02/21/2025: Introduction to Functions

CSCI 246: Discrete Structures

Textbook reference: Ch 11.5, Hampkins

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Announcements

- I have not yet completed requested regrades that were handed to me last class.

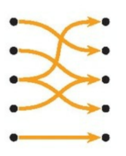
Today's Agenda

- Reading + problems quizzes (15 mins)
- Mini-lecture ($\approx$ 15 mins)
- Group exercises ($\approx$ 15 mins)

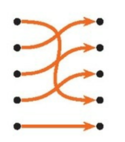
Reading Quiz (Extra Credit)

State whether each relationship below is a function or not. If it is a function, state whether the function is injective (1-to-1), surjective (onto), both, or neither.

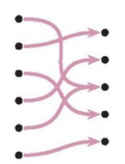
(a)



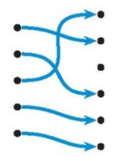
(b)



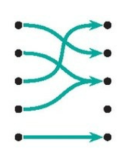
(c)



(d)



(e)



Problems Quiz (Quantifiers and Set Operations)

1. Label each sentence below about the integers as true or false.

   (a) $\exists x, \exists y : x + y = 0$.

   (b) $\forall x, \forall y, x + y = 0$.

   (c) $\exists x : \forall y, x + y = 0$.

   (d) $\forall x, \exists y : x + y = 0$

   (e) $\exists x, \exists y : xy = 0$.

   (f) $\forall x, \forall y, xy = 0$.

   (g) $\forall x, \exists y : xy = 0$.

   (h) $\exists x : \forall y, xy = 0$.

2. How many integers in the range 1 to 1000 (inclusive) are divisible by 2 or by 5? Justify your answer. **Hint:** If A and B be finite sets. Then $|A| + |B| = |A \cup B| + |A \cap B|$.

Additional notes on relations

Reminder of Definition

Given sets A and B the **Cartesian Product** $A \times B$ is just the set of ordered pairs (a, b) where $a \in A, b \in B$.

Example

Suppose $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$. Then the set $A \times B$ is given by

$$\begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5) \end{Bmatrix}$$

Example

Suppose $A = \{a, b, c\}$ and $B = \{a, b, c, d, e, f\}$. Then the set $A \times B$ is given by

$$\begin{Bmatrix} (a,a), & (a,b), & (a,c), & (a,d), & (a,e), & (a,f), \\ (b,a), & (b,b), & (b,c), & (b,d), & (b,e), & (b,f), \\ (c,a), & (c,b), & (c,c), & (c,d), & (c,e), & (c,f) \end{Bmatrix}$$

Perspective on Relations

Any *relation* can be considered as a subset of $A \times B$.

Example

Suppose $A = B = \{1, 2, 3, 4, 5\}$. Then the relation $\boxed{" \leq "}$ (more formally $\{(a, b) : a \leq b\}$) includes (is satisfied by) the elements below shaded in green.

$$\left\{ \begin{array}{ccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5) \end{array} \right\}$$

Perspective on Relations

Any *relation* can be considered as a subset of $A \times B$.

Example

Suppose $A = B = \{1, 2, 3, 4, 5\}$. Then the relation $\boxed{" \geq "}$ (more formally $\{(a, b) : a \geq b\}$) includes (is satisfied by) the elements below shaded in green.

$$\left\{ \begin{array}{ccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5) \end{array} \right\}$$

Remark

Whenever A and B are finite sets, we can draw (or imagine) a matrix of all possible ordered pairs (a, b), and think about a relation in terms of properties of this matrix.

Example

If a relationship is **reflexive**, it must include all diagonal elements.

$$\left\{ \begin{array}{ccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5) \end{array} \right\}$$

The yellow cells could be either included or excluded.

Remark

Whenever A and B are finite sets, we can draw (or imagine) a matrix of all possible ordered pairs (a, b), and think about a relation in terms of properties of this matrix.

Example

If a relationship is **symmetric**, then if some element is included, its transposed element must be included. For example,

$$\left\{ \begin{array}{ccccc} (1,1), & \boxed{(1,2),} & (1,3), & (1,4), & (1,5), \\ \boxed{(2,1),} & (2,2), & (2,3), & \boxed{(2,4),} & (2,5), \\ (3,1), & (3,2), & \boxed{(3,3),} & (3,4), & (3,5), \\ (4,1), & \boxed{(4,2),} & (4,3), & (4,4), & (4,5), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), \end{array} \right\}$$

Functions: A special kind of relation

Remark

A **function** is a special kind of relation.

