Monday 01/27/2025: Boolean Algebra

CSCI 246: Discrete Structures

Textbook reference: Sec. 7, Scheinerman

New way to get quizzes back!

Each student may line up either before class (2:00-2:10) or after class (3:00-3:10) to collect graded quizzes from the previous class from Fatima.

This will hopefully free up Fatima's time during class to assist with group exercises.

Today's Agenda

- Reading quiz (5 mins)
- Mini-lecture ($\approx$ 15 mins)
  - Review Friday quizzes.
  - Go over Sec. 6 group problems
- Group exercises ($\approx$ 25 mins)

Reading Quiz

Reading Quiz (Sec. 7 - Boolean Algebra)

Use a truth table to prove that the expressions $x \implies y$ and $(\neg x) \vee y$ are logically equivalent.

Notation reminder

- $\implies$ means `implies`
- $\neg$ means `not`
- $\vee$ means `or`

Mini-lecture

Feedback on your reading for Sec. 6 (counterexample)

Poll

Is the following statement true or false?

$$-5|5 = -1$$

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Alert!

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Reminder of Definition 3.2 (Divisible)

Let a and b be integers. We say that a is *divisible* by b provided there is an integer c such that $bc = a$. The notation for this is $b|a$.

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Solution to poll: The statement is **incorrect**! Why?

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Solution to poll: The statement is **incorrect**! Why? $-5|5$ is a **proposition**, so it can't equal -1. We would instead write just $-5|5$, or perhaps $-5|5 = \texttt{TRUE}$. How do we justify the proposition?

Feedback on your reading for Sec. 6 (counterexample)

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Solution to poll: The statement is **incorrect**! Why? $-5|5$ is a **proposition**, so it can't equal -1. We would instead write just $-5|5$, or perhaps $-5|5 = \texttt{TRUE}$. How do we justify the proposition? We have $-5|5$ because if $a = 5$ and $b = -5$, there exists an integer (namely $c = -1$) such that $bc = a$.

Feedback on Sec. 6 reading

Reading Quiz (Sec. 6 - Counterexamples)

Disprove the following conjecture: *Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.* You can disprove the conjecture by providing a counterexample. **Make sure to show that your counterexample satisfies the hypothesis (the "if" statement).**

Common errors

1. **Not justifying the counterexample.** Many people correctly gave a counterexample, e.g. $a = 5, b = -5$, but did not justify why $-5|5$.
2. **Incorrect use of notation.** Many people wrote $-5|5 = -1$. (However, I didn't take off points for this error.)

Reading Quiz Scores

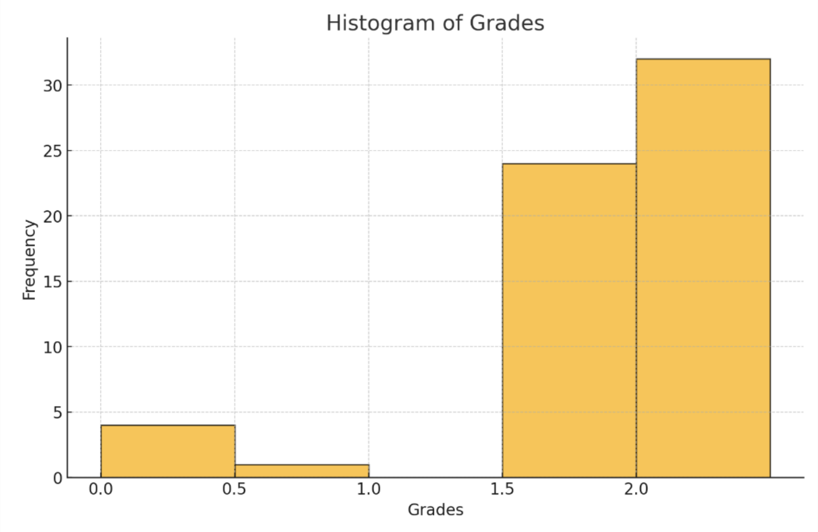

Figure 1: Reading Quiz (Sec. 6)

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Tips for Reading Math Textbooks

Work through examples.

- Don't just skim (or worse, skip!) the examples. Instead, devote more of your focus to the examples. Working through them and verbalizing the main ideas behind them is a great way to test to make sure you're understanding what you're reading.
- Read with a pencil and paper in your hand and when you get to an example, work it out! Try not to look at the solution until you are done.
- When checking your work, make sure you understand each step and why.
- Practice the examples BEFORE you attempt homework problems and then try to do your homework without referring back to the example

Source: https://learningcenter.unc.edu/tips-and-tools/readingmathtexts/.

My solution to reading Quiz (Sec 6 - Counterexamples)

Disprove the following conjecture: *Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.*

Annotation	Main Text
Structure	Let $a = 5$, $b = -5$. First, we show that the hypothesis holds [i.e., that $(5|-5)$ and $(-5|5)$].
Unravel defn.	$a|b$ means there is an integer x such that $ax = b$. Likewise $b|a$ means there is an integer y such that $by = a$.
The "Glue"	Substituting for a and b, we need to show that there are integers x and y such that $5x = -5$ and $-5y = 5$. We see these equations hold by taking $x = -1$ and $y = -1$.
Structure	Hence $5|-5$ and $-5|5$, so the hypothesis is met.
Structure	Now we show that the conclusion fails [i.e. that $-5 \neq 5$.]
The "Glue"	This is immediately clear.

