01/31/2025: Induction

CSCI 246: Discrete Structures

Textbook reference: Ch. 4, Hampkins

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Anonymous (Old School) Poll

When are picking up your quizzes, please

1. Find the glass jar in the front of the room
2. Tear off a sheet of paper
3. Write the typical number of hours you spend per week doing work for this class (NOT counting attending class)
4. Put the paper in the jar

Today's Agenda

- Reading & problems quizzes (10 mins)
- Mini-lecture ($\approx$ 15 mins)
  - Induction
  - Boolean Algebra properties
- Group exercises ($\approx$ 20 mins)

Today's Quiz

Logistics Alert

Please write your last name on the back of the page.

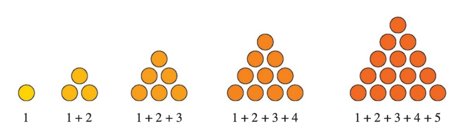
Problems Quiz (Secs. 5 and 6 - Proofs and Counterexamples)

Is the sum of two odd numbers always even? If so, provide a proof. If not, provide a counterexample.

Reading Quiz (Induction)

Prove that the n-th triangular number is $n(n+1)/2$. That is, prove

$$1 + 2 + 3 + \cdots + n = n(n+1)/2.$$



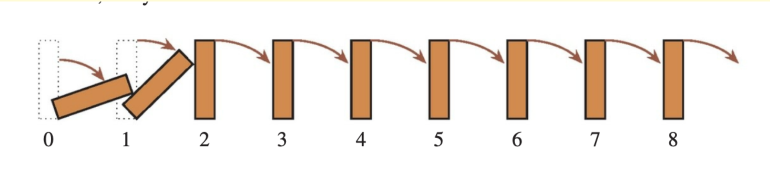
The Induction Strategy to Proving Things

Reading Quiz (Induction)

Prove that the n-th triangular number is $n(n+1)/2$. That is, prove

$$1 + 2 + 3 + \cdots + n = n(n+1)/2.$$

Induction Strategy



Base. Show the equation holds for some starting point (usually 0 or 1).
Induction step. Show that if the equation holds for natural number n, then it holds for natural number $n+1$.

The strategy in the yellow box lets us conclude the equation holds for all n.

Solving the Reading Quiz By Induction

Reading Quiz (Induction)

Prove that the n-th triangular number is $n(n+1)/2$. That is, prove

$$\boxed{1+2+3+\cdots+n} = \boxed{n(n+1)/2}. \tag{1}$$

Base. We show Eq. (1) holds for $n = 1$. In that case, the RHS of Eq. (1) is

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1,$$

which equals the LHS of Eq. (1).

Induction step. We show that if the equation holds for some n, then it also holds for $n+1$. In other words, we assume Eq. (1), and we must show

$$\boxed{1+2+3+\cdots+n} + n + 1 = (n+1)(n+2)/2. \tag{2}$$

We have

$$\begin{aligned} \text{LHS of Eq. (2)} &= \boxed{\tfrac{n(n+1)}{2}} + (n+1) && \text{(by assumption)} \\ &= \frac{(n+1)(n+2)}{2} && \text{(after some algebra)} \end{aligned}$$

Boolean Algebra Properties

Theorem 7.2

- $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$. (Commutative properties)
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$. (Associative properties)
- $x \wedge \text{TRUE} = x$ and $x \vee \text{FALSE} = x$. (Identity elements)
- $\neg(\neg x) = x$.
- $x \wedge x = x$ and $x \vee x = x$.
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. (Distributive properties)
- $x \wedge (\neg x) = \text{FALSE}$ and $x \vee (\neg x) = \text{TRUE}$.
- $\neg(x \wedge y) = (\neg x) \vee (\neg y)$ and $\neg(x \vee y) = (\neg x) \wedge (\neg y)$. (DeMorgan's Laws)

Figure 1: Boolean Algebra Properties

Example application

We can show that $(x \vee y) \vee (x \vee \neg y)$ is a tautology as follows

$$\begin{aligned}(x \vee y) \vee (x \vee \neg y) &= (x \vee x) \vee (y \vee \neg y) && \text{(commutative, associative props.)}\\ &= x \vee \texttt{True} && \text{(unnamed props \#5,7)}\\ &= \texttt{True} && \text{(unnamed prop \#7)}\end{aligned}$$

