04/11/2025: Subgraphs

CSCI 246: Discrete Structures

Textbook reference: Sec 48, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Today's Agenda

- Problems and reading quiz (15 mins)
- Mini-lecture ($\approx$ 10 mins)
- Group exercises ($\approx$ 20 mins)

Feedback on Wednesday's Quiz

Reading Quiz Scores

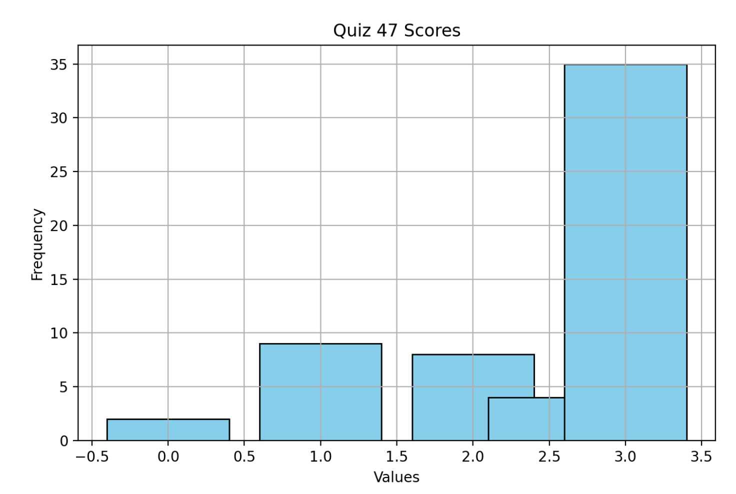


Figure 1: Median Score = 3/3 (100%)

Today's quiz

Problems Quiz (recurrence, big O notation, algorithm efficiency)

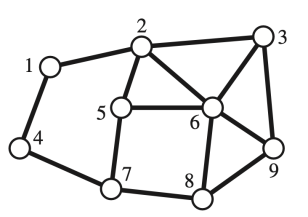
1. Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$ with initial conditions $a_0 = 5$ and $a_1 = 1$ to give an explicit formula for a_n.
2. Let $a_n = 3n^2 + 7$. Prove that $a_n = \Theta(n^2)$.

Reference material about second-order recurrences

To solve a recurrence of the form $a_n = s_1 a_{n-1} + s_2 a_{n-2}$, solve the quadratic formula $x^2 - s_1 x - s_2 = 0$ to find roots r_1 and r_2. If $r_1 \neq r_2$, then $a_n = c_1 r_1^n + c_2 r_2^n$. If $r_1 = r_2 \triangleq r$, then $a_n = c_1 r^n + c_2 n r^n$. Then find c_1 and c_2.

Reading Quiz (Subgraphs)

Name one clique and one independent set from the graph below.



Thoughts On Subgraphs

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Poll. How would you describe a **subgraph** in words?

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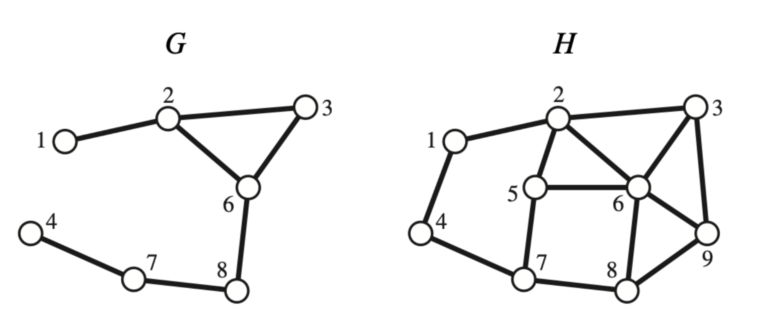
Example: G is a subgraph of H