Remark. I scored out of 2 points, and gave 1.5 points for a correct counterexample, and 0.5 points for any argument that the hypothesis holds.

An interesting question

Reading Quiz (Sec. 6 - Counterexamples)

Disprove the following conjecture: *Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.*

Poll

Is $a = 1, b = 0$ a valid counterexample?

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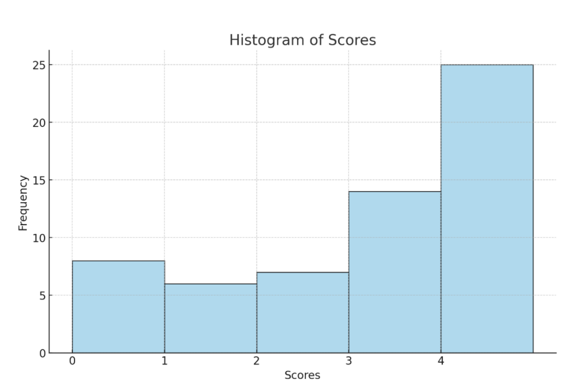
Solution to poll: We have $a|b = 1|0$, since there is an integer c such that $ac = b$ (that is, $1 \cdot c = 0$). However, we do not have $b|a = 0|1$, since there is no integer d such that $bd = a$ (that is, there is no integer d such that $0 \cdot d = 1$). That is, the hypothesis doesn't hold, so the statement doesn't apply.

Problems Quiz Scores



Most common error: Most students with 3/4 correctly wrote the truth table columns for $A \implies B, B \implies A$ and $A \iff B$, but but didn't argue why/how $A \iff B$ is identical to $(A \implies B)\,\texttt{and}\,(B \implies A)$.

Solution: The question was exactly a group exercise from Sec. 4 (Theorems). See slides from that section for solution.

Review solutions to Sec. 6 group exercises

Sec. 7 (Boolean algebra) group work!

Students are randomly assigned into groups of 3 on the next slide.

Each group gets $\frac{1}{2}$ of a white board.

If the $\frac{1}{2}$ white board is inconvenient, feel free to write on a window!

Group 1: justice.mosso,owen.obrien,nicholas.harrington1
Group 2: joseph.triem,connor.yetter,anthony.mann
Group 3: peter.buckley1,evan.barth,jeremiah.mackey
Group 4: devon.maurer,carsten.brooks,jacob.ketola
Group 5: kaden.price,michael.oswald,blake.leone
Group 6: emmeri.grooms,griffin.short,carver.wambold
Group 7: caitlin.hermanson,connor.mizner,connor.graville
Group 8: jack.fry,samuel.mosier,tyler.broesel
Group 9: ethan.johnson18,lucas.jones6,conner.reed1
Group 10: micaylyn.parker,peyton.trigg,jacob.shepherd1
Group 11: samuel.hemmen,cameron.wittrock,nolan.scott1
Group 12: yebin.wallace,alexander.knutson,colter.huber
Group 13: pendleton.johnston,reid.pickert,jada.zorn
Group 14: luke.donaldson1,joseph.mergenthaler,jonas.zeiler
Group 15: ryan.barrett2,william.elder1,samuel.rollins
Group 16: jacob.ruiz1,aaron.loomis,lynsey.read
Group 17: zeke.baumann,delaney.rubb,james.brubaker
Group 18: erik.moore3,derek.price4,sarah.periolat
Group 19: alexander.goetz,tristan.nogacki,jett.girard
Group 20: mason.barnocky,jakob.kominsky,luka.derry
Group 21: julia.larsen,bridger.voss,evan.schoening
Group 22: john.fotheringham,adam.wyszynski,matthew.nagel, timothy.true

Group exercises

1. DeMorgan's laws are:
$$\neg(x \wedge y) = (\neg x) \vee (\neg y) \quad \text{and} \quad \neg(x \vee y) = (\neg x) \wedge (\neg y)$$
Prove the first of these (using truth tables). Then use DeMorgan's law to show how to disprove an if-and-only-if statement.
2. A **tautology** is a Boolean expression that evaluates to `TRUE` for all possible values of its variables. For example, the expression $x \vee \neg x$ evaluates to `TRUE` both when $x = \texttt{TRUE}$ and $x = \texttt{FALSE}$. Use truth tables to show the following are tautologies:
   (a) $(x \vee y) \vee (x \vee \neg y)$
   (b) $x \implies x$
   (c) $\texttt{FALSE} \implies x$
   (d) $(x \implies y) \wedge (y \implies z) \implies (x \implies z)$
3. A **contradiction** is a Boolean expression that evaluates to `FALSE` for all possible values of its variables. For example, the expression $x \wedge \neg x$ is a contradiction. Use truth tables to show that the following are contradictions:
   (a) $(x \vee y) \wedge (x \vee \neg y) \wedge \neg x$
   (b) $x \wedge (x \implies y) \wedge (\neg y)$.
4. Reprove the items in #2 and #3 using the properties in Theorem 7.2 and the fact from Prop 7.3 that $x \implies y$ is equivalent to $(\neg x) \vee y$.