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Measuring intransitivity

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Abstract

A framework for measures of non-transitivity in finite tournaments is presented. Axiomatic characterizations for five well-known intransitivity measures are derived. The characterizing conditions are obtained by formalizing and generalizing amongst other things principles of weighting or non-weighting of elementary preference reversals and ways of diminishing the intransitivity. © 1997 Elsevier Science B.V.

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1. Introduction

Preference relations evolving from pairwise decisions, individually or collectively, may exhibit cycles. As is illustrated by for instance Tversky (1969) and Fishburn (1991), even after careful reflections transitivity is not always obtained. At first sight, these cycles seem to be inconsistent and irrational, but as is argued in Delver et al. (1991), they can be seen as results of a two levelled analysis. On the local binary level, preferences between two alternatives emerge, regardless of the other alternatives, yielding cycles on the global level, where preferences between two alternatives may depend on a third.

Nevertheless, practical considerations may force us to make choices in correspondence with the assumption of transitivity. In order to determine the deviation from

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transitivity, many intransitivity measures have been proposed, see for example Slater (1961) or Kendall, Babington Smith (1940). These measures compare preference relations with respect to their intransitivity.

In this paper the question of measuring intransitivity is studied. An intransitivity measure expresses the ‘amount’ of intransitivity in a preference relation in a non-negative number. There are various ways to measure intransitivity. To illustrate this, consider the following two sets of pairwise comparisons between five objects a_1, $a_2, \ldots, a_5$. We assume that a_i is preferred to a_j if i is smaller than j, except that in the first set a_5 is preferred to a_1 and in the second set a_3 to a_1.

Two directed graphs R_1 and R_2 below show these sets of preferences, where non-depicted arrows go down.

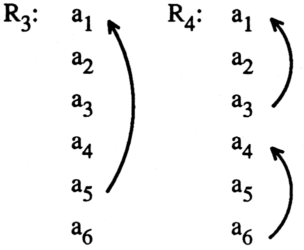
Fig. 1. The directed graphs R_1 and R_2 (non-depicted arrows go down).

One way to measure intransitivity is by counting the minimal number of preference reversals (on the binary level) which are needed to resolve all intransitivities (on the global level). This measure is called Slater’s *i*. In the example above the measure according to Slater’s *i* equals one, for both sets of preference relations. So, according to this measure both sets are equally intransitive.

On the other hand, the comparison ‘a_5 is preferred to a_1’ in R_1 introduces more intransitive patterns than does ‘a_3 is preferred to a_1’ in R_2. Indeed, in the first set there are precisely three circular triads: $<a_1, a_2, a_5, a_1>$, $<a_1, a_3, a_5, a_1>$, $<a_1, a_4, a_5, a_1>$, whereas in the second there is only one: $<a_1, a_2, a_3, a_1>$. So, according to this principle, which actually boils down to Kendall and Babington Smith’s measure, R_1 is more intransitive than R_2.

A great part of the discussion on measuring intransitivity, revolves around the justification of this weighting and, moreover, concentrates on statistical and probabilistic aspects, e.g. compare Bezembinder (1981), David (1988) or Remage, Thompson (1966). If we assume that choices in each comparison are made at random, then weighting is not justified. On the other hand, for most reasonable alterative hypotheses to randomness, a slight upset ‘a_3 is preferred to a_1’ is more likely to occur than the major upset ‘a_5 is preferred to a_1’.

We do not enter the debate on whether or not the hypothesis that choices are made at random must be disproved before any other hypothesis, for example that the probability of choosing one object over another is monotonically increasing with the difference between their respective scale values, needs to be conceded. Instead, this paper provides a systematic and axiomatic-like study of intransitivity measures, see also Kazanskaya (1982). By formalizing the two principles described above, weighting versus non-weighting and ways of diminishing intransitivity, and examining salient features of the various measures we were able to characterize well-known intransitivity measures. We characterized Slater's i, Kendall and Babington Smith's λ, the measure ν based on the difference between the outscore vector of a tournament and that of a linear ordering, compare Fulkerson (1965) or Ryser (1964), the measure δ, which equals the proportion of the total number of preferences that are embedded in a cycle, see Bezembinder (1981), and Bezembinder (1981) ρ, which measures the dimension of the cycle space of a tournament. These measures are discussed in later sections. They illustrate the variety existing in the measurement of intransitivity. As the reader may verify using table 1.1, not only the various intransitivity values of the same preference relation may differ considerably, but also the ordering on the set of tournaments, from transitive to most intransitive, depends upon the measure that is taken.



	i	λ	ν	δ	ρ
R_3 :	1	3	1	10/15	6
R_4 :	2	2	2	6/15	2

Table 1

We conclude the introduction with an overview of the various measures and conditions considered. In Section 2, some intuitive appealing properties are introduced. In a manner of speaking, these guarantee that renaming the alternatives has no impact on the intransitivity measure. Furthermore, it is required that only transitive relations have intransitivity measure zero. If a preference relation is such that all elements within a subset Y of alternatives are preferred to all those not in Y, then reversing precisely these preferences has no effect on the amount of intransitivity. At the end of this list, there is a extensiveness condition, which requires that in situations described above the intransitivity is the sum of the intransitivities in the parts in Y and its complement. In this paper these more or less natural conditions (compare Rubinstein, 1980) serve as a framework for intransitivity measures.

In addition to these properties, we introduce conditions that describe the impact of a preference reversal at the local binary tournament level, on the global level of intransitivity. Let γ be an intransitivity measure. First of all, we introduce the condition of *equability* for γ in Section 4. It formalizes the idea that there does not exist a context so that some preference reversals are more sweeping and therefore should receive more weight. Equability states that all elementary preference reversals are to be weighted uniformly, because the difference in intransitivity caused by such a reversal is supposed to be independent of the specific tournament structure in which it is embedded. Together with the *diminishability property*, which guarantees that the intransitivity as measured by γ, can be diminished by means of an elementary preference reversal, equability characterizes positive multiples of Slater's i. We call them i-like measures. We study variations on the conditions of equability and diminishability, which will be the leitmotif of Sections 5 and 6. The measure λ, which assigns to tournaments its number of 3-cycles, is not equable. This may be deduced from Fig. 1. As may be shown by computation, the change of intransitivity measured by λ, caused by an elementary preference reversal, is related to the difference in outscore between the two alternatives, see Monsuur (1994). The larger the difference, the more the intransitivity will change. So, contrary to Slater's i, for λ there exists a context so that some reversals receive more weight, while other reversals have to receive the same weight. In Section 5, it is shown that *echelon-equability* alone is enough to characterize λ-like measures. This condition states that reversing a preference between two alternatives belonging to the same echelon with respect to the outscore, yields a fixed deviation of intransitivity.

Next, we consider a condition that is stated more in terms of tournament structures. Let $T \in \mathbb{T}$ be such that there are two alternatives a and b which are involved in just one 3-cycle $<a, b, c, a>$. Further, assume that a and b play the same role with respect to the alternatives $y \in X\text{-}\{a, b, c\}$: $<a, y> \in T$ if, and only if, $<b, y> \in T$. An intransitivity measure γ is said to *change uniformly at adjacent equivalent reversals*, if the preference reversals between a and b yields a deviation of intransitivity, which is independent of the tournament structure on $X\text{-}\{a, b, c\}$. Together with the *independence of 3-cycle orientation*, it characterizes the λ-like measures. Independence of 3-cycle orientation means that reversing all preferences along a 3-cycle, does not alter the intransitivity. In Section 3 it appears that this independence condition precisely characterizes the class of intransitivity measures which only depend on outscores.

In Section 6, we consider the measure ν which is based on the difference between the outscore vector of a preference relation and that of a linear order. It is shown that ν is equable. Therefore, because of the characterization of Slater's i, we deduce that ν is not diminishable by elementary preference reversals. Nevertheless, ν may be seen as a 'Slater-related' measure: ν measures the least possible upsets needed to make any tournament, with a given outscore vector, transitive. If we replace the diminishability condition by independence of cycle orientation and the *diminishability by top–bottom corrections*, we obtain a characterization of ν-like measures. To define the notion of a top–bottom correction, let t and b be two elements. Take a tournament T and an irreducible part of it, say R. Then $<t, b>$ is a top–bottom correction if $<b, t> \in R$ and t has maximal outscore at R while b has minimal outscore at R.

Despite the fact that ν is equable, contrary to Slater's i, it matches a more global level of analysis: ν satisfies the *uniform changing at top–bottom corrections*, a salient feature of ν. An intransitivity measure is said to change uniform at top–bottom corrections, if the top–bottom reversal $<t, b>$ causes equal change in intransitivity for all tournaments for which $<t, b>$ is a top–bottom correction. Together with the independence of 3-cycle orientation, it characterizes the ν-like measures. The conditions of uniform changes are equability conditions, but are stated more in terms of tournament structures.

In Section 7, we study quadratic measures. A quadratic measure belongs to the class of quadratic functions of the sizes of the irreducible parts of a tournament. They are characterized by the *average reduction* condition. Roughly speaking, this condition states that successive reductions of intransitivity by chipping off linear parts of irreducible tournaments, yields an arithmetic sequence. The measures δ and Bezembinder's ρ do satisfy this condition. The measure δ equals the proportion of the total number of preferences that are part of a cycle, while ρ is the dimension of the cycle space. By observing that δ measures a four cycle *twice* as much as a three cycle (there are two three cycles in a four cycle) and that ρ measures a four cycle *three* times as much as a three cycle (there are three cycles in a four cycle, one in a three cycle), we obtain our final two characterizations. The results are summarized in table 8.1.

2. Intransitivity measures

After some basic denotations, a framework consisting of four conditions for intransitivity measures is provided. The model presented here is for a fixed set of objects. So, all binary relations taken into consideration here are defined on this fixed set of objects. At the end of this section we show that there is a unique extension of every intransitivity measure to binary relations on arbitrary subsets, supposing that this extension satisfies natural extended conditions of the framework.

