04/01/2025: Recurrence

CSCI 246: Discrete Structures

Textbook reference: Sec 23, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Friday's Problems Quiz

The problems quiz on Friday (04/02) will cover:

- Conditional Probability and Independence
- Random Variables
- Expectations

Today's Agenda

- Reading quiz (5 mins)
- Q & A on Group Exercises (10 mins)
- Mini-lecture ($\approx$ 10 mins)
- Group exercises ($\approx$ 20 mins)

Feedback on Monday’s Quiz

Reading Quiz Scores

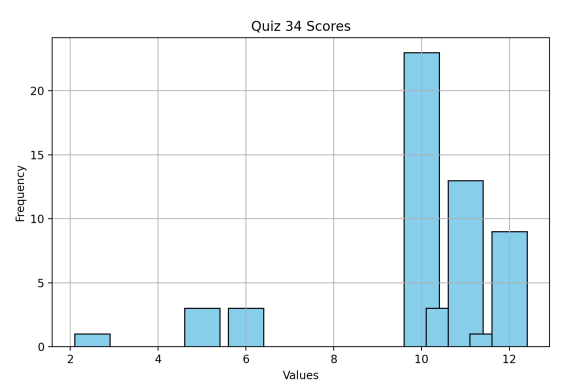

Figure 1: Median Score = 10/10 (100%)

Grading Rubric.

1. 10 point for question on expectations.
2. 2 points extra credit for question on variance.

Today's quiz

Reading Quiz (Recurrence Relations)

Solve the recurrence relation $a_n = 5a_{n-1} + 3$ with initial condition $a_0 = 1$.

A proposition which may be useful for the reading quiz is given below.

Proposition 23.1 from Scheinerman

All solutions to the recurrence relation $a_n = sa_{n-1} + t$ where $s \neq 1$ have the form

$$a_n = c_1 s^n + c_2,$$

where c_1 and c_2 are specific numbers.

Thoughts on Recurrence

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Appeal. This is the best situation to be in. Clear and quick. The textbook gives techniques for obtaining these.

