03/12/2025: Intro to Probability (Part 1)

CSCI 246: Discrete Structures

Textbook reference: Sec 30, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Announcement: Friday 03/14 will be a review day

1. No reading assignment, reading quiz, or problem quiz.
2. We review group exercises from binomial coefficients, inclusion/exclusion, and intro to probability 1.
3. Time permitting, we will also have an open Q & A. Feel free to bring questions on any topic we've covered so far.

Today's Agenda

- Reading quiz (5 mins)
- Mini-lecture ($\approx$ 25 mins)
- Group exercises ($\approx$ 15 mins)

Feedback on Monday's Quiz

Reading Quiz

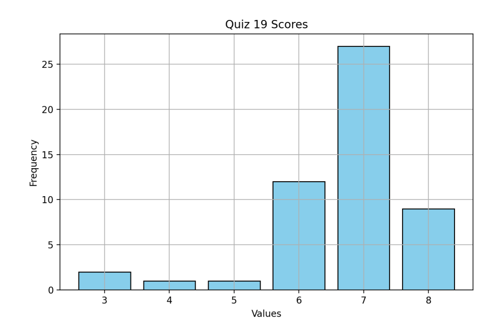

Figure 1: Median Score = 7/8 (87.5%)

Rubric.

1. 6 points. One point for each sign.
2. 1 point.
3. 1 point.

Some nice creative thinking in answering Question #2

Question. (True or False.) Consider the length-k lists whose elements are chosen from the set $\{1, 2, \ldots, n\}$. The number of lists which use all of the elements in $\{1, 2, \ldots, n\}$ at least once is n^k.

Student answers.

- False. Let $n = 3$, so we're choosing elements from $\{1, 2, 3\}$. Let $k = 3$. Then the lists which contain all the elements are
$$\Big\{(1,2,3),(1,3,2),(2,3,1),(2,1,3),(3,2,1),(3,1,2)\Big\}.$$
There are 6 items in this list, and $6 \neq n^k = 3^3 = 27$.
- False. By the question $k < n$ is valid, and if $k < n$ then no lists can contain all elements.

Reading quiz

Reading Quiz

A *hand* of poker is a five-element subset of the standard deck of 52 cards. Assuming that the deck is well-shuffled, what is the probability of drawing the hand $6\clubsuit - K\heartsuit - 3\diamondsuit - A\spadesuit - 7\clubsuit$?

Additional thought on inclusion/exclusion

Definition

A **derangement** is a permutation of the elements of a set in which no element appears in its original position. That is, a derangement is a list of length n using the elements of $\{1, \ldots, n\}$ such that the number j does not occupy position j of the list for any $j = 1, \ldots n$.

Proposition (Scheinerman pp.114)

$$\#\text{derangements} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Poll

Suppose that a professor gave a test to 4 students – A, B, C, and D – and wants to let them grade each other's tests. Of course, no student should grade their own test. How many ways could the professor hand the tests back to the students for grading, such that no student receives their own test back?

Solution

Enumeration Solution

Out of 24 possible permutations (4!) for handing back the tests, there are only 9 derangements (shown in blue italics below). In every other permutation of this 4-member set, at least one student gets their own test back (shown in bold red).

ABCD, **AB**DC, **A**CB**D**, **A**CDB, **A**DBC, **A**D**C**B,
BA**CD**, *BADC*, BCA**D**, *BCDA*, *BDAC*, BD**C**A,
CAB**D**, *CADB*, C**B**A**D**, C**B**DA, *CDAB*, *CDBA*,
DABC, DA**C**B, D**B**AC, D**BC**A, *DCAB*, *DCBA*.

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Formulaic Solution

$$
\begin{aligned}
\#\text{derangements} &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} && \text{(General formula)} \\
&= 4! \sum_{k=0}^{4} \frac{(-1)^k}{k!} && \text{(Substitute } n = 4\text{)} \\
&= 4! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right) && \text{(Expand sum)} \\
&= 4! - 4! + 4 \cdot 3 - 4 + 1 && \text{(Distribute 4!, simplify)} \\
&= 12 - 4 + 1 = 9
\end{aligned}
$$

Introduction to Probability: Samples and Events

Poll

Imaging tossing two coins and observing whether 0, 1, 2 heads are obtained. Will each of these events occur about 1/3 of the time?

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TABLE 9.1.1 Experimental Data Obtained from Tossing Two Quarters 50 Times

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	𝍸 𝍸 |	11	22%
1 head obtained	𝍸 𝍸𝍸 𝍸𝍸 ||	27	54%
0 heads obtained	𝍸𝍸 ||	12	24%

Empirically, 1 head is obtained about twice as often as the other options. Why?

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Empirically, 1 head is obtained about twice as often as the other options. Why? We'll clarify this matter in the next two slides.

Definition

A (finite) **probability space** is a pair (S, P) where S is a finite, nonempty set and P is a function $S \to \mathbb{R}$ such that $P(s) \geq 0$ for all $s \in S$ and

$$\sum_{s \in S} P(s) = 1$$

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Remark: What is a sample space?

