On Tournament Matrices

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ABSTRACT

Let T be a tournament of order n with adjacency matrix M. We find several conditions that are equivalent to M being singular. A correlation between the number of 3-cycles in T and the rank of M is established. It is shown that asymptotically at least $\frac{1}{2}$ of the tournament matrices are nonsingular. We also derive bounds on the spectral radius of tournament matrices with a given row-sum vector.

1. INTRODUCTION

A *tournament matrix* of order n is an n-by-n $(0, 1)$ matrix $M = [m_{ij}]$ which satisfies

$$M + M^T = J_n - I_n,$$

where J_n denotes the n-by-n matrix of all 1's and I_n denotes the n-by-n identity matrix. Thus M is "combinatorially skew-symmetric" in the sense that each of its diagonal entries equals 0 and an off-diagonal entry m_{ij} equals 1 if and only if the entry m_{ji} equals 0. It is known [see de Caen (1988), de Caen and Hoffman (1989)] that the rank of a tournament matrix of order n (over the real field) is at least $n - 1$. A *tournament* of order n is a digraph obtained by

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arbitrarily orienting each edge of the complete graph on n vertices. Thus a tournament matrix is merely the adjacency matrix of a tournament. Tournament matrices and their ranks arise naturally in the study of biclique partitions of complete graphs.

Let K_n denote the complete graph with vertices $1, \dots, n$. A *biclique* is a complete bipartite edge subgraph of K_n. A *biclique partition* of K_n is a collection of bicliques $B_1, \dots, B_l$ such that each edge of K_n is an edge of exactly one of $B_1, \dots, B_l$. The well-known theorem of Graham and Pollak (1971) implies that K_n cannot be partitioned into fewer than $n-1$ bicliques. If U and V are disjoint nonempty subsets of $\{1, \dots, n\}$, then $B(U, V)$ denotes the biclique with edge set equal to $\{\{u, v\} \mid u \in U,\ v \in V\}$. If S is a subset of $\{1, \dots, n\}$, then the *characteristic vector* of S is denoted by $\vec{S}$ and is defined by

$$\vec{S} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}, \qquad \text{where} \quad s_i = \begin{cases} 1 & \text{if} \quad i \in S, \\ 0 & \text{if} \quad i \notin S. \end{cases}$$

Suppose that $B(X_1, Y_1), \dots, B(X_l, Y_l)$ is a biclique partition of K_n. Let $M = [m_{ij}]$ be the n-by-n $(0,1)$ matrix defined by $m_{ij} = 1$ if and only if $i \in X_k$ and $j \in Y_k$ for some k. Then M is a tournament matrix of order n and

$$M = \vec{X}_1 \vec{Y}_1^T + \cdots + \vec{X}_l \vec{Y}_l^T.$$

Let X (respectively, Y) be the n-by-l matrix with columns $\vec{X}_1, \dots, \vec{X}_l$ (respectively, $\vec{Y}_1, \dots, \vec{Y}_l$). Then $M = XY^T$. Hence a partition of K_n into l bicliques gives a factorization of a tournament matrix of order n as the product of $(0,1)$ matrices of orders n by l and l by n. It is also clear that any such factorization of a tournament matrix of order n determines a biclique partition of K_n. Note in particular that the rank of M is at most l. Because the rank of a tournament matrix of order n is at least $n-1$, we have that $l \geqslant n-1$, which agrees with the bound given by the Graham-Pollak theorem. Thus we conclude that a biclique partition of K_n has at least $n-1$ bicliques and that any partition of K_n into $n-1$ bicliques gives rise to a singular tournament matrix.

The severe restriction on the possible ranks of a tournament matrices of order n and the connection with biclique partitions of K_n led de Caen (1988) to consider the possibility of a "combinatorial classification" of singular tournament matrices. Such a classification appears difficult to find. However, the problem becomes more tractable when we restrict our attention to tournament matrices with the same score vector.

Let $M = [m_{ij}]$ be a tournament matrix of order n, and let T be the tournament of order n with $i \to j$ if and only if $m_{ij} = 1$. The row-sum vector of M is called its *score vector*. The row-sum vector of M^T is called the *coscore vector of* M. Thus the score vector (respectively, the coscore vector) of M is the outdegree sequence (respectively, the indegree sequence) of the vertices in T. Let $s = (s_1, \dots, s_n)^T$ be an n-by-1 vector whose entries are nonnegative integers. Then the class of all tournament matrices with score vector s is denoted by $\mathscr{T}(s)$. If $\mathscr{T}(s)$ is nonempty, then s is a *valid* score vector. The vector s is *monotone* provided

$$s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n.$$

Because the rows and columns of M may be simultaneously permuted to obtain a tournament matrix of the same rank which has a monotone score vector, we may without loss of generality consider only the classes $\mathscr{T}(s)$ where s is monotone. Under this assumption, Landau (1953) has shown that s is a valid score vector if and only if

$$\sum_{i=1}^{k} s_i \geqslant \binom{k}{2} \qquad (k = 1, \dots, n)$$

and equality holds for $k = n$. Throughout the remainder of this paper we shall assume without mention that all of our score vectors are monotone valid score vectors. Note that if s and r are the score and coscore vectors of a tournament matrix of order n, then they satisfy

$$r = (n-1)\mathbf{1} - s,$$

$$s^T\mathbf{1} = \binom{n}{2},$$

$$r^T\mathbf{1} = \binom{n}{2},$$

$$r^T r = s^T s,$$

where $\mathbf{1}$ denotes the n-by-1 vector of all ones. The tournament T is *strong* provided T is a strongly connected digraph. Recall that a (0, 1) matrix A of order n is *reducible* if there exists a permutation matrix P of order n such that P^TAP has the block form

$$P^TAP = \begin{bmatrix} B_1 & O \\ B_2 & B_3 \end{bmatrix}$$

where B_1 and B_3 are nonvacuous square matrices. If no such permutation matrix exists, then A is *irreducible*. It is well known [see p. 19 of Varga (1962)] that the matrix $A = [a_{ij}]$ is irreducible if and only if the digraph with arcs

$$i \to j \quad \text{if and only if} \quad a_{ij} = 1$$

is strongly connected. Thus we conclude that T is a strong tournament if and only if M is an irreducible matrix. Ryser (1964) has shown that either every matrix in $\mathscr{T}(s)$ is irreducible or no matrix in $\mathscr{T}(s)$ is irreducible. Thus we say that the score vector s is *strong* if every tournament matrix in $\mathscr{T}(s)$ is irreducible.

Let s be a valid score vector. Each of the remaining sections is devoted to certain properties of the tournament matrices in the class $\mathscr{T}(s)$. We now summarize the content of each section.

Assume n is an odd positive integer. Let u_n be the n-by-1 vector each of whose entries equals $(n-1)/2$. Then u_n is a valid score vector, and an element of $\mathscr{T}(u_n)$ is a *regular tournament* matrix of order n. It is known (de Caen and Hoffman, 1989) that every regular tournament matrix of order $n \geqslant 3$ is nonsingular. In Section 2, we generalize this to show that if s is a score vector whose entries are "sufficiently close together" then every tournament matrix in $\mathscr{T}(s)$ is nonsingular.

Let p_n denote the probability that a random tournament matrix of order n is nonsingular. Maybee and Pullman (1990) have conjectured that $\lim_{n\to\infty} p_n = 1$. Let T be a tournament of order n with adjacency matrix M. In Section 3, we show that if T has more than

$$\frac{1}{4}\binom{n}{3}$$

cycles of length three then M is nonsingular. Combining this with known results of Moran (1947) on the distribution of the number of 3-cycles in a random tournament, we show that

$$\overline{\lim_{n\to\infty}}\, p_n \geqslant \frac{1}{2}.$$

We consider spectral properties of tournament matrices in Section 4. It is shown that the spectral radius of a singular tournament matrix of order n is bounded above by $(n-2)/2$. Tournament matrices achieving this bound are investigated. We also find bounds on the spectral radius of a singular tournament matrix with score vector s.

In Section 5, we consider the problem of determining the score vectors s for which every matrix in $\mathscr{T}(s)$ is singular. This involves studying the polytope $\mathscr{T}_+(s)$ of all nonnegative matrices M of order n which satisfy

$$M + M^T = J_n - I_n,$$

$$M\mathbf{1} = s.$$

Let M be a tournament matrix of order n. Because M has rank at least $n - 1$, the nullspace of M is spanned by a single integral vector v. In Section 6 we discuss properties of v.

