Fair Seeding in Knockout Tournaments

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We investigated the existence of fair seeding in knockout tournaments. We define two fairness criteria, both adapted from the literature: envy-freeness and order preservation. We show how to achieve the first criterion in tournaments whose structure is unconstrained, and prove an impossibility result for balanced tournaments. For the second criterion we have a similar result for unconstrained tournaments, but not for the balanced case. We provide instead a heuristic algorithm which we show through experiments to be efficient and effective. This suggests that the criterion is achievable also in balanced tournaments. However, we prove that it again becomes impossible to achieve when we add a weak condition guarding against the phenomenon of tournament dropout.

Categories and Subject Descriptors: I.2.11 [**Artificial Intelligence**]: Multiagent Systems

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1. INTRODUCTION

Knockout tournaments constitute a common format of sporting events, which attract millions of viewers and billions of dollars annually, and also model a specific type of election scheme (namely, sequential pairwise elimination election). In this simple and familiar format, the players are placed at the leaf nodes of a binary tree (which is often, but not always, balanced). The players at sibling nodes will compete against each other in pairwise matches and the winners will move up the tree. The winner of the tournament is the player who reaches the root node of the tournament tree. The arrangement of the players at the leaf nodes is called the seeding of the tournament.

The seeding can significantly influence the result of a tournament. This has been demonstrated in several papers, but most of them focus on how to find a seeding that maximizes the winning probability of the strongest player (so-called predictive power) [Appleton 1995; Horen and Riezman 1985; Ryvkin 2005; Vu and Shoham 2010]. This usually means giving the strongest player the easiest schedule, while making it harder for the remaining players. An example of such seedings is given in Figure 1. In this example, players are numbered based on their strengths, with player 1 being the strongest player. This seeding is rarely seen in practice, however, because it seems to be very unfair to other players, especially the strong ones such as 2 or 3.

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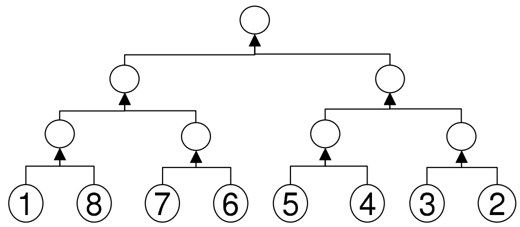

Fig. 1. A biased seeding that maximizes winning probability of player 1.

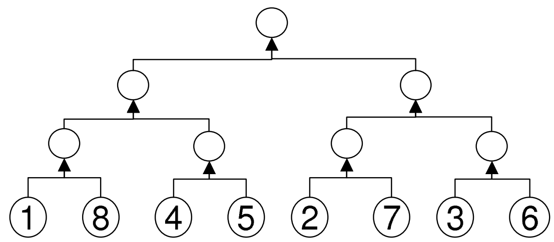

Fig. 2. A popular seeding with 8 players.

The most popular seeding actually used in major sport tournaments is the one that pairs up in the first stage the strongest player with the weakest player, the second strongest player with the second weakest player, so on and so forth. An example for 8 players is shown in Figure 2. One possible rationale is that it seems fair: confrontations between strong players are delayed until later rounds, and that helps increase the chance that one of those strong players will win the tournament. This raises several questions: How do we capture this intuition about fairness? Is there any other seeding that is also fair? Can we always find such a fair seeding?

To answer these questions, we consider two alternative fairness criteria adapted from the literature: envy-freeness and order preservation. We analyze separately each criterion across different tournament settings, providing both possibility and impossibility results. To explain our results we need to lay out the settings with all their variants, which we do in the next section. In Section 3 we summarize the existing results from the literature. We then analyze envy-freeness and order preservation in Section 4 and 5, respectively, and conclude in Section 6.

2. SETTINGS

We consider two different types of tournament structures and two different models of winning probabilities between the players. Regarding the tournament structure, we consider both balanced and unconstrained tournaments (as long as the tournament structure is a binary).

Among the unconstrained tournaments, we want to draw attention to a special structure called "caterpillar" tree. In a tournament with a caterpillar structure, the players are arranged in some order. The first two players will compete against each other. The winner of the match will then advance forward and compete with the third player, so on and so forth. An example is shown in Figure 3. This structure has been studied in the social choice literature as a voting tree structure and has been shown to have interesting properties [Fischer et al. 2009].

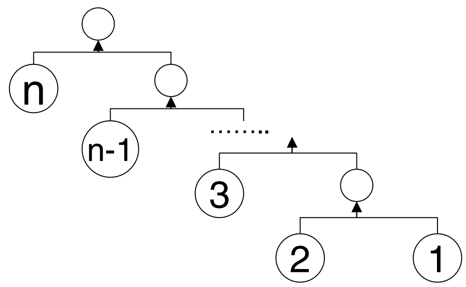

Fig. 3. A caterpillar tournament for n players.

Among balanced tournaments, S_n^* is a seeding formed by matching in the first stage the strongest player with the weakest player, the second strongest player with the second weakest player, and so on and so forth. For the successive rounds, the subtree containing the strongest player is combined with the subtree containing the weakest players possible, so on and so forth.

$$S_n^* = \left[[1, n], \ldots, [i, (n-i+1)], [(\frac{n}{2} - i + 1), (\frac{n}{2} + i)], \ldots\right].$$

Here we encode the seeding by a permutation of the numbers between 1 and n. The kth number is the ranking of the player that will be placed at the kth leaf node of the binary tournament tree, from left to right. An example of S_n^* when $n = 8$ is $S_8^* = [1, 8, 4, 5, 2, 7, 3, 6]$, as shown in Figure 2.

