Markov's ≤

Motivation

Intuition

Formulation

Proof

Example

Extensions

Probability Bounds

Goal Bound probability of events (often bad)

Excessive rain | Heavy traffic | Large loss | Disease outbreak

Now Markov's Inequality

Later Stronger bounds

Start Intuitive definition

Markov's Meerkats

Average meerkat height is 10”

Can half the meerkats be ≥ 40” tall? ✗

No If half of meerkats were ≥ 40” tall — average would be ≥ ½ x 40” = 20”

F_{40} fraction of meerkats that are ≥ 40” tall

If $F_{40} \cdot 40 > 10$, average would be > 10

$F_{40} \cdot 40 \leq 10$ $F_{40} \leq 10/40 = 1/4$

General μ $F_{4\cdot\mu} \cdot (4\cdot\mu) \leq \mu$ $F_{4\cdot\mu} \leq 1/4$

Markov's ≤

X - nonnegative discrete or continuous r.v. with finite mean μ

Two forms

Intuitive, memorable

$$\forall\, \alpha \geq 1 \qquad P(X \geq \alpha\mu) \leq \frac{1}{\alpha}$$

A nonnegative r.v. is at least α times ≥ its mean with probability ≤ 1/ α

Direct proof, easier to apply, more common

Just use value of interest

$$a = \alpha\mu \qquad \forall\, a \geq \mu \qquad P(X \geq a) \leq \frac{\mu}{a}$$