2. SCORE VECTORS WITH NEARLY EQUAL ENTRIES

Let s be an n-by-1 score vector. In this section we establish a criterion for the nonsingularity of a tournament matrix. This criterion is then applied to generalize the fact that every regular tournament matrix of order $n \geqslant 3$ is nonsingular. Throughout this paper $\mathbf{1}$ denotes a column vector of all ones of appropriate size.

We begin by establishing several properties that are equivalent to the singularity of a tournament matrix.

LEMMA 2.1. *Let M be a tournament matrix of order n. Suppose $x \in \mathscr{R}^n$ with $Mx = 0$. Then $(x^T\mathbf{1})^2 = x^Tx$.*

Proof.

$$\begin{aligned} 0 &= x^T\left(M + M^T\right)x \\ &= x^T\left(J_n - I_n\right)x \\ &= \left(x^T\mathbf{1}\right)^2 - x^Tx. \end{aligned}$$ ■

We denote by $\operatorname{Col}(M)$ the vector space spanned by the columns of the matrix M.

THEOREM 2.2. *Let M be a tournament matrix of order $n \geqslant 2$. Let s and r be the score and coscore vectors of M. Then the following statements are* *equivalent*:

(i) M *is singular*.
(ii) M *has rank* $n - 1$.
(iii) *There exists a unique column vector* y *with* $y^T M = 0$, $y^T y = 1$, *and* $y^T \mathbf{1} = 1$.
(iv) $\mathbf{1} \notin \mathrm{Col}(M)$.
(v) *For* $x \in \mathrm{Col}(M)$, $(x - \mathbf{1})^T (x - \mathbf{1}) \geqslant 1$ *and equality holds for exactly one* x.
(vi) *For* $x \in \mathrm{Col}(M)$, $(n-1) x^T x \geqslant (\mathbf{1}^T x)^2$.
(vii) $(n-1) M^T M - rr^T$ *is a positive semidefinite matrix*.
(viii) $\mathrm{Col}(M) \cap \mathrm{Col}(M^T)$ *is a vector space of dimension* $n - 2$.

Proof. Assume M is singular. Let u and v be two nonzero vectors with $Mu = 0$ and $Mv = 0$. By Lemma 2.1, $u^T \mathbf{1} \neq 0$. Let $\lambda = v^T \mathbf{1} / u^T \mathbf{1}$. Then $M(\lambda u - v) = 0$ and $(\lambda u - v)^T \mathbf{1} = 0$. Lemma 2.1 implies that $v = \lambda u$. Hence we conclude that the nullspace of M has dimension 1 and that M has rank $n - 1$. It now follows that there is a unique vector y with $y^T M = 0$, $y^T y = 1$, and $y^T \mathbf{1} \geqslant 0$. By Lemma 2.1 (applied to M^T), we conclude that $y^T \mathbf{1} = 1$ and that (i) implies (iii). Because M has rank $n - 1$, y is a unit normal to the hyperplane spanned by the columns of M. The distance between $\mathbf{1}$ and this hyperplane equals $y^T \mathbf{1}$. Hence (i) implies (iv) and (v). It is clear that each of (ii)–(v) implies (i). Thus we have established the equivalence of (i)–(v).

Let x be a nonzero vector in $\mathrm{Col}(M)$, and let l be the line through the origin in the direction of x. Then the distance between the line l and the point $\mathbf{1}$ equals

$$\sqrt{\left(\mathbf{1} - \frac{\mathbf{1}^T x}{x^T x} x\right)^T \left(\mathbf{1} - \frac{\mathbf{1}^T x}{x^T x} x\right)} = \sqrt{n - \frac{\left(\mathbf{1}^T x\right)^2}{x^T x}} .$$

It follows that statement (vi) is equivalent to the statement that all lines through the origin that lie in $\mathrm{Col}(M)$ are at least 1 unit from $\mathbf{1}$. Hence (v) and (vi) are equivalent.

Let v be any column vector, and let $x = Mv$. Then

$$v^T\left((n-1) M^T M - rr^T\right) v \geqslant 0 \quad \text{if and only if} \quad (n-1) x^T x \geqslant \left(\mathbf{1}^T x\right)^2.$$

Thus (vi) and (vii) are equivalent statements.

Clearly (viii) implies (i). Finally the implication (ii) $\rightarrow$ (viii) follows from

$$\mathrm{Col}(M) + \mathrm{Col}(M^T) \supseteq \mathrm{Col}(M + M^T) = \mathrm{Col}(J_n - I_n) = \mathscr{R}^n$$

and

$$\begin{aligned}\dim[\mathrm{Col}(M)+\mathrm{Col}(M^T)] &= \dim \mathrm{Col}(M)+\dim \mathrm{Col}(M^T)-\dim[\mathrm{Col}(M)\cap \mathrm{Col}(M^T)]\\ &= \operatorname{rank} M+\operatorname{rank} M^T-\dim[\mathrm{Col}(M)\cap \mathrm{Col}(M^T)].\end{aligned}$$

Hence all the statements are equivalent. ∎

The equivalence of statements (i) and (v) in Theorem 2.2 provides a useful criterion for the nonsingularity of a tournament matrix M. Namely, M is nonsingular if and only if there exists a vector in its column space whose Euclidean distance from $\mathbf{1}$ is less than 1. For example, if M is a regular tournament matrix of order $n > 1$, then $\mathbf{1} \in \mathrm{Col}(M)$ and thus M is nonsingular. We now use this criterion to generalize the fact that every regular tournament matrix is nonsingular.

COROLLARY 2.3. *Let $s = (s_1, s_2, \dots, s_n)^T$ be a score vector with $n > 1$. If $s^Ts < n^2(n-1)/4$, then every matrix in $\mathcal{T}(s)$ is nonsingular.*

Proof. Suppose $s^Ts < n^2(n-1)/4$ and $M \in \mathcal{T}(s)$. Because s is a score vector,

$$\mathbf{1}^Ts = \binom{n}{2}.$$

Thus

$$\begin{aligned}(n-1)s^Ts &< \frac{n^2(n-1)^2}{4}\\ &= (\mathbf{1}^Ts)^2.\end{aligned}$$

Since $s \in \mathrm{Col}(M)$, Theorem 2.2 implies that M is nonsingular. ∎

Let n be a positive integer, and let u_n be the n-by-1 vector with each entry equal to $(n-1)/2$. The following is immediate upon noting that for an n-by-1 score vector s we have $s^Tu_n = n(n-1)^2/4$.

COROLLARY 2.4. *Let s be an n-by-1 score vector with $n > 1$. If $(s-u_n)^T(s-u_n) < n(n-1)/4$, then every matrix in $\mathcal{T}(s)$ is nonsingular.*

Corollary 2.4 shows that every matrix in $\mathscr{T}(s)$ is nonsingular provided s is sufficiently close to the vector u_n. For example, suppose n is even, and let s be the score vector whose first $n/2$ entries equal $(n-2)/2$ and whose remaining entries equal $n/2$. A matrix in $\mathscr{T}(s)$ is a *nearly regular* tournament matrix. Because $(s-u_n)^T(s-u_n) = n/4$, every nearly regular tournament matrix of order $n \geqslant 4$ is nonsingular. We show the usefulness of Corollary 2.3 by considering the classes of tournament matrices for strong score vectors with $3 \leqslant n \leqslant 6$. Table 1 lists the strong score vectors and the square of the length of each score vector.

Thus the only possible score vectors for a singular irreducible tournament matrix of order n with $n \leqslant 6$ are $(1,1,2,3,4,4)^T$ and $(1,1,3,3,3,4)^T$. In Section 4, we show that the singular tournament matrices with score vector $(1,1,2,3,4,4)^T$ are the following matrices:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

Note that up to simultaneous row and column permutations there is only one such matrix.

Suppose $M = [m_{ij}]$ is a singular tournament matrix with score vector equal to $(1,1,3,3,3,4)^T$. Then $(\frac{1}{3},\frac{1}{3},1,1,1,\frac{4}{3})^T \in \mathrm{Col}(M)$ and is exactly 1 unit from 1. It follows from Theorem 2.2 that $(\frac{2}{3},\frac{2}{3},0,0,0,-\frac{1}{3})^T$ is a unit normal to $\mathrm{Col}(M)$. Thus, $(2,2,0,0,0,-1)^T$ is orthogonal to the columns of M. Since the entries of M are 0's and 1's, both $m_{12}=0$ and $m_{21}=0$, contrary to the assumption that M is a tournament matrix. Hence, every tournament matrix with score vector equal to $(1,1,3,3,3,4)^T$ is nonsingular.