Most knockout tournaments used in practice are balanced. Although the caterpillar tournament approximates some real-world competition structures (for example the world championship organized by the World Boxing Association), the main reason to consider unbalanced tournaments in addition to balanced ones is to highlight the special properties of the latter.

Regarding the winning probability models, we assume that for any pairwise match, the probability of one player winning against the other is known. This probability can be obtained from past statistics or from some learning models. In the first model, we do not place any constraints on the probabilities besides the fundamental properties.

Definition 2.1 (*General Winning Probability Model*). Given a set of n players, the winning probabilities between the players form a matrix P such that p_{ij} denotes the probability that player i will win against player j, $\forall (i \neq j) : 1 \leq i, j \leq n$, and P satisfies the following constraints:

(1) $p_{ij} + p_{ji} = 1$
(2) $0 \leq p_{ij}, p_{ji} \leq 1$

Note that the winning probabilities might not be transitive, e.g., we can have $p_{ij} > 0.5$, $p_{jk} > 0.5$, and $p_{ki} > 0.5$.

The second winning probability model is the monotonic model which is a special case of the general model. This model is popular and well known in the literature [Horen and Riezman 1985; Hwang 1982; Moon and Pullman 1970; Schwenk 2000; Vu et al. 2009]. The players are assumed to have unknown but fixed intrinsic strengths or abilities. They are numbered from 1 to n in descending order of their strengths and the winning probabilities between the players reflect these rankings.

Definition 2.2 (*Monotonic Winning Probability Model*). Given a set of n players, the winning probabilities between the players form a matrix P such that p_{ij} denotes the probability that player i will win against player j, $\forall (i \neq j) : 1 \leq i, j \leq n$, and P satisfies the following constraints:

(1) $p_{ij} + p_{ji} = 1$
(2) $0 \leq p_{ij}, p_{ji} \leq 1$
(3) $p_{ij} \leq p_{i(j+1)}$
(4) $p_{ji} \geq p_{(j+1)i}$ (which is actually implied by (1) and (3))

We define a knockout tournament as the following.

Definition 2.3 (*General Knockout Tournament*). Given a set N of players and a winning probability matrix P, a knockout tournament $KT_N = (T, S)$ is defined by:

(1) A tournament structure T which is a binary tree with $|N|$ leaf nodes
(2) A seeding S which is a one-to-one mapping between the players in N and the leaf nodes of T

Note that we will write KT_N as KT when the context is clear.

Given the winning probability matrix P for a set of players N and a tournament $KT_N = (T, S)$, the probability of player k winning the tournament, denoted $q(k, KT_N)$, is calculated by the following recursive formula.

(1) If $N = \{j\}$, then $q(k, KT_N) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$

(2) If $|N| \geq 2$, let $KT_{N1} = (T_1, S_1)$ and $KT_{N2} = (T_2, S_2)$ be the two sub-tournaments of KT_N such that T_1 and T_2 are the two subtrees connected to the root node of T, and N_1 and N_2 are the set of players assigned to the leaf nodes of T_1 and T_2 by S_1 and S_2 respectively. If $k \in N_1$ then

$$q(k, KT_N) = \sum_{i \in N_2} q(k, KT_{N1}) \times q(i, KT_{N2}) \times p_{ki}$$

and symmetrically for $k \in N_2$.

Using the formula above, we can efficiently calculate the winning probabilities of the players in time $O(|N|^2)$.

3. PAST RESULTS

There are two main approaches in tournament design. The first one is qualitative approach: maximizing a specific quantity of the tournament. In Hazon et al. [2007] and Vu et al. [2009], the objective function is maximizing the winning probability of a given target player. In other works [Groh et al. 2009; Horen and Riezman 1985; Ryvkin 2005; Vu and Shoham 2010], the authors assume the players are ordered based on their intrinsic strengths, and the winning probabilities are monotonic with regards to this ordering. The quantities being maximized in these papers include the winning probability of the strongest player, the expected strength of the winner, or the effort of the players.

The second approach is axiomatic. Different properties are proposed for a "good" seeding. Most of the work within this approach focuses on balanced knockout tournaments with the monotonic winning probability model. In Schwenk [2000], three axioms are proposed to specify what a good seeding should satisfy. The axioms are called delayed confrontation, sincerity rewarded, and favoritism minimized.

— *Delayed Confrontation*. Two players rated among the top 2^j players shall never meet until the number of players has been reduced to 2^j or fewer.
— *Sincerity Rewarded*. A higher-ranked player should never be penalized by being given a schedule more difficult than that of any lower ranked.
— *Favoritism Minimized*. The schedule should minimize favoritism to any particular rank.

In order to achieve these axioms, Schwenk [2000] introduces a seeding method in which for $n = 2^m$ players, they are divided into m cohorts with cohort C_i $(1 \leq i \leq m)$ consisting of players in the set $\{2^{i-1} + 1, ..., 2^i\}$. The players within each cohort will then be randomly placed at the pre-assigned positions of the cohort. The randomized seeding is shown to satisfy all three axioms on expectation. However, for a chosen seeding after the randomization, it can be grossly unfair.

Alternatively, in Hwang [1982], a different property called monotonicity is used as a requirement for a good seeding.

— *Monotonicity*. The probability of a given player winning the tournament is higher than all of the weaker players.

To achieve this property, the players are reseeded after each round such that the strongest player remained faces the weakest, the second-strongest faces the second-weakest, so on and so forth.