Let $X=\{x_1, x_2, \ldots, x_n\}$ be a set of n objects or alternatives. Denoting cardinality of a set Y by $\#Y$, it follows that $\#X=n$. We assume that X is finite and that there are at least three elements. So

$$3 \leq \#X < \infty.$$

These objects are ordered by <u>binary relations</u>, denoted by R, i.e. subsets of the cartesian product $X \times X$. Let $Y \subset X$, then the <u>restriction</u> of R to Y is denoted by $R|_Y$. It is defined as follows

$$R|_Y := \{ <x, y> \in Y \times Y | <x, y> \in R \}.$$

A binary relation is a <u>tournament</u>, if it is asymmetric[1] and weakly complete[2]. A binary relation is a <u>linear ordering</u>, if in addition it is transitive[3]. The set of tournaments is denoted by $\mathbb{T}$ and that of linear orderings by $\mathbb{L}$. By definition, $\mathbb{L} \subset \mathbb{T}$. A preference relation S is a <u>subtournament</u> if there is a non-empty subset Y of X and a tournament R, such that S is the restriction of R to Y. In that case Y is called the <u>domain</u> of S. The set of all subtournaments is denoted by $\tilde{\mathbb{T}}$. The set of linear subtournaments is indicated by $\tilde{\mathbb{L}}$. Two subtournaments are called <u>disjoint</u> if their domains are disjoint. Let R and S be two disjoint subtournaments with domains Y and Z respectively, then the <u>concatenation</u> of R and S, denoted by $R \gg S$, is the following subtournament

$$R \gg S := \{ <x, y> \in X \times X | <x, y> \in R, <x, y> \in S \text{ or } x \in Y \text{ and } y \in Z \}.$$

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[1] $<x, y> \in R$ implies $<y, x> \notin R$.
[2] $x \neq y$, $<x, y> \notin R$ implies $<y, x> \in R$.
[3] $<x, y>, <y, z> \in R$ implies $<x, z> \in R$.

Clearly $R \gg S$ is a subtournament with domain $Y \cup Z$.

Let S be a subtournament with domain Y. Then a tournament R is a linear extension of S, if

- $R|_Y = S$,
- $R|_{X-Y} \in \tilde{\mathbb{L}}$ and
- for all $x \in X-Y$ either for $y \in Y$ $<x, y> \in R$ or for $y \in Y$ $<y, x> \in R$.

Let the subtournament on a singleton $x \in X$ be identified with x. Let R be a subtournament with domain $X-Y$, where $Y=\{y_1, y_2, \ldots, y_k\}$. Then for all $t \in \{0, \ldots, k\}$ $y_1 \gg y_2 \gg \cdots \gg y_t \gg R \gg y_{t+1} \gg \cdots \gg y_k$ is a linear extension of R.

Let R be a subtournament with domain Y and let π be a permutation of Y. The permutation of R is defined as $\pi R := \{<\pi(x), \pi(y)> \mid <x, y> \in R\}$. Note that πR is a subtournament with domain Y. Let T be a tournament and $<x, y> \in T$. Then $yx\,T = (T - \{<x, y>\}) \cup \{<y, x>\}$. So yxT is obtained from T by reversing the preference between x and y.

Concatenation, reversion, restriction and permutation are operations which are used in Storcken, de Swart (1992) in order to characterize the phenomenon ordering. Concatenation is also called summation (see Rosenstein, 1982) or lexicographic product (see Jónsson, 1982).

Definition 2.1. Intransitivity measure: An intransitivity measure is a function γ from $\mathbb{T}$ to the set of non-negative reals, $[0, \infty)$, which satisfies

(2.1.1) γ indicates intransitivity, that is $\gamma(R) \neq 0$ for all $R \in \mathbb{T} - \mathbb{L}$.

In this paper we restrict ourselves to measures that satisfy (2.1.2) up to (2.1.4)

(2.1.2) γ is independent of renaming, that is $\gamma(R) = \gamma(\pi R)$ for all $R \in \mathbb{T}$ and all permutations π on X.

(2.1.3) γ is independent of commutations in concatenation, that is $\gamma(R \gg S) = \gamma(S \gg R)$ for all disjoint partial tournaments S and R, such that the union of their domains is X.

(2.1.4) γ is extensive, that is $\gamma(R \gg S) = \gamma(R \gg S') + \gamma(R' \gg S)$ for all disjoint partial tournaments R and S and all disjoint partial linear orderings R' and S' such that the domains of R and R' are equal, the domains of S and S' are equal, and the union of these domains is X.

So, an intransitivity measure assigns a non-negative number to every tournament. Intransitivity indicating means that this number is positive whenever the tournament is intransitive. Independence of renaming means that renaming the objects and likewise the local preferences does not alter the intransitivity. In order to illustrate independency of commutations in concatenation, let Y be a non-trivial subset of X, R a subtournament on Y and S a subtournament on $X-Y$. The concatenation of R and S forms a linear ordering of R and S. Hence, the concatenation operation should not introduce intransitivity. So, in this paper the intransitivity of $R \gg S$ is taken to be the sum of the intransitivities of R and S. This condition is called extensiveness. Because the domain X of the tournaments on which the measure γ is defined is fixed, we are not able to express this equality directly. Instead, we imposed condition (2.1.3) and (2.1.4). Condition (2.1.3) means that it does not matter how subtournaments are linearly arranged with respect to each other. This is obviously true under extensiveness. Condition (2.1.4) is an adoption of extensiveness to the model discussed here. Instead of measuring R, a linear extension, $R \gg S'$, of it is measured. This implies that the intransitivity of this extension is located in R and that the linear addition S' has no influence on it. This is precisely what one would expect. Moreover, theorem 2.4 shows that (2.1.3) and (2.1.4) form a good adoption of extensiveness to the fixed domain case. Therefore (2.1.4) is called extensiveness. It is straightforward to show that condition (2.1.1) up to (2.1.4) are logically independent.

There are measures that do not satisfy all these extra conditions. For example, consider the transitive dimension of a tournament T, $\tau(T)$, see Barthélemy (1990). It equals the minimal number s such that T is the lexicographic sum of partial orders P_0, $P_1, \ldots, P_s$. It is straightforward to show that $\tau(R \gg S) = \max(\tau(R \gg S'), \tau(R' \gg S))$, so τ does not satisfy (2.1.4).

Also, one may find (2.1.3) more or less disputable, because one can imagine that an individual will react differently to intransitive choices between alternatives that are high and low on his preference scale.

In this paper (2.1.1) up to (2.1.4) will serve as a framework for measures of intransitivity. All the measures that we are going to characterize do satisfy (2.1.1) up to (2.1.4). It is natural to expect that there is no intransitivity in a linear ordering. This, however, is a consequence of (2.1.4), as is shown in the following proposition.

Proposition 2.2. *Let γ be an intransitivity measure. Then $\gamma(R) = 0$ for all linear orderings R.*

Proof. Let $R \in \mathbb{L}$. Let Y be a non-trivial subset of X such that $R = R|_Y \gg R|_{X-Y}$. By extensiveness it follows that

$$\gamma(R) = \gamma(R|_Y \gg R|_{X-Y}) + \gamma(R|_Y \gg R|_{X-Y}) = 2\gamma(R).$$

Hence $\gamma(R) = 0$. □

The following proposition reads that all linear extensions of a subtournament yield the same level of intransitivity.

Proposition 2.3. *Let γ be an intransitivity measure. Let S_1 and S_2 be two linear extensions of a subtournament R. Then $\gamma(S_1) = \gamma(S_2)$.*

Proof. Let Y be the domain of R. Let $X - Y = \{z_1, \ldots, z_k\}$. It is sufficient to prove that

$$\gamma(R \gg z_1 \gg \cdots \gg z_k) = \gamma(z_1 \gg \cdots \gg z_t \gg R \gg z_{t+1} \gg \cdots \gg z_k).$$

By the independence of commutations in concatenations,

$$\gamma(z_1 \gg \cdots \gg z_t \gg R \gg z_{t+1} \gg \cdots \gg z_k) = \gamma(R \gg z_{t+1} \gg \cdots \gg z_k \gg z_1 \gg \cdots \gg z_t).$$

Applying the independence of renaming yields the desired result. □

Let $\tilde{\gamma}$ be a function from $\tilde{\mathbb{T}}$ to $[0, \infty)$. Then $\tilde{\gamma}$ is said to be an extension of the intransitivity measure γ, if

- $\tilde{\gamma}(R) = \gamma(R)$ for all $R \in \mathbb{T}$ and
- $\tilde{\gamma}(R \gg S) = \tilde{\gamma}(R) + \tilde{\gamma}(S)$ for all disjoint subtournaments R and S in $\tilde{\mathbb{T}}$.

The following theorem reads that there is a unique correspondence between intransitivity measures and extended intransitivity measures.

Theorem 2.4. *Let γ be an intransitivity measure. Then there is a unique extension of the intransitivity measure γ, say $\tilde{\gamma}$. If, in addition, $\tilde{\gamma}$ is an extension of an intransitivity measure γ', then $\gamma = \gamma'$.*

Proof. Define $\tilde{\gamma}(R) := \gamma(S)$ for all $R \in \tilde{\mathbb{T}}$, where S is a linear extension of R. Then $\tilde{\gamma}$ is well-defined by proposition 2.3. We prove that $\tilde{\gamma}$ is an extension of γ. By definition $\tilde{\gamma}(R) = \gamma(R)$ for all $R \in \mathbb{T}$. Let R and S be two disjoint subtournaments. Let Y be the domain of R and Z that of S. Let $Y = \{y_1, \dots, y_k\}$, $Z = \{z_1, \dots, z_l\}$ and $X - (Y \cup Z) = \{x_1, \dots, x_m\}$. Then

$$\begin{aligned} \tilde{\gamma}(R \gg S) &= \gamma(R \gg S \gg x_1 \gg \cdots \gg x_m) \\ &= \gamma(R \gg z_1 \gg \cdots \gg z_l \gg x_1 \gg \cdots \gg x_m) \\ &\quad + \gamma(y_1 \gg \cdots \gg y_k \gg S \gg x_1 \gg \cdots \gg x_m) \qquad \text{(by (2.1.4))} \\ &= \tilde{\gamma}(R) + \tilde{\gamma}(S) \qquad \text{(by definition).} \end{aligned}$$

