Monday 01/27/2025: Boolean Algebra

CSCI 246: Discrete Structures

Textbook reference: Sec. 7, Scheinerman

New way to get quizzes back!

Each student may line up either before class (2:00-2:10) or after class (3:00-3:10) to collect graded quizzes from the previous class from Fatima.

This will hopefully free up Fatima's time during class to assist with group exercises.

Today's Agenda

- Reading quiz (5 mins)
- Mini-lecture ($\approx$ 15 mins)
  - Review Friday quizzes.
  - Go over Sec. 6 group problems
- Group exercises ($\approx$ 25 mins)

Reading Quiz

Reading Quiz (Sec. 7 - Boolean Algebra)

Use a truth table to prove that the expressions $x \implies y$ and $(\neg x) \vee y$ are logically equivalent.

Notation reminder

- $\implies$ means `implies`
- $\neg$ means `not`
- $\vee$ means `or`

Mini-lecture

Feedback on your reading for Sec. 6 (counterexample)

Poll

Is the following statement true or false?

$$-5|5 = -1$$

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Alert!

In mathematical reasoning, you always need to **refer back to the definitions**.

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Reminder of Definition 3.2 (Divisible)

Let a and b be integers. We say that a is *divisible* by b provided there is an integer c such that $bc = a$. The notation for this is $b|a$.

Feedback on your reading for Sec. 6 (counterexample)

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Solution to poll: The statement is **incorrect**! Why?

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Solution to poll: The statement is **incorrect**! Why? $-5|5$ is a **proposition**, so it can't equal -1. We would instead write just $-5|5$, or perhaps $-5|5 = \texttt{TRUE}$. How do we justify the proposition?

Feedback on your reading for Sec. 6 (counterexample)

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Is the following statement true or false?

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Let a and b be integers. We say that a is *divisible* by b provided there is an integer c such that $bc = a$. The notation for this is $b|a$.

Solution to poll: The statement is **incorrect**! Why? $-5|5$ is a **proposition**, so it can't equal -1. We would instead write just $-5|5$, or perhaps $-5|5 = \texttt{TRUE}$. How do we justify the proposition? We have $-5|5$ because if $a = 5$ and $b = -5$, there exists an integer (namely $c = -1$) such that $bc = a$.

Feedback on Sec. 6 reading

Reading Quiz (Sec. 6 - Counterexamples)

Disprove the following conjecture: *Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.* You can disprove the conjecture by providing a counterexample. **Make sure to show that your counterexample satisfies the hypothesis (the "if" statement).**

Common errors

1. **Not justifying the counterexample.** Many people correctly gave a counterexample, e.g. $a = 5, b = -5$, but did not justify why $-5|5$.
2. **Incorrect use of notation.** Many people wrote $-5|5 = -1$. (However, I didn't take off points for this error.)

Reading Quiz Scores

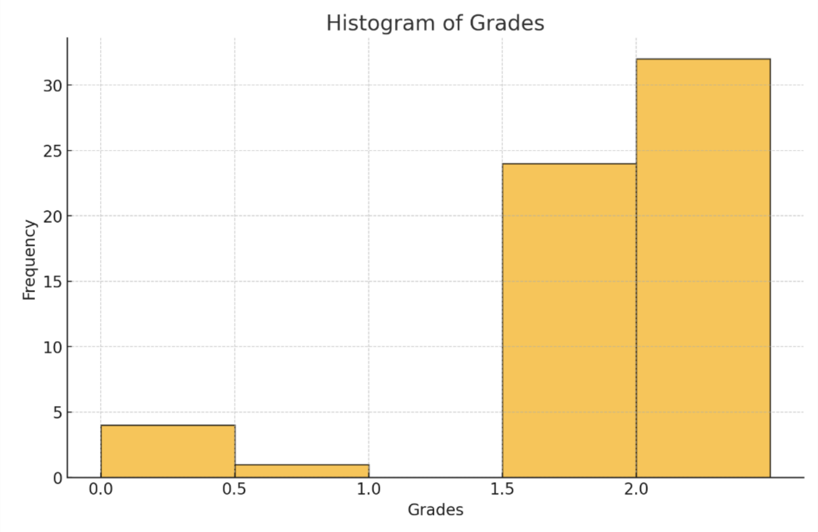

Figure 1: Reading Quiz (Sec. 6)

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Tips for Reading Math Textbooks

Work through examples.