We will generalize sincerity rewarded and monotonicity so that they can also be applied for the general winning probability model. Unlike past work which uses probabilistic seedings [Schwenk 2000] or dynamic seedings [Hwang 1982], we focus on deterministic and static seedings instead, since they are more often used in practice. Interestingly, in Horen and Riezman [1985] and Groh et al. [2009] it is shown that for a balanced tournament of size 4 with monotonic winning probability model, the seeding [1 4 2 3] always achieves "Monotonicity." Tournament of size 4 is a special case due to its extremely small size. For our paper, we focus on tournaments of sizes 8 or larger.

Other tournament formats such as round robin or Swiss-system are also addressed in the literature (see Kendall et al. [2010] for an extensive survey). Most of the work here focuses on graph-theoretic and combinatorial properties of the tournament (e.g., no team plays on the same court two consecutive rounds, or no team plays against extremely weak or extremely strong teams in consecutive rounds) as opposed to the winning probabilities of the players.

4. FIRST FAIRNESS CRITERION: ENVY-FREENESS

In Schwenk [2000], it is argued that a good seeding should not give any strong player a harder tournament schedule than a weaker player. One of the main reasons is that if strong players are given harder schedules, they will have incentives to under-perform during the preseason period or lie about their strengths in order to get lower rankings instead. We formalize this intuition to define envy-freeness as the first criterion for a seeding to be fair: no player envies the seeding position of another player weaker than him.

First we need to specify when a player is considered to be stronger than another. In the monotonic winning probability model, the notion of a stronger player is clear. For any pair of players (i, j) such that $i < j$, player i is always stronger than player j. In the general model, that is no longer the case. Player i might be stronger than j while playing against some opponents but not the others. Similarly, the winning probability between i and j by itself is not a good indicator either. Player i may have much higher chance to win in the match against j, but has much lower chance than j while playing against other players. In this case, it is unclear which player should be considered stronger.

To avoid these disputable cases, we define a player to be stronger between the two if he dominates the second player when playing against all other players.

Definition 4.1 (*Dominance*). Player i dominates another player j if and only if the following holds: (a) $p_{ij} \geq p_{ji}$, and (b) $\forall k : 1 \leq k \leq n$ $(k \neq i, j)$, it is the case that $p_{ik} \geq p_{jk}$.

This definition is natural and intuitive. Notice that it also applies to the monotonic model; here domination is equivalent to being higher ranked.

When there is a tie between two players (i.e., all of their winning probabilities are equal), we need a tie-breaking rule. Our results hold for any monotonic tie-breaking rule. WLOG, we assume the tie is resolved based on lexicographic ordering: for any pair of players (i, j) such that $\forall k : 1 \leq k \leq n$ $(k \neq i, j)$, $p_{ik} = p_{jk}$ and $p_{ij} = p_{ji} = 0.5$, i is considered to dominate j if and only if $i < j$.

Given the definition of Dominance, we define Envy-freeness as the following.

Definition 4.2 (*Envy-Free Seeding*). Given a set N of players, a winning probability matrix P, and a knockout tournament $KT_N = (T, S)$, the seeding S is envy-free if and only if $\forall(i, j) \in N$ such that i dominates j, player i does not have a higher probability of winning the tournament by swapping his position in the seeding S with j's, i.e., $q(i, (T, S)) \geq q(i, (T, S' = S_{i \leftrightarrow j}))$.

Notice that in the general winning probability model, if there is no pair of players such that one dominates the other, any seeding will trivially satisfy the requirement above. On the other extreme, in the monotonic model, an envy-free seeding will have to satisfy the requirement for all pairs of players.

4.1. Envy-Freeness for Unconstrained Tournaments

When there is no constraint on the structure of the tournament, envy-freeness can be achieved by the following tournament structure and seeding method. Note that the result holds for the general model (and hence the monotonic model also).

THEOREM 4.3. *Given a set N of players, a winning probability matrix P, envy-freeness can be achieved by the knockout tournament with the caterpillar structure and the seeding S such that for every pair of players (i, j), if i dominates j then j is placed below i on the tournament tree.*

The seeding above can be achieved easily by first constructing a directed acyclic graph in which each node represents a player and there is a directed edge from player i to player j $(\forall(i, j) \in N)$ if and only if i dominates j. We then perform a topological sort of the graph and use this ordering to place players on the tournament tree. We are then guaranteed that dominated players are placed lower on the tournament tree and the above condition is satisfied.

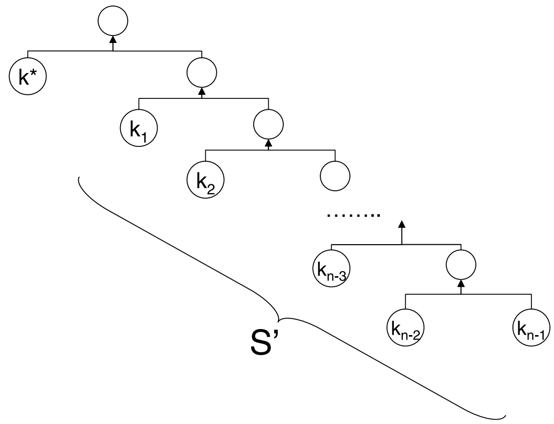

Fig. 4. The seeding S composed of k^* and S'.

PROOF OF THEOREM 4.3. We prove the theorem by induction on the number of players.

Base case. When $|N| = 2$, there is only one possible binary tree with 2 leaf nodes which is trivially envy-free.

Inductive case. Assume that the theorem holds for all N' such that $|N'| = n - 1$, we need to show for all N such that $|N| = n$, the seeding we construct as above is envy-free.

Let S be a seeding which satisfies the condition that for any pair of players (i, j), if i dominates j then j is placed below i on the tournament tree. Let k^* be the player at the root node of the tournament tree using the seeding S. Let $N' = N \setminus \{k^*\}$. Due to the special property of S, there is no player $k' \in N$ such that k' dominates k^*.