By Corollary 2.3 every tournament matrix of order n in a class $\mathscr{T}(s)$ where $s^Ts < n^2(n-1)/4$ is nonsingular. The previous example gives a score vector s with $s^Ts \geqslant n^2(n-1)/4$ for which every tournament in $\mathscr{T}(s)$ is nonsingular. Thus, Corollary 2.3 is not tight. However, the next result shows that Corollary 2.3 is tight provided we consider *real* matrices A of order n with $A + A^T = J_n - I_n$.

TABLE 1

n	s^T	s^Ts	$n^2(n-1)/4$
3	(1, 1, 1)	3	4.5
4	(1, 1, 2, 2)	10	12
5	(1, 1, 2, 3, 3)	24	25
	(2, 2, 2, 2, 2)	20	25
	(1, 2, 2, 2, 3)	22	25
6	(1, 1, 2, 3, 4, 4)	47	45
	(1, 1, 3, 3, 3, 4)	45	45
	(1, 2, 2, 3, 3, 4)	43	45
	(1, 2, 3, 3, 3, 3)	41	45
	(1, 2, 2, 2, 4, 4)	41	45
	(1, 2, 2, 3, 3, 4)	43	45
	(1, 2, 3, 3, 3, 3)	41	45
	(2, 2, 2, 3, 3, 3)	39	45

LEMMA 2.5. *Suppose $n \geqslant 2$. Let s be a real n-by-1 vector with*

$$s^T\mathbf{1} = \binom{n}{2}.$$

Then there exists a vector y with $y^Ty = 1$, $y^T\mathbf{1} = 1$, and $y^Ts = 0$ if and only if $s^Ts \geqslant n^2(n-1)/4$.

Proof. First assume that the entries of s are equal. Then

$$s = \frac{n-1}{2}\mathbf{1}.$$

Thus

$$s^Ts = \frac{n(n-1)^2}{4},$$

and if $y^T\mathbf{1} = 1$ then $y^Ts = (n-1)/2$. Hence the lemma holds when all the entries of s are equal. Now assume that not all of the entries of s are equal.

Then the system of equations

$$\begin{aligned} \sum_{i=1}^{n} w_i &= 1, \\ \sum_{i=1}^{n} w_i s_i &= 0 \end{aligned} \tag{3}$$

in the variables $w_1, \dots, w_n$ has a solution set of dimension $n - 2$. Let $w = (w_1, \dots, w_n)^T$ be a particular solution to (3), and let S be the solution space of the homogeneous system. It suffices to show that $s^Ts \geqslant n^2(n-1)/4$ if and only if the Euclidean distance from the origin to $w + S$ is less than or equal to 1. Let $\{s\}^{\perp}$ denote the subspace consisting of all vectors orthogonal to s. Now $w + S$ is an affine hyperplane in $\{s\}^{\perp}$ and has normal

$$p = \mathbf{1} - \frac{n(n-1)}{2s^Ts}s.$$

Thus the distance from the origin to the set $w + S$ is given by

$$\frac{w^Tp}{\sqrt{p^Tp}} = \frac{1}{\sqrt{n - \dfrac{n^2(n-1)^2}{4s^Ts}}},$$

and the lemma follows. ∎

THEOREM 2.6. *Suppose $n \geqslant 2$. Let s be a real n-by-1 vector with*

$$s^T\mathbf{1} = \binom{n}{2} \quad \text{and} \quad s^Ts \geqslant \frac{n^2(n-1)}{4}.$$

Then there exists a real matrix A of order n which is singular and satisfies $A + A^T = J_n - I_n$ and $A\mathbf{1} = s$.

Proof. Let $y = (y_1, y_2, \dots, y_n)^T$ be a vector with the properties guaranteed by Lemma 2.5. Without loss of generality we may assume that $y_{n-1} \neq y_n$. A matrix $A = [a_{ij}]$ satisfying

$$\begin{aligned} A + A^T &= J_n - I_n, \\ y^TA &= 0, \\ A\mathbf{1} &= s \end{aligned}$$

is constructed one row at a time. To begin, let $(a_{21}, a_{31}, \ldots, a_{n1})^T$ be an arbitrary solution to

$$\sum_{l=2}^{n} a_{l1} y_l = 0,$$

$$\sum_{l=2}^{n} a_{l1} = n - 1 - s_1.$$

Let $a_{1i} = 1 - a_{i1}$ $(i = 2, 3, \ldots, n)$ and $a_{11} = 0$. Suppose that the a_{ij} and a_{ji} have been defined for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$ such that

$$a_{ij} + a_{ji} = 1 \qquad (i \neq j),$$

$$a_{ii} = 0 \qquad (i = 1, 2, \ldots, k),$$

$$\sum_{l=1}^{n} a_{il} = s_i \qquad (i = 1, 2, \ldots, k),$$

$$\sum_{l=1}^{n} a_{lj} y_l = 0 \qquad (j = 1, 2, \ldots, k).$$

IF $k < n - 2$, we define $(a_{k+2,k+1}, a_{k+3,k+1}, \ldots, a_{n,k+1})^T$ to be an arbitrary solution of

$$\sum_{l=k+2}^{n} a_{l,k+1} y_l = -\sum_{l=1}^{k} a_{l,k+1} y_l,$$

$$\sum_{l=k+2}^{n} a_{l,k+1} = n - 1 - s_{k+1} - \sum_{l=1}^{k} a_{l,k+1}.$$

Now set $a_{k+1,k+1} = 0$ and $a_{k+1,j} = 1 - a_{j,k+1}$ for $j = k + 2, \ldots, n$. Defining the $a_{i,j}$ in this manner forces

$$\sum_{l=1}^{k+1} a_{k+1,l} = s_{k+1} \quad \text{and} \quad \sum_{l=1}^{n} a_{l,k+1} y_l = 0.$$

Continuing until $k = n - 2$, we have defined all but the elements $a_{n-1,n}$, $a_{n,n-1}$, $a_{n-1,n-1}$, and $a_{n,n}$. We note that it is possible to take $a_{ij} = 0$

$(1 \leqslant i < j \leqslant n-2)$. Set

$$\begin{aligned} a_{n-1,n-1} &= 0, \\ a_{n,n} &= 0, \\ a_{n-1,n} &= s_{n-1} - \sum_{l=1}^{n-2} a_{n-1,l}, \\ a_{n,n-1} &= 1 - a_{n-1,1}. \end{aligned}$$

Then $A + A^T = J_n - I_n$ and $A\mathbf{1} = s$. We have

$$\begin{aligned} 2y^TAy &= y^T(A + A^T)y \\ &= y^T(J_n - I_n)y \\ &= \left(y^T\mathbf{1}\right)^2 - y^Ty \\ &= 0. \end{aligned}$$

Let $A_1, A_2, \dots, A_n$ be the columns of A. Because y is orthogonal to $A_1, A_2, \dots, A_{n-2}$ and to s, y is orthogonal to $A_{n-1} + A_n$ and

$$y^TAy = y_{n-1}y^TA_{n-1} + y_n y^TA_n.$$

Hence,

$$\begin{aligned} 0 &= y^TA_{n-1} + y^TA_n, \\ 0 &= y_{n-1}y^TA_{n-1} + y_n y^TA_n. \end{aligned}$$

It follows that both y^TA_{n-1} and y^TA_n equal 0. Therefore, $y^TA = 0$ and A is singular. ■

Let $s = (1,1,3,3,3,4)^T$. As we have seen, every matrix in $\mathcal{T}(s)$ is nonsingular. However, by Theorem 2.6 there does exist a real matrix A with row-sum vector equal to s which satisfies $A + A^T = J_6 - I_6$ and is singular. The matrix

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

is one such matrix.