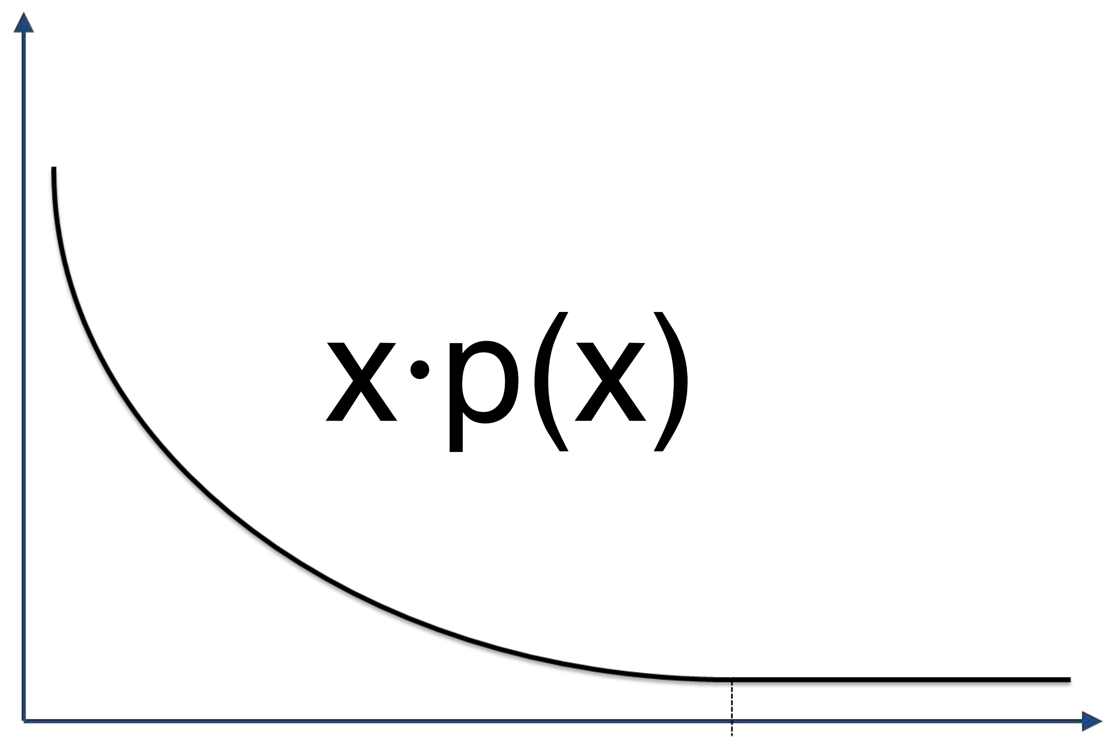
`&\ge \sum_{x\ge a} a\cdot p(x)\\`

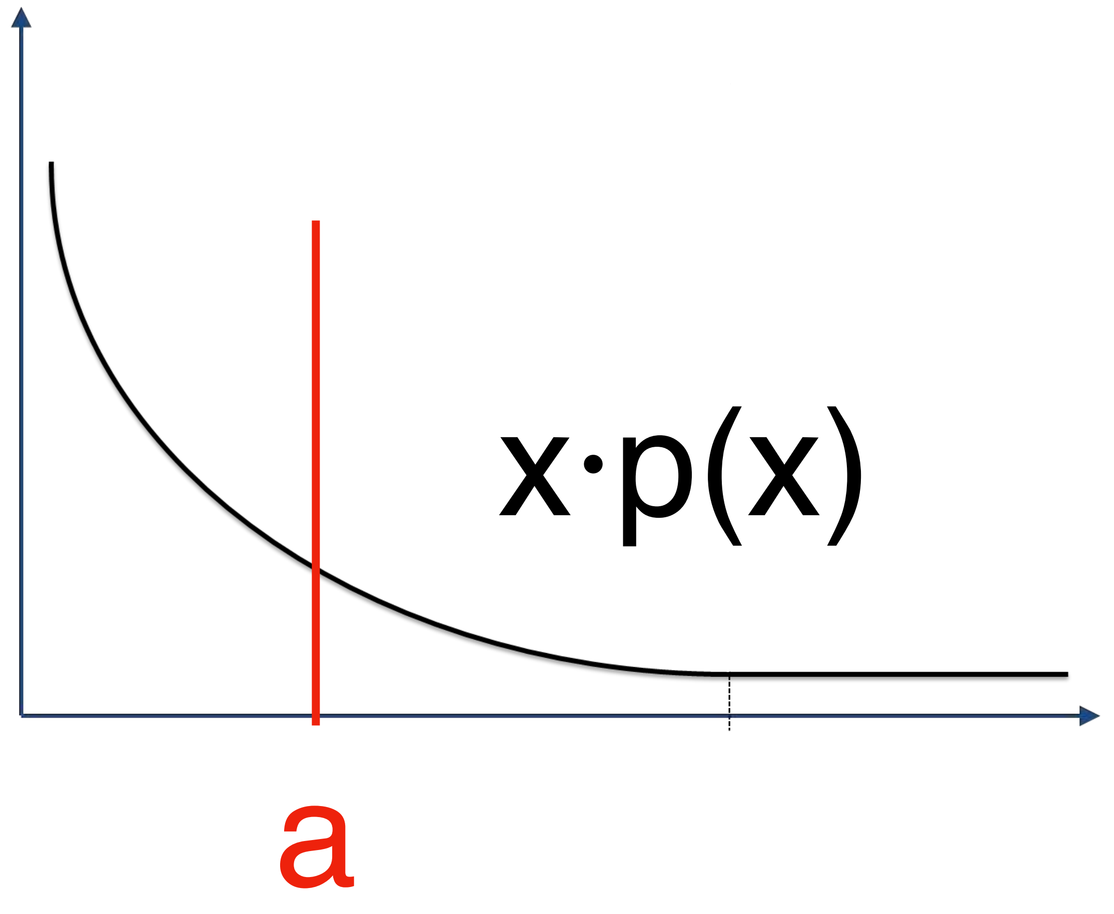
Proof

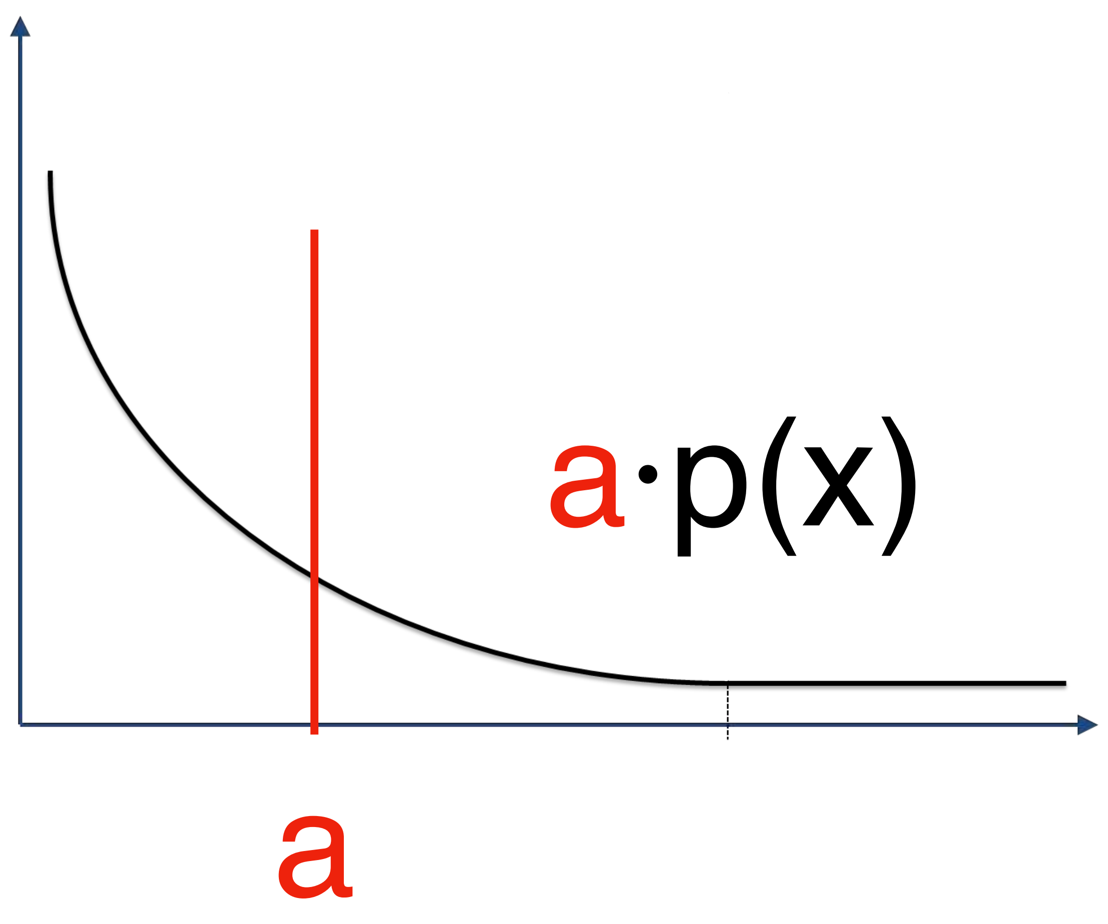
$$P(X \geq a) \leq \frac{\mu}{a}$$

Prove for discrete r.v.'s, same proof works for continuous, just $\sum \rightarrow \int$

$$\mu = \sum_{x} x \cdot p(x) \quad \geq \sum_{x \geq a} x \cdot p(x) \quad \geq \sum_{x \geq a} a \cdot p(x) \quad = a \cdot P(X \geq a)$$







