1. Shapes of closed curves

Our interest is in the completion of shapes of partially observed closed curves, and their classification. This first requires us to adopt a suitable representation for the shape of a fully observed curve. We adopt a parametric representation of a closed curve by representing as an absolutely continuous function[1] $C : \mathbb{S} \to \mathbb{R}^2$, thus automatically ensuring that the curve is closed. The notion of a shape of such a curve requires invariances to transformations that represent nuisance information. Specifically, if $\Gamma := \{\gamma : S \to S$ is an orientation-preserving diffeomorphism$\}$ and $SO(2)$ is the rotation group in $\mathbb{R}^3$, the shape of a parameterized curve $C : D \to \mathbb{R}^2$ is defined to be the equivalence class

$$[C] := \Big\{\sigma OC(\gamma(t)) + a, \gamma \in \Gamma, O \in SO(2), a \in \mathbb{R}^2, \sigma > 0\Big\}.$$

Thus $[C]$ is the set of all possible curves that can be obtained through a translation $(C + a)$, rotation (OC), scale change (σC), a reparametrization $(C(\gamma))$, or a combination of the transformations, of the curve C. In words, the shape of a curve C is what is left once variations due scale, translation, rotation and reparametrisation has been accounted for.

A key ingredient in several classification methods (e.g. linear/quadratic discriminant analysis; kernel-based methods) for functional data is the notion of similarity or distance between curves. A popular choice is the distance induced by the $\mathbb{L}^2$ norm of a Hilbert space of square-integrable functions. However, it is well known that the $\mathbb{L}^2$ is unsuitable for comparing curves in the presence of parameterisation variability (**??**). To this end we employ a suitable representation (transformation) of a curve C that allows us to compute distances easily while accounting for the necessary invariances.

1.1. **Notation.** A closed planar curve C is always an absolutely continuous mapping $C : \mathbb{S} \to \mathbb{R}^2$, and the set of planar closed curves will be denoted by $\mathcal{C}$. The set $\mathcal{C}$ is equipped with the norm $\|x\|_{\mathbb{L}^2} := [\int_{\mathbb{S}} \|x(t)\|_2 dt]^{1/2}$ where $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^2$. $SO(2)$ is the special orthogonal group of rotation matrices of $\mathbb{R}^2$, and Γ is group of orientation-preserving diffeomorphisms of $\mathbb{S}$. The set Γ_I will denote the group $\{\gamma : [0,1] \to [0,1], \gamma' > 0, \gamma(0) = 0, \gamma(1) = 1\}$.

1.2. **Square-root velocity transform.** For a detailed introduction to the transform, its properties and advantages we refer to the reader to Chapter 6 of the book by **?**. Here we briefly outline the important concepts required for our purposes. For an absolutely continuous curve $C : \mathbb{S} \to \mathbb{R}^2$ consider the transformation

$$C \mapsto \frac{C'}{\|C'\|_{\mathbb{L}^2}} =: Q_C,$$

[1] Upon identification of $\mathbb{S}$ with $[0,1] \cong \mathbb{R}/2\pi\mathbb{Z}$, a curve $C : [0,1] \to \mathbb{R}^2$ is absolutely continuous if and only if there exists an integrable function $g : [0,1] \to \mathbb{R}^2$ such that $C(t) - C(0) = \int_0^t g(u)du, \forall t \in [0,1]$.

where C' is the (vector) derivative of $C(t)$ with respect to t, and $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^2$.

The unique (up to translations) inverse of a square-root transformed curve Q_C is $\int_o^t Q_C(s)\|Q_C(s)\|_2 ds$. To ensure that the curves are closed we need to impose an additional constraint: $\int_{\mathbb{S}} \|C'(t)\|_2 dt = \int_{\mathbb{S}} Q_C(t)\|Q_C(t)\|_2 = 0$.