Since N' has size $n - 1$, based on the inductive hypothesis, the seeding S' for N' (as shown in Figure 4) is envy-free. Thus to show that S is also envy-free, we just need to show that k^* is not envious of any player placed below him. In other words, k^* will not have a higher winning probability when swapping his seeding position with any player $k_i \in N'$.

Using the same notation as in Section 2, we denote S as:

$$[k^*, k_1, k_2, ..., k_{i-1}, k_i, k_{i+1}, ..., k_{n-1}].$$

Let S_i be the seeding created from S by swapping k^* with k_i:

$$S_i = [k_i, k_1, k_2, ..., k_{i-1}, k^*, k_{i+1}, ..., k_{n-1}].$$

Notice that in S_i, regardless of the ordering of the players k_j on the left of k^*, $\forall 1 \leq j \leq i$, the winning probability of k^* will stay the same.

Formally, let $p(k_j)$, $\forall j : i < j \leq n - 1$, be the probability that player k_j will get to play against k^* (in other words, the probability that k_j will win the sub-tournament among $\{k_{i+1}, ..., k_{n-1}\}$). The probability that k^* will win the tournament with the seeding S_i ($q(k^*, S_i)$), is the same as with the seeding $S'_i = [k_1, k_2, ..., k_{i-1}, k_i, k^*, k_{i+1}, ..., k_{n-1}]$ because:

$$q(k^*, S_i) = \sum_{j:i<j\leq n-1} \left[p(k_j) \times p_{k^*,k_j}\right] \times \prod_{l:1\leq l\leq i} p_{k^*k_l} = q(k^*, S'_i).$$

Table I. The Winning Probabilities between 8 Players

	2	3	4	5	6	7	8
1	0.5	0.5	0.5	$0.5 + 2\epsilon$	$0.5 + 2\epsilon$	1	1
2	-	0.5	0.5	0.5	$0.5 + \epsilon$	1	1
3	-	-	0.5	0.5	$0.5 + \epsilon$	1	1
4	-	-	-	0.5	$0.5 + \epsilon$	1	1
5	-	-	-	-	0.5	1	1
6	-	-	-	-	-	1	1
7	-	-	-	-	-	-	0.5

Let us consider the winning probability of k^* in S'_{i-1}, which is created by swapping k^* and k_i (i.e., $S'_{i-1} = [k_1, k_2, ..., k_{i-1}, k^*, k_i, k_{i+1}, ..., k_{n-1}]$).

$$\begin{aligned}
q(k^*, S'_{i-1}) &= \sum_{j:i<j\leq n-1} [p(k_j)p_{k_jk_i}p_{k^*k_j} + p(k_j)p_{k_ik_j}p_{k^*k_i}] \times \prod_{l:1\leq l\leq i-1} p_{k^*k_l} \\
&= \sum_{j:i<j\leq n-1} p(k_j)[p_{k_jk_i}p_{k^*k_j} + p_{k_ik_j}p_{k^*k_i}] \times \prod_{l:1\leq l\leq i-1} p_{k^*k_l} \\
&\geq \sum_{j:i<j\leq n-1} p(k_j) \times \min(p_{k^*k_j}, p_{k^*k_i}) \times \prod_{l:1\leq l\leq i-1} p_{k^*k_l} \\
&\geq \sum_{j:i<j\leq n-1} p(k_j) \times p_{k^*k_j} \times p_{k^*k_i} \times \prod_{l:1\leq l\leq i-1} p_{k^*k_l} \\
&= \sum_{j:i<j\leq n-1} p(k_j) \times p_{k^*k_j} \times \prod_{l:1\leq l\leq i} p_{k^*k_l} \\
&= q(k^*, S'_i).
\end{aligned}$$

The first inequality holds because $p_{k_jk_i} + p_{k_ik_j} = 1$. The second inequality holds because $0 \leq p_{k^*k_j}, p_{k^*k_i} \leq 1$. This means when we swap the position of k^* with the position of the player right in front of k^* in the seeding, the winning probability of k^* does not decrease. Thus if we keep swapping the position of k^* with k_j in S'_j sequentially for $j = (i-1) \rightarrow 1$, we have the following sequence of inequalities:

$$q(k^*, S_i) = q(k^*, S'_i) \leq q(k^*, S'_{i-1}) \leq ... \leq q(k^*, S'_2) \leq q(k^*, S'_1) \leq q(k^*, S).$$

This shows that k^* will not have a higher winning probability by swapping its seeding position with player k_i. Thus S is envy-free. □

4.2. Envy-Freeness for Balanced Tournaments

When the tournament structure has to be a balanced binary tree, it is no longer possible to always achieve envy-freeness even when the winning probabilities between players are monotonic. Here we assume n, the number of players, is a power of 2.

THEOREM 4.4. *For any $n = 2^r \geq 8$, there exists a set of n players with a monotonic winning probability matrix P such that it is not possible to find an envy-free seeding S for the balanced knockout tournament between these n players.*

PROOF. We first prove the result for the case when $n = 8$, and then for general n.

The case of $n = 8$. We construct a tournament between 8 players with the winning probability matrix as shown in Table I (note that we show only the top half of the matrix).

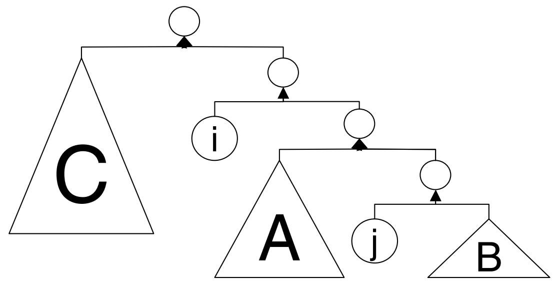

Fig. 5. The *reduced* tournament structure that does not satisfy envy-freeness.

