# 03/10/2025: Inclusion/Exclusion

CSCI 246: Discrete Structures

Textbook reference: Sec 19, Scheinerman

### Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

### Today's Agenda

- Reading quiz (5 mins)
- Mini-lecture ($\approx$ 20 mins)
- Group exercises ($\approx$ 20 mins)

# Reading Quiz

## Reading Quiz (Inclusion-Exclusion)

1. Replace each ? with a $+$ or - to make the equation correct.

$$|A \cup B \cup C| = |A| \; ? \; |B| \; ? \; |C| \; ? \; |A \cap B| \; ? \; |A \cap C| \; ? \; |B \cap C| \; ? \; |A \cap B \cap C|$$

2. (True or False.) Consider the length-$k$ lists whose elements are chosen from the set $\{1, 2, \ldots, n\}$. The number of lists which use all of the elements in $\{1, 2, \ldots, n\}$ at least once is $n^k$.
3. (True or False.) Consider the length-$n$ lists whose elements are chosen from the set $\{1, 2, \ldots, n\}$ without repetition. A list is called a *derangement* if the number $j$ does not occupy position $j$ in the list for any $j = 1, 2, \ldots, n$. The number of derangements is $n!$.

# Feedback on Friday's Quiz

# Problems Quiz (Equiv. Relations, Partitions, Functions)

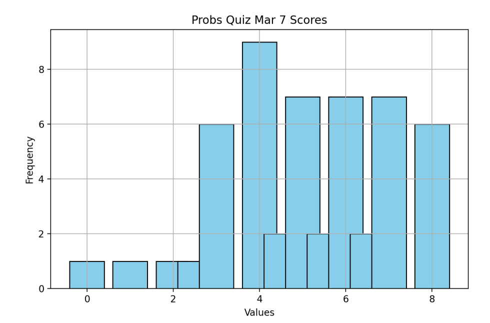


**Figure 1:** Median Score = 5/8 (62.5%)

**Notes.**

1. 3 points. Most people got full credit.
2. 2 points. Many people struggled here.
3. 3 points. Most people struggled on this one. A common mistake was proving that $R$ was an equivalence relation.

# Overview of inclusion-exclusion

## The ski lift operator is absolutely vibing

Out of 40 people ski lift operators at Bridger Bowl,

- 18 like Led Zeppelin;
- 16 like the Grateful Dead;
- 12 like Taylor Swift;
- 7 like Zeppelin and the Dead;
- 5 like the Zeppelin and Taylor;
- 3 like the Dead and Taylor;
- 2 like all three

How many lift operators don't like any of these performers?

## Poll

Find the problems with the following claims:

- **Claim.** The answer is # operators - (# fans Zep + # fans Dead + # fans Swift) $= 40 - (18+16+12)$.

**Solution**: $40 - (18+16+12) + (7+5+3) - 2 = 7$.

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- **Claim.** The answer is # operators - (# fans Zep + # fans Dead + # fans Swift) =40 -(18+16+12). **Analysis.** This gives -6. So clearly something is wrong. But what? We subtracted some people twice – e.g. those that like both Zeppelin and the Dead.
- **Claim.** We have to add back in the people who like two performers. So the answer is 40 -(18+16+12) + (7+5+3).

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**Solution**: 40 - (18+16+12) + (7+5+3) - 2 = 7.

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**Solution**: 40 - (18+16+12) + (7+5+3) - 2 = 7.

## Formulaic solution

Let $Z$ be the set of lift operators who like Zeppelin, $D$ be the set of lift operators who like the Dead; and $T$ be the set of lift operators who like Taylor swift. Then the number of lift operators who like at least one of those performers is given by

$$\begin{aligned}|Z \cup D \cup T| = {} & |Z| + |D| + |T| \\ & - |Z \cap D| - |Z \cap T| - |D \cap T| \\ & + |Z \cap D \cap T|\end{aligned}$$

Substituting in known values, we find 33 operators like at least one of those performers, and so 7 operators don't like any of them.