For a curve C, by taking its derivative and dividing by its length $\|C'\|_{\mathbb{L}^2}$, the transform accounts for translation and scale variabilities. Thus the image of the set of absolutely continuous closed curves with fixed lengths l, $\{C : \mathbb{S} \to \mathbb{R}^2 : \int_{\mathbb{S}} \|C'(t)\|_2 dt = l\}$, under the square-root transform map $C \mapsto Q_C$ is the set

$$\mathcal{Q} := \Big\{Q_C : \int_{\mathbb{S}} \|Q_C\|_2^2 = 1, \int_{\mathbb{S}} Q_C(t)\|Q_C(t)\|_2 dt = 0\Big\},$$

since $\|Q_C\|_{\mathbb{L}^2} = \|C'\|_{\mathbb{L}^2}^{1/2}$. Thus the set $\mathcal{Q}$ is a subset of $\mathbb{L}^2(\mathbb{S}, \mathbb{R}^2) := \{Q : \mathbb{S} \to \mathbb{R}^2 : \int_{\mathbb{S}} \|Q(t)\|_2^2 < \infty\}$. It is referred to as the pre-shape space corresponding to the curves, since variations due to rotation and parameterization are yet to be accounted for. It is not a linear space and is a manifold (**?**).

Before defining the shape space, we discuss the actions of groups Γ and $SO(2)$ on the set $\mathcal{Q}$. The set Γ of reparameterisations (or warp maps) of $\mathbb{S}$ is group with group action given by composition. Its action on $\mathcal{Q}$ is defined by $(Q_C, \gamma) \mapsto Q_C(\gamma)\sqrt{|\gamma'|}$, where γ' is the derivative of γ (see Chapters 5 and 6 **?** for more details). The derivative of $\gamma : \mathbb{S} \to \mathbb{S}$ needs to be viewed as a derivative of $\gamma : [0, 1] \to [0, 1]$ based on the identification $\mathbb{S} \cong \mathbb{R}/2\pi\mathbb{Z}$, and hence $|z|$ is just the absolute value of the real number z. The action of the rotation group $SO(2)$ is defined in the usual way as the map $SO(2) \times \mathcal{Q} \to \mathcal{Q}$ with $(O, Q_C) \mapsto \{OQ_C(t) : t \in \mathbb{S}\}$.

Two important ramifications of the described framework, motivating its use in our work for analyzing shapes of curves, are the following. Under the square-root velocity framework:

1. The actions of $SO(2)$ and Γ on $\mathcal{Q}$ commute, i.e. they can be applied to a curve in any order. This ensures that their combined action is given by the product group $\Gamma \times SO(2)$.
2. The action of $\Gamma \times SO(2)$ on $\mathcal{Q}$ is by isometries: Given two curves C_1 and C_2, we have $\|OQ_{C_1}(\gamma)\sqrt{\gamma'} - OQ_{C_2}(\gamma)\sqrt{\gamma'}\|_{\mathbb{L}^2} = \|Q_{C_1} - Q_{C_2}\|_{\mathbb{L}^2}$, for every $(\gamma, O) \in \Gamma \times SO(2)$. This ensures that if two square-root velocity transformed curves are rotated and reparameterized the same way, their distance remains unchanged.

Starting with a curve C we can now define its shape to be the equivalence class or its orbit of its corresponding square-root transform:

$$[Q_C] = \text{closure}\{OQ_C(\gamma)\sqrt{\gamma'} : (\gamma, O) \in \Gamma \times SO(2)\},$$

where the closure is with respect to the norm $\|\cdot\|_{\mathbb{L}^2}$ on Q. The shape space consequently is defined as $\mathcal{Q}_s := \{[Q_C] : Q_C \in \mathcal{Q}\}$. Property (2) ensures that the metric induced by the norm $\|\cdot\|_{\mathbb{L}^2}$ on $\mathcal{Q}$ descends onto a metric d

on the shape space (quotient space) $\mathcal{Q}_s$ in a natural way. Given two curves C_1 and C_2, the shape distance between them is defined as

$$
\begin{aligned}
(1) \quad d(C_1, C_2) &:= \inf_{(\gamma, O) \in \Gamma \times SO(2)} \|Q_{C_1} - OQ_{C_2}(\gamma)\sqrt{\gamma'}\|_{\mathbb{L}^2} \\
&= \inf_{(\gamma, O) \in \Gamma \times SO(2)} \Big[\int_{\mathbb{S}} \Big\| Q_{C_1}(t) - OQ_{C_2}(\gamma(t))\sqrt{\gamma'(t)} \Big\|_2^2 dt \Big]^{1/2} \\
&= \inf_{(\gamma, O) \in \Gamma \times SO(2)} \|OQ_{C_1}(\gamma)\sqrt{\gamma'} - Q_{C_2}\|_{\mathbb{L}^2}.
\end{aligned}
$$

The symmetry with respect to the action either on Q_{C_1} or Q_{C_2} is an attractive feature and will be used profitably in the sequel.