Since there are only 315 unique seedings for a tournament of 8 players, we can exhaustively check all of them using a computer program. Given this winning probability matrix, for each of those 315 seedings, there is always a pair of players whose probabilities of winning the tournament violate the criterion.

The case of $n = 2^r > 8$. To construct a tournament between n players, we add $(n-8)$ dummy players to the given set of 8 players. The dummy players lose to the original 8 players with probability 1. Note that player 7 and 8 can be regarded as the dummy players to the top 6 players since player 7 and 8 lose with probability 1 to these players. Let M denote the set of the top 6 players from 1 to 6. Essentially the tournament between n players is reduced to a tournament between the players in M as if we take out all of the dummy players and player 7 and 8.

Let us consider all possible tournament trees T_M for 6 players. We first prove the following property of T_M: For any seeding to be envy-free, in the *reduced* tournament tree T_M of the players in M, there exists no pair of players $(i, j) \in M$ such that the difference between their depths is greater than 1, as in Figure 5.

This is true since we can always swap j with the dummy player who plays against i in the original tournament and improve the winning probability of j. To show this, let us first calculate the winning probability of j before the swap:

$$Q_1 = \sum_{l\in B}[q(l, B) \times p_{jl}] \times \sum_{k\in A}[q(k, A) \times p_{jk}] \times p_{ji} \times \sum_{h\in C}[q(h, C) \times p_{jh}]$$

Recall that $q(k, A)$ is the probability that player k will win the sub-tournament A.

After the swap, the winning probability of j becomes:

$$\begin{aligned} Q_2 &= \sum_{l\in B}\sum_{k\in A} q(l, B) \times q(k, A)[p_{jl}p_{lk} + p_{jk}p_{kl}] \times p_{ji} \times \sum_{h\in C}[q(h, C) \times p_{jh}] \\ &\geq \sum_{l\in B, k\in A} q(l, B) \times q(k, A) \times \min(p_{jl}, p_{jk}) \times p_{ji} \times \sum_{h\in C}[q(h, C) \times p_{jh}] \\ &> \sum_{l\in B, k\in A} q(l, B) \times q(k, A) \times (p_{jl} \times p_{jk}) \times p_{ji} \times \sum_{h\in C}[q(h, C) \times p_{jh}] \\ &= \sum_{l\in B} q(l, B) \times p_{jl} \times \sum_{k\in A} q(k, A) \times p_{jk} \times p_{ji} \times \sum_{h\in C}[q(h, C) \times p_{jh}] = Q_1. \end{aligned}$$

The first inequality holds because $p_{lk} + p_{kl} = 1$. The second inequality is strict because for j, l, and k in the top 6 players, p_{jl} and p_{jk} is less than 1. Note that we do not assume any relationship between i and j here since our focus is on j being envious of the dummy player who got matched up with i in the original tournament.

There are only 6 possible knockout tournament trees for 6 players, as shown in Figure 6. However, only the top two trees satisfy the condition that there is no pair

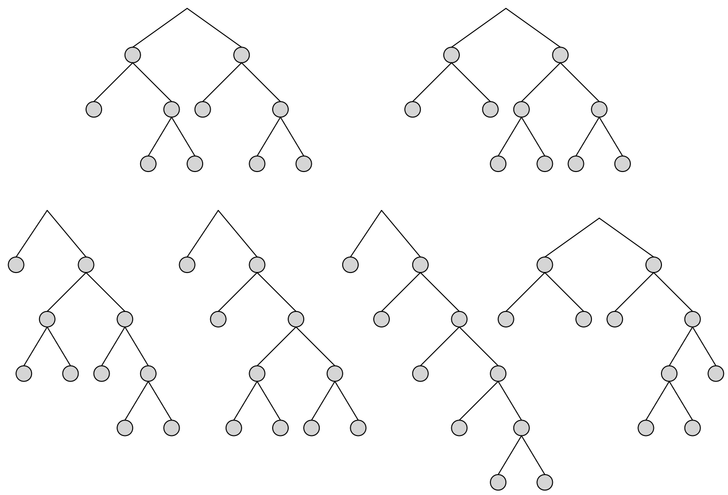

Fig. 6. Possible tournament trees for 6 players.

of players with a difference in their depths greater than 1. Thus if the seeding of the original tournament is envy-free, the reduced tournament tree T_M has to be one of the top two trees.

For each of these trees, notice that the two players who are placed at a smaller depth than the rest must have played against dummy players in the original tournament before they can advance to their current round. Notice that with the current winning probability matrix, player 7 and 8 play the same role as the dummy players. Thus if there existed an envy-free seeding for the original tournament, we would be able to find an envy-free seeding for the set of the top 8 players. However, we have shown in the first part that there does not exist such a seeding. This implies there is no envy-free seeding for the tournament of size $n > 8$ either. □

Note that the same result does not apply for unconstrained tournaments. In the previous proof, we make use of the fact that any player must play against some player (which can be a dummy player) to advance to the next round. In the case of unconstrained tournament, a player can advance directly to the final round without competing.

Since the monotonic model is a special case of the general model, we have the following corollary.

COROLLARY 4.5. *For any given $n = 2^r \geq 8$, with the general winning probability model, it is not always possible to find an envy-free seeding for a balanced knockout tournament of size n.*

Given this impossibility result for balanced tournaments, in the next section we discuss an alternative fairness criterion that seems intuitively weaker weaker condition, with the hope that it can be more easily satisfied.

