02/19/2025: Introduction to Relations

CSCI 246: Discrete Structures

Textbook reference: Ch 11.1-11.2, Hampkins

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Today's Agenda

- Reading quiz (5 mins)
- Mini-lecture ($\approx$ 20 mins)
- Group exercises ($\approx$ 20 mins)

Definition

Suppose that R is a binary relation on a set A.

1. The relation R is **reflexive** on A if $a \, R \, a$ for all $a \in A$.
2. The relation R is **symmetric** on A if, whenever $a \, R \, b$, then also $b \, R \, a$.
3. The relation R is **transitive** on A if, whenever $a \, R \, b$ and $b \, R \, c$, then also $a \, R \, c$.

Reading Quiz (Relations)

Give an example of

1. A binary relation that is reflexive and symmetric but not transitive.
2. A binary relation that is reflexive and transitive but not symmetric.
3. A binary relation that is symmetric and transitive but not reflexive.

Notes on relations

Definition

Let A and B be two sets. A **binary relation** (or simply a **relation**) R from A to B is a subset of the Cartesian product $A \times B$.

Example

Let $A = \{2,3,4,5,6\}$ and $B = \{3,6,9\}$ and relation $R = \{(a,b) : a|b, a \in A, b \in B\}$. Then $R = \{(2,6),(3,3),(3,6),(3,9),(6,6)\}$.

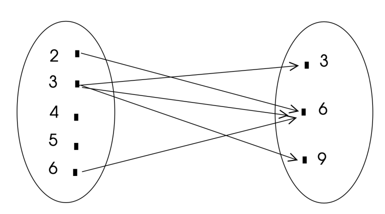


Figure 1: Set diagram representation of the relation R

Definition

Let A and B be two sets. A **binary relation** (or simply a **relation**) R from A to B is a subset of the Cartesian product $A \times B$.

Example

Let $A = \{2, 3, 4, 5, 6\}$ and $B = \{3, 6, 9\}$ and relation $R = \{(a, b) : a|b, a \in A, b \in B\}$. Then $R = \{(2, 6), (3, 3), (3, 6), (3, 9), (6, 6)\}$.

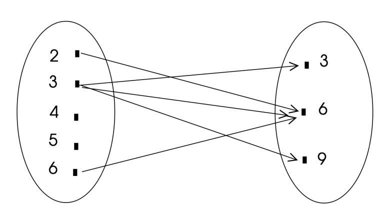


Figure 1: Set diagram representation of the relation R

Poll

Can this relation be reflexive, symmetric, and/or transitive? Why or why not?

Remark

The reading considers relations where the two underlying sets are the same. That is, it considers relations R from A to A, where A is a set. Hence, the definitions of reflexive, symmetric, and transitive are relevant.

Poll

Now consider the relation $R = \{(a, b) : a|b\}$ on the set $\mathbb{Z}$ of integers.
Is this relation

- Reflexive?

Remark

The reading considers relations where the two underlying sets are the same. That is, it considers relations R from A to A, where A is a set. Hence, the definitions of reflexive, symmetric, and transitive are relevant.

Poll

Now consider the relation $R = \{(a, b) : a|b\}$ on the set $\mathbb{Z}$ of integers.
Is this relation

- Reflexive? ✓ Every number divides itself: $a|a$, since $a \cdot 1 = a$.
- Symmetric?

Remark

The reading considers relations where the two underlying sets are the same. That is, it considers relations R from A to A, where A is a set. Hence, the definitions of reflexive, symmetric, and transitive are relevant.

Poll

Now consider the relation $R = \{(a, b) : a|b\}$ on the set $\mathbb{Z}$ of integers.
Is this relation

- Reflexive? ✓ Every number divides itself: $a|a$, since $a \cdot 1 = a$.
- Symmetric? ✗ For example, $3|6$ but it is not the case that $6|3$.
- Transitive?

Remark

The reading considers relations where the two underlying sets are the same. That is, it considers relations R from A to A, where A is a set. Hence, the definitions of reflexive, symmetric, and transitive are relevant.

