02/24/2025: Relations

CSCI 246: Discrete Structures

Textbook reference: Sec 14, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Announcements

- Regrades from last Friday (2/14) are available.

Today's Agenda

- Reading quiz (5 mins)
- Mini-lecture ($\approx$ 10 mins)
- Group exercises ($\approx$ 30 mins)

Reading Quiz (Relations)

For each of the following relations defined on the integers, please state whether the relation is reflexive, symmetric, and/or transitive.

1. $=$ (equality)
2. $\leq$ (less than or equal to)
3. $<$ (strict less than)
4. $|$ (divides)

Feedback on Friday's Quiz

Scores On Problems Quiz (Quantifiers and Set Operations)

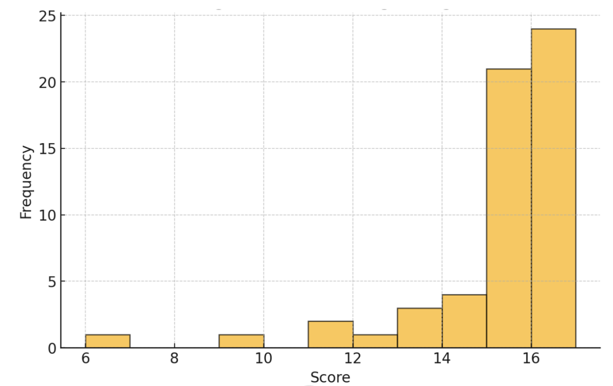


Figure 1: Median Score = 15/16 (93.7%)

Rubric.

1. Quantifiers question (8 points total): 1 point for each subquestion (a)-(h)
2. Set operations question (8 points total): 4 points for the correct answer. 4 points for a correct explanation.

Scores On Reading Quiz (Intro to Functions) (Extra Credit)

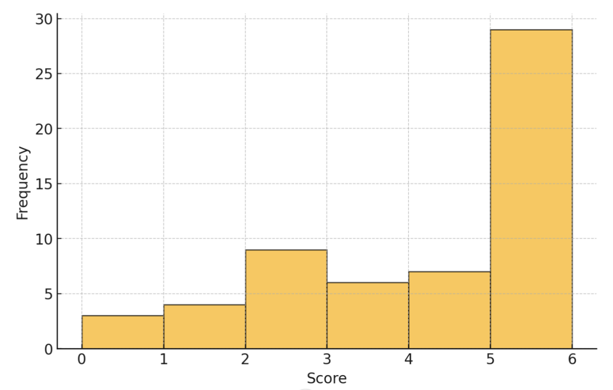
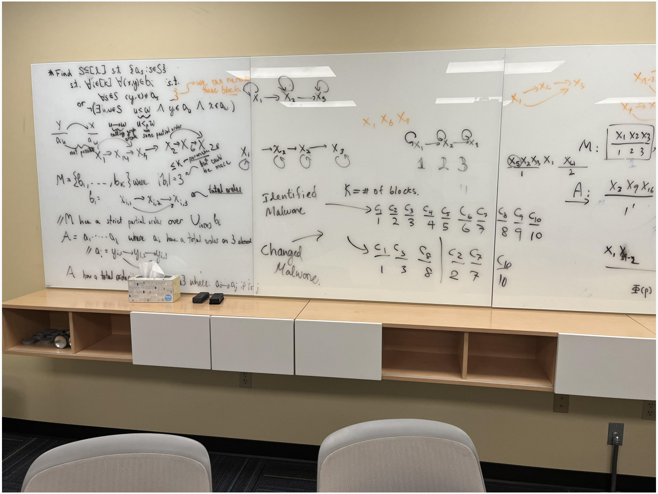


Figure 2: Median Score = 4.5/5 (90%)

Rubric. 1 point for each diagram if correct.

High-level re-motivation

Cybersecurity



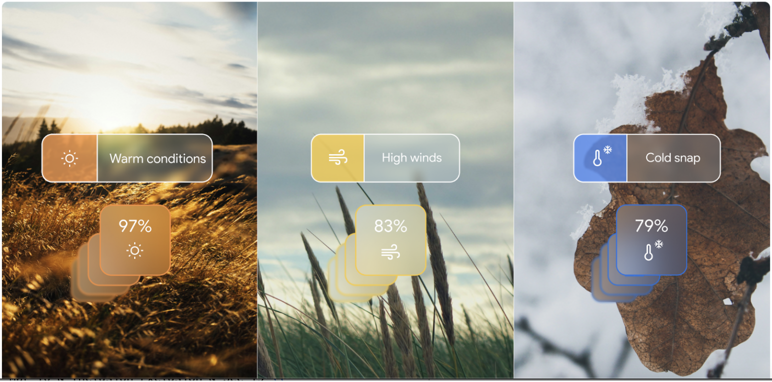
Machine Learning

GenCast predicts weather and the risks of extreme conditions with state-of-the-art accuracy

