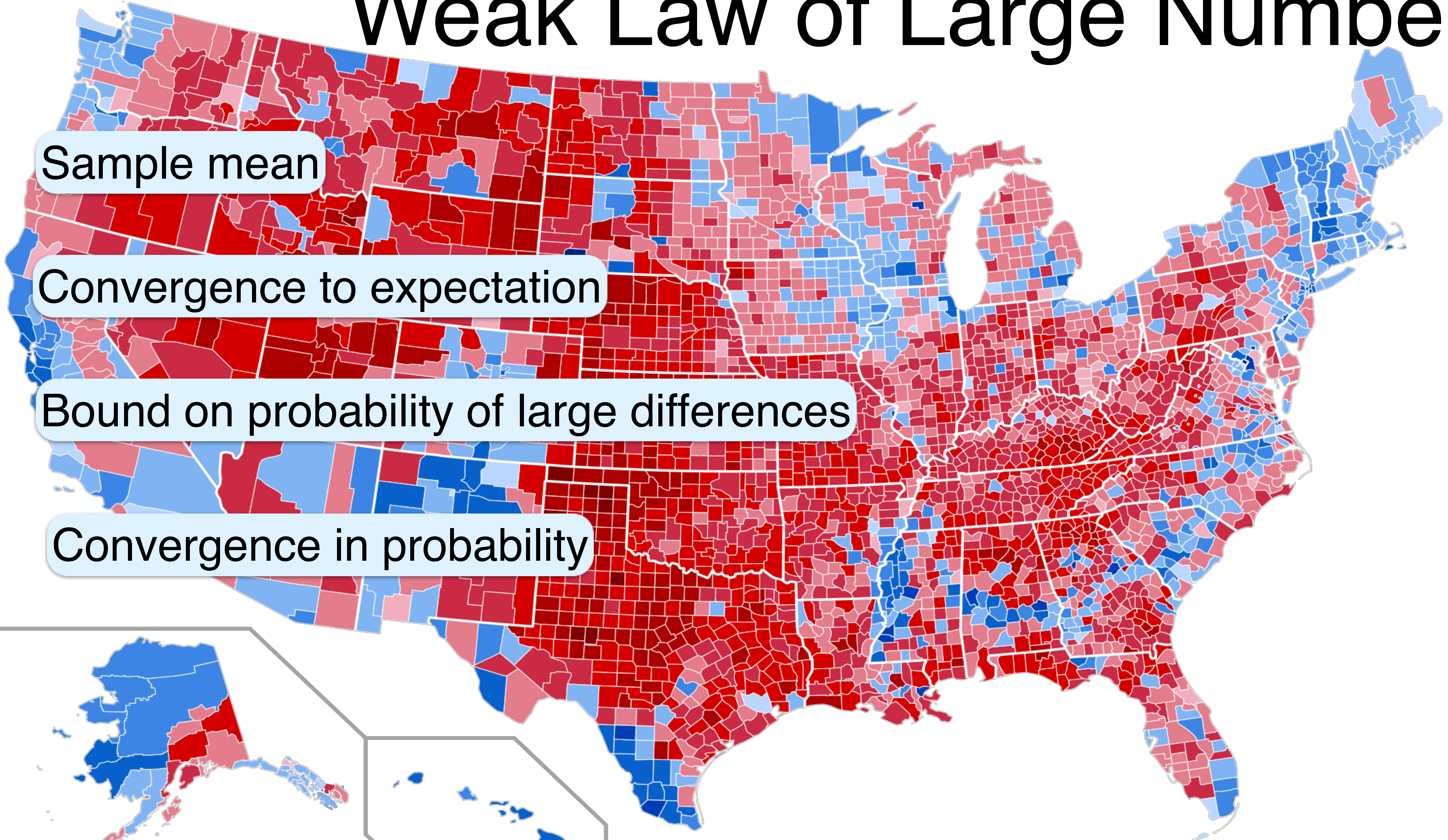
Weak Law of Large Numbers

Sample mean

Convergence to expectation

Bound on probability of large differences

Convergence in probability



Motivation

Probability theory based on sample averages converging to expectation

Flip many fair coins, fraction of heads converges to 1/2

Roll many fair dice, average value converges to 3.5

← Intuition

→ Rigorous

Sample Mean

Sequence abbreviation $x^n \stackrel{\text{def}}{=} x_1, x_2, \ldots, x_n$

Mean $\overline{x^n} \stackrel{\text{def}}{=} \frac{x_1+x_2+\ldots+x_n}{n}$

$n = 4$ $x^4 = 3, 1, 4, 2$ $\overline{x^4} = \frac{3+1+4+2}{4} = 2.5$

n samples from a distribution $X^n = X_1, X_2, \ldots, X_n$

Sample mean $\overline{X^n} \stackrel{\text{def}}{=} \frac{X_1 + \ldots + X_n}{n}$

