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A comparison of the effectiveness of tournaments*

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1. INTRODUCTION AND SUMMARY

When it is desired to compare t treatments in a paired-comparison experiment the simplest balanced design requires $(t-1)$ units of each treatment. Each replication, comprising $\frac{1}{2}t(t-1)$ comparisons, is analogous to a round robin tournament involving t players, with each comparison corresponding to a game between two contestants. This analogy was first pointed out by Kendall (1955). If we are primarily concerned with picking the best treatment it may be practicable to use a technique requiring less than $(t-1)$ units of each treatment per replication. The knock-out tournament (or cup-tie procedure) has been suggested for this purpose by Maurice (1958) and also by David (1959). Maurice compares the balanced design or round robin tournament with the cup-tie procedure for the case of four populations with means in the most unfavourable configuration (one population has a mean larger by δ than the common mean of the other three populations). She bases the comparison on the amount of replication (and hence the sampling cost) required to ensure detection of the difference δ with probability at least P, and concludes that the cup-tie procedure is the more economical. In effecting comparisons with the balanced design she uses tables provided by Bechhofer (1954). David investigates some properties of knock-out and round robin tournaments and obtains an expression for the probability with which the best player wins a knock-out tournament, certain assumptions having been made about the strengths of the players.

In this paper six tournament types are investigated for their effectiveness in selecting the best one of four players. The types, together with abbreviations used for convenience in referring to them, are as follows: R = round robin tournament, K_0 = single knock-out tournament with a random draw and each pair playing one game, K_1 = single knock-out tournament with a random draw and each pair playing until one of the players has won two games, K_2 = combination of two knock-out tournaments with separate random draws. K_3 = combination of two knock-out tournaments with seeding in the second draw based on the outcome of the first tournament, and D = double elimination tournament, in which no player is eliminated until he has lost two games. The K_0 type requires only three games, whereas the other types each require a minimum of six games.

Representing the four players by C_1, C_2, C_3 and C_4, we write π_{ij} for the probability with which C_i defeats C_j. We assume that every game results in the selection of a winner, so that

$$\pi_{ji} = 1-\pi_{ij}.$$

In § 2 we describe the tournament types and discuss the evaluation of the probability that a specified player emerges as winner, after a play-off if necessary, in each type. Without loss of generality we take C_1 as the specified player. The evaluation of the expected number of games in each type is discussed in § 3. In § 4 a best player is defined and certain assumptions

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are made regarding the parameters π_{ij}. Two criteria are used for the comparison of the effectiveness of the tournament types, viz. (*a*) the probability that the best player wins, and (*b*) the expected number of games. Comparisons are based on three series of assigned parameter values. The special case in which all but one of the players are of equal strength (counterpart of the most unfavourable configuration discussed by Maurice) is considered in § 5.

The comparisons in §§ 4 and 5 indicate that if sufficient emphasis is placed on criterion (*a*), K_1 is the best of the six procedures. Otherwise K_0 is preferable. This conclusion is in general agreement with that obtained by Maurice.

2. Evaluation of the probability that C_1 wins

2·1. *Round robin tournament*

In the round robin tournament every player meets every other player once. For four players this yields 64 possible outcomes which may be classified in terms of the partitions

$$[3210], \quad [31^3], \quad [2^30], \quad [2^21^2],$$

occurring respectively in the following numbers of different ways:

$$24, \quad 8, \quad 8, \quad 24.$$

The notation [3210] is used to indicate that one player has won three games, another two, a third one, and the fourth none. The other partitions are similarly defined. Outcomes related to the players in specified order will be enclosed in parentheses. Thus (3210) will represent the outcome in which C_1 has won three games, C_2 has won two games, and so on.

In evaluating the probability that C_1 emerges as winner, one must consider, in addition to the probability of an outright win, the probabilities that C_1 ties with one or two other players and wins the resulting play-off. The probability that C_1 wins outright is readily seen from the outcomes (3210) and (31^3) to be given by $\pi_{12}\pi_{13}\pi_{14}$.