- Don't just skim (or worse, skip!) the examples. Instead, devote more of your focus to the examples. Working through them and verbalizing the main ideas behind them is a great way to test to make sure you're understanding what you're reading.
- Read with a pencil and paper in your hand and when you get to an example, work it out! Try not to look at the solution until you are done.
- When checking your work, make sure you understand each step and why.
- Practice the examples BEFORE you attempt homework problems and then try to do your homework without referring back to the example

Source: https://learningcenter.unc.edu/tips-and-tools/readingmathtexts/.

My solution to reading Quiz (Sec 6 - Counterexamples)

Disprove the following conjecture: *Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.*

Annotation	Main Text
Structure	Let $a = 5$, $b = -5$. First, we show that the hypothesis holds [i.e., that $(5|-5)$ and $(-5|5)$].
Unravel defn.	$a|b$ means there is an integer x such that $ax = b$. Likewise $b|a$ means there is an integer y such that $by = a$.
The "Glue"	Substituting for a and b, we need to show that there are integers x and y such that $5x = -5$ and $-5y = 5$. We see these equations hold by taking $x = -1$ and $y = -1$.
Structure	Hence $5|-5$ and $-5|5$, so the hypothesis is met.
Structure	Now we show that the conclusion fails [i.e. that $-5 \neq 5$.]
The "Glue"	This is immediately clear.

Remark. I scored out of 2 points, and gave 1.5 points for a correct counterexample, and 0.5 points for any argument that the hypothesis holds.

An interesting question

Reading Quiz (Sec. 6 - Counterexamples)

Disprove the following conjecture: *Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.*

Poll

Is $a = 1, b = 0$ a valid counterexample?

An interesting question

Reading Quiz (Sec. 6 - Counterexamples)

Disprove the following conjecture: *Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.*

Poll

Is $a = 1, b = 0$ a valid counterexample?

Reminder of Definition 3.2 (Divisible)

Let a and b be integers. We say that a is *divisible* by b provided there is an integer c such that $bc = a$. The notation for this is $b|a$.

An interesting question

Reading Quiz (Sec. 6 - Counterexamples)

Disprove the following conjecture: *Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.*

Poll

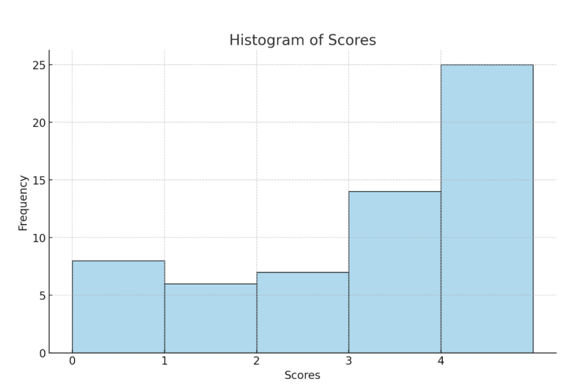
Is $a = 1, b = 0$ a valid counterexample?

Reminder of Definition 3.2 (**Divisible**)

Let a and b be integers. We say that a is *divisible* by b provided there is an integer c such that $bc = a$. The notation for this is $b|a$.

Solution to poll: We have $a|b = 1|0$, since there is an integer c such that $ac = b$ (that is, $1 \cdot c = 0$). However, we do not have $b|a = 0|1$, since there is no integer d such that $bd = a$ (that is, there is no integer d such that $0 \cdot d = 1$). That is, the hypothesis doesn't hold, so the statement doesn't apply.

Problems Quiz Scores



Most common error: Most students with 3/4 correctly wrote the truth table columns for $A \implies B, B \implies A$ and $A \iff B$, but but didn't argue why/how $A \iff B$ is identical to $(A \implies B)$ `and` $(B \implies A)$.

Solution: The question was exactly a group exercise from Sec. 4 (Theorems). See slides from that section for solution.

Review solutions to Sec. 6 group exercises

Sec. 7 (Boolean algebra) group work!

Students are randomly assigned into groups of 3 on the next slide.

Each group gets $\frac{1}{2}$ of a white board.

If the $\frac{1}{2}$ white board is inconvenient, feel free to write on a window!