&\ge \sum_{x\ge a} a\cdot p(x)\\

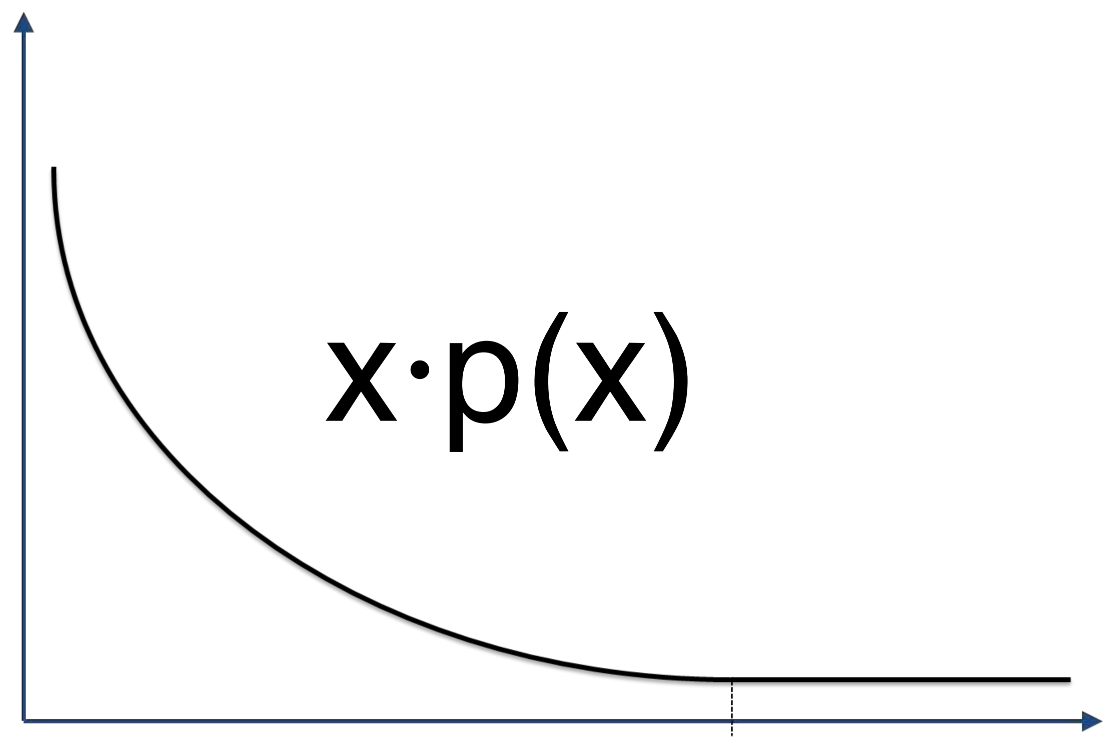
Proof