3. THREE-CYCLES IN TOURNAMENTS

Let $x_1, x_2, \ldots, x_n$ be n brands of a certain product. The method of paired comparisons consists of testing the brands two at a time. Thus, for each i and j with $i \neq j$, a test determines which product is better. If upon comparison i is judged to be better than j, then we say i *dominates* j. Using the results of the $\binom{n}{2}$ paired comparisons, it is desired to rank the brands. The paired comparisons define a tournament T of order n where $i \to j$ if and only if i dominates j. The presence of cycles in T indicates inconsistencies in the judging. Intuitively, a cycle of shorter length points to a deeper inconsistency than one of longer length. Kendall and Smith (1940) use the number of 3-cycles in a tournament (suitably normalized to equal 1 when no 3-cycles are present and 0 when there are as many 3-cycles as possible) as a measure of the consistency of the paired comparisons.

In this section the rank of a tournament matrix is related to the number of 3-cycles in its corresponding tournament. Let M be a tournament matrix of order n with score vector $s = (s_1, \ldots, s_n)^T$, and let T be the tournament associated with M. A *triple* is a set of three vertices. The triple $\{i, j, k\}$ is a *cyclic* triple of T provided the vertices i, j, and k form a 3-cycle in T. Denote the number of 3-cycles in the tournament T by $c(T)$. By $c(M)$ we shall mean $c(T)$. The following well-known proposition [see Moon (1968)] shows that $c(T)$ is easily computable in terms of n and s.

PROPOSITION 3.1. *Let M be a tournament matrix of order n with score vector $s = (s_1, \ldots, s_n)^T$ and associated tournament T. Then*

$$c(T) = \binom{n}{3} - \sum_{i=1}^{n} \binom{s_i}{2}$$

$$= \frac{n(n-1)(2n-1)}{12} - \frac{s^T s}{2}.$$

Proof. Each triple of T either is cyclic or has a unique vertex which dominates the other 2 vertices. Since vertex i dominates s_i vertices, there are exactly $\binom{s_i}{2}$ triples which contain i and two vertices dominated by i. Hence, there are

$$\sum_{i=1}^{n} \binom{s_i}{2}$$

noncyclic triples of T and exactly

$$\binom{n}{3} - \sum_{i=1}^{n} \binom{s_i}{2}$$

cyclic triples. The proposition follows upon recalling that

$$s^T \mathbf{1} = \binom{n}{2}. \qquad \blacksquare$$

Corollary 2.3 and Proposition 3.1 can be combined to relate the number of cyclic triples in a tournament and the rank of its tournament matrix.

COROLLARY 3.2. *Let M be a tournament matrix of order n with associated tournament T. If more than one-fourth of the triples of T are cyclic, then M is nonsingular.*

Proof. Let $s = (s_1, \dots, s_n)^T$ be the score vector of M. By Proposition 3.1,

$$c(T) = \frac{n(n-1)(2n-1)}{12} - \frac{s^T s}{2}.$$

Assume that

$$c(T) > \frac{1}{4}\binom{n}{3}.$$

Then

$$\begin{aligned} s^T s &< \frac{n(n-1)(2n-1)}{6} - \frac{n(n-1)(n-2)}{12} \\ &= \frac{n^2(n-1)}{4}. \end{aligned}$$

By Corollary 2.3, M is nonsingular. $\blacksquare$

Using the following results of Moran (1947), it is possible to give an asymptotic estimate of the fraction of tournaments in which more than one-fourth of the triples are cyclic.

PROPOSITION 3.3. *The expected number of* 3*-cycles in a random tournament matrix of order* n *is equal to*

$$\frac{1}{4}\binom{n}{3}.$$

Proof. For each triple $\{i, j, k\}$ there exist exactly $2 \times 2^{\binom{n}{2}-3}$ tournaments in which the triple is cyclic. There are $\binom{n}{3}$ triples. Thus the expected number μ of 3-cycles satisfies

$$\mu = \frac{2 \times 2^{\binom{n}{2}-3} \times \binom{n}{3}}{2^{\binom{n}{2}}} = \frac{1}{4}\binom{n}{3}. \qquad \blacksquare$$

In light of Proposition 3.3, we can rephrase Corollary 3.2 as follows: If a tournament matrix T has more than the expected number of 3-cycles, then its adjacency matrix is nonsingular. The following theorem of Moran (1947) states that if the set of tournament matrices is viewed as a sample space with uniform probability, then the distribution of the number of 3-cycles is asymptotically normal. A proof may be found in Moon (1968).

THEOREM 3.4. *Let* c_n *denote the number of* 3*-cycles in a random tournament* T_n *of order* n. *Then the distribution* $(c_n - \mu)/\sigma$ *tends to the normal distribution* d_n *with zero mean and unit variance, where*

$$\mu = \frac{1}{4}\binom{n}{3} \quad \textit{and} \quad \sigma^2 = \frac{3}{16}\binom{n}{3}.$$

Let M_n denote a random tournament matrix of order n, and let p_n equal the probability that M_n is nonsingular. By Corollary 3.2,

$$\begin{aligned} p_n &\geqslant \Pr\left(c(M_n) > \frac{1}{4}\binom{n}{3}\right) \\ &= \Pr(c_n > \mu) \\ &= \Pr\left(\frac{c_n - \mu}{\sigma} > 0\right). \end{aligned}$$

By Theorem 3.4,

$$\lim_{n\to\infty} \Pr\left(\frac{c_n - \mu}{\sigma} > 0\right) = \Pr(d_n > 0) = \frac{1}{2}.$$

In summary we have shown the following:

Corollary 3.5. *Let $\epsilon > 0$. Then there exists a positive integer n_0 such that if $n > n_0$, then the fraction of tournament matrices of order n that are nonsingular is at least $\frac{1}{2} - \epsilon$.*

Let $\mathscr{T}_n^*$ be the set of real matrices A of order n with $A + A^T = J_n - I_n$ and whose row-sum vector s satisfies $s^Ts < n^2(n-1)/4$. Arguments similar to those used for Corollary 2.3 show that every matrix in $\mathscr{T}_n^*$ is nonsingular. $\mathscr{T}_n^*$ is clearly a convex set in $\mathscr{R}^{n^2}$ and therefore is a connected set. Because the determinant is a continuous function which does not vanish on any matrix in $\mathscr{T}_n^*$, the matrices in $\mathscr{T}_n^*$ all have determinants of the same sign. Let $A_n = \frac{1}{2}(J_n - I_n)$. Since $A_n \in \mathscr{T}_n^*$ and $\det A_n = (-1)^{n-1}(n-1)/2$, we conclude:

Corollary 3.6. *If M is a tournament matrix of order n with score vector s and $s^Ts > n^2(n-1)/4$, then $(-1)^{n-1}\det M > 0$.*

Combining Theorem 3.4 and Corollary 3.6, we obtain

Corollary 3.7. *Let $\epsilon > 0$. Then for n sufficiently large the fraction of tournament matrices of order n whose determinant has sign equal to the sign of $(-1)^{n-1}$ is at least $\frac{1}{2} - \epsilon$.*

As noted in the introduction, it has been conjectured that "almost all" tournament matrices are nonsingular. Komlós (1967) has shown that almost all square (0, 1) matrices are nonsingular. Thus the conjecture for tournament matrices is reasonable. Corollary 3.7 lends more support to the conjecture. Indeed, if one could show that the likelihood that a random tournament matrix has positive determinant is asymptotically the same as the likelihood that it has a negative determinant, then Corollary 3.7 would imply that almost all tournament matrices are nonsingular.

4. SPECTRAL PROPERTIES

Theorem 2.2 shows that the geometric multiplicity of the eigenvalue 0 for a tournament matrix is either 0 or 1. In this section other properties of the eigenvalues of tournament matrices are investigated. We begin by stating a result of Brauer and Gentry (1968).

PROPOSITION 4.1. *Let M be a tournament matrix of order n, and let λ be an eigenvalue of M. Then*

$$\frac{n-1}{2} \geqslant \operatorname{Re} \lambda \geqslant -\frac{1}{2}.$$

Moreover, $\operatorname{Re} \lambda = (n-1)/2$ *if and only if* $\lambda = (n-1)/2$ *and M is a regular tournament matrix.*

Maybee and Pullman (1990) have noted that like 0, any eigenvalue λ of a tournament matrix with $\operatorname{Re} \lambda > -\frac{1}{2}$ has geometric multiplicity 1.

PROPOSITION 4.2. *Let M be a tournament matrix of order n. Suppose λ is an eigenvalue of M with* $\operatorname{Re} \lambda > -\frac{1}{2}$. *Then the geometric multiplicity of λ equals* 1.