2. Curve completion and classification

Suppose we are given closed planar curves $C_j^m : \mathbb{S} \to \mathbb{R}^2, j = 1 \ldots, n$ each of which has been observed only on a region $\mathcal{R}_j \subset \mathbb{S}$, where $\mathbb{S}$ is the unit circle in $\mathbb{R}^2$. We assume that the curves are absolutely continuous. Additionally, a training sample $\{(y_i, C_i^o), 1, \ldots, N\}$ consisting of class labels $y_i \in \{0, 1\}$ and $C_i^o : \mathbb{S} \to \mathbb{R}^2$, fully observed on a common domain $\mathbb{S}$, is provided. The set of fully observed curves are elements of the set $\mathcal{C}$; denote by $\mathcal{C}^m$ the set of all partially observed curves.

The problem at hand is to model the shape of and complete each partially observed curve C_j^m and assign it to one of G groups. We consider two approaches: one based on a variational formulation, and the other using kernel-based classifier. Before describing our approaches, some comments on the set Γ are in order.

2.1. The set of reparameterizations Γ.

Elements of the group Γ of diffeomorphisms of $\mathbb{S}$ can be viewed in the following manner. The unit circle $\mathbb{S}$ can be identified with the quotient group $\mathbb{R}/2\pi\mathbb{Z} \cong [0, 1]$. Through this identification, every continuous mapping $\beta : \mathbb{R} \to \mathbb{R}$ induces a continuous mapping of $\mathbb{S}$ onto itself such that $\beta(t + 1) = \beta(t) + 1$ for all $t \in \mathbb{R}$. If β is monotone increasing, we say that the induced map on $\mathbb{S}$ is orientation-preserving (based on a choice of clockwise or anti-clockwise orientation).

Consider now the set

$$\Gamma_{\mathbb{R}} := \{\beta : \mathbb{R} \to \mathbb{R} : \beta(t + 1) = \beta(t) + 1, \text{ continuous and increasing}\}.$$

Each member β of $W_{\mathbb{R}}$ induces a warp map $\tilde{\beta} : \mathbb{S} \to \mathbb{S}$ with $\tilde{\beta}(e^{2\pi i t}) = e^{2\pi i \beta(t)}$, where β is referred to as the lift of $\tilde{\beta}$. This β satisfies $\beta(t + 1) = \beta(t) + 1$ for all $t \in [0, 1]$, and consequently we have, for $t \in [0, 1]$, $\beta(t) = \gamma(t) + c$, where γ is a warp map of $[0, 1]$ and $c \in (0, 1]$ (through the identification of $[0, 1]$ with $\mathbb{R}/2\pi\mathbb{Z}$). This procedure can be viewed as one that produces a warp map of $\mathbb{S}$ by 'unwrapping' $\mathbb{S}$ at a chosen point s and generating a warp map of $[0, 1]$. If $\Gamma_I := \{\gamma : [0, 1] \to [0, 1] : \gamma' > 0, \gamma(0) = 0, \gamma(1) = 1\}$ is the group of diffeomorphisms of $[0,1]$, the map $\Gamma \mapsto \mathbb{S} \times \Gamma_I$ is a bijection[2]. We will hence employ the product group $\mathbb{S} \times \Gamma_I$ in place of Γ. This ensures that the domain of each curve C_j^m and C_i^o can be identified with $[0,1]$ upon unwrapping the circle.