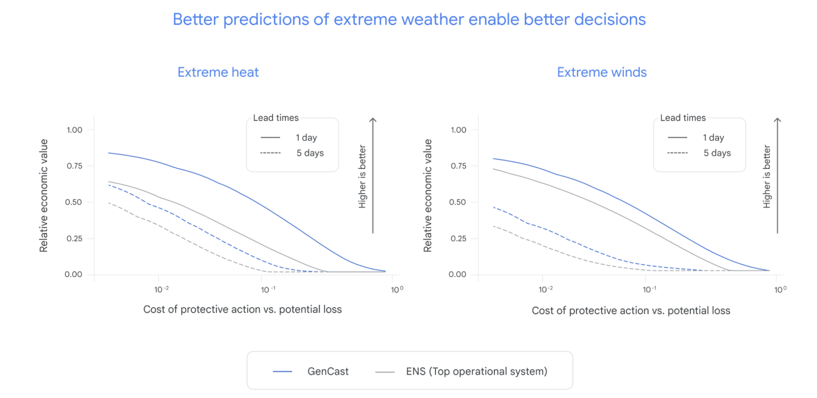
4 DECEMBER 2024

Ilan Price and Matthew Willson

Share



Machine Learning



B VE, VP AND SUB-VP SDES

Below we provide detailed derivations to show that the noise perturbations of SMLD and DDPM are discretizations of the Variance Exploding (VE) and Variance Preserving (VP) SDEs respectively. We additionally introduce sub-VP SDEs, a modification to VP SDEs that often achieves better performance in both sample quality and likelihoods.

First, when using a total of N noise scales, each perturbation kernel $p_{\sigma_i}(\mathbf{x} \mid \mathbf{x}_0)$ of SMLD can be derived from the following Markov chain:

$$\mathbf{x}_i = \mathbf{x}_{i-1} + \sqrt{\sigma_i^2 - \sigma_{i-1}^2}\mathbf{z}_{i-1}, \quad i = 1, \cdots, N, \tag{20}$$

where $\mathbf{z}_{i-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, $\mathbf{x}_0 \sim p_{\text{data}}$, and we have introduced $\sigma_0 = 0$ to simplify the notation. In the limit of $N \to \infty$, the Markov chain $\{\mathbf{x}_i\}_{i=1}^N$ becomes a continuous stochastic process $\{\mathbf{x}(t)\}_{t=0}^1$, $\{\sigma_i\}_{i=1}^N$ becomes a function $\sigma(t)$, and $\mathbf{z}_i$ becomes $\mathbf{z}(t)$, where we have used a continuous time variable $t \in [0, 1]$ for indexing, rather than an integer $i \in \{1, 2, \cdots, N\}$. Let $\mathbf{x}\left(\frac{i}{N}\right) = \mathbf{x}_i$, $\sigma\left(\frac{i}{N}\right) = \sigma_i$, and $\mathbf{z}\left(\frac{i}{N}\right) = \mathbf{z}_i$ for $i = 1, 2, \cdots, N$. We can rewrite Eq. (20) as follows with $\Delta t = \frac{1}{N}$ and $t \in \{0, \frac{1}{N}, \cdots, \frac{N-1}{N}\}$:

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \sqrt{\sigma^2(t + \Delta t) - \sigma^2(t)}\, \mathbf{z}(t) \approx \mathbf{x}(t) + \sqrt{\frac{\mathrm{d}\left[\sigma^2(t)\right]}{\mathrm{d}t}\Delta t}\, \mathbf{z}(t),$$

where the approximate equality holds when $\Delta t \ll 1$. In the limit of $\Delta t \to 0$, this converges to

$$\mathrm{d}\mathbf{x} = \sqrt{\frac{\mathrm{d}\left[\sigma^2(t)\right]}{\mathrm{d}t}}\mathrm{d}\mathbf{w}, \tag{21}$$

which is the VE SDE.

