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# How random are team sports leagues?

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ABSTRACT

Competitive sports success is often attributed to a blend of luck and skill, with ongoing debates about the precise balance. This study analyzed historical rankings of major team sports leagues (three football, two basketball, and one ice hockey) to assess the extent to which competition outcomes are influenced by randomness. We created an agent-based model for sports teams, where team strength derives from the cumulative strengths of players. Disparities in team strengths influence game outcomes in the numerical simulations, mirroring Bonabeau's concept of self-organized hierarchies in combat among animals. Like the archetype, our model exhibits a phase transition between deterministic and random phases. We calibrated the model using collected comprehensive sports data and discovered that real leagues operate near the boundary between the phases, suggesting a balanced impact of random factors and players' abilities, with a slight advantage for the latter.

## 1. Introduction

Sports events would not hold their appeal if their outcomes were solely determined by chance or completely foreseeable. The essence of sports lies in the unpredictability, where despite our attempts to forecast winners based on historical data, each new result remains uncertain. This naturally prompts an inquiry into the degree of randomness inherent in sports competitions. This study delves into this very question across diverse sports leagues.

Formally established in the 19th century, sports leagues have become ubiquitous across various sports. The growing popularity of sports has captured the scientific community's attention, leading to the collection of comprehensive data from team-based competitions. While pioneering works primarily relied on clinical trials and surveys [1–3], the increasing wealth of data has enabled researchers to employ more sophisticated methodologies from statistics, mathematical modeling, and machine learning [4–8]. These studies often aim to elucidate or predict diverse aspects, ranging from single event outcomes [9–12], player performance [12–14], end-of-season standings [15–17], to team scores [18]. Other investigations delve into modeling and quantifying the competitiveness within team sports leagues [19–25], devising unambiguous ranking methods [26,27], or gauging the impact of home advantage prior [28,29] and during the COVID-19 pandemic [30]. A recent study employed network science techniques to conduct a temporal analysis of football passing networks for F.C. Barcelona and their adversaries [31].

It is widely recognized that triumph in competitive sports is a blend of both luck and skill, although debates about the precise proportions of these components persist [32,33]. Models in sports statistics that view matches as random processes boast a rich history, exemplified by the Poisson model [34–37], random point processes [38], Bayesian networks [39], and lately agent-based models [40–42]. The issue of predicting sports outcomes holds significance not only for the betting market but also for the broader sports economy. As a result, it forms the basis of a dedicated research field known as *competitive balance*. Alwell [43] defines

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competitive balance as "a measure of league parity, a condition of being equal and fair in a contest". In contrast, Fort and Maxcy [44] outline two avenues within this field: one centered around competitive balance itself and another focused on *Contest Outcome Uncertainty* (COU). The exploration of competitive balance seeks to quantify how closely matched teams are within a season (concentration level) and the extent to which certain teams persistently triumph over multiple seasons (dominance level) [45]. COU research examines the impact of competitive balance on fans and tests *the Uncertainty of Outcome Hypothesis* (UOH), which posits that spectators prefer events with relatively uncertain outcomes [46,47].

This study is dedicated to unraveling the extent of randomness and unpredictability inherent in team sports leagues. To this end, we analyze historical ranking tables across six prominent team sports leagues: three football (the Premier League, Bundesliga, La Liga), two basketball (Liga ABC, Basketbol Süper Ligi), and one ice hockey (Svenska Hockeyligan). Our investigation delves into the predictability of outcomes through the computation of the mean absolute changes in the team ranks $\overline{\rho}$ between consecutive seasons. This assessment encompasses both the entire league and the individual rankings of each team. In the case of a league, $\overline{\rho}$ corresponds to the adjusted churn [48], a recognized metric in the realm of competitive balance [45]. Calculating average rank changes $\overline{\rho}$ for each individual rank gives us a more comprehensive understanding, improving the insights gained from the conventional approach. This approach not only sheds light on the dynamic interplay between teams, but also retains a holistic view of seasonal shifts.

We introduce an agent-based model that simulates team sports leagues to further interpret our findings. The model is inspired by Bonabeau's model of self-organizing hierarchies [49]. Our algorithm encapsulates key events that occur during real league competitions, including matches where team strength differentials influence the probability of victory, player exchanges between teams, and dynamic player skill levels over time. The impact of player and team strength on results can be adjusted using a single value, the determinism parameter $\beta$. Other model hyperparameters include the number of teams in a league, the number of players per team, and the number of rounds within an observed season.

The key contributions and findings of our study are as follows:

- adapting the Bonabeau model for modeling team sports leagues,
- discovering a continuous transition between the deterministic and random phases in our model,
- finding that real football, basketball, and ice hockey leagues operate in the critical region near the boundary between the phases.

The remainder of the manuscript is organized as follows. Section 2 outlines the Bonabeau hierarchy model, Section 3 introduces our team sports league model, Section 4 presents our model's numerical analysis and discusses factors influencing system behavior, Section 5 compares it to real data, and Section 6 concludes our studies, discusses the limitations, and suggests future work.

## 2. Bonabeau model

The Bonabeau hierarchy model [49], introduced by Eric Bonabeau, provides a framework for understanding how hierarchical structures emerge organically within groups through self-organization rather than top-down imposition. Inspired by complex systems theory and agent-based modeling, the model explains how social and organizational stratifications can arise from decentralized interactions and competition among individuals or agents within a network. This emergent view of hierarchy aligns with real-world systems, where structures develop and adapt through continuous, dynamic interactions among participants.

In the Bonabeau model, each agent competes for limited resources, status, or influence, creating an environment where differences in power and rank naturally emerge. At the core of the model is the concept of winning and losing interactions. The agents, which are randomly distributed over a regular lattice with a density $\rho$, are characterized by a time-dependent variable $h_i(t)$, where $i = 1, 2, 3, \dots N$, which is called the status. Initially, $h_i(0) = 0$ for all agents (egalitarian situation).

At each time step, a randomly chosen agent $i$ moves into a neighboring site of the lattice. If the target site is empty, the agent takes it. If it is already occupied by an agent $j$, a fight decides which of them will stay in this place (the loser goes to the free site). Agent $i$ wins against agent $j$ with the probability

$$P_{ij}(t) = \frac{1}{1 + e^{\eta(h_j(t) - h_i(t))}}, \tag{1}$$

where $\eta > 0$ is a free parameter. The winner also increases its status by 1, while the loser decreases by $F \geq 1$. After $N$ movement processes (with or without combat), all agents update their status by a relaxation factor:

$$h_i' = (1 - \mu) h_i, \tag{2}$$

where $\mu \in (0, 1)$, and $h_i'$ is the new value of the agent's status. Interestingly, Eq. (1) closely resembles the formula for the expected score in the Elo rating system [50,51]. This similarity is probably coincidental, as Eric Bonabeau did not mention the Elo rating system in his publications. The fundamental difference between the Elo method and the model in question lies in the way ratings are updated after the encounter. In the Bonabeau model, the difference between the winner and the loser updates is constant and equal to $\Delta h_{win} - \Delta h_{los} = 1 - (-F) = F + 1$, while in the Elo method, this difference depends nonlinearly on the rankings of rivals. If a powerful competitor faces a very weak one and wins the match as expected, the rankings of both players will not change. Conversely, if an underdog unexpectedly wins the encounter, his status will increase significantly, and the ranking points of the favorite will be reduced. In the Elo method, players can increase their status only by winning against a higher-rated opponent. On the other hand, the relaxation of statuses (Eq. (2)) occurs only in the Bonabeau model.