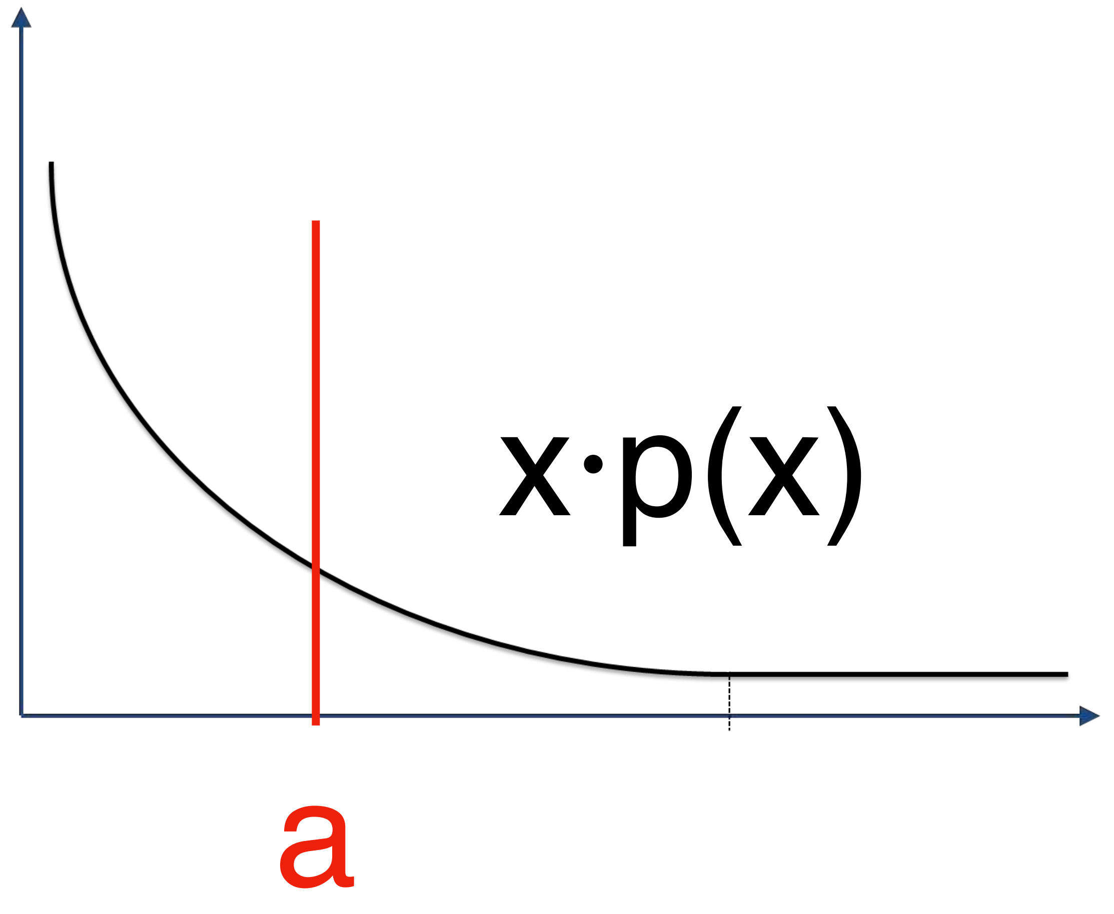
$$\mu = \int_x x \cdot p(x) \ge \int_{x \ge a} x \cdot p(x)$$

