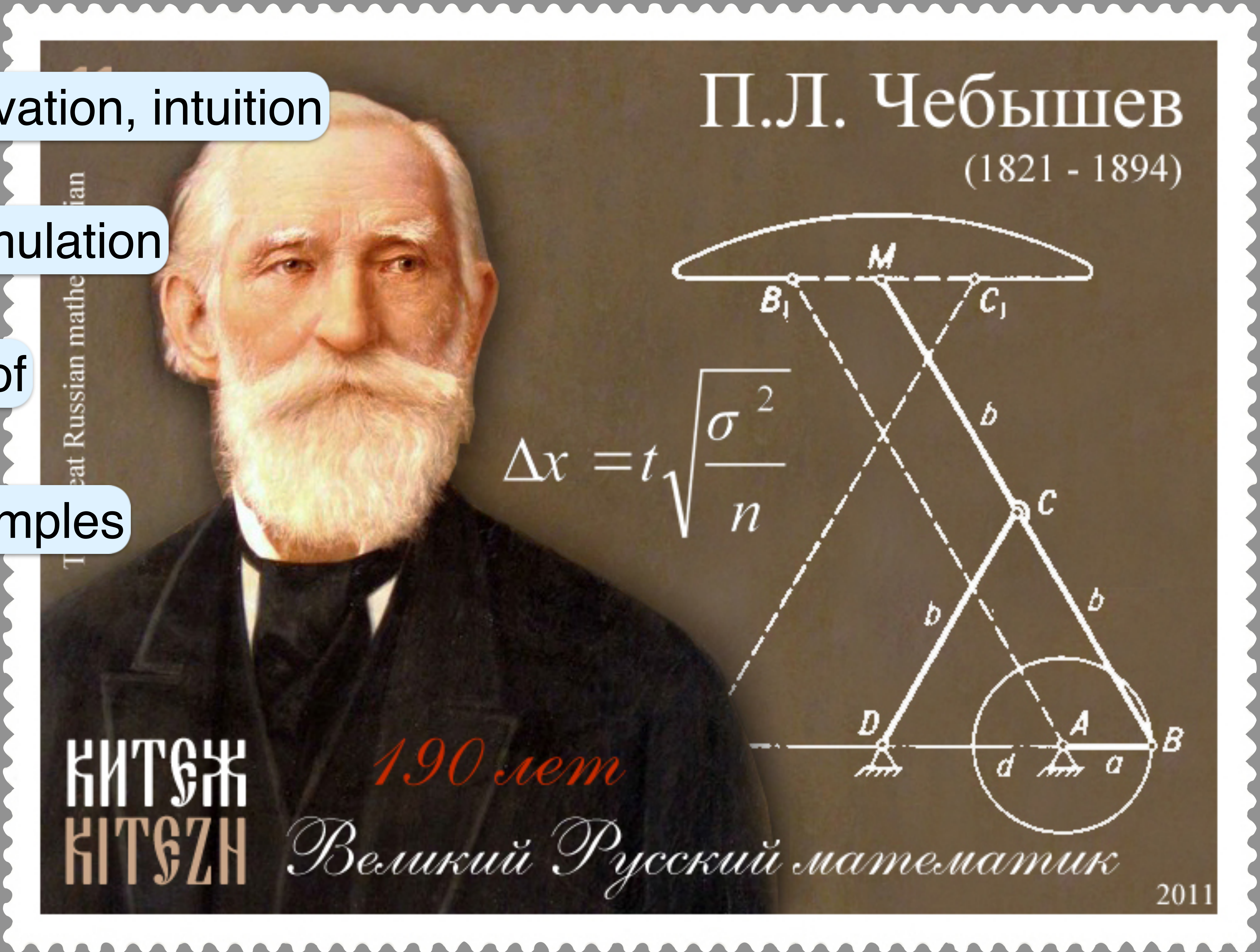
Chebyshev's VI

Motivation, intuition

Formulation

Proof

Examples



Family

Destined → 

Walking difficulty  → 

Many results

$\forall$ n < $\exists$ prime < 2n

Probability theory

Legendary teacher

Punctual  

Advocated applied math

Flowery language

Pafnuty Chebyshev

1821-1894

To isolate mathematics from practical sciences is to shut the cow away from the bulls

Famous students

Markov

Lyapunov

> 12K “descendents”