5. SECOND FAIRNESS CRITERION: ORDER PRESERVATION

Instead of focusing on the seeding positions of the players, we place a requirement on their probabilities of winning the tournament instead: stronger players must have higher probabilities of winning the tournament than weaker players. This requirement seems to be less restrictive than the envy-freeness criterion, yet it still encourages the players to perform accordingly to their strengths.

Definition 5.1 (*Order-Preserving Seeding*). Given a set N of players, a winning probability matrix P, and a knockout tournament $KT_N = (T, S)$, the seeding S is order preserving if and only if $\forall (i, j) \in N$ such that i dominates j, player i does not have a lower probability of winning the tournament than player j, that is, $q(i, (T, S)) \geq q(j, (T, S))$.

The criterion is adapted and generalized from the notion of Monotonicity first introduced in Hwang [1982].

5.1. Order Preservation for Unconstrained Tournaments

When there is no constraint on the structure of the tournament, it is easy to see that the same tournament structure and seeding in Theorem 4.3 will achieve order preservation.

THEOREM 5.2. *Given a set N of players, a winning probability matrix P, order preservation can be achieved by the knockout tournament with the caterpillar structure and the seeding S such that for every pair of players (i, j), if i dominates j then j is placed below i on the tournament tree.*

PROOF. Recall from Theorem 4.3 that the seeding S created as above is envy-free. Using proof by contradiction, we assume S is not order preserving, and show that this implies it will not be envy-free either.

If S is not order preserving, there must exist two player i, and j such that i dominates j, and the winning probability of i is smaller than j, i.e., $q(i, S) < q(j, S)$. Since i dominates j, i must be placed above j on the tournament tree. Let A be the set of players placed above i on the tournament tree, B between i and j, and C below j. The winning probabilities of j given the seeding S can be calculated as the following:

$$q(j, S) = \prod_{k \in A} p_{jk} \times p_{ji} \times \prod_{k \in B} p_{jk} \times \prod_{k \in C} [q(k, C) \times p_{jk}].$$

Here we use $q(k, C)$ to denote the probability that player $k \in C$ will win the subtournament between the players in C and advance to the match against player j.

Let S' be the seeding created by swapping the position of i and j in S. Since S is envy-free, the winning probability of i with the seeding S must not be smaller than with S', i.e., $q(i, S) \geq q(i, S')$. The winning probability of i with S' can be calculated as the following:

$$q(i, S') = \prod_{k \in A} p_{ik} \times p_{ij} \times \prod_{k \in B} p_{ik} \times \prod_{k \in C} [q(k, C) \times p_{ik}].$$

However, since i dominates j, $\forall k \in N$, $p_{ik} \geq p_{jk}$ and $p_{ij} \geq p_{ji}$. Therefore we have:

$$q(i, S') \geq \prod_{k \in A} p_{jk} \times p_{ji} \times \prod_{k \in B} p_{jk} \times \prod_{k \in C} [q(k, C) \times p_{jk}] = q(j, S) > q(i, S).$$

This is a contradiction. Thus S must be also order-preserving. □

Notice that the method above does not take into consideration the actual values of the winning probabilities between players. The criterion can be achieved as long as the relative ordering of the players are known. It would be very useful if the same result applied for the case of balanced tournament. Unfortunately, as we show in the next section, no fixed seeding is always order preserving for balanced tournaments, even when the winning probabilities are monotonic.

5.2. Order Preservation for Balanced Tournaments

If the seeding S_n^* as in Figure 2 could be shown to always satisfy order preservation, this would justify its popularity in practice. However, as we show in the following theorem, there are cases in which this seeding violates the criterion.

THEOREM 5.3. *For any* fixed *seeding S of a balanced tournament of size* $n = 2^r \geq 8$, *there exists a monotonic winning probability matrix P such that S is not order preserving.*

PROOF. We first show by induction that if there exists a fixed seeding S that always satisfies order preservation, then S has to be S_n^* (as defined in Section 2).

Base case. When $n = 2$, $S_2^* = [1\ 2]$ is the only seeding.

Inductive case. Assume that the claim holds for $n = m$, we need to show that it also holds for $n = 2m$. We will present a sequence of winning probability matrices that all impose different constraints on an order preserving seeding. Since we are looking for a fixed seeding that is order preserving for *any* given winning probabilities, any seeding that does not satisfy all of these constraints can be discarded.

Let's first consider the following winning probabilities between the players:

— $\forall i, j : 1 \leq i, j < 2m,\ p_{ij} = 0.5$.
— $\forall i : 1 \leq i < 2m,\ p_{i(2m)} = 1$.

Note that due to monotonic tie-breaking rule, player 1 dominates all other players here. The only order preserving seedings in this case are the ones that pair up player 1 and $2m$ in the first round. Otherwise whoever gets to play against player $2m$ will have a winning probability of $\frac{2}{m}$, whereas player 1 only wins with probability $\frac{1}{m}$. Thus we can now only consider the seedings that match up player 1 and $2m$ in the first round.

Let us consider another set winning probabilities between the players:

— $\forall i, j : 1 \leq i, j < 2m - 1,\ p_{ij} = 0.5$.
— $\forall i : 1 \leq i < 2m - 1,\ p_{i(2m)} = p_{i(2m-1)} = 1$.
— $p_{(2m)(2m-1)} = 0.5$.

Since we already know that the remaining seedings under consideration must match up player 1 with player $2m$, player 2 must then be matched up with player $2m - 1$ to satisfy order preservation.