The outcomes in which C_1 ties with two other players are (2220), (2202), and (2022). In writing down the associated probabilities it is helpful to note that the player with no wins has lost to each of the others, while the latter have each won once in games among themselves. Thus, for example, we have

$$P(2220) = \pi_{14}\pi_{24}\pi_{34}(\pi_{12}\pi_{23}\pi_{31} + \pi_{13}\pi_{21}\pi_{32}).$$

The play-off for outcomes in this category takes the form of a round robin tournament with three players, with the possible outcomes [210] and $[1^3]$. If $[1^3]$ occurs the play-off must be repeated. Thus the probability that a given player wins the play-off is conditional upon the non-occurrence of $[1^3]$. For example, the probability that C_1 wins a play-off with C_2 and C_3 may be written as

$$P_{123} = \pi_{12}\pi_{13}/(\pi_{12}\pi_{13} + \pi_{21}\pi_{23} + \pi_{31}\pi_{32}).$$

The complete probability that C_1 ties with two other players and wins the resulting play-off is evaluated from the expression

$$P(2220)\,P_{123} + P(2202)\,P_{124} + P(2022)\,P_{134}.$$

Each of the outcomes (2211), (2121), and (2112), in which C_1 ties with one other player, may occur in four ways, the respective probabilities of which are easily written down. The complete probability that C_1 ties with one other player and wins the resulting play-off is evaluated from the expression

$$P(2211)\pi_{12}+P(2121)\pi_{13}+P(2112)\pi_{14}.$$

2·2. *Single knock-out tournament (K_0 type)*

We designate by K_0 the single knock-out tournament involving four players and consisting of three games, or two rounds of play. For the first round the players are selected into pairs at random. The two winners in the first round play off in the second, so that the final winner will have won two games, another player will have won one, and the other two none. The draw, or selection of the pairs for the first round, may be made in three ways, each of which leads to eight possible outcomes, or 24 outcomes in all. Each of these is the partition $[210^2]$. There are 12 possible outcomes associated with players in specified order, each of which may occur in two ways, i.e. in two out of the three possible draws. Since the three draws are equally likely, the probability (before the draw) that C_i wins two games and C_j wins one in the K_0 tournament is given by the general relation

$$\pi_{ij}(\pi_{ik}\pi_{jm}+\pi_{im}\pi_{jk})/3,$$

where C_k and C_m are the two players with no wins. The probabilities of the 12 outcomes are evaluated from this relation by assigning all possible ordered pairs obtainable from the numbers 1, 2, 3 and 4 to the ordered pair (i, j), and in each case letting k and m be the two numbers not in the pair.

2·3. *Single knock-out tournament (K_1 type)*

One way of introducing replication into a single knock-out tournament is to have each pair play more than one game before going on to the next round. We designate by K_1 the single knock-out tournament in which each pair plays until one of the players has won two games (a 'best two out of a possible three' criterion). K_1 is effectively the simple tournament with three games played in each contest, the third being unnecessary when the outcome is determined by the first two. By specifying an odd number of games one takes care of the problem of ties in the individual contests.

The probability that C_i wins two out of a possible three games with C_j is readily seen to be given by

$$Q_{ij} = \pi_{ij}^2(1+2\pi_{ji}).$$

The probabilities associated with the various outcomes in the K_1 tournament are evaluated by following the procedure described in § 2·2 except that all π_{ij}, $i, j = 1, 2, 3, 4$ are replaced by the corresponding Q_{ij}.

2·4. *Combination of two knock-out tournaments (K_2 type)*

Replication may also be introduced by repeating the entire tournament, the winner being declared as the player with the largest total number of wins, possibly after a play-off. The simplest way of repeating the knock-out tournament is to start with a fresh random draw for each repetition. We designate by K_2 the combination of two knock-out tournaments based on separate random draws. The two tournaments are evidently independent events. The outcomes of the combination may be classified in terms of the partitions $[420^2]$, $[41^20]$, $[3210]$, $[3^20^2]$, $[2^21^2]$, $[2^30]$, the first three of which correspond to an outright win for one of the players, while the next two represent ties involving two players, and the last one represents a tie involving three players.