**Table 1**
Summary of model parameters (top six rows) and pivotal variables.

| Symbol | Definition | Values |
|---|---|---|
| $K$ | Number of players in a team | $K \in \mathbb{N}$ |
| $L$ | Player base size multiplier | $L > 1$ |
| $R$ | Number of rounds in a season | $R \in \mathbb{N}$ |
| $S$ | Number of seasons in one simulation | $S \in \mathbb{N}$ |
| $T$ | Number of teams in a league | $T \in \mathbb{N}$ |
| $\beta$ | Parameter of determinism | $\beta \in \langle 0; 1 \rangle$ |
| $h_k$ | Strength of a player $k$ | $h_k \in \langle 0; 1 \rangle$ |
| $H_m$ | Strength of a team $m$ | $H_m \in \langle 0; K \rangle$ |
| $r_i(s)$ | Rank of team $i$ after season $s$ | $r_i(s) \in 1, 2, \ldots, T$ |
| $\rho_i(s)$ | The absolute rank change of team $i$ between seasons $s$ and $s+1$ | $\rho_i(s) \in 0, 1, \ldots, T-1$ |

Using the mean-field approximation, it can be shown that the system undergoes a phase transition between an egalitarian and a hierarchical phase. If the density of agents is below the critical value, i.e., $\rho < \rho_c = \frac{2\mu}{\eta(1-\mu)(1+F)}$ then interactions between agents occur too rarely and the relaxation factor $(1-\mu)$ has a dominant influence on the system. Otherwise, the model dynamics are driven by frequent fights, in which agents with higher status gain an advantage in future combats. This feedback loop, where success breeds further success, leads to a self-organized system where hierarchy becomes more rigid over time.

In this way, the Bonabeau hierarchy model reflects real-world situations, such as in organizational or social settings, where competition and interaction shape leadership roles and status. The model has been widely applied in fields ranging from organizational science [52] to social behavior studies [53] and computational biology [54], offering a powerful tool for analyzing decentralized systems where structured order arises from individual interactions rather than explicit design.

## 3. Team sports league model

Since hierarchies are inherent to professional sports, it is appealing to apply the Bonabeau model to such systems. This is also justified by the fact that the Elo rating system [50], based on a similar formula, is widely used in many leagues (e.g., the United States Chess Federation, the European Go Federation, the FIFA World Rankings, the Universal Tennis Ranking) for calculating the relative skill levels of teams. However, it is not straightforward due to some essential differences between animal fights for territory and sports rivalry. In the latter case, there is no spatial element; therefore, it is impossible to define the agent density responsible for the frequency of duels. This is predetermined by the league fixtures calendar. Also, the team or player does not alter their force after each match (except for slight fluctuations); it would be unrealistically often. The third key element of the Bonabeau model is the formula for the probability of winning a fight (Eq. (1)). It can be easily adapted to describe a match between sports teams; therefore, it constitutes the core of our model.

Our investigation centers on team sports leagues, where teams of uniform size compete in matches, each competing against all others at least once within a given season (round-robin tournament). Each match involves two teams, and a definitive winner emerges; draws are excluded. Since not all team sports allow for ties (e.g., basketball, volleyball, baseball), we decided that a model without ties would be more general and simpler. At the season's conclusion, a final standings table is constructed based on the tally of match victories. In scenarios where multiple teams share an identical win count, they share the same final ranking, resulting in rank gaps. For instance, if two teams claim the highest win count, they will secure the first rank, with the following team attaining the third rank.

The sports' dynamics are characterized by their hyperparameters: the number of players constituting a team ($K$), the number of teams constituting the league ($T$), the number of rounds in a season ($R$), a player base size multiplier ($L$), and a determinism parameter $\beta \geq 0$. The parameter $\beta$ influences how much a match outcome hinges on the difference in player abilities between the opposing teams (similarly to the work of Jerdee and Newmann [55], although they interpret this parameter as the *complexity* or *depth of a game*). Elevated $\beta$ values amplify this influence, anchoring match outcomes more firmly to skill disparities. In contrast, lower $\beta$ values heighten the randomness of match results.

In streamlining the model, we adopt a simplifying assumption: the teams constituting the league are unchangeable and encompass $K$ players who consistently engage in every match. In this context, substitutes are absent. However, teams' compositions can shift because the number of active participants in the sport surpasses those participating in the league. Consequently, transfers become feasible at each season's conclusion, as detailed below.

Let us define a player base of size $N = KTL$, which encompasses all sportsmen wishing to play in a given league. Within the player base, $KT$ players partake in the league, while $KT(L-1)$ individuals comprise a draft list, where $L > 1$ governs the player base's scope. International fame, high salaries, and high competitiveness characterize leagues with high $L$ values, while low values of $L$ correspond to local, low-level leagues. In all our numerical simulations, $L$ is 100 unless otherwise stated. A player $k$ is defined by his strength attribute $h_k \in \langle 0; 1 \rangle$. Initial values of $h_k$ can be generated via various probability distributions (inspired by previous studies [40–42], we use a truncated normal distribution with the mean $\mu = 0.5$ and the standard deviation $\sigma = 0.125$). Please refer to Table 1 for a summary of model parameters and pivotal variables.

### 3.1. League creation

A pivotal stage in the league's establishment entails the draft phase, a critical element in team formation. This process involves teams sequentially selecting players from the entire player base. Subsequently, following each selection, the sequence of teams is reversed, with the subsequent player being chosen. This sequence continues until each team has amassed the requisite number of players. Notably, the likelihood of selecting a given player hinges on the roulette method – a mechanism predicated on players' strengths. This selection probability scales linearly with each player's strength. A team's overall strength, labeled as $H_n$, can be expressed in a simplified way as the sum of the strengths of its individual players:

$$H_n = \sum_{k=1}^{K} h_{n,k}. \tag{3}$$

Here, $h_{n,k}$ signifies the strength of the $k$th player within the team $n$. We realize that the real strength of a team is not the simple sum of the strengths of its players because interactions between players play a key role. However, the nature of these interactions varies between sports, and our goal was to create a model that was as general as possible. Therefore, we decided to use a simple sum as a compromise between drawing team strengths directly from the statistical distribution (omitting player strengths) and building a complex model that considers players' fit within a team.

### 3.2. Seasons

During a season, each pair of teams engages in $R$ matches, with ties excluded. The likelihood of team $n$ prevailing over team $m$ is captured by the formula:

$$P_{n,m} = \frac{1}{1 + e^{\beta(H_m - H_n)}} \tag{4}$$

This expression, rooted in the Bonabeau model [49], serves as the cornerstone for assessing the probability of victory between teams. Critically, this distribution confers a finite probability of winning the match to each team, regardless of their relative strengths and the value of $\beta$. The probability that team $m$ secures a victory against team $n$ is denoted as $P_{m,n} = 1 - P_{n,m}$. After each pair of teams has played $R$ matches, the end-of-season ranking is created based on the number of victories. To introduce the evolution of teams to the model, two processes are taken into account: player transfers and random fluctuations in players' strengths. Both are resolved at the end of a season.