We can take this argument and generalize it to show for each value of $k = 3 \to m$, player k has to be matched up with player $(2m - k + 1)$ by using the following set of winning probabilities:

— $\forall i, j : 1 \leq i, j \leq 2m - k,\ p_{ij} = 0.5$.
— $\forall i, j : 1 \leq i \leq 2m - k,\ 2m - k + 1 \leq j \leq 2m,\ p_{ij} = 1$.
— $\forall i, j : 2m - k + 1 \leq i, j \leq 2m,\ p_{ij} = 0.5$.

In other words, the top $(2m - k)$ players tie with each other, and win with probability 1 against the bottom k players. The bottom k players also tie among themselves. In this case, since the top $k - 1$ players are already matched up with the bottom $k - 1$ players, player k has to be matched up with player $(2m - k + 1)$ in order to have a higher winning probability than the players from $(k + 1)$ to $(2m - k)$.

Now let us consider the case in which the top m players win against the bottom m players with probability 1. We have shown that player k, $\forall k : 1 \leq k \leq m$, has to be paired up with player $(2m - k + 1)$. Therefore, all the top m players will advance to the next round with probability 1. The current tournament essentially becomes a tournament between these m players. Based on the inductive hypothesis, they must be seeded by S^*_m. This proves that the seeding for $2m$ players must be S^*_{2m}.

In the preceding, we have used induction to show that the fixed seeding that satisfies order preservation for n players could only possibly be S^*_n. Now we just need to find a winning probability matrix such that S^*_n violates the criterion. Consider the following winning probabilities.

— $\forall i, j : 1 \leq i < 6 < j \leq n$, $p_{ij} = 1$.
— $\forall i : 9 \leq i \leq n$, $p_{7i} = p_{8i} = 1$.
— $\forall i, j : 1 \leq i, j \leq 5$, $p_{ij} = 0.5$.
— $p_{36} = p_{46} = p_{56} = 0.5$, and $p_{16} = p_{26} = 1$.
— All other winning probabilities are 0.5.

In other words, the top 8 players win against all other players with probability 1. The top 5 players tie with each other and win against player 7, and 8 with probability 1. Player 6 loses to 1 and 2 with probability 1 but ties with player 3, 4, and 5.

Let's consider the last 3 rounds of the tournament. Only the top 8 players make to these rounds and they are seeded by $S^*_8 = [1, 8, 4, 5, 2, 7, 3, 6]$. Intuitively, since player 3 has only 50% chance of winning against player 6, player 2 has more chance of getting into the final match than player 1. If we carry out the calculations, we have $q(1, S^*_n) = 0.25 < q(2, S^*_n) = 0.375$, which violates order preservation. Thus S^*_n is not always order preserving either. □

Since the monotonic model is a special case of the general model, we have the following corollary.

COROLLARY 5.4. *For any given $n = 2^r \geq 8$, there is no* fixed *seeding S that is always order preserving for any balanced tournaments of size n with the general winning probability model.*

The results above do not preclude the existence of a fair seeding for any *given* set of players and winning probabilities. While such existence remains an important open problem, we propose a heuristic algorithm to find an order preserving seeding and show through experiments that the algorithm is indeed efficient and effective. The pseudocode is provided in Algorithm 1.

The recursive algorithm attempts to find a fair seeding for each half of the initial seeding. It then checks to see if there are any pairs of players that violate the order preserving condition. If there are, it will pick the pair with the maximum difference in ranking (on the dominance graph) between the players and swap the seeding positions of those players. It will then repeat the whole process.

To test the algorithm, we generate 100k test cases of monotonic winning probability model for each of the values of n between 8 and 256. For each test case, we randomly generate the winning probability matrix for n players and then use the algorithm to find a seeding that satisfies order preservation. We use S^*_n (defined in Section 2) as the initial seeding input.

It is not trivial to generate the monotonic winning probability matrix since we need to make sure the matrix satisfies the monotonicity while ensuring certain randomness. The process is composed of the following steps.

(1) Generate the winning probabilities of player 1 by sampling $(n-1)$ values uniformly from [0.5,1] and sort them in ascending order.
(2) Assign $p_{ii} = 0.5$ $\forall i$.

Algorithm 1 Find-fair-seed (S: Initial seeding)

if S satisfies fairness **then**
  Return S;
end if
done $\leftarrow$ false;
while !*done* **do**
  done $\leftarrow$ true;
  Find-fair-seed (the first half of S);
  Find-fair-seed (the second half of S);
  $N \leftarrow$ the set of players in S;
  $wp \leftarrow$ winning probabilities of the players in N;
  $max\_dif \leftarrow 0$;
  for $i, j \in N : j - i > max\_dif$ **do**
    if $wp(i) < wp(j)$ **then**
      done $\leftarrow$ false;
      $max\_dif \leftarrow j - i$;
      $i^* \leftarrow i$;
      $j^* \leftarrow j$;
    end if
  end for
  if !*done* **then**
    Swap the seeding positions of i^*, j^* in S;
  end if
end while
Return S;

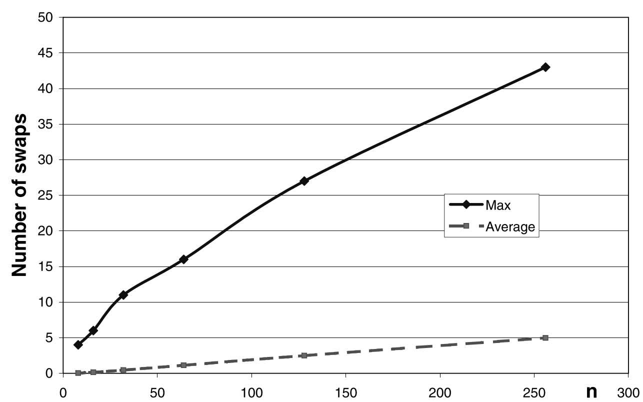

Fig. 7. The average and maximum number of swaps over 100k tournaments for each n.