$\overline{X^n}$ is a random variable

Independent Samples

Independent random variables with the *same* distribution are
Independent identically distributed (iid)

Independent $B_{0.3}$ r.v.'s are iid $B_{0.3}$, or iid

X_1, X_2, X_3 are iid $B_{0.3}$

Each X_i is $B_{0.3}$ selected $\perp\!\!\!\perp$ of all others

$$P\,(X_1 = 1, X_2 = 0, X_3 = 1) = 0.3 \cdot 0.7 \cdot 0.3 = 0.063$$

Weak Law of Large Numbers

As # samples increases, the sample mean → distribution mean

$X^n = X_1,\ldots,X_n$ iid samples from distribution with finite mean μ and finite std σ

As $n \to \infty$

$\overline{X^n}$ approaches μ

P(sample mean differs from μ by any given amount) ↘ 0 with n

$$P\left(|\overline{X^n} - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2} \cdot \frac{1}{n}$$

$\overline{X^n}$ "converges in probability" to μ

Polling Error

2016 Presidential elections

Poll 100,000 people

Assuming every person voted for Trump independently w. probability p

Bound the probability that off by more than 1%

WLLN

$$P\left(|\overline{X^n} - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2} \cdot \frac{1}{n}$$

$$\sigma^2 = p(1-p) \leq \tfrac{1}{4}$$

$$P(|\overline{X^{100{,}000}} - p| \geq 0.01) \leq \frac{1/4}{0.01^2 \cdot 100{,}000} = 2.5\%$$

Proof of WLLN $P\left(|\overline{X^n} - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{n \cdot \epsilon^2}$

$X_1, X_2, \ldots,$ iid with finite μ and σ, sample mean $\overline{X^n} \stackrel{\text{def}}{=} \frac{1}{n} \sum X_i \qquad \sum = \sum_{i=1}^{n}$

Expectation

$$E\left(\overline{X^n}\right) = E\left(\tfrac{1}{n} \sum X_i\right) = \tfrac{1}{n} \sum E(X_i) = \tfrac{1}{n} \sum \mu = \mu$$

Variance

$$V\left(\overline{X^n}\right) = V\left(\tfrac{1}{n} \sum X_i\right) = \tfrac{1}{n^2} V\left(\sum X_i\right) = \tfrac{1}{n^2} \sum V(X_i) = \tfrac{1}{n^2} \sum \sigma^2 = \frac{\sigma^2}{n}$$

Chebyshev

$$\forall \epsilon > 0 \qquad P\left(|\overline{X^n} - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{n \cdot \epsilon^2} \searrow 0$$

$$n \to \infty$$

Sensors

n sensors measure temperature t

Each reads $T_i = t + Z_i$ $\quad\quad$ Z_i - noise with zero mean and variance ≤ 2

How many sensors needed to estimate t to ± ½ with probability ≥ 95%

$$P\left(|\overline{X^n} - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2} \cdot \frac{1}{n}$$

$$P(|\overline{T}^n - t| \geq 0.5) \leq \frac{2}{\frac{1}{4}n} \quad \leq 0.05$$

$$n \geq \frac{2}{\frac{1}{4} \cdot 0.05} \quad = 2 \cdot 4 \cdot 20 = 160$$

Generalization

Same proof works when means μ_i and σ_i differ.

Just let $\mu \stackrel{\text{def}}{=} \frac{1}{n} \sum \mu_i$ and $\sigma^2 \stackrel{\text{def}}{=} \frac{1}{n} \sum \sigma_i^2$

$$P\left(|\overline{X^n} - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2} \cdot \frac{1}{n}$$

Convergence in Probability

$X_1, X_2, \ldots$ infinite sequence of random variables

X_n converges in probability to a random variable Y $\quad X_n \xrightarrow{p} Y$