“Father” of modern Russian Mathematics

Most famous for spellings

Chebychev, *Chebysheff*, Chebychov, *Chebyshov*, *Tchebychev*, *Tchebycheff*, *Tschebyschev*, *Tschebyschef*, *Tschebyscheff*, …

Markov → Chebyshev

Markov

Probability (non-neg X is ≥ $\alpha\ \mu$) ≤ $1/\alpha$

Chebyshev

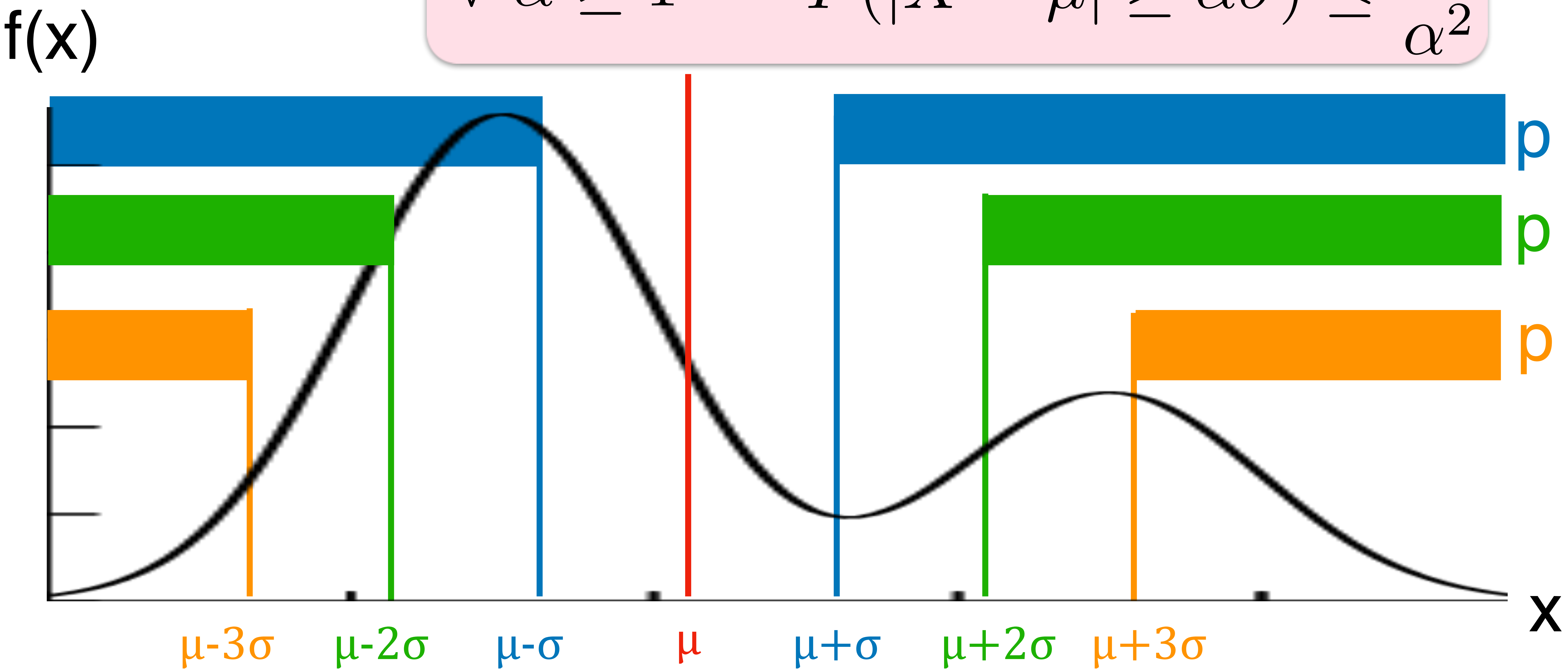
Probability (any X is ≥ $\alpha\ \sigma$ away from μ) ≤ $1/\alpha^2$

Chebyshev's Inequality

Same two versions **1** Easier to illustrate, understand, remember

X any discrete or continuous r.v. with finite mean μ and std σ

$$\forall\ \alpha \geq 1 \qquad P(|X-\mu| \geq \alpha\sigma) \leq \frac{1}{\alpha^2}$$