The *spectral radius* $\rho(M)$ of a tournament matrix M of order n is the maximum modulus of an eigenvalue of M. From the theory of nonnegative matrices, $\rho(M)$ is itself an eigenvalue of M. Proposition 4.1 gives an upper bound for the spectral radius of a tournament matrix of order n. This bound can be improved when the tournament matrix is singular.

PROPOSITION 4.3. *Let M be a singular tournament matrix of order n. Then* $\rho(M) \leqslant (n-2)/2$.

Proof. Let $\rho(M) = \lambda_1$, $0 = \lambda_2, \lambda_3, \dots, \lambda_n$ be the eigenvalues of M. By Proposition 4.1, $\operatorname{Re} \lambda_i \geqslant -\frac{1}{2}$. Because $\operatorname{tr} M = 0$,

$$0 = \rho(M) + \sum_{i=3}^{n} \operatorname{Re} \lambda_i \geqslant \rho(M) - \frac{n-2}{2}.$$

Thus $(n-2)/2 \geqslant \rho(M)$. ∎

The tournament matrices of order n whose spectral radius achieves the upper bound in Proposition 4.1 all have the same score vector. This is not the case for singular tournament matrices of order n whose spectral radius equals the upper bound of Proposition 4.3. Indeed, later in this section examples of such singular tournament matrices of order 8 with score vectors $(3,3,3,3,3,3,3,7)^T$ and $(1,3,3,3,3,5,5,5)^T$ will be given. However, we now show that the score vector s of a singular tournament matrix of order n with spectral radius $(n-2)/2$ satisfies $s^T s = n^2(n-1)/4$.

THEOREM 4.4. *Let M be a singular tournament matrix of order n. Suppose that the spectral radius of M equals $(n-2)/2$. Then the score vector s and the coscore vector r of M satisfy $s^Ts = r^Tr = n^2(n-1)/4$.*

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of M, and assume that $\lambda_1 = (n-2)/2$ and $\lambda_2 = 0$. Let $\lambda_j = a_j + b_j i$ for $j = 3, 4, \dots, n$. Because $\operatorname{tr} M = 0$ and $\operatorname{Re} \lambda_j \geqslant -\frac{1}{2}$, we must have $a_j = -\frac{1}{2}$ for $j = 3, 4, \dots, n$. It is clear that $\operatorname{tr} M^l$ equals the number of closed directed paths of length l in the tournament corresponding to M. Thus $\operatorname{tr} M^2 = 0$ and

$$0 = \sum_{i=1}^{n} \lambda_i^2 = \left(\frac{n-2}{2}\right)^2 + \frac{n-2}{4} - \sum_{i=3}^{n} b_j^2 - i\sum_{j=3}^{n} b_j.$$

In particular,

$$\sum_{j=3}^{n} b_j^2 = \frac{n^2-3n+2}{4}. \tag{4}$$

Now

$$\begin{aligned}
\operatorname{tr} M^3 &= \sum_{j=1}^{n} \lambda_j^3 \\
&= \left(\frac{n-2}{2}\right)^3 + \sum_{j=3}^{n}\left(-\tfrac{1}{2} + b_j i\right)^3 \\
&= \left(\frac{n-2}{2}\right)^3 - \left(\frac{n-2}{8}\right) + \frac{3}{4} i \sum_{j=3}^{n} b_j + \frac{3}{2}\sum_{j=3}^{n} b_j^2 - i\sum_{j=3}^{n} b_j^3.
\end{aligned}$$

Equating real parts and using (4), we have

$$\operatorname{tr} M^3 = \left(\frac{n-2}{2}\right)^3 - \left(\frac{n-2}{8}\right) + \frac{3}{2}\left(\frac{n^2-3n+2}{4}\right) \tag{5}$$

$$= \frac{n(n-1)(n-2)}{8}. \tag{6}$$

As noted, $\operatorname{tr} M^3 = 3c(M)$, where $c(M)$ denotes the number of 3-cycles in M. Proposition 3.1 and (6) now imply that $s^Ts = n^2(n-1)/4$. The result follows upon recalling that $r^Tr = s^Ts$. ∎

We note that Theorem 2.2, Corollaries 2.3 and 2.4, and Propositions 4.1 and 4.2 can be extended to include real matrices M of order n which satisfy $M + M^T = J_n - I_n$. The following theorem uses strongly the fact that the entries of a tournament matrix are 0 or 1.

THEOREM 4.5. *Suppose $M = [m_{ij}]$ is a tournament matrix of order n with score vector $s = (s_1, s_2, \dots, s_n)^T$ and coscore vector $r = (r_1, r_2, \dots, r_n)^T$. Let t_i equal the number of directed paths in M of length 2 which begin at vertex i, and let $\Delta = s^Ts - n^2(n-1)/4$. Suppose M is singular. Then*

$$\frac{n-2}{2}s_i - \sqrt{\frac{s_ir_i\Delta}{n-1}} \leqslant t_i \leqslant \frac{n-2}{2}s_i + \sqrt{\frac{s_ir_i\Delta}{n-1}} \qquad (i = 1, \dots, n).$$

Proof. By Theorem 2.2 (applied to M^T), $(n-1)MM^T - ss^T$ is a positive semidefinite matrix. Let ϵ_i be the (0, 1) column vector whose only 1 is in the ith row. For $\lambda \in \mathscr{R}$,

$$\begin{aligned}
&(n-1)(1-\lambda\epsilon_i)^T MM^T(1-\lambda\epsilon_i) - (1-\lambda\epsilon_i)^T ss^T(1-\lambda\epsilon_i) \\
&\quad = (n-1)1^TMM^T1 - 2(n-1)\lambda 1^TMM^T\epsilon_i \\
&\qquad + \lambda^2(n-1)\epsilon_i^TMM^T\epsilon_i - 1^Tss^T1 + 2\lambda 1^Tss^T\epsilon_i - \lambda^2\epsilon_i^Tss^T\epsilon_i \\
&\quad = (n-1)rr^T - 2(n-1)\lambda r^TM^T\epsilon_i + \lambda^2(n-1)\sum_{j=1}^{n}(m_{ij})^2 \\
&\qquad - \frac{n^2(n-1)^2}{4} + n(n-1)\lambda s_i - \lambda^2 s_i^2 \\
&\quad = \lambda^2\left[(n-1)\sum_{j=1}^{n}(m_{ij})^2 - s_i^2\right] + \lambda(n-1)\left(ns_i - 2\epsilon_i^TMr\right) + (n-1)\Delta \\
&\quad \geqslant 0.
\end{aligned}$$

Because M is a (0, 1) matrix

$$\sum_{j=1}^{n}(m_{ij})^2 = \sum_{j=1}^{n}m_{ij} = s_i. \tag{7}$$

Also,

$$\begin{aligned}\epsilon_i^T M r &= \sum_{i\to j} r_j \\ &= \sum_{i\to j}(n-1-s_j) \\ &= (n-1)s_i - \sum_{i\to j} s_j \\ &= (n-1)s_i - t_i. \end{aligned} \tag{8}$$

After substituting (7) and (8) we obtain the quadratic inequality

$$\lambda^2 s_i r_i + \lambda(n-1)[2t_i - (n-2)s_i] + (n-1)\Delta \geqslant 0 \qquad (\lambda \in \mathscr{R}).$$

Thus,

$$(n-1)^2[2t_i - (n-2)s_i]^2 \leqslant 4(n-1)\Delta s_i r_i,$$

and the theorem follows. ■

We illustrate Theorem 4.5 with some examples. First, we reconsider the score vector $s = (1,1,3,4,4,4,4)^T$. In this instance $s^T s = 75$ and $\Delta = 1.5$. Suppose $M = [m_{ij}] \in \mathscr{T}(s)$. Let t_i be as in the previous theorem. Because either $m_{12} = 1$ or $m_{21} = 1$, one of t_1 or t_2 equals 1. The bounds given by Theorem 4.5 for t_1 and t_2 are

$$1 < \tfrac{5}{2} - \sqrt{\tfrac{5}{4}} \leqslant t_1, t_2 \leqslant \tfrac{5}{2} + \sqrt{\tfrac{5}{4}} < 4.$$

Hence, we have another proof that every matrix in $\mathscr{T}(s)$ is nonsingular.