2.2. **Curve completion.** The observed region $\mathcal{R}_j$ associated with a partially observed curve C_j^m is the subinterval $[0, t_j]$ with $t_j < 1$ for all $j = 1, \ldots, N$. Suppose that $C_j^m(0) = \mathbf{a}_j := (a_{1j}, a_{2j})^T$ and $C_j^m(t_j) = \mathbf{b}_j := (b_{1j}, b_{2j})^T$. Then the set of curves comprising the missing segment of curve C_j^m is

$$\mathcal{X}_j := \{X : [t_j, 1] \to \mathbb{R}^2 : X(t_j) = \mathbf{b}, X(1) = \mathbf{a}\}.$$

For a partially observed $C_j^m : [0, t_j]$ and an $X \in \mathcal{X}_j$ with $j = 1, \ldots, n$, define its completion to be the concatenated closed curve

$$C_j \circ X(t) := C_j^m(t)\mathbb{I}_{t \in [0,t_j]} + X(t)\mathbb{I}_{(t_j,1]}.$$

Denote by $Q_{C_j^m \circ X}$ the square-root transform of $C_j^m \circ X$. For each $j = 1, \ldots, n$, let $\Theta_j := \mathbb{S} \times \Gamma_I \times SO(2) \times \mathcal{X}_j$, and recall that $\mathcal{C}^o$ and $\mathcal{C}^m$ denote the sets of fully and partially observed curves, respectively. For a fixed $C \in \mathcal{C}$, for each $j = 1, \ldots, n$ define the cost functional $\Phi_{\theta_j} : \mathcal{C}^m \times \mathcal{C} \to \mathbb{R}$ by

$$\begin{aligned}\Phi_{\theta_j}(C_j^m, C) &:= d^2(C_j^m \circ X_j, OC(\gamma)), \quad \theta_j \in \Theta_j \\ &= \inf_{(s,\gamma,O) \in \mathbb{S} \times \Gamma_I \times SO(2)} \|Q_{C_j^m \circ X_j}, OQ_C(\gamma)\sqrt{\gamma'}\|^2_{\mathbb{L}^2}.\end{aligned}$$

The optimal shape completion of a partially observed curve $Q_{C_j^m}, j = 1, \ldots, n$ is $Q_{C_j^m \circ X_j^*}$, where X_j^* is obtained from the solution set of:

$$\theta_j^* := (s_j^*, \gamma_j^*, O_j^*, X_j^*) = \operatorname*{argmin}_{\theta_j \in \Theta_j} \Phi_{\theta_j}(C_j^m, C). \tag{2}$$

In the expression above s_j^* corresponds to the optimal point at which $\mathbb{S}$ was unwrapped in order to identify Γ with $\mathbb{S} \times \Gamma_I$; $\gamma_j^* : [0,1] \to [0,1]$ and O_j^* represents the optimal reparameterization of the curve $C_j^m \circ X_j^*$. The use of a valid distance on the quotient shape space $\mathcal{Q}_s$ allows us to apply the shape transformations on $Q_{C_i}, i = 1 \ldots, n$ in the variational problem with introducing any arbitrariness. The (product) group structure of $\mathbb{S} \times \Gamma_I \times SO(2)$, and its action on $\mathcal{Q}$, ensures that if the transformations were to have been applied to $Q_{C_j^m \circ X^*}$, the resulting solution set would instead contain the corresponding group inverses. (INSERT ILLUSTRATIVE FIGURE).

2.3. **Classification.** We outline two approaches for classification of partially observed curves $C_j^m, j = 1, \ldots, N$.

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[2]Technically, this is not a bijection since for a $\gamma \in \Gamma$, the corresponding $\beta \in \Gamma_I$ has a jump discontinuity at the point $t_c \in [0,1]$ where $\beta(t_c) + c = 1$. This can be circumvented by assuming that the members of Γ and Γ_I are absolutely continuous (as opposed to diffeomorphims); then the map between to the sets is bijective a.e.

2.3.1. *Combining completion and classification.* The variational formulation for the completion of each curve C_j^m with respect to a fixed curve C can be augmented to address the classification task in the following manner. Using the training sample $\{(y_i, C_i^o), 1, \dots N\}$ partition the set $\{C_i^o, i = 1, \dots, N\}$ into $\bigcup_{g \in G} \{C_{ig}^o, i = 1 \dots, N_g\}$ with $N_1 + \cdots + N_g = N$. Recall that the shape space of a given set of curves is the quotient metric space with the metric d in (**??**). Such a structure allows us to define the sample Fréchet mean shape curve for a given set of curves. Formally, for each $g \in G$, consider the sample Fréchet functional define on the set $\mathcal{C}$ as

$$F_g : C \to \mathbb{R}, \quad C \ni C \mapsto F_g(C) := \sum_{i=1}^{N_g} d^2(C, C_i^0).$$

The (local) minimizer of F_g is referred to as the Fréchet mean set, since d is a distance between two equivalence classes. We then select a member $\hat{M}_g : \mathbb{S} \to \mathbb{R}^2$ from set this and refer to it as the Fréchet mean of the group $\{C_{ig}^o, i = 1 \dots, N_g\}$ with $g = 1, \dots, G$.

