04/14/2025: Connection

CSCI 246: Discrete Structures

Textbook reference: Sec 49, Scheinerman

Problems Quiz On Wednesday

Hence, our final problems quiz will be on WEDNESDAY 04/16. The topics that will be covered are:

- Fundamentals of Graph Theory (Scheinerman Sec 47; see group exercises and reading quiz from 04/09)
- Subgraphs (Scheinerman Sec 48; see group exercises and reading quiz from 04/11)
- Connection (Scheinerman Sec 49; we will cover this today)

Today's Agenda

- Reading quiz (5 mins)
- Review solutions to previous group exercises ($\approx$ 10 mins)
- New group exercises ($\approx$ 20 mins)
- Review solutions to new group exercises ($\approx$ 10 mins)

Feedback on Friday's Quizzes

Reading Quiz Scores

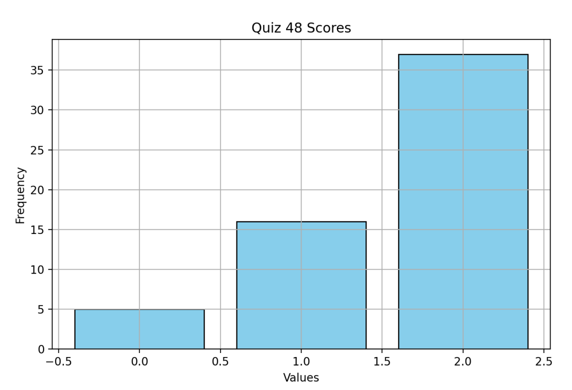


Figure 1: Median Score = 2/2 (100%)

Grading Rubric:

1. (1 point) Clique.
2. (1 point) Independent set.

Problem Quiz Scores

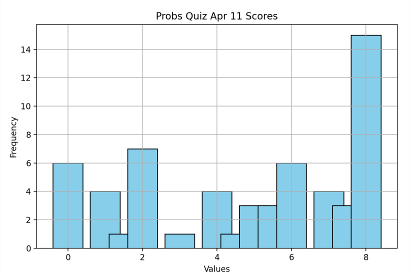


Figure 2: Median Score = 5/8 (68.75%)

Grading Rubric:

1. (4 points) 1 point for getting the roots correct, 1 point for choosing the correct form out of the two listed, 1 point for solving for c_1, c_2 correctly, 1 point for the final equation being stated correctly (given c_1 and c_2).
2. (4 points) If you appealed to the definitions: 2 points for correctly proving Big O, 2 points for correctly proving Big Omega. Note, however, there is a shortcut solution, which is to simply reference the theorem on polynomial orders. That is also a perfectly acceptable answer worth 4 points.

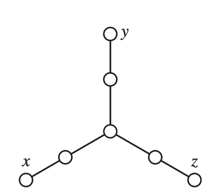
Today's quiz

Reading quiz (Connection)

The argument below is from the text. Is it right or wrong? If it's wrong, what is the problem with it?

Reference passage from text

Is the is-connected-to relation transitive? Suppose, in a graph G, we know that x is connected to y and that y is connected to z. We want to prove that x is connected to z.



Since x is connected to y, there must be an (x, y)-path; let's call it P. And since y is connected to z, there must be a (y, z)-path. Let's call it Q. Notice that the last vertex of P is the same as the first vertex of Q (it's y). Therefore, we can form the concatenation $P + Q$, which is an (x, z)-path. Therefore x is connected to z.

Q&A On Previous Group Exercises

Group exercises

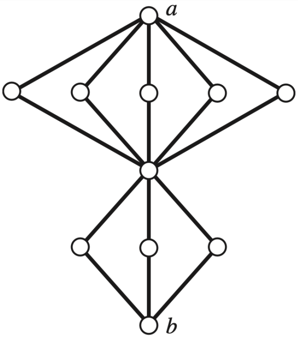
aaron.loomis: 13
adam.wyszynski: 9
alexander.knutson: 20
anthony.mann: 15
blake.leone: 14
bridger.voss: 1
caitlin.hermanson: 15
cameron.wittrock: 4
carsten.brooks: 17
carver.wambold: 12
colter.huber: 2
conner.reed1: 1
connor.mizner: 8
connor.yetter: 16
derek.price4: 8
devon.maurer: 17
emmeri.grooms: 18
erik.moore3: 16
ethan.johnson18: 16
evan.barth: 6
evan.schoening: 2

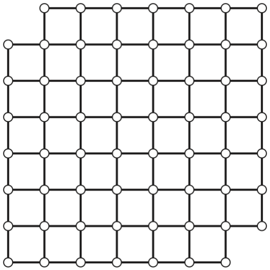
griffin.short: 9
jack.fry: 14
jacob.ketola: 8
jacob.shepherd1: 12
jada.zorn: 2
jakob.kominsky: 5
james.brubaker: 3
jeremiah.mackey: 15
jett.girard: 10
john.fotheringham: 3
jonas.zeiler: 11
joseph.mergenthaler: 6
joseph.triem: 17
julia.larsen: 18
justice.mosso: 5
kaden.price: 19
lucas.jones6: 21
luka.derry: 4
luke.donaldson1: 7

lynsey.read: 19
mason.barnocky: 10
matthew.nagel: 11
micaylyn.parker: 14
michael.oswald: 5
nolan.scott1: 4
owen.obrien: 13
pendleton.johnston: 9
peter.buckley1: 13
reid.pickert: 19
ryan.barrett2: 11
samuel.hemmen: 1
samuel.mosier: 7
samuel.rollins: 6
sarah.periolat: 20
timothy.true: 10
tristan.nogacki: 3
tyler.broesel: 18
william.elder1: 12
yebin.wallace: 20
zeke.baumann: 7