This proves that $\tilde{\gamma}$ is an extension of γ. In order to prove that it is the only extension let $\bar{\gamma}$ be an extension of γ as well. Let R be a subtournament. We prove that $\bar{\gamma}(R) = \tilde{\gamma}(R)$. Since this is obviously true if the domain Y of R equals X, we assume that $Y \neq X$. Therefore, let $Y = \{y_1, \dots, y_k\}$ and $X - Y = \{x_1, \dots, x_l\}$. Now

$$\begin{aligned} 0 &= \gamma(y_1 \gg y_2 \gg \cdots \gg y_k \gg x_1 \gg \cdots \gg x_l) \\ &= \bar{\gamma}(y_1 \gg \cdots \gg y_k \gg x_1 \gg \cdots \gg x_l) = \bar{\gamma}(y_1 \gg \cdots \gg y_k) \\ &\quad + \bar{\gamma}(x_1 \gg \cdots \gg x_L). \end{aligned}$$

Hence, $\bar{\gamma}(x_1 \gg \cdots \gg x_l) = 0$. Therefore

$$\begin{aligned} \bar{\gamma}(R) &= \bar{\gamma}(R) + \bar{\gamma}(x_1 \gg \cdots \gg x_l) = \bar{\gamma}(R \gg x_1 \gg \cdots \gg x_l) \\ &= \gamma(R \gg x_1 \gg \cdots \gg x_l) = \tilde{\gamma}(R). \end{aligned}$$

Finally, let $\tilde{\gamma}$ be an extension of an intransitivity measure γ. Then $\gamma = \tilde{\gamma}|_{\mathbb{T}} = \gamma'$. □

The foregoing theorem shows that the confinement to a fixed set of objects is not essential. As will be shown in proposition 2.5, if $\tilde{\gamma}$ is the unique extension of an intransitivity measure γ, then it satisfies natural extensions of condition (2.1.1) up to (2.1.4).

Proposition 2.5. *Let $\tilde{\gamma}$ be the unique extension of an intransitivity measure γ. Then*

(i) $\tilde{\gamma}(R) \neq 0$ *for all* $R \in \tilde{\mathbb{T}} - \tilde{\mathbb{L}}$
(ii) $\tilde{\gamma}(R) = \tilde{\gamma}(\pi R)$ *for all subtournaments R and all permutations π on the domain of R*
(iii) $\tilde{\gamma}(R \gg S) = \tilde{\gamma}(R) + \tilde{\gamma}(S)$ *for all disjoint subtournaments R and S in* $\tilde{\mathbb{T}}$.

Proof. As is proved in theorem 2.4, $\tilde{\gamma}(R) = \gamma(S)$, where S is a linear extension of R.

(i) If $R \notin \tilde{\mathbb{L}}$, then any linear extension S of R is not in $\mathbb{L}$. So, then $\tilde{\gamma}(R) = \gamma(S) \neq 0$.
(ii) Let $R \in \tilde{\mathbb{T}}$ and π a permutation on the domain of R, say Y. Let $X - Y = \{z_1, \ldots, z_k\}$. Let $\tilde{\pi}$ be the permutation on X such that $\tilde{\pi}(y) = \pi(y)$ for all $y \in Y$ and $\tilde{\pi}(z) = z$ for all $z \in X - Y$. Then

$$\begin{aligned}\tilde{\gamma}(R) &= \gamma(R \gg z_1 \gg \cdots \gg z_k) = \gamma(\tilde{\pi}(R \gg z_1 \gg \cdots \gg z_k)) \\ &= \gamma(\pi R \gg z_1 \gg \cdots \gg z_k) = \tilde{\gamma}(\pi R).\end{aligned}$$

(iii) This is true by definition. □

In the sections that follow, we will give characterizations of various intransitivity measures. To this end, we introduce the notion of γ-like measures.

Definition 2.6. Let γ and γ' be intransitivity measures. Then γ' is said to be γ-like, if there is a real positive scalar a such that $\gamma' = a\gamma$.

Of course, if γ' is γ-like, then γ is γ'-like.

3. Outscore dependent measures

In a round-robin tournament in which each player contests every other player in precisely one game, the tournament winner is often determined by the number of wins. This number of wins is usually called the outscore. In this section intransitivity measures depending on the outscores are studied. This dependency presupposes that the paired comparisons are comparable to each other. The win in one game can be balanced by the loss in another or the strict preference to one object can be balanced by the dominance of another.

It is shown that cycle orientation independency, which will be introduced in definition 3.3, characterizes this type of measure. In Sections 5–7 special subtypes of outscore dependent measures are studied. These types are the well-known three cycle count of Kendall and Babington Smith, the proportion of the total number of preferences which are on a circuit, the sum of absolute differences between the outscores of a tournament and those of a linear ordering and finally, quadratic measures. Let R be a tournament and x an object in X. The set of elements which are preferred by x at R is

$$\mathrm{Out}(x, R) := \{y \in X \mid \langle x, y\rangle \in R\}.$$

The <u>outscore</u> of x <u>at</u> R is just the cardinality of this set

$$\mathrm{out}(x, R) := \#\mathrm{Out}(x, R)$$

which is often referred to as the Copeland score. The <u>vector of outscores</u> of R, out(R), is the ordered n-tuple (out(x_1, R), out(x_2, R), ...,out(x_n, R)) where we assume that the alternatives of X are labelled in such way that out $(x_1, R) \le \mathrm{out}(x_2, R) \le \cdots \le \mathrm{out}(x_n, R)$.

Definition 3.1. An intransitivity measure γ is said to be <u>outscore dependent</u> if for all tournaments R and R', $\mathrm{out}(R) = \mathrm{out}(R')$ implies $\gamma(R) = \gamma(R')$.

Consider the set of vectors corresponding to an outscore vector

$$V := \{\underline{x} \in \mathbb{R}^n \mid \text{there is a tournament } R \in \mathbb{T} \text{ such that } \mathrm{out}(R) = \underline{x}\}.$$

It is well-known, see for instance Moon (1968) or Ford, Fulkerson (1962), that a vector $\underline{x} \in \mathbb{R}^n$, such that $x_1 \le x_2 \le \cdots \le x_n$, is in V if, and only if,

$$\Sigma_{i=1}^{k} x_i \ge \binom{k}{2} \text{ for all } k \in \{1, 2, \ldots, n\} \text{ and } \Sigma_{i=1}^{n} x_i = \binom{n}{2}.$$

Obviously, if a measure γ is outscore dependent, then there exists a function f such that γ is the composition of f and the function out: $\gamma = f_o$ out. Hence, f can be taken as a function from V to $[0, \infty)$.

In order to determine all functions f such that f_o out is an intransitivity measure, some further notations are needed. Let $\tilde{V}$ denote the set of outscore vectors of a subtournament:

$$\tilde{V} := \left\{\underline{x} \mid \text{there is a } k \in \{1, \ldots, n\} \text{ such that } \underline{x} \in R^k, \Sigma_{i=1}^{l} x_i \ge \binom{l}{2} \text{ for all } 1 \le l \le k \text{ and } \Sigma_{i=1}^{k} x_i = \binom{k}{2}\right\}.$$

Let $K \subset \{1, 2, \ldots, n\} = N$. Let $\underline{x} \in \mathbb{R}^n$ then $\underline{x}|_K := \langle x_i \rangle_{i \in K}$ is the restriction of $\underline{x}$ to the coordinates in K. As usual for $\alpha \in R$ and $\underline{x}$, $\underline{y} \in \mathbb{R}^k$, multiplication of $\underline{x}$ by a scalar α is denoted by $\alpha\underline{x}$ and the sum or difference vector of $\underline{x}$ and $\underline{y}$ is denoted by $\underline{x} + \underline{y}$ or $\underline{x} - \underline{y}$ respectively. Let $\mathrm{e} \in \mathbb{R}^n$ be the vector whose coordinates are all 1.

Theorem 3.2. *Let f be a function from $\tilde{V}$ to $[0, \infty)$. Then $\tilde{\gamma} = f_o$ out is an (outscore dependent) extended intransitivity measure if, and only if,*

(3.2.1) *for all $\underline{x} \in \tilde{V}$: $f(\underline{x}) = 0$ if and only if $\underline{x} = \langle 0, 1, \ldots, k-1 \rangle$, for some $k \le n$*

(3.2.2) *$f(\underline{x}) = f(\underline{x}|_K) + f(\underline{x}|_{N-K} - k(\underline{e}|_{N-K}))$ for all $\underline{x} \in V$ such that*

$$K = \{1, 2, \ldots, k\} \text{ and } \Sigma_{i=1}^{k} x_i = \binom{k}{2}.$$

Proof. (Only if) (3.2.1) follows because of proposition 2.2, intransitivity indication and the unique correspondence between intransitivity measures and extended intransitivity measures. Finally, because $\tilde{\gamma}$ is extensive, (3.2.2) follows. (if) It is straightforward to show that $\gamma = f_o$ out: $\mathbb{T} \to [0, \infty)$ is an intransitivity measure. Let $\tilde{\gamma} = f_o$ out. Then $\tilde{\gamma}$ is the unique extension of γ. □

Let R be a tournament. Let $x_1, x_2, \ldots, x_k \in X$. Then $<x_1, x_2, \ldots, x_k, x_1>$ is a cycle in R, if $<x_1, x_2>, <x_2, x_3>, \ldots, <x_{k-1}, x_k>$ and $<x_k, x_1>$ are all in R. In that case k is the length of the cycle, therefore it is also called a k-cycle. Consider R' obtained from R by only reversing the pairs on the cycle. Then out(R) and out(R') are equal. Hence, if a measure is outscore dependent, the orientation of cycles does not matter. To formally state the converse result, we introduce the following definition.

Definition 3.3. Let γ be an intransitivity measure. Then γ is said to be <u>independent of k-cycle orientation</u>, if $\gamma(R) = \gamma(R')$ for all tournaments R and R', which can be obtained from each other by reversing preferences of a k-cycle. A measure γ is called <u>independent of cycle orientation</u>, if for all $k \geq 3$ it is independent of k-cycle orientation.

The following lemma is used to characterize all outscore dependent intransitivity measures. It states that tournaments with identical outscore vectors can be obtained from each other by a sequence of 3-cycle reversals. As this result is well-known, see for instance Moon (1968) page 73 theorem 35, it will be stated without proof.

Lemma 3.4. *Let R and S be tournaments such that out(R) = out(S). Then by means of 3-cycle reversals a sequence $R^0, R^1, \ldots, R^t$ of tournaments can be obtained, such that* $R^0 = R$ and $R^t = S$. □

This lemma yields the following consequence. Its proof is left to the reader.