### 3.3. Transfers

During the transfer phase, each team has the opportunity to exchange one of its current players with a player who was not part of the league during the current season. The process involves the following steps:

1. A list of potential transfer candidates is created that consists of players who were not involved in the current season. The selection of candidates is based on a roulette method, where the probability of a player being on the list is proportional to their strength.
2. Teams that are participating in the transfer process choose transfer candidates in order of their previous season's rankings. If there are tied teams, their order is randomized. This order mimics a reward structure, where higher-ranked teams have priority in choosing from the available candidates.
3. For each team, the following actions occur:
   (a) The team selects the best available candidate from the list based on their strength.
   (b) The team also selects one of its current players to be potentially replaced. The probability of selecting a player for replacement is inversely proportional to their strength.
   (c) A player exchange occurs between the selected candidate $B$ and the chosen player $A$ from the team. The decision to exchange players depends on a distribution similar to Eq. (4):
   $$P_{B,A} = \frac{1}{1 + e^{\beta(h_A - h_B)}} \tag{5}$$
   (d) If the exchange happens, player $B$ becomes a member of the team, and player $A$ is released.
4. The transfer process continues with the next team in the order of rankings, excluding candidates already chosen or released by other teams. These players will be available in the next transfer window after the following season.

To control the frequency of potential transfers, one can set the transfer window to occur every $w \in \mathbf{N}^{+}$ seasons. This process simulates the dynamics of player transfers and their impact on team compositions.

### 3.4. Player strength changes

After the transfer phase, the model simulates the natural skill changes of the players by introducing small random changes to their strengths. Each player's strength is updated on the basis of the following equation:

$$h_k(s) = h_k(s-1) \pm \Delta h \tag{6}$$

Here, $h_k(s)$ represents the strength of player $k$ in season $s$, and $\Delta h$ is a small random value distributed uniformly within the interval $\langle -\varepsilon; \varepsilon\rangle$. The value of $\varepsilon$ is a small positive number predetermined at the simulation's start ($\varepsilon = 0.05$ in all our simulations). If the updated player strength is outside the range $\langle 0; 1\rangle$, it is overwritten by a closer boundary value. The same rule applies to the initial strengths of the players.

### 3.5. Rank of a team

We define the rank of team $i$ after season $s$ as $r_i(s)$, and denote the absolute rank change of team $i$ between two consecutive seasons as $\rho_i(s) = |r_i(s+1) - r_i(s)|$. Over $S$ seasons, we calculate the average and standard deviation of absolute rank changes between successive seasons as follows:

$$\bar{\rho} = \frac{1}{T(S-1)} \sum_{i=1}^{T} \sum_{s=1}^{S-1} \rho_i(s), \tag{7}$$

$$\sigma_\rho = \sqrt{\frac{1}{T(S-1)} \sum_{i=1}^{T} \sum_{s=1}^{S-1} (\rho_i(s) - \bar{\rho})^2}. \tag{8}$$

Furthermore, let $\delta_r(s) = |r(s) - r(s-1)|$ denote the absolute rank change of any team that holds rank $r$ before season $s$, e.g., $\delta_5(3) = 2$ means that after three seasons ($s = 3$) a team from fifth position ($r = 5$) moved up or down two positions. The average absolute rank change between two consecutive seasons for a team previously ranked $r$ is defined as:

$$\overline{\rho_r} = \frac{1}{S-1} \sum_{s=2}^{S} \delta_r(s). \tag{9}$$

Importantly, the symbol $i$ in $\rho_i(s)$ refers to the team index, whereas the symbol $r$ in $\delta_r(s)$ pertains to the team's rank.

### 3.6. Promotion and relegation

The relegation system in sports leagues adds an extra layer of complexity to dynamics, as teams from the lower division can replace teams from the higher division based on their performance. However, it is reasonable to simplify the model by assuming that the skill levels of teams promoted to the higher league are not much higher (on average) than those of teams relegated to the lower league. In this case, replacing one team with another would not significantly impact the dynamics of the model. This assumption allows for a more straightforward representation of the model while still capturing the essential dynamics of team sports leagues. Moreover, our model includes a stochastic component for the replacement of individual players in every team. Thus, this part of the dynamics can, to some extent, mirror stochastic contributions related to the promotion and relegation process. The potential implications of including the promotion/relegation mechanism in our model are discussed in more detail in Section 6.

## 4. Parameter of determinism

Let us now explore the influence of the determinism parameter $\beta$ on various measures while maintaining consistent hyperparameter values comparable to certain European football leagues ($T = \{18, 20\}, K = 11, R = 2, L = 100$). Fig. 1a illustrates the average number of points per team, assuming that a win is rewarded with one point. Notably, this average remains unaltered with varying $\beta$, as each team engages in $R(T-1)$ matches with an average point outcome of 0.5 points. In contrast, the standard deviation of this average increases with $\beta$. In sports characterized by high determinism values, even a minor disparity in team strengths can decisively sway match outcomes. Consequently, stronger teams amass significantly more points throughout the season than their weaker counterparts, culminating in a broader point distribution and an amplified standard deviation.

A low $\beta$ corresponds to matches primarily influenced by random elements, yielding comparable (though not identical) point totals at the season's conclusion for all teams. Such circumstances manifest in a lower standard deviation across the distribution of scores achieved by various teams. A sport with $\beta = 0$ equates to coin tossing, wherein both outcomes bear equal probability. Notably, a smooth yet observable transitional region emerges for $0.2 < \beta < 5$, bridging the two system behaviors. Beyond this range, changes in the standard deviation prove negligible. For higher $\beta$ values, matches exhibit greater predictability, manifesting in the average absolute rank changes $\rho_i(s)$ (Eq. (7)) of team $i$ between consecutive seasons $s$ and $s+1$, along with their associated standard deviation (Eq. (8)). Elevated $\beta$ values correspond to diminished averages ($\bar{\rho}$) and standard deviations ($\sigma_\rho$) of absolute rank changes (as seen in Fig. 1b), indicating teams' propensity to maintain similar ranks across successive seasons. Strong teams sustain their dominance, and weaker teams remain in lower standings. Up to now, we have considered changes in the ranks of a given team $i$ in consecutive seasons $\rho_i(s)$. Now, we study another measure: changes in a team's ranks at a given rank level $r$. Examining average rank

