03/31/2025: Expectation

CSCI 246: Discrete Structures

Textbook reference: Sec 34, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Today's Agenda

- Reading quiz (5 mins)
- Mini-lecture ($\approx$ 20 mins)
- Group exercises ($\approx$ 20 mins)

Feedback on Friday's Quizzes

Reading Quiz Scores

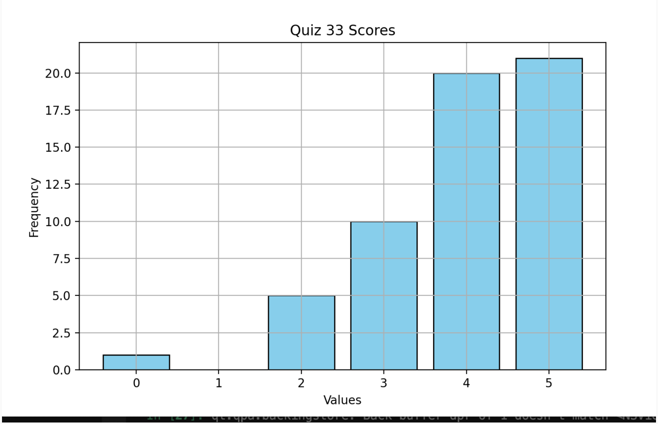


Figure 1: Median Score = 4/5 (80%)

Grading Rubric.

1. 1 point for stating that a *random variable* is a function on the sample space S.
2. 4 points (1 per subpart). Correct answers: True, True, True, True.

Problem Quiz Scores

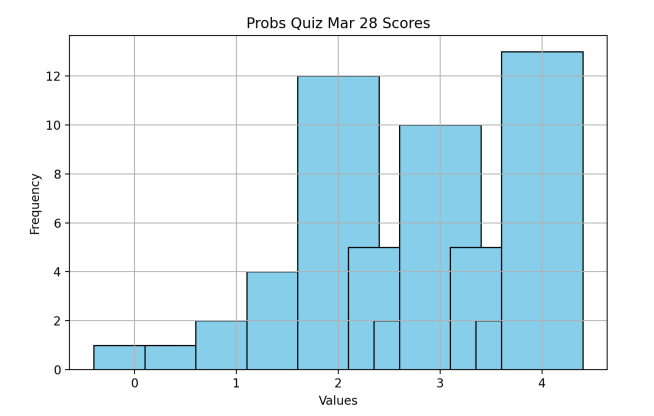


Figure 2: Median Score = 3/4 (75%)

Grading Rubric.

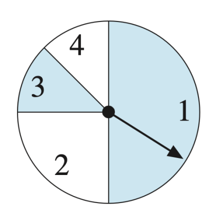
1. 2 points for correctly providing the # of subsets in 2 different ways.
2. 2 points for correctly providing the probability of a full house. (Let's review this.)

Today's quiz

Reading Quiz (Expectation)

Answer the questions below. You do **not** need to simplify your answers!

1. Consider the spinner shown in the figure. Suppose that the likelihood of landing in each region is proportional to the area of the region. Let X be the number that appears on the spinner. Compute the expected value of X.
2. (Extra credit.) Let X be the number that appears on a random toss of a die. Compute the variance of X.



Thoughts on Expectation

Suppose we roll a pair of dice. What is the *expected value* of the sum?

Formally, let X be the sum of the number on the two dice. We must compute $E[X]$.

Computing An Expectation

Table 1: *Possible outcomes when rolling a pair of dice.* Each table entry gives the sum of the values on the two dice.

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Method 1.