Recurrent computer programs are slow

Consider the program to solve $a_n = 3a_{n-1} - 2a_{n-2}$ with initial conditions $a_0 = 1, a_1 = 5$:

```
procedure get_term(n)
   if (n < 0)
      print 'Illegal argument'
      exit
   end

   if (n == 0)
      return 1
   end

   if (n == 1)
      return 5
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   return 3*get_term(n-1) - 2*get_term(n-2)
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Poll. How many times is this program called in order to calculate a_n?

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Solution. From the last line, the program is called

$$b_n = b_{n-1} + b_{n-2} + 1$$

times to get term a_n.

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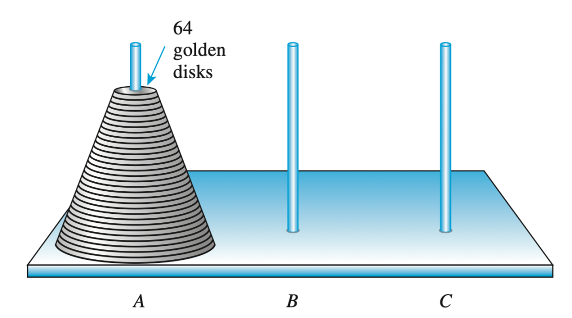
Problem. This program is expensive to run! To get the 50-th term a_{50}, the program must be called $b_{50} \approx 40.7$ billion times!

Tower of Hanoi

In 1883, French mathematician Édouard Lucas invented a puzzle, based on an old Indian legend, called the **Tower of Hanoi**.

Prize. The puzzle offered 10,000 francs (about $45,000 USD today) to anyone who could solve the puzzle with 64 disks.

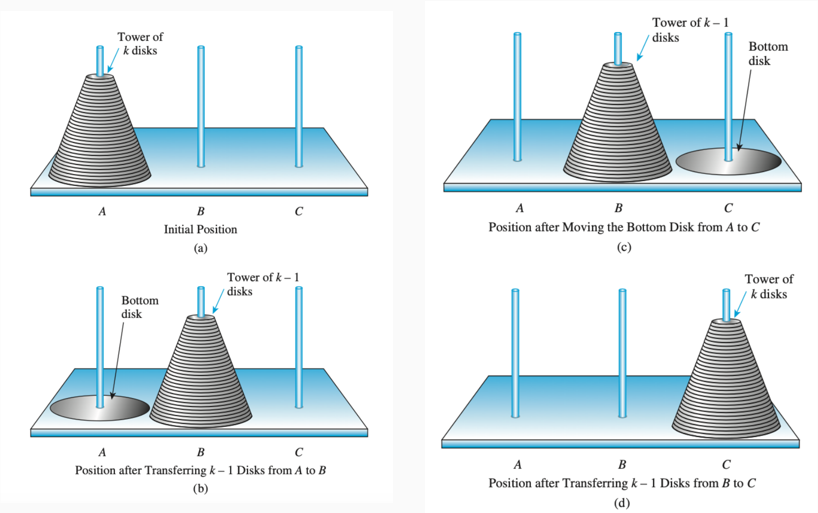
Tower of Hanoi



Rules.

- You must move all disks one by one from one pole to another.
- You must never place a larger disk on top of a smaller one.

Tower of Hanoi: Solution As A Recursion



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Let m_k be the minimum number of moves needed to transfer a tower of k disks from one pole to another.

Poll. Can you provide a recursive formula for m_k?

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Poll. Can you provide a recursive formula for m_k?

Solution. From the last slide, we have

$$\begin{aligned} m_k &= m_{k-1} + 1 + m_{k-1} \\ &= 2m_{k-1} + 1 \end{aligned}$$

Remark. By solving this recursion, we find $m_{64} \approx 1.88 \times 10^{19}$. In other words, if you work rapidly, and move one disk every second, the total number of time it would take to solve the puzzle with 64 disks is: 584.5 billion years!

Examples of recursions

They come up everywhere!

Some examples from textbooks:

- Fibonnaci sequence (e.g. for population growth)
- Compound interest

Some examples from real problems:

- Reinforcement learning (e.g. computer chess)
- Early cancer detection

Group exercises

aaron.loomis: 15
adam.wyszynski: 9
alexander.goetz: 17
alexander.knutson: 1
anthony.mann: 19
blake.leone: 13
bridger.voss: 15
caitlin.hermanson: 20
cameron.wittrock: 13
carsten.brooks: 4
carver.wambold: 8
colter.huber: 20
conner.reed1: 4
connor.mizner: 3
connor.yetter: 10
derek.price4: 18
devon.maurer: 11
emmeri.grooms: 7
erik.moore3: 10
ethan.johnson18: 6
evan.barth: 9
evan.schoening: 16
griffin.short: 9
jack.fry: 14
jacob.ketola: 15
jacob.ruiz1: 12
jacob.shepherd1: 6
jada.zorn: 16
jakob.kominsky: 21
james.brubaker: 2
jeremiah.mackey: 14
jett.girard: 20
john.fotheringham: 16
jonas.zeiler: 5
joseph.mergenthaler: 17
joseph.triem: 5
julia.larsen: 3
justice.mosso: 13
kaden.price: 10
lucas.jones6: 8
luka.derry: 14
luke.donaldson1: 8
lynsey.read: 12
mason.barnocky: 3
matthew.nagel: 21
micaylyn.parker: 12
michael.oswald: 19
nolan.scott1: 7
owen.obrien: 11
pendleton.johnston: 11
peter.buckley1: 19
reid.pickert: 2
ryan.barrett2: 18
samuel.hemmen: 1
samuel.mosier: 17
samuel.rollins: 4
sarah.periolat: 7
timothy.true: 2
tristan.nogacki: 21
tyler.broesel: 5
william.elder1: 1
yebin.wallace: 18
zeke.baumann: 6

Group exercises

1. Some so-called intelligence tests often include problems in which a series of numbers is presented and the subject is required to find the next term of the sequence. For example, the sequence might begin 1,2,4,8. No doubt the examiner is looking for 16 as the next term.

   Show how to "outsmart" the intelligence test by finding a polynomial expression (of degree 3) for a_n such that $a_0 = 1, a_1 = 2, a_2 = 4, a_3 = 8$, but $a_4 = 15$.
2. Solve each of the following recurrence relations by giving an explicit formula for a_n. For each, please calculate a_9.

Problem	Recurrence relation	Initial condition
2a.	$a_n = 3a_{n-1} - 1$	$a_0 = 10$
2b.	$a_n = -a_{n-1}$	$a_0 = 5$
2c.	$a_n = 8a_{n-1} - 15a_{n-2}$	$a_0 = 1,\ a_1 = 4$
2d.	$a_n = -2a_{n-1} - a_{n-2}$	$a_0 = 5,\ a_1 = 1$