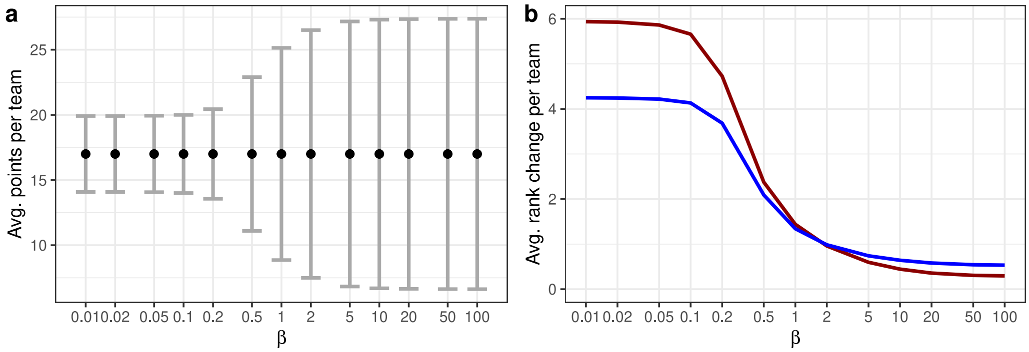


**Fig. 1.** **(a)** The average number of points earned by each team at the end of a season. Error bars represent standard deviations whose magnitude results from the differences between the number of points scored by individual teams. There is a transition region $0.2 < \beta < 5$ for values with low and high deviations. **(b)** Average (dark red) $\bar{\rho}$ and standard deviation (blue) $\sigma_\rho$ of the absolute rank changes $\rho_i(s)$ between subsequent seasons for each team. These values of $\bar{\rho}$ and $\sigma_\rho$ are the result of averaging over the last 80 seasons. Both drawings show results for $T = 18, K = 11, R = 2, S = 100$.

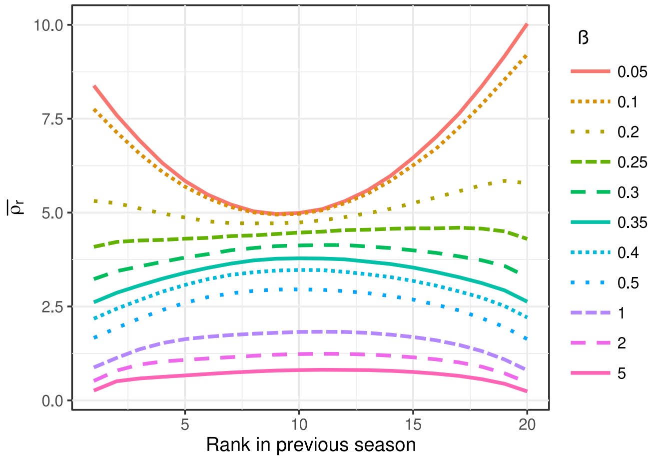


**Fig. 2.** Average rank change $\overline{\rho_r}$ between two subsequent seasons $s-1$ and $s$ vs rank in season $s-1$ for different values of $\beta$. One can see that for low values of determinism, there are significant rank changes for league leaders and losers, while for high values of determinism, these groups of teams have the smallest rank changes. Other hyperparameters: $K = 11, T = 20, R = 2$.

changes $\overline{\rho_r}$ within two consecutive seasons $s-1$ and $s$ for each individual rank (Eq. (9)), Fig. 2 reveals distinct behavior patterns for teams at the ends of the ranking table compared to those in the middle. For $\beta \leq 0.2$, teams positioned at the table's extremities tend to undergo greater rank fluctuations between successive seasons. This phenomenon stems from the fact that end-ranked teams have more ranks to gain or lose. Given the heightened influence of random factors over skills for $\beta \leq 0.2$, such changes are more probable.

This dynamic shifts notably for $\beta \geq 0.3$. Top and bottom teams now exhibit diminished ranking shifts compared to middle-ranked teams, suggesting competitive imbalance [56]. This trend might be encapsulated as *elitism*, where superior teams dominate over multiple seasons, or as overall class formation [57,58]. Parameter $\beta$ also affects the process of player transfers. In high $\beta$ scenarios, teams exhibit greater rationality and tend to retain stronger players, albeit still susceptible to "misjudgments" in strength assessments. Fig. 3 illustrates the evolution of team strengths, averaged across numerous instances, with the selection criterion being the ranking position post the initial season. For low values of $\beta$, all teams experience comparable strength increments. Conversely, if $\beta > 0.3$, top teams exhibit an accelerated initial growth rate, leading to substantial inter-team strength disparities. These findings underscore the presence of two distinct phases: one characterized by uniform team strength and unpredictable match outcomes and another where team hierarchies emerge. Remarkably, both phases witness the progression of all teams' strengths, even those with modest starting standings at the league's commencement (a sole exception being a team ranked 20th for $\beta = 3$). This phenomenon arises due to the abundance of proficient players in an extensive draft list encompassing $KT(L-1)$ players.

To pinpoint the precise values of critical $\beta^*$ signifying the transition between low and high determinism phases, we computed inflection points of $\bar{\rho}$ (Eq. (7)) and $\sigma_\rho$ (Eq. (8)) utilizing the bisection extremum surface (BESE) and bisection distance estimator (BEDE) methods [59]. We have used the R package `inflection` [60], developed by Demetris T. Christopoulos, the original author of these methods. Fig. 4 depicts how the critical value $\beta^*_{AVG}$ decreases following a power law with the number of players $K$, number of

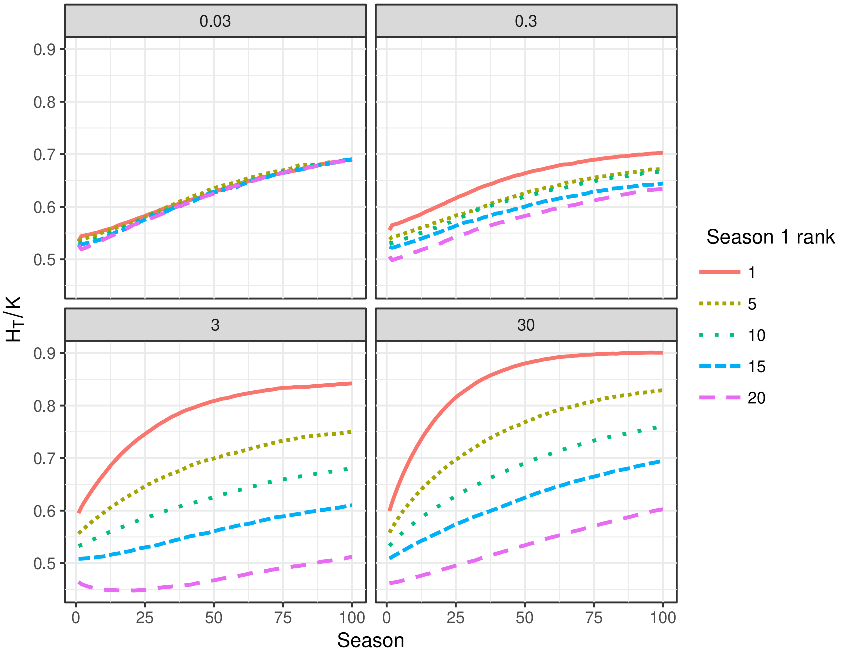


**Fig. 3.** Evolution of teams' strength ($K = 11, T = 20, R = 2, L = 100$). The parameter $\beta$ values are shown at the top of each panel. All teams develop similarly for low values of $\beta$, while for high $\beta$, the strongest teams strengthen much faster.

