MGF Examples & Applications

Bernoulli

$X \sim B_p$ $\quad p(0) = 1 - p$ $\quad p(1) = p$ $\quad M(t) = 1 + p(e^t - 1)$

$$M(t) = E(e^{tX}) = p(0) \cdot e^{t \cdot 0} + p(1) \cdot e^{t \cdot 1} = (1-p) + pe^t$$

$$E(X^n) = M^{(n)}(0)$$

$M'(t) = pe^t$ $\qquad E(X) = M'(0) = p$

$M^{(n)}(t) = pe^t$ $\qquad E(X^n) = M^{(n)}(0) = p$ $\qquad X^n = X$

Binomial

$B_{p,n} \quad 0 \leq p \leq 1, \;\; n \geq 0$

$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$M(t) = \;\; [1 + p(e^t - 1)]^n$

$$M(t) \;\; = \;\; E(e^{tX}) \quad = \sum_{k=0}^{n} p(k) \cdot e^{tk}$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{tk}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$$

$$= \;\; (pe^t + 1 - p)^n$$

Use….. Binomial Thm

$$= \;\; [1 + p(e^t - 1)]^n$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \; a^k \; b^{n-k}$$

Moments

$$M(t) = \ [1 + p(e^t - 1)]^n$$

$E(X)$

$$M'(t) = n(pe^t + 1 - p)^{n-1} \cdot pe^t$$

$$M'(0) = n(pe^0 + 1 - p)^{n-1} \cdot pe^0 = np$$

Poisson

$P_\lambda,\ \lambda > 0$

$p(k) = e^{-\lambda}\,\frac{\lambda^k}{k!}$

$M(t) = e^{\lambda(e^t-1)}$

$$M(t) = E(e^{tX}) = \sum_{k=0}^{\infty} p(k)\cdot e^{tk}$$

$$= \sum_{k=0}^{\infty} e^{-\lambda}\,\frac{\lambda^k}{k!}\cdot e^{tk}$$

$$= \sum_{k=0}^{\infty} e^{-\lambda}\,\frac{(\lambda e^t)^k}{k!}$$

$$= e^{-\lambda}\cdot\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!} = e^{-\lambda}\cdot e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

Standard Normal

$\mathcal{N}(0,1)$ $\quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\quad M(t) = e^{\frac{t^2}{2}}$

$$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$\int_{-\infty}^{\infty} \to \int$

$$= \int e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int \exp[-\frac{x^2-2tx}{2}]\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int \exp[-\frac{(x-t)^2}{2} + \frac{t^2}{2}]\,dx$$

$$= e^{\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int \exp[-\frac{(x-t)^2}{2}]\,dx$$

$$= e^{\frac{t^2}{2}}$$

"Abnormal" Normal

Standard normal $Z \sim \mathcal{N}(0,1)$ $M_Z(t) = e^{\frac{t^2}{2}}$

General normal $X \sim \mathcal{N}(\mu, \sigma^2)$ $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$X = \sigma Z + \mu$

$$M_X(t) = M_{\sigma Z + \mu}(t) = e^{\mu t} M_Z(\sigma t)$$

$$= e^{\mu t} \cdot e^{\frac{\sigma^2 t^2}{2}}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Properties

Informally

If $M_X(t) = M_Y(t)$ then X and Y have the same distribution

Can invert $M_X(t)$ to obtain X

If $M_{X_n}(t) \rightarrow M_X(t)$ then the $F_{X_n} \rightarrow F_X$

Poisson and Binomial

Binomial

$$B_{p,n}(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$M_{p,n}(t) = [1 + p(e^t - 1)]^n$$

Poisson

$$P_\lambda(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

$$M_\lambda(t) = e^{\lambda(e^t - 1)}$$

Convergence

$$B_{\frac{\lambda}{n},n} \to P_\lambda$$

$$M_{\frac{\lambda}{n},n}(t) \to M_\lambda(t) \ ?$$

$$(1 + \tfrac{1}{n})^n \to e$$

$$(1 + \tfrac{a}{n})^n \quad = \quad [(1 + \tfrac{a}{n})^{\frac{n}{a}}]^a \quad \to \quad e^a$$

$$M_{\frac{\lambda}{n},n}(t) \ = \ [1 + \tfrac{\lambda}{n}(e^t - 1)]^n \ \to \ e^{\lambda(e^t-1)} \ = \ M_\lambda(t)$$