The variational classifier based on the augmented optimization problem in (**??**) is defined by the following rule:

$$\text{Assign } C_j^m \text{ to group } g^* \text{ where } (g^*, \theta_j^*) = \underset{g \in \{1, \dots, G\} \times \theta_j \in \Theta_j}{\operatorname{argmin}} \Phi_{\theta_j}(C_j^m, \hat{M}_g).$$

The classifier assigns C_j^m to the group whose Fréchet mean is closest to C_j^m. The advantage of this approach lies in the fact that completion and classification of partially observed curves are unified under the same metric (distance), and in a certain sense carried out simultaneously.

2.3.2. *Kernel-based classifier.* Consider curves $C_j^m \circ X_j : \mathbb{S} \to \mathbb{R}^2, j = 1 \dots, n$ that have been completed using the variational formulation in (**??**). For each curve $C_j^m \circ X_j$ a nonparametric estimate of the (conditional) group probabilities $\pi_g := P(y_j = g | C_j^m \circ X_j), g = 1, \dots, G$ can be constructed using the distance d on the shape space of curves. Using the estimate a curve $C_j^m \circ X_j$ is assigned to the group with the largest probability. We eschew the cumbersome notation involving the completed curves and outline the methodology for a generic curve $C : \mathbb{S} \to \mathbb{R}^2$ to belong to two groups based on the training sample; hence $g = 1, 2$ with labels 0 and 1. Extension to more than two groups is routine.

Assume that training sample $\{(y_i, C_i^0, i = 1 \dots, N\}$ consists of independent realizations of the random element $(\mathbf{y}, \mathbf{C})$ taking values in $\{0, 1\} \times \mathcal{C}$. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the random element $\mathbf{C}$ is the mapping $(t, \omega) \mapsto \mathbf{C}(t, \omega)$. Thus $\mathbf{C}$ is an $\mathbb{R}^2$-valued stochastic process $\{\mathbf{C}(t), t \in \mathbb{S}\}$. The kernel-based estimate (see for e.g. Chapter 8 of **?**) of the probability of assignment for a curve C to group 1, $\pi(C)$, is given by

$$\hat{\pi}_N(C) = \frac{\sum_{i=1}^N y_i K_{i,h_N}}{\sum_{i=1}^N K_{i,h_N}}, \tag{3}$$

where $K_{i,h_N} := K(h_N^{-1} d(C, C_i^o))$ for a given kernel K and a positive bandwidth sequence h_N. The distance d is defined on the quotient space of shape curves in (**??**).

The asymptotic properties of the estimator $\hat{\pi}_N(C)$ are intimately related to the (shifted) small-ball probability of the process $\mathbf{C}$ under the metric d:

$$\begin{aligned}\phi(Q_C, h_N) &:= \mathbb{P}(d(\mathbf{C}, C) < h_N), \quad C \in \mathcal{C}, h_N > 0\\ &= \mathbb{P}\left(\inf_{(O,\gamma)\in SO(2)\times\Gamma} \|\mathbf{Q_C} - OQ_C(\gamma)\sqrt{|\gamma'|}\|_{\mathbb{L}^2} < h_N\right),\end{aligned}$$

where $\mathbf{Q_C}$ is the (pathwise) square-root transform of the random curve $\mathbf{C}$. For a detailed account of small-ball and shifted small-ball probabilities of processes, and their role in kernel-based estimators involving functional data, see **????** and references therein. To the best of our knowledge results regarding small-ball probabilities are available only for processes with values in linear function spaces (e.g. Hilbert space with $\mathbb{L}^2$ norm or Banach space with supremum norm). The process $\mathbf{Q_C} = \frac{\mathbf{C}'}{\|\mathbf{C}'\|_{\mathbb{L}^2}}$ takes values in an infinite-dimensional manifold (pre-shape space) $\mathcal{Q} \subset \mathbb{L}^2(\mathbb{S}, \mathbb{R}^2)$, defined earlier. Moreover, d is on the quotient shape space, and is defined between orbits of $\mathbf{Q_C}$ and Q_C (with respect to the action of Γ and $\mathbb{S}$).

