02/14/2025: Set operations

CSCI 246: Discrete Structures

Textbook reference: Sec. 12, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Today's Agenda

- Reading and problems quizzes (15 mins)
- Mini-lecture ($\approx$ 15 mins)
- Group exercises ($\approx$ 20 mins)

Reading Quiz (Set Operations)

How many integers in the range 1 to 1000 (inclusive) are divisible by 2 or by 5? Justify your answer. **Hint:** Use Proposition 12.4.

Proposition 12.4 (Scheinerman)

Let A and B be finite sets. Then $|A| + |B| = |A \cup B| + |A \cap B|$.

Problems Quiz (Lists, Factorials, and Intro to Sets)

1. A candy shop is preparing two Valentine's Day gift bags by selecting sweets from a collection of 100 different types of sweets (including lollipops, truffles, gummies, etc.) There are many of each of the 100 types of sweets. The first gift bag should contain 10 candies, each of a different type. The second gift bag should contain 5 candies, each of a different type. A candy type can appear in both bags, but each bag must have only unique types of candy within it. In how many different ways can the shop fill the two gift bags?

2. Prove that the following two sets are equal:

$$E = \{x \in \mathbb{Z} : x \text{ is even}\}, \text{and}$$
$$F = \{x \in \mathbb{Z} : x = a + b, \text{where } a \text{ and } b \text{ are both odd}\}$$

Feedback on Wednesday's Reading Quizzes on Quantifiers

Scores on Reading Quiz (Quantifiers)

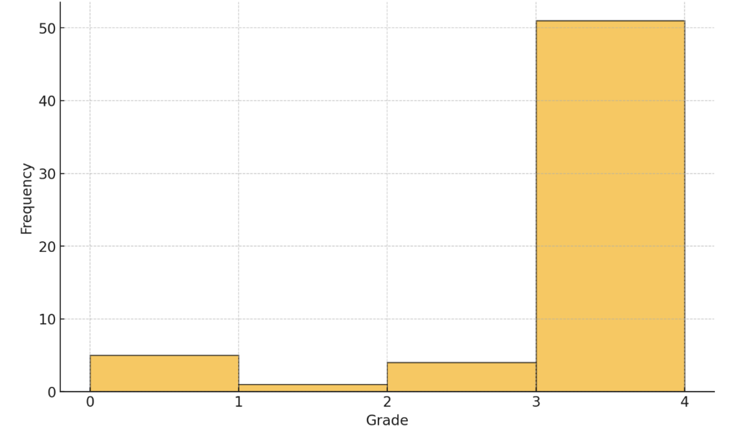

Figure 1: Median = 4.0 (100%)

Rubric: I used the standard 4-point grading scale for proofs. (Exception: Due to the existence of participation grades, I am not giving 1 point just for writing something down.)

Reading Quiz (Quantifiers)

Let $A = \{x \in \mathbb{Z} : 6|x\}$. Prove that $\forall x \in A$, x is even.

Puzzle

Evaluate the (partial) student solution below.

(Partial) Student Solution

$$A = \{x \in \mathbb{Z} : 6|x\} \tag{1}$$

$$\exists y, \forall x,\ x = 6y \tag{2}$$

Reading Quiz (Quantifiers)

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Puzzle

Evaluate the (partial) student solution below.

(Partial) Student Solution

$$A = \{x \in \mathbb{Z} : 6|x\} \tag{1}$$

$$\exists y, \forall x, \ x = 6y \tag{2}$$

Comments

Line 2 in the solution is `False`.

$$\underbrace{\exists}_{\text{there exists}} y, \underbrace{\forall}_{\text{for all}} x, \ x = 6y$$

(No matter what you pick for y, 6 times y can't represent all integers).

Solution Sketch

General Observation

The primary issue (even to students who scored 4/4) was **communication** – that is structuring the mathematical argument clearly.

Solution Using The If-Then Proof Template.

Proposition. Let $A = \{x \in \mathbb{Z} : 6|x\}$. Then $\forall x \in A$, x is even.

Proof.

Annotation	**Main Text**
Convert Prop. to "if-then" form	We show that if $x \in A$, then x is even.
State assumption	Suppose $x \in A$.
Unravel defs.	So $6|x$. That is, there is an integer c such that $x = 6c$.
*** The glue ***	We can write $x = 6c = 2(3c)$.
Unravel defs.	So there is an integer $d (= 3c)$ such that $x = 2d$. In other words, $2|x$.
State conclusion.	So, x is even.

Notes on set operations

Intuition about set operations

We can build intuition about the set properties by creating an application.
For example, one of the distributive properties is:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Suppose we work for Rosauers and we are studying how likely customers are to make a purchase in different parts of the store. Let

- P: The set of Rosauers customers who made a purchase.
- R: The set of Rosauers customers who were shopping for roses.
- C: The set of Rosauers customers who were shopping for candy.