Group 1: justice.mosso,owen.obrien,nicholas.harrington1
Group 2: joseph.triem,connor.yetter,anthony.mann
Group 3: peter.buckley1,evan.barth,jeremiah.mackey
Group 4: devon.maurer,carsten.brooks,jacob.ketola
Group 5: kaden.price,michael.oswald,blake.leone
Group 6: emmeri.grooms,griffin.short,carver.wambold
Group 7: caitlin.hermanson,connor.mizner,connor.graville
Group 8: jack.fry,samuel.mosier,tyler.broesel
Group 9: ethan.johnson18,lucas.jones6,conner.reed1
Group 10: micaylyn.parker,peyton.trigg,jacob.shepherd1
Group 11: samuel.hemmen,cameron.wittrock,nolan.scott1
Group 12: yebin.wallace,alexander.knutson,colter.huber
Group 13: pendleton.johnston,reid.pickert,jada.zorn
Group 14: luke.donaldson1,joseph.mergenthaler,jonas.zeiler
Group 15: ryan.barrett2,william.elder1,samuel.rollins
Group 16: jacob.ruiz1,aaron.loomis,lynsey.read
Group 17: zeke.baumann,delaney.rubb,james.brubaker
Group 18: erik.moore3,derek.price4,sarah.periolat
Group 19: alexander.goetz,tristan.nogacki,jett.girard
Group 20: mason.barnocky,jakob.kominsky,luka.derry
Group 21: julia.larsen,bridger.voss,evan.schoening
Group 22: john.fotheringham,adam.wyszynski,matthew.nagel, timothy.true

Group exercises

1. DeMorgan's laws are:
$$\neg(x \wedge y) = (\neg x) \vee (\neg y) \quad \text{and} \quad \neg(x \vee y) = (\neg x) \wedge (\neg y)$$
Prove the first of these (using truth tables). Then use DeMorgan's law to show how to disprove an if-and-only-if statement.
2. A **tautology** is a Boolean expression that evaluates to `TRUE` for all possible values of its variables. For example, the expression $x \vee \neg x$ evaluates to `TRUE` both when $x = \texttt{TRUE}$ and $x = \texttt{FALSE}$. Use truth tables to show the following are tautologies:
   (a) $(x \vee y) \vee (x \vee \neg y)$
   (b) $x \implies x$
   (c) $\texttt{FALSE} \implies x$
   (d) $(x \implies y) \wedge (y \implies z) \implies (x \implies z)$
3. A **contradiction** is a Boolean expression that evaluates to `FALSE` for all possible values of its variables. For example, the expression $x \wedge \neg x$ is a contradiction. Use truth tables to show that the following are contradictions:
   (a) $(x \vee y) \wedge (x \vee \neg y) \wedge \neg x$
   (b) $x \wedge (x \implies y) \wedge (\neg y)$.
4. Reprove the items in #2 and #3 using the properties in Theorem 7.2 and the fact from Prop 7.3 that $x \implies y$ is equivalent to $(\neg x) \vee y$.

Solutions to group exercises

Solution to group exercise #1a

Problem. DeMorgan's laws are:

$$\neg(X \wedge Y) = (\neg X) \vee (\neg Y) \quad \text{and} \quad \neg(X \vee Y) = (\neg X) \wedge (\neg Y)$$

Prove the first of these (using truth tables).

Solution.

X	Y	$X \wedge Y$	$\neg(X \wedge Y)$	$\neg X$	$\neg Y$	$(\neg X) \vee (\neg Y)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The 4th and 7th columns have the same truth values. Hence, $\neg(X \wedge Y) = (\neg X) \vee (\neg Y)$.

Solution to group exercise #1b

Problem. Use DeMorgan's law to show how to disprove an if-and-only-if statement.

Solution. In Group Exercise #2 from the Theorems day, we showed that

$$A \iff B \quad = \quad (A \implies B) \land (B \implies A). \tag{1}$$

That is, A-if-and-only-if B is identical to if-A-then-B `and` if-B-then-A.

To disprove an if-and-only-if statement, we need to establish $\neg(A \iff B)$. Now note that:

$$\begin{aligned} \neg(A \iff B) &= \neg\Big((A \implies B) \land (B \implies A)\Big) && \text{(by substituting Eq. (1)).} \\ &= \neg(A \implies B) \lor \neg(B \implies A) && \text{(by DeMorgan's law)} \end{aligned}$$

So we can disprove and if-and-only-if statement *either* by showing that $A \implies B$ fails *or* by showing that $B \implies A$ fails.

Solution to group exercise #2a

Problem. Use truth tables to show the following is a tautology:

$$(X \vee Y) \vee (X \vee \neg Y)$$

Solution.