The probabilities associated with the various outcomes may be written by appropriately associating the expressions related to the single tournament described in § 2·2. Play-offs for the ties involving two and three players are, of course, the same as those described in § 2·1.

2·5. *Combination of two knock-out tournaments (K_3 type)*

Instead of using separate random draws, if the tournaments are played in succession, one may use the outcome of the first in setting up the draw for the second. At the end of the first tournament some evidence has been provided as to the relative strengths of the players. The two who have won games appear to be stronger than the two with no wins. We wish to investigate the possibility that this information may be used to improve the effectiveness of the combination of two tournaments.

In the K_3 type we select the draw for the second tournament so that the two players with wins in the first tournament remain in opposite halves while the positions of their opponents are interchanged. The first tournament is based on a random draw, and its outcome together with the particular draw used uniquely determine the draw for the second tournament. We wish to keep the two players who have appeared best in the first tournament apart in the first round of the second. Only two of the three possible draws will accomplish this, and the arrangement which we have proposed selects the one of these which was not used in the first tournament. Thus the only alternative seeding arrangement would involve repeating the first draw. Further reference to this point will be made in § 4.

The possible outcomes of the K_3 combination are the same as those of K_2, but the tournaments are not independent events. The conditional probabilities involved are obtained by appropriate association of the probabilities of the outcomes of the K_0 tournament together with the draws from which they arise.

2·6. *Double elimination tournament*

In the double elimination tournament no player is eliminated until he has lost two games. The first round is the same as for the single knock-out tournament (K_0), but in the second round, in addition to the two winners, the two losers are entered in play. Of these, one will be a two-time loser at the end of the second round, and is then eliminated from further play. This leaves one two-time winner and two with a win and a loss, which we shall designate respectively by WW, WL, and LW, letting W stand for a win and L for a loss. In the fifth game WL and LW play off for the elimination of the loser. The winner, either WLW or LWW, meets WW in the sixth game. Should WW win, his opponent is eliminated and the tournament ends, the winner being represented by WWW. However, should WW lose, this being his first loss, he then has the record WWL whereas his opponent in the sixth game now has the record $WLWW$ or $LWWW$ as the case may be. A seventh game is then required to determine the final winner, represented by $WWLW$, $WLWWW$, or $LWWWW$ as the case may be.

The probability that C_1 wins is the sum of the probabilities that he achieves one of the winning outcomes WWW, $WWLW$, $WLWWW$, or $LWWWW$. This is easily evaluated for a given draw and the result for each of the other two possible draws may be inferred by appropriate interchanges of subscripts. For the random draw the resulting values are averaged over the three possible draws.

3. Evaluation of the expected number of games

3·1. *Round robin tournament*

The round robin tournament with four players ends in six games only if one player achieves an outright win. Play-offs involving three players require initially three additional games, three more if $[1^3]$ occurs, and so on, leading to an infinite geometric progression with common ratio $P(1^3)$. Thus, for example, the expected number of additional games in a play-off involving C_1, C_2, and C_3 may be written as

$$E_{123} = 3/(\pi_{12}\pi_{13}+\pi_{21}\pi_{23}+\pi_{31}\pi_{32}).$$

The expected number of additional games due to ties involving three players is evaluated from the expression

$$P(2220)\,E_{123}+P(2202)\,E_{124}+P(2022)\,E_{134}+P(0222)\,E_{234}.$$

Each of the outcomes in the classification $[2^2 1^2]$ leads to a single-game play-off. Thus the expected number of additional games due to ties involving two players is the sum of the probabilities of the outcomes (2211), (2121), (2112), (1221), (1212), and (1122).

3·2. *Single knock-out tournaments (K_0 and K_1 types)*

The K_0 tournament will always consist of three games, since no tie-breaking play-offs are required. However, the K_1 type requires a minimum of six games, since at least two are needed in each of the three contests. A third game is required whenever one player wins the first game and the other the second. Thus in a given contest the probability of a third game is the probability that a given player wins the first game and loses the second, or vice-versa. For a given draw the excess over six of the expected number of games is the sum of the probabilities of third games in all contests evaluated for all possible outcomes. For the random draw these values are averaged over the three possible draws.

