02/12/2025: Quantifiers

CSCI 246: Discrete Structures

Textbook reference: Sec. 11, Scheinerman

Graded Quiz Pickup

Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

Today's Agenda

- Reading quiz (5 mins)
- Mini-lecture ($\approx$ 15 mins)
- Group exercises ($\approx$ 25 mins)

Reading Quiz (Quantifiers)

Let $A = \{x \in \mathbb{Z} : 6|x\}$. Prove that $\forall x \in A$, x is even.

Results on Monday’s Reading Quizzes

Reading Quiz Scores: Sets, Part I

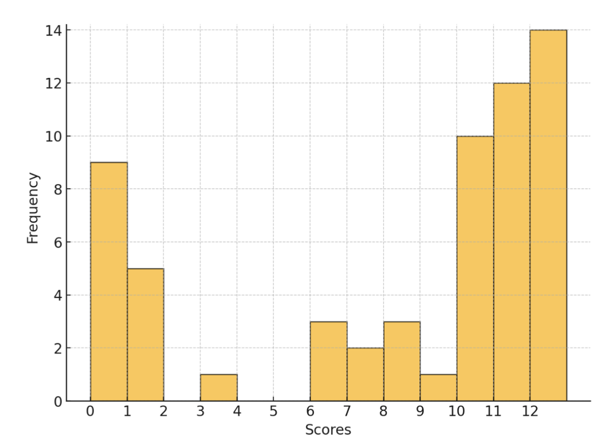

Figure 1: Median Total Score = 10.3/10 (103%)

Formula for total score. A total score was given out of 10. This was obtained by taking 5 times your score on the reading quiz question (2 pts total) and adding half of your extra credit score (scored out of 4).

Reading Quiz: Extra Credit Scores

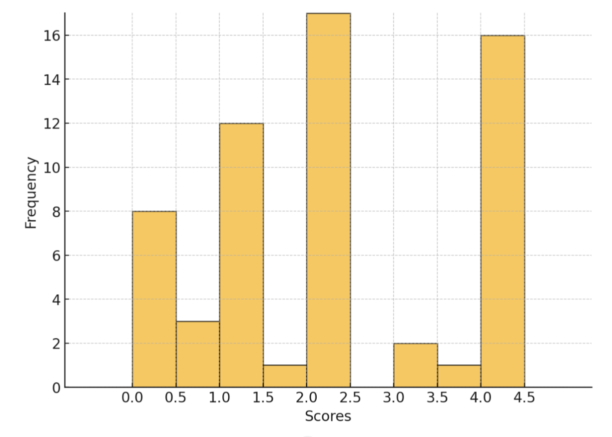

Figure 2: Median E.C. Score = 2/4 (50%)

Remark. Most people chose the set equality question. There were various issues with this that we will discuss in class today. ★ Make sure you can answer the set equality question correctly for Friday's problems quiz ★ .

Feedback on Sets I

Problem

Prove that the following two sets are equal:

$$E = \{x \in \mathbb{Z} : x \text{ is even}\}, \text{and}$$
$$F = \{x \in \mathbb{Z} : x = a + b, \text{where } a \text{ and } b \text{ are both odd}\}$$

Puzzle

Evaluate the student solution below.

Student Solution

If a and b are both odd, then we can write $a = 2c + 1$ and $b = 2d + 1$, where both c and d are integers. Now

$$a + b = (2c + 1) + (2d + 1) = 2c + 2d + 2 = 2(c + d + 1).$$

That is, $a + b = 2e$, where $e \triangleq c + d + 1$. Hence, $a + b$ is even. Hence, the sets E and F are equal.

How to show two sets are equal

Proving two sets are equal

To show $A = B$, we show $A \subseteq B$ and $B \subseteq A$.

- $\boxed{A \subseteq B.}$ We show that if $x \in A$, then $x \in B$.
- $\boxed{B \subseteq A.}$ We show that if $x \in B$, then $x \in A$.

How to show two sets are equal

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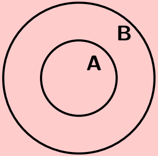
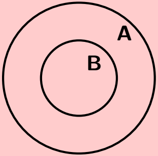
Visualization

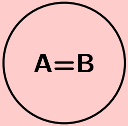
The first bulletpoint shows



The second bulletpoint shows



Therefore



Solution to problem

Problem. Prove that the following two sets are equal:

$$E = \{x \in \mathbb{Z} : x \text{ is even}\}, \text{and}$$
$$F = \{x \in \mathbb{Z} : x = a + b, \text{where } a \text{ and } b \text{ are both odd}\}$$

Solution. To show $E = F$, we show $E \subseteq F$ and $F \subseteq E$.