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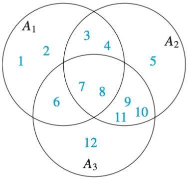
## General formula

**Theorem 19.1** **(Inclusion-Exclusion)** Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = {} & |A_1| + |A_2| + \cdots + |A_n| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\ & - \cdots + \cdots\cdots\cdots \\ & \pm |A_1 \cap A_2 \cap \cdots \cap A_n|.\end{aligned}$$

# Why does inclusion-exclusion work?




| El't | $A_1$ | $A_2$ | $A_3$ | $A_1 \cap A_2$ | $A_1 \cap A_3$ | $A_2 \cap A_3$ | $A_1 \cap A_2 \cap A_3$ |
|---|---|---|---|---|---|---|---|
| 1 | + | | | | | | |
| 2 | + | | | | | | |
| 3 | + | + | | − | | | |
| 4 | + | + | | − | | | |
| 5 | | + | | | | | |
| 6 | + | | + | | − | | |
| 7 | + | + | + | − | − | − | + |
| 8 | + | + | + | − | − | − | + |
| 9 | | + | + | | | − | |
| 10 | | + | + | | | − | |
| 11 | | + | + | | | − | |
| 12 | | | + | | | | |

Let us form a chart. The rows are elements and the columns are terms in the formula.

- If an item is not in a set, the entry is blank.
- If the item is a member of the set,
  - We put a a $+$ sign if the column label is an intersection of an odd number of sets.
  - We put a $-$ sign if the column label is an intersection of an even number of sets.

**Poll**

We want to count each element once. So each row should contain what?

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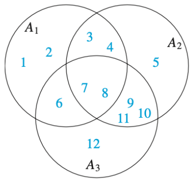


| El't | $A_1$ | $A_2$ | $A_3$ | $A_1 \cap A_2$ | $A_1 \cap A_3$ | $A_2 \cap A_3$ | $A_1 \cap A_2 \cap A_3$ |
|---|---|---|---|---|---|---|---|
| 1 | + | | | | | | |
| 2 | + | | | | | | |
| 3 | + | + | | − | | | |
| 4 | + | + | | − | | | |
| 5 | | + | | | | | |
| 6 | + | | + | | − | | |
| 7 | + | + | + | − | − | − | + |
| 8 | + | + | + | − | − | − | + |
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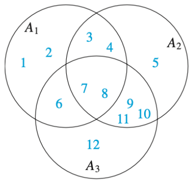
**Poll**

We want to count each element once. So each row should contain what?
Exactly one more $+$ than $-$.

# Why does inclusion-exclusion work?

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|---|---|---|---|---|---|---|---|
| 1 | + | | | | | | |
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| 7 | + | + | + | − | − | − | + |
| 8 | + | + | + | − | − | − | + |
| 9 | | + | + | | | − | |
| 10 | | + | + | | | − | |
| 11 | | + | + | | | − | |
| 12 | | | + | | | | |

Each element $x$ belongs to exactly $k$ sets. And

$$\text{the number of } +\text{s is } \binom{k}{1} + \binom{k}{3} + \binom{k}{5} + \dots$$

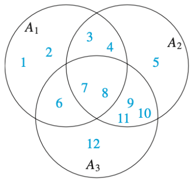
$$\text{the number of } -\text{s is } \binom{k}{2} + \binom{k}{4} + \binom{k}{6} + \dots$$

If we sum up these (signed) binomial coefficients, we always get exactly 1!

**Exercise**

Verify that this is true with three different elements.

# Why does inclusion-exclusion work?

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|---|---|---|---|---|---|---|---|
| 1 | + | | | | | | |
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| 3 | + | + | | − | | | |
| 4 | + | + | | − | | | |
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Why does this property always hold?

## Proposition (Scheinerman Exercise 17.15)

For any integer $n > 0$,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots \pm \binom{n}{n} = 0$$

**Proposition (Scheinerman Exercise 17.15)**

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**Remark**

Moving all the negative terms over to the right-hand side gives

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

In other words, *the number of subsets of an n-set with an even number of elements is the same as the number of subsets with an odd number of elements*

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## Group activity (5 minutes)

Explain why this formula holds.