Now let $s = (1,1,2,3,4,4)^T$. We use Theorem 4.5 to determine all the singular tournament matrices in $\mathscr{T}(s)$. In this case $\Delta = 2$. Let $M = [m_{ij}] \in \mathscr{T}(s)$ be a singular matrix. By simultaneous row and column permutations we may assume that $m_{12} = 1$ and $m_{56} = 1$. Thus the other entries of row 1 equal 0, and the other entries of row 6 equal 1. Theorem 4.5 implies that $t_3 \geqslant 4 - \sqrt{\tfrac{12}{5}} > 2$. If $m_{32} = 1$, then $t_3 = 2$. Hence $m_{32} = 0$. This implies that $m_{23} = 1$ and the other entries in the second row are 0's. By Theorem 4.5, $t_4 \geqslant 6 - \sqrt{\tfrac{12}{5}} > 4$. If $m_{43} = 1$, then $t_4 = 1 + 1 + 2$. Hence, $m_{43} = 0$. The matrix is now completely determined, and

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

It is easily verified that M is singular. The four matrices in (1) and (2) can be obtained by simultaneously interchanging the first two rows and columns or the last two rows and columns. We conclude that these are the only singular matrices in $\mathscr{T}(s)$.

Recall that a singular tournament matrix of order n with score vector s satisfies $s^Ts \geqslant n^2(n-1)/4$. In Theorem 4.5, $\Delta = 0$ if and only if $s^Ts = n^2(n-1)/4$. The singular tournament matrices of order n whose score vector s satisfies $s^Ts = n^2(n-1)/4$ have many interesting properties.

COROLLARY 4.6. *Let s be a score vector of order n. Assume that $s^Ts = n^2(n-1)/4$. Then $M \in \mathscr{T}(s)$ is singular if and only if*

$$Ms = \frac{n-2}{2}s.$$

Proof. Let $M \in \mathscr{T}(s)$. First assume that $Ms = [(n-2)/2]s$. Then

$$M\left(s - \frac{n-2}{2}\mathbf{1}\right) = 0.$$

Because

$$s^T\mathbf{1} = \binom{n}{2},$$

$s \neq [(n-2)/2]\mathbf{1}$ and thus M is singular.

Now suppose M is singular. Because $s^Ts = n^2(n-1)/4$, Theorem 4.5 implies that

$$t_i = \frac{n-2}{2}s_i \qquad (i = 1, \dots, n).$$

But t_i equals the entry in row i of $M^2\mathbf{1}$. Hence $Ms = M^2\mathbf{1} = [(n-2)/2]s$, as desired. ∎

We summarize with the following corollary.

COROLLARY 4.7. *Let M be a singular tournament matrix of order n with spectral radius equal to $(n-2)/2$. Then n is even, and the score and coscore* *vector of M satisfy*

$$Ms = \left(\frac{n-2}{2}\right)s,$$

$$r^TM = \left(\frac{n-2}{2}\right)r^T,$$

$$s^Ts = \frac{n^2(n-1)}{4},$$

$$r^Tr = \frac{n^2(n-1)}{4}.$$

Proof. Since M is a $(0,1)$ matrix, the eigenvalues of M are algebraic integers. Hence $(n-2)/2$ is an integer and n is even. The result now follows from Theorems 4.4 and Corollary 4.6. ∎

Let n be a positive even integer. A trivial example of a score vector s of length n which satisfies $s^Ts = n^2(n-1)/4$ is

$$s = \left(n-1, \frac{n-2}{2}, \frac{n-2}{2}, \dots, \frac{n-2}{2}\right)^T.$$

Any matrix $A \in \mathcal{T}(s)$ has the form

$$A = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{bmatrix},$$

where B is a regular tournament matrix of order $n-1$. Since A is singular, Corollary 4.7 implies that $As = [(n-2)/2]s$, and this fact is easily verified. The next two propositions give more interesting examples of such score vectors.

PROPOSITION 4.8. *Suppose k is a positive integer with $n = 4k$. Let s be the monotone score vector of length n which has a single score equal to* 1, $2k$ *scores equal to* $2k-1$, *and* $2k-1$ *scores equal to* $2k+1$. *Then s is a strong score vector with $s^Ts = n^2(n-1)/4$, and $M \in \mathcal{T}(s)$ is singular if and only if there* *exists a permutation matrix P such that*

$$PMP^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & & & & & & \\ \vdots & \vdots & & C & & & D & \\ 1 & 0 & & & & & & \\ 1 & 1 & & & & & & \\ \vdots & \vdots & & J - D^T & & & F & \\ 1 & 1 & & & & & & \end{bmatrix},$$

where C and F are regular tournament matrices of order $2k - 1$, D is a $(0, 1)$ matrix of order $2k - 1$ with exactly $k - 1$ ones in each row and column, and J is the all-1's matrix of approximate size.

Proof. Clearly,

$$\begin{aligned} s^T s &= 1 + 2k(2k-1)^2 + (2k-1)(2k+1)^2 \\ &= 1 + (8k^3 - 8k^2 + 2k) + (8k^3 + 4k^2 - 2k - 1) \\ &= 16k^3 - 4k^2 \\ &= 4k^2(4k-1) \\ &= \frac{n^2(n-1)}{4}. \end{aligned}$$

Let

$$E = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & & & & & & \\ \vdots & \vdots & & L + L^2 + \cdots + L^{k-2} & & & L + \cdots + L^{k-2} & \\ 1 & 0 & & & & & & \\ 1 & 1 & & & & & & \\ \vdots & \vdots & & J - (L + \cdots + L^{k-2})^T & & & L + \cdots + L^{k-2} & \\ 1 & 1 & & & & & & \end{bmatrix},$$

where L is a permutation matrix of order $2k - 1$ corresponding to the permutation

$$1 \to \cdots \to 2k - 1 \to 1$$

and J is a matrix of all 1's of appropriate size. Then E belongs to $\mathscr{T}(s)$ and its digraph is strongly connected, because

$$1 \to 2 \to 3 \to \cdots \to n \to 1$$

in the tournament corresponding to E. It follows that s is a strong score vector.

Let $M = [m_{ij}] \in \mathscr{T}(s)$. By Corollary 4.6, M is singular if and only if $Ms = [(n-2)/2]s$, or equivalently, if and only if

$$M\left(s - \frac{n-2}{2}\mathbf{1}\right) = 0.$$

Let

$$x = \frac{1}{2}\left(s - \frac{n-2}{2}\mathbf{1}\right).$$

Then x is the column vector whose first entry equals $1-k$, whose last $2k-1$ entries equal 1, and whose remaining entries equal 0. Assume that M is singular. Because $m_{11} = 0$ and $Mx = 0$,

$$m_{1,2k+2} = \cdots = m_{1,4k} = 0.$$

By simultaneously permuting rows and columns we may without loss of generality assume that $m_{12} = 1$. Since $m_{21} = 0$ and $Mx = 0$,

$$m_{2,2k+2} = \cdots = m_{2,4k} = 0.$$

The second row has exactly $2k-1$ ones, and therefore

$$m_{2,3} = m_{2,4} = \cdots = m_{2,2k+1} = 1.$$

Thus M has the form

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & & & & & & \\ \vdots & \vdots & & C & & D & & \\ 1 & 0 & & & & & & \\ 1 & 1 & & & & & & \\ \vdots & \vdots & & J - D^T & & F & & \\ 1 & 1 & & & & & & \end{bmatrix},$$

where C and F are tournament matrices of order $2k - 1$.

$Mx = 0$ implies that D has constant row sums equal to $k - 1$. Similarly, F has constant row sums equal to $k - 1$. It follows that C and F are regular tournament matrices and that D has constant row and column sums equal to $k - 1$. The converse is easily verified. ■

PROPOSITION 4.9. *Let k be an odd integer and let $n = k^2 + 1$. Let s be the monotone score vector of length n with half of its scores equal to $\binom{k}{2}$ and the other half equal to $\binom{k+1}{2}$. Then s is a strong score vector with $s^T s = n^2(n - 1)/4$, and $M \in \mathcal{T}(s)$ is singular if and only if M has the form*

$$M = \begin{bmatrix} B & C \\ J - C^T & D \end{bmatrix}$$

where B and D are regular tournament matrices of order $n/2$, C is a $(0, 1)$ matrix of order $n/2$ with exactly $(k - 1)^2/4$ ones in each row and column, and J is the all-1's matrix of appropriate size.

