03/28/2025: Random Variables

CSCI 246: Discrete Structures

Textbook reference: Sec 33, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Today's Agenda

- Feedback from Wednesday ($\approx$ 5 mins)
- Reading and problems quiz (15 mins)
- Mini-lecture ($\approx$ 10 mins)
- Group exercises ($\approx$ 15 mins)

Feedback on Wednesday’s Quiz

Reading Quiz Scores

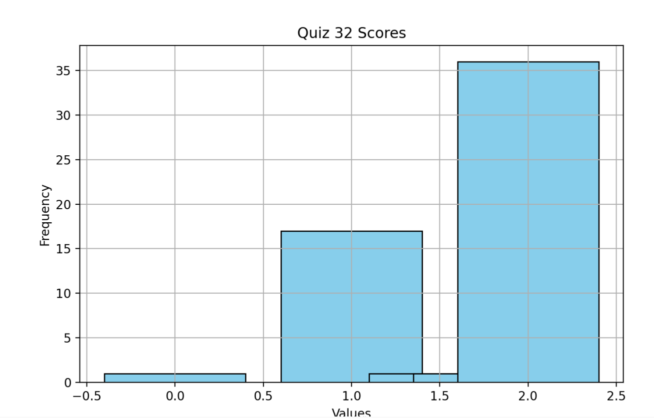


Figure 1: Median Score = 2/2 (100%)

Grading Rubric.

1. 1 point. Correct computation.
2. 1 point. Correctly stating *independence*.

Reminder: Reading Quiz (Conditional Probability and Independence)

Let A and B be events in a probability space (S, P). Under what condition does

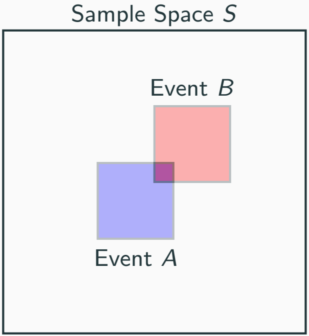
$$P(A \cap B) = P(A)\, P(B) \qquad (1)$$

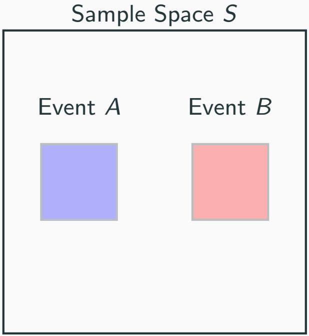
?

Poll: What do you think about this answer?

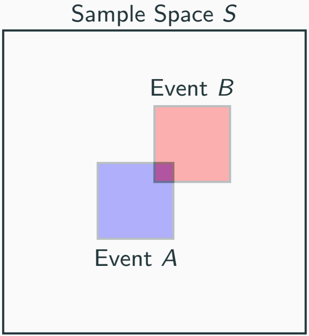
Eq. (1) holds when $P(A)$ and $P(B)$ have the same probability of happening. E. g. $P(A) = 1/2$ and $P(B) = 1/2$.

Subpoll. Consider the two scenarios below, where we represent probabilities as areas. In both cases $P(A) = P(B) = \frac{1}{16}$. But in one case A, B are independent, and in the other they are dependent. Which is which? Why?



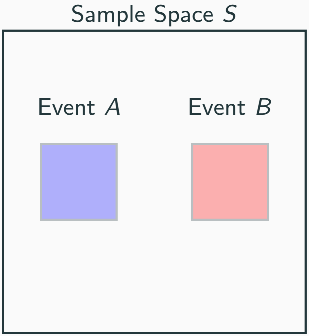


Subpoll. Consider the two scenarios below, where we represent probabilities as areas. In both cases $P(A) = P(B) = \frac{1}{16}$. But in one case A, B are independent, and in the other they are dependent. Which is which? Why?



