02/03/2025: More Induction

CSCI 246: Discrete Structures

Textbook reference: Ch. 4, Hampkins

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Review of Make Up Policy / Grading Policy

- No makeup quizzes for absences. But 3 reading quizzes / 3 participations / 1 problems quiz are dropped.
- What about protection against occasionally getting a low quiz score? As of last week, I am sprinkling in extra credit to some quizzes.

Today's Agenda

- Reading & problems quizzes (5 mins)
- Mini-lecture ($\approx$ 15 mins)
  - Review Boolean Algebra
  - Induction
- Group exercises ($\approx$ 30 mins)

Today's Quiz

Logistics Alert

Please write your last name on the back of the page.

Reading Quiz (Induction)

Prove that every positive integer has a prime factorization (i.e. can be expressed as the product of prime numbers).

Bonus Alert

This quiz is extra credit.

Problems Quiz Scores: Proofs and Counterexamples

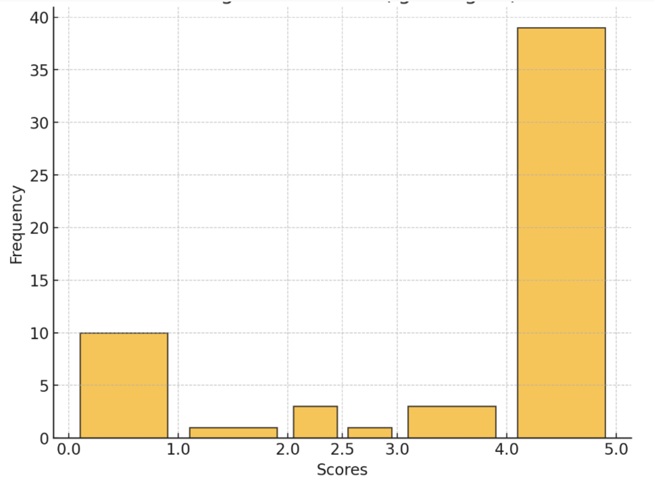

Figure 1: Median Score = 5.0 / 5.0

Reading Quiz Scores: Induction

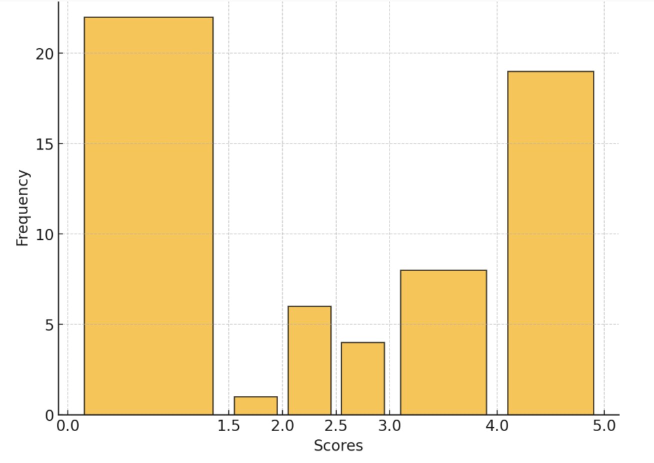

Figure 2: Median Score = 2.5 / 5.0

Review Boolean Algebra Group Exercises.

Strong Induction

Strong Induction

Assume that A is a set of natural numbers with the property that, for every natural number n, if every number smaller than n is in A, then n itself is in A. Then every natural number is in A.

When to Use Strong Induction

Sometimes there are scenarios where the induction implication goes naturally not from n to $n + 1$, but:

- From some smaller number to $n + 1$, or
- From several smaller numbers

Example: Every positive integer has a prime factorization

- Suppose every number smaller than n has a prime factorization.
- The number n must either be prime (in which case we are done) or composite.
- If n is composite, then $n = ab$ for smaller number $a, b < n$. But a, b have their own factorizations by the induction hypothesis. These can be put together to make a prime factorization of n.

Group 1: aaron.loomis,lucas.jones6,jonas.zeiler
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Group 3: nolan.scott1,jack.fry,connor.mizner
Group 4: adam.wyszynski,julia.larsen,carsten.brooks
Group 5: peter.buckley1,jacob.ruiz1,colter.huber
Group 6: alexander.knutson,conner.reed1,pendleton.johnston
Group 7: michael.oswald,evan.schoening,joseph.triem
Group 8: samuel.hemmen,connor.yetter,anthony.mann
Group 9: lynsey.read,owen.obrien,jakob.kominsky
Group 10: cameron.wittrock,connor.graville,tyler.broesel
Group 11: delaney.rubb,jett.girard,evan.barth
Group 12: derek.price4,alexander.goetz,ryan.barrett2
Group 13: zeke.baumann,griffin.short,william.elder1
Group 14: jada.zorn,matthew.nagel,ethan.johnson18
Group 15: jacob.ketola,luke.donaldson1,yebin.wallace
Group 16: jeremiah.mackey,jacob.shepherd1,erik.moore3
Group 17: justice.mosso,carver.wambold,samuel.rollins
Group 18: joseph.mergenthaler,james.brubaker,john.fotheringham
Group 19: micaylyn.parker,devon.maurer,reid.pickert
Group 20: timothy.true,caitlin.hermanson,kaden.price
Group 21: peyton.trigg,samuel.mosier,blake.leone
Group 22: tristan.nogacki,bridger.voss,luka.derry