Consequence 3.5. *Let γ be an intransitivity measure. Then γ is outscore independent if, and only if, γ is independent of* (3)-*cycle orientation*. □

4. Slater's i

The intransitivity measure Slater's i, introduced by Slater (1961), can be defined for all tournaments $T \in \mathbb{T}$ as follows

$$i(T) := \min\left\{\frac{1}{2} \#(T \,\Delta\, L) : L \in \mathbb{L}\right\}$$

where Δ denotes symmetric difference.

So, $i(T)$ is equal to the minimal number of preference reversals in order to obtain a transitive tournament. It is straightforward to prove that i is an intransitivity measure. Slater's i has been studied in other contexts as well. See for example Junger (1985) and

Ali et al. (1986). For all $L \in \mathbb{L}$ the difference $\#(yxT \,\Delta\, L) - \#(T \,\Delta\, L)$ is either -2 or 2. From this it follows that i satisfies the following condition.

Definition 4.1. An intransitivity measure γ is said to be equable, if for all tournaments T, T' and all objects $x, y \in X$:

$$|\gamma(yxT) - \gamma(T)| = |\gamma(yxT') - \gamma(T')| \text{ if } \langle x, y \rangle \in T \cap T' \text{ and } \gamma(T) \neq \gamma(yxT),\ \gamma(T') \neq \gamma(yxT').$$

Equability states that all elementary preference reversals are to be weighted uniformly, because the difference in intransitivity caused by such a reversal is supposed to be independent of the specific tournament structure in which it is embedded.

Besides equability, Slater's i satisfies the following condition.

Definition 4.2. An intransitivity measure γ is said to be diminishable (by elementary preference reversals) if, for all $T \in \mathbb{T} - \mathbb{L}$, there is a pair $\langle x, y \rangle \in T$ such that $\gamma(yxT) < \gamma(T)$.

To demonstrate that Slater's i is diminishable, take $T \in \mathbb{T} - \mathbb{L}$ and $\langle x, y \rangle \in T - L^*$, where L^* minimizes $\frac{1}{2}\#T \,\Delta L$ for $L \in \mathbb{L}$. Then $\#(yxT \,\Delta L^*) = \#(T \,\Delta\, L^*) - 2$, which proves diminishability. It is clear that if γ satisfies definitions 4.1 and 4.2, then so does each γ-like measure. Now we can state a characterization for Slater's i.

Theorem 4.3. *Let γ be an intransitivity measure. Then γ is i-like if, and only if, γ is diminishable and equable.*

Proof. The 'only-if-part' is shown above. In order to prove the 'if-part' assume that γ is diminishable and equable. Let $\hat{T}$ be a linear extension of a 3-cycle, say $\langle x, y, z, x \rangle$. Because $\hat{T} \notin \mathbb{L}$, $\gamma(\hat{T}) := \alpha > 0$. Let $T \in \mathbb{T}$ and $\langle a, b \rangle \in T$ such that $\gamma(T) \neq \gamma(ba\ T)$. By independence of renaming, we may assume that $\langle x, y \rangle = \langle a, b \rangle$. Then, using equability,

$$|\gamma(baT) - \gamma(T)| = |\gamma(ba\hat{T}) - \gamma(\hat{T})| = \alpha. \tag{*}$$

By induction to k, we will prove that for all $T \in \mathbb{T}$, $i(T) = k$ precisely when $\gamma(T) = \alpha k$. The basis $k = 0$ follows from proposition 2.2. For the proof of the induction step, take $T \in \mathbb{T}$ with $i(T) = k + 1$. By diminishability of the measure i, there is a pair $\langle a, b \rangle \in T$, such that $i(T) > i(baT) = k$. So, using the induction hypothesis, $\gamma(T) \neq \gamma(baT) = \alpha k$. Next, from (*) we deduce that either $\gamma(T) = \alpha + \gamma(baT) = \alpha + \alpha k = \alpha(k+1)$ or $\gamma(T) = \gamma(baT) - \alpha = \alpha(k-1)$. But $\gamma(T) = \alpha(k-1)$ would imply that $i(T) = k - 1$, which is not the case. Conversely, let $\gamma(T) = \alpha(k+1)$. Because of the diminishability of γ, there is a pair $\langle a, b \rangle \in T$, such that $\gamma(T) > \gamma(baT)$. From (*), we deduce that $|\gamma(baT) - \gamma(T)| = \alpha$, so $\gamma(baT) = \alpha k$. Because of the induction hypothesis, $i(baT) = k$. But then, again using the induction hypothesis, we conclude that $i(T) \geq k + 1$, so $i(T) = k + 1$. □

As already is demonstrated in the introduction the 3-cycle count λ is not equable. It will be shown in Section 5 that λ is diminishable. Moreover, it will be shown in Section 6, that the measure ν is equable, but not diminishable. So, the conditions of theorem 4.3 are independent. Further, note that the transitive dimension of a tournament, introduced in Section 2, although not fitting our framework for intransitivity measures is equable. This may be proved by observing that $\tau(yxT) \leq \tau(T)+1$ for all tournaments T and all pairs $<x, y> \in T$.

The equability condition applies to every elementary preference reversal disregarding the global structure in which such a reversal takes place. Therefore a local level of analysis was possible in the proof of theorem 4.3. In the following sections other 'equability' conditions involving more global surroundings of the elementary preference reversals will be considered. Consequently, characterizations are based on more global levels of analysis.

5. The three cycle count

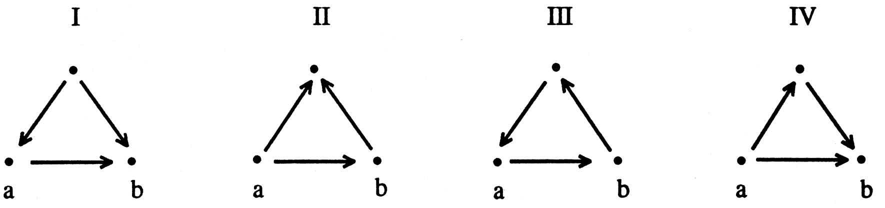
Having the characterization of all outscore dependent measures from Section 3, attention will be focused on some special subclasses. First we consider the three cycle count of Kendall and Babington Smith which they introduced in 1940. This measure, denoted by λ, is equal to the number of transitivity violations in a tournament. To derive a nice well-known (see Kendall, Babington Smith, 1940) expression of λ, note that the number of transitive triads at a tournament R is equal to

$$\Sigma_{x \in x} \binom{\text{out}(x, R)}{2}.$$

Because $\binom{n}{3}$ equals the number of triads, it follows that

$$\lambda(R) = \binom{n}{3} - \Sigma_{x \in X} \binom{\text{out}(x, R)}{2}.$$

From this formula, we deduce that λ is outscore dependent. Furthermore, $\lambda(R) \leq (n^3 - n)/24$ if n is odd and $\lambda(R) \leq (n^3 - 24n)/24$ if n is even. See for instance Moon (1968). In Kadane (1966) it is proved that $\lambda(R)$ is the number of preference reversals necessary to break all ties in the outscore vector of R. Clearly, if there are no ties in out(R), then $R \in \mathbb{L}$. Because the two characterizations of λ, presented here, employ the argument of Kadane's result, let us reconsider this underlying argument. It also shows that λ is reducible. Let R be a tournament and $a, b \in X$ such that $<a, b> \in R$ and out(a, R) = out(b, R). In order to study the impact of this reversal on the measure λ, it is sufficient to consider triads involving a and b. There are four types of such triads. They can be pictured as follows.



Reversing the preference $<a, b>$ in a type I or II triad does not change λ. In a type III λ decreases by 1 and a type IV increases λ by 1. Hence, if there are k type III triads, in which case by the assumption that out(a, R) = out(b, R), there are $k-1$ type IV triads, then it follows that $\lambda(baR) = \lambda(R) + k - 1 - k = \lambda(R) - 1$. This proves Kadane (1966) theorem from 1966, but also makes clear that λ satisfies the following condition, which is related to the equability property (definition 4.1).

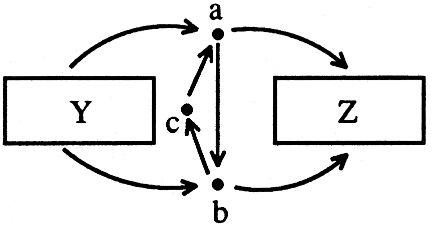
Definition 5.1. An intransitivity measure γ is said to be echelon-equable if $\gamma(baR) - \gamma(R) = \gamma(baR') - \gamma(R')$ for all $R, R' \in \mathbb{T}$ and all $a, b \in X$, with $<a, b> \in R$, $<a, b> \in R'$ and out(a, R) = out(b, R) = out(a, R') = out(b, R').

Echelon-equability means that reversing a preference between two objects, having the same outscore hence belonging to the same outscore echelon, yields a fixed deviation of the intransitivity measure, which does not depend on the further structure of the tournaments R and R'. The following condition is based on Out(.) instead of out(.).

Definition 5.2. An intransitivity measure γ is said to change uniformly at adjacent equivalent reversals, if $\gamma(R) - \gamma(baR) = \gamma(R') - \gamma(baR')$ for all $R, R' \in \mathbb{T}$ and all triples of objects $a, b, c \in X$ such that

- $<a, b>, <b, c>, <c, a> \in R \cap R'$, and
- Out(a, R) − {b} = Out(b, R) − {c} = Out(a, R') − {b} = Out(b, R') − {c}.

The following diagram may help to understand the previous definition.



Here $Z = \text{Out}(a, R) - \{b\} = \text{Out}(a, R') - \{b\} = \text{Out}(b, R) - \{c\} = \text{Out}(b, R') - \{c\}$ and $Y = X - (Z \cup \{a, b, c\})$. Disregarding the ordering on $Y \cup Z \cup \{c\}$. we have that the preference between a and b is adjacent equivalent. The alternatives a and b have the same outscore and are involved in precisely one 3-cycle. If we reverse this preference yielding baR and baR', then we may hope that this causes an equal change in intransitivity, for both change R to baR and change R' to baR'. Because out(a, R) = out(b, R) = out(a,

$R')=\text{out}(b, R')$, it follows that echelon-equability implies uniform changing at adjacent equivalent reversals. The following theorem provides two characterizations of λ.