These events are **independent**.

$$P(A \cap B) = P(A)\, P(B) = \frac{1}{16} \cdot \frac{1}{16} = \frac{1}{256}$$

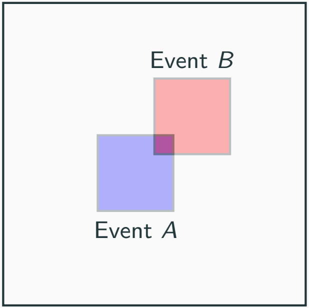


These events are **dependent**.

$$P(A \cap B) = 0 \neq P(A)\, P(B) = \frac{1}{16} \cdot \frac{1}{16}$$

Subpoll. Consider the two scenarios below, where we represent probabilities as areas. In both cases $P(A) = P(B) = \frac{1}{16}$. But in one case A, B are independent, and in the other they are dependent. Which is which? Why?

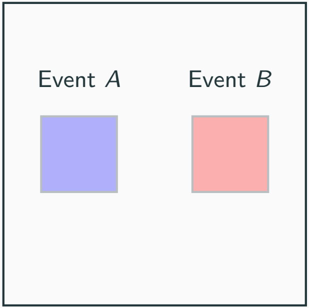
Sample Space S



These events are **independent**.

$$\begin{aligned} P(A \cap B) &= P(A)\, P(B) \\ &= \frac{1}{16} \cdot \frac{1}{16} = \frac{1}{256} \end{aligned}$$

Sample Space S



These events are **dependent**.

$$\begin{aligned} P(A \cap B) &= 0 \\ &\neq P(A)\, P(B) = \frac{1}{16} \cdot \frac{1}{16} \end{aligned}$$

Remark. Dependence doesn't need to be so extreme. The left figure would depict dependence even if we just pulled the blue and red squares apart from each other ever so slightly. Likewise if we pushed them closer together slightly.

Note on overall grades

The averages posted on Brightspace are cumulative.

Grade Item	Points	Grade
Average Reading Quiz Grade As Of 02-09-2025	69 / 100	69 %
Average Problems Quiz Grade as of 02-09-2025	111 / 100	111 %
Average Reading Quiz Grade As Of 03-06-2025	82.6 / 100	82.6 %
Average Problems Quiz Grade as of 03-06-2025	100 / 100	100 %

Today’s quiz

Reading Quiz (Random Variables)

1. Let (S, P) be a probability space. What is the name for a function defined on S?

2. Let (S, P) be the probability space formed by rolling a pair of die. Let X be a random variable defined on this space as the sum of the values of the two dice.

   a. (True/false) $P(X = 8) = P(\{s \in S : X(s) = 8\})$.
   b. (True/false) $P(X = 8) = P(\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\})$.
   c. (True/false) $P(X = 8) = \frac{5}{36}$.
   d. (True/false) $P(X \geq 8) = P(\{s \in S : X(s) \geq 8\})$.

Problems Quiz (Binomial Coefficients, Incl/Excl, Intro to Probability)

1. Determine the number of 2 element subsets of $\{a, b, c, d, e\}$ in two different ways.

2. What is the probability of getting a full house in a 5-card poker hand? A full house is three cards with one common rank and two other cards of another common rank, such as three queens and two 4s. (Note: you do not need to justify your answer or simplify binomial coefficients.)

Thoughts on random variables

Random Variables

Key concept. A **random variable** is a function on the sample space of a probability space.

Example. Suppose we flip four coins. The sample space consists of 16 outcomes (see below). We could define a random variable (function) that takes the value green whenever exactly two of the coins are heads, and red otherwise.

Sample Space S

HHTT	THTT	HTTT	TTTT
HHHT	THHT	HTHT	TTHT
HHTH	THTH	HTTH	TTTH
HHHH	THHH	HTHH	TTHH

Binomial Random Variables

Problem (Quality Control in Manufacturing)

A factory produces **LED bulbs**, and 1% of them are defective. A random sample of 20 bulbs is shipped to a customer.

1. What is the probability that **exactly 0** of the 20 bulbs are defective?
2. What is the probability that **no more than 1** bulb is defective?

Tool: Binomial Random Variables

A binomial random variable X gives the number of successes in a sequence of n independent experiments, each asking a yes–no question, where success occurs with probability p.

