Friday 01/24/2025: Counterexample

CSCI 246: Discrete Structures

Textbook reference: Sec. 6, Scheinerman

Quiz Set up

- **Sheet of paper**: Please bring your own sheet of paper to class each day for quizzes if possible. However, if you don't have any, you are welcome to take a blank sheet of paper from the stack in the front of the room.
- **Rules for quizzes**: For all quizzes in the course, you should use only paper and pencil. Please close your computers and textbooks, and put away your cellphones.

Today's Agenda

- Reading quiz / problems quiz (10 mins)
- Mini-lecture ($\approx$ 15 mins)
  - Go over Sec. 5 group problems
  - Comments on Sec. 5 reading quiz
- Group exercises ($\approx$ 25 mins)

Friday Quiz

Reading Quiz (Sec. 6 - Counterexamples)

Disprove the following conjecture:

Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.

Note: You can disprove the conjecture by providing a counterexample. Make sure to show that your counterexample satisfies the hypothesis (the "if" statement), but not the conclusion (the "then" statement).

Problems Quiz (Sec. 4 - Theorems)

Two propositions are considered *equivalent* if they have the same truth table values. Show that the biconditional $A \iff B$ is equivalent to $(A \implies B)\,\texttt{and}\,(B \implies A)$.

Mini-lecture

Review solutions to Sec. 5 group exercises

Observation on Sec. 5 (Proofs) reading quiz

Observation

A number of students wrote out expressions that look something like

$$2|x + 2|y = 2|z$$

Warning

This expression has no meaning. Note from the definition of divisibility (see below) that $2|x$ is itself *already* an equation. The notation $2|x$ *means* that there is an integer a such that $2a = x$.

Definition (**Divisible**)

Let a and b be integers. We say that a is *divisible* by b (notated as $b|a$) provided there is an integer c such that $bc = a$.

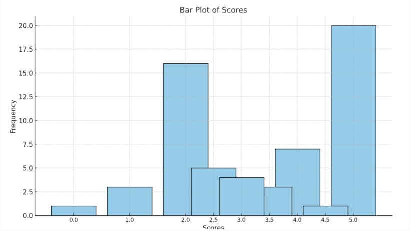
How I graded the reading quiz for Sec. 5 (Proofs)

Proposition. The sum of two even integers is even.

Proof.

Points	Role	Text
	Convert Prop. to "if-then" form	We show that if x and y are even integers, then $x + y$ is even.
1	State "if"	Let x and y be even integers
1	Unravel defs.	Then by Defs. 3.1 and 3.2, there exist integers a, b such that $x = 2a$ and $y = 2b$.
1	*** The glue ***	Hence, $x + y = 2a + 2b = 2(a + b)$.
1	Unravel defs.	So there is an integer $c = a + b$ such that $x + y = 2c$.
1	State "then"	Hence, $x + y$ is even.

Sec 5. Reading Quiz Scores



Great job to 1/3 of the class!

Thoughts if your scores are lower than you want:

- **Strategy** - Read *actively*. Try to work out examples on own while reading
- **Time** - Can you increase the time you spend on the reading?
- **Patience and persistence** - You may still be adjusting to mathematical thinking (and/or the quiz formats). Hang in there. Play the long game.
    - Each reading quiz is less than 1% of your total grade. You just need to catch on eventually.
    - The material will be retested (problems quiz and final), and that carries about *twice* as much weight.
- **Recruit help** - Get help from me, Fatima (TA), Kelly Joyce (tutor), or elsewhere.

Sec. 6 (counterexample) group work!

Students are randomly assigned into groups of 3 on the next slide.

Each group gets $\frac{1}{2}$ of a white board.

If the $\frac{1}{2}$ white board is inconvenient, feel free to write on a window!