Theorem 5.3. *Let γ be an intransitivity measure. Then* (i), (ii) *and* (iii) *are equivalent, where*

(i) *γ is λ-like*
(ii) *γ is echelon-equable*
(iii) *γ changes uniformly at adjacent equivalent reversals and is independent of cycle orientation.*

Proof. Consider $\hat{T}$, a linear extensions of a 3-cycle $<x, y, z, x>$. Then $\gamma(\hat{T}):=\alpha>0$. Then, by independence of renaming and proposition 2.3, $\gamma(E)=\alpha$ for any linear extension of a 3-cycle $<a, b, c, a>$. The following implications will be proved.

(i)$\rightarrow$(iii). This is obvious, since $\lambda(R)-\lambda(baR)=\lambda(R')-\lambda(baR')=1$.
(ii)$\rightarrow$(i). We prove that $\gamma(T)=\alpha\lambda(T)$.

Observe that if $T\notin\mathbb{L}$, then there is a 3-cycle, say $<a, b, c, a>$ in T, such that $\text{out}(a, T)=\text{out}(b, T)$. Consider E, a linear extension of the 3-cycle $<a, b, c, a>$, such that $\text{out}(a, E)=\text{out}(b, E)=\text{out}(a, T)=\text{out}(b, T)$. Because of echelon-equability, $\gamma(T)-\gamma(baT)=\gamma(E)-\gamma(baE)=\gamma(E)=\alpha$. Therefore, we have that for all tournaments T and $p, q\in X$, if $<p, q>\in T$ and $\text{out}(p, T)=\text{out}(q, T)$, then $\gamma(T)-\gamma(qpT)=\alpha$. But then, by Kadane's result, we obtain that $\gamma(T)=\alpha\lambda(T)$, which is also true if $T\in\mathbb{L}$.

(iii)$\rightarrow$(ii). In order to prove the echelon-equability, let $T\in\mathbb{T}$ and $a, b\in X$ such that $<a, b>\in T$ and $\text{out}(a, T)=\text{out}(b, T)$. Then there is an alternative c, such that $<a, b, c, a>$ is a 3-cycle in T.
We construct two tournaments T^1 and E. Take T^1 as follows: $T^1|_{X-\{b\}}=T|_{X-\{b\}}$, $<a, b>$, $<b, c>\in T^1$ and $<b, x>\in T^1$ if, and only if, $<a, x>\in T$ for all $x\in X-\{b, c\}$. Since $\text{out}(T^1)=\text{out}(T)$, we have by independence of cycle orientation and consequence 3.5 that $\gamma(T)=\gamma(T^1)$ and $\gamma(baT)=\gamma(baT^1)$. Let E be a linear extension of the 3-cycle $<a, b, c, a>$ and $\text{Out}(a, E)=\text{Out}(a, T^1)$. Then obviously, $\text{Out}(a, E)-\{\text{b}\}=\text{Out}(a, T^1)-\{\text{b}\}=\text{Out}(b, E)-\{c\}=\text{Out}(b, T^1)-\{c\}$. So, by uniform changing at adjacent equivalent reversals,

$$\gamma(E)=\gamma(E)-\gamma(baE)=\gamma(T^1)-\gamma(baT^1),$$

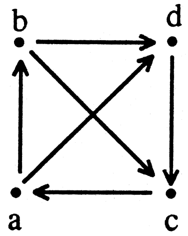
which equals $\gamma(T)-\gamma(baT)$. Since $\gamma(E)=\alpha$, we are done □

Independence of conditions in theorem 5.3 (iii). With respect to the independence of the characterizing conditions consider the following measure ν: $\mathbb{T}\rightarrow[0, \infty)$, discussed in Section 6 which is defined for all $R\in\mathbb{T}$ as follows

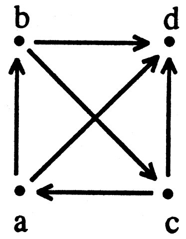
$$\nu(R):=\frac{1}{2}\|\text{out}(R)-<0, 1, 2, \ldots, n-1>\|_1.$$

Here for all vectors $\underline{x} \in R^n$, $\|\underline{x}\|_1 := \Sigma_{i=1}^{n} |x_i|$. The norm $\| \ldots \|_1$ is known as the L_1-norm. So ν is based on the difference between the outscore vector of a tournament R and that of a linear ordering. The more out(R) resembles the outscore vector of a linear ordering, the smaller $\nu(R)$ will be. The measure ν is outscore dependent and therefore independent of cycle orientation. But it does not change uniformly at adjacent equivalent reversals. To prove this consider the following tournaments.

R



R'



Obviously $\nu(R) - \nu(baR) = 1 - 1 = 0$ and $\nu(R') - \nu(baR') = 1 - 0 = 1$. So, one may easily verify that ν does not change uniformly at adjacent equivalent reversals. Now we construct an intransitivity measure which is not independent of cycle orientation but which changes uniformly at adjacent equivalent reversals. Consider the following measure.

$$\gamma'(R) := \begin{cases} n^3 & \text{if } R \in W \\ \lambda(R) & \text{otherwise} \end{cases}$$

where W is a set of tournaments R such that there is a numbering of the alternatives in X, say $a_1, a_2, \ldots, a_{2k}$ if n is even and $a_1, a_2, \ldots, a_{2k+1}$ if n is odd and R is a linear extension of the following tournament on $a_1, a_2, \ldots, a_{2k}$:

$$R' := \{ <a_i, a_j> \in X \times X | i + 1 \le j \le i + k + 1 \text{ and } j \ne i + k - 1 \text{ and } j \le 2k \text{ or } j < i \text{ and } <a_i, a_j> \notin R' \}.$$

Let a tournament R be represented by a matrix $[a_{ij}]_{i=1}^{n}{}_{j=1}^{n}$, where $a_{ij} \in \{0, 1\}$ while $a_{ij} = 1$ means that $<a_i, a_j> \in R$. Then the following matrices represent elements from W.

$n = 6$

$$\begin{matrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

$n = 7$

$$\begin{matrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

It is straightforward to show that in case $n \ge 6$ for every ordering $R' \in W$ and every tuple of alternative $a, b \in X$ the symmetric difference of Out(a, R') and Out(b, R') consists of more than two alternatives. Hence, uniform changing at adjacent equivalent reversals is not applicable for tournaments $R' \in W$. Furthermore, considering out(R'), it follows that there are no tournaments $R'' \in \mathbb{T}$ and alternatives $x, y \in X$, such that $yxR'' = R'$ and out(x, R'') = out(y, R''). So, resolving ties of the outscore vector does not yield tournaments in W. Now it is straightforward to show that γ' is an intransitivity measure which is not independent of cycle orientation, but which changes uniformly at adjacent equivalent reversals.

In Monsuur (1994) it is shown that λ can also be characterized by a condition called outscore-equability, which is stronger than the echelon-equability. There it is shown that if an intransitivity measure γ is outscore-equable, there exists a number $k > 0$, such that for all $R \in \mathbb{T}$ and all $x, y \in X$, if $<x, y> \in R$ then $\gamma(yxR) - \gamma(R) = k[\text{out}(x, R) - \text{out}(y, R) - 1]$. In case $\gamma = \lambda$, $k = 1$. So, the larger the difference in outscores, the larger is the impact of the preference reversal.

Thus outscore-equability and echelon-equability, contrary to equability, match a more global level of analysis, which consists of balancing and weighting pairwise comparisons.

6. The measure ν

Now we will discuss two characterizations of the intransitivity measure ν, which was introduced in the previous section and is defined by $\nu(R) = \frac{1}{2}\|\text{out}(R) - <0, 1, 2, \ldots, n-1>\|_1$. Because ν is outscore dependent, it is independent of cycle orientation. As will be shown, this is one of the characterizing properties. Another property is again one which expresses a uniform measure change caused by specific preference reversals. These reversals concern top and bottom elements. In order to define this notion we recall the well-known irreducibility property. See for instance Moon (1968). A partial tournament P on Y is said to be irreducible, if there is no set $Z \subset Y$, $Z \neq \emptyset$ and $Z \neq Y$, such that $P = P|_Z \gg P|_{Y-Z}$. Hence, P is irreducible if it is not the concatenation of two other subtournaments. These irreducible tournaments are strongly connected. Let R be a tournament, then there exist irreducible subtournaments P^1, $P^2, \ldots, P^k$ on $Y^1, \ldots, Y^k$ respectively such that

$$R = P^1 \gg P^2 \gg \cdots \gg P^k = R|_{Y^1} \gg R|_{Y^2} \gg \cdots \gg R|_{Y^k}.$$

We say that $P^1, \ldots, P^k$ are the irreducible components of R. Let P be a subtournament on Y. An alternative with maximal outscore at P is called a top-element of P. If the outscore is minimal at P it is called a bottom-element of P. Note that if P is transitive, a top-element is the best alternative with respect to P and a bottom-element the worst. Let t be a top-element of P and b a bottom-element such that $<b, t> \in P$. Then tbP is called a top-bottom correction.

Definition 6.1. An intransitivity measure γ changes uniformly at a top–bottom correction (of an irreducible component), if $\gamma(R) - \gamma(tbR) = \gamma(R') - \gamma(tbR')$ for all tournaments $R, R' \in T$ and $t, b \in X$ such that tbP and tbP' are top–bottom corrections of irreducible components P and P' of R and R' respectively.