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

Solution

Let:

- $n = 20 \rightarrow$ total bulbs tested.
- $p = 0.01 \rightarrow$ probability of a bulb being defective.
- $X \rightarrow$ the number of defective bulbs, which follows a **binomial distribution**

1. The probability that **exactly 0** of the 20 bulbs are defective is

$$P(X = 0) = \binom{20}{0}(0.01)^0(0.99)^{20} = 0.8179$$

2. The probability that **no more than 1** bulb is defective is

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \binom{20}{0}(0.01)^0(0.99)^{20} + \binom{20}{1}(0.01)^1(0.99)^{19} \\ &= 0.8179 + 0.1652 = 0.9831 \end{aligned}$$

Group exercises

aaron.loomis: 8
adam.wyszynski: 13
alexander.goetz: 15
alexander.knutson: 17
anthony.mann: 4
blake.leone: 21
bridger.voss: 6
caitlin.hermanson: 2
cameron.wittrock: 10
carsten.brooks: 19
carver.wambold: 3
colter.huber: 18
conner.reed1: 13
connor.mizner: 6
connor.yetter: 11
derek.price4: 21
devon.maurer: 9
emmeri.grooms: 15
erik.moore3: 9
ethan.johnson18: 12
evan.barth: 4
evan.schoening: 20
griffin.short: 16
jack.fry: 7
jacob.ketola: 5
jacob.ruiz1: 19
jacob.shepherd1: 17
jada.zorn: 1
jakob.kominsky: 12
james.brubaker: 9
jeremiah.mackey: 14
jett.girard: 15
john.fotheringham: 14
jonas.zeiler: 20
joseph.mergenthaler: 3
joseph.triem: 2
julia.larsen: 3
justice.mosso: 1
kaden.price: 2
lucas.jones6: 8
luka.derry: 5
luke.donaldson1: 1
lynsey.read: 7
mason.barnocky: 10
matthew.nagel: 19
micaylyn.parker: 7
michael.oswald: 11
nolan.scott1: 10
owen.obrien: 11
pendleton.johnston: 18
peter.buckley1: 16
reid.pickert: 13
ryan.barrett2: 16
samuel.hemmen: 21
samuel.mosier: 5
samuel.rollins: 18
sarah.periolat: 6
timothy.true: 12
tristan.nogacki: 4
tyler.broesel: 17
william.elder1: 14
yebin.wallace: 8
zeke.baumann: 20

Group exercises

1. A fair coin is flipped three times. This is modeled by a probability space (S, P) where S contains the eight lists from (H, H, H) to (T, T, T), all with probability $\frac{1}{8}$. Let X denote the number of times we see TAILS.
   a. Write X explicitly as a function defined on S.
   b. Write the event "X is odd" as a set.
   c. Calculate $P(\text{X is odd})$.
2. A basketball player has a 75% chance of making a free throw. During a game, the player takes 10 free throws. Model the number of free throws made as a binomial random variable.
   a. What is the probability that the player makes exactly 8 free throws?
   b. What is the probability that the player makes at least 7 free throws?
3. For each situation below, determine whether X and Y independent. First provide the intuition, and then provide a mathematical argument.
   a. A fair coin is flipped three times. Let X be the number of heads and Y be the number of tails.
   b. A card is drawn at random from a standard deck of 52 cards. Let X be the rank of the card (from 2 to ace) and Y be the suit of the card.
   c. Two cards are drawn at random (without replacement) from a standard deck of 52 cards. Let X be the rank of the first card and Y be the rank of the second card.

Solution to group exercise #1

Problem. A fair coin is flipped three times. This is modeled by a probability space (S, P) where S contains the eight lists from (H, H, H) to (T, T, T), all with probability $\frac{1}{8}$. Let X denote the number of times we see TAILS.

a. Write X explicitly as a function defined on S.
b. Write the event "X is odd" as a set.
c. Calculate P(X is odd).

Solution.

a. The function $X : S \to \mathbb{N}$ is given by

$$\begin{aligned} X(HHH) &= 0 & \quad X(THH) &= 1 \\ X(HHT) &= 1 & \quad X(THT) &= 2 \\ X(HTH) &= 1 & \quad X(TTH) &= 2 \\ X(HTT) &= 2 & \quad X(TTT) &= 3 \end{aligned}$$

b. Let A be the event "X is odd". Then $A = \{HHT, HTH, THH, TTT\}$.
c. By the equally likely probability formula,

$$P(A) = \frac{|A|}{|S|} = \frac{4}{8}.$$

Solution to group exercise #2

Problem. A basketball player has a 75% chance of making a free throw. During a game, the player takes 10 free throws. Model the number of free throws made as a binomial random variable. (a) What is the probability that the player makes exactly 8 free throws? (b) What is the probability that the player makes at least 7 free throws?