Group 1: joseph.mergenthaler,pendleton.johnston,blake.leone
Group 2: connor.yetter,jacob.ketola,mason.barnocky
Group 3: michael.oswald,carsten.brooks,connor.graville
Group 4: emmeri.grooms,luka.derry,ryan.barrett2
Group 5: connor.mizner,kaden.price,anthony.mann
Group 6: nolan.scott1,bridger.voss,jack.fry
Group 7: lynsey.read,yebin.wallace,ethan.johnson18
Group 8: william.elder1,colter.huber,sarah.periolat
Group 9: john.fotheringham,jonas.zeiler,nicholas.harrington1
Group 10: peyton.trigg,tyler.broesel,micaylyn.parker
Group 11: jakob.kominsky,james.brubaker,alexander.knutson
Group 12: devon.maurer,jett.girard,samuel.mosier
Group 13: samuel.rollins,cameron.wittrock,jacob.ruiz1
Group 14: caitlin.hermanson,conner.reed1,owen.obrien
Group 15: julia.larsen,reid.pickert,alexander.goetz
Group 16: zeke.baumann,jacob.shepherd1,jeremiah.mackey
Group 17: evan.schoening,griffin.short,joseph.triem
Group 18: samuel.hemmen,delaney.rubb,derek.price4
Group 19: adam.wyszynski,carver.wambold,justice.mosso
Group 20: jada.zorn,lucas.jones6,timothy.true
Group 21: matthew.nagel,luke.donaldson1,peter.buckley1
Group 22: aaron.loomis,evan.barth,tristan.nogacki, erik.moore3

Group exercises

1. Disprove: If a and b are integers with $a|b$, then $a \leq b$.
2. Disprove: If p and q are prime, then $p + q$ is composite.
3. Disprove: An integer x is positive if and only if $x + 1$ is positive.
4. What does it mean for an if-and-only-if statement to be false? What properties should a counterexample for an if-and-only-if statement have?

(Optional.) If you have extra time, you might try these for extra practice:

a. Disprove: If a,b, and c are positive integers with $a|bc$, then $a|b$ or $a|c$.

b. Disprove: If p is prime, then $2^p - 1$ is also prime.

Group exercise #1: Solution

Problem. Disprove: If a and b are integers with $a|b$, then $a \leq b$.

Solution (longer). Let $a = -5$ and $b = 5$. We will show that for this choice of a and b, the hypothesis holds (i.e. $b|a$), but the conclusion doesn't (i.e. $b > a$). By definition of divisibility, $b|a$ means that there is an integer c such that $ac = b$. In this case, we need to show that there is an integer c such that $5c = -5$. Indeed, the equation holds for $c = -1$. Therefore, $b|a$, and the hypothesis holds. However, clearly $b > a$, and so the conclusion fails.

Solution (shorter). Let $a = -5$ and $b = 5$. We will show that the hypothesis holds (i.e. $5| -5$), but the conclusion doesn't (i.e. $5 > -5$). To verify the hypothesis $5| -5$, note that there is an integer $c = -1$ such that $5c = -5$. We immediately see that $5 > -5$, and so the conclusion fails.

Remark. In here and the following solutions , I provide longer solutions to clarify the logic for students who are struggling. However, in practice, feel free to provide shorter solutions, such as the one above.

Group exercise #2: Solution

Problem. Disprove: If p and q are prime, then $p+q$ is composite.

Solution (longer). Let $p=2$ and $q=3$. We will show that for the counterexample, the hypothesis holds (i.e. 2 and 3 are prime), but the conclusion doesn't (i.e. $r = 2 \cdot 3 = 5$ is not composite). By definition of prime, an integer s is prime if $s > 1$ and the only positive divisors of s are 1 and s. When $p = 2$, we have that $2 > 1$ and the only positive divisors are 2 and 1, hence it is prime. A similar statement shows that q and r are prime. Hence, p and q are prime, and the hypothesis holds. Moreover, r is prime, and therefore not composite, and so the conclusion fails.

Group exercise #3: Solution

Problem. An integer x is positive if and only if $x + 1$ is positive.

Solution (longer). Let A be the proposition that an integer x is positive, and B be the proposition that $x + 1$ is positive. We can show that $A \iff B$ fails by showing that $B \implies A$ fails. We show that there exists a case where B is true, but A is false. In particular, take $x = 0$. Then B is true (since $x + 1 = 1$ is positive), but A fails (since $x = 0$ is not positive.)

Group exercise #4: Solution

Problem. What does it mean for an if-and-only-if statement to be false? What properties should a counterexample for an if-and-only-if statement have?

Solution. Recall from the group exercises of Sec. 4 (Theorems) that $A \iff B$ is identical to $(A \implies B)\,\texttt{and}\,(B \implies A)$. Hence, we can show that $A \iff B$ fails by showing that either $A \implies B$ fails or $B \implies A$ fails. (For more information on this strategy, see the group exercise on DeMorgan's law in Sec. 7, Boolean Algebra.)