01/29/2025: Multiple Proofs

CSCI 246: Discrete Structures

Textbook reference: Ch. 2, Hampkins

Brightspace announcement highlights

- Friday's problems quiz will only cover Secs. 5 & 6 (Proofs & Counterexamples). Some of today's material will be relevant to Proofs.
- Starting tomorrow, Joyce Kelly's office hours are Thursdays 4-6 pm.

New quiz return method

Quizzes are grouped into four bins (A-G, H-L, M-R, S-Z) by last name.
The quizzes are upside down with your last name on the back.
Come find yours before, during, or after class.
Only turn the quiz over if it's yours.

Today's Agenda

- Reading quiz (5 mins)
- Mini-lecture ($\approx$ 15 mins)
  - Additional proofs practice (via Hamkins).
  - Resolving a puzzle from earlier
- Group exercises ($\approx$ 30 mins)
  - Continue the Boolean Algebra exercises.

Reading Quiz

Logistics Alert

Please setup your quizzes in the standard way, but **also** write your last name on the back of the page.

Reading Quiz (Multiple Proofs - Hampkins CH 2)

Use one of the seven methods in the textbook, or perhaps your own method, to prove the theorem below. For `extra credit`, provide two different proofs.

Theorem. *For any natural number n, the number $n^2 - n$ is even.*

Definition

The **natural numbers** are the non-negative integers: $0, 1, 2, 3, \ldots$.

Multiple Proofs

Quote of the Day

"Perhaps the greatest pleasure of mathematics is that, as you try to solve problems, you find that you come to understand new techniques. You see the problem **from a different perspective**, and that's when you make progress."

– Andrew Wiles (who famously solved Fermat's Last Theorem)

Role of this chapter

- To demonstrate *perspective shifting* in mathematics: its existence, beauty, and value.
- Another chance to practice theorems and proofs.

Exercise: Hamkins Ex. 2.1 (First part)

Proposition. The sum, difference, and product of two even numbers is even. (Try it!)

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Proof.

Annotation	Main Text
Convert Prop. to "if-then" form	We show that if x and y are even integers, then $x+y$, $x-y$, and xy are even.
State "if"	Let x and y be even integers
Unravel defs.	Then by the definition of even, there exist integers a, b such that $x = 2a$ and $y = 2b$.
*** The glue ***	What goes here?!?!
Unravel defs.	So there are integers c, d, e such that $x + y = 2c$, $x - y = 2d$, and $xy = 2e$.
State "then"	Hence, $x + y, x - y$ and xy are even.

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Unravel defs.	Then by the definition of even, there exist integers a, b such that $x = 2a$ and $y = 2b$.
*** The glue ***	We have: $x + y = 2a + 2b = 2\underbrace{(a+b)}_{\triangleq c}$ $x - y = 2a - 2b = 2\underbrace{(a-b)}_{\triangleq d}$ $xy = 2a \cdot 2b = 2\underbrace{(2ab)}_{\triangleq e}$
Unravel defs.	So there are integers c, d, e such that $x + y = 2c$, $x - y = 2d$, and $xy = 2e$.
State "then"	Hence, $x + y, x - y$ and xy are even.

Exercise: Hamkins Ex. 2.1 (Second part)

Proposition. The sum and difference of two odd numbers is even, but the product of odd numbers is odd.

Proof.

Annotation	Main Text
Convert Prop. to "if-then"	We show that if x and y are odd integers, then $x+y$ and $x-y$ are even, but xy is odd.
State "if"	Let x and y be odd integers.
Unravel defs.	Then by the definition of odd, there exist integers a, b such that $x = 2a+1$ and $y = 2b+1$.
* The glue *	We have: $x + y = (2a+1) + (2b+1) = 2a + 2b + 2 = 2\underbrace{(a+b+1)}_{\triangleq\, c}$ $x - y = (2a+1) - (2b+1) = 2a - 2b = 2\underbrace{(a-b)}_{\triangleq\, d}$ $xy = (2a+1)(2b+1) = 4ab + 2a + 2b + 1 = 2\underbrace{(2ab+a+b)}_{\triangleq\, e} + 1$
Unravel defs.	So there are integers c, d, e such that $x+y = 2c$, $x-y = 2d$, and $xy = 2e+1$.
State "then"	Hence, $x+y$ and $x-y$ are even, but xy is odd.

Resolving a puzzle from earlier

Puzzle

In Sec. 5 (Proofs), we showed that *zero is divisible by zero* ($0|0$). How can we make sense of this? Many of us have been told that we "can't divide by 0".