Solution. Let X be the number of free throws made. By the binomial assumption, the probability that the random variable X takes on the specific value $x \in \{0, 1, \ldots, 10\}$ is given by

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{10}{x} .75^x (.25)^{10-x}$$

a. By the above

$$P(X = 8) = \binom{10}{8} .75^8 (.25)^2 \approx 0.28$$

So there is an approximately 28% chance that the player makes 8 of the 10 free throws.

b. Using similar computations as above, we can compute

$$\begin{aligned} P(X \geq 7) &= P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10) \\ &\approx 0.25 + 0.28 + 0.18 + 0.05 = 0.76. \end{aligned}$$

Solution to group exercise #3a

Problem. A fair coin is flipped three times. Let X be the number of heads and Y be the number of tails. Determine whether X and Y independent. First provide the intuition, and then provide a mathematical argument.
Solution.

- *Intuition.* If X is the number of heads, then we know the number of tails Y exactly (since $Y = 3 - X$). So these random variables can't be independent. In fact, they are as dependent as two random variables can possibly be, since they are deterministically related.
- *Mathematical argument.* By Def 33.6 in Scheinerman, X and Y are independent if for all x, y
$$P(X = x \text{ and } Y = y) = P(X = x)\, P(Y = y).$$
By the above, if $x = 0$, then we know $y = 3$. So let's consider $x = 0$ and $y \neq 3$. In particular, let's assume $x = 0$ and $y = 0$. Then
$$P(X = 0 \text{ and } Y = 0) = 0$$
(since it's impossible for 3 coin tosses to give 0 heads and 0 tails). However, by the equally likely probability formula,
$$P(X = 0) = \frac{1}{8} \qquad \text{and} \qquad P(Y = 0) = \frac{1}{8}.$$
Hence, we have found values x and y such that
$$P(X = x \text{ and } Y = y) \neq P(X = x)\, P(Y = y).$$
Hence, X and Y are not independent.

Solution to group exercise #3b

Problem. A card is drawn at random from a standard deck of 52 cards. Let X be the rank of the card (from 2 to ace) and Y be the suit of the card. Determine whether X and Y independent. First provide the intuition, and then provide a mathematical argument.

Solution.

- *Intuition.* These random variables should be independent because knowing the suit of the card doesn't provide any information about the rank. For instance, knowing your card is a $\heartsuit$ doesn't change the probability that it is a King.
- *Mathematical argument.* By Def 33.6 in Scheinerman, X and Y are independent if for all x, y

  $$P(X = x \text{ and } Y = y) = P(X = x)\, P(Y = y). \tag{2}$$

  Let x be any value in the set $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$. Let y be any value in $\{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}$. Then $P(X = x) = \frac{1}{13}$ and $P(Y = y) = \frac{1}{4}$. Moreover, $P(X = x \wedge Y = y) = \frac{1}{52}$. Thus, Eq. (2) is satisfied (since $\frac{1}{52} = \frac{1}{13} \times \frac{1}{4}$). Hence, X and Y are independent random variables.

Solution to group exercise #3c

Problem. Two cards are drawn at random (without replacement) from a standard deck of 52 cards. Let X be the rank of the first card and Y be the rank of the second card. Determine whether X and Y independent. First provide the intuition, and then provide a mathematical argument.
Solution.

- *Intuition.* These random variables should be dependent because knowing the rank of the first card provides information about the probabilities of getting various ranks for the second card. For instance, if your first card is a king, then there is a lower probability you'll get a king on the second card (since there's one fewer king in the deck).
- *Mathematical argument.* By the equally likely probability formula,

$$P(X = \texttt{King}) = \frac{4}{52} \qquad \text{and} \qquad P(Y = \texttt{King}) = \frac{4}{52}.$$

On the other hand, we have

$$P(X = \texttt{King} \wedge Y = \texttt{King}) = \frac{4}{52} \times \frac{3}{51}$$

Hence, we have found values x and y such that

$$P(X = x \text{ and } Y = y) \neq P(X = x)\, P(Y = y).$$