3·3. *Combinations of two knock-out tournaments (K_2 and K_3 types)*

In the combinations of two knock-out tournaments if any player achieves an outright win the series ends in six games. Ties involving two players and represented by the partitions $[3^2 0^2]$ and $[2^2 1^2]$ each require a single-game play-off. The expected number of additional games on account of these is therefore the sum of the probabilities of the appropriate outcomes. The number of additional games due to ties involving three players is evaluated in the manner described in § 3·1.

3·4. *Double elimination tournament*

The minimum number of games required in the double elimination tournament with four players is six, since three players must lose two games each. Thus the expected number of games is six plus the probability of a seventh game. Such a game is required when WW loses in the sixth game. Using the record of C_1 as a basis of classification we may say that a seventh game is required in the event of each of the following outcomes: $WWLW$, $WWLL$, $WLWWW$, $WLWWL$, $WWWW$, and $LWWWL$. Since the probability of a seventh game is not affected by the winning or losing of C_1 in that game, we may group these outcomes in pairs to give WWL, $WLWW$, and $LWWW$. In addition, some of the outcomes in which C_1 is eliminated earlier in the tournament may lead to a seventh game. They are WLL, LWL, and LL, a seventh game being required in these cases whenever the player with no previous losses loses in the sixth game. Using this classification system, one may determine the probability of a seventh game for a given draw. The result for each of the other two possible draws may be inferred by appropriate interchanges of subscripts. For the random draw the resulting values are averaged over the three possible draws.

4. Comparisons in series of numerical examples

For the purpose of comparing the effectiveness of the six tournament types described above, we define a best player as one having a probability greater than $\frac{1}{2}$ of defeating each of the others. Without loss of generality we may refer to the players in decreasing order of strength, C_1 being strongest and C_4 being weakest. The possibility that some of the players C_2, C_3 and C_4 may be equally strong is not excluded. We shall consider only situations in which circular triads amongst the parameters do not occur; accordingly we assume that the six parameters satisfy the following inequalities:

$$\pi_{14} \geqslant \pi_{13} \geqslant \pi_{12} > \tfrac{1}{2}, \quad \pi_{24} \geqslant \pi_{23} \geqslant \tfrac{1}{2}, \quad \pi_{13} \geqslant \pi_{23} \geqslant \tfrac{1}{2}, \quad \pi_{14} \geqslant \pi_{24} \geqslant \pi_{34} \geqslant \tfrac{1}{2}. \tag{4·1}$$

Table 4·1. *Assigned parameter values in three series of examples*

	Example no.	π_{23}	π_{24}	π_{34}
Series 1	1·1	0·5000	0·5000	0·5000
$\pi_{12} = 0·5400$	1·2	·6400	·8500	·6200
$\pi_{13} = ·6500$	1·3	·6127	·8396	·7679
$\pi_{14} = ·8600$	1·4	·6400	·8500	·8200
	1·5	·6500	·8600	·8400
Series 2	2·1	0·5000	0·5000	0·5000
$\pi_{12} = 0·7000$	2·2	·5758	·7247	·6598
$\pi_{13} = ·7600$	2·3	·7500	·8200	·7200
$\pi_{14} = ·8600$	2·4	·7500	·8400	·8000
	2·5	·7500	·8500	·8400
Series 3	3·1	0·5000	0·5000	0·5000
$\pi_{12} = 0·8000$	3·2	·5862	·6923	·6136
$\pi_{13} = ·8500$	3·3	·7000	·7500	·7000
$\pi_{14} = ·9000$	3·4	·8000	·8300	·7500
	3·5	·8400	·8700	·8500

Comparisons have been made by assigning values to the six parameters and evaluating the two criteria, viz. (*a*) the probability that C_1 wins, and (*b*) the expected number of games for each of the tournament types. Typical results in the form of three series of examples are presented below. The assigned parameter values, satisfying the inequalities (4·1) are given in Table 4·1. In the examples 1·3, 2·2, and 3·2 the values of π_{23}, π_{24}, and π_{34} have been determined from the assigned values of π_{12}, π_{13}, and π_{14} so as to satisfy the mathematical model proposed by Bradley & Terry (1952). For the three series and each of the six tournament types the probabilities that C_1 wins are given in Table 4·2, listed in generally increasing order of the probabilities involved. The same order of presentation is used in Table 4·3, which lists the expected numbers of games for all types except K_0 (which always requires three games).

