04/09/2025: Graph Theory Fundamentals

CSCI 246: Discrete Structures

Textbook reference: Sec 47, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Today's Agenda

- Reading quiz (5 mins)
- Review problems quizzes (15 mins)
- Mini-lecture ($\approx$ 10 mins)
- Group exercises ($\approx$ 15 mins)

Feedback on Monday's Quiz

Reading Quiz Scores

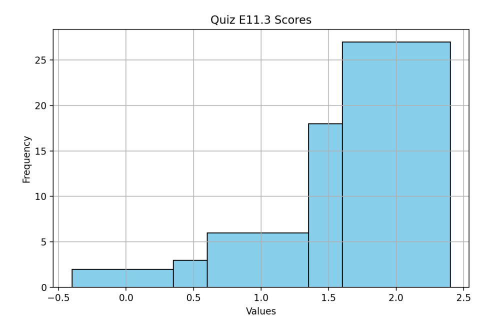


Figure 1: Median Score = 1.75/2 (87.5%)

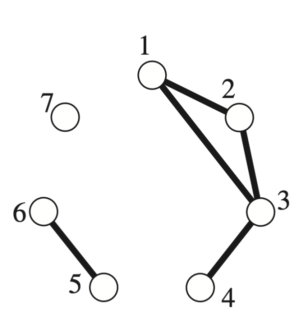
Grading Rubric:

1. (1 point) Needed to give reasonable answer to number of elementary operations (for the WHOLE algorithm snippet).
2. (1 point) Stating the order (ideally as Big Theta, but Big O was accepted)

Today’s reading quiz

Reading Quiz (Graph Theory Fundamentals)

1. What is the degree of vertex 1 in the graph below?



2. Let $G = (V, E)$. The sum of the degree of vertices in G is *how many* times the number of edges? That is, if we write

$$\sum_{v \in V} d(v) = C|E|,$$

what is C?

3. Give a brief explanation for your answer to number 2.

Review for Friday's Problems Quiz

Thoughts On Graph Theory Fundamentals

Graphs as abstractions: The Bridge of Königsberg.

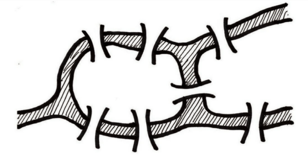


Figure 2: Bridge of Königsberg.

Claim. There was a pub on the center island, with challenges and late-night attempts to "walk the bridges," to make a tour of the town crossing every bridge exactly once. Despite the accompanying boasts, rarely reproducible in the sober morning, this was a difficult task. Most who attempted found that they had missed a bridge or that they crossed a bridge twice.

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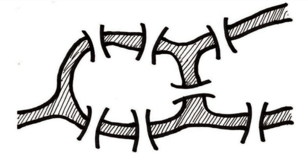

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Question. Can you find a tour of the town that traverses each of the seven bridges exactly once?

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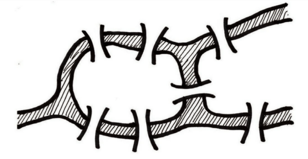

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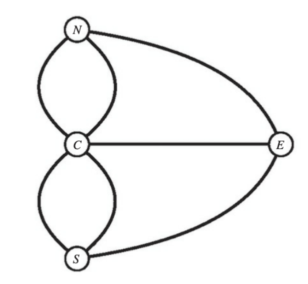

Figure 3: Graph representation

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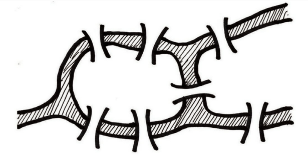

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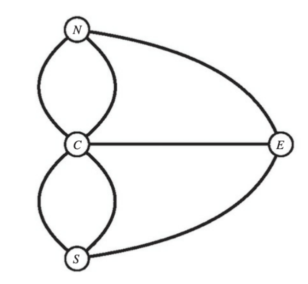

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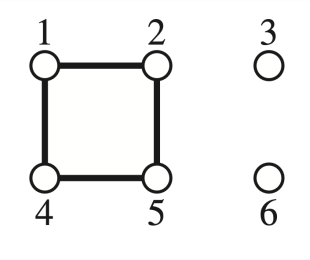
Click here for short video.