Group exercises

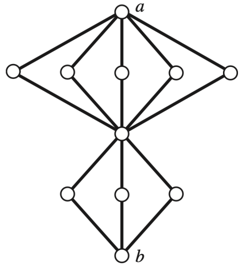
1. Let G be the graph in the top figure. (a) How many different paths are there from a to b? (b) How many different walks are there from a to b?
2. Let G be a graph. A path P in G that contains all the vertices of G is called a *Hamiltonian Path*. Prove that the graph in the bottom figure does not have a Hamiltonian path.
3. How many Hamiltonian paths does a complete graph on $n \geq 2$ vertices have?
4. Let G be a graph with $n \geq 2$ vertices.
    a. Prove that if G has at least $\binom{n-1}{2} + 1$ edges, then G is connected.
    b. Show that the result in (a) is best possible; that is, for each $n \geq 2$, prove there is a graph with $\binom{n-1}{2}$ edges that is not connected.





Solution to group exercise #1

Problem. Let G be the graph in the top figure. (a) How many different paths are there from a to b? (b) How many different walks are there from a to b?



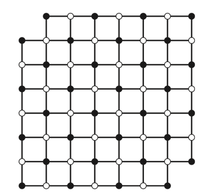
Solution.

a. There are 5x3=15 different paths from a to b.
b. There are infinitely many walks from a to b.

Solution to group exercise #2

Problem. Let G be a graph. A path P in G that contains all the vertices of G is called a *Hamiltonian Path*. Prove that the graph in the bottom figure of the group exercises does not have a Hamiltonian path.

Solution. Color the graph as follows.



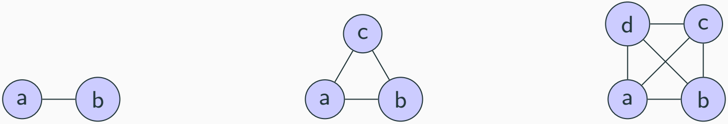
Suppose a Hamilton path P exists. We can think of P as list of vertices where each one is adjacent to the next. However, note that neighbors of a white vertex are always black, and neighbors of a black vertex are always white. Thus P must alternate between black and white vertices. Now, there are $2(3+5+7) = 30$ white vertices and $2(2+4+6)+8 = 32$ black vertices. Hence, any enumeration of the 62 vertices must contain at least two consecutive vertices that have the same color. This is a contradiction.

Solution to group exercise #3

Problem. How many Hamiltonian paths does a complete graph on $n \geq 2$ vertices have?

Solution. $n!$, since there are $n!$ ways to permute the n vertices.

Remark. Let's make the argument more concrete. Let K_n be the complete graph on n vertices. By enumeration, K_2 has two Hamiltonian paths: $a \sim b$ and $b \sim a$. Also by enumeration, K_3 has six Hamiltonian paths: $a \sim b \sim c$, $a \sim c \sim b$, $b \sim c \sim a$, $b \sim a \sim c$, $c \sim a \sim b$, and $c \sim b \sim a$. For K_4, enumeration is starting to become unwieldy, so we think more abstractly: we have 4 choices for where to start (a, b, c or d), then 3 choices of where to go next, then 2 choices for after that, and then the destination spot is determined.



K_2 K_3 K_4

Solution to group exercise #4a

Problem. Let G be a graph with $n \geq 2$ vertices. Prove that if G has at least $\binom{n-1}{2} + 1$ edges, then G is connected.

Solution. We proceed by contraposition. Suppose G is not connected. Then there is a vertex v not connected to any other vertex. Thus, there must be at least $n-1$ edges missing from the maximum possible number $\binom{n}{2}$. That is, there can be no more than $\binom{n}{2} - (n-1)$ edges. But

$$\begin{aligned} \binom{n}{2} - (n-1) &= \binom{n}{2} - \binom{n-1}{1} \\ &= \binom{n-1}{2} && \text{(By Pascal's Identity)} \end{aligned}$$

So there are at most $\binom{n-1}{2}$ edges.

Remark 1. Recall that a proof by contraposition proves $A \implies B$ by proving $\neg B \implies \neg A$.

Remark 2. From Pascal's identity, we have

$$\binom{n}{2} = \binom{n-1}{1} + \binom{n-1}{2}$$

Solution to group exercise #4b

Problem. Let G be a graph with $n \geq 2$ vertices. Show that the result in #4a is best possible; that is, for each $n \geq 2$, prove there is a graph with $\binom{n-1}{2}$ edges that is not connected.

Solution.

b. Select any vertex (we'll call it v^*) to isolate. Of the remaining $n-1$ vertices, form the complete graph K_{n-1}. The subgraph K_{n-1} has $\binom{n-1}{2}$ edges. However, the full graph G is disconnected, since there is no path that connects v^* to any other vertex.