02/10/2025: Sets

CSCI 246: Discrete Structures

Textbook reference: Sec. 10, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Today's Agenda

- Reading and problems quiz (12 mins)
- Mini-lecture ($\approx$ 15 mins)
- Group exercises ($\approx$ 20 mins)

Reading Quiz (Sets I)

Answer **both** of the following questions

1. How many subsets does $A = \{1, 2, 3\}$ have?
2. Let A be any finite set. How many subsets does A have?

Extra Credit for Reading Quiz (Sets I)

Choose **one** of the following and answer it.

1. Prove that the following two sets are equal:

$$E = \{x \in \mathbb{Z} : x \text{ is even}\}, \text{and}$$
$$F = \{x \in \mathbb{Z} : x = a + b, \text{where } a \text{ and } b \text{ are both odd}\}$$

   Note: If we have done an exercise in the past which provides part of the solution, you may write "as we showed in a previous exercise (...)"

2. Let P be the set of Pythagorean triples; that is,

$$P = \{(a, b, c) : a, b, c \in \mathbb{Z} \text{ and } a^2 + b^2 = c^2\}$$

   and let T be the set

$$T = \{(p, q, r) : p = x^2 - y^2, q = 2xy, r = x^2 + y^2 \text{ where } x, y \in \mathbb{Z}\}$$

   Prove T $\subseteq$ P.

Results on Friday's Quizzes

Problems Quiz Scores: Induction and Boolean Algebra

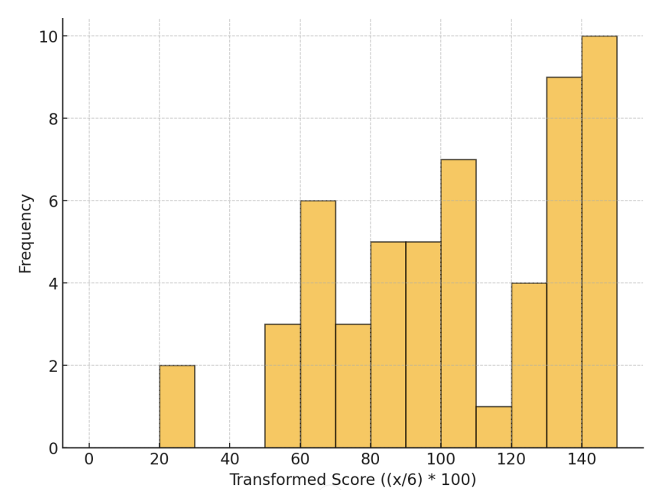

Figure 1: Median Score = 100%, Mean Score=103.7%

Reading Quiz Scores: Induction

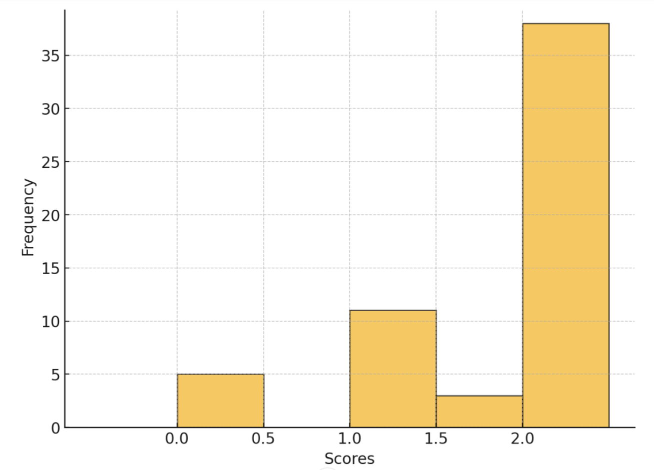

Figure 2: Median Score = 2/2 (100%), Mean Score=1.6/2 (80%)

Another quick look at the List Group Exercises

Some notes on Sets I (Introduction, Subsets)

Three special sets of numbers

Definition

The **integers** are the positive whole numbers, the negative whole numbers, and zero. That is, the set of integers, denoted by $\mathbb{Z}$ is

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3 \dots\}.$$

Definition

The **natural numbers** (denoted $\mathbb{N}$) are the non-negative integers; that is

$$\mathbb{N} = \{0, 1, 2, 3 \dots\}.$$

Definition

The **rational numbers** (denoted $\mathbb{Q}$) are the numbers formed by dividing two integers a/b, where $b \neq 0$. That is,

$$\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\}.$$

Counting Subsets (Decision Tree Perspective)