We can view the class probability π (conditional expectation of $\mathbf{y}$ given $\mathbf{C}$) as a map from $\mathcal{C}$ to $[0, 1]$. We make the following assumptions.

A1. The kernel K is supported on $[0, 1]$ and bounded away from 0 and 1.
A2. The bandwidth $h_N \to 0$ as $N \to \infty$.
A3. $\phi(C, h_N) > 0$ for every $h_N > 0$ with $N\phi(C, h_N) \to \infty$ as $N \to \infty$.
A4. The conditional probability $\pi : \mathcal{C} \to [0, 1]$ is α-Lipschitz, i.e. there exists a $\lambda > 0$ such that for every $\tilde{C} \in \mathcal{C}$, $|\pi(C) - \pi(\tilde{C})| \leq \lambda\|C - \tilde{C}\|^\alpha$.

The following result relates the behaviour of $\phi(Q_C, h_N)$ to the small-ball probability $\phi(C, h_N) = \mathbb{P}(\|\mathbf{C} - C\| < h_N)$ of the process $\mathbf{C}$ (taking values in a linear space), and establishes consistency and rate of convergence of the estimate $\hat{\pi}_N$, as $N \to \infty$.

Theorem 1. *Under assumptions A1-A3, $\phi(Q_C, h_N) > 0$ for every $h_N > 0$ and $N\phi(Q_C, h_N) \to \infty$ as $N \to \infty$. As a consequence, $N \to \infty$:*

1. $\hat{\pi}_N$ converges in probability to π;
2. $\hat{\pi}_N - \pi = O_{\mathbb{P}}(h_N^\beta)$.

Proof. The key argument is to demonstrate that $\phi(Q_C, h_N) > 0$ and $N\phi(Q_C, h_N) \to \infty$ under assumptions A1-A3. Proofs of consistency and rate of convergence follow using almost identical arguments as in the proofs of Theorems 6.1 (p. 63) and 8.2 (p. 123) of **?**, and are omitted.

The shifted small-ball probability satisfies

$$\begin{aligned}
\phi(Q_C, h_N) &= \mathbb{P}(d(\mathbf{C}, C) < h_N), \quad C \in \mathcal{C}, h_N > 0 \\
&= \mathbb{P}\left(\inf_{(O,\gamma)\in SO(2)\times\Gamma} \|\mathbf{Q_C} - OQ_C(\gamma)\sqrt{|\gamma'|}\|_{\mathbb{L}^2} < h_N\right) \\
&= \mathbb{P}\left(\inf_{(O,\gamma)\in SO(2)\times\Gamma} \|\mathbf{Q_C} - OQ_C(\gamma)\sqrt{|\gamma'|}\|^2_{\mathbb{L}^2} < h_N^2\right) \\
&= \mathbb{P}\left(\|\mathbf{Q_C} - \tilde{O}Q_C(\tilde{\gamma})\sqrt{|\tilde{\gamma}'|}\|^2_{\mathbb{L}^2} < h_N^2 \text{ for some } (\tilde{O},\tilde{\gamma}) \in SO(2)\times\Gamma\right)
\end{aligned}$$

under the assumption that the (unique) infimum is attained at $(\tilde{O},\tilde{\gamma}) \in SO(2)\times\Gamma$. The infimum will be attained if the orbits of the elements of $\mathcal{Q}$ are closed under the action of the product group $SO(2)\times\Gamma$. While the orbit under $SO(2)$ is closed, the same isn't generally true for Γ. A technical adjustment in the definition of Γ rectifies this; for details see **?**.[3] Observe that $\{\omega \in \Omega : \|\mathbf{Q_C}(\omega) - Q_C(\omega)\|^2_{\mathbb{L}^2} < h_N^2\} \subseteq \{\omega \in \Omega : \|\mathbf{Q_C}(\omega) - \tilde{O}Q_C(\omega)(\tilde{\gamma})\sqrt{\tilde{\gamma}'}\|^2_{\mathbb{L}^2} < h_N^2 \text{ for some } (\tilde{O},\tilde{\gamma}) \in SO(2)\times\Gamma\}$. This can be seen by noting that the relationship is trivially true if $\tilde{\gamma}$ is the identity map in Γ. Thus we have that

$$\phi(Q_C, h_N) \geq \mathbb{P}\left(\|\mathbf{Q_C} - Q_C\|^2_{\mathbb{L}^2} < h_N^2\right).$$