So, uniformly changing at top–bottom corrections, means that any top–bottom correction gives rise to the same measure alteration. The intransitivity measure ν satisfies this condition. To see this consider a top–bottom correction tbP of an irreducible component P of a tournament $R \in \mathbb{T}$. Let P be a subtournament on B. Suppose without loss of generality that

$$R = R|_A \gg P \gg R|_C . \quad (*)$$

Let $A=\{a_1, \ldots, a_k\}$, $B=\{b_1, \ldots, b_l\}$ and $C=\{c_1, \ldots, c_m\}$. Furthermore, let $\mathrm{out}(a_1, R) \geq \cdots \geq \mathrm{out}(a_k, R) \geq \mathrm{out}(b_1, R) \geq \cdots \geq \mathrm{out}(b_l, R) \geq \mathrm{out}(c_1, R) \geq \cdots \geq \mathrm{out}(c_m, R)$ and $b_1 = t$ and $b_l = b$. Because of (*) and because P is irreducible,

$$\mathrm{out}(b_1, R) < l + m - 1, \qquad (**)$$

$$\mathrm{out}(b_l, R) \geq m + 1. \qquad (***)$$

Applying the definition of ν yields

$$\begin{aligned}\nu(P) - \nu(tbP) = \frac{1}{2}\,(&|\mathrm{out}(b_1, R) - (m + l - 1)| + |\mathrm{out}(b_l, R) - (m - 1)| \\ &- |\mathrm{out}(b_1, R) + 1 - (m + l - 1)| - |\mathrm{out}(b_l, R) - 1 - (m - 1)|).\end{aligned}$$

Using (**) and (***) this yields

$$\begin{aligned}\nu(P) - \nu(tbP) = \frac{1}{2}\,(&m + l - 1 - \mathrm{out}(b_1, R) + \mathrm{out}(b_l, R) - m + 1 \\ &- (m + l - 1 - \mathrm{out}(b_1, R) - 1) - (\mathrm{out}(b_l, R) - m)) = 1.\end{aligned}$$

This proves that ν changes uniformly at top–bottom corrections. The three cycle-count λ does not change uniformly at top–bottom corrections: a top–bottom correction at an irreducible 3-cycle yields a change of 1, where at a 4-cycle this change is 2.

The following lemma is used in the first characterization of ν. It says that there exist top–bottom correctable tournaments for any given outscore vector.

Lemma 6.2. *Let* $\underline{x} \in V - \{<0, 1, 2, \ldots, n-1>\}$. *Then there is a tournament* R *and elements* a, b *in a non-empty subset* Y *of* X *such that*

$$\mathit{out}(R) = \underline{x},\ <b, a> \in R,\ a \textit{ is a top} - \textit{element of } R|_Y \text{ and } b \textit{ is a bottom} \\ - \textit{element of } R|_Y, \textit{ where } Y \textit{ is an irreducible part of } R.$$

Proof. Because $\underline{x} \in V$, there is a tournament R' in T such that $\mathrm{out}(R') = \underline{x}$. Let $R' = R'|_{Y^1} \gg R'|_{Y^2} \gg \cdots \gg R'|_{Y^k}$, where $R'|_{Y^1}, \ldots, R'|_{Y^k}$ are the irreducible components of R'. Because $\underline{x} \neq <0, 1, \ldots, n-1>$, $R' \notin \mathbb{L}$. Hence, $\#Y^L \geq 3$ for at least one of the subsets Y^1 up to Y^k. Let a be a top-element and b a bottom-element of $R'|_{Y^l}$. If $<b, a> \in R'$, then we are ready.

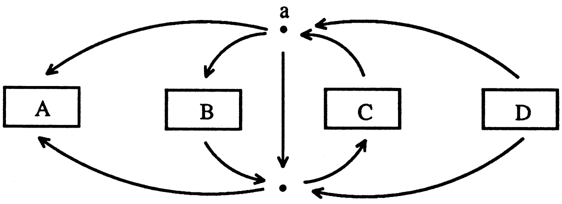


Fig. 2.

Therefore suppose $<a, b>\in R'$. Let for $x\in\{a, b\}$. $O_x = \text{Out}(x, R'|_{Y^l}) - \{a, b\}$ and $I_x=(Y^l-O_x)-\{a, b\}$. Furthermore, let $A=O_a\cap O_b$, $B=O_a\cap I_b$, $C=I_a\cap O_b$ and $D=I_a\cap I_b$. Then we can draw the following diagram with respect to $R'|_{Y^l}$, see Fig. 2.

If there is a $c\in C$, then reversing the cycle $<a, b, c, a>$ at R' yields the desired result. Therefore, suppose $C=\emptyset$. Because $R'|_{Y^l}$ is irreducible, $A\neq\emptyset$ and $D\neq\emptyset$ and there is a sequence, say, $y^1, y^2, \ldots, y^m\in Y^l$ such that $<y^i, y^{i+1}>\in R'$ for all $i\in\{1, \ldots, m-1\}$, $y^1\in A$ and $y^m\in D$. Now reversing the cycle $<a, b, y^1, \ldots, y^m, a>$ at R' yields the desired result. □

We now can state and prove our first characterization of the measure ν.

Theorem 6.3. *Let γ be an intransitivity measure. Then* (i) *and* (*ii) are equivalent, where*
(i) *γ is ν-like*
(ii) *γ is independent of cycle orientation and changes uniformly at top–bottom corrections.*

Proof.
(i)⇒(ii) Is shown above.
(ii)⇒(i) Let T be the linear extension of a three cycle, say $<x, y, z, x>$. Let $\alpha:=\gamma(\hat{T})$, then $\alpha>0$.

With induction on $k\geq 0$, we prove that for all $R\in\mathbb{T}$, $\nu(R)\leq k$ implies $\gamma(R)=\alpha\nu(R)$.

Basis $k=0$. This is evident by proposition 2.2.

Induction step. Suppose $\nu(R)=k+1$. Then we have to prove that $\gamma(R)=\alpha(k+1)$. By lemma 6.2 there is a tournament R' such that $\text{out}(R)=\text{out}(R')$ and at R' there is a top–bottom correction possible, say between t and b. Now by independency of cycle orientation, it follows that $\gamma(R)=\gamma(R')$. Because of independency of renaming, we can assume without loss of generality that $x=b$ and $y=t$. So, b is a bottom and t is a top-element of the irreducible component $\hat{T}|_{\{x,y,z\}}$ of $\hat{T}$. Hence, by uniform changing at top–bottom corrections it follows that

$$\gamma(R') - \gamma(tbR') = \gamma(\hat{T}) - \gamma(tb\hat{T}) = \alpha,$$

because $tb\hat{T}\in\mathbb{L}$. So

$$\begin{aligned}\gamma(R) = \gamma(R') = \gamma(tbR') + \alpha &= \alpha\nu(tbR') + \alpha \quad \text{(induction hypothesis)}\\ &= \alpha k + \alpha = \alpha(k+1). \quad \square\end{aligned}$$

Independence of the conditions in theorem 6.3 (ii): We showed earlier that λ is not uniformly changing at top–bottom corrections. In order to prove the independence of both characterizing conditions in the previous theorem, it is sufficient to construct a measure which is not independent of cycle orientation but which changes uniformly at top–bottom corrections. Consider the following intransitivity measure

$$\gamma(R)=\begin{cases}\lambda(R) \text{ if } R\in Z\\ \nu(R) \text{ if } R\notin Z\end{cases}$$

where $R\in Z\subseteq\mathbb{T}$ is such that there is a numbering of the alternatives in X, say a_1, $a_2,\ldots,a_n$, with $R=a_na_{n-2}(a_{n-2}a_1L)$ or $R=a_3a_1(a_na_3)L$, $L=a_1\gg\cdots\gg a_n$. So, $\text{out}(R)=<1, 1, 2,\ldots,n-3, n-2, n-2>$ and R is irreducible. It is easy to verify that $R\in Z$ does not have any top–bottom corrections. Further, for all $R\notin Z$, $tbR\notin Z$ for all top–bottom corrections tb of R. This may be deduced by observing that if $tbR\in Z$, this would imply that R is irreducible and consequently tbR must have a unique alternative with highest outscore, which is obviously not true. So, γ changes uniformly at top–bottom corrections. But, since $R=a_na_1L\notin Z$ while $\text{out}(R)=<1, 1, 2,\ldots,n-3, n-2, n-2>$, γ is not independent of cycle orientation.

We end this section with a second characterization of ν, which is based on the fact that ν measures the least possible upsets needed to make a tournament, with a given ordered outscore vector, transitive. So $\nu(T)=\min\{i(T'): \text{out}(T')=\text{out}(T)\}$, See Ryser (1964) or Fulkerson (1965).

Observe that the measure ν is equable. In order to prove this let $T\in\mathbb{T}$, $x, y\in X$ and $<x, y>\in T$. Consider the outscore vectors of T and yxT. The rearrangements of the elements of X, corresponding to the orderings of the outscores in $\text{out}(T)$ and $\text{out}(yxT)$ can be taken as the same for both T and yxT, supposing that $\text{out}(x, T)\neq\text{out}(y, T)+1$. So, in that case, we may deduce that for some i, j

$$2(\nu(T)-\nu(yxT))=|\text{out}(x, T)-(i-1)|+|\text{out}(y, T)-(j-1)|-|\text{out}(x, yxT)-(i-1)|-|\text{out}(y, yxT)-(j-1)|,$$

from which straightforwardly follows that $\nu(T)-\nu(yxT)\in\{-1, 0, 1\}$. If $\text{out}(x, T)=\text{out}(y, T)+1$, then $\nu(T)=\nu(yxT)$. This proves the equability of ν. Altogether, it shows that ν may be seen as a Slater's i related measure. But ν is not diminishable, which follows from theorem 4.3, and also from the following example. Let $L=a_1\gg a_2\gg a_3\gg a_4\gg a_5$ and $R=a_5a_3(a_4a_1L)$. Then $\nu(R)=1$, while for all $x, y\in\{a_1, a_2,\ldots,a_5\}$, $yxR\notin L$.

In order to obtain a characterization of ν, which resembles theorem 4.3, we introduce the following condition, which clearly holds for ν-like measures.

Definition 6.4. An intransitivity measure γ is diminishable by top–bottom corrections, if for all top–bottom corrections tbT of a tournament T, $\gamma(tbT)<\gamma(T)$.

The following theorem characterizes ν-like measures as the only equable, cycle orientation independent intransitivity measures which in addition are diminishable by top–bottom corrections.