Computing An Expectation

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Method 1. Sum over outcomes in the sample space.

$$E[X] = \sum_{s \in S} X(s) P(s)$$

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$$E[X] = \sum_{s \in S} X(s) P(s)$$

Application.

$$\begin{aligned} E[X] &= 1{+}1 \cdot \frac{1}{36} + 1{+}2 \cdot \frac{1}{36} + 2{+}1 \cdot \frac{1}{36} + 3{+}1 \cdot \frac{1}{36} + 2{+}2 \cdot \frac{1}{36} + 1{+}3 \cdot \frac{1}{36} + \ldots + 6{+}6 \cdot \frac{1}{36} \\ &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{36} + 3 \cdot \frac{1}{36} + 4 \cdot \frac{1}{36} + 4 \cdot \frac{1}{36} + 4 \cdot \frac{1}{36} + \ldots + 12 \cdot \frac{1}{36} \end{aligned}$$

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$$E[X] = \sum_{s \in S} X(s)P(s)$$

Application.

$$E[X] = 1{+}1 \cdot \frac{1}{36} + 1{+}2 \cdot \frac{1}{36} + 2{+}1 \cdot \frac{1}{36} + 3{+}1 \cdot \frac{1}{36} + 2{+}2 \cdot \frac{1}{36} + 1{+}3 \cdot \frac{1}{36} + \ldots + 6{+}6 \cdot \frac{1}{36}$$

$$= 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{36} + 3 \cdot \frac{1}{36} + 4 \cdot \frac{1}{36} + 4 \cdot \frac{1}{36} + 4 \cdot \frac{1}{36} + \ldots + 12 \cdot \frac{1}{36}$$

Remark. We must sum over **many** (36) terms.

Computing An Expectation

Table 2: *Possible outcomes when rolling a pair of dice.* Each table entry gives the sum of the values on the two dice.

Dice Values	1	2	3	4	5	6
1	2	3	4	5	6	7
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Method 2.

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Method 2. Sum over possible values of the random variable.

$$E[X] = \sum_{a \in \mathbb{R}} a \, P(X = a)$$

Computing An Expectation

Table 2: *Possible outcomes when rolling a pair of dice.* Each table entry gives the sum of the values on the two dice.

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$$E[X] = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \ldots + 12 \cdot \frac{1}{36}$$

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$$E[X] = \sum_{a \in \mathbb{R}} a \, P(X = a)$$

Application.

$$E[X] = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \ldots + 12 \cdot \frac{1}{36}$$

Remark. We must sum over **fewer** (12) terms.

Computing An Expectation

Table 3: *Possible outcomes when rolling a pair of dice.* Each table entry gives the sum of the values on the two dice.

Dice Values	1	2	3	4	5	6
1	2	3	4	5	6	7
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Method 3.

Computing An Expectation

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Method 3. Use linearity of expectation.

$$E[X] = E[D_1] + E[D_2]$$

where D_n is the value on the n-th die.

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Application. We have

$$E[X] = E[D_1] + E[D_2] = 3.5 + 3.5 = 7.$$

since, by the previous methods, each $E[D_n] = \frac{1}{6} \cdot (1 + 2 + 3 + 4 + 5 + 6) = 3.5$.

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Remark. Here we can sum over **even fewer** (7) terms. (Note: we can only use Method 3 in cases where we can express X as a sum of other RVs).

Summary on computing an expectation

Some ways of computing the expectation are more efficient than others.

Foreshadowing our upcoming section on computational complexity, we can make a more general statement:

Remark: Scalability of different methods for computing an expectation

Let X be the sum of n random variables each from the same sample space S. Then the number of operations needed to compute $E[X]$ is

- Method 1: $\mathcal{O}(|S|^n)$, i.e. **exponential** in n.
- Method 2: $\mathcal{O}(n\,|S|)$, i.e. **linear** in n.
- Method 3: $\mathcal{O}(|S|)$, i.e. **independent** of n.

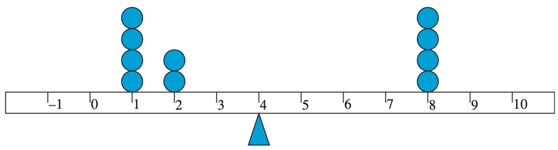
We will investigate computational complexity (and define $\mathcal{O}$) in a couple of class meetings.

Expectation as center of probability mass

Consider a probability space (S, P) where $S = \{1, 2, \ldots, 10\}$ and $P(s) = \frac{1}{10}$ for all outcomes $s \in S$. Define a random variable X as below:

s	$X(s)$	s	$X(s)$
1	1	6	2
2	1	7	8
3	1	8	8
4	1	9	8
5	2	10	8

Now we make a seesaw, placing a weight $P(X = a)$ for each outcome a that the random variable can take on:



What is the expected value?