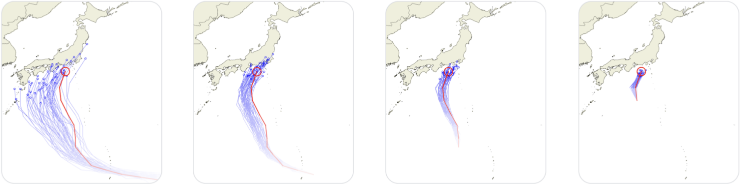
For the perturbation kernels $\{p_{\alpha_i}(\mathbf{x} \mid \mathbf{x}_0)\}_{i=1}^N$ used in DDPM, the discrete Markov chain is

$$\mathbf{x}_i = \sqrt{1 - \beta_i}\mathbf{x}_{i-1} + \sqrt{\beta_i}\mathbf{z}_{i-1}, \quad i = 1, \cdots, N, \tag{22}$$

where $\mathbf{z}_{i-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$. To obtain the limit of this Markov chain when $N \to \infty$, we define an auxiliary set of noise scales $\{\bar{\beta}_i = N\beta_i\}_{i=1}^N$, and re-write Eq. (22) as below

$$\mathbf{x}_i = \sqrt{1 - \frac{\bar{\beta}_i}{N}}\mathbf{x}_{i-1} + \sqrt{\frac{\bar{\beta}_i}{N}}\mathbf{z}_{i-1}, \quad i = 1, \cdots, N. \tag{23}$$

GenCast forecasts for the path of Typhoon Hagibis



7-day forecast · 5-day forecast · 3-day forecast · 1-day forecast

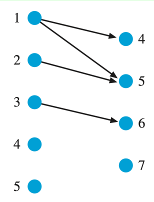
Drawing pictures of relations

Drawing relations from A to B

To draw a picture of relation R from A to B, we draw two collections of dots. The first collection of dots corresponds to elements in A, and we place these on the left side of the figure. The dots for B go on the right. We then draw an arrow from $a \in A$ to $b \in B$ just when (a, b) is in the relation.

Example

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7\}$, and let R be the relation $\{(1, 4), (1, 5), (2, 5), (3, 6)\}$. A picture of the relation R is given in the figure below.



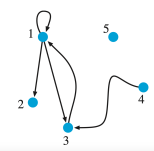
Drawing relations on A

To draw a picture of relation R on A, we have two options

1. We can make a diagram in which each element of A is represented by a dot. If aRb, then we draw an arrow from dot a to dot b. If it happens that b is also related to a, then we draw another arrow from b to a. If aRa, then we draw a looping arrow form a to itself.
2. We can consider this as a relation R from A to A, and use the method of the previous slide.

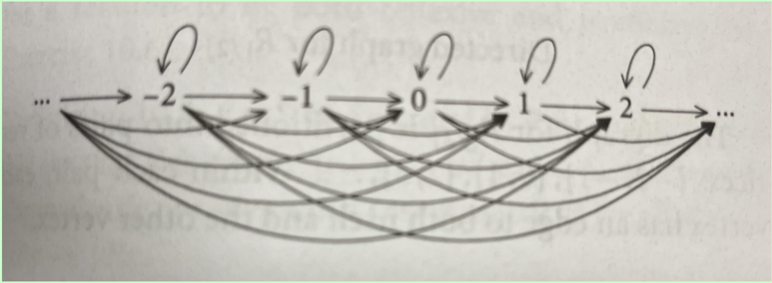
Example

Let $A = \{1, 2, 3, 4, 5\}$ and $R = \{(1, 1), (1, 2), (1, 3), (4, 3), (3, 1)\}$. A picture of the relation R (using option #1) is given in the figure below.



An Example Using a Set with Infinitely Many Elements

Let $A = \mathbb{Z}$ and R be the $\leq$ relation. A picture of the relation is given below



Properties of relations via pictures

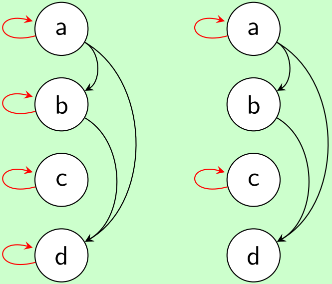
Reflexivity

- A relation is **reflexive** if every element has a self-loop.
- A relation is **not reflexive** if at least one element has no self-loop.

Example

Reflexive $\forall\, x \in A, xRx$

Not reflexive $\exists\, x \in A : x \not{R} x$



When one or more red connection drops out, the relation loses reflexivity.

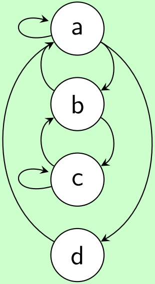
Symmetric

- A relation is **symmetric** if any connection to a distinct element is paired with a connection in the opposite direction.
- A relation is **not symmetric** if at least one connection to a distinct element lacks a connection in the opposite direction.

