For instance, an organization with a hierarchical management structure can make use of decisionmaking techniques for incomplete rankings. The nature of the hierarchical structure ensures that some managers will not have sufficient exposure to rank all alternatives. However the method in §8 gives the highest level manager a fair representation of rankings from all people in the organization below him. This can be extremely useful when trying to analyse input from many employees regarding proposed alternatives to working conditions and other important areas.

Alternatively, an organization with a matrix management structure often finds it difficult to rank the priorities of all projects passing through the service departments. The method of §7 offers a way of measuring the combined priority of all the projects, according to the service departments. This can be useful in allocating time of both managers and employees to particular projects.

10. Summary

The purpose of this paper has been two-fold in providing a review of existing techniques for ranking problems that are available, and in detailing new methods which are based on recent theory and have been shown through experiments to give superior results.

A number of different ranking problems have been considered—tournaments, generalized tournaments, consensus group decisionmaking and the use of incomplete results.

§9 has briefly suggested how these methods can be effectively used by organizations in their own decisionmaking processes.

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