Second Formulation

X any discrete or continuous r.v. with finite mean μ and std σ

1 Easier to visualize, understand, remember

$$\forall\ \alpha \geq 1 \qquad P(|X-\mu| \geq \alpha\sigma) \leq \frac{1}{\alpha^2}$$

2 Easier to prove, use

$$a = \alpha\sigma$$

$$\forall\ a \geq \sigma \qquad P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

Just use value of interest

Reduce to Markov

Markov $\forall\ a \geq \mu \qquad P(X \geq a) \leq \frac{\mu}{a}$

same

Chebyshev $\forall\ a \geq \sigma \quad P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$

different

$$|X - \mu| \geq a$$

$$\Updownarrow$$

$$(X - \mu)^2 \geq a^2$$

$$P((X - \mu)^2 \geq a^2) \leq \frac{\sigma^2}{a^2}$$

same

Apply Markov to $(X - \mu)^2$

Proof

$$Z \geq 0$$

Markov's ≤

Proof

$$\boxed{P(|X - \mu_X| \geq a) \leq \frac{\sigma_X^2}{a^2}}$$

Y
Soon

X - any random variable $\quad \mu_X = EX \quad \sigma_X^2 = V(X) = E(X - \mu_X)^2$

$$Y = (X - \mu_X)^2 \qquad Y \geq 0 \qquad \mu_Y = E(X - \mu_X)^2 = \sigma_X^2$$

$$P(|X - \mu_X| \geq a) = P((X - \mu_X)^2 \geq a^2)$$

$$= P(Y \geq a^2)$$

Markov

$$\leq \frac{\mu_Y}{a^2} = \frac{\sigma_X^2}{a^2}$$

Citations

X - # paper citations

$\mu = 8$

Suppose $\sigma = 5$

$P(X \geq 28)$?

Markov

$P(X \geq 28) \leq 8/28 \approx 29\%$

$$P(X \geq a) \leq \frac{\mu}{a}$$

Chebyshev

$P(X \geq 28) = P(X - \mu \geq 20)$

$\leq P(|X - \mu| \geq 20)$

$\leq \left(\frac{\sigma}{20}\right)^2 = \left(\frac{5}{20}\right)^2 = \frac{1}{16} \approx 6.3\%$

$$P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

Markov: $\leq 0.02\%$

$P(X \geq 40{,}000) = P(X - \mu \geq 39{,}992)$

$\leq P(|X - \mu| \geq 39{,}992)$

$\leq \left(\frac{\sigma}{39{,}992}\right)^2 = \left(\frac{5}{39{,}992}\right)^2 = 1.6 \times 10^{-6}\,\%$

Inside View

Survey Responses

Survey expected to result in μ = 1M responses with σ = 50K

Bound P(0.8M < # responses < 1.2M)

$0.8M = \mu - 4\sigma$ $1.2M = \mu + 4\sigma$

$P(\mu - 4\sigma < X < \mu + 4\sigma) = P(|X - \mu| < 4\sigma)$

$= 1 - P(|X - \mu| \geq 4\sigma)$

$\geq 1 - 1/16$

$= 15/16$

Mark x Che

	Formula	Applies	Input	Range	Decreases
Markov	$P(X \geq a) \leq \frac{\mu}{a}$	X≥0	μ	a ≥ μ	Linearly
Chebyshev	$P(\|X - \mu\| \geq a) \leq \frac{\sigma^2}{a^2}$	Any X	μ and σ	a ≥ σ	Quadratically