Theorem 6.5. *Let γ be an intransitivity measure. Then γ is ν-like if, and only if, γ is equable, independent of cycle orientation and diminishable by top–bottom corrections.*

Proof. Only the 'if-part' is proved, because the 'only if-part' is discussed above. Equability and diminishability by top–bottom corrections imply uniform changes at top–bottom corrections. Now we are done by theorem 6.3. □

Independence of the conditions in theorem 6.5: As for the independence of these three conditions consider λ, τ and μ. The measure τ is defined by

$$\tau(R) = \begin{cases} \nu(R) & \text{if } R \notin W \\ \nu(R) - 1 & \text{if } R \in W, \end{cases}$$

where $R \in W$ if $\text{out}(R) = <\frac{1}{2}(n-1), \ldots, \frac{1}{2}(n-1)>$ if n is odd and $\text{out}(R) = <\frac{1}{2}n, \ldots, \frac{1}{2}n, \frac{1}{2}n - 1, \ldots, \frac{1}{2}n - 1>$ if n is even, where we take $n \geq 5$. The measure μ is defined by again using the set Z defined after the proof of theorem 6.3:

$$\mu(R) = \begin{cases} \nu(R) & \text{if } R \notin Z \\ \nu(R) + 1 (= 2) & \text{if } R \in Z. \end{cases}$$

Before it was argued that λ is independent of cycle orientation, but not equable. It is straightforward to show that λ is diminishable by top–bottom corrections. The measure τ is an intransitivity measure and it is easy to verify that it is independent of 3-cycle orientation, is equable, since $\nu(yx\ R) \in \{\nu(R) - 1,\ \nu(R)\}$ for all $R \in W$, but is not diminishable by top–bottom corrections when $R \in W$. The measure μ is also an intransitivity measure that is not independent of 3-cycle orientations, but is diminishable by top–bottom corrections. It is also equable, because ν is equable and if $R \in Z$, then $yxR \notin \mathbb{L}$ for all $(x, y) \in R$.

7. Quadratic measures

Here we study measures which only depend on the orders ("sizes") of the irreducible components. Moreover, this dependency is quadratic. So, as a matter of fact these measures form a special subclass of outscore dependent measures. After an introduction and a characterization of this class two special measures belonging to this class are discussed. These are the measure δ based on the portion of preferences which are part of a cycle and Bezembinder's ρ (see Bezembinder (1981) and David (1988)) which is equal to the dimension of the cycle space of a tournament.

Let γ be an intransitivity measure. Let T be a tournament with irreducible components $P^1, P^2, \ldots, P^k$ such that $T = P^1 \gg P^2 \gg \cdots \gg P^k$. Let t_i denote the cardinality of P^i for all $i \in \{1, 2, \ldots, k\}$. So, $t_i = \#P^i$. Let T^i be a linear extension of P^i for all $i \in \{1, 2, \ldots, k\}$. Then by extensiveness we have $\gamma(T) = \gamma(T^1) + \gamma(T^2) + \cdots + \gamma(T^k)$. Now γ is said to be a <u>quadratic measure</u> if there is a polynomial $p(x)$ of degree 2 such that for all $i \in \{1, \ldots, k\}$

$$\gamma(T^i) = p(t_i) := At_i^2 + Bt_i + C, \text{ where } A, B \text{ and } C \text{ are real numbers, } A \neq 0.$$

Hence,

$$\gamma(T) = A\Sigma_{i=1}^{k} t_i^2 + B\Sigma_{i=1}^{k} t_i + Ck = A\Sigma_{i=1}^{k} t_i^2 + Bn + Ck.$$

We say that γ is based on the polynomial $p(x)$. So, a quadratic measure γ is a function of the sum of squares of the sizes of the irreducible parts and the number of these parts of a tournament. Of course not all triples $<A, B, C>$ constitute a quadratic measure. If for instance P^i is a singleton and consequently T^i a linear ordering, then $\gamma(T^i)=0$. So, 1 is a root of polynomial $p(x)$. So, the polynomial has two real roots say 1 and r and $p(x)=c(x-1)(x-r)$, where c is a real constant different from zero. Furthermore, $p(x)\geq 0$ for $x\in\{1, 2, \ldots, n\}$. Hence, if $c>0$, then $r<3$ and if $c<0$, then $r>n$. These conditions characterize all polynomials that constitute a quadratic measure.

Theorem 7.1. *Let $p(x)$ be a polynomial of degree 2. Then there exists a quadratic measure based on $p(x)$ if, and only if, there are real numbers r and c, with $c\neq 0$, such that $p(x)=c(x-1)(x-r)$, $r<3$ if $c>0$ and $\mathrm{r}>n$ if $c<0$.*

Proof. The 'only if-part' is discussed above. In order to prove the 'if-part' let c and r be as described above. Define γ: $\mathbb{T}\to[0, \infty)$ for all tournaments T with irreducible component orders $t_1, t_2, \ldots, t_k$ as follows

$$\gamma(T) = \Sigma_{i=1}^{k} c(t_i - 1)(t_i - r).$$

Because $t_i\in\{1, \ldots, n\}$ and $r<3$ if $c>0$ and $r>n$ if $c<0$, it follows that γ is well-defined and intransitivity indicating. Because renaming does not change the size of irreducible components, γ is independent of renaming. The same holds for independence of commutations in concatenations. In order to prove extensiveness let $R\gg S\in\mathbb{T}$, where R has l irreducible components with sizes $r_1, r_2, \ldots, r_l$ and S has m irreducible components with sizes $s_1, \ldots, s_m$. Let, furthermore, $R\gg R'$ be a linear extension of R and $S'\gg S$ one of S. Then R' has $l':=\Sigma_{i=1}^{m} s_i$ irreducible components all of size 1 and S' has $m':=\Sigma_{i=1}^{l} r_i$ of these components also of size 1. Now by the definition of γ

$$\gamma(R \gg S) = \Sigma_{i=1}^{l} c(r_i - 1)(r_i - r) + \Sigma_{i=1}^{m} c(s_i - 1)(s_i - r),$$

$$\gamma(R \gg R') = \Sigma_{i=1}^{l} c(r_i - 1)(r_i - r) + \Sigma_{i=1}^{l'} c(1 - 1)(1 - r) = \Sigma_{i=1}^{l} c(r_i - 1)(r_i - r),$$

$$\gamma(S' \gg S) = \Sigma_{i=1}^{m'} c(1 - 1)(1 - r) + \Sigma_{i=1}^{m} c(s_i - 1)(s_i - r) = \Sigma_{i=1}^{m} c(s_i - 1)(s_i - r).$$

Hence, extensiveness holds. □

If γ is a quadratic measure based on polynomial $c(x-1)(x-r)$, then we also say that γ is based on c and r.

An intransitivity measure γ is <u>linear</u> if $\gamma(T^i)=Bt_i+C$. If P_i is a singleton, T^i is a linear ordering, so $\gamma(T^i)=0$, implying that $C=-B$. Hence $\gamma(T)=B(n-k)$, $B>0$.

The δ measure indicates the proportion of the total number of preferences that are part of a cycle. Let T be a tournament and $P_1, P_2, \ldots, P_k$ irreducible parts of T, so that $T=P_1\gg P_2\gg\cdots\gg P_k$. Now a preference $<x, y>\in T$ is on a cycle if $<x, y>\in P_i$

for some $i \in \{1, 2, \dots, k\}$. Let t_i be the size on which P_i is a partial tournament. Then obviously

$$\delta(T) = \frac{1}{n(n-1)} \sum_{i=1}^{k} t_i(t_i - 1).$$

So, δ is a quadratic measure based on $c = [1/n(n-1)]$ and $r = 0$.

Bezembinder's ρ measures the dimension of the cycle space of a tournament. Unlike λ, it takes into account all cycles. So, it is obvious that ρ measures four cycles three times as much as three cycles. It is well-known, see Bezembinder (1981), that for a tournament T

$$\rho(T) = \frac{1}{2} n(n-1)\delta(T) - n + k,$$

where k is the number of irreducible components of T. From this it follows that ρ is quadratic and based on $c = \frac{1}{2}$ and $r = 2$. The following condition for intransitivity measures characterizes the class of linear and quadratic measures.

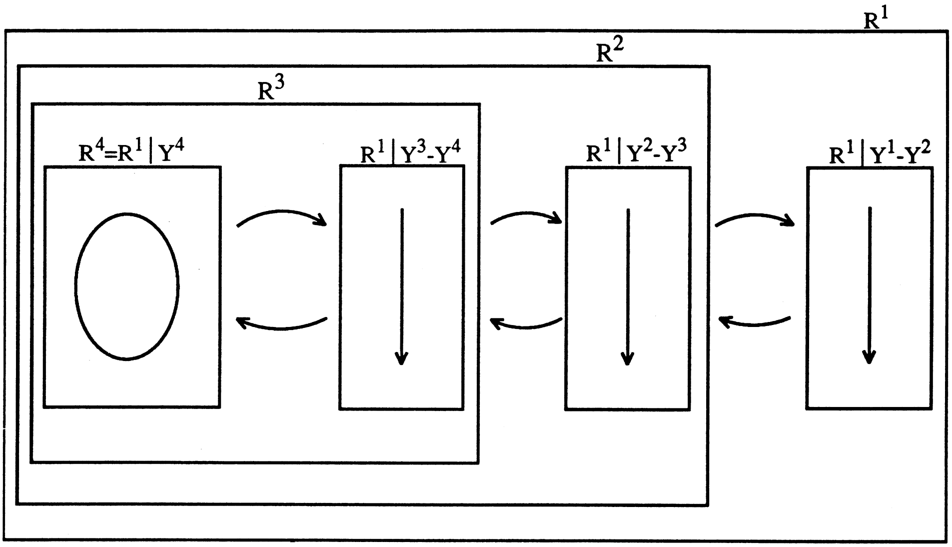
Definition 7.2. An intransitivity measure γ is said to average reduction if

$$[\gamma(T^1) - \gamma(T^2)] + [\gamma(T^3) - \gamma(T^4)] = 2[\gamma(T^2) - \gamma(T^3)], \quad (*)$$

for all tournaments T^1, T^2, T^3 and T^4 such that

– T^i is a linear extension of an irreducible partial tournament R^i on say Y^i for all $i \in \{1, 2, 3, 4\}$,
– $Y^1 \supset Y^2 \supset Y^3 \supset Y^4$ and $\#(Y^1 - Y^2) = \#(Y^2 - Y^3) = \#(Y^3 - Y^4)$,
– $R^i = R^1|_{Y^i}$ for $i \in \{2, 3, 4\}$, and
– $R^1|_{Y^i - Y^{i+1}} \in \tilde{\mathbb{L}}$ for $i \in \{1, 2, 3\}$.

In order to explain average reduction consider the following picture.