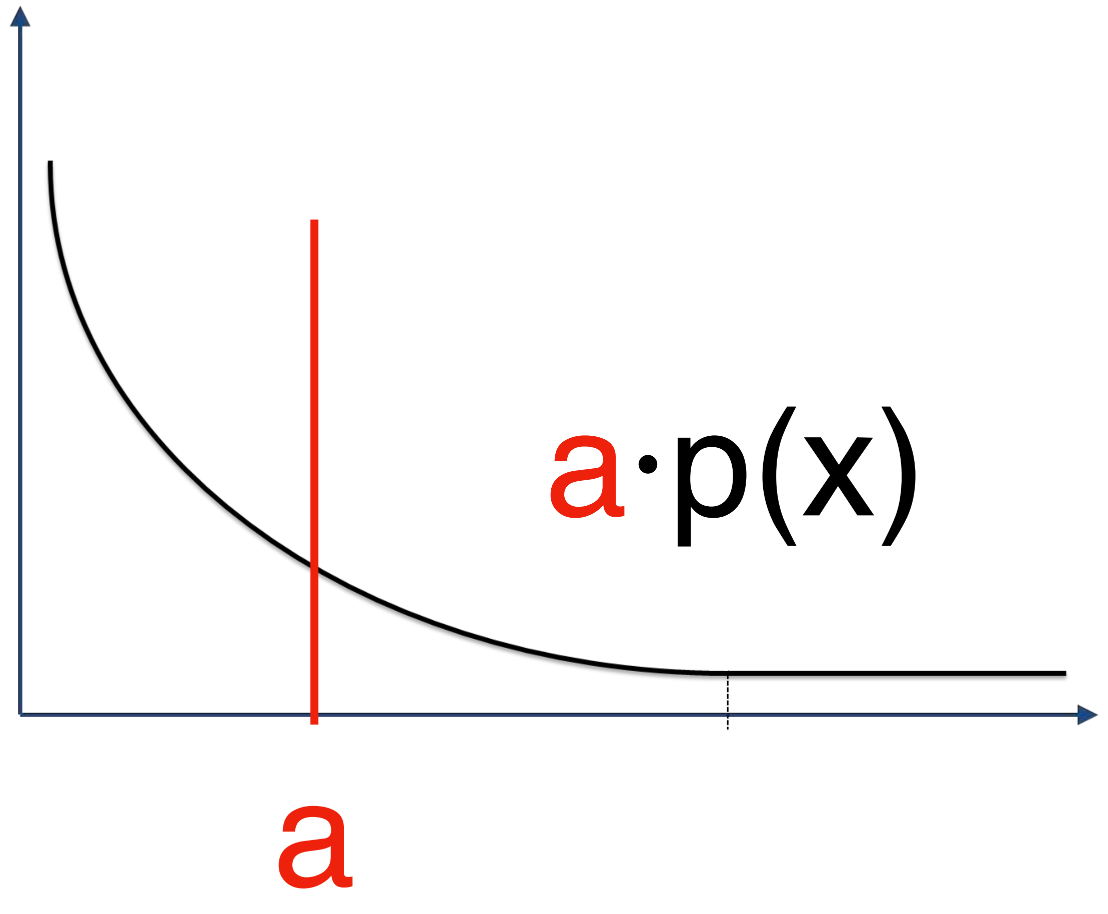
$$P(X \ge a) \le \frac{\mu}{a}$$

Prove for discrete r.v.'s, same proof works for continuous, just $\sum \rightarrow \int$

$$\mu = \sum_x x \cdot p(x) \ge \sum_{x \ge a} x \cdot p(x) \ge \sum_{x \ge a} a \cdot p(x) = a \cdot P(X \ge a)$$







Citation Counts

A journal paper is cited 8 times on average

Roughly right

Popular (multiple) hypothesis-testing paper

Cited ≥ 40,000 times

J. R. Statist. Soc. B (1995)
57, No. 1, pp. 289-300

Controlling the False Discovery Rate: a Practical and Powerful Approach to Multiple Testing

By YOAV BENJAMINI† and YOSEF HOCHBERG

Bound probability that a paper gets cited ≥ 40,000 times

X: # paper citations

$X \geq 0$

$\mu = 8$

Markov

$$P(X \geq a) \leq \frac{\mu}{a}$$

$P(X \geq 40{,}000) \leq \mu / 40K = 8 / 40K = 0.02\%$

So Far

$P(X \geq a)$

X | Outside | [0,a)

Upper bound | $P \leq \mu / a$

View from outside [0,a)

The View from Within

$P(X \leq a)$

X Inside $[0,a)$

Lower bound $P > 1 - \mu / a$

Inside Outside Almost same Opposite Depends on point of view

Add Example

$P\ (X<a) \geq$

Strengths

Weaknesses

Applies to *all* nonnegative random variables

Can always be used

Used to derive other inequalities

Chebyshev

Chernoff

Applies to *all* nonnegative random variables

Wait! How did a strength turn into a weakness?