It will be observed in Table 4·1 that each series is characterized by a common set of values of π_{12}, π_{13}, and π_{14} in the five examples. Within series the values of π_{23}, π_{24}, and π_{34} have been assigned in a generally increasing order of magnitude in the ranges defined by (4·1), starting in each case with the lower limit of $\frac{1}{2}$. With this arrangement one finds that the probability that C_1 wins decreases throughout each series for all six tournament types. This is because we are giving an increasing advantage to C_2, and to some extent to C_3, relative to the weakest player C_4, thus tending to increase the probability that C_1 will meet later in the tournament or in a play-off a player that he has the relatively smaller chance of defeating.

Table 4·2. *Probabilities that C_1 wins in three series of examples*

Example no.	Tournament type K_0	R	K_2	K_3	D	K_1
1·1	0·4581	0·5239	0·5279	0·5284	0·5307	0·5374
1·2	·4249	·4615	·4641	·4669	·4711	·4789
1·3	·4209	·4564	·4581	·4621	·4672	·4737
1·4	·4173	·4503	·4502	·4551	·4618	·4688
1·5	·4156	·4472	·4463	·4516	·4593	·4667
2·1	0·5958	0·6910	0·6974	0·7036	0·7212	0·7406
2·2	·5817	·6637	·6714	·6795	·6992	·7179
2·3	·5734	·6438	·6452	·6564	·6860	·7063
2·4	·5707	·6393	·6390	·6514	·6826	·7034
2·5	·5694	·6370	·6357	·6488	·6810	·7022
3·1	0·7216	0·8190	0·8255	0·8341	0·8592	0·8751
3·2	·7133	·8028	·8119	·8246	·8491	·8652
3·3	·7089	·7924	·7996	·8115	·8436	·8602
3·4	·7044	·7803	·7832	·7977	·8381	·8562
3·5	·7014	·7734	·7726	·7894	·8347	·8540

Table 4·3. *Expected numbers of games in three series of examples*

Example no.	Tournament type R	K_2	K_3	D	K_1
1·1	6·8175	6·7796	6·7630	6·4543	7·3301
1·2	6·7320	6·6978	6·6780	6·4761	7·2401
1·3	6·7740	6·7174	6·6908	6·4812	7·2255
1·4	6·7757	6·7121	6·6831	6·4818	7·1962
1·5	6·7769	6·7097	6·6794	6·4811	7·1808
2·1	6·6588	6·6298	6·6215	6·4072	7·2080
2·2	6·6406	6·6139	6·6026	6·4250	7·1741
2·3	6·6197	6·5699	6·5649	6·4267	7·0836
2·4	6·6298	6·5724	6·5629	6·4280	7·0502
2·5	6·6355	6·5745	6·5620	6·4286	7·0298
3·1	6·5120	6·4496	6·4514	6·3218	7·0202
3·2	6·4683	6·4453	6·4569	6·3361	6·9984
3·3	6·4611	6·4331	6·4337	6·3419	6·9468
3·4	6·4523	6·4109	6·4154	6·3458	6·8685
3·5	6·4547	6·4048	6·4055	6·3473	6·7936

In Table 4·2 we observe that K_2 is slightly better than R in the early part of each series. but that this effect tends to be reversed for the larger values of π_{23}, π_{24}, and π_{34}. The seeding arrangement (K_3) produces only a small improvement over the use of separate random draws (K_2). It was remarked in § 2·5 that the only alternative seeding arrangement would involve making a single random draw and using it for both tournaments. This has been found in a number of examples to give results which are intermediate between those for K_2 and K_3. The latter thus appears to be the optimum available seeding arrangement. In terms of the probability that C_1 wins, K_0 is least desirable, D is somewhat better than R, K_2, and K_3, but K_1 is superior to all of the others.