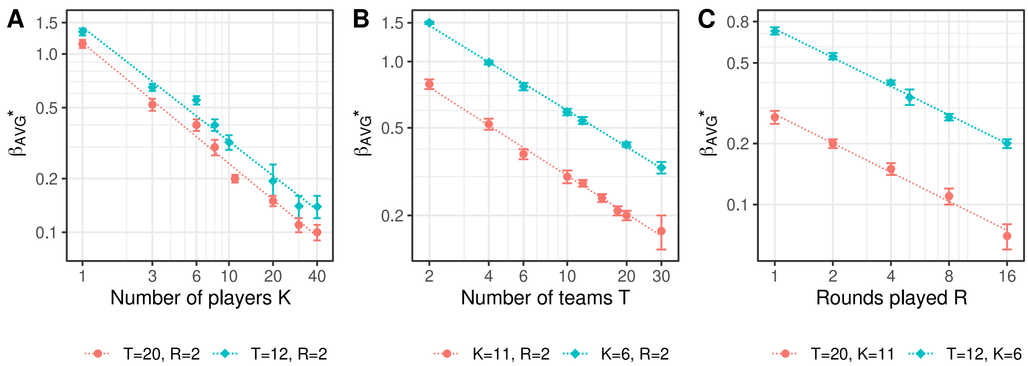


**Fig. 4.** Critical values of $\beta^*_{AVG}$ for varying hyperparameters. Plots are on a log–log scale. Dashed lines represent fits obtained with linear regression (details in text and Table 2).

**Table 2**
Scaling factors $\gamma$ estimated using linear regression using data from each plot in Fig. 4 (type-A uncertainties are shown in the parentheses; all p-values are below $5 \cdot 10^{-4}$).

| Plot | Constants | Variable | $\gamma$ |
|---|---|---|---|
| A | $T = 20, R = 2$ | $K$ | 0.68(3) |
| A | $T = 12, R = 2$ | $K$ | 0.64(3) |
| B | $K = 11, R = 2$ | $T$ | 0.57(1) |
| B | $K = 6, R = 2$ | $T$ | 0.55(1) |
| C | $T = 20, K = 11$ | $R$ | 0.48(3) |
| C | $T = 12, K = 6$ | $R$ | 0.47(2) |

teams $T$, and the number of rounds $R$. The characteristic exponents $\gamma$ were derived through linear regression: $\ln(\beta^*_{AVG}) = -\gamma \ln(X) + C$, where $X = \{T, K, R\}$. Table 2 provides the computed $\gamma$ values along with p-values reflecting the statistical significance of the fits.

Let us now consider a simplified model scenario, where player transfers to teams are entirely random or disabled, the term $H_m - H_n$ in Eq. (4) represents the difference between two sums of $K$ random variables from a truncated normal distribution. In the absence of truncation, such a difference would scale proportionally to $\sqrt{K}$, given that $H_m - H_n = K(\langle h_{m,k}\rangle - \langle h_{n,k}\rangle)$, and

**Table 3**
Estimated parameters of determinism $\beta_{AVG}$ and $\beta_{SD}$ of real sports leagues (see Fig. 5). Type-A uncertainties are shown in parentheses. For comparison, the last two columns show the critical values $\beta^*_{AVG}$ and $\beta^*_{SD}$ (see Fig. 1b) observed in models with matching hyperparameters $K, T, R$. See Fig. 6 for a comparison of model data and real sports leagues.

| League | $\overline{\rho}$ | $\sigma_\rho$ | $\beta_{AVG}$ | $\beta_{SD}$ | $\beta^*_{AVG}$ | $\beta^*_{SD}$ |
|---|---|---|---|---|---|---|
| Premier League | 3.32(14) | 3.15 | 0.36(2) | 0.31(1) | 0.20(2) | 0.23(2) |
| Bundesliga | 3.85(14) | 3.18 | 0.28(2) | 0.27(1) | 0.21(2) | 0.24(2) |
| La Liga | 3.79(13) | 2.98 | 0.30(2) | 0.33(1) | 0.20(2) | 0.23(2) |
| Liga ACB | 2.97(12) | 2.54 | 0.67(4) | 0.67(4) | 0.35(2) | 0.44(2) |
| Basketbol Süper Ligi | 3.12(17) | 2.69 | 0.58(4) | 0.55(5) | 0.39(2) | 0.45(2) |
| SHL | 3.15(21) | 2.44 | 0.49(9) | 0.52(2) | 0.34(2) | 0.40(2) |

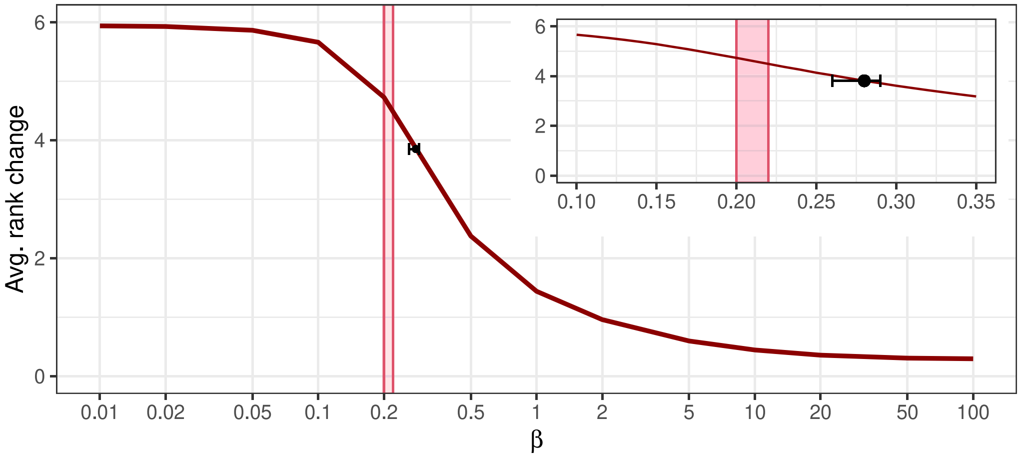


**Fig. 5.** Value of determinism level $\beta_{AVG}$ derived from the average rank change $\overline{\rho}$ in Bundesliga (black point) relative to overall rank change curve for appropriate model ($T = 18, K = 11, R = 2$). The red rectangular area represents the critical value $\beta^*_{AVG}$ with uncertainties for models with hyperparameters taken from Bundesliga data. The main plot's horizontal axis is on a logarithmic scale. The inset plot shows a close-up of a region near both $\beta_{AVG}$ and $\beta^*_{AVG}$ with a linear horizontal scale.

fluctuations in the difference $\langle h_{m,k}\rangle - \langle h_{n,k}\rangle \sim 1/\sqrt{K}$. Consequently, an anticipated relationship emerges $\beta^* \sim K^{-1/2}$. In particular, the observed scaling exponent $\gamma_K \approx 0.65$ exceeds expectations, which can be potentially attributed to biases in the transfer phase and the truncated nature of the individual strength distribution $h_k$. A similar rationale extends to other dependencies. The expected rank $E[r_i(s)]$ of the team $i$ after the season $s$ coincides with its initial strength-based ranking. Meanwhile, fluctuations $\delta r_i(s)$ decrease as the determinism parameter $\beta$ increases. In contrast, a team's count of victories within a season $s$, which determines $r_i(s)$, results from $R(T-1)$ random experiments (matches). Thus, we anticipate that $\delta r_i(s)$ will decrease with $T$ and $R$. Systems with minor fluctuations require lower values of $\beta^*$ to transition to the deterministic phase. This finding is consistent with previous works [18,21,61–64], in which comparisons were made between regular leagues and single-elimination tournaments or between leagues with varying season lengths. The results presented in Figs. 1–4 refer to small ($T < 30$) systems with the extensive player base ($L = 100$). Please refer to Appendices A and B for the analysis of systems of larger sizes or different values of a player base size multiplier $L$.

