Tournaments

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Abstract

We evaluate multiple tournament formats to assess their effectiveness in accurately reproducing the true underlying ranking of participating teams. Using repeated simulations for each structure, we compare teams' true performance ranks to their simulated outcomes. To quantify the accuracy of each format, we calculate weighted mutual information between true and simulated rankings across a large number of replicates. "Remarkable" weights are applied to look at only the top n teams/competitors in a tournament. This information-theoretic approach allows us to objectively compare formats and identify which structures most effectively preserve rank order. Results are visualized through comparative plots, providing clear insights into the trade-offs and strengths of each tournament design.

| Keywords: tournament structures, mutual information

1 Introduction

There are hundreds of different tournament structures that dictate how a set of teams compete against each other to determine an overall ranking. Different tournament structures have unique strengths and weaknesses, affecting how well they reflect the true rankings of the teams. The goal of a tournament is to find the best teams, but how can we quantify how well a tournament performs? In some instances, the tournament organizers only award the overall winner, but other times, the top three or ten teams are awarded. Because of this, organizers may prefer one structure over another based on their needs. Tournament organizers must also take into account factors like cost, timeliness, entertainment value, and fairness when choosing a tournament structure.

To evaluate a tournament's effectiveness, we propose a numerical metric that quantifies how accurately it orders teams based on their true rankings. Tournament results can be viewed as a "message" attempting to convey the true team rankings. However, various factors introduce noise, leading to information loss. We use principles from information theory to measure this information loss across multiple tournament simulations to assess the reliability of different formats.

2 Methods

Consider a set of n teams indexed from $i = 1, 2, \cdots, n$ each with an associated true strength parameter $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ such that $(\theta_1 > \theta_2 > \ldots > \theta_n)$. Next, let $r(\boldsymbol{\theta})$ be the indexes of $\boldsymbol{\theta}$ when theta is sorted from largest to smallest so that $r(\boldsymbol{\theta}) = \{1, 2, \ldots, n-1, n\}$. Next define $T(\phi, s)$ as a tournament with schedule ϕ and seeding structure s. ϕ contains all the information about the scheduling of teams, which could be fully known prior to the tournament (i.e. round robin) or determined as the tournament progresses (i.e. single elimination tournament). We then let $\hat{r}(T(\phi, s), \boldsymbol{\theta})$ be an n- dimensional vector valued random variable that gives the results of a tournament as a vector of the indexes of the vector $\boldsymbol{\theta}$. The index in the first position of the vector $\hat{r}(T(\phi, s), \boldsymbol{\theta})$ indicates the index of the team that finished first, the index in the second position of the vector $\hat{r}(T(\phi, s), \boldsymbol{\theta})$ indicates the index of the team that finished second, and so on.

For example, in a 4 team tournament, $\hat{r}(T(\phi, s), \boldsymbol{\theta}) = (3,\ 2,\ 1,\ 4)$ indicates that the team with index 3 (i.e. the true third best team) won the tournament, the true second best team finished second, the true best team finished third and the true 4th team finished 4th in that particular tournament. If $\hat{r}(T(\phi, s), \boldsymbol{\theta}) = (1,\ 2,\ 3,\ 4)$, this means that the random outcome of the tournament was the same as the true ordering of the teams.

If one views the true ranking of the teams in a tournament as a message to be sent to a receiver and the outcome of the tournament as a message that is received, we can measure the "goodness" of a tournament in terms of its ability to accurately transmit the true ranking. We can then use concepts from information theory to assess the ability of a tournament to correctly rank teams. Specifically, we start with the concept of mutual information (CITE). For two random variables X and Y mutual information is defined as:

$$I(X;Y) = \sum_{y \in Y} \sum_{x \in X} p(x, y) \log \frac{p(x, y)}{p(x)\, p(y)}$$

In our setting, we replace X and Y with $r(\boldsymbol{\theta})$ and $\hat{r}(T(\phi, s), \boldsymbol{\theta})$, and we could seek to estimate $I(r(\boldsymbol{\theta}), \hat{r}(T(\phi, s), \boldsymbol{\theta}))$. For simplicity, we drop the arguments from the functions r and $\hat{r}$ for ease of exposition.

We note here that r is not a random variable, however, since the indexing of the teams in the vector $\boldsymbol{\theta}$ is arbitrary (we use 1, 2, 3, etc. for convenience, but any indexing is valid), we can view this a random variable where all permutation so of the n teams are equally likely. Therefore when calculating the mutual information in this setting, we set $p(r) = \frac{1}{n!}$. By a similar argument we set $p(\hat{r}) = \frac{1}{n!}$.