(3) Generate p_{ij} ($\forall(i, j) : i < j$) by sampling uniformly between $p_{i(j-1)}$ and $p_{(i-1)j}$, and then assign $p_{ji} = 1 - p_{ij}$.

We show in Figure 7 a graph of the average and maximum number of swaps that are made before a fair seeding is found. The graph shows that both quantities grow linearly with n. And even in the case of $n = 256$ players, the maximum number of swaps needed is still very small.

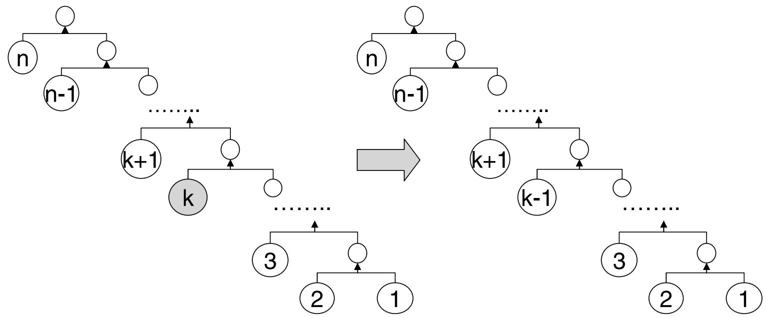

Fig. 8. A tournament robust to the dropout of any player k.

Based on the empirical success of the algorithm, we conjecture that order-preservation is always achievable. However, the criterion becomes provably impossible to achieve when a relatively weak condition is added. We discuss this additional condition and how it interacts with the order preservation criterion in the next section.

5.3. Robustness against Dropout

The experimental results of the heuristic algorithm in the previous section suggest that perhaps it is easy and always possible to find an order preserving seeding given the winning probability matrix. However, this optimism is quite fragile. For example, in this section we show that when we add a reasonable requirement of robustness against dropout, the criterion becomes provably impossible to achieve in certain cases.

A dropout occurs when a player forfeits his participation in a tournament due to exogenous factors such as injuries or illness, or in order to manipulate the outcome of the tournament. This phenomenon can significantly influence the winning probabilities of the remaining players. We define robustness against dropout as the following: there does not exist a pair of players (i, j) such that i dominates j, and j gains while i loses in winning probability after some player drops out. We allow the re-seeding of the remaining players after a dropout and if the tournament has to be balanced, a dummy player will be added.

Definition 5.5 (*Robustness against Dropout*). Given a set N of players, a winning probability matrix P, and a knockout tournament $KT = (T, S)$. The seeding S is robust against dropout if for any dropout of player $k \in N$, there exists a seeding S' for the remaining $|N| - 1$ players such that $\forall (i, j) \in N \setminus \{k\}$, if i dominates j and $q(i, (T, S')) - q(i, (T, S)) \leq 0$ then $q(j, (T, S')) - q(j, (T, S)) \leq 0$.

It is easy to show that given a winning probability matrix, the robustness requirement is achievable for both balanced and unconstrained tournaments (e.g., the tournament shown in Figure 8). Indeed the condition is quite weak when used alone. There are other stronger conditions that can be used. For example, no weak player gains in winning probability more than the players stronger than him (this in fact implies the requirement we proposed). However, when coupled with the order preservation requirement, this weak requirement is already enough to make it no longer always possible to find a seeding that can satisfy simultaneously both (even when the winning probabilities are monotonic).

THEOREM 5.6. *For any $n = 2^r \geq 8$, there exists a set of n players with a monotonic winning probability matrix P such that it is not possible to find a seeding S such* *that S satisfies simultaneously the order preservation requirement and the condition for robustness against dropout.*

PROOF. For a given n, we show how to construct the winning probability matrix P. There will be 3 top players and $n-3$ dummy players. The winning probabilities between the top 3 players are: $p_{1,2} = 0.5$, $p_{1,3} = 0.6$, and $p_{2,3} = 0.5$. The dummy players will lose against the top 3 players with probability 1. Thus the tournament is reduced to a tournament between these 3 players. There are only three possible seedings: [1 *vs.* 2] *vs.* 3, [1 *vs.* 3] *vs.* 2, or [2 *vs.* 3] *vs.* 1. The first two seedings are not order preserving (e.g., for the first seeding S_1, $q(1, S_1) = 0.3 < 0.45 = q(3, S_1)$). The last seeding does not satisfy the condition of robustness: when player 3 drops out, the winning probability of player 2 increases from 0.25 to 0.5, whereas the winning probability of player 1 decreases from 0.55 to 0.5. Note that this proof works for both balanced and unbalanced tournament. □

6. CONCLUSION

Fairness in tournament designs is an important consideration and to date the literature on tournament designs has not adequately addressed it. We introduced two alternative fairness criteria: envy-freeness and order preservation. We showed that when there is no constraint on the structure of the tournament, fairness can be achieved easily for both criteria. For balanced tournaments, we provided two impossibility results: one for any seeding under the first criterion, and the other for any fixed seeding under the second criterion. When the seeding can vary depending on the actual values of the winning probabilities, we proposed a heuristic algorithm to find a seeding that satisfies order preservation. We showed that the algorithm is efficient and effective through experiments.

However, our hope of being able to find a fair seeding in all cases was dimmed by the fact that when we included a requirement of robustness, the criterion became provably impossible to satisfy for all cases. This suggests one should try to find the conditions of the winning probabilities between players that will guarantee the existence of a fair seeding. This remains an interesting open problem.

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