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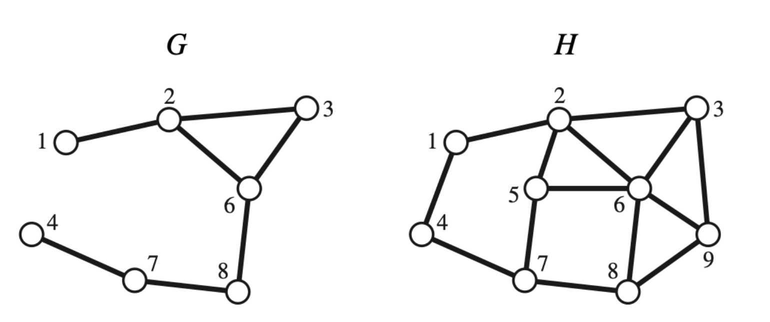

Solution. Possibly remove some vertices (and edges that touch them, so that you still have a graph). Then possibly remove some additional edges.

Induced subgraphs

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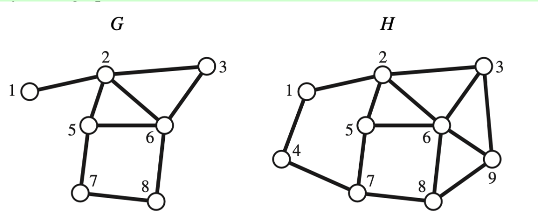

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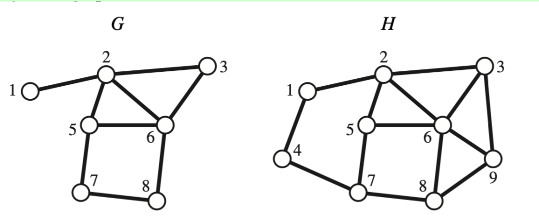

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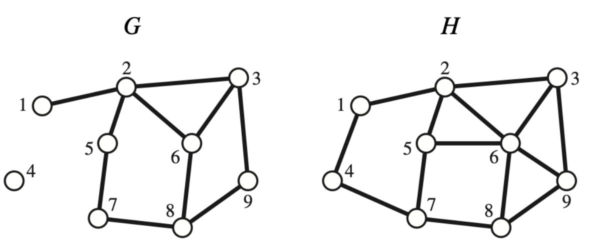

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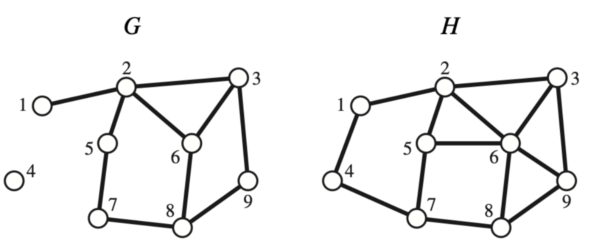

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Group exercises

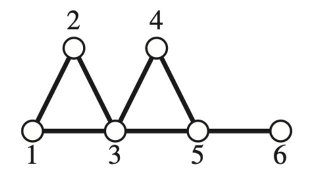
aaron.loomis: 18
adam.wyszynski: 4
alexander.knutson: 2
anthony.mann: 3
blake.leone: 8
bridger.voss: 5
caitlin.hermanson: 3
cameron.wittrock: 20
carsten.brooks: 5
carver.wambold: 10
colter.huber: 6
conner.reed1: 12
connor.mizner: 11
connor.yetter: 20
derek.price4: 6
devon.maurer: 8
emmeri.grooms: 1
erik.moore3: 12
ethan.johnson18: 10
evan.barth: 4
evan.schoening: 19

