Moment Generating Functions

- Moments
- Single function to rule them all
- Moment generating function (MGF)
- Basic examples
- Properties
- Raison d'être

EX_MACHINA


Massage Envy®

The Mathematician’s Revenge

Engineers have machines they can sell

Computer Scientists have software

Food makers have production lines (mcdonalds)

Even service providers franchise (massage Envy)

Mathematicians need to think about every problem

Math envy

Moments

Expectations of powers of X are called Moments

$E(X)$ $E(X^2)$ $E(X^3)$

Sometimes called raw moments to distinguish from central moments

$E(X-\mu)^n$

Determine | Mean | Variance | … | All together → distribution itself

General method to find all moments

Moment Generating Function (MGF)

Maps a random variable X to a real function M

M: $\mathbb{R} \rightarrow \mathbb{R}$

$$M(t) \stackrel{\text{def}}{=} M_X(t) \stackrel{\text{def}}{=} E[e^{tX}] = \begin{cases} \sum p(x)\, e^{tx} & \text{discrete} \\ \int f(x)\, e^{tx}\, dx & \text{continuous} \end{cases}$$

Determined by distribution p or f

X more convenient

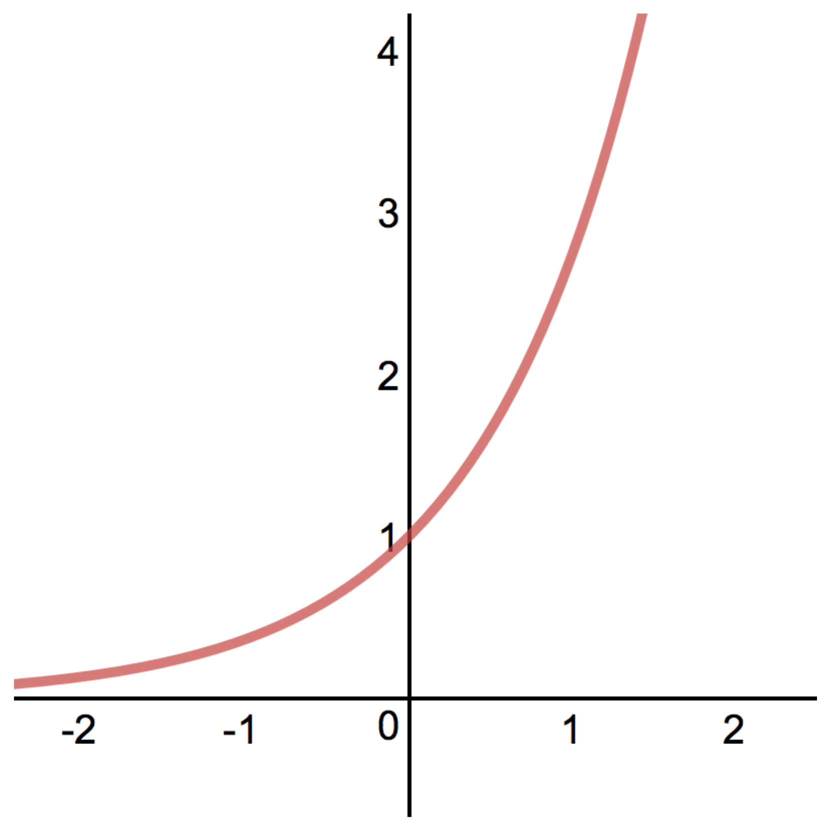
Discrete

Continuous

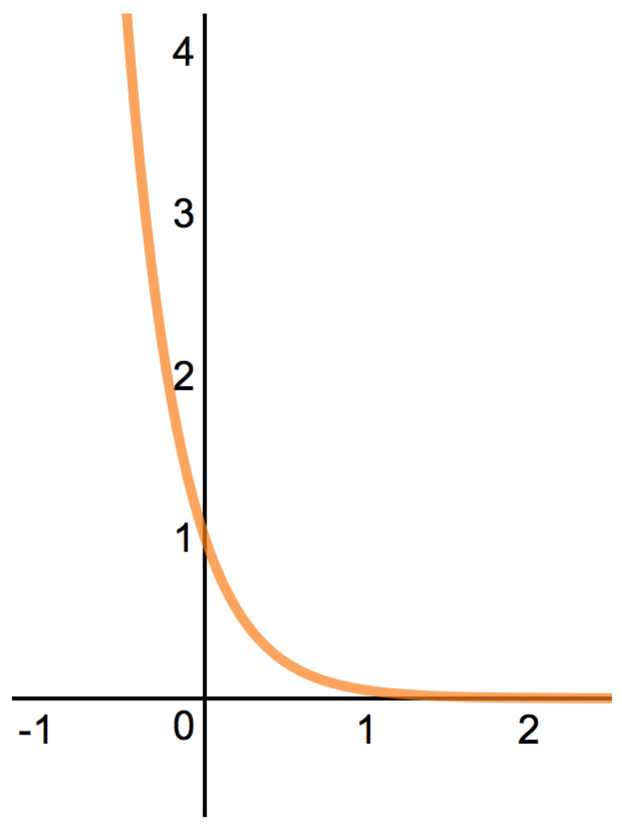
One Value

Constant random variable $X = c$

$$M_c(t) = E(e^{tX}) = \sum_x p(x) \cdot e^{tx} = p(c) \cdot e^{ct} = e^{ct}$$