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Reminder of Definition 3.2 (Divisible)

Let a and b be integers. We say that a is *divisible* by b provided there is an integer c such that $bc = a$. The notation for this is $b|a$.

Observations

- Although $\frac{b}{a}$ is a **number**, the above shows that $b|a$ is a **proposition**.
- We don't have $0|1$ (`False`), but we do have $0|0$ (`True`).

Resolution to Puzzle

- $\frac{x}{0}$ is problematic, since not all x are divisible by 0. ($0|x$ is `False` for all integers x except $x = 0$.)
- If we take $x = 0$, we do have $0|0$, but we don't know what number $\frac{0}{0}$ equals (i.e. c in the definition could be any integer).

Sec. 7 Reading Quiz Solution

Proposition 7.3 The expressions $x \to y$ and $(\neg x) \vee y$ are logically equivalent.

Proof. We construct a truth table for both expressions.

x	y	$x \to y$	$\neg x$	y	$(\neg x) \vee y$
TRUE	TRUE	TRUE	FALSE	TRUE	TRUE
TRUE	FALSE	FALSE	FALSE	FALSE	FALSE
FALSE	TRUE	TRUE	TRUE	TRUE	TRUE
FALSE	FALSE	TRUE	TRUE	FALSE	TRUE

The columns for $x \to y$ and $(\neg x) \vee y$ are the same, and therefore these expressions are logically equivalent. ■

Figure 1: Solution to Reading Quiz (Sec. 7)

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Figure 1: Solution to Reading Quiz (Sec. 7)

Scoring rubric

- 10/10 points if truth tables for $x \implies y$ and $(\neg x) \vee y$ were both given correctly, and *there was some explanation of how the latter was determined.*
- 7.5/10 points if truth tables for $x \implies y$ and $(\neg x) \vee y$ were both given correctly.

Sec. 7 Reading Quiz Scores

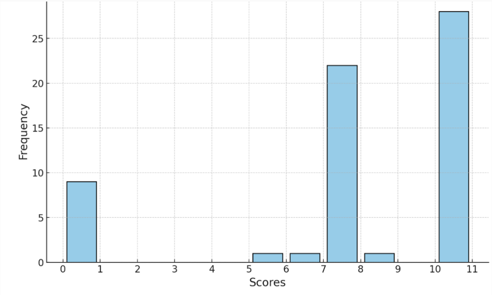

Figure 2: Reading Quiz (Sec. 7)

Group work

Announcements about group work

- Ideas if you get stuck during group exercises:
  - (a) Get my/Fatima's attention.
  - (b) Find other group to give a hint/lead.
  - (c) Use textbook as resource.
- It's okay to get stuck! That's a <u>natural</u> part of learning!

Quote of the Semester

"The best way to learn is to do; the worst way to teach is to talk."

– Paul Halmos, a renowned mathematician and expositor

Group work (Boolean algebra)

Students are randomly assigned into groups of 3 on the next slide.

Each group gets $\frac{1}{2}$ of a white board.

If the $\frac{1}{2}$ white board is inconvenient, feel free to write on a window!

Group 1: tyler.broesel,ryan.barrett2,jada.zorn
Group 2: jacob.ruiz1,mason.barnocky,jacob.ketola
Group 3: devon.maurer,peter.buckley1,cameron.wittrock
Group 4: alexander.knutson,delaney.rubb,peyton.trigg
Group 5: tristan.nogacki,jett.girard,anthony.mann
Group 6: evan.barth,owen.obrien,nicholas.harrington1
Group 7: connor.graville,luka.derry,pendleton.johnston
Group 8: lucas.jones6,william.elder1,luke.donaldson1
Group 9: lynsey.read,connor.yetter,blake.leone
Group 10: alexander.goetz,jakob.kominsky,samuel.mosier
Group 11: colter.huber,samuel.rollins,justice.mosso
Group 12: conner.reed1,jonas.zeiler,james.brubaker
Group 13: joseph.triem,bridger.voss,matthew.nagel
Group 14: julia.larsen,timothy.true,emmeri.grooms
Group 15: jacob.shepherd1,jack.fry,nolan.scott1
Group 16: michael.oswald,carsten.brooks,john.fotheringham
Group 17: evan.schoening,jeremiah.mackey,erik.moore3
Group 18: derek.price4,carver.wambold,aaron.loomis
Group 19: joseph.mergenthaler,yebin.wallace,adam.wyszynski
Group 20: samuel.hemmen,reid.pickert,griffin.short
Group 21: micaylyn.parker,sarah.periolat,zeke.baumann
Group 22: ethan.johnson18,connor.mizner,kaden.price, caitlin.hermanson