Definition

A relation $f \subset A \times B$ is a **function** from A to B, written $f : A \to B$, if it exhibits the *function property*: for every $a \in A$, there is a unique $b \in B$ with $(a, b) \in f$.

Example

Here is a relationship that exhibits the function property

$$\left\{ \begin{array}{ccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), \end{array} \right\}$$

Each row has *exactly* one cell shaded green.

Function Properties from the Matrix Point of View

Surjective

- A function is **surjective** (also called *onto*) if every column includes at least one shaded element

Example

$$\left\{ \begin{array}{cccc} (1,1), & (1,2), & (1,3), & (1,4), \\ (2,1), & (2,2), & (2,3), & (2,4), \\ (3,1), & (3,2), & (3,3), & (3,4), \\ (4,1), & (4,2), & (4,3), & (4,4), \\ (5,1), & (5,2), & (5,3), & (5,4) \end{array} \right\}$$

Anti-Example

$$\left\{ \begin{array}{ccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5) \end{array} \right\}$$

Injective

- A function is **injective** (also called *one-to-one*) if columns include at most one shaded element.

Example

$$
\left\{ \begin{array}{ccccc}
(1,1), & (1,2), & (1,3), & (1,4), & (1,5), \\
(2,1), & (2,2), & (2,3), & (2,4), & (2,5), \\
(3,1), & (3,2), & (3,3), & (3,4), & (3,5), \\
(4,1), & (4,2), & (4,3), & (4,4), & (4,5),
\end{array} \right.
$$

Anti-Example

$$
\left\{ \begin{array}{cccc}
(1,1), & (1,2), & (1,3), & (1,4), \\
(2,1), & (2,2), & (2,3), & (2,4), \\
(3,1), & (3,2), & (3,3), & (3,4), \\
(4,1), & (4,2), & (4,3), & (4,4), \\
(5,1), & (5,2), & (5,3), & (5,4)
\end{array} \right\}
$$

Bijective

- A function is **bijective** (also called *a one-to-one correspondence*) if it is both injective and surjective: that is, if each column includes *exactly* one shaded element.

Example

$$\left\{\begin{array}{ccccc} \colorbox{lightgreen}{$(1,1),$} & (1,2), & (1,3), & (1,4), & (1,5), \\ (2,1), & (2,2), & (2,3), & \colorbox{lightgreen}{$(2,4),$} & (2,5), \\ (3,1), & (3,2), & \colorbox{lightgreen}{$(3,3),$} & (3,4), & (3,5), \\ (4,1), & (4,2), & (4,3), & (4,4), & \colorbox{lightgreen}{$(4,5),$} \\ (5,1), & \colorbox{lightgreen}{$(5,2),$} & (5,3), & (5,4), & (5,5), \end{array}\right\}$$

- Because it's **bijective**, each column has *exactly* one cell shaded green.
- (And recall that because it's a **function**, each row has *exactly* one cell shaded green.)

Anti-Examples

Every function shown on the previous two slides is an anti-example.

Applications of Injectivity in Computer Science

Main Idea

An injective function (one-to-one function) ensures that distinct inputs map to distinct outputs. This property is crucial to avoid collisions and loss of information. That is, a non-injective transformation can't be reversed (without loss).

Application: Data Compression (Lossless Encoding)

Huffman coding and arithmetic coding in data compression rely on injective mappings to ensure that no two distinct data sequences are encoded identically. This ensures **accurate reconstruction of the original data**.

Application: Memory Addressing

In virtual memory management, an injective mapping from virtual addresses to physical addresses ensures that each virtual address corresponds to a unique physical location, **preventing data overwriting**.

Applications of Surjectivity in Computer Science

Application: Compiler Design and Code Optimization

In **compiler design**, surjective functions appear in register allocation, where a compiler assigns a set of variables (inputs) to a limited number of CPU registers (outputs).

A surjective mapping ensures that every available register is used efficiently—i.e., each register stores at least one variable, **preventing wasted resources.**

Application: Hash Functions

In a **hash function**, a set of inputs (e.g., file names, user IDs, code commits) is mapped to a set of outputs (hash values – like `EE25G1`).

A **surjective hash function** ensures that every possible output (hash value) has at least one corresponding input, effectively utilizing the full range of available hash values.

In Distributed Hash Tables (DHTs) (used in peer-to-peer networks like BitTorrent and blockchain applications), surjective functions ensure that data is evenly distributed among nodes. This helps in **load balancing** and **efficient retrieval of data**.

Feedback on last reading quiz (relations)

The question asked to generate relations that satisfied some properties.