Consider the square-root map $\mathfrak{C} : \mathcal{C} \to \mathcal{Q}$, $\mathfrak{C}(C) = Q_C$. The map is a bijection between the two spaces (**?**). It is however a complicated map between two Hilbert spaces. Instead of directly dealing with the map in order to relate $\phi(Q_C, \cdot)$ to $\phi(C, \cdot)$, we adopt the following strategy.

Denote by $\mathbb{S}^\infty$ the unit sphere in $\mathbb{L}^2(\mathbb{S}, \mathbb{R}^2)$. Note that the pre-shape space $\mathcal{Q} = \left\{Q_C : \int_{\mathbb{S}} \|Q_C\|_2^2 = 1, \int_{\mathbb{S}} Q_C(t)\|Q_C(t)\|_2 dt = 0\right\}$ is proper subset of $\mathbb{S}^\infty$. Thus we can view $\mathbf{C}(C) = \mathbf{Q_C}$ and $\mathfrak{C}(C) = Q_C$ as random elements taking values in $\mathbb{S}^\infty$. Consider the radial map in a Hilbert space $\mathfrak{R} : \mathcal{C} \to \mathbb{S}^\infty$ given by

$$\mathfrak{R}C = \begin{cases} C & \text{if } \|C\| \leq 1; \\ \frac{C}{\|C\|} & \text{if } \|C\| > 1. \end{cases}$$

The map $\mathfrak{R}$ is the unique metric projection of $\mathcal{C}$ onto $\mathbb{S}^\infty$ and is 1-Lipschitz and nonexpansive, that is,

$$\|\mathfrak{R}C_1 - \mathfrak{R}C_2\| \leq \|C_1 - C_2\|, \quad C_1, C_2 \in \mathcal{C}.$$

Since the image of $\mathfrak{C}$ (i.e. $\mathcal{Q}$) is contained in the image of $\mathfrak{R}$ (i.e. $\mathbb{S}^\infty$), and the noting that $\mathfrak{C}$ is bijective and $\mathfrak{R}$ is the unique projection on $\mathbb{S}^\infty$, for $C \in \mathcal{C}$ we necessarily have $\mathfrak{C}(C) = \mathfrak{R}(C)$.

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[3]The group Γ needs to extended to the semi-group $\tilde{\Gamma}$ that allows for derivatives to be 0 at some points. We could then alter the action of $\tilde{\Gamma}$ on $\mathcal{Q}$ to be just the composition $Q_C(\gamma), \gamma \in \tilde{\Gamma}$, and the arguments in the proof remain valid.

From the definitions of the maps $\mathfrak{C}$ and $\mathfrak{R}$, from equation (**??**) we have,

$$\begin{aligned} \phi(Q_C, h_N) &\geq \mathbb{P}\left(\|\mathfrak{C}(\mathbf{C}) - \mathfrak{C}(C)\|_{\mathbb{L}^2}^2 < h_N^2\right) \\ &= \mathbb{P}\left(\|\mathfrak{R}(\mathbf{C}) - \mathfrak{C}(C)\|_{\mathbb{L}^2}^2 < h_N^2\right) \\ &\geq \mathbb{P}\left(\|\mathbf{C} - C\|_{\mathbb{L}^2}^2 < h_N^2\right) \\ &= \phi(C, h_N), \end{aligned}$$

since for a fixed N, $\{\omega \in \Omega : \|\mathbf{C}(\omega) - C\|_{\mathbb{L}^2} < h_N\} \subseteq \{\omega \in \Omega : \|\mathfrak{R}(\mathbf{C}(\omega)) - \mathfrak{R}(C)\|_{\mathbb{L}^2} < h_N|\}$ with $\mathfrak{R}$ being 1-Lipschitz. This completes the proof. $\square$