A **sample space** is the set of all possible outcomes of a random process or experiment.

Scheinerman refers to (S, P) as a *sample space*. However, most people use *sample space* to refer to S alone. (We will follow the latter convention.)

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Example Of A Probability Space

In the coin example, we can construct the probability space as

$$S = \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}$$

and $P(s) = 1/4$ for all $s \in S$.

Overview

An **event** is a subset of a sample space.

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Let (S, P) be a probability space. Then an **event** E is a subset of S (i.e. $E \subseteq S$). The probability of an event E, denoted $P(E)$ is

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In the coin example, what is the event of observing one head? What is its probability? The event of observing one head is $E = \{\mathtt{HT}, \mathtt{TH}\}$. The probability of the event is $P(E) = P(\{\mathtt{HT}\}) + P(\{\mathtt{TH}\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

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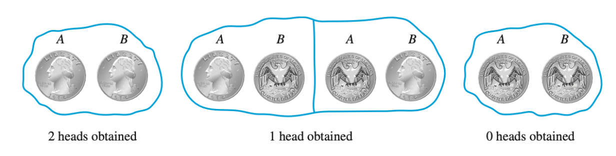

Equally Likely Probability Formula

If (P, S) is a probability space in which all outcomes are equally likely and E is an event in S, then the probability of E is

$$P(E) = \frac{|E|}{|S|} = \frac{\text{the number of outcomes in } E}{\text{the number of outcomes in } S}$$

Introduction to Probability: Some Applications

Background on cards

An ordinary deck of cards contains 52 cards divided into four *suits*. The *red suits* are diamonds ($\diamondsuit$) and hearts ($\heartsuit$), and the *black suits* are clubs ($\clubsuit$) and spades ($\spadesuit$). Each suit contains 13 cards of the following *denominations*: 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king), and A (ace). The cards J, Q, and K are called *face cards*.

Poll

Suppose we draw a card randomly from a shuffled deck. (1) What is the sample space of outcomes? (2) What is the event that the chosen card is a black face card. (3) What is the probability that the chosen card is a black face card?

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2. $$E = \{J\spadesuit, Q\spadesuit, K\spadesuit, J\clubsuit, Q\clubsuit, K\clubsuit\}$$

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Solution

1. The sample space S are the 52 cards in the deck.
2. $$E = \{J\spadesuit, Q\spadesuit, K\spadesuit, J\clubsuit, Q\clubsuit, K\clubsuit\}$$
3. By the equally likely probability formula,
$$P(E) = \frac{|E|}{|S|} = \frac{6}{52} \approx 11.5\%$$

Poll

A bag contains 20 marbles. These marbles are identical, except that they are labeled with the integers 1 through 20. Five marbles are drawn randomly from the bag. What is the probability space?

Solution(s): The solution depends on what's meant by "randomly"!

There are a few ways to think about this.

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$$S = \{\text{partial permutations of } \{1, \ldots, 20\} \text{ of size } 5\},$$
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$$S = \{1, \ldots, 20\}^5, |S| = 20^5, P(s) = \frac{1}{20^5} \ \forall s \in S$$

Group exercises

aaron.loomis: 19
adam.wyszynski: 3
alexander.goetz: 9
alexander.knutson: 1
anthony.mann: 20
blake.leone: 21
bridger.voss: 13
caitlin.hermanson: 6
cameron.wittrock: 2
carsten.brooks: 8
carver.wambold: 7
colter.huber: 21
conner.reed1: 2
connor.mizner: 20
connor.yetter: 11
derek.price4: 11
devon.maurer: 5
emmeri.grooms: 13
erik.moore3: 14
ethan.johnson18: 3
evan.barth: 18
evan.schoening: 6
griffin.short: 14
jack.fry: 9
jacob.ketola: 8
jacob.ruiz1: 9
jacob.shepherd1: 15
jada.zorn: 7
jakob.kominsky: 21
james.brubaker: 8
jeremiah.mackey: 18
jett.girard: 20
john.fotheringham: 17
jonas.zeiler: 7
joseph.mergenthaler: 17
joseph.triem: 14
julia.larsen: 5
justice.mosso: 12
kaden.price: 19
lucas.jones6: 4
luka.derry: 3
luke.donaldson1: 18
lynsey.read: 10
mason.barnocky: 1
matthew.nagel: 11
micaylyn.parker: 2
michael.oswald: 12
nolan.scott1: 4
owen.obrien: 17
pendleton.johnston: 10
peter.buckley1: 16
reid.pickert: 15
ryan.barrett2: 16
samuel.hemmen: 19
samuel.mosier: 4
samuel.rollins: 1
sarah.periolat: 5
timothy.true: 16
tristan.nogacki: 6
tyler.broesel: 10
william.elder1: 12
yebin.wallace: 15
zeke.baumann: 13