Poll

The following are answers that some students provided to the question. Are these things relations? Why or why not?

a. Divisibility, i.e. $a|a$ for all $a \in A$.

The question asked to generate relations that satisfied some properties.

Poll

The following are answers that some students provided to the question. Are these things relations? Why or why not?

a. Divisibility, i.e. $a|a$ for all $a \in A$.
   - **Response:** This can't be a relation, because relations are defined on *two* elements of a set (not just one).

b. Multiplication, i.e. $a = 2b$.

The question asked to generate relations that satisfied some properties.

Poll

The following are answers that some students provided to the question. Are these things relations? Why or why not?

a. Divisibility, i.e. $a|a$ for all $a \in A$.
   - **Response:** This can't be a relation, because relations are defined on *two* elements of a set (not just one).

b. Multiplication, i.e. $a = 2b$.
   - **Response:** This can't be a relation, because relations are *binary*. That is, relations can only map a pair of elements (a, b) to the values 0 and 1 (or `Yes` and `no`, or `True` and `False`) designating whether the pair of elements satisfies the condition or not.

c. Intersection, i.e. $A \cap B$.

The question asked to generate relations that satisfied some properties.

Poll

The following are answers that some students provided to the question. Are these things relations? Why or why not?

a. Divisibility, i.e. $a|a$ for all $a \in A$.
   - **Response:** This can't be a relation, because relations are defined on *two* elements of a set (not just one).

b. Multiplication, i.e. $a = 2b$.
   - **Response:** This can't be a relation, because relations are *binary*. That is, relations can only map a pair of elements (a, b) to the values 0 and 1 (or `Yes` and `no`, or `True` and `False`) designating whether the pair of elements satisfies the condition or not.

c. Intersection, i.e. $A \cap B$.
   - **Response:** This can't be a relation, because relations are defined on two *elements* of a set, not the sets themselves.

Scores On Reading Quiz (Extra Credit)

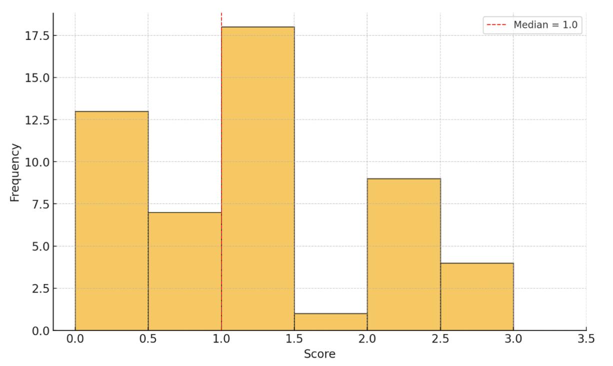


Figure 1: Median Extra Credit Score = 1/3

Group Exercises: Introduction to Functions

1. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Write down all functions $f : A \to B$. Indicate which are one-to-one and which are onto B.
2. Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Write down all functions $f : A \to B$. Indicate which are one-to-one and which are onto B.
3. For each of the following relations, answer the following questions: (i) Is it a function? If not, explain why and stop. Otherwise, continue onward. (ii) What are its domain, codomain, and range? (iii) Is the function injective? (iv) Is it surjective?
   a) $\{(x, y) : x, y \in \mathbb{Z}, y = 2x\}$
   b) $\{(x, y) : x, y \in \mathbb{Z}, xy = 0\}$
   c) $\{(x, y) : x, y \in \mathbb{Z}, y = x^2\}$
4. Suppose $f : A \to B$ is a function where A and B are finite sets. Write psuedocode of an algorithm to test whether f is bijective (a one-to-one correspondence).

Solution to Group Exercise #1

Problem. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Write down all functions $f : A \to B$. Indicate which are one-to-one and which are onto B.

Solution. There are 8 possible functions:

$$F_1 \triangleq \left\{(1,4),(2,4),(3,4)\right\} \qquad F_5 \triangleq \left\{(1,5),(2,4),(3,4)\right\}$$

$$F_2 \triangleq \left\{(1,4),(2,4),(3,5)\right\} \qquad F_6 \triangleq \left\{(1,5),(2,4),(3,5)\right\}$$

$$F_3 \triangleq \left\{(1,4),(2,5),(3,4)\right\} \qquad F_7 \triangleq \left\{(1,5),(2,5),(3,4)\right\}$$

$$F_4 \triangleq \left\{(1,4),(2,5),(3,5)\right\} \qquad F_8 \triangleq \left\{(1,5),(2,5),(3,5)\right\}$$

Of these, all are onto except F_1 and F_8. None of them are one-to-one.