Poll

Now consider the relation $R = \{(a, b) : a|b\}$ on the set $\mathbb{Z}$ of integers.
Is this relation

- Reflexive? ✓ Every number divides itself: $a|a$, since $a \cdot 1 = a$.
- Symmetric? ✗ For example, $3|6$ but it is not the case that $6|3$.
- Transitive? ✓ Suppose $a|b$ and $b|c$. We need to show $a|c$. Since $a|b$, there is an integer q such that $b = aq$. By definition of $b|c$, there is an integer r such that $c = br$. Hence we have

$$c = br = (aq)r = a(qr)$$

Hence there is an integer $s = qr$ such that $c = as$. Hence $a|c$.

Example of Relations in Computer Science

Example

Let's consider a "friendship" relation in a **social network** like Facebook. We can define a relation R on a set of users where: $(x, y) \in R$ if and only if x is a friend of y.

Poll

Is this relation

- Reflexive?

Example of Relations in Computer Science

Example

Let's consider a "friendship" relation in a **social network** like Facebook. We can define a relation R on a set of users where: $(x, y) \in R$ if and only if x is a friend of y.

Poll

Is this relation

- Reflexive? ✗ Users don't usually friend themselves in social networks.
- Symmetric?

Example of Relations in Computer Science

Example

Let's consider a "friendship" relation in a **social network** like Facebook. We can define a relation R on a set of users where: $(x, y) \in R$ if and only if x is a friend of y.

Poll

Is this relation

- Reflexive? ✗ Users don't usually friend themselves in social networks.
- Symmetric? ✓ (-ish.) In most social networks, a friendship connection is bidirectional. If Alice is a friend of Bob, then Bob is a friend of Alice. However, in platforms like X (formerly Twitter), where following is *one-way*, the "follows" relation is *not symmetric*.
- Transitive?

Example of Relations in Computer Science

Example

Let's consider a "friendship" relation in a **social network** like Facebook. We can define a relation R on a set of users where: $(x, y) \in R$ if and only if x is a friend of y.

Poll

Is this relation

- Reflexive? ✗ Users don't usually friend themselves in social networks.
- Symmetric? ✓ (-ish.) In most social networks, a friendship connection is bidirectional. If Alice is a friend of Bob, then Bob is a friend of Alice. However, in platforms like X (formerly Twitter), where following is *one-way*, the "follows" relation is *not symmetric*.
- Transitive? ✗ If Alice is friends with Bob and Bob is friends with Charlie, it does not necessarily mean that Alice is friends with Charlie.

Why do relations matter?: Some examples from social networks

- **Social Influence & Virality**
  - If relationships in a network are transitive, messages, trends, and viral content can spread faster.
- **Friend-of-a-Friend Recommendations**
  - Even if friendship is not strictly transitive, many recommendation algorithms use approximate transitivity to suggest new friends (e.g., LinkedIn's "People You May Know" feature).
- **Algorithm Optimization**
  - If a relation is symmetric, storage can be optimized by only storing one direction of the connection.
  - Search algorithms (e.g., shortest path) can be optimized by reducing redundant checks.

Equivalence relations

Definition

An **equivalence relation** is a relation that is

1. Reflexive ✓
2. Symmetric ✓
3. Transitive ✓

Remark

Q: Why do equivalence relations matter?

Remark

Q: Why do equivalence relations matter?

A: They can make problem solving easier.

Remark

Q: Why do equivalence relations matter?

A: They can make problem solving easier.

Example

Q: Why is $n^{70} - n^{22}$ always even?

Remark

Q: Why do equivalence relations matter?

A: They can make problem solving easier.

Example

Q: Why is $n^{70} - n^{22}$ always even?

Useful Point-Of-View (POV): Let us define the "mod-2" equivalence relation (written $\equiv_2$). Here two numbers are equivalent if they have the same remainder when divided by 2. So

$$x \equiv_2 \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$$

Now multiplication and addition is well-defined mod-2. (For justification, see Hampkins Ch2 or today's group exercises). So we can just use "0" and "1" as representatives in any multiplication or addition problem about odds and evens.