Example

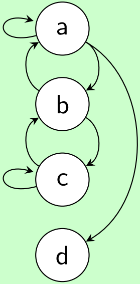
Symmetric

$xRy \implies yRx$



Not symmetric

$\exists\, x, y \in A : xRy$ but $y \not{R} x$



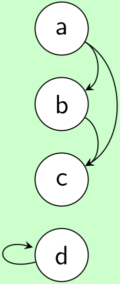
When we dropped the connection from d to a, the relationship lost symmetry.

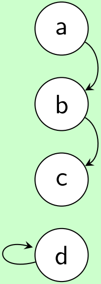
Transitivity

- A relation is **transitive** if any two-hop trip can also be done in a single-hop.
- A relation is **not transitive** if there is at least one two-hop trip that cannot be done in a single hop.

Example

Transitive	**Not transitive**
$xRy, yRz \implies xRz$	$\exists\, x, y, z \in A : xRy$ and yRz but $x \not{R} z$





When we dropped the connection from a to c, the relationship lost transitivity.

Group exercises

aaron.loomis: 3
adam.wyszynski: 13
alexander.goetz: 21
alexander.knutson: 9
anthony.mann: 19
blake.leone: 1
bridger.voss: 12
caitlin.hermanson: 11
cameron.wittrock: 14
carsten.brooks: 6
carver.wambold: 1
colter.huber: 0
conner.reed1: 16
connor.graville: 20
connor.mizner: 3
connor.yetter: 15
delaney.rubb: 4
derek.price4: 7
devon.maurer: 20
emmeri.grooms: 21
erik.moore3: 8
ethan.johnson18: 15
evan.barth: 14
evan.schoening: 13
griffin.short: 11
jack.fry: 7
jacob.ketola: 13
jacob.ruiz1: 2
jacob.shepherd1: 0
jada.zorn: 16
jakob.kominsky: 3
james.brubaker: 8
jeremiah.mackey: 10
jett.girard: 19
john.fotheringham: 20
jonas.zeiler: 0
joseph.mergenthaler: 15
joseph.triem: 2
julia.larsen: 7
justice.mosso: 8
kaden.price: 16
lucas.jones6: 12
luka.derry: 11
luke.donaldson1: 1
lynsey.read: 5
mason.barnocky: 17
matthew.nagel: 9
micaylyn.parker: 4
michael.oswald: 9
nolan.scott1: 14
owen.obrien: 19
pendleton.johnston: 6
peter.buckley1: 18
peyton.trigg: 17
reid.pickert: 12
ryan.barrett2: 2
samuel.hemmen: 6
samuel.mosier: 5
samuel.rollins: 21
sarah.periolat: 5
timothy.true: 4
tristan.nogacki: 17
tyler.broesel: 10
william.elder1: 18
yebin.wallace: 10
zeke.baumann: 18

Group Exercises: Relations

1. *(Relations as ordered pairs.)* Write the following relations on the set $\{1, 2, 3, 4, 5\}$ as sets of ordered pairs: (a) is-less-than, (b) is-divisible-by, (c) is-equal-to, (d) has-the-same-parity-as. (Note: When we say two numbers have the same parity, we mean they are both odd or both even.)
2. *(Drawing pictures of relations.)* Draw pictures of the following relations on A.
    a. Let $A = \{a \in \mathbb{N} : a|10\}$ and let R be the relation $|$ (divides) restricted to A.
    b. Let $A = \{1, 2, 3, 4, 5\}$ and let R be the relation $=$ (equals) restricted to A.
3. *(Drawing pictures of relations.)* Draw pictures of the following relations on $A \times B$.
    a. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4\}$. Let R be the relation $\geq$ (greater than or equal to) from A to B.
4. *(Properties of relations.)* The property *irreflexive* is not the same as being not reflexive. To illustrate this, do the following: (a) Give an example of a relation on a set that is neither reflexive nor irreflexive. (b) Give an example of a relation on a set that is both reflexive and irreflexive. Part (a) is not too hard, but for (b), you will need to create a rather strange example.
5. *(Properties of relations.)* For each of the following relations on the set of all human beings, determine whether the relation is reflexive, irreflexive, symmetric, anti-symmetric, and/or transitive: (a) is-taller-than, (b) has-the-same-parents-as (i.e., same mother and father), (c) is-the-child-of, (d) is-married-to.

Solution to Group Exercise #1

Problem. *(Relations as ordered pairs.)* Write the following relations on the set $\{1,2,3,4,5\}$ as sets of ordered pairs: (a) is-less-than, (b) is-divisible-by, (c) is-equal-to, (d) has-the-same-parity-as. (Note: When we say two numbers have the same parity, we mean they are both odd or both even.)