## 5. Comparison with real data

We amassed historical data samples from three football leagues (German Bundesliga [1991–2021], English Premier League [1996–2020], Spanish La Liga [1988–2021]), two basketball (Liga ABC [1996–2023], Basketbol Süper Ligi [2006–2023]) and one ice hockey league (Swedish SHL [1988–1999]). We have selected data from leagues that meet specific criteria, which are explained in detail in Appendix C. Subsequently, we computed the average $\overline{\rho}$ and standard deviation $\sigma_\rho$ of absolute rank changes throughout successive seasons within these leagues (see Table 3). To ascertain the determinism parameter for each actual league, we run multiple iterations of our agent-based model using various $\beta$ values in alignment with the pertinent hyperparameter values. Through this iterative process, we determined the $\beta$ values for which the $\overline{\rho}$ and $\sigma_\rho$ values from the simulations match the values derived from the data. Given the slight discrepancies between the optimal $\beta$ values for $\overline{\rho}$ and $\sigma_\rho$, we employed the symbols $\beta_{AVG}$ and $\beta_{SD}$ to denote these two values, respectively. Fig. 5 serves as an illustrative instance of estimating $\beta_{AVG}$ using the average rank change $\overline{\rho}$ for the Bundesliga.

Upon inspection of Fig. 6, it becomes evident that the estimated values $\beta$ derived from the actual data closely align with the transition points $\beta^*$ extracted from the corresponding models. This observation intimates that real sports leagues, when perceived as multi-agent dynamical systems, inhabit a realm proximate to the boundary separating deterministic and random phases, as outlined within our model. This discovery harmonizes with our collective understanding, recognizing that even in the upper echelons of

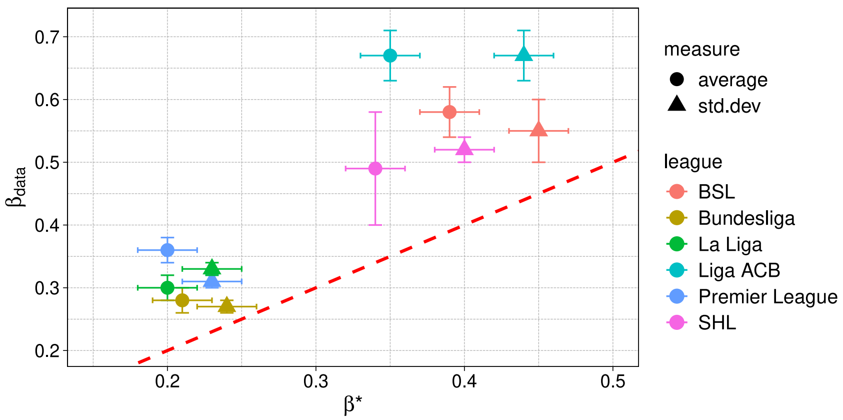


**Fig. 6.** Comparison of determinism parameter $\beta_{\text{data}}$ ($\beta_{AVG}$ and $\beta_{SD}$) calculated from real data (vertical axis) and critical values of this parameter $\beta^*$ received from corresponding models. The dashed line shows the diagonal $\beta_{\text{data}} = \beta^*$. Since all estimated values lie close to critical model parameters but above this diagonal, we consider that team leagues operate in the critical region but in the deterministic phase of our model.

competitive sports, where player skills wield immense significance, random variables such as injuries and weather remain pervasive influencers. The aggregate ratios, computed for both the average and standard deviation values in all leagues, fall within the range $\beta/\beta^* \in (1.13, 1.97)$. Fig. 6 underscores a consistent trend: the $\beta$ values calculated from empirical data exceed the estimated critical values $\beta^*$ derived from the models. In essence, the leagues under consideration reside on the deterministic side of the phase transition boundary. When comparing the values of $\beta$ for different sports, it can be seen that hockey and basketball are more deterministic than football. This is in line with the other studies [37]. Instead of estimating $\beta_{data}$ for the whole time range, one can consider estimating it in shorter periods. Such an approach is presented in Appendix D.

## 6. Discussion

The core objective of this study was to examine the degree of randomness and unpredictability present within team sports leagues. To address this inquiry, we introduced an agent-based model inspired by Bonabeau's description of self-organized hierarchical systems [49] that emerge in competitive animal encounters. We then calibrated its parameters to reproduce the dynamics of six real team sports leagues: three football (Premier League, Bundesliga, La Liga), two basketball (Liga ABC, Basketbol Süper Ligi), and one ice hockey (Svenska Hockeyligan). Using historical ranking data from these leagues, we estimate their respective determinism parameters $\beta$ and observe their proximity to the critical values $\beta^*$, placing them squarely within the deterministic domain of the transition boundary. An intriguing question arises as to why the system resides closely with the critical line. One plausible hypothesis is that opposing mechanisms tied to the deterministic and stochastic aspects of sports competitions compel sports leagues toward the transition line, mirroring the dynamics observed in self-organizing criticality [65–67]. This would imply that this dynamic equilibrium optimally serves the interests of sports organizations. Were the leagues overly deterministic, they might fail to garner sufficient audience engagement, given that the outcomes would lack the potential for emotional involvement. In contrast, if the leagues were entirely random, the audience might still remain disinterested, as a completely unpredictable sport would become mere gambling. We believe that compliance with this stylized fact gives credence to our model. For clarity, we are not suggesting that the system is precisely at the critical point, as in the case of, e.g., the sandpile or the forest-fire models, but we think it is close to the transition between fully deterministic and random behavior. Since sports leagues are systems driven by many factors (including socio-economic impacts), it is difficult to compare them to self-organized critical systems observed in simpler physical experiments.

Our model boasts versatility and is adaptable to a diverse range of sports leagues by selecting suitable values for key parameters such as the number of teams $T$, players $K$ per team, and rounds $R$ per season. The model's behavior hinges on the value of $\beta$, which categorizes it into one of two phases: a random (egalitarian) phase or a deterministic (hierarchical) phase. Within the random phase, the final standings of teams exhibit unpredictability, with substantial fluctuations in rankings from season to season. Conversely, the deterministic phase yields relatively stable rankings, accentuating the establishment of a fixed team hierarchy.