In Table 4·3 we observe that the expected number of games is less for the double elimination tournament than for each of the R, K_2, and K_3. Thus, for the cases considered, the round robin and the two knock-out combinations K_2 and K_3 always compare unfavourably with the double elimination procedure in terms of both criteria. However, the K_1 type apparently accomplishes a higher probability that C_1 wins at the expense of a higher expected number of games, while K_0, although yielding the lowest probability, requires only three games.

In order to effect reasonable comparisons between D, K_0, and K_1 we set up the cost function

$$\text{expected cost} = q[1-P(C_1 \text{ wins})] + \text{expected number of games}, \tag{4·2}$$

where q is a factor representing the assessed cost of a wrong decision, expressed in units of the cost per game.

Using (4·2) one may find for each example three critical values of q, viz.

q_1: above which D is preferable to K_0;
q_2: above which K_1 is preferable to D;
q_3: above which K_1 is preferable to K_0.

If $q_1 < q_2$ the interval between these two values is a range of q for which D is better than K_0 and K_1. However, if $q_1 \geqslant q_2$ there is no value of q for which D is to be preferred to K_0 and K_1.

Table 4·4. *Critical values of q in three series of examples*

Example no.	q_1	q_2	q_3	Example no.	q_1	q_2	q_3	Example no.	q_1	q_2	q_3
1·1	47	130	54	2·1	27	41	29	3·1	24	43	26
1·2	75	97	78	2·2	29	40	30	3·2	24	41	26
1·3	75	114	80	2·3	30	32	30	3·3	24	36	26
1·4	78	102	81	2·4	30	29	30	3·4	25	28	25
1·5	79	94	81	2·5	30	28	30	3·5	25	23	24

For the three series of examples considered in this section the critical values q_1, q_2, and q_3, rounded to the nearest integer below the exact value, are given in Table 4·4. We observe that in Examples 2·4, 2·5, and 3·5 there is no value of q for which D is preferable to K_0 and K_1. In the other cases, in which there is a range of q where D is preferable to K_0 and K_1, that range is observed to be, in general, relatively short. Furthermore, upon evaluating the advantage of using D instead of K_1, for $q = q_3$ and also for some other intermediate values in the range between q_1 and q_2, it has been found that the largest advantage amounts to only 0·51 (Example 1·1). That is, the advantage of using D instead of K_1 in a range in which D is preferable seems to be only about half the cost of one game. In view of these considerations it would seem to be best to use K_0 for $q < q_3$ and K_1 for $q > q_3$. Since K_1 is effectively the simple tournament with replication in each contest, we conclude that it seems to be better to vary the amount of experimentation than to depart from the ordinary (and simplest) form of tournament.

5. Comparisons in a special case

For the special case in which C_2, C_3, and C_4 are equal in strength we may write

$$\pi_{12} = \pi_{13} = \pi_{14} = \pi \quad \text{and} \quad \pi_{23} = \pi_{24} = \pi_{34} = \tfrac{1}{2}. \tag{5·1}$$

Expressions for the probability that C_1 wins and the expected number of games in each of the six tournament types are found in the manner indicated in §§ 2 and 3. If we denote the probability that C_1 wins by P and use the abbreviations for the tournament types as subscripts, we find that

$$\left.\begin{aligned} P_R &= \pi^3(2-\pi)+\pi^3(1-\pi)(2+\pi+2\pi^2)/4(1-\pi+\pi^2),\\ P_{K_0} &= \pi^2,\\ P_{K_1} &= \pi^3(2-\pi)+\pi^3(1-\pi)(-2+8\pi-4\pi^2),\\ P_{K_2} &= \pi^3(2-\pi)+2\pi^4(1-\pi)(3-2\pi+\pi^2)/3(1-\pi+\pi^2),\\ P_{K_3} &= \pi^3(2-\pi)+\pi^4(1-\pi)(4-3\pi+2\pi^2)/2(1-\pi+\pi^2),\\ \text{and} \qquad P_D &= \pi^3(2-\pi)+2\pi^4(1-\pi). \end{aligned}\right\} \tag{5·2}$$