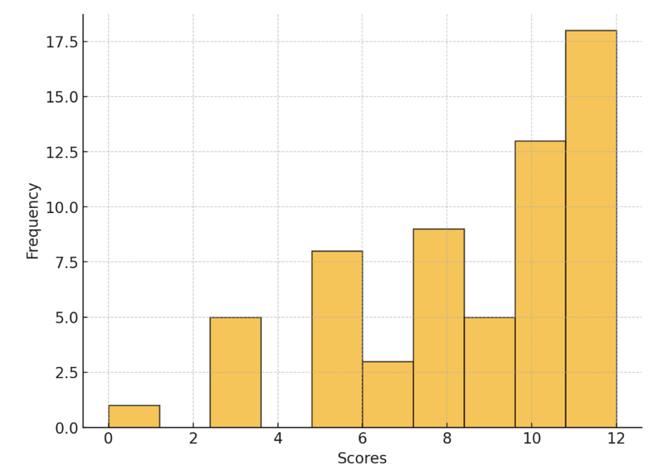
instead of using truth tables.

Reading Quiz Scoring Rubric: Multiple Proofs

Scoring rubric (out of 10 points)

Description	Main	E.C.
Correct and well-written.	10	2
Good work but some mathematical or writing errors that need addressing.	7.5	1
Some good intuition, but there is at least one serious flaw.	5	0
I don't understand this, but I see that you did work on it.	2.5	0
No work is evident.	0	0

Reading Quiz Scores: Multiple Proofs



Factorial notation

For group work, you will need the definition (really just notation) below.

Definition

The **factorial** of a non-negative integer n (denoted by $n!$) is the product of all positive integers less than or equal to n.

That is,

$$n! \triangleq n \times (n-1) \times \cdots \times 2 \times 1.$$

Example

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Convention

By convention, we define

$$0! = 1$$

Group 1: samuel.hemmen,derek.price4,peter.buckley1
Group 2: carver.wambold,blake.leone,evan.schoening
Group 3: james.brubaker,cameron.wittrock,jacob.ketola
Group 4: evan.barth,anthony.mann,lynsey.read
Group 5: joseph.triem,alexander.goetz,connor.mizner
Group 6: ryan.barrett2,alexander.knutson,luka.derry
Group 7: mason.barnocky,micaylyn.parker,aaron.loomis
Group 8: nolan.scott1,matthew.nagel,samuel.mosier
Group 9: yebin.wallace,john.fotheringham,ethan.johnson18
Group 10: lucas.jones6,jeremiah.mackey,colter.huber
Group 11: sarah.periolat,luke.donaldson1,owen.obrien
Group 12: tyler.broesel,michael.oswald,erik.moore3
Group 13: joseph.mergenthaler,jett.girard,reid.pickert
Group 14: jacob.ruiz1,kaden.price,devon.maurer
Group 15: pendleton.johnston,julia.larsen,griffin.short
Group 16: justice.mosso,bridger.voss,jacob.shepherd1
Group 17: caitlin.hermanson,jakob.kominsky,carsten.brooks
Group 18: tristan.nogacki,zeke.baumann,connor.graville
Group 19: connor.yetter,delaney.rubb,jonas.zeiler
Group 20: jack.fry,jada.zorn,samuel.rollins
Group 21: william.elder1,timothy.true,peyton.trigg
Group 22: adam.wyszynski,emmeri.grooms,conner.reed1

Group exercises: Induction, and finishing up Boolean Algebra

Show by induction that $2^n < n!$ for all $n \geq 4$.

Reprove the below items using the `properties` in Thm. 7.2 and the fact from Prop 7.3 that $x \implies y$ is equivalent to $(\neg x) \vee y$.

1. A **tautology** is a Boolean expression that evaluates to `TRUE` for all possible values of its variables. For example, the expression $x \vee \neg x$ evaluates to `TRUE` both when $x = \texttt{TRUE}$ and $x = \texttt{FALSE}$. Prove the following:

   (a) $x \implies x$

   (b) $\texttt{FALSE} \implies x$

   (c) $(x \implies y) \wedge (y \implies z) \implies (x \implies z)$

2. A **contradiction** is a Boolean expression that evaluates to `FALSE` for all possible values of its variables. For example, the expression $x \wedge \neg x$ is a contradiction. Prove the following:

   (a) $(x \vee y) \wedge (x \vee \neg y) \wedge \neg x$

   (b) $x \wedge (x \implies y) \wedge (\neg y)$.