X	Y	$\neg Y$	$X \vee Y$	$X \vee (\neg Y)$	$(X \vee Y) \vee (X \vee \neg Y)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	F	T	T

The last column shows that $(X \vee Y) \vee (X \vee \neg Y)$ is always true, and hence a tautology.

Solution to group exercise #2b

Problem. Use truth tables to show the following is a tautology:

$$X \implies X.$$

Solution. Recall the truth table for implication.

X	Y	$X \implies Y$
T	T	T
T	F	F
F	T	T
F	F	T

Here we have Y=X. So only the first and last row of the truth table ever occur. So $X \implies X$ is always true. Thus, $X \implies X$ is a tautology.

Solution to group exercise #2 c

Problem. Use truth tables to show the following is a tautology:

$$\texttt{FALSE} \implies X$$

Solution. The truth table for implication is

W	X	$W \implies X$
T	T	T
T	F	F
F	T	T
F	F	T

But here, we know W is $\texttt{FALSE}$. That is, only the last two rows of the truth table ever occur. So $\texttt{FALSE} \implies X$ is always true. Hence, $\texttt{FALSE} \implies X$ is a tautology.

Solution to group exercise #2 d

Problem. Use truth tables to show the following is a tautology:

$$(X \implies Y) \land (Y \implies Z) \implies (X \implies Z)$$

Solution.

<table>
<tr><th>Proposition</th><th>Nickname</th><th colspan="8">Values</th></tr>
<tr><td>X</td><td></td><td>T</td><td>T</td><td>F</td><td>F</td><td>T</td><td>T</td><td>F</td><td>F</td></tr>
<tr><td>Y</td><td></td><td>T</td><td>F</td><td>T</td><td>F</td><td>T</td><td>F</td><td>T</td><td>F</td></tr>
<tr><td>Z</td><td></td><td colspan="4">T</td><td colspan="4">F</td></tr>
<tr><td>$X \implies Y$</td><td>A</td><td>T</td><td>F</td><td>T</td><td>T</td><td>T</td><td>F</td><td>T</td><td>T</td></tr>
<tr><td>$Y \implies Z$</td><td>B</td><td colspan="4">T</td><td>F</td><td>T</td><td>F</td><td>T</td></tr>
<tr><td>$(X \implies Y) \land (Y \implies Z)$</td><td>$C \triangleq A \land B$</td><td>T</td><td>F</td><td>T</td><td>T</td><td>F</td><td>F</td><td>F</td><td>T</td></tr>
<tr><td>$X \implies Z$</td><td>D</td><td colspan="4">T</td><td>F</td><td>F</td><td>T</td><td>T</td></tr>
<tr><td>$(X \implies Y) \land (Y \implies Z) \implies (X \implies Z)$</td><td>$C \implies D$</td><td colspan="4">T</td><td colspan="4">T</td></tr>
</table>

Solution to group exercise #3a

Problem. Use truth tables to show the following is a contradiction:

$$(X \vee Y) \wedge (X \vee \neg Y) \wedge \neg X$$

Solution.

Proposition	Nickname	Truth Values			
X		T	T	F	F
Y		T	F	T	F
$X \vee Y$	A	T	T	T	F
$\neg Y$		F	T	F	T
$X \vee (\neg Y)$	B	T	T	F	T
$\neg X$	C	F	F	T	T
$(X \vee Y) \wedge (X \vee \neg Y) \wedge \neg X$	$A \wedge B \wedge C$	F	F	F	F

Solution to group exercise #3b

Problem. Use truth tables to show the following is a contradiction:

$$X \land (X \implies Y) \land (\neg Y)$$

Solution.

Proposition	Nickname	Truth Values			
X	A	T	T	F	F
Y		T	F	T	F
$X \implies Y$	B	T	F	T	T
$\neg Y$	C	F	T	F	T
$X \land (X \implies Y) \land (\neg Y)$	$A \land B \land C$	F	F	F	F

Solution to group exercise #4

The solutions to group exercise #4 refer to the properties from the textbook below.