Similarly, using E to denote the expected number of games, we have

$$\left.\begin{aligned} E_R &= 7-3\pi/2+3\pi^2/2-\pi^3+9\pi^2(1-\pi)/4(1-\pi+\pi^2),\\ E_{K_0} &= 3,\\ E_{K_1} &= 7+2\pi-7\pi^2/2+7\pi^3-10\pi^4+4\pi^5,\\ E_{K_2} &= 41/6-\pi+\pi^2/6+2\pi^3/3-2\pi^4/3+2\pi^2(1-\pi)(2-\pi)/(1-\pi+\pi^2),\\ E_{K_3} &= 27/4-\pi/2-\pi^2/4+\pi^3-\pi^4+3\pi^2(1-\pi)(2-\pi)/2(1-\pi+\pi^2),\\ \text{and} \qquad E_D &= 13/2-\pi^2/2+2\pi^3-2\pi^4. \end{aligned}\right\} \tag{5·3}$$

For the special case, the inequalities (4·1) reduce to

$$1 > \pi > \tfrac{1}{2}. \tag{5·4}$$

From the expressions (5·2) together with the condition (5·4) it may be shown that

$$P_{K_1} > P_D > P_{K_3} > P_{K_2} > P_R > P_{K_0}. \tag{5·5}$$

Since it may be shown that E_D is less than each of E_R, E_{K_1}, E_{K_2}, and E_{K_3}, the double elimination tournament is preferable to each of R, K_2, and K_3 in terms of both criteria. However, as in § 4, it may be observed that K_1 yields the highest probability that C_1 wins at the expense of the highest expected number of games, while K_0, although giving the lowest probability, requires only three games.

For the comparison of D, K_0, and K_1 we again use the cost function (4·2). The critical values q_1, q_2, and q_3 have been evaluated for nine values of π, uniformly spaced over the range defined in (5·4), and are given, rounded to the nearest integer below the exact value, in Table 5·1. Inspection of these values indicates that for $\pi < 0·70$ there is no range of values of q for which D is preferable to K_0 and K_1. For $\pi \geq 0·75$ there is such a range, the length of which tends to increase as π increases. However, upon evaluating the advantage of using D instead of K_1 for $\pi = 0·95$ and $q = q_3$, it has been found that the advantage is only about 0·40, or less than half the cost of one game. For smaller values of π the advantage is less than this. Thus we have again the conclusion that it would seem to be best to use K_0 for $q < q_3$

and K_1 for $q > q_3$, the choice therefore depending on the emphasis to be placed on the criterion of reliability relative to that of sampling cost. It would appear that, at least for four players, there is not much to be gained from varying the ordinary tournament rules, so long as one allows variation in the specified number of wins.

Table 5·1. *Critical values of q in the special case*

π	q_1	q_2	q_3
0·55	165	147	161
·60	75	70	74
·65	47	46	47
·70	34	36	34
·75	27	31	28
·80	24	31	25
·85	23	36	25
·90	26	53	28
·95	39	144	44

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REFERENCES

BECHHOFFER, R. E. (1954). A single-sample multiple decision procedure for ranking means of normal populations with known variances. *Ann. Math. Statist.* **25**, 16–39.

BRADLEY, R. A. & TERRY, M. E. (1952). Rank analysis of incomplete block designs. I. The method of paired comparisons. *Biometrika*, **39**, 324–45.

DAVID, H. A. (1959). Tournaments and paired comparisons. *Biometrika*, **46**, 139–49.

KENDALL, M. G. (1955). Further contributions to the theory of paired comparisons. *Biometrics*, **11**, 43–62.

MAURICE, RITA J. (1958). Selection of the population with the largest mean when comparisons can be made only in pairs. *Biometrika*, **45**, 581–6.