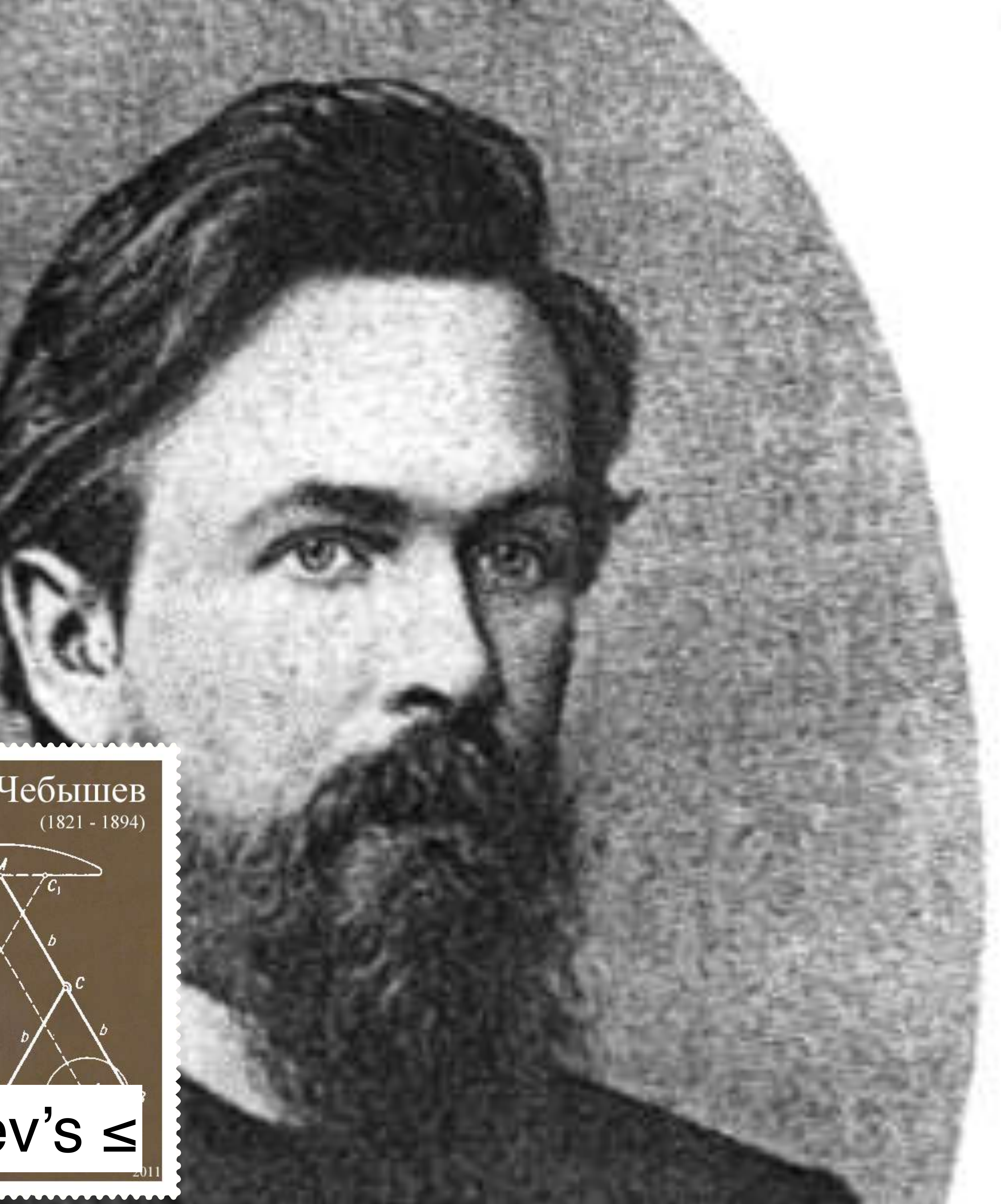
Limited to inequalities that hold for *all* distributions