Proof. It is easily verified that s is a strong score vector. Let $M \in \mathcal{T}(s)$. By Corollary 4.6, M is singular if and only if $Ms = [(n - 2)/2]s$. Let

$$x = s - \frac{n - 2}{2}\mathbf{1}.$$

Then the first $n/2$ entries of x equal $-(k - 1)/2$, and the remaining entries equal $(k + 1)/2$. M is singular if and only if $Mx = 0$. Write

$$M = \begin{bmatrix} B & C \\ J - C^T & D \end{bmatrix},$$

where B and D are both of order $n/2$. Let b_i and c_i $(i = 1, 2, \ldots, n/2)$ denote the row sums of B and C, respectively. Similarly let d_i and e_i $(i = 1, 2, \ldots, n/2)$ denote the row sums of $J - C^T$ and D, respectively. Then

$$b_i + c_i = \binom{k}{2} \quad \text{and} \quad d_i + e_i = \binom{k+1}{2}.$$

Furthermore, M is singular if and only if

$$\left(\frac{k+1}{2}\right)c_i - \left(\frac{k-1}{2}\right)b_i = 0$$

and

$$\left(\frac{k+1}{2}\right)d_i - \left(\frac{k-1}{2}\right)e_i = 0$$

for $i = 1, 2, \ldots, n/2$. Hence M is singular if and only if

$$b_i = \frac{k^2-1}{4}, \qquad c_i = \frac{(k-1)^2}{4},$$

$$e_i = \frac{(k+1)^2}{4}, \qquad d_i = \frac{k^2-1}{4}$$

for $i = 1, 2, \ldots, n/2$. The proposition now follows. ∎

Let M be an irreducible tournament matrix of order n. Let x be an n-by-1 vector each of whose entries is positive, and suppose that λ and μ are real numbers such that $\lambda x \leqslant Mx \leqslant \mu x$, where $\leqslant$ between vectors means entrywise ordering. Then Perron-Frobenius theory implies that $\lambda \leqslant \rho(M) \leqslant \mu$. This fact along with Proposition 4.3 and Theorem 4.5 yields the following corollary.

COROLLARY 4.10. *Let M be an irreducible tournament matrix of order n with monotone score vector $s = (s_1, \ldots, s_n)^T$ and coscore vector $r = (r_1, \ldots, r_n)^T$. Let $\Delta = s^T s - n^2(n-1)/4$. If M is singular, then*

$$\frac{n-2}{2} - \sqrt{\frac{r_1 \Delta}{s_1(n-1)}} \leqslant \rho(M) \leqslant \frac{n-2}{2}.$$

Corollary 4.10 only provides information on the spectral radius of singular tournament matrices. We now discuss bounds on the spectral radius for arbitrary tournament matrices. Moon (1968) studies the diameters of strong tournaments and obtains the following result:

THEOREM 4.11. *For any* $\epsilon > 0$ *almost all tournament matrices* M_n *(that is, all but a fraction that tends to zero as n tends to infinity) satisfy*

$$\left(\tfrac{1}{4} - \epsilon\right)(n-2)(J_n - I_n) \leqslant M_n^2 \leqslant \left(\tfrac{1}{4} + \epsilon\right)(n-2)(J_n - I_n), \tag{9}$$

where $\leqslant$ *between matrices means entrywise ordering.*

Suppose M_n satisfies (9); then

$$\left(\tfrac{1}{4} - \epsilon\right)(n-2)(n-1)\mathbf{1} \leqslant M^2\mathbf{1} \leqslant \left(\tfrac{1}{4} + \epsilon\right)(n-2)(n-1). \tag{10}$$

If M_n is irreducible, then Perron-Frobenius theory and (10) imply that

$$\left(\tfrac{1}{4} - \epsilon\right)(n-2)(n-1) \leqslant \left[\rho(M_n)\right]^2 \leqslant \left(\tfrac{1}{4} + \epsilon\right)(n-2)(n-1).$$

Moon and Moser (see Moon, 1968) have shown that almost all tournaments are strong, and thus we conclude:

COROLLARY 4.12. *For any* $\epsilon > 0$ *the spectral radii of almost all tournament matrices* M_n *of order n satisfy*

$$\frac{1-\epsilon}{2}\sqrt{(n-1)(n-2)} \leqslant \rho(M_n) \leqslant \frac{1+\epsilon}{2}\sqrt{(n-1)(n-2)}\,.$$

5. THE POLYTOPE $\mathscr{T}_+(s)$

We have exhibited examples of score vectors s for which every matrix in $\mathscr{T}(s)$ is nonsingular. In this section we consider score vectors s for which every matrix in $\mathscr{T}(s)$ is singular. Let $s = (s_1, \dots, s_n)^T$ be a monotone strong score vector with $\mathscr{T}(s) \neq \varnothing$. Let $\mathscr{T}^*(s)$ [respectively, $\mathscr{T}_+(s)$] denote the class of real [respectively, nonnegative] matrices M of order n which satisfy

$$M + M^T = J_n - I_n,$$
$$M\mathbf{1} = s.$$

Throughout this section we view the set of matrices of order n as vectors in $\mathscr{R}^{n^2}$.

PROPOSITION 5.1. *$\mathscr{T}^*(s)$ is a flat of dimension* $\binom{n-1}{2}$ *in* $\mathscr{R}^{n^2}$.

Proof. Consider the following system of linear equations in the variables x_{ij} $(1 \leqslant i \leqslant j \leqslant n)$:

$$\begin{gathered}(i-1)-\sum_{k=1}^{i-1} x_{ki}+\sum_{k=i+1}^{n} x_{ik}=s_i \qquad (i=1,\dots,n-1), \\ x_{ii}=0 \qquad (i=1,2,\dots,n).\end{gathered} \tag{11}$$

A solution x_{ij} $(1 \leqslant i \leqslant j \leqslant n)$ to (11) determines the tournament matrix $M=[m_{ij}]$, where

$$m_{ij}=\begin{cases} x_{ij} & \text{if } 1 \leqslant i<j \leqslant n, \\ 0 & \text{if } i=j, \\ 1-x_{ij} & \text{if } 1 \leqslant j<i \leqslant n. \end{cases}$$

Clearly, $M \in \mathscr{T}^*(s)$. Conversely, a matrix $M=[m_{ij}] \in \mathscr{T}^*(s)$ gives rise to the solution $x_{ij}=m_{ij}$ $(1 \leqslant i \leqslant j \leqslant n)$. The equations in (11) are easily seen to be independent. Thus the solution set of (11) is $\mathscr{T}^*(s)$ and has dimension

$$\binom{n+1}{2}-n-(n-1)=\binom{n-1}{2}. \qquad \blacksquare$$

PROPOSITION 5.2. *$\mathscr{T}_+(s)$ is a convex polytope of full dimension in $\mathscr{T}^*(s)$, and its extreme points are precisely the elements of $\mathscr{T}(s)$.*

Proof. $\mathscr{T}_+(s)$ is the polytope defined by the inequalities

$$\begin{aligned} 0 \leqslant m_{ij} &\leqslant 1 && (i \neq j), \\ m_{ii} &= 0 && (i=1,\dots,n), \\ m_{ij}+m_{ji} &= 1 && (i \neq j), \\ \sum_{j=1}^{n} m_{ij} &= s_i && (i=1,\dots,n). \end{aligned}$$

Any matrix of $\mathscr{T}(s)$ is clearly an extreme point of $\mathscr{T}_+(s)$.

Conversely, suppose $M = [m_{ij}]$ is an extreme point of $\mathscr{T}_+(s)$. Let G be the graph with vertices $\{1, \dots, n\}$ where i and j are adjacent if and only if $0 < m_{ij} < 1$. Suppose G has a simple cycle γ. Let A be a (0, 1) matrix of order n such that $A + A^T$ is the adjacency matrix of the graph consisting of precisely the edges of γ. Then for ϵ sufficiently small, both $M + \epsilon(A - A^T)$ and $M - \epsilon(A - A^T)$ lie in $\mathscr{T}_+(s)$. This contradicts the extremality of M. Hence G has no cycles. Because s has integer entries, no vertex of G has degree equal to 1. We conclude that G has no edges and hence $M \in \mathscr{T}(s)$.