Solution.

a.

$$\begin{aligned} R = \Big\{ \quad & (1,2),(1,3),(1,4),(1,5), \\ & (2,3),(2,4),(2,5), \\ & (3,4),(3,5),(4,5) \quad \Big\} \end{aligned}$$

b.

$$R = \Big\{(1,1),(2,1),(3,1),(4,1),(5,1),(2,2),(4,2),(3,3),(4,4),(5,5)\Big\}$$

c.

$$R = \Big\{(1,1),(2,2),(3,3),(4,4),(5,5)\Big\}$$

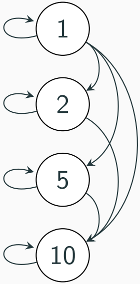
d.

$$\begin{aligned} R = \Big\{ \quad & (1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5), \\ & (2,2),(2,4),(4,4),(4,2) \quad \Big\} \end{aligned}$$

Solution to Group Exercise #2a

Problem. *(Drawing pictures of relations.)* Draw pictures of the following relation on A: Let $A = \{a \in \mathbb{N} : a|10\}$ and let R be the relation $|$ (divides) restricted to A.

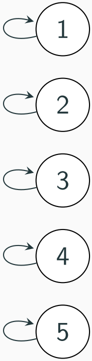
Solution. First note that $A = \{1, 2, 5, 10\}$. Now we draw



Solution to Group Exercise #2b

Problem. *(Drawing pictures of relations.)* Draw pictures of the following relation on A: Let $A = \{1, 2, 3, 4, 5\}$ and let R be the relation $=$ (equals) restricted to A.

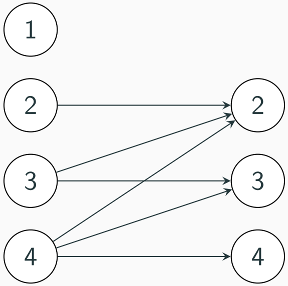
Solution. We draw



Solution to Group Exercise #3

Problem. *(Drawing pictures of relations.)* Draw pictures of the following relations on $A \times B$: Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4\}$. Let R be the relation $\geq$ (greater than or equal to) from A to B.

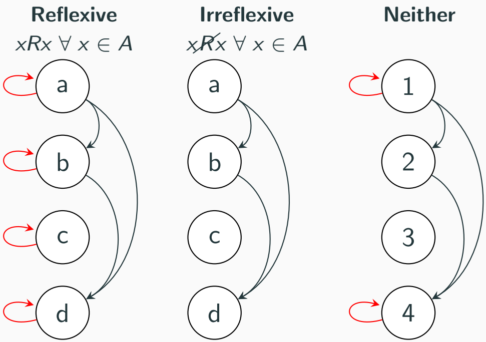
Solution. We draw



Solution to Group Exercise #4a

Problem. *(Properties of relations.)* The property *irreflexive* is not the same as being not reflexive. To illustrate this, do the following: (a) Give an example of a relation on a set that is neither reflexive nor irreflexive. (b) Give an example of a relation on a set that is both reflexive and irreflexive. Part (a) is not too hard, but for (b), you will need to create a rather strange example.

Solution to (a). The solution is determined by the red loops below.



Solution to Group Exercise #4b

Problem. *(Properties of relations.)* The property *irreflexive* is not the same as being not reflexive. [...] Give an example of a relation on a set that is both reflexive and irreflexive.

Solution to (b). Let R be any relation on the empty set $\emptyset$. Then the conditions of reflexive and irreflexive both hold vacuously. If this seems puzzling, let's recall the truth table for implication.

Original propositions		**New proposition**
P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

We can write the condition for reflexivity as

$$\underbrace{a \in A}_{\text{P}} \implies \underbrace{aRa}_{\text{Q}}$$

and the condition for irreflexivity as

$$\underbrace{a \in A}_{\text{P}} \implies \underbrace{a \not R a}_{\text{Q}}$$

Now if $A = \emptyset$, the hypothesis P is always false, so both statements are tautologies.

Solution to Group Exercise #5

Problem. *(Properties of relations.)* For each of the following relations on the set of all human beings, determine whether the relation is reflexive, irreflexive, symmetric, anti-symmetric, and/or transitive: (a) is-taller-than, (b) has-the-same-parents-as (i.e., same mother and father), (c) is-the-child-of, (d) is-married-to.

Solution.

a. Irreflexive, anti-symmetric (vacuously), transitive.
b. Reflexive, symmetric, transitive.
c. Irreflexive, anti-symmetric (vacuously)
d. Irreflexive, symmetric, transitive (vacuously).