Group exercises: Induction

1. Show by induction that $n^2 - n$ is even for any natural number n (that is, for $n = 0, 1, 2, \ldots$).
2. Show by induction that $2^n < n!$ for all $n \geq 4$.
3. Show by induction that $f_0 + \cdots + f_n = f_{n+2} - 1$ in the Fibonacci sequence.
4. Show by induction that $f_n < 2^n$ in the Fibonacci sequence.

Definition

The **Fibonacci sequence** is the sequence given by the following recursive rules:

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+2} = f_n + f_{n+1}$$

The sequence is: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

Logistics Alert

#1 has been carried over from Friday.

Solutions to group exercises.

Exercise 1

Show by induction that $n^2 - n$ is even for any natural number n (that is, for $n = 0, 1, 2, \ldots$).

Solution

Induction base. Take $n = 0$. Then $n^2 - n = 0^2 - 0 = 0$, which is even. So $n^2 - n$ is even when $n = 0$.

Induction step. Now we assume the proposition holds at index n (this is called the induction hypothesis), and we need to show it holds at index $n+1$. That is, we assume $n^2 - n$ is even, and we need to show that $(n+1)^2 - (n+1)$ is even. We have

$$\begin{aligned}
(n+1)^2 - (n+1) &= (n^2 + 2n + 1) - (n+1) \\
&= n^2 + n \\
&= \underbrace{(n^2 - n)}_{\text{even by induct. hypoth.}} + \underbrace{(2n)}_{\text{even by def. even}} \qquad \text{(subtract, add } n\text{)}
\end{aligned}$$

And the sum of two even numbers is even.
Hence, $n^2 - n$ is even $\implies (n+1)^2 - (n+1)$ is even.

Exercise 2

Show by induction that $2^n < n!$ for all $n \geq 4$.

Solution

Induction base. Take $n = 4$. Then $2^n = 2^4 = 16$ and $n! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. So $2^n < n!$ when $n = 4$.

Induction step. Now we <u>assume</u> $2^n < n!$ (this is called the induction hypothesis), and we <u>need to show</u> $2^{n+1} < (n+1)!$ We have

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &< 2 \cdot n! && \text{(by the induction hypothesis)} \\ &< (n+1) \cdot n! && \text{(since } n \geq 4\text{)} \\ &= (n+1)! && \text{(by def. factorial)} \end{aligned}$$

Hence, $2^n < n! \implies 2^{n+1} < (n+1)!$

Exercise 3

Show by induction that $f_0 + \cdots + f_n = f_{n+2} - 1$ in the Fibonacci sequence.

Solution

Induction base. Take $n = 0$. Then $f_0 + \cdots + f_n = f_0 = 0$ and $f_{n+2} - 1 = f_2 - 1 = 1 - 1 = 0$. So $f_0 + \cdots + f_n = f_{n+2} - 1$ when $n = 0$.

Induction step. Now we assume the proposition holds at index n (this is called the induction hypothesis), and we need to show it holds at index $n + 1$. That is, we assume $f_0 + \cdots + f_n = f_{n+2} - 1$, and we need to show that $f_0 + \cdots + f_{n+1} = f_{n+3} - 1$. We have

$$\begin{aligned} f_0 + \cdots + f_{n+1} &= (f_0 + \cdots + f_n) + f_{n+1} \\ &= (f_{n+2} - 1) + f_{n+1} && \text{(by the induction hypothesis)} \\ &= (f_{n+1} + f_{n+2}) - 1 \\ &= f_{n+3} - 1 && \text{(by def. Fibonacci number)} \end{aligned}$$

Hence, if the proposition holds at index n, it holds at index $n + 1$.

Exercise 4

Show by induction that $f_n < 2^n$ in the Fibonacci sequence.

Solution

Induction base. Take $n = 0$. Then $f_n = f_0 = 0$ and $2^n = 2^0 = 1$. So $f_n < 2^n$ when $n = 0$.

Induction step. Here we use strong induction. We assume the proposition holds at indices $1, 2, \ldots n$ (this is the (strong) induction hypothesis), and we need to show it holds at index $n + 1$. That is, we assume $f_k < 2^k$ for $k = 1, \ldots, n$, and we need to show that $f_{n+1} < 2^{n+1}$. We have

$$\begin{aligned}
f_{n+1} &= f_n + f_{n-1} && \text{(by def. Fibonacci number)} \\
&< 2^n + 2^{n-1} && \text{(by the (strong) induction hypothesis)} \\
&< 2^n + 2^n \\
&= 2 \cdot 2^n \\
&= 2^{n+1}.
\end{aligned}$$

Hence, $f_n < 2^n \implies f_{n+1} < 2^{n+1}$.