Remark

Q: Why do equivalence relations matter?

A: They can make problem solving easier.

Example

Q: Why is $n^{70} - n^{22}$ always even?

Useful Point-Of-View (POV): Let us define the "mod-2" equivalence relation (written $\equiv_2$). Here two numbers are equivalent if they have the same remainder when divided by 2. So

$$x \equiv_2 \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$$

Now multiplication and addition is well-defined mod-2. (For justification, see Hampkins Ch2 or today's group exercises). So we can just use "0" and "1" as representatives in any multiplication or addition problem about odds and evens.

A: By the above POV, we only need to consider two cases:

$$\begin{aligned} n^{70} - n^{22} &= 0^{70} - 0^{22} = 0 - 0 = 0 && \text{if } n = 0 \ (n \text{ is even}) \\ n^{70} - n^{22} &= 1^{70} - 1^{22} = 1 - 1 = 0 && \text{if } n = 1 \ (n \text{ is odd}) \end{aligned}$$

Either way, the result is 0 (the result is even).

Take home

The purpose of equivalences

The purpose of equivalences is similar to the purpose of the set identities we did last class: they can help us see things from different perspectives.

Whether it's looking at each side of an equality of an identity, or creatively constructing new equivalence relations, these tools can help us to solve problems that aren't otherwise solvable (quickly or at all).

Take home

The purpose of equivalences

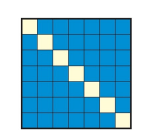
The purpose of equivalences is similar to the purpose of the set identities we did last class: they can help us see things from different perspectives.

Whether it's looking at each side of an equality of an identity, or creatively constructing new equivalence relations, these tools can help us to solve problems that aren't otherwise solvable (quickly or at all).

The purpose of multiple points of view

Why see things from different points of view? Recall from Hampkins Ch2 (Multiple Proofs), different points of view suggest different generalizations.

For example, consider how hard it would be to answer the question about $n^{70} - n^{22}$ if, instead of using mod-2 equivalences, we tried the extending the picture proof we used for $n^2 - n$:



Group 1: erik.moore3,samuel.hemmen,julia.larsen
Group 2: jacob.shepherd1,connor.graville,luke.donaldson1
Group 3: derek.price4,griffin.short,luka.derry
Group 4: cameron.wittrock,tristan.nogacki,jack.fry
Group 5: connor.yetter,conner.reed1,john.fotheringham
Group 6: james.brubaker,caitlin.hermanson,devon.maurer
Group 7: pendleton.johnston,evan.schoening,adam.wyszynski
Group 8: timothy.true,blake.leone,joseph.mergenthaler
Group 9: jonas.zeiler,anthony.mann,colter.huber
Group 10: justice.mosso,carver.wambold,peter.buckley1
Group 11: aaron.loomis,nolan.scott1,sarah.periolat
Group 12: matthew.nagel,samuel.rollins,carsten.brooks
Group 13: owen.obrien,ryan.barrett2,jett.girard
Group 14: samuel.mosier,tyler.broesel,evan.barth
Group 15: jacob.ruiz1,delaney.rubb,peyton.trigg
Group 16: zeke.baumann,yebin.wallace,mason.barnocky
Group 17: jeremiah.mackey,alexander.goetz,bridger.voss
Group 18: lucas.jones6,reid.pickert,jakob.kominsky
Group 19: emmeri.grooms,kaden.price,william.elder1
Group 20: jacob.ketola,lynsey.read,connor.mizner
Group 21: joseph.triem,michael.oswald,micaylyn.parker
Group 22: alexander.knutson,ethan.johnson18,jada.zorn