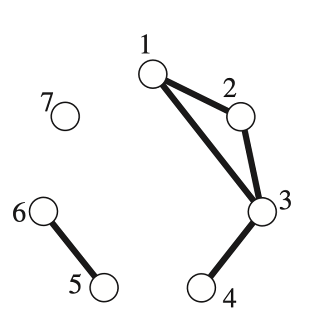
Group exercises

aaron.loomis: 14
adam.wyszynski: 9
alexander.goetz: 11
alexander.knutson: 7
anthony.mann: 1
blake.leone: 20
bridger.voss: 20
caitlin.hermanson: 3
cameron.wittrock: 6
carsten.brooks: 2
carver.wambold: 17
colter.huber: 2
conner.reed1: 15
connor.mizner: 17
connor.yetter: 4
derek.price4: 18
devon.maurer: 19
emmeri.grooms: 7
erik.moore3: 8
ethan.johnson18: 13
evan.barth: 10
evan.schoening: 15
griffin.short: 14
jack.fry: 10
jacob.ketola: 12
jacob.shepherd1: 16
jada.zorn: 13
jakob.kominsky: 11
james.brubaker: 14
jeremiah.mackey: 19
jett.girard: 3
john.fotheringham: 3
jonas.zeiler: 21
joseph.mergenthaler: 10
joseph.triem: 8
julia.larsen: 8
justice.mosso: 6
kaden.price: 5
lucas.jones6: 9
luka.derry: 12
luke.donaldson1: 4
lynsey.read: 13
mason.barnocky: 11
matthew.nagel: 6
micaylyn.parker: 1
michael.oswald: 21
nolan.scott1: 5
owen.obrien: 7
pendleton.johnston: 1
peter.buckley1: 9
reid.pickert: 19
ryan.barrett2: 18
samuel.hemmen: 2
samuel.mosier: 16
samuel.rollins: 5
sarah.periolat: 4
timothy.true: 12
tristan.nogacki: 20
tyler.broesel: 18
william.elder1: 16
yebin.wallace: 17
zeke.baumann: 15

Group exercises

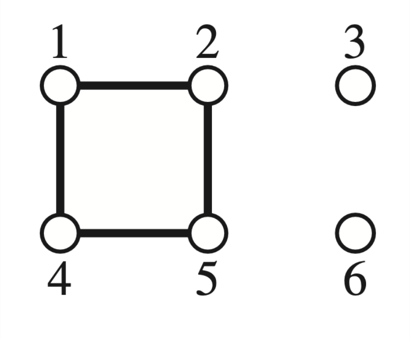
1. Write the graph in the top figure as a pair of sets (V, E).
2. Draw a picture of the graph below $\left(\{a, b, c, d, e\}, \left\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, e\}, \{c, d\}\right\}\right)$.
3. Construct a graph for which the is-adjacent relation, $\sim$, is transitive.
4. How many edges are in K_n, a complete graph with n vertices?
5. How many different graphs can be formed with vertex set $V = \{1, 2, 3, \ldots, n\}$?
6. Prove that in every graph, the number of vertices with odd degree is even. (For example, the graph from the reading quiz, reproduced in the bottom figure, has exactly four vertices of odd degree.)





Solution to group exercise #1

Problem. Write the graph below as a pair of sets (V, E).



Solution.

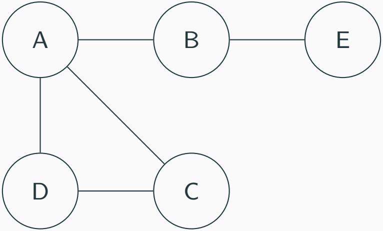
$$\left(\{1,2,3,4,5,6\}, \Big\{\{1,2\},\{2,5\},\{5,4\},\{1,4\}\Big\}\right)$$

Solution to group exercise #2

Problem. Draw a picture of the graph below

$$\left(\{a, b, c, d, e\},\ \left\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, e\}, \{c, d\}\right\}\right).$$

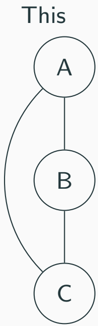
Solution. There are many possible solutions. For example:

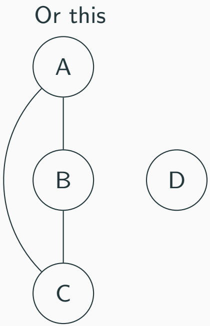


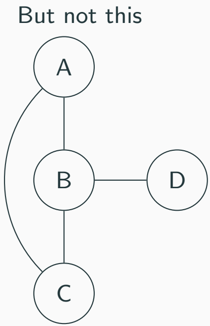
Solution to group exercise #3

Problem. Construct a graph for which the is-adjacent relation, $\sim$, is transitive.

Solution. There are many possible solutions. For example:



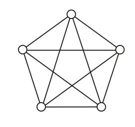




Solution to group exercise #4

Problem. How many edges are in K_n, a complete graph with n vertices?

Example. K_5 is shown to the right.



Solution. In a complete graph, all pairs of vertices are adjacent. Since there are $\binom{n}{2} = \frac{n(n-1)}{2}$ ways to choose pairs of n vertices, there must be $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

Alternate Solution (From A Student). We can use Theorem 47.5, which says

$$\sum_{v \in V} d(v) = 2|E|. \tag{1}$$

In a complete graph, $d(v) = n - 1$ for all vertices $v \in V$, since each vertex must be adjacent to all other vertices. Since there are n vertices all together, we have

$$\sum_{v \in V} d(v) = \sum_{v \in V} (n-1) = |V|(n-1) = n(n-1)$$

Substituting this into Eq. (1) and solving for $|E|$ yields the solution:

$$|E| = \frac{n(n-1)}{2}.$$

Solution to group exercise #5

Problem. How many different graphs can be formed with vertex set $V = \{1, 2, 3, \ldots, n\}$?

Solution. $2^{\binom{n}{2}}$. To justify this, note:

- When forming a graph from n vertices, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ candidate edges (since there are $\binom{n}{2}$ ways to choose pairs from a set of n items).
- For each edge, we make a free binary decision to include or exclude the edge in the graph.
- Thus, by the multiplication principle (and specifically Scheinerman Theorem 8.6), there are $2^{\binom{n}{2}}$ possible graphs that can be formed. [Imagine forming a list of length $\binom{n}{2}$, where each entry in the list takes on one of 2 values (e.g. `include` or `exclude`) corresponding to whether that particular edge is included or excluded in the graph.]

Solution to group exercise #6 (Slide 1/2)

Problem. Prove that in every graph, the number of vertices with odd degree is even.

Remark. If you'd like some motivation for this problem, check out Section 7.1 (pp.125-129) of the book by László Lovász and others. It's a really fun read!

Solution (by László Lovász). One way of proving the theorem is to build up the graph one edge at a time, and observe how the parities of the degrees change. An example is shown in Figure 7.4. We start with a graph with no edge, in which every degree is 0, and so the number of nodes with odd degree is 0, which is an even number.

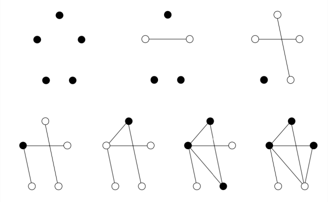
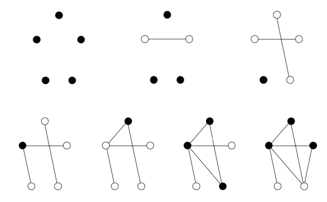


Figure 4: Building up a graph one edge at a time. Black circles mark nodes of even degree.

(See next slide for more.)

Solution to group exercise #6 (Slide 2/2)



Now if we connect two nodes by a new edge, we change the parity of the degrees at these nodes. In particular,

- if both endpoints of the new edge had even degree, we increase the number of nodes with odd degree by 2;
- if both endpoints of the new edge had odd degree, we decrease the number of nodes with odd degree by 2;
- if one endpoint of the new edge had even degree and the other had odd degree, then we don't change the number of nodes with odd degree.

Thus if the number of nodes with odd degree was even before adding the new edge, it remained even after this step. This proves the theorem. (Note that this is a proof by induction on the number of edges.)