Remark. How did we know there were 8 possible functions?

Solution to Group Exercise #1

Problem. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Write down all functions $f : A \to B$. Indicate which are one-to-one and which are onto B.

Solution. There are 8 possible functions:

$$F_1 \triangleq \Big\{(1,4),(2,4),(3,4)\Big\} \qquad F_5 \triangleq \Big\{(1,5),(2,4),(3,4)\Big\}$$

$$F_2 \triangleq \Big\{(1,4),(2,4),(3,5)\Big\} \qquad F_6 \triangleq \Big\{(1,5),(2,4),(3,5)\Big\}$$

$$F_3 \triangleq \Big\{(1,4),(2,5),(3,4)\Big\} \qquad F_7 \triangleq \Big\{(1,5),(2,5),(3,4)\Big\}$$

$$F_4 \triangleq \Big\{(1,4),(2,5),(3,5)\Big\} \qquad F_8 \triangleq \Big\{(1,5),(2,5),(3,5)\Big\}$$

Of these, all are onto except F_1 and F_8. None of them are one-to-one.

Remark. How did we know there were 8 possible functions? By the multiplication principle, there are $|B|^{|A|} = 2^3 = 8$ possible functions, because there are $|B|$ possible outputs for each of the $|A|$ inputs. (When A, B are finite, you can consider a function $A \to B$ as a list with $|A|$ entries and $|B|$ choices for each entry.)

Solution to Group Exercise #2

Problem. Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Write down all functions $f : A \to B$. Indicate which are one-to-one and which are onto B.

Solution. There are 9 possible functions:

$$F_1 \triangleq \Big\{(1,3),(2,3)\Big\} \qquad F_4 \triangleq \Big\{(1,4),(2,3)\Big\} \qquad F_7 \triangleq \Big\{(1,5),(2,3)\Big\}$$

$$F_2 \triangleq \Big\{(1,3),(2,4)\Big\} \qquad F_5 \triangleq \Big\{(1,4),(2,4)\Big\} \qquad F_8 \triangleq \Big\{(1,5),(2,4)\Big\}$$

$$F_3 \triangleq \Big\{(1,3),(2,5)\Big\} \qquad F_6 \triangleq \Big\{(1,4),(2,5)\Big\} \qquad F_9 \triangleq \Big\{(1,5),(2,5)\Big\}$$

Of these, none are onto. All are one-to-one except F_1, F_5, and F_9.

Remark. By the multiplication principle, we know there are $|B|^{|A|} = 3^2 = 9$ possible functions.

Solution to Group Exercise #3

Problem. For each of the following relations, answer the following questions: (i) Is it a function? If not, explain why and stop. Otherwise, continue onward. (ii) What are its domain, codomain, and range? (iii) Is the function injective? (iv) Is it surjective?

a) $f_1 \triangleq \{(x, y) : x, y \in \mathbb{Z}, y = 2x\}$
b) $f_2 \triangleq \{(x, y) : x, y \in \mathbb{Z}, xy = 0\}$
c) $f_3 \triangleq \{(x, y) : x, y \in \mathbb{Z}, y = x^2\}$

Solution.

(i) Is it a function? f_1 and f_3 are functions because each x maps to exactly one y. f_2 is not a function because $x = 0$ maps to more than one (in fact, infinitely many) y's.

(ii) For all three functions, the domain and codomain are the integers. (This was explicitly stated in the problem.) The range for f_1 is the integers. The range for f_3 is the natural numbers (i.e., $\{0, 1, 2, 3, \ldots\}$).

(iii) f_1 is injective, but f_2 is not. (Think: you can recover x from y for f_1, but not f_2).

(iv) f_1 is surjective, but f_2 is not. This is immediately clear from (ii), because a function is surjective if and only if its range equals its codomain. (Think: the realized function values covers the entire space of the codomain for f_1, but not f_2.)

Solution to Group Exercise #4

Problem. Suppose $f : A \to B$ is a function where A and B are finite sets. Write psuedocode of an algorithm to test whether f is bijective (a one-to-one correspondence).

Psuedocode to test a function is bijective

Input: A function $f : A \to B$, sets A (domain), B (codomain)
Output: Determines if f is bijective
Initialize: An empty set `seen_values` ;

```
/* Check for injectivity                                    */
for each x in A do
    y ← f(x);
    if y is in seen_values then
        Output: "Not injective";
        Exit;
    Add y to seen_values;
/* Check for surjectivity                                   */
if seen_values contains all elements of B then
    Output: "Function is bijective"
else
    Output: "Not surjective"
```