- $\boxed{E \subseteq F.}$ We show that if $x \in E$, then $x \in F$. Let $x \in E$. Then $x = 2a$ for some integer a. Now we write

$$x = 2a = \underbrace{(2a+1)}_{\text{odd (by def.)}} + \underbrace{(-1)}_{\text{odd}}$$

  Hence x is the sum of two odd numbers. Hence $x \in F$.
- $\boxed{F \subseteq E.}$ We show that if $x \in F$, then $x \in E$. Let $x \in F$. Hence $x = a + b$ for odd integers a, b. By the definition of odd, we can write $a = (2c+1)$ and $b = (2d+1)$ for integers c, d. Hence

$$x = a + b = (2c+1) + (2d+1) = 2c + 2d + 2 = 2(c + d + 1)$$

  That is, $a + b = 2e$, where $e \triangleq c + d + 1$. Hence, $a + b$ is even.

Q&A on the Group Exercises for Sets, Part I

Group 1: jack.fry,caitlin.hermanson,luka.derry
Group 2: joseph.triem,alexander.goetz,mason.barnocky
Group 3: lucas.jones6,evan.schoening,ethan.johnson18
Group 4: ryan.barrett2,jonas.zeiler,alexander.knutson
Group 5: anthony.mann,kaden.price,jett.girard
Group 6: nolan.scott1,devon.maurer,jacob.ruiz1
Group 7: jacob.shepherd1,joseph.mergenthaler,zeke.baumann
Group 8: peter.buckley1,jacob.ketola,derek.price4
Group 9: timothy.true,griffin.short,tyler.broesel
Group 10: jakob.kominsky,erik.moore3,yebin.wallace
Group 11: adam.wyszynski,connor.yetter,john.fotheringham
Group 12: james.brubaker,colter.huber,matthew.nagel
Group 13: aaron.loomis,delaney.rubb,conner.reed1
Group 14: carver.wambold,lynsey.read,blake.leone
Group 15: tristan.nogacki,luke.donaldson1,samuel.mosier
Group 16: michael.oswald,justice.mosso,pendleton.johnston
Group 17: jada.zorn,emmeri.grooms,micaylyn.parker
Group 18: samuel.hemmen,bridger.voss,carsten.brooks
Group 19: sarah.periolat,connor.mizner,cameron.wittrock
Group 20: evan.barth,reid.pickert,connor.graville
Group 21: william.elder1,peyton.trigg,samuel.rollins
Group 22: jeremiah.mackey,julia.larsen,owen.obrien

Group Exercises: Quantifiers

1. Label each sentence below about the integers as true or false.

   a. $\forall x, \forall y, x + y = 0$.

   b. $\forall x, \exists y : x + y = 0$.

   c. $\exists x : \forall y, x + y = 0$.

   d. $\exists x, \exists y : x + y = 0$.

   e. $\forall x, \forall y, xy = 0$.

   f. $\forall x, \exists y : xy = 0$.

   g. $\exists x : \forall y, xy = 0$.

   h. $\exists x, \exists y : xy = 0$.

2. The notation $\exists!$ can be read "there is a unique." Label each sentence below true or false.

   a. $\exists! x \in \mathbb{N} : x^2 = 4$.

   b. $\exists! x \in \mathbb{Z} : x^2 = 4$.

   c. $\exists! x \in \mathbb{N} : x^2 = 3$.

   d. $\exists! x \in \mathbb{Z} : \forall y \in \mathbb{Z}, xy = x$.

   e. $\exists! x \in \mathbb{Z} : \forall y \in \mathbb{Z}, xy = y$.

3. For each sentence below, write the negation of the sentence, but place the $\neg$ symbol as far to the right as possible. Then rewrite the negation in English. The first problem is done for you.

   a. $\forall x \in \mathbb{Z}, x$ is odd. Solution: $\exists x \in \mathbb{Z} : \neg(x$ is odd.). In English: "There is an integer which is not odd."

   b. $\forall x \in \mathbb{Z}, x < 0$.

   c. $\exists x \in \mathbb{Z} : x = x + 1$.

   d. $\exists x \in \mathbb{N} : x > 10$.

   e. $\forall x \in \mathbb{N}, x + x = 2x$.

   f. $\exists x \in \mathbb{Z} : \forall y \in \mathbb{Z}, x > y$.

   g. $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x = y$.

   h. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : x + y = 0$.