Theorem 7.2

- $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$. (Commutative properties)
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$. (Associative properties)
- $x \wedge \text{TRUE} = x$ and $x \vee \text{FALSE} = x$. (Identity elements)
- $\neg(\neg x) = x$.
- $x \wedge x = x$ and $x \vee x = x$.
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. (Distributive properties)
- $x \wedge (\neg x) = \text{FALSE}$ and $x \vee (\neg x) = \text{TRUE}$.
- $\neg(x \wedge y) = (\neg x) \vee (\neg y)$ and $\neg(x \vee y) = (\neg x) \wedge (\neg y)$. (DeMorgan's Laws)

Figure 2: Boolean Algebra Properties

Solution to group exercise #4 - Redo of 2a

Problem. Show the following is a tautology:

$$(X \vee Y) \vee (X \vee \neg Y)$$

Solution.

$$\begin{aligned}(X \vee Y) \vee (X \vee \neg Y) &= (X \vee X) \vee (Y \vee \neg Y) && \text{(commutative, associative props.)}\\ &= X \vee \texttt{True} && \text{(unnamed props \#5,7)}\\ &= \texttt{True} && \text{(unnamed prop \#7)}\end{aligned}$$

Solution to group exercise #4 - Redo of #2b

Problem. Show the following is a tautology:

$$X \implies X.$$

Solution.

$$\begin{aligned} X \implies X &= (\neg X \vee X) && \text{(Prop 7.3)} \\ &= \texttt{True} && \text{(unnamed prop \#7)} \end{aligned}$$

Solution to group exercise #4 - Redo of #2 c

Problem. Show the following is a tautology:

$$\texttt{FALSE} \implies X$$

Solution.

$$\begin{aligned}
\texttt{FALSE} \implies X &= (\neg\texttt{FALSE}) \vee X && \text{(Prop 7.3)} \\
&= \texttt{TRUE} \vee X \\
&= \texttt{TRUE}
\end{aligned}$$

Solution to group exercise #4 - Redo of #2 d

Problem. Show the following is a tautology:

$$(X \implies Y) \wedge (Y \implies Z) \implies (X \implies Z)$$

$$\begin{aligned}
&\underbrace{(X \implies Y) \wedge (Y \implies Z)}_{\triangleq A} \implies \underbrace{(X \implies Z)}_{\triangleq B} & \\
=&\neg A \vee B & \text{(Prop 7.3)} \\
=&\Big(\neg\big[X \implies Y\big] \vee \big[\neg(Y \implies Z)\big]\Big) \vee (X \implies Z) & \text{(DeMorgan's Law)} \\
=&(X \vee \neg Y) \vee (Y \vee \neg Z) \vee (\neg X \vee Z) & \text{(Prop 7.3, DeMorgan's Law)} \\
=&(X \vee \neg X) \vee (Y \vee \neg Y) \vee (Z \vee \neg Z) & \text{(Associative, commutative props.)} \\
=&\mathtt{TRUE} \vee \mathtt{TRUE} \vee \mathtt{TRUE} & \text{Unnamed Prop \#7} \\
=&\mathtt{TRUE} &
\end{aligned}$$

Solution to group exercise #4 - Redo of #3a

Problem. Show the following is a contradiction:

$$(X \vee Y) \wedge (X \vee \neg Y) \wedge \neg X$$

Solution.

$$\begin{aligned}
&(X \vee Y) \wedge (X \vee \neg Y) \wedge \neg X \\
=&(X \wedge X \wedge \neg X) \vee (X \wedge \neg Y \wedge \neg X) \vee (Y \wedge X \wedge \neg X) \vee (Y \wedge \neg Y \wedge \neg X) && \text{Distributive prop.} \\
=&\texttt{FALSE} \vee \texttt{FALSE} \vee \texttt{FALSE} \vee \texttt{FALSE} && \text{Unnamed prop. \# 7} \\
=&\texttt{FALSE}
\end{aligned}$$

Remark

A tricky part of applying the properties is using the distributive law correctly. For intuition, recall how multiplication distributes over addition [e.g. $4(3+5) = 4 \cdot 3 + 4 \cdot 5$].

Solution to group exercise #4 - Redo of #3b

Problem. Show the following is a contradiction:

$$X \wedge (X \implies Y) \wedge (\neg Y)$$

Solution.

$$\begin{aligned}
& X \wedge (X \implies Y) \wedge (\neg Y) & \\
=& X \wedge (\neg X \vee Y) \wedge (\neg Y) & \text{Prop 7.3} \\
=& (X \wedge \neg X \wedge \neg Y) \vee (X \wedge Y \wedge \neg Y) & \text{Distributive prop.} \\
=& \texttt{FALSE} \vee \texttt{FALSE} & \text{Unnamed prop. \# 7} \\
=& \texttt{FALSE} &
\end{aligned}$$