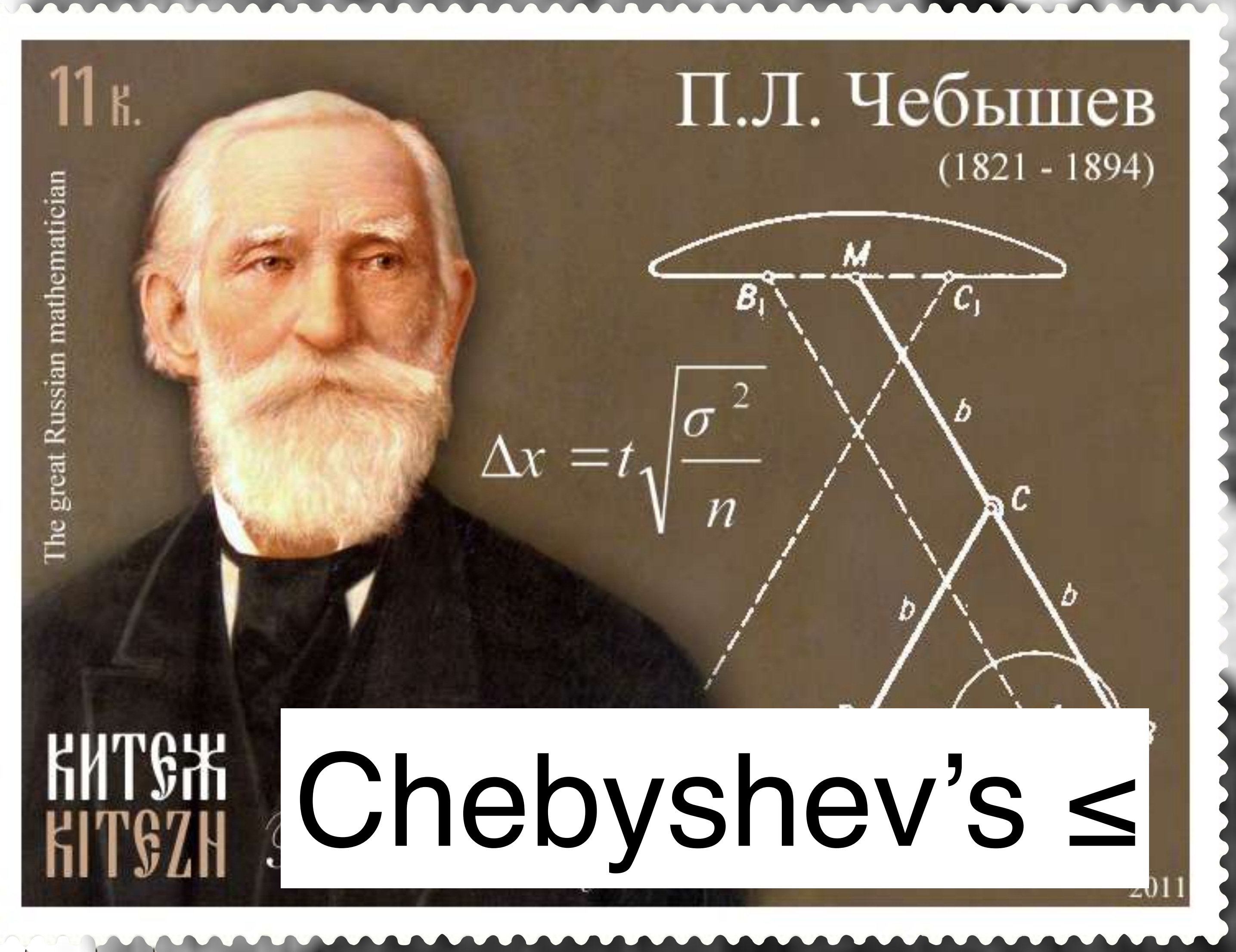
Strengthen

Supplement assumptions

Markov's ≤

- Motivation, intuition
- Formulation
- Proof
- Example
- Extensions





Chebyshev's ≤

Generalize?

Can the Markov ≤ be: Generalized (conditions relaxed)?

Strengthened?

Generalization attempt: Can we remove non-negativity?

If X can be negative, then P(X ≥ a) can be close to 1 for any a

- large 0 a

$$p(x) = \begin{cases} 1-\epsilon & x = a \\ \epsilon & x = \frac{\mu-(1-\epsilon)a}{\epsilon} \end{cases}$$

$$EX = \mu \qquad p(X \geq a) = p(a) \approx 1$$ ✗

Strengthen?

Can we strengthen $P(X \geq a) \leq \dfrac{\mu}{a}$?

Can the ≤ hold with equality?

$$\mu = \sum_x x \cdot p(x) \quad \geq \sum_{x \geq a} x \cdot p(x) \quad \geq \sum_{x \geq a} a \cdot p(x) \quad = a \cdot P(X \geq a)$$

$$\boxed{\forall x \in (0, a), \quad p(x) = 0} \qquad \boxed{\forall x > a, \quad p(x) = 0}$$

$$X \in \{0, a\}$$

No sweeping improvements… continue looking…