The methodologies employed in this study possess broader implications for competitive balance research [45]. We introduce an enhancement to the established adjusted churn metric [48] – specifically, the average rank change for each team's rank between consecutive seasons. This extension offers a more comprehensive grasp of the distinctions in rank alterations among the top, bottom, and mid-tier teams. The phase in which the league operates significantly influences the degree of average rank changes for teams situated at opposite ends of the ranking spectrum. This nuanced insight could not be discerned solely through adjusted churn or analogous measures. We are aware of other, not rank-based, measures of competitive balance, such as the actual standard deviation (ASD) or ratio of standard deviations (RSD) of points ratios [61]. However, applying these measures to our model is not straightforward since it does not consider ties where both teams gain points.

Furthermore, our agent-based model showcases the capacity to generate synthetic data of exceptional quality, affording the flexibility to manipulate input parameters. Such synthetic data hold substantial potential for subsequent competitive balance investigations. By leveraging this model, researchers can delve into the intricate dynamics of competitive balance within varying sports leagues. The strategies outlined in this study can catalyze advancements in competitive balance analysis, ensuring a more profound understanding of how different factors, ranging from the interplay of skill and randomness to determinism parameters, affect the equilibrium and competitiveness of team sports leagues.

The model presented here omits some features of real sport leagues, such as the promotion/relegation system, ties, and home advantage. The impact of the first one was studied by numerical simulations by Puterman and Wang [68]. They showed that the promotion/relegation system increases competitiveness by reducing the stratification of teams' strengths in the league. It leads to less predictable outcomes compared to the league without a promotion/relegation system. Home advantage and ties also increase uncertainty about outcomes but in a different way. If the home team is weaker, home advantage reduces the difference in strength between the teams. Draws, on the other hand, naturally increase the pool of possible results and also reduce the average point difference between teams for their participation in the match. It can therefore be assumed that ties and home advantage will result in less dispersion of points in the end-of-season table. To sum up, after implementing all three features in the model, the values of critical $\beta^*$ would probably increase, because a higher $\beta$ value would be needed for the league to reach a deterministic phase. Still, the effects of these features are rather subtle (especially home advantage [69,70]), and therefore, taking them into account is unlikely to change the main conclusions of our work.

The model design embraces simplicity but retains the potential for further enhancements. Although its primary objective is to capture essential dynamics, notable extensions can be envisioned to accommodate additional complexities within sports leagues. The model does not readily apply to leagues featuring internal divisions or splits, such as the NBA with Eastern and Western conferences. The disparity in match counts due to subgroup differences presents a challenge. Nevertheless, the model's foundational structure could serve as a basis for modifications addressing such intricacies. Beyond that, the model could be enriched with more intricate reward systems, enabling a more refined reflection of various league-related factors. It is commonly known that a team's performance in top leagues (especially in football) is primarily driven by two goals: to avoid relegation (for the weakest teams) and to qualify for the lucrative international tournaments (where only top teams are allowed to enter), meaning there is no big difference in reward for mid-tier teams [71]. Including player exchange mechanisms between teams and a more intricate player development aspect can add depth to the model. These enhancements would contribute to a more nuanced representation of the multifaceted dynamics present in real-world sports leagues. Ultimately, while simplicity remains a virtue, the model's adaptability and potential for extensions open avenues for addressing a broader range of league structures, scenarios, and competitive dynamics.

## CRediT authorship contribution statement

**Maciej Pawlik:** Writing – original draft, Visualization, Software, Investigation, Data curation, Conceptualization. **Robert Paluch:** Writing – review & editing, Supervision, Methodology, Investigation. **Michał Boruta:** Software. **Janusz A. Hołyst:** Writing – review & editing, Supervision.

## Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the authors used Grammarly, Writefull, and ChatGPT to correct the grammar and style of the manuscript. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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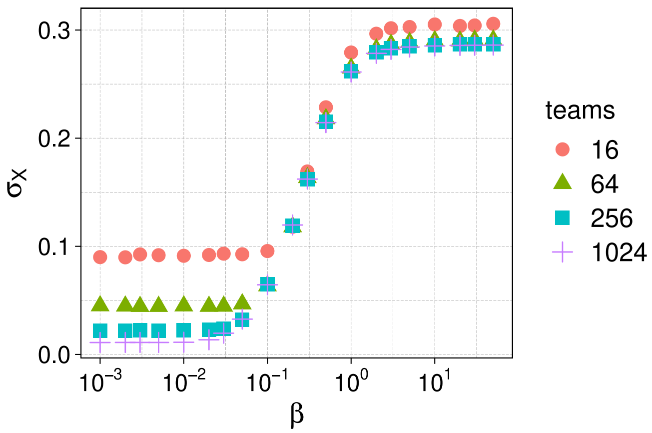


**Fig. A.7.** The standard deviation of the average fraction of matches won by a team versus the determinism parameter $\beta$. The results were obtained from simulations with the following parameters: $K = 11$, $R = 2$, $S = 200$. Only the last 100 seasons were used to calculate $\sigma_X$.

## Appendix A. Model behavior as system size increases

Sports leagues are naturally small systems, usually having no more than two dozen teams. To investigate the phase transition, we performed simulations for much larger systems. Since the number of points scored by a team depends on the number of teams in the league, we study the fraction of matches won by a team, i.e., $X_i = \frac{W_i}{W_i+L_i}$, where $W_i(L_i)$ are the number of wins (losses) of the team $i$ during a given season. A similar approach was proposed by Bonabeau et al. [49].

Fig. A.7 shows the value of the standard deviation $\sigma_X = \sqrt{\frac{\sum_{i=1}^{T}(X_i-0.5)^2}{T}}$ as a function of the determinism parameter $\beta$ for various number of teams in the league. The critical $\beta$ value at which the transition from random (low $\sigma_X$) to deterministic (high $\sigma_X$) phase occurs depends on the number of teams in the league, which is consistent with Fig. 4. However, it can be seen that the results for $T = 256$ and $T = 1024$ almost overlap, suggesting that the critical value of $\beta$ is independent of $T$ in the limit of large $T$. As the number of teams in the league increases, the value of $\sigma_X$ tends to zero in the random phase and to a constant value in the deterministic phase.

## Appendix B. Model sensitivity

The player base size multiplier $L$ is a free parameter that cannot be fitted from the real data. We checked how much influence this parameter has on the dynamics of our model. Fig. B.8 shows the value of the standard deviation $\sigma_X = \sqrt{\frac{\sum_{i=1}^{T}(X_i-0.5)^2}{T}}$ as a function of the determinism parameter $\beta$ for two different sports ($K = \{5, 11\}$) and a wide range of $L$ values. Please refer to Appendix A for the definition of $X_i$. One can see that data points for $L \geq 10$ overlap, which means that in this case, the model behavior is not affected by $L$. For smaller values of $L$, we observe the shift of data points towards lower values of $\sigma_X$ (for $\beta$ = const) or towards higher values of $\beta$ (for $\sigma_X$ = const). Let us stress that small values of $\sigma_X$ correspond to the random (egalitarian) phase when all teams perform similarly. This is because with a small pool of players (low $L$), statistically, fewer players will be very talented (skill levels are drawn from a normal distribution). This negatively impacts the top teams in the league that have a priority in selecting players to transfer. The strengths of the teams do not separate as quickly as for high $L$ and therefore, the outcomes of matches are more unexpected.