Group exercises

1. Let (S, P) be the probability space in which $S = \{1, 2, 3, 4\}$. Suppose $P(1) = x$, $P(2) = 2x$, $P(3) = 3x$, and $P(4) = 4x$. Find x.
2. Suppose that that (S, P_1) and (S, P_2) are two probability spaces that have the same set of outcomes, S. Is it possible that each outcome is less likely in the first probability space than it is in the second? That is, can we have $\forall s \in S, P_1(s) < P_2(s)$?
3. A dart is thrown blindly at the target shown in the figure. The probability that the dart lands in one of the four concentric regions is proportional to the area of the region. The radii of the circles in the figure are 1,2,3, and 4 units, respectively. Please note that region 2 consists of just the annular region from radius 1 to 2, and does not include the enclosed circular region 1. Let (S, P) be a probability space modeling this situation. The set S consists of the four outcomes: hitting region 1,2,3 or 4. Please find $P(1), P(2), P(3)$, and $P(4)$.

Solution to group exercise #1

Problem. Let (S, P) be the probability space in which $S = \{1, 2, 3, 4\}$. Suppose $P(1) = x, P(2) = 2x, P(3) = 3x$, and $P(4) = 4x$. Find x.

Solution.

$$
\begin{aligned}
\sum_{s \in S} P(s) &= 1 && \text{(Def. probability space)} \\
\implies P(1) + P(2) + P(3) + P(4) &= 1 && \text{(Expand sum)} \\
\implies x + 2x + 3x + 4x &= 1 && \text{(By assumption)} \\
\implies 10x &= 1 \\
\implies x &= \frac{1}{10}
\end{aligned}
$$

Remark.

Therefore,

$$P(1) = \frac{1}{10}, \quad P(2) = \frac{2}{10}, \quad P(3) = \frac{3}{10}, \quad P(4) = \frac{4}{10}.$$

Solution to group exercise #2

Problem. Suppose that that (S, P_1) and (S, P_2) are two probability spaces that have the same set of outcomes, S. Is it possible that each outcome is less likely in the first probability space than it is in the second? That is, can we have $\forall s \in S, P_1(s) < P_2(s)$?

Solution. No. By definition of a (finite) probability space, we have

$$\sum_{s \in S} P_1(s) = \sum_{s \in S} P_2(s) = 1 \tag{1}$$

But since $P_1(s) < P_2(s)$ for all s, we have

$$\sum_{s \in S} P_1(s) < \sum_{s \in S} P_2(s) \tag{2}$$

Substituting Eq. (1) into Eq. (2) , we obtain

$$1 < 1$$

which is a contradiction.

Remark on group exercise #2

Proof Template 12 Proof by contradiction

To prove "If A, then B":

We assume the conditions in A.
Suppose, for the sake of contradiction, not B.
Argue until we reach a contradiction.
$\Rightarrow\Leftarrow$ ■

(The symbol $\Rightarrow\Leftarrow$ is an abbreviation for the following: Thus we have reached a contradiction. Therefore the supposition (not B) must be false. Hence B is true.)

Figure 2: *A template for proof by contradiction.* Source: Scheinerman Sec. 20 (pp. 121)

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Remark

This is an example of proof by contradiction (Scheinerman Sec. 20).

Proposition A:	(S, P_1) and (S, P_2) are two probability spaces that have the same set of outcomes, S.
Proposition B:	$\forall s \in S, P_1(s) < P_2(s)$

We prove $A \implies (\texttt{not } B)$ by assuming A and B and obtaining a contradiction.

Remark on group exercise #3

The areas of the circles are given by

$$C_1 = \pi 1^2 = \pi \qquad C_3 = \pi 3^2 = 9\pi$$

$$C_2 = \pi 2^2 = 4\pi \qquad C_4 = \pi 4^2 = 16\pi$$

Hence, the areas of the annular regions are given by

$$A_1 = C_1 = \pi \qquad A_3 = C_3 - C_2 = 5\pi$$

$$A_2 = C_2 - C_1 = 3\pi \qquad A_4 = C_4 - C_3 = 7\pi$$

By assumption, the probability of the dart landing in each annular region is

$$P(1) \propto A_1, \quad P(2) \propto A_2, \quad P(3) \propto A_3, \quad P(4) \propto A_4,$$

where $\propto$ means "is proportional to". In other words,

$$P(1) = \frac{A_1}{Z}, \quad P(2) = \frac{A_2}{Z}, \quad P(3) = \frac{A_3}{Z}, \quad P(4) = \frac{A_4}{Z},$$

where Z is some unknown "normalizing constant". We solve for Z by using that the probabilities over all outcomes must sum to unity:

$$\sum_{s \in S} P(s) = 1 \implies \frac{A_1}{Z} + \frac{A_2}{Z} + \frac{A_3}{Z} + \frac{A_4}{Z} = 1$$

$$\implies A_1 + A_2 + A_3 + A_4 = Z \implies Z = 16\pi.$$

Hence

$$P(1) = \frac{1}{16}, \quad P(2) = \frac{3}{16}, \quad P(3) = \frac{5}{16}, \quad P(4) = \frac{7}{16}.$$