**Hint:** Revisit some of theorems and propositions from Sec. 17 (Binomial Coefficients).

# A justification via Pascal's Triangle

$$\binom{0}{0}$$

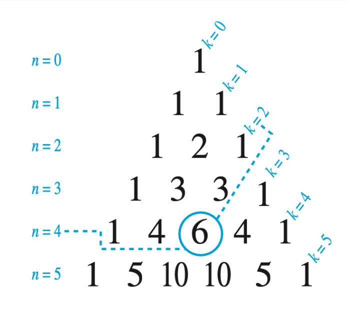
$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$




## Remark

The intermediate number in any row is formed by adding the two numbers just to its left and just to its right in the previous row.

## Pascal's Identity

Let $k$ and $n$ be integers with $0 < k < n$. Then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

# Group exercises

aaron.loomis: 12
adam.wyszynski: 17
alexander.goetz: 8
alexander.knutson: 14
anthony.mann: 12
blake.leone: 1
bridger.voss: 10
caitlin.hermanson: 13
cameron.wittrock: 13
carsten.brooks: 15
carver.wambold: 17
colter.huber: 7
conner.reed1: 1
connor.mizner: 10
connor.yetter: 11
derek.price4: 18
devon.maurer: 3
emmeri.grooms: 7
erik.moore3: 9
ethan.johnson18: 4
evan.barth: 7
evan.schoening: 19
griffin.short: 20
jack.fry: 14
jacob.ketola: 13
jacob.ruiz1: 2
jacob.shepherd1: 21
jada.zorn: 4
jakob.kominsky: 8
james.brubaker: 2
jeremiah.mackey: 2
jett.girard: 16
john.fotheringham: 5
jonas.zeiler: 18
joseph.mergenthaler: 21
joseph.triem: 19
julia.larsen: 12
justice.mosso: 21
kaden.price: 6
lucas.jones6: 1
luka.derry: 14
luke.donaldson1: 15
lynsey.read: 17
mason.barnocky: 3
matthew.nagel: 4
micaylyn.parker: 5
michael.oswald: 20
nolan.scott1: 6
owen.obrien: 8
pendleton.johnston: 5
peter.buckley1: 3
reid.pickert: 9
ryan.barrett2: 11
samuel.hemmen: 18
samuel.mosier: 11
samuel.rollins: 20
sarah.periolat: 16
timothy.true: 16
tristan.nogacki: 19
tyler.broesel: 10
william.elder1: 6
yebin.wallace: 15
zeke.baumann: 9

# Group exercises

1. A professor in a discrete mathematics class passes out a form asking students to check all the math and computer science courses they have recently taken. She found that, out of a total of 50 students in the class,

   30 took precalculus;
   18 took calculus;
   26 took Python;
   9 took both precalculus and calculus
   16 took both precalculus and Python;
   8 took both calculus and Python;
   47 took at least one of the three courses;

   a. How many students did not take any of the three courses?
   b. How many students took all three courses?
   c. How many students took precalculus and calculus but not Python? How many students took precalculus but neither calculus not Python?

2. Of the integers between 1 and 1,000,000 (inclusive), how many are *not* divisible by 2, 3, or 5?

3. The squares of a $4 \times 4$ checkerboard are colored black or white. Use inclusion-exclusion to find the number of ways the checkerboard can be colored so that no row is entirely one color.