## Appendix C. Real data

Certain criteria were followed to select the real leagues to compare the results of the agent-based model with real-world data. The data collected had to satisfy specific requirements to ensure meaningful comparisons. These requirements included:

1. Multiple subsequent seasons: The data should cover results from multiple consecutive seasons to analyze the dynamics and trends over time.
2. Consistent league size and scoring system: The number of teams within the league and the scoring system should remain relatively consistent throughout the seasons. Although points may not always accurately represent the quality of the teams [72], consistency in scoring systems allows a meaningful comparison of rank changes.
3. Absence of internal divisions: The league should not be divided into confederations, subregions, or classes that could impact the dynamics of rank changes within the league. If lower-class leagues are associated with the main league, these lower-class leagues should not be considered, as they might have different dynamics. This ensures that the analysis is focused on a single competitive environment.

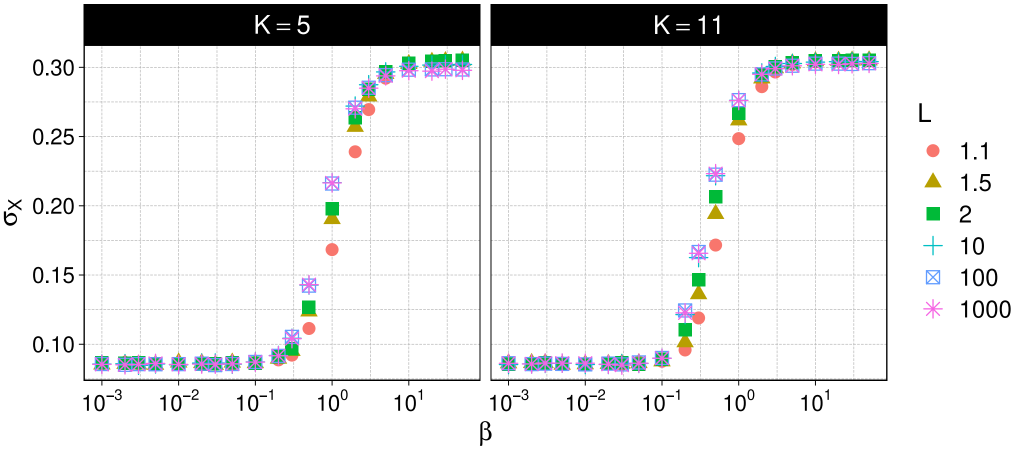


**Fig. B.8.** The standard deviation of the average fraction of matches won by a team versus the determinism parameter $\beta$. The results were obtained from simulations with the following parameters: $T = 18$, $R = 2$, $S = 200$. Only the last 100 seasons were used to calculate $\sigma_X$.

**Table C.4**
Summary of the collected datasets.

| Name | Sport | Country | Seasons | Teams |
|---|---|---|---|---|
| Premier League | Football | England | 1996–2020 | 20 |
| Bundesliga | Football | Germany | 1991–2021 | 18 |
| La Liga | Football | Spain | 1988–2021 | 20 |
| Liga ACB | Basketball | Spain | 1996–2023 | 18 |
| Basketbol Süper Ligi | Basketball | Turkey | 2006–2023 | 16 |
| Svenska Hockeyligan | Hockey | Sweden | 1988–1999 | 12 |

Based on these criteria, data from six real leagues were collected for comparison with the model: three football (the Premier League, Bundesliga, La Liga), two basketball (Liga ABC, Basketbol Süper Ligi), and one ice hockey (Svenska Hockeyligan). The data collected from these leagues were used to analyze and evaluate how well the agent-based model's predictions align with real-world outcomes. The details of the collected datasets are summarized in Table C.4.

In the context of the collected real-world data, there were certain changes in the number of participating teams in some leagues for specific seasons:

- Bundesliga: There was an increase in the number of participating teams for a particular season. This change occurred in the 1991/92 season when teams from former East Germany were allowed to participate in the Bundesliga. This league expansion resulted in an increase in the number of teams competing in that season.
- La Liga: Similarly, the number of participating teams in La Liga also increased for a specific duration. In this case, the increase occurred over two seasons. The expansion occurred in the 1987/88 and 1988/89 seasons.
- Liga ACB: Fewer matches were played in the 2019/2020 season due to the Covid-19 pandemic (only 22–23 per team). There was one extra team in the league in the 2020/2021 season.
- Basketbol Süper Ligi: In 2018, only 15 teams participated in the league, while in 2019/2020, the number of matches played was lower due to the Covid-19 pandemic.

## Appendix D. The evolution of the determinism parameter

One can consider estimating the value of the determinism parameter $\beta$ using position tables from only two seasons and construct a time series of this parameter. Fig. D.9 shows the evolution of $\beta_{AVG}$ for two basketball leagues. One can see that the values fluctuate around the mean value throughout the time range. The significant uncertainties are due to low statistics, as leagues usually have no more than 16 teams. A possible way to reduce imprecision is to estimate the parameter $\beta$ in intervals of several years, possibly overlapping.

## Appendix E. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.physa.2025.130814.

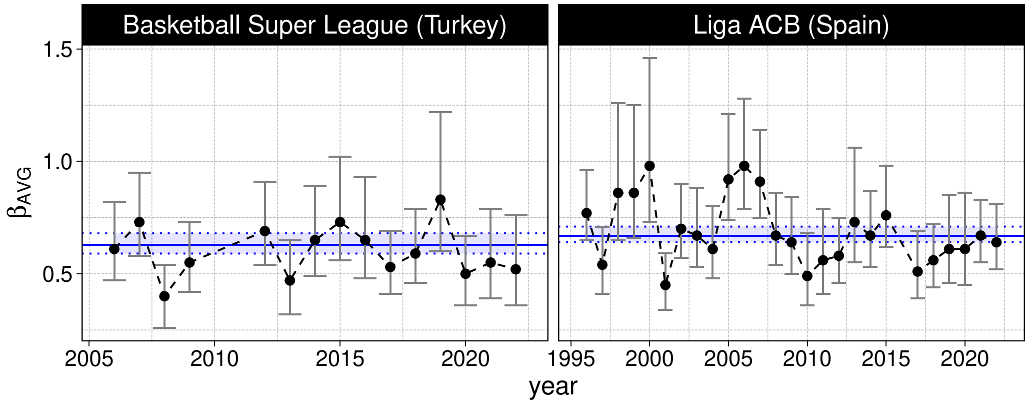


**Fig. D.9.** The evolution of the determinism parameter $\beta_{AVG}$ over time for the basketball leagues. Horizontal blue lines show the average value computed for all seasons, and the blue ribbons present standard errors. One outlier for the Liga ACB (2016) and two outliers for the BSL (2010–2011) were removed from the plots.

## Data availability

The datasets, including the historic ranking tables of selected sports leagues, were collected from public websites and included in supplementary information files.

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