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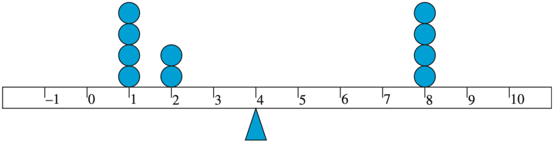
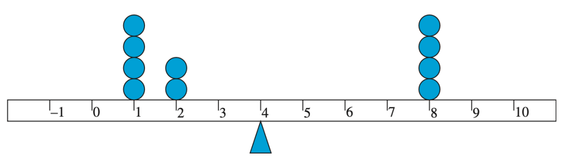


What is the expected value? The **balancing point** ( ) of the seesaw!

Expectation as center of probability mass



To be concrete, the expected value for this problem is given by

$$\begin{aligned} E[X] &= \sum_{a \in \mathbb{R}} a\, P(X = a) \\ &= 1 \times 0.4 + 2 \times 0.2 + 8 \times 0.4 \\ &= 4 \end{aligned}$$

Variance as measure of spread

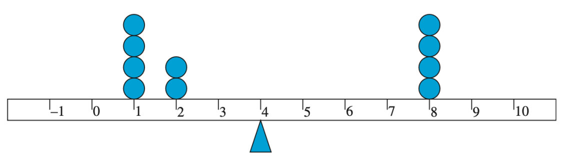
Definition

Let X be a real-valued random variable on a probability space (S, P). Let $\mu \triangleq E[X]$. Then the **variance** of X is

$$\text{Var}(X) = \mathbb{E}\left[(X - \mu)^2\right]$$

Remark

The variance is the **expected squared deviation** from the mean.



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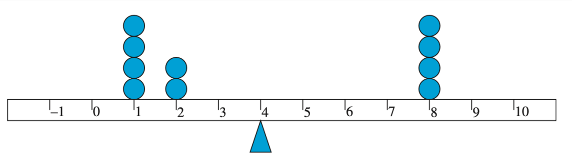
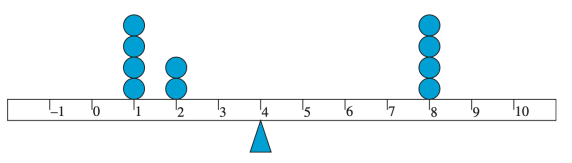


Poll. Can anybody here explain how to compute the variance for this problem?

Expectation vs. variance



Here, the **expected value** $E[X]$ (a.k.a. the mean μ) is given by

$$\begin{aligned} E[X] &= \sum_{a \in \mathbb{R}} a \; P(X = a) \\ &= 1 \times 0.4 + 2 \times 0.2 + 8 \times 0.4 \\ &= 4 \end{aligned}$$

The **variance** $Var(X)$ (i.e. the expected squared deviation) is given by

$$\begin{aligned} Var(X) &= \sum_{a \in \mathbb{R}} (a - \mu)^2 \; P(X = a) \\ &= \text{(1-4)\^{}2} \times 0.4 + \text{(2-4)\^{}2} \times 0.2 + \text{(8-4)\^{}2} \times 0.4 \\ &= 9 \times 0.4 + 4 \times 0.2 + 16 \times 0.4 \\ &= 10.8 \end{aligned}$$

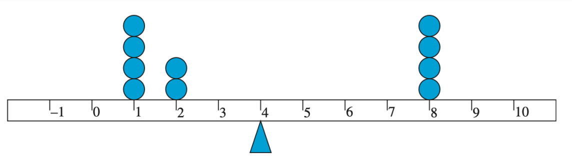
Remark

When computing the variance, we can use

$$\mathrm{Var}(X) = \sum_{s\in S}[X(s) - \mu]^2\ P(s)$$

or

$$= \sum_{a\in\mathbb{R}}(a - \mu)^2\ P(X = a)$$



Remark

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$$\mathrm{Var}(X) = \sum_{s \in S} [X(s) - \mu]^2 \, P(s)$$

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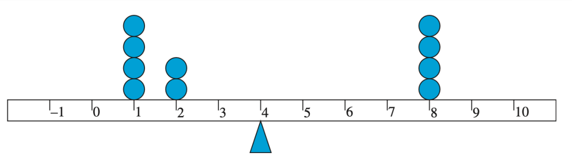

Poll. Can anybody explain how two apply these different methods to this problem?