$P(X_n$ differs from Y by any given fixed amount$) \searrow 0$ with n

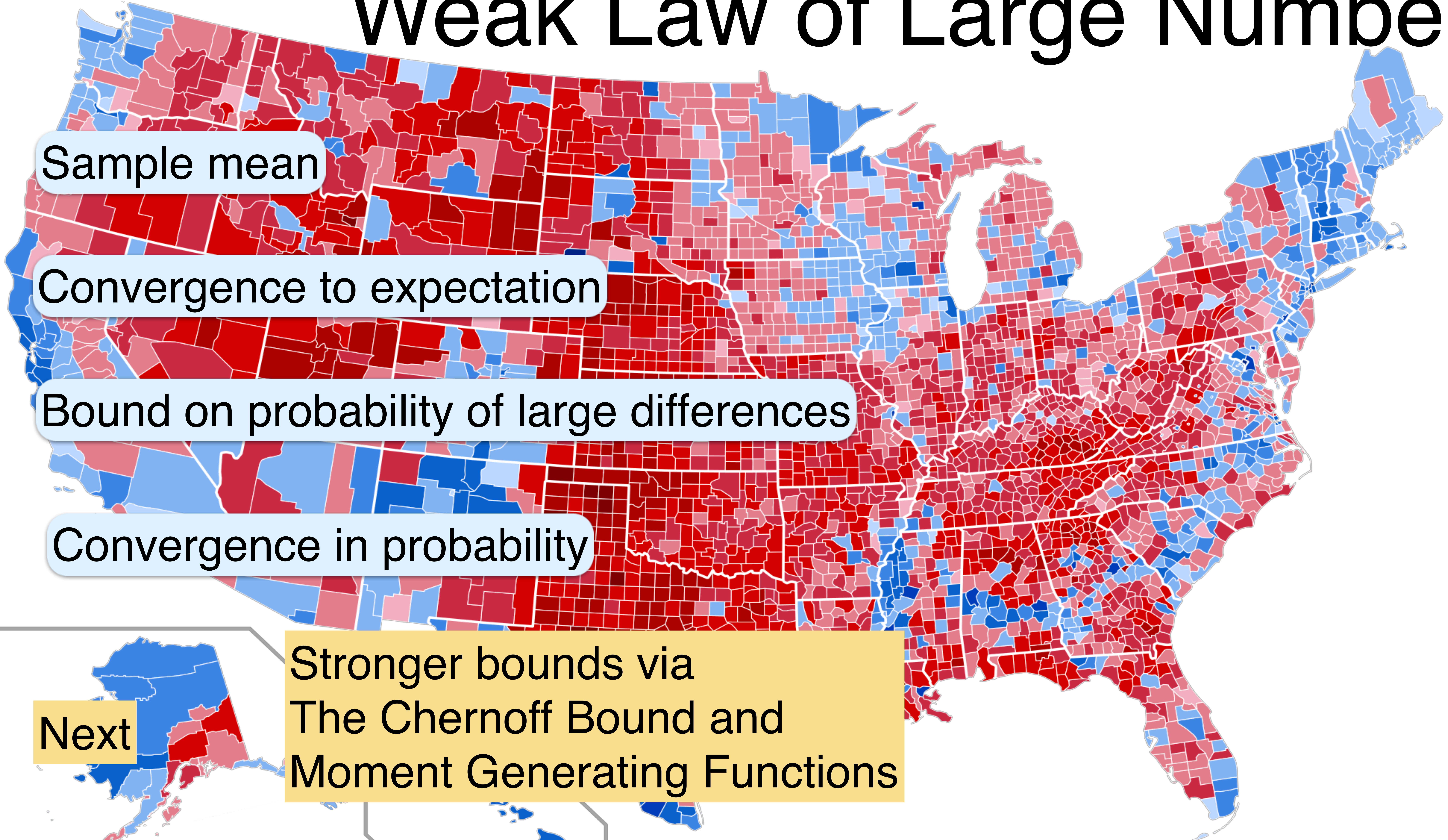
For every $\delta > 0$ $\quad P(|X_n - Y| \geq \delta) \searrow 0$ with n

For every $\delta>0$ and $\varepsilon>0$ there is an N s.t for all $n \geq N$

$P(|X_n - Y| \geq \delta) < \varepsilon$

WLLN: $\overline{X^n}$ converges in probability to μ $\quad \overline{X^n} \xrightarrow{p} \mu$

Weak Law of Large Numbers



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Next

Stronger bounds via
The Chernoff Bound and
Moment Generating Functions

Coin Flips

Most basic convergence to average is B(p)

Flip n B(p) coins, average # 1’s will approach np

Probability of a sequence with k 1’s and n-k 0’s is $p^k q^{n-k}$

Wolog assume p>0.5, then most likely is 1^n

Yet by WLLN with probability →1 we see roughly pn 1’s and qn 0’s

Why do we observe these sequences and not the most likely ones?

Strength in #s. # sequences of a given composition increases near 1/2

pn balances # x probability.