We define the mutual information of r and $\hat{r}$ to be:

$$I(r, \hat{r}) = \sum_{r} \sum_{\hat{r}} p(r, \hat{r}) \log \frac{p(r, \hat{r})}{p(r) p(\hat{r})} = n! \sum_{\hat{r}} p(r, \hat{r}) \log \frac{p(r, \hat{r})}{(\frac{1}{n!})^2}$$

While there are $n!$ different permutations for the result of r, we don't need to sum across these as any specific choice of r is just an arbitrary labeling of the true strength parameters vector $\boldsymbol{\theta}$. So given r, the distribution of $\hat{r}$ is the same up to the labeling. Therefore, we only need to consider a single permutation of r, we compute the the quantity inside the summation, and then multiple by $n!$ (effectively summing across r).

In order to compute this quantity, we need to compute $p(r, \hat{r})$. This probability is found by calculating the probability of a given permutation of $\hat{r}$ and r is assumed to be the permutation from $\{1, 2, \ldots, n\}$. We estimate $p(r, \hat{r})$ empirically through simulation.

However, using mutual information in this form does not suit our needs in this setting. The problem is that mutual information in this form will yield high values of mutual information when the output from the tournament $\hat{r}$ is highly consistent *even if the ranking from the tournament is incorrect.* As an example, if $n = 4$ and the true order of $\boldsymbol{\theta}$ is $r = \{1, 2, 3, 4\}$ and $p(\hat{r} = \{4, 3, 2, 1\}) = 1$ will be the same mutual information as when $r = \{1, 2, 3, 4\}$ and $p(\hat{r} = \{1, 2, 3, 4\}) = 1$ and in both of these cases the mutual information will be maximized at:

$$I(r, \hat{r}) = n! \sum_{\hat{r}} p(r, \hat{r}) \log \frac{p(r, \hat{r})}{\left(\frac{1}{n!}\right)^2} = n! log((n!)^2)$$

and for the specific case when $n = 4$ would be:

$$4! log_2(4!^2) = 24 * log_2(24^2) = 220.0782$$

In order to alleviate this problem, we instead consider *weighted* mutual information. We want to give more weight to permutations from $\hat{r}$ that are "closer" to r. There are any number of reasonable choices for this weighting function, and we choose to use squared error. Specifically,

$$w(r, \hat{r}) = \begin{cases} \frac{1}{\hat{r}' r}, & r \neq \hat{r} \\ 1, & r = \hat{r} \end{cases}$$

In general, though any loss function $l(r, \hat{r})$ can be used:

$$w(r, \hat{r}) = \begin{cases} \frac{1}{l(r, \hat{r})}, & r \neq \hat{r} \\ 1, & r = \hat{r} \end{cases}$$

Note that one is 1/N!

And this does work because consistently getting the order incorrect is perfect information.

However, because the number of teams that need to be accurately ordered differs based on the tournament organizers, a second weighting function was found to portray the importance, or remarkability, of each value in $\hat{r}$. We call these "remarkable weights" ($m(r)$) to differentiate from the other weighting function. "Remarkable weights" are set to allow for a subjective belief of what determines a good tournament. For a four-team tournament, if the user decides that only the winner is important, then $m(r) = \{1, 0, 0, 0\}$. However, if the organizers want an accurate leaderboard with the winner as the most important, then a potential $m(r)$

could be $\{3, 1, 1, 0\}$. With the addition of the weighting functions, the formula for mutual information is updated:

$$I(X;Y) = \sum_{r}\sum_{\hat{r}} w(r,\hat{r}) * m(r) * p(r,\hat{r}) \log \frac{p(r,\hat{r})}{p(r)p(\hat{r})}$$

Because this mutual information is unitless, the formula can be standardized to be between 0 and 1 using the complement of Rajski's distance:

$$1 - (1 - \frac{I(X,Y)}{\max\{H(X), H(Y)\}})$$

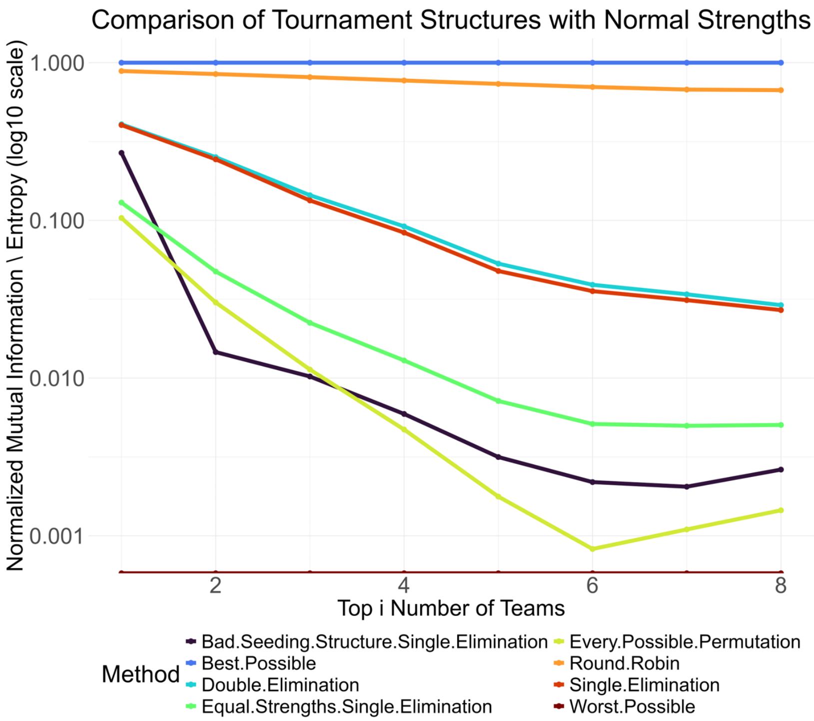
where $H(X,Y)$ is entropy given by:

$$max\{H(X), H(Y)\} = \sum_{r} p(r) \log p(r) = n! \log n!$$

The complement is taken so that the more accurate tournaments are closer to 1 and the less accurate are closer to 0.

3 Results

In order to fairly compare tournament structures, we assume the true strengths of the teams follow a normal distribution, where each team is assigned an equally spaced quantile and each team's strength is the z-score of its respective quantile. Using this assumption, 10,000 simulations, all with 8 total teams, were ran for a single game round robin (each team plays each other once), a single elimination structure with the usual seeding structure (1 vs 8, 4 vs 5, 3 vs 6, 2 vs 7), a single elimination structure with a poor seeding structure (1 vs 2, 3 vs 4, 5 vs 6, 7 vs 8). For a better understanding of the results, the best possible results (every structure has the best in rank 1, the 2nd best in rank 2, etc), the worst possible results (every structure has the worst in rank 1, the 2nd worst in rank 2, etc), every possible permutation of the ranks once, and a single elimination tournament with the usual seeding structure where every team is of equal strengths. Because the metric found is unitless, we were able to normalize the values using the best and worst results to make the range from 0 to 1, with 0 as the worst and 1 as the best. After normalizing, we get the following graph (note the y-axis is on a log10 scale):



Based on the graph, the intuitive best tournament, round robin, does show to be the best. Double elimination is slightly better than single elimination, with the gap growing wider as the top number of teams looked at increases. However, those are under the assumption that the true strengths were ordered correctly in the seeding. If there is a poor seeding structure, as seen in the graph, it can drastically impact the effectiveness of the tournament. In the extreme case seen, the tournament can actually become worse than deciding games by random chance (like flipping a coin). Another important discovery is that the single elimination structure with equal strengths is better than every possible permutation. The line for every possible permutation would likely be similar to a round robin structure with all equal strengths, so perhaps the closer the teams are in true strength, the closer round robin and single elimination become.

4 Conclusion

This study introduced a weighted mutual information framework to evaluate the effectiveness of different tournament formats in preserving the true underlying rankings of teams. By simulating tournaments with varying seeding structures and applying an information-theoretic approach, we quantified how accurately each format conveys ranking information. Our findings confirm the intuitive advantage of round robin tournaments, which consistently outperformed single and double elimination formats in ranking accuracy. Additionally, we observed that seeding plays a critical role: poor seeding structures can significantly degrade performance, in some cases making outcomes less informative than random chance.

These results underscore the trade-off between tournament accuracy and practical constraints such as time, cost, and entertainment value. While round robin offers the highest fidelity to true rankings, it is often impractical for large tournaments, highlighting the need for hybrid or adaptive structures that balance accuracy with logistical feasibility.

Future work should extend this analysis to scenarios with more tournament structures, more teams, incorporate probabilistic strength models reflecting real-world uncertainty, and explore alternative weighting schemes for different competitive priorities (e.g., top-4 accuracy versus full ranking). Applying these methods to actual tournament data could further validate their usefulness for organizers aiming to design fair and informative competitions.

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Supplementary Material

All supplementary material available at https://github.com/gjm112/tournaments.

5 References

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