$M_1(t) = e^t$



$M_{-2}(t) = e^{-2t}$

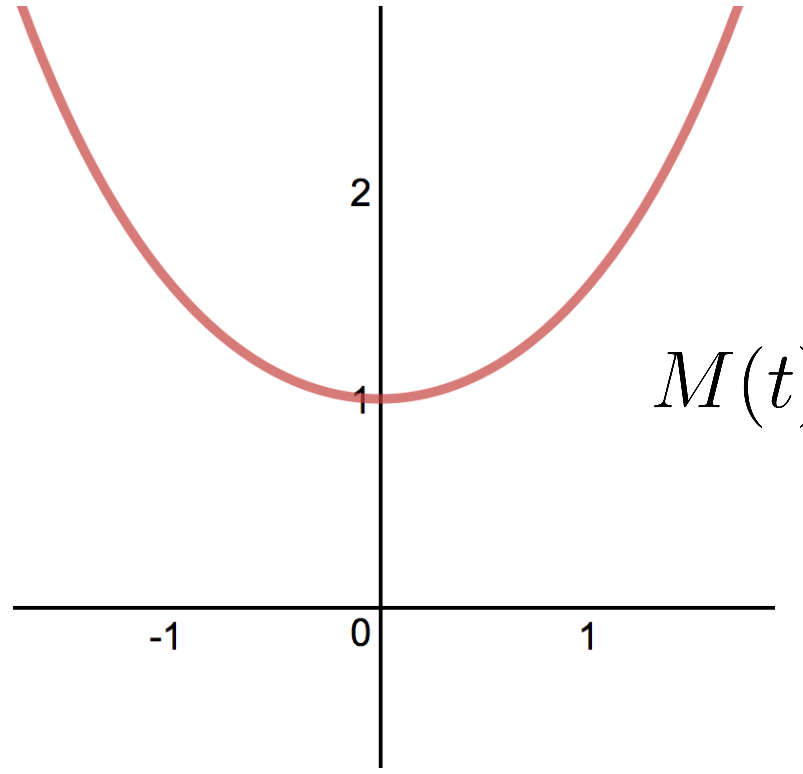
Two Values

Arbitrary values

$$p_{-1} = p_1 = \frac{1}{2}$$

$$p(c_1) = p_1 \quad p(c_2) = p_2$$

$$M(t) \;=\; E(e^{tX}) \;=\; p_1\, e^{c_1 t} + p_2\, e^{c_2 t}$$



$M(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$

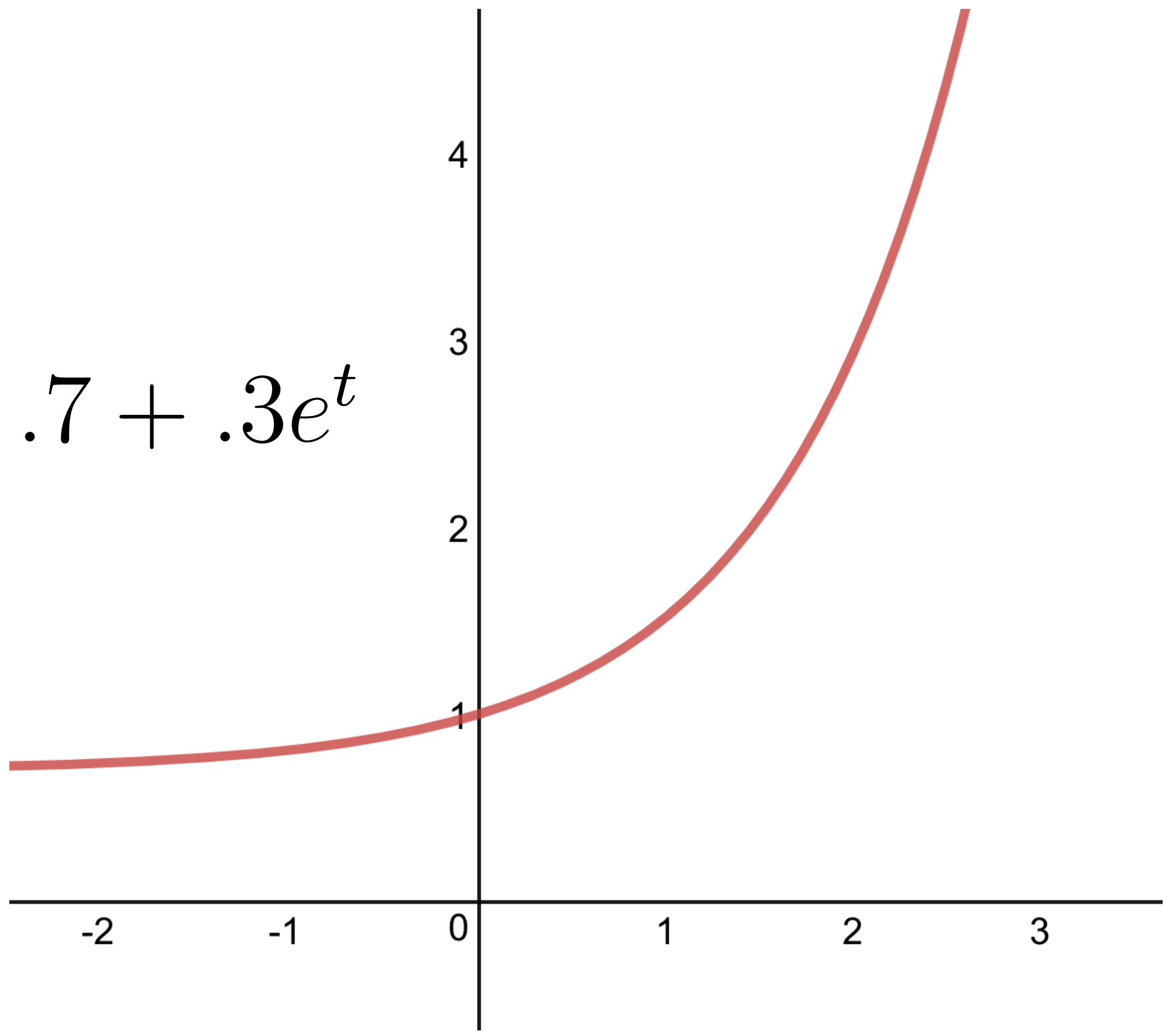
0, 1 values

Bernoulli

$X \sim B_p$ $\quad p_0 = 1 - p \quad p_1 = p$

$B_{.3}$

$M(t) = .7 + .3e^t$



$$M(t) = \;(1-p)e^{t\cdot 0} + pe^{t\cdot 1} = \;(1-p) + pe^t$$

Basic Properties

say what if xmax >0, =0, <0
similar for min

Positive $$M(t) = E(e^{tX}) > 0$$

0 → 1 $$M(0) = E(e^{0X}) = E(e^{0}) = 1$$

Finite support X

As t → ∞ $$M(t) \sim p(x_{\max}) \cdot e^{t \cdot x_{\max}}$$

Translation and Scaling

Translation X → X+b

$$M_{X+b}(t) \;=\; E(e^{t(X+b)}) \;=\; E(e^{tX}\cdot e^{tb}) \;=\; e^{tb}\cdot E(e^{tX}) \;=\; e^{tb}M_X(t)$$

Scaling X → aX

$$M_{aX}(t) \;=\; E(e^{t(aX)}) \;=\; E(e^{atX}) = M_X(at)$$

Translation and Scaling X → aX+b

$$M_{aX+b}(t) \;=\; e^{bt}\cdot M_{aX}(t) \;=\; e^{bt}\cdot M_X(at)$$

$M_c(t) = e^{ct} \longrightarrow$ $M_{a\cdot c+b} = e^{bt}\cdot M_c(at) = e^{bt}\cdot e^{cat} = e^{(ac+b)\cdot t}$ ✅

Independent Addition

Independent variables