If we chip off the linear part $R^1|_{Y^1-Y^2}$ from R^1 yielding the irreducible partial tournament R^2, the measure of intransitivity may reduce by $\gamma(T^1)-\gamma(T^2)$, where T^i is a linear extension of R^i. Similarly this holds for $\gamma(T^2)-\gamma(T^3)$ and $\gamma(T^3)-\gamma(T^4)$. Now, although the parts that are chipped off are identical, because they are partial linear tournaments on subsets of equal size, it may happen that these reductions are not equal. For instance in the case of the 3-cycle count λ these reductions are not always equal. Averaging reduction now imposes that in these situations the middle reduction is the average of the outer ones.

From (*) it follows that $\gamma(T^1)-\gamma(T^4)=3(\gamma(T^2)-\gamma(T^3))$. This means that total reduction caused by the three chips, equals three times the middle reduction. Finally, (*) may by rearrangement yield

$$[\gamma(T^1)-\gamma(T^2)]-[\gamma(T^2)-\gamma(T^3)]=[\gamma(T^2)-\gamma(T^3)]-[\gamma(T^3)-\gamma(T^4)].$$

This means that the reduction of reductions is uniform.

It is easy to verify that a linear measure averages reduction. Let γ be a quadratic measure based on c and r. In order to prove that it averages reduction let T^1, T^2, T^3 and T^4 be as in definition 7.2. Let $\#Y^1=k$, $\#Y^2=k-m$, $\#Y^3=k-2m$ and $Y^4=k-3m$. Then

$$\begin{aligned}[\gamma(T^1)-\gamma(T^2)]+[\gamma(T^3)-\gamma(T^4)] &= c[(k-1)(k-r)-(k-m-1)(k-m-r) \\ &\qquad +(k-2m-1)(k-2m-r) \\ &\qquad -(k-3m-1)(k-3m-r)] \\ &= 2c[(2k-1-r)m-3m^2].\end{aligned}$$

Furthermore,

$$\begin{aligned}[\gamma(T^2)-\gamma(T^3)] &= c[(k-m-1)(k-m-r)-(k-2m-1)(k-2m-r)] \\ &= c[(2k-1-r)-3m^2].\end{aligned}$$

From this, equation (*) straightforwardly follows. Conversely, as is proved in theorem 7.3, if γ averages reduction, then it is a linear or quadratic measure.

Theorem 7.3. *Let $\#X\geq 7$. The intransitivity measure γ is linear or quadratic if, and only if, γ averages reductions.*

Proof. The ‘only-if’ part is argued above. In order to prove the ‘if’ part, let γ average reductions. First, we will prove that linear extensions of equal sized irreducible partial tournaments are equally intransitive. Let R and S be two irreducible partial tournaments on Y and Z, such that $\#Y=\#Z$. Because independence of renaming preserves irreducibility, we may assume that $Y=Z$. We prove that $\gamma(T^R)=\gamma(T^S)$, where T^R is a linear extension of R and T^S is one of S. If $\#Y\in\{1, 3, 4\}$, then because of independence of renaming, we may assume that $R=S$ and then we are finished by proposition 2.3.

From Moon (1968), page 6, we may deduce that if T is an irreducible partial tournament on U, with size greater than or equal to four, then there is an alternative $x\in U$, such that $T|_{U-\{x\}}$ is irreducible. Applying this argument successively we may find

$y_1, y_2, \ldots, y_l \in Y$ $z_1, z_2, \ldots, z_l \in Z$, $Y_1, Y_2, \ldots, Y_l \subset Y$ and $Z_1, Z_2, \ldots, Z_l \subset Z$, where $l = \# Y = \# Z$, such that $Y_k = \{y_1, \ldots, y_k\}$ and $Z_k = \{z_1, z_2, \ldots, z_k\}$ for all $k \in \{1, \ldots, l\}$ and $R|_{Y_t}$ and $S|_{Z_t}$ are irreducible for all $t \in \{3, \ldots, l\}$. Let T_t^R be a linear extension of $R|_{Y_t}$ and T_t^S one of $S|_{Z_t}$ for all $t \in \{3, \ldots, l\}$. Now by averaging reduction it follows that

$$\gamma(T_{t+3}^R) = 3\gamma(T_{t+2}^R) - 3\gamma(T_{t+1}^R) + \gamma(T_t^R) \text{ and} \tag{1}$$

$$\gamma(T_{t+3}^S) = 3\gamma(T_{t+2}^S) - 3\gamma(T_{t+1}^S) + \gamma(T_t^S) \text{ for all } t \in \{3, \ldots, l-3\}. \tag{2}$$

Above we have shown that $\gamma(T_3^S) = \gamma(T_3^R)$ and $\gamma(T_4^S) = \gamma(T_4^R)$. So, if we have proved that $\gamma(T_5^S) = \gamma(T_5^R)$, then by (1) and (2) with $t = 3$ we have $\gamma(T_6^S) = \gamma(T_6^R)$. Using (1) and (2) with $t = 4$ we would obtain that $\gamma(T_7^S) = \gamma(T_7^R)$. Repeating this successively yields finally that $\gamma(T_l^S) = \gamma(T_l^R)$. But as $T_l^3 = T^S$ and $T_l^R = T^R$, this would prove our desired result. So, we only have to prove that $\gamma(T_5^S) = \gamma(T_5^R)$. Henceforth, it suffices to prove that $\gamma(T^R) = \gamma(T^S)$ if $\# Y = 5$.

Let R be an irreducible tournament on $\{a, b, c, d, e\}$ and S an irreducible tournament on $\{a, b, c, d, f\}$. Now by Moon's argument there is an alternative $x \in \{a, b, c, d, e\}$ and $y \in \{a, b, c, d, f\}$ such that $R|_{\{a,b,c,d,e\}-\{x\}}$ and $S|_{\{a,b,c,d,f\}-\{y\}}$ are irreducible. Without loss of generality let $x = e$ and let $<a, b, c, d, a>$ be a cycle in R. As by renaming any irreducible tournament on four alternatives can be transformed in any other such tournament, we may assume without loss of generality that $y = f$, $R|_{\{a,b,c,d\}} = S|_{\{a,b,c,d\}}$ and $<a, b, c, a>$ is a cycle in R and S. Now consider $P = R \cup S \cup \{<e, f>\}$. Then P is an irreducible tournament on $\{a, b, c, d, e, f\}$. Let T^4, T^3 be linear extensions of $P|_{\{a,b,c,d\}}$, $P|_{\{a,b,c\}}$ respectively. Further, we take T^P to be a linear extension of P. By averaging reduction, we obtain

$$[\gamma(T^P) - \gamma(T^R)] + [\gamma(T^4) - \gamma(T^3)] = 2[\gamma(T^R) - \gamma(T^4)]$$

So,

$$\gamma(T^R) = \frac{1}{3}\gamma(T^P) + \gamma(T^4) - \frac{1}{3}\gamma(T^3).$$

Likewise

$$\gamma(T^S) = \frac{1}{3}\gamma(T^P) + \gamma(T^4) - \frac{1}{3}\gamma(T^3).$$

Thus

$$\gamma(T^R) = \gamma(T^S).$$

Hence, we have just proved that there are numbers $\alpha_1, \alpha_2, \ldots$ such that for all k, if R is an irreducible subtournament on Y, with $\# Y = k$, then $\gamma(T^R) = \alpha_k$ where T^R is the linear extension of R. Now, by averaging reduction, for all $k \geq 3$, we obtain $\alpha_{k+3} - 3\alpha_{k+2} + 3\alpha_{k+1} - \alpha_k = 0$. This equation is a homogeneous third order linear difference equation with constant coefficients. It has the following general solution $\alpha_k = Ak^2 + Bk + C$, $k \geq 3$, where $A, B, C \in \mathbb{R}$. So, γ is a linear or quadratic measure. □

Now we characterize δ-like and ρ-like measures. As proportionality is disregarded by theorem 7.1 this boils down to fixing the second root r of the polynomial $c(x-1)(x-r)$. Therefore, another condition yielding an extra condition for r is introduced. It is straightforward to show that δ satisfies the following condition.

Definition 7.4. An intransitivity measure γ is said to measure a four cycle twice as much as a three cycle if, $\gamma(T^4)=2\gamma(T^3)$, where T^4 is a linear extension of a four cycle and T^3 is one of a three cycle.

This condition takes into account that there is only one type of three and four cycle and that the latter is build of two three cycles which join one preference. Similarly measuring a four cycle three times as much as a three cycle can be defined. This condition takes into account that there are three circuits in a four cycle and just one in a three cycle. It is satisfied by the measure ρ. Now we have the following characterization.

Theorem 7.5. *Let $\#X \geq 7$. Let γ be an intransitivity measure. Then γ is δ-like if, and only if, γ averages reductions and it measures four cycles twice as much as three cycles.*

Proof. The 'only if-part' is argued before. Now the 'if-part' is proved. Consider such a measure. Since clearly γ is not linear, by theorem 7.3 there are c, $r \in \mathbb{R}$ such that γ is quadratic and based on c and r. Now let T^4 and T^3 be as in definition 7.4.
Then

$$\gamma(T^4) = c(4-r)3,$$

$$\gamma(T^3) = c(3-r)2 \text{ and}$$

$$\gamma(T^4) = 2\gamma(T^3).$$

These three yield $r=0$. So, γ is δ-like. □

Similarly to theorem 7.5 the following can be proved.

Theorem 7.6. *Let $\#X \geq 7$. Let γ be an intransitivity measure. Then γ is ρ-like if, and only if, γ averages reductions and measures four cycles three times as much as three cycles.* □

Because the independence of the conditions in all the theorems of this section is quite obvious we end the discussion on quadratic measures.

8. Summary of results

TABLE 8.1
Table of properties

	equability		uniform change at		reduction by		
	equability	echelon equability	adjacent eq. reversals	top-bottom corrections	elementary reversals	top-bottom corrections	independence of cycle orientation
i	(■)	□	□	□	(■)	□	□
λ	□	(■)		□	■	□	
λ			(■)				(■)
ν	(■)	□	□		□	(■)	(■)
ν				(■)			(■)

	averaging reduction	measures a 4-cycle twice as much as a 3-cycle	measures a 4-cycle three times as much as a 3-cycle
δ	(■)	(■)	
ρ	(■)		(■)

□ : not satisfied

■ : satisfied

(■) : characterizing properties, to be taken together in each row

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