Then

$$(R \cup C) \cap P \underbrace{=}_{\text{commutative}} P \cap (R \cup C) \underbrace{=}_{\text{distributive}} (P \cap R) \cup (P \cap C).$$

This says that the set of Rousauers customers who were shopping for roses or candy and made a purchase is the same as the set of Rosauers customers who were shopping for roses and made a purchase combined with the set of Rosauers customers who were shopping for candy and made a purchase.

Why set operations matter

Question: Why should we care about the the properties of set operations? Aren't they obvious (as the previous slide might suggest)?

Answer: Sometimes it's easier to do things one way than another. Today's reading quiz is a great example.

Reading Quiz (Set Operations)

How many integers in the range 1 to 1000 (inclusive) are divisible by 2 or by 5? Justify your answer.

Sketch of Solution

Let

$$A \triangleq \{x \in \mathbb{Z} : 2|x\}$$
$$B \triangleq \{x \in \mathbb{Z} : 5|x\}$$

We want $|A \cup B|$. That's hard to calculate.

However, we have

$$|A| = 500 \qquad \text{(by skip counting)}$$
$$|B| = 200 \qquad \text{(by skip counting)}$$
$$|A \cap B| = \left|\{x \in \mathbb{Z} : 10|x\}\right| = 100 \qquad \text{(by skip counting)}$$

Now we apply the identity to find

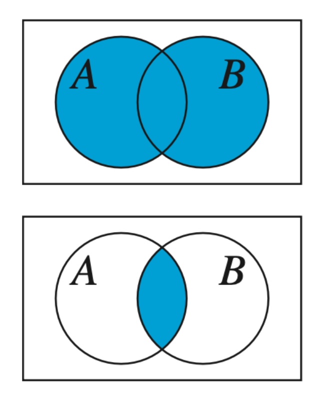
$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 500 + 200 - 100 = 600 \end{aligned}$$

Intuition: Avoid double-counting

Proposition 12.4 (Scheinerman)

Let A and B be finite sets. Then

$$|A| + |B| = |A \cup B| + |A \cap B|.$$



Application: Database management

One application of set theory in computer science is in **database management systems** (DBMS), particularly in **SQL query optimization**.

Example

Imagine a company has two database tables:

- **Customers** (Set C): Contains all registered customers.
- **Orders** (Set O): Contains all customer orders.

A database query like:

```sql
SELECT * FROM Customers
WHERE CustomerID NOT IN (SELECT CustomerID FROM Orders);
```

is using the set difference $C - O$ to find customers who have never placed an order.

Key Set Operations in Databases

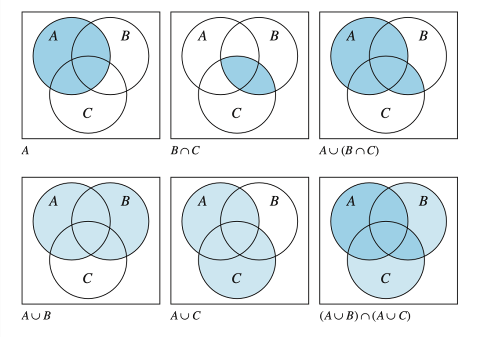
- Union ($A \cup B$) $\rightarrow$ `UNION` in SQL (Combining results from multiple queries)
- Intersection ($A \cap B$) $\rightarrow$ `INTERSECT` in SQL (Finding common data between queries)
- Set Difference ($A - B$) $\rightarrow$ `EXCEPT` or `NOT IN` in SQL (Finding items in one table but not the other)

Q&A on the Group Exercises for Quantifiers

Group 1: ethan.johnson18,joseph.triem,jett.girard
Group 2: carsten.brooks,samuel.rollins,pendleton.johnston
Group 3: william.elder1,aaron.loomis,delaney.rubb
Group 4: owen.obrien,julia.larsen,alexander.goetz
Group 5: evan.schoening,griffin.short,jacob.ruiz1
Group 6: adam.wyszynski,ryan.barrett2,caitlin.hermanson
Group 7: justice.mosso,peyton.trigg,evan.barth
Group 8: colter.huber,yebin.wallace,james.brubaker
Group 9: jonas.zeiler,jeremiah.mackey,sarah.periolat
Group 10: jada.zorn,timothy.true,jakob.kominsky
Group 11: devon.maurer,michael.oswald,mason.barnocky
Group 12: connor.mizner,bridger.voss,carver.wambold
Group 13: matthew.nagel,nolan.scott1,tristan.nogacki
Group 14: zeke.baumann,lucas.jones6,peter.buckley1
Group 15: joseph.mergenthaler,micaylyn.parker,derek.price4
Group 16: connor.yetter,jacob.ketola,blake.leone
Group 17: samuel.hemmen,luka.derry,john.fotheringham
Group 18: jack.fry,conner.reed1,connor.graville
Group 19: jacob.shepherd1,luke.donaldson1,erik.moore3
Group 20: cameron.wittrock,tyler.broesel,lynsey.read
Group 21: samuel.mosier,emmeri.grooms,reid.pickert
Group 22: kaden.price,anthony.mann,alexander.knutson