Three Random Variables

Poll. These three random variables all have the same mean. Which has the largest variance?

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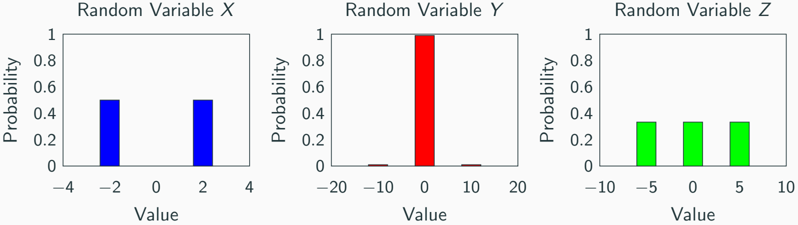

Solution. Z has the largest variance. (See textbook for more info.)

Group exercises

aaron.loomis: 16
adam.wyszynski: 9
alexander.goetz: 8
alexander.knutson: 14
anthony.mann: 4
blake.leone: 10
bridger.voss: 12
caitlin.hermanson: 19
cameron.wittrock: 5
carsten.brooks: 18
carver.wambold: 20
colter.huber: 8
conner.reed1: 7
connor.mizner: 17
connor.yetter: 5
derek.price4: 19
devon.maurer: 1
emmeri.grooms: 7
erik.moore3: 15
ethan.johnson18: 20
evan.barth: 15
evan.schoening: 1
griffin.short: 8
jack.fry: 17
jacob.ketola: 21
jacob.ruiz1: 9
jacob.shepherd1: 14
jada.zorn: 12
jakob.kominsky: 10
james.brubaker: 21
jeremiah.mackey: 17
jett.girard: 11
john.fotheringham: 12
jonas.zeiler: 6
joseph.mergenthaler: 16
joseph.triem: 16
julia.larsen: 3
justice.mosso: 15
kaden.price: 5
lucas.jones6: 3
luka.derry: 11
luke.donaldson1: 11
lynsey.read: 7
mason.barnocky: 4
matthew.nagel: 21
micaylyn.parker: 13
michael.oswald: 14
nolan.scott1: 2
owen.obrien: 3
pendleton.johnston: 20
peter.buckley1: 18
reid.pickert: 2
ryan.barrett2: 19
samuel.hemmen: 13
samuel.mosier: 6
samuel.rollins: 9
sarah.periolat: 2
timothy.true: 4
tristan.nogacki: 13
tyler.broesel: 10
william.elder1: 1
yebin.wallace: 18
zeke.baumann: 6

Group exercises

1. *Simplified stock market.* Suppose there are three kinds of days: `GOOD`, `GREAT`, and `ROTTEN`. The following chart gives the frequency of each of these types of days and the effect on the price of a certain stock that day.

Type of day	Frequency	Change in stock value
`GOOD`	60%	+2
`GREAT`	10%	+5
`ROTTEN`	30%	-4

   Let X_n be the change in the value of the stock after n consecutive days.

   a. Find the expected change in stock price after one day. (That is, find $E[X_1]$.)
   b. Find the variance in stock price after one day. (That is, find $Var[X_1]$.)
   c. Find $E[Y_5]$.
   d. Find $Var[Y_5]$.

2. A pair of 25-sided dice is rolled. Let X be the sum of the two numbers and Y be the product. Find $E[X]$ and $E[Y]$.

3. The term *expected value* can be a bit deceiving. Sometimes it is not what someone might expect! (a) Give an example of a random variable X where $E[X] = 1$ but $P(X = 1) = 0$. (b) Give an example of a random variable X where $E[X] < 0$ but the probability that X is positive is nearly 100%.