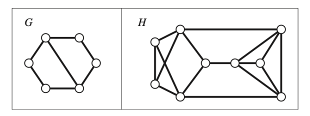
griffin.short: 9
jack.fry: 8
jacob.ketola: 6
jacob.shepherd1: 14
jada.zorn: 1
jakob.kominsky: 14
james.brubaker: 19
jeremiah.mackey: 14
jett.girard: 16
john.fotheringham: 4
jonas.zeiler: 1
joseph.mergenthaler: 15
joseph.triem: 9
julia.larsen: 13
justice.mosso: 18
kaden.price: 11
lucas.jones6: 2
luka.derry: 16
luke.donaldson1: 9

lynsey.read: 16
mason.barnocky: 20
matthew.nagel: 2
micaylyn.parker: 7
michael.oswald: 19
nolan.scott1: 13
owen.obrien: 18
pendleton.johnston: 13
peter.buckley1: 17
reid.pickert: 3
ryan.barrett2: 15
samuel.hemmen: 10
samuel.mosier: 11
samuel.rollins: 16
sarah.periolat: 5
timothy.true: 17
tristan.nogacki: 12
tyler.broesel: 7
william.elder1: 15
yebin.wallace: 7
zeke.baumann: 17

Group exercises

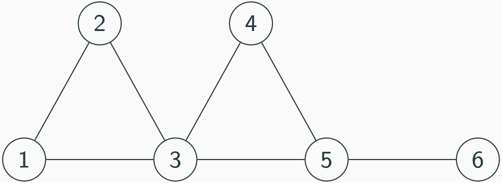
1. Let G be the graph in the top figure. Draw pictures of the following subgraphs: (a) $G - 1$, (b) $G - \{5, 6\}$, (c) $G[\{3, 4, 6\}]$, (d) $G[\{2, 4, 6\}]$.
2. Which of the various properties of relations does the is-a-subgraph-of relation exhibit? Is it reflexive? Symmetric? Transitive?
3. Let G be a complete graph with n vertices. (a) How many spanning subgraphs does G have? (b) How many induced subgraphs does G have?
4. Let G and H be the two graphs in the bottom figure. Please find $\alpha(G), \omega(G), \alpha(H), \omega(H)$. (Recall that $\alpha(\cdot)$ is the size of a largest independent set and $\omega(\cdot)$ is the size of a largest clique.)



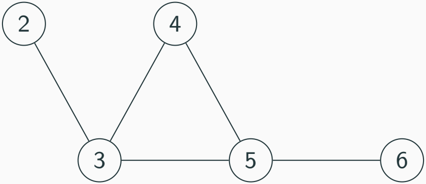


Solution to group exercise #1a

Problem. Let G be the graph in the figure below. Draw a picture of the subgraph $G - 1$.

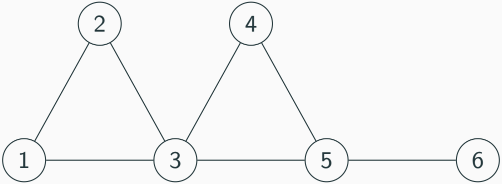


Solution.

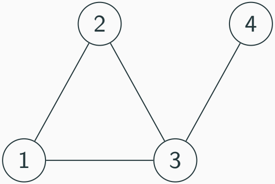


Solution to group exercise #1b

Problem. Let G be the graph in the figure below. Draw a picture of the subgraph $G - \{5, 6\}$.

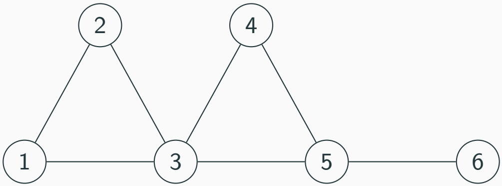


Solution.

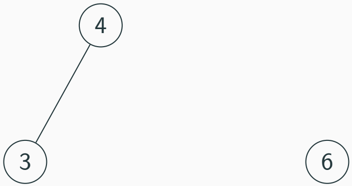


Solution to group exercise #1c

Problem. Let G be the graph in the figure below. Draw a picture of the subgraph $G[\{3, 4, 6\}]$.