Sum of ⫫ Gaussians

Alternative proof

Sum of ⫫ normals is normal

$$X_1 \sim N(\mu_1, \sigma_1^2) \qquad M_{X_1}(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}}$$

$$X_2 \sim N(\mu_2, \sigma_2^2) \qquad M_{X_2}(t) = e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}}$$

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \cdot e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1+\mu_2)t + \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}$$

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

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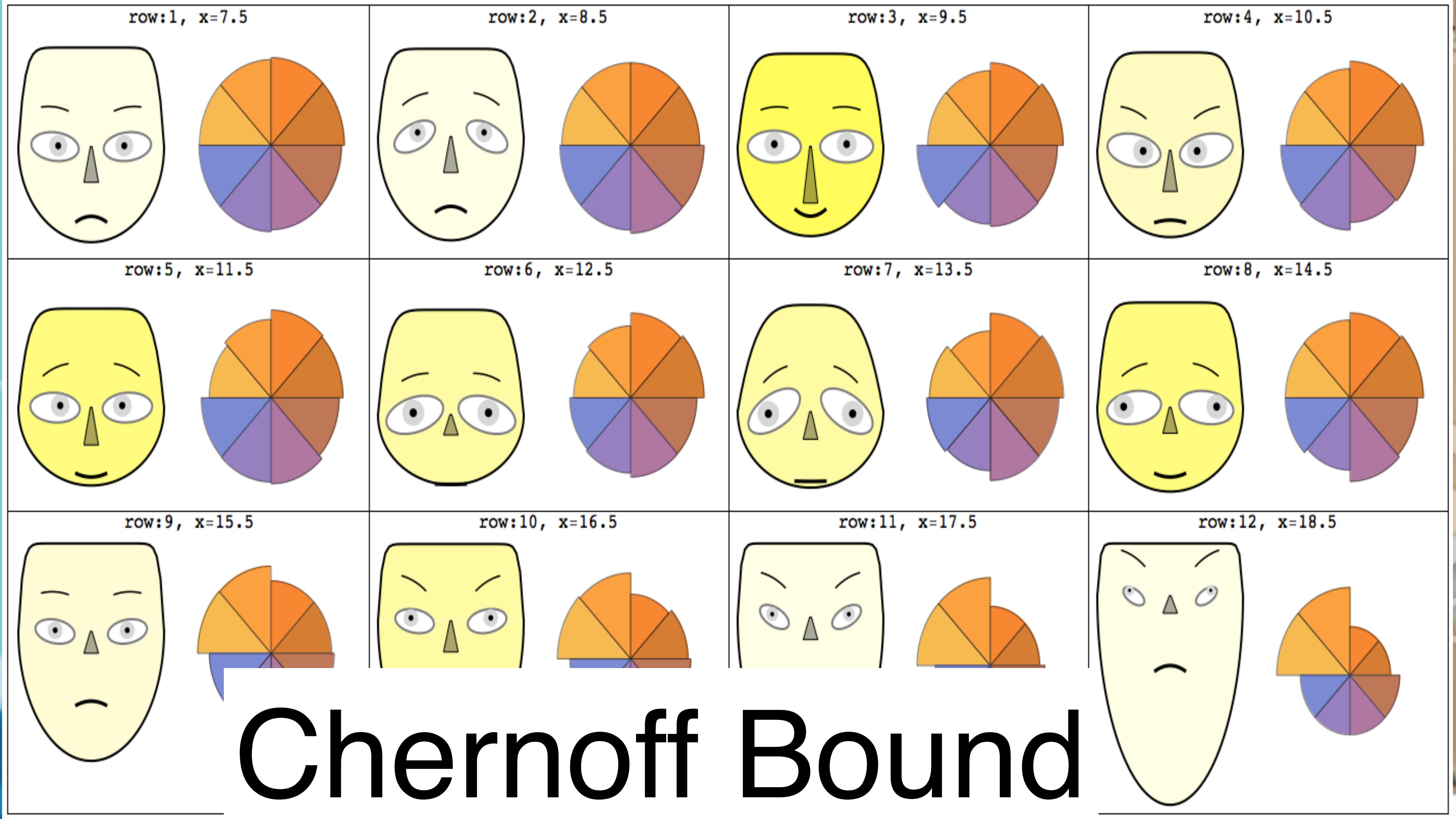

Chernoff Bound