Solutions to Group Exercises #1

Label each sentence below about the integers as true or false.

a. $\forall x, \forall y, x + y = 0$. False. (Counter-example: $x = 1, y = 1$.)
b. $\forall x, \exists y : x + y = 0$. True. (Set $y = -x$.)
c. $\exists x : \forall y, x + y = 0$. False. (Counter-example: If $x = 0$, set $y = 1$. If $x \neq 0$, set $y = 0$.)
d. $\exists x, \exists y : x + y = 0$. True. (Set $x = 1, y = -1$.)
e. $\forall x, \forall y, xy = 0$. False. (Counterexample: set $x = y = 1$.)
f. $\forall x, \exists y : xy = 0$. True. (Set $y = 0$.)
g. $\exists x : \forall y, xy = 0$. True. (Set $x = 0$.)
h. $\exists x, \exists y : xy = 0$. True. (Set $x = y = 0$.)

Solutions to Group Exercises #2

The notation $\exists!$ can be read "there is a unique." Label each sentence below true or false.

a. $\exists! x \in \mathbb{N} : x^2 = 4$. True. ($x = 2$ is the only solution to the equation in the natural numbers.)
b. $\exists! x \in \mathbb{Z} : x^2 = 4$. False. ($x = 2$ and $x = -2$ are two different integers which solve the equation.)
c. $\exists! x \in \mathbb{N} : x^2 = 3$. False. (There is no solution to the equation in the set of natural numbers.)
d. $\exists! x \in \mathbb{Z} : \forall y \in \mathbb{Z}, xy = x$. True. ($x = 0$ is the only solution to the equation in the integers.)
e. $\exists! x \in \mathbb{Z} : \forall y \in \mathbb{Z}, xy = y$. True. ($x = 1$ is the only solution to the equation in the integers.)

Solutions to Group Exercises #3 a-d

For each sentence below, write the negation of the sentence, but place the $\neg$ symbol as far to the right as possible. Then rewrite the negation in English.

a. $\forall x \in \mathbb{Z}, x$ is odd. Solution: $\exists x \in \mathbb{Z} : \neg(x$ is odd$)$. In English: "There is an integer which is not odd." Alternatively, we could write the negated proposition as $\exists x \in \mathbb{Z} : (x$ is even$)$. In English: "There is an integer which is even." Note: the negated proposition is True.

b. $\forall x \in \mathbb{Z}, x < 0$. Solution: $\exists x \in \mathbb{Z} : \neg(x < 0.)$. In English: "There is an integer which is not less than zero." Alternatively, we could write $\exists x \in \mathbb{Z} : x \geq 0$. In English: "There is an integer which is greater than or equal to zero." Note: the negated proposition is True.

c. $\exists x \in \mathbb{Z} : x = x + 1$. Solution: $\forall x \in \mathbb{Z} : \neg(x = x + 1)$. That is, $\forall x \in \mathbb{Z} : (x \neq x + 1)$. In English: "No integer equals one plus itself." Note: the negated proposition is True.

d. $\exists x \in \mathbb{N} : x > 10$. Solution: $\forall x \in \mathbb{N} : \neg(x > 10)$. Alternatively, $\forall x \in \mathbb{N} : x \leq 10$. In English: "Each natural number is less than or equal to 10." Note: the negated proposition is False.

Solutions to Group Exercises #3 e-h

For each sentence below, write the negation of the sentence, but place the $\neg$ symbol as far to the right as possible. Then rewrite the negation in English.

e. $\forall x \in \mathbb{N}, x + x = 2x$. Solution: $\exists x \in \mathbb{N} : \neg(x + x = 2x)$. That is, $\exists x \in \mathbb{N} : x + x \neq 2x)$. In English: "There is a natural number such that the sum of the number with itself does not equal double itself." Note: the negated proposition is False.

f. $\exists x \in \mathbb{Z} : \forall y \in \mathbb{Z}, x > y$. Solution: $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : \neg(x > y)$. That is, $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : x \leq y$ In English: "For all integers, there is another integer which is at least as large as it." Note: the negated proposition is True .

g. $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x = y$. Solution: $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z} : \neg(x = y)$. That is, $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z} : x \neq y$. In English: "There are two integers which do not equal each other." Note: the negated proposition is True .

h. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : x + y = 0$. Solution: $\exists x \in \mathbb{Z} : \forall y \in \mathbb{Z} : \neg(x + y = 0)$. That is, $\exists x \in \mathbb{Z} : \forall y \in \mathbb{Z} : (x + y \neq 0)$. In English: "There is an integer such that the sum with all other integers does not equal zero." Note: the negated proposition is False .