# Solution to group exercise #1

a. *How many students did not take any of the three courses?* Since 50 students are in the class and 47 took at least one course, the number of students who didn't take any course is

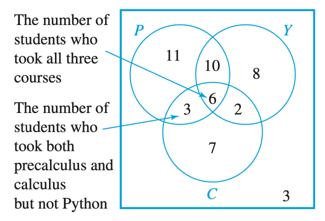
$$50 - 47 = 3.$$

b. *How many students took all three courses?* Let $P$ be the set of students who took precalculus, $C$ be the the set of students who took calculus, and $Y$ be the set of students who took Python. Then

$$|P \cup C \cup Y| = |P| + |C| + |Y| - |P \cap C| - |P \cap Y| - |C \cap Y| + |P \cap C \cap Y| \qquad \text{(Incl.-Excl.)}$$
$$\implies 47 = 30 + 18 + 26 - 9 - 16 - 8 + |P \cap C \cap Y| \qquad \text{(Substitution)}$$

Solving for $|P \cap C \cap Y|$ gives $|P \cap C \cap Y| = 6$.

c. *How many students took precalculus and calculus but not Python?* We fill in the diagram below (called a Venn diagram). It is easiest to start in the middle and work outward.



# Solution to group exercise #2

**Problem.** Of the integers between 1 and 1,000,000 (inclusive), how many are *not* divisible by 2, 3, or 5?

**Solution.** Let $S = \{x \in \mathbb{Z} : 1 \leq x \leq 1,000,000\}$. Let $A$ be the subset of numbers in $S$ that are divisible by 2, $B$ be the subset of numbers in $S$ that are divisible by 3, and $C$ be the subset of numbers in $S$ that are divisible by 5. We want to find $|S| - |A \cup B \cup C| = 1,000,000 - |A \cup B \cup C|$. By inclusion/exclusion,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

By skip-counting, we calculate each set size as

$$|A| = \left\lfloor \frac{1,000,000}{2} \right\rfloor = 500,000 \qquad |A \cap B| = \left\lfloor \frac{1,000,000}{6} \right\rfloor = 166,666$$

$$|B| = \left\lfloor \frac{1,000,000}{3} \right\rfloor = 333,333 \qquad |A \cap C| = \left\lfloor \frac{1,000,000}{10} \right\rfloor = 100,000$$

$$|C| = \left\lfloor \frac{1,000,000}{5} \right\rfloor = 200,000 \qquad |B \cap C| = \left\lfloor \frac{1,000,000}{15} \right\rfloor = 66,666$$

$$|A \cap B \cap C| = \left\lfloor \frac{1,000,000}{30} \right\rfloor = 33,333$$

where $\lfloor \cdot \rfloor$ is the floor operator:

$$\lfloor x \rfloor = \text{the greatest integer } n \text{ such that } n \leq x$$

Substituting into the inclusion/exclusion formula, we find $|A \cup B \cup C| = 733,334$, and so the solution is $1,000,000 - 733,334 = 266,666$.

# Solution to group exercise #3

**Problem.** The squares of a $4 \times 4$ checkerboard are colored black or white. Use inclusion-exclusion to find the number of ways the checkerboard can be colored so that no row is entirely one color.

**Solution.** Let $B_i$ be the set of colorings in which row $i$ is entirely of one color. The set of "bad" colorings is $B_1 \cup \cdots \cup B_4$. The number of "good" colorings is $2^{16} - |B_1 \cup \cdots \cup B_4|$. Now by inclusion-exclusion,

$$|B_1 \cup \cdots \cup B_4| = \sum_i |B_i| - \sum_{i<j} |B_i \cap B_j| + \sum_{i<j<k} |B_i \cap B_j \cap B_k| - |B_1 \cap B_2 \cap B_3 \cap B_4|.$$

$$\begin{aligned} |B_1 \cup \cdots \cup B_4| &= \binom{4}{1} \cdot 2^1 \cdot 2^{12} - \binom{4}{2} \cdot 2^2 \cdot 2^8 \\ &\quad + \binom{4}{3} \cdot 2^3 \cdot 2^4 - \binom{4}{4} \cdot 2^4 \cdot 2^0 \end{aligned}$$

Each summand above is the product of three terms. The first term gives the number of subsets in the intersection, the second term gives the number of ways to assign a uniform color to all squares in each of the designated rows, and the third term gives the number of ways to assign colors to squares in other rows.

**Remark.** In simplified form, the solution is $14^4$, as can be verified either with a calculator or via an application of the Binomial Theorem [interpreting the right-hand-side of the equation above as the binomial expansion of $(16-2)^4$.]