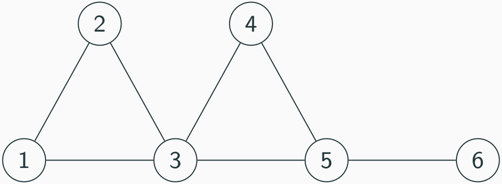


Solution.

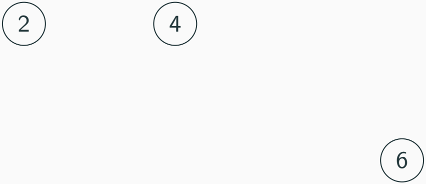


Solution to group exercise #1d

Problem. Let G be the graph in the figure below. Draw a picture of the subgraph $G[\{2, 4, 6\}]$.



Solution.



Solution to group exercise #2

Problem. Which of the various properties of relations does the is-a-subgraph-of relation exhibit? Is it reflexive? Symmetric? Transitive?

Solution.

- **Reflexive?** Yes, $G = (V, E)$ is a subgraph of itself, since $V \subseteq V$ and $E \subseteq E$.
- **Symmetric?** No. Let $G = (V_G, E_G)$ be a subgraph of $H = (V_H, E_H)$ where either $V_G \subsetneq V_H$ or $E_G \subsetneq E_H$. In the first case, $V_H \subseteq V_G$ fails, and in the second case $E_H \subseteq E_G$ fails. Either way, G is not a subgraph of H.
- **Transitive?** Yes. Let $F = (V_F, E_F)$ be a subgraph of $G = (V_G, E_G)$ and G be a subgraph of $H = (V_H, E_H)$. Then $V_F \subseteq V_G \subseteq V_H$ and $E_F \subseteq E_G \subseteq E_H$ by definition of subgraph (and transitivity of the subset operation). In particular, $V_F \subseteq V_H$ and $E_F \subseteq E_H$. Hence F is a subgraph of H.

Solution to group exercise #3

Problem. Let G be a complete graph with n vertices. (a) How many spanning subgraphs does G have? (b) How many induced subgraphs does G have?

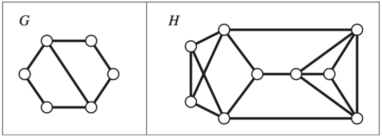
Solution.

(a). The solution is $2^{\binom{n}{2}}$. A complete graph with n vertices has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges (since there are $\binom{n}{2}$ ways to choose pairs from a set of n items). A spanning subgraph keeps all the original vertices and deletes some number of the edges (possibly none, possibly all). In other words, for each edge, we make a free binary decision to keep or delete the edge. Thus, there are $2^{\binom{n}{2}}$ possible spanning subgraphs. (Note that the conclusion can be justified via Scheinerman Theorem 8.6, if we imagine forming a list of length $\binom{n}{2}$, where each element of the list is chosen from a pool of 2 choices.)

(b). There are 2^n different ways to form subsets of n vertices. Each subset determines an induced subgraph (since an induced subgraph is determined by keeping a subset of vertices, and then removing exactly the edges that touch at least one vertex that has been discarded.) Hence, there are 2^n possible induced subgraphs of G. (Note that this argument applies to any graph G with n vertices, not just complete graphs.)

Solution to group exercise #4

Problem. Let G and H be the two graphs in the figure below. Please find $\alpha(G), \omega(G), \alpha(H), \omega(H)$. (Recall that $\alpha(\cdot)$ is the size of a largest independent set and $\omega(\cdot)$ is the size of a largest clique.)



Solution.

$$\omega(G) = 2, \quad \alpha(G) = 3, \quad \omega(H) = 3\,?, \quad \alpha(H) = ???$$

The main point of this exercise is that it's annoying to try to compute these by brute force as the number of vertices n grows. For example, to find a brute force solution for $\alpha(\cdot)$, you would need to try all subsets of vertices and check which are independent. However, this procedure has time complexity $O(2^n)$, so it is only practical for small graphs. Further study of graph theory would introduce algorithms to compute these quantities.