MGF of sum is product of MGF's

Two variables

X ⫫ Y

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX} \cdot e^{tY}] = E[e^{tX}] \cdot E[e^{tY}] = M_X(t) \cdot M_Y(t)$$

n variables

$X_1, X_2, \ldots, X_n$ ⫫

$$X \stackrel{\text{def}}{=} X_1 + X_2 + \ldots + X_n$$

$$M_X(t) = \prod_{i=1}^n M_{X_i}(t)$$

Average

$X_1, X_2, \ldots, X_n$ ⫫

$$\overline{X} \stackrel{\text{def}}{=} \frac{X_1+X_2+\ldots+X_n}{n}$$

$$M_{\overline{X}}(t) = \prod_{i=1}^n M_{X_i}\left(\frac{t}{n}\right)$$

What's in the Name?

$M_X(t)$ Moment Generating Function of X Why?

Determines ("generates") all moments

$E(X^n)$ n^{th} "raw" moment of X

Moment Generation

Taylor series $e^y = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$

$$M_X(t) = E(e^{tX})$$

$$= E(1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \ldots)$$

$$= 1 + \tfrac{t}{1!}E(X) + \tfrac{t^2}{2!}E(X^2) + \tfrac{t^3}{3!}E(X^3) + \ldots$$

$$M_X(0) = 1 = E(X^0)$$

$$M'_X(t) = E(X) + \tfrac{t}{1!}E(X^2) + \tfrac{t^2}{2!}E(X^3) + \ldots$$

$$M'_X(0) = E(X)$$

$$M''_X(t) = E(X^2) + \tfrac{t}{1!}E(X^3) + \ldots$$

$$M''_X(0) = E(X^2)$$

$$M_X^{(n)}(0) = E(X^n)$$

Moment Generating Functions

Moments

Single function that determines all

Moment generating function (MGF)

Definition

Examples

Properties