Theorem 1.3.1 *A set with n elements has 2^n subsets.*

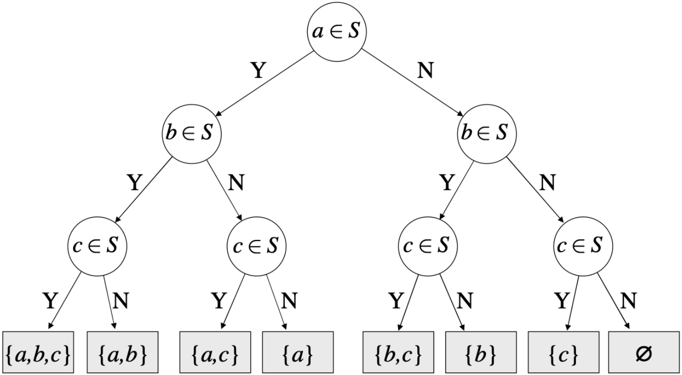

FIGURE 1.2. A decision tree for selecting a subset of $\{a, b, c\}$.

Source: Lovasz, Pelikan, Vesztergombi. *Discrete Mathematics: Elementary and Beyond.*

Counting Subsets (Enumeration Perspective)

$$\begin{array}{ccccccc}
0 & \Leftrightarrow & 0_2 & \Leftrightarrow & 000 & \Leftrightarrow & \emptyset \\
1 & \Leftrightarrow & 1_2 & \Leftrightarrow & 001 & \Leftrightarrow & \{c\} \\
2 & \Leftrightarrow & 10_2 & \Leftrightarrow & 010 & \Leftrightarrow & \{b\} \\
3 & \Leftrightarrow & 11_2 & \Leftrightarrow & 011 & \Leftrightarrow & \{b,c\} \\
4 & \Leftrightarrow & 100_2 & \Leftrightarrow & 100 & \Leftrightarrow & \{a\} \\
5 & \Leftrightarrow & 101_2 & \Leftrightarrow & 101 & \Leftrightarrow & \{a,c\} \\
6 & \Leftrightarrow & 110_2 & \Leftrightarrow & 110 & \Leftrightarrow & \{a,b\} \\
7 & \Leftrightarrow & 111_2 & \Leftrightarrow & 111 & \Leftrightarrow & \{a,b,c\}
\end{array}$$

Source: Lovasz, Pelikan, Vesztergombi. *Discrete Mathematics: Elementary and Beyond.*

Remark. Using a code like this, you could determine that the 233rd subset of a 10-element set $\{a_1, \ldots, a_{10}\}$ consists of elements $\{a_3, a_4, a_5, a_7, a_{10}\}$. (That's because 233 in binary is 11101001, which corresponds to the code 0011101001.)

Subtopics from reading

Scheinerman Sec. 10 (Sets I: Introduction, Subsets) covers the following subtopics

1. How to read and write set notation,
2. How to count the number of subsets of a set,
3. How to show one set is a subset of the other, and
4. How to show two sets are equal.

Today's group exercises provide additional practice on these subtopics.

Group 1: jacob.shepherd1,samuel.hemmen,lucas.jones6
Group 2: jonas.zeiler,alexander.goetz,julia.larsen
Group 3: james.brubaker,peter.buckley1,alexander.knutson
Group 4: joseph.mergenthaler,zeke.baumann,peyton.trigg
Group 5: jeremiah.mackey,jack.fry,jacob.ketola
Group 6: matthew.nagel,luka.derry,ryan.barrett2
Group 7: timothy.true,micaylyn.parker,aaron.loomis
Group 8: carsten.brooks,jacob.ruiz1,owen.obrien
Group 9: justice.mosso,connor.graville,delaney.rubb
Group 10: pendleton.johnston,blake.leone,adam.wyszynski
Group 11: bridger.voss,reid.pickert,evan.schoening
Group 12: tyler.broesel,carver.wambold,sarah.periolat
Group 13: connor.yetter,tristan.nogacki,jakob.kominsky
Group 14: griffin.short,emmeri.grooms,kaden.price
Group 15: colter.huber,jett.girard,luke.donaldson1
Group 16: devon.maurer,evan.barth,anthony.mann
Group 17: cameron.wittrock,caitlin.hermanson,connor.mizner
Group 18: erik.moore3,yebin.wallace,nolan.scott1
Group 19: jada.zorn,michael.oswald,samuel.mosier
Group 20: conner.reed1,mason.barnocky,samuel.rollins
Group 21: john.fotheringham,derek.price4,william.elder1
Group 22: ethan.johnson18,joseph.triem,lynsey.read

Group exercises: Sets

1. Find the cardinality of the following sets
   a) $\{x \in \mathbb{Z} : |x| \leq 10\}$
   b) $\{x \in \mathbb{Z} : 1 \leq x^2 \leq 2\}$
   c) $\{x \in \mathbb{Z} : x \in \emptyset\}$
   d) $2^{2^{\{1,2,3\}}}$
   e) $\{x \in 2^{\{1,2,3,4\}} : |x| = 1\}$
   f) $\{\{1,2\},\{3,4,5\}\}$
2. Let $A = \{x \in \mathbb{Z} : 4|x\}$ and $B = \{x \in \mathbb{Z} : 2|x\}$. Prove that $A \subseteq B$.
3. Let $A = \{0,1,2,3,4\}$ and $B = \{x \in \mathbb{N} : x^2 < 17\}$. Prove that $A = B$.

