References

1. G. Birkhoff and S. Mac Lane, A Survey of Modern Algebra, New York, 1953.
2. N. Bourbaki, Éléments de Mathématique, Book II, Algèbre, Ch. II, Algèbre linéaire, Paris, 1947.
3. N. Jacobson, Lectures in Abstract Algebra, Volume II, Linear Algebra, New York, 1953.
4. R. D. Sheffield, A general theory for linear systems, this MONTHLY, vol. 65, 1958, pp. 109–111.
5. R. M. Thrall and L. Tornheim, Vector Spaces and Matrices, New York, 1957.

A TOURNAMENT PROBLEM

FRANCIS SCHEID, Boston University

Four bridge players (or tennis players [1]) wanting to play a fair "round robin" tournament would have no difficulty at all in arranging a schedule. Using the letters A, B, C, D, to denote the players they would automatically play the three matches

A and B against C and D
A and C against B and D
A and D against B and C.

This schedule has the following features:

(a) Each player plays exactly *once with* each other player.
(b) Each player plays exactly *twice against* each other player.
(c) No one plays *against* himself.
(d) No one plays *with* himself.

It is also true that each team plays the only possible team of opponents. When there are more players, this additional feature (each team playing all possible opponents) requires a very long tournament. The question arises, "Can a schedule be arranged for v players so that at least features (a), (b), (c), (d) are satisfied?"

For $v=5$ a little effort produces a successful result such as

$$AB - CD, \qquad AC - DE, \qquad AD - BE, \qquad AE - BC, \qquad BD - CE,$$

and further effort shows that, aside from relettering of the players, no other solution is possible.

Before considering larger values of v it is helpful to deduce the number of matches necessary, *assuming* that a solution is possible. Our problem is, of course, similar to that of constructing a block design. In the usual notation for the parameters of a block design we let

v = number of players,
$\lambda = 3$ (number of matches involving a given pair),

$k = 4$ (four players to a match),
$b =$ number of matches,
$r = v - 1$ (number of times each individual plays),

and have at once the familiar

$$b = \{v(v-1)\}/4. \tag{1}$$

This makes it evident that solutions cannot exist except possibly for

$$v = 4, 5, 8, 9, 12, 13, \cdots.$$

The first two cases were covered at the outset. Some additional solutions follow, and others may be found in [4].

$v = 8$

AB – CD	BC – EH
AC – FH	BD – EG
AD – FG	BF – DH
AE – CG	BG – CF
AF – BE	CE – DF
AG – BH	CH – DG
AH – DE	EF – GH

$v = 9$

AB – CD	AH – CF	BI – EH
AC – EI	AI – FG	CG – DF
AD – HI	BC – FI	CH – DE
AE – DG	BD – GI	CI – GH
AF – BE	BF – DH	EF – DI
AG – BH	BG – CE	EG – FH

$v = 12$*

AB – HI	AL – EH	AK – DG
EF – BG	CE – LF	KI – EJ
GH – DK	AE – IC	HK – EL
BI – CG	CD – EI	AF – JD
AC – KJ	FH – BK	DE – FJ
FG – CH	DF – IG	GK – CL
LJ – FB	AD – BL	AI – FK
CJ – DH	BD – KE	BC – DL
LI – GJ	BH – IJ	HL – DI
CK – BJ	EG – HJ	KL – FI
AG – BE	AJ – GL	AH – CF

$v = 13$

BM – IF	CL – DK	EJ – GH
CA – JG	DM – EL	FK – HI
DB – KH	EA – FM	GL – IJ
EC – LI	FB – GA	HM – JK
FD – MJ	GC – HB	IA – KL
GE – AK	HD – IC	JB – LM
HF – BL	IE – JD	KC – MA
IG – CM	JF – KE	LD – AB
JH – DA	KG – LF	ME – BC
KI – EB	LH – MG	AF – CD
LJ – FC	MI – AH	BG – DE
MK – GD	AJ – BI	CH – EF
AL – HE	BK – CJ	DI – FG

$v = 16$*

AB – GN	AP – CH	AC – HP	BC – MJ	AH – CP	GH – OE
BO – LI	MP – GI	CF – EI	AE – LJ	PJ – KO	MN – EK
CD – NO	GL – CK	LN – FP	AD – KI	NJ – HI	CJ – BM
KM – EN	AG – BN	FG – DJ	EH – GO	IJ – HN	BF – HK
IO – BL	DL – HM	AM – FO	BD – PE	AO – FM	DH – LM
KL – CG	AF – MO	KN – EM	DF – JG	LP – FN	PI – GM
AN – BG	DM – HL	OC – DN	EG – OH	AK – ID	NI – HJ
DO – CN	JO – KP	NP – FL	AL – JE	DE – BP	GK – CL
BE – DP	BJ – CM	FH – BK	AI – DK	AJ – EL	MI – GP
CE – IF	PO – KJ	DG – FJ	FK – BH	EF – IC	BI – LO

* The results for $v=12$ and $v=16$ were obtained by D. La Dage, Boston University.

An obvious generalization of our problem involves teams of m players. With the same notation as before and $k=2m$ we replace (a) and (b) by

(a^1) Each player plays exactly $m-1$ times with each other player.
(b^1) Each player plays exactly m times against each other player.

This implies $\lambda=2m-1$ and leads at once to

$$b = \{v(v - 1)\}/(2m) \tag{2}$$

Again solutions are possible only for certain sequences. The following examples were obtained by manual rearrangement of standard block designs [3].

$m = 3, v = 7$	$m = 3, v = 9$	$m = 4, v = 8$
ABC – DEF	ABC – DEF	ABCD – EFGH
ADE – BFG	ABE – DGH	ABEF – CDGH
AFG – BCE	ADG – CFI	ABGH – CDEF
BDG – ACF	AFH – BDI	ACEG – BDFH
BEF – CDG	AEH – CFG	ACFH – BDEG
CDF – AEG	AFI – BEG	ADEH – BCFG
CEG – ABD	ACD – EHI	ADFG – BCEH
	AGI – BCH	
	BFH – CEI	
	BDI – CEG	
	BFG – CDH	
	DEF – GHI	

It is interesting to note that the first of these is, as printed, a pair of Steiner triple systems of order seven. The second, however, cannot be so divided. Perhaps some valiant or fortunate reader will discover a design for $m=3$, $v=10$ or $m=4$, $v=9$. A few moments is sufficient, however, to try the various combinations for $m=3$, $v=6$ and discover that no solution is possible even though (2), for these values, suggests $b=5$.

This impossibility raises the general question of existence. No effort has been made to discover the answer. However, R. C. Bose has suggested that his method of Galois fields [2] can be modified to cover certain sequences; indeed, the design listed earlier for $v=13$ was found by his method. Should the military forces of any friendly nation develop a lively interest in tournaments it is very likely that many hands will become available for the task. To further these prospects this brief paper should perhaps have been titled "On the testing of individual ability in group competition."

References

1. Lewis Carroll, On the scheduling of lawn tennis tournaments (Complete works of Lewis Carroll, Modern Library).

2. R. C. Bose, On the construction of balanced incomplete block designs, Ann. of Eugenics, vol. 9, 1939, pp. 353–399.

3. R. A. Fisher and Frank Yates, Statistical Tables for Biological, Agricultural, and Medical Research, London, 1938 (5th ed., 1957).

4. M. Kraitchik, Mathematical Recreations, New York, 1942.