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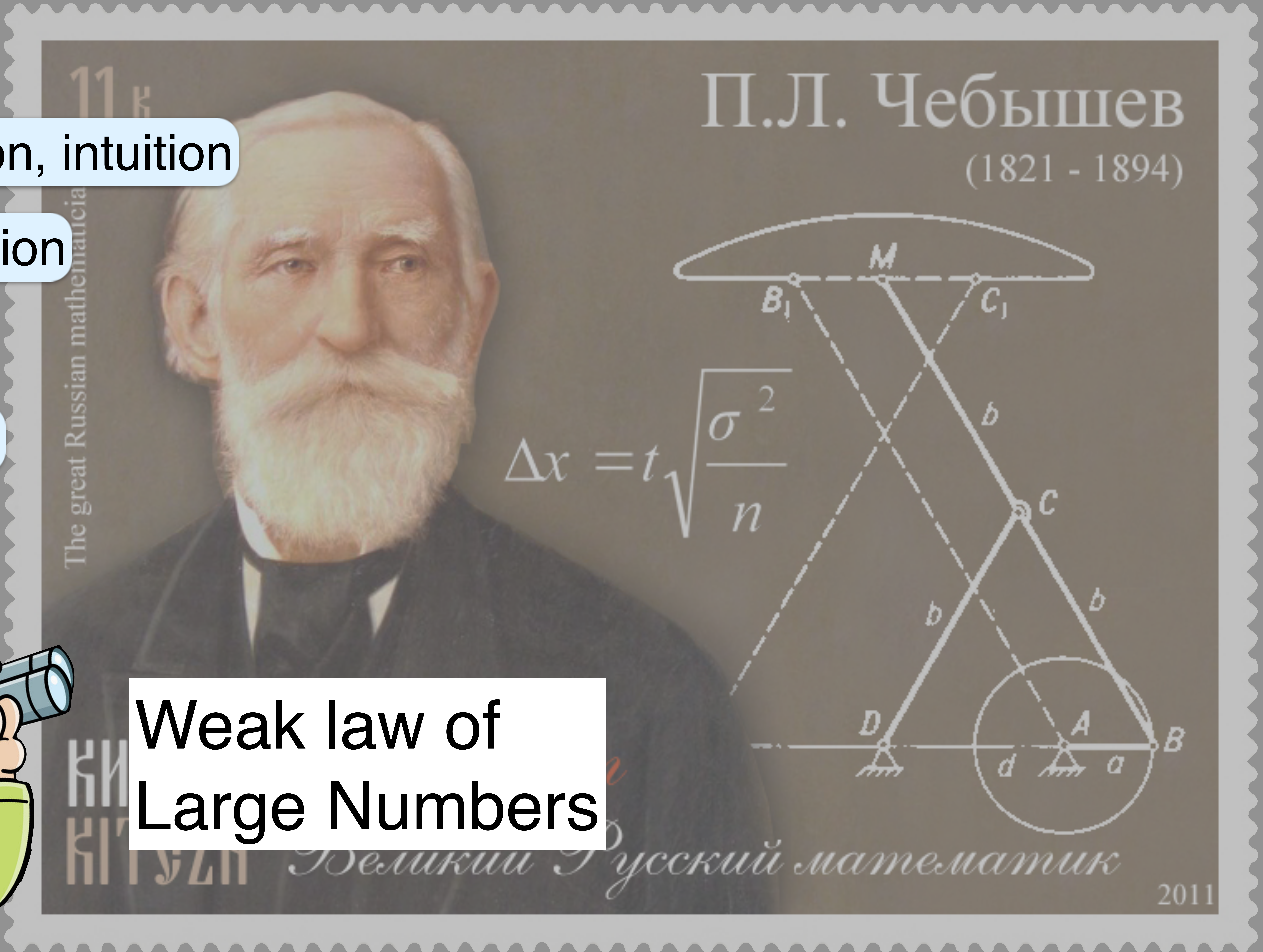
Example



Weak law of Large Numbers

One-Sided Chebyshev

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I have a couple of proofs at http://www.se16.info/hgb/cheb.htm#OTProof and http://www.se16.info/hgb/cheb2.htm

One of these, loosely based on *Probability and Random Processes* by Grimmett and Stirzaker, would give a proof like this:

With $a > 0$, for any $b \geq 0$

$$P(X \geq a) = P(X + b \geq a + b) \leq E\left[\frac{(X+b)^2}{(a+b)^2}\right] = \frac{\sigma^2 + b^2}{(a+b)^2}$$

But treating $\dfrac{\sigma^2 + b^2}{(a+b)^2}$ as a function of b, the minimum occurs at $b = \sigma^2/a$, so

$$P(X \geq a) \leq \frac{\sigma^2 + (\sigma^2/a)^2}{(a + \sigma^2/a)^2} = \frac{\sigma^2(a^2 + \sigma^2)}{(a^2 + \sigma^2)^2} = \frac{\sigma^2}{\sigma^2 + a^2}.$$

	chev	chov	sheff	shov	schef	scheff	schev
Che							
Tche							

Alternative spellings: *Chebychev*, *Chebysheff*, Chebychov, *Chebyshov, Tchebychev*, *Tchebycheff, Tschebyschev*, *Tschebyschef*, *Tschebyscheff*