Let

$$B = \frac{1}{|\mathscr{T}(s)|} \sum_{m \in \mathscr{T}(s)} M$$

be the barycenter of $\mathscr{T}_+(s)$. Because s is strong, each off-diagonal entry of B is positive. For $i < j < k$, let $\Delta_{i,j,k}$ denote the $(0, 1, -1)$ matrix of order n with 1's in positions (i, j), (j, k) and (k, i), with -1's in positions (j, i), (k, j), and (i, k), and with 0's elsewhere. Then there exists $\rho > 0$ such that

$$B + \epsilon\Delta_{i,j,k} \in \mathscr{T}_+(s) \qquad (i < j < k \text{ and } -\rho < \epsilon < \rho).$$

Because $\mathscr{T}_+(s)$ is convex, there exists $\delta > 0$ such that

$$B + v \in \mathscr{T}_+(s) \qquad \text{for any} \quad v \in \operatorname{span}\{\Delta_{i,j,k}\} \quad \text{with } v^T v < \delta.$$

It is an easy exercise to show that the vectors $\Delta_{i,j,k}$ $(1 \leqslant i < j < k \leqslant n)$ span the solution space of the homogeneous system corresponding to (11). Hence, $\mathscr{T}_+(s)$ has full-dimension in $\mathscr{T}^*(s)$. ■

THEOREM 5.3. *Let s be a strong score vector such that $\mathscr{T}(s)$ is nonempty. Then there exists a nonsingular matrix in $\mathscr{T}_+(s)$.*

Proof. Let $A = [a_{ij}]$ be the matrix of order n where

$$a_{ij} = 0 \qquad (1 \leqslant i \leqslant j \leqslant n-1),$$

$$a_{ij} = 1 \qquad (1 \leqslant j < i \leqslant n-1),$$

$$a_{in} = s_i - (i-1) \qquad (1 \leqslant i \leqslant n-1),$$

$$a_{ni} = i - s_i \qquad (1 \leqslant i \leqslant n-1),$$

$$a_{nn} = 0.$$

Then $A \in \mathscr{T}^*(s)$ and is nonsingular. On $\mathscr{T}^*(s)$ the determinant can be expressed as a polynomial in $\binom{n-1}{2}$ variables. Because $\mathscr{T}_+(s)$ has full dimension in $\mathscr{T}^*(s)$, this polynomial vanishes on $\mathscr{T}_+(s)$ if and only if it is identically zero. The result now follows. ■

Now let $s = (s_1, \dots, s_n)^T$ be a monotone, but not necessarily strong, score vector with $\mathscr{T}(s)$ nonempty. Ryser (1964) has shown that there exist positive integers l and $n_1, \dots, n_l$ such that every matrix $M \in \mathscr{T}(s)$ has the block form

$$M = \begin{bmatrix} M_1 & & & 0\text{'s} \\ & M_2 & & \\ & & \ddots & \\ 1\text{'s} & & & M_l \end{bmatrix}$$

where M_i is a tournament matrix of order n_i and is strong if $n_i \geqslant 2$. Clearly, M is nonsingular if and only if each M_i is nonsingular. An immediate consequence of Theorem 5.3 is the following:

COROLLARY 5.4. *Every matrix in $\mathscr{T}_+(s)$ is singular if and only if some n_i equals* 1.

We have yet to find a strong score vector s for which every matrix in $\mathscr{T}(s)$ is singular. Hence we are led to believe:

CONJECTURE 5.5. Every matrix in $\mathscr{T}(s)$ is singular if and only if some n_i equals 1.

6. ANNIHILATING VECTORS

Let M be a singular tournament matrix of order n. By statement (iii) of Theorem 2.2 there exists a vector $y = (y_1, \dots, y_n)^T$ in the null space of M such that $y^Ty = (y^T\mathbf{1})^2$ and $y^T\mathbf{1} > 0$. Because the entries of M are integral, we may take y to have entries which are integers with greatest common divisor equal to 1. A nonzero integral vector $w = (w_1, \dots, w_n)^T$ whose entries are relatively prime with $w^T\mathbf{1} > 0$ and which is in the null space of some tournament matrix is called an *annihilating vector*. This section discusses properties of annihilating vectors. Annihilating vectors were studied in Maybee and Pullman (1990).

By Lemma 2.2, an n-by-1 annihilating vector w must satisfy $w^T w = (w^T \mathbf{1})^2$. A stronger requirement is given by:

PROPOSITION 6.1. *Let $w = (w_1, \dots, w_n)^T$ be an annihilating vector. Then either w has exactly one nonzero entry or for each j there exists a set $W_j \subseteq \{1, \dots, n\} - \{j\}$ such that $w_j = -\sum_{i \in W_j} w_i$.*

Proof. Let $M = [m_{ij}]$ be a tournament matrix of order n with $Mw = 0$. Assume w has at least two nonzero entries. Because M has rank equal to $n-1$ and $Mw = 0$, no column of M is a zero column. Thus for each j, there exists an integer i with $m_{ij} = 1$. But then

$$0 = \sum_{k=1}^{n} m_{ik} w_k.$$

Hence, we may take $W_j = \{i : i \neq j \text{ and } m_{ij} = 1\}$. ■

Note that in particular Proposition 6.1 implies that if w has at least two nonzero entries, then the greatest common divisor of $\{w_1, \dots, w_n\} - \{w_j\}$ equals 1 ($j = 1, \dots, n$). This is a result of Maybee and Pullman (1990). The next proposition determines the annihilating vectors each of whose entries has absolute value equal to 1.

PROPOSITION 6.2. *Let $v = (\overbrace{1, \dots, 1}^{k}, \overbrace{-1, \dots, -1}^{l})^T$ with $k > l$. Then v is an annihilating vector if and only if there exists a positive integer m with*

$$l = \binom{m}{2} \quad \text{and} \quad k = \binom{m+1}{2}.$$

Proof. Suppose that v is an annihilating vector. Then $v^T v = (v^T \mathbf{1})^2$ and thus $k + l = (k - l)^2$. It follows that

$$k = l + \frac{1}{2} + \frac{\sqrt{8l+1}}{2}.$$

Let m equal the integer $\frac{1}{2} + \sqrt{8l+1}/2$. It is easy to show that $l = \binom{m}{2}$. But $k - l = m$ and thus $k = \binom{m+1}{2}$.

For the converse, we give an inductive construction. Let v_m denote the m^2-by-1 vector whose first $\binom{m+1}{2}$ entries are 1 and whose remaining entries are -1. If n is a nonnegative integer, then let R_{2n+1} denote a regular tournament matrix of order $2n+1$. The matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has annihilating vector $v_2 = (1, 1, 1, -1)^T$. Suppose that v_m is the annihilating vector of

$$M = \begin{bmatrix} A & B \\ J - B^T & C \end{bmatrix},$$

where A has order $\binom{m+1}{2}$, C has order $\binom{m}{2}$, and J denotes a matrix of all ones. Then v_{m+1} is the annihilating vector of the matrix

$$\begin{bmatrix} R_{2m+1} & J & J \\ O & C & J - B^T \\ O & B & A \end{bmatrix},$$

where O denotes a zero matrix. ■

We conclude this short section by giving a few examples of annihilating vectors.

Let $n = k^2 + 1$, where k is an odd integer, and let s be the monotone score vector with half its scores equal to $\binom{k}{2}$ and the other half equal to $\binom{k+1}{2}$. The proof of Proposition 4.9 shows that

$$v = \Bigg(\overbrace{-\frac{k-1}{2}, \dots, -\frac{k-1}{2}}^{2k}, \overbrace{\frac{k+1}{2}, \dots, \frac{k+1}{2}}^{2k} \Bigg)^T$$

is the annihilating vector of every singular tournament matrix with score vector s.

Another interesting example is the annihilating vector of the tournament matrix M_l of order $2l$ whose 1's above the main diagonal occur in the positions $(2i, 2i+2)$ $(i = 1, \dots, l-1)$. Let f_i denote the ith term in the Fibonacci sequence. Then it can be shown that the annihilating vector of M_l is given by

$$v_l = \left(-f_0, f_0, -f_1, f_1, \dots, -f_{l-2}, f_{l-2}, f_{l-2}, f_{l-1}\right)^T.$$

Because v_l is an annihilating vector, $v^T v = (v^T \mathbf{1})^2$. Hence

$$(f_{l-2} + f_{l-1})^2 = 2\left(\sum_{i=1}^{l-2} f_i^2\right) + f_{l-2}^2 + f_{l-1}^2.$$

It follows that

$$f_{l-2} f_{l-1} = \sum_{i=1}^{l-2} f_i^2,$$

which is a well-known identity of the Fibonacci numbers.

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