Group Exercises: Introduction to Relations

1. Consider the collection of numerical expressions for rational numbers, like $\frac{3}{4}$ or $-\frac{6}{12}$. Let us consider these expressions not as numbers but as syntactic expressions $\frac{p}{q}$ – that is, as pairs of integers, a numerator p and nonzero denominator q – so that we count $\frac{1}{2}$ as a different expression than $\frac{2}{4}$. Define the relation $\frac{p}{q} \approx \frac{r}{s}$ for such expressions if they represent the same rational number, which happens precisely when $ps = rq$ in the integers. Prove this is an equivalence relation.
2. A relation R is defined on $\mathbb{Z}$ by $a\,R\,b$ if $a + b$ is even. Show that R is an equivalence relation.
3. (Bonus.) Show that both addition and multiplication are well-defined with respect to congruence modulo n, for every positive integer n.

Solution to Group Exercise #1

Solution. We verify the three properties of an equivalence relationship below.

- *Reflexivity*: We need to check
$$\frac{p}{q} \approx \frac{p}{q},$$
which happens when $pq = pq$. This is always true. ✓ .
- *Symmetry*: We need to check that if $\frac{p}{q} \approx \frac{r}{s}$, then $\frac{r}{s} \approx \frac{p}{q}$. That is, we need to check that if $ps = qr$, then $rq = sp$. This is true by commutativity of multiplication. ✓ .
- *Transitivity*: We need to check that if $\frac{p}{q} \approx \frac{r}{s}$ and $\frac{r}{s} \approx \frac{t}{u}$, then $\frac{p}{q} \approx \frac{t}{u}$. That is, we need to check that if $ps = qr$ and $ru = st$, then $pu = qt$. Now note
$$\begin{aligned} ps &= qr && \text{By hypothesis} \\ ps(ru) &= qr(st) && \text{Multiply both sides by same amount} \\ p\cancel{s}(\cancel{r}u) &= q\cancel{r}(\cancel{s}t) && \text{Cancel identical terms} \\ pu &= qt, \end{aligned}$$
which is what we needed to show. ✓

Solution to Group Exercise #2

Problem. A relation R is defined on $\mathbb{Z}$ by $a \, R \, b$ if $a + b$ is even. Show that R is an equivalence relation.

Solution. We verify the three properties of an equivalence relationship below.

- *Reflexivity*: We need to check aRa, which means that $a + a$ must be even. But $a + a = 2a$, which is even by definition of even. ✓
- *Symmetry*: We need to check that if aRb , then bRa. That is, we need to check that if $a + b$ is even, then $b + a$ is even. This is true by commutativity of addition. ✓
- *Transitivity*: We need to check that if aRb and bRc , then aRc. That is, we need to check that if $a + b$ and $b + c$ are even, then $a + c$ is even. Now by definition of even, $a + b = 2q$ and $b + c = 2r$ for some integers q and r. Thus

$$
\begin{aligned}
(a + b) + (b + c) &= 2q + 2r \\
\implies a + 2b + c &= 2q + 2r \\
\implies a + c &= 2q + 2r - 2b = 2(q + r - b),
\end{aligned}
$$

  and so $a + c$ is even by the definition of even. ✓

Solution to Group Exercise #3

Problem (Bonus.) Show that both addition and multiplication are well-defined with respect to congruence modulo n, for every positive integer n.

Remark 1. By definition of congruence modulo n, if $x \equiv_n x'$, then $x = x' + nr$ for some integer r.

Solution to problem. Suppose $x \equiv_n x'$ and $y \equiv_n y'$. Then by Remark 1, $x = x' + nr$ for some integer r and $y = y' + nq$ for some integer q. Now note

$$\begin{aligned} x + y &= x' + y' + nr + nq \\ &= x' + y' + n(r + q) \end{aligned}$$

Hence $x + y \equiv_n x' + y'$. ✓

Moreover,

$$\begin{aligned} xy &= (x' + nr)(y' + nq) \\ &= x'y' + x'nq + y'nr + n^2rq \\ &= x'y' + n(x'q + y'r + nrq) \end{aligned}$$

Hence $xy \equiv_n x'y'$. ✓

Remark 2. This problem generalizes Thm 88 on pp.124 of Hampkins.