Solution to group exercise #1 (a-d)

Solution.

a) $$\left|\{x \in \mathbb{Z} : |x| \le 10\}\right| = \left|\{-10, \ldots, -1, 0, 1, \ldots, 10\}\right| = 21$$

b) $$\left|\{x \in \mathbb{Z} : 1 \le x^2 \le 2\}\right| = \left|\{-1, 1\}\right| = 2$$

c) $$\left|\{x \in \mathbb{Z} : x \in \emptyset\}\right| = \left|\emptyset\right| = 0$$

d) To find $\left|2^{2^{\{1,2,3\}}}\right|$, first recall that for any finite set A, 2^A refers to the set of all subsets of A. So setting $A \triangleq \{1, 2, 3\}$, we have

$$B \triangleq 2^A = 2^{\{1,2,3\}} = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}.$$

Now 2^B is the set of all subsets of *that*, so

$$2^B = \Big\{\emptyset, \big\{\{1\}\big\}, \big\{\{2\}\big\}, \big\{\{3\}\big\}, \big\{\{1\}, \{2\}\big\}, \ldots \big\{\{1\}, \{1, 2, 3\}, \{2\}, \{1, 2\}\big\}, \ldots\Big\}$$

Notably, $B = 2^A$ has $2^{|A|} = 2^3 = 8$ elements, and so 2^B has $2^{|B|} = 2^8$ elements. Hence, $\left|2^{2^{\{1,2,3\}}}\right| = 2^8$.

Solution to group exercise #1 (e-f)

Solution.

e) To find $\left|\{x \in 2^{\{1,2,3,4\}} : |x| = 1\}\right|$, we proceed similarly as in part (d). The set $B \triangleq 2^{\{1,2,3,4\}}$ refers to the set of all subsets of $\{1,2,3,4\}$. That is,

$$B = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}, \ldots, \{1,2,3,4\}\Big\}.$$

In particular B has $2^{|\{1,2,3,4\}|} = 2^4 = 16$ elements. Only four of those elements have cardinality 1, namely $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$. Hence $\left|\{x \in 2^{\{1,2,3,4\}} : |x| = 1\}\right| = 4$.

f) $\left|\{\{1,2\}, \{3,4,5\}\}\right| = 2$

Solution to group exercise #2

Problem. Let $A = \{x \in \mathbb{Z} : 4|x\}$ and $B = \{x \in \mathbb{Z} : 2|x\}$. Prove that $A \subseteq B$.

Solution.

Annotation	**Main Text**
Convert Prop. to "if-then" form	We show that if $x \in A$, then $x \in B$.
State assumption ("if")	Let $x \in A$.
Unravel defs.	Since, $x \in A$, we know that $4|x$. So by the definition of divisibility, there is some $n \in \mathbb{Z}$ such that $x = 4n$.
*** The glue ***	We write $x = 4n = 2(2n)$
Unravel defs.	So there is an integer $c = 2n$ such that $x = 2c$. So $2|x$.
State conclusion	Hence, $x \in B$.

Solution to group exercise #3 (Big picture)

Problem. Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{x \in \mathbb{N} : x^2 < 17\}$. Prove that $A = B$.

Structure of solution. To show $A = B$, we show $A \subseteq B$ and $B \subseteq A$.

- $\boxed{A \subseteq B.}$ We show that if $x \in A$, then $x \in B$.
- $\boxed{B \subseteq A.}$ We show that if $x \in B$, then $x \in A$.

Solution to group exercise #3

Problem. Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{x \in \mathbb{N} : x^2 < 17\}$. Prove that $A = B$.

Solution. To show $A = B$, we show $A \subseteq B$ and $B \subseteq A$.

- $\boxed{A \subseteq B.}$ We show that if $x \in A$, then $x \in B$. Let $x \in A$. Then $x \in \{0, 1, 2, 3, 4\}$. Then $x^2 \in \{0, 1, 4, 9, 16\}$. So $x^2 < 17$. Hence $x \in B$.
- $\boxed{B \subseteq A.}$ We show that if $x \in B$, then $x \in A$. Let $x \in B$. Hence $x \in \mathbb{N}$ and $x^2 < 17$. By taking the square root of each side, $x^2 < 17$ implies $|x| < \sqrt{17}$. The only natural numbers satisfying $|x| < \sqrt{17}$ are $0, 1, 2, 3$ and 4. Hence $x \in A$.