Group Exercises: Set Operations

1. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7\}$. Please compute (a) $A \cup B$, (b) $A \cap B$, (c) $A - B$, (d) $B - A$, (e) $A \triangle B$, (f) $A \times B$, and (g) $B \times A$.
2. The distributive properties for sets are
   $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \qquad \text{and} \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
   The Venn diagram below illustrates the first distributive property. Make a Venn diagram to illustrate the second distributive property.



3. Prove that the first identity in #2 holds. Hint: You may want to refer back to Theorem 7.2.

Solution to Group Exercise #1 (a-e)

Problem. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7\}$. Please compute (a.) $A \cup B$, (b.) $A \cap B$, (c.) $A - B$, (d.) $B - A$, and (e.) $A \triangle B$.

Solution.

(a.) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$.

(b.) $A \cap B = \{4, 5\}$.

(c.) $A - B = \{1, 2, 3\}$.

(d.) $B - A = \{6, 7\}$.

(e.) $A \triangle B = \{1, 2, 3, 6, 7\}$. [Note: $A \triangle B = (A - B) \cup (B - A)$.]

Solution to Group Exercise #1 (f-g)

Problem. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7\}$. Please compute (f.) $A \times B$ and (g.) $B \times A$.
Solution.

(f.)

$$\begin{aligned} A \times B = \Big\{ &(1,4),(1,5),(1,6),(1,7), \\ &(2,4),(2,5),(2,6),(2,7), \\ &(3,4),(3,5),(3,6),(3,7), \\ &(4,4),(4,5),(4,6),(4,7), \\ &(5,4),(5,5),(5,6),(5,7) \Big\} \end{aligned}$$

(g.)

$$\begin{aligned} B \times A = \Big\{ &(4,1),(4,2),(4,3),(4,4),(4,5) \\ &(5,1),(5,2),(5,3),(5,4),(5,5) \\ &(6,1),(6,2),(6,3),(6,4),(6,5) \\ &(7,1),(7,2),(7,3),(7,4),(7,5) \Big\} \end{aligned}$$

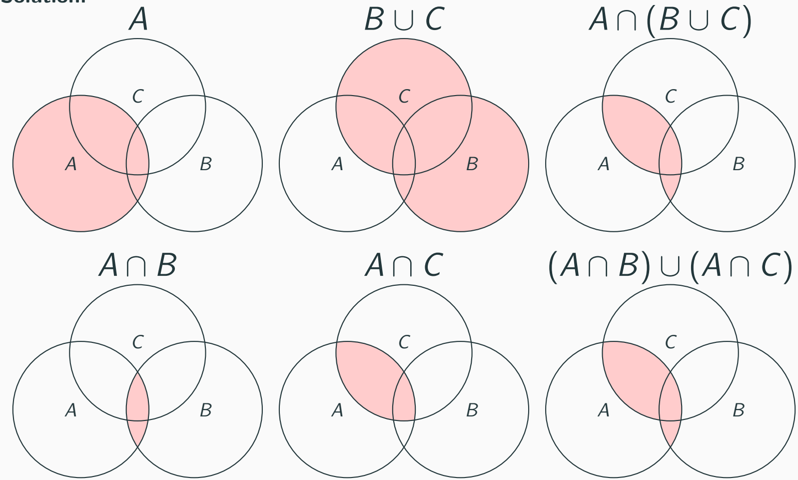
Solution to Group Exercise #2

Problem. The distributive properties for sets are

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \qquad \text{and} \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Make a Venn diagram to illustrate the second distributive property.

Solution.



Solution to Group Exercise #3

Problem. The distributive properties for sets are

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \qquad \text{and} \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Prove that the first identity above holds. Hint: You may want to refer back to Theorem 7.2.

Solution.

$$\begin{aligned}
A \cup (B \cap C) &= \{x : x \in A \text{ or } x \in B \cap C\} && \text{(by def. union } \cup\text{)} \\
&= \{x : x \in A \text{ or } (x \in B \text{ and } x \in C)\} && \text{(by def. intersection } \cap\text{)} \\
&= \{x : (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\} && \text{(by distributive prop.; Thm 7.2)} \\
&= \{x : (x \in A \cup B) \text{ and } (x \in A \cup C)\} && \text{(by def. union } \cup\text{)} \\
&= (A \cup B) \cap (A \cup C) && \text{(by def. intersection } \cap\text{)}
\end{aligned}$$

Remark. We used the distributive property for *Boolean Algebra* (from Theorem 7.2) to prove the distributive property for *sets*.