Friday 01/17/2025: Theorems

CSCI 246: Discrete Structures

Quiz

Replace each ? with a checkmark ✓ if the combination of truth values for propositions A and B is *possible* under the given logical connective. Replace it with a ✗ if the combination is *impossible*.

Propositions		**Logical Connectives**		
A	B	if A then B	if B then A	A if and only if B
T	T	?	?	?
T	F	?	?	?
F	T	?	?	?
F	F	?	?	?

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F	F	?	?	?

See whiteboard for solution.

Propositional logic

Original propositions		New propositions		
A	B	if A then B	if B then A	A if and only if B
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

- The column headings show 3 new propositions, formed from the original propositions by **logical connectives**.
- The first two columns combined with one remaining column gives the **truth table** for that logical connective.
- Each logical connective can be thought of as a **function** or mapping $\{T, F\} \times \{T, F\} \to \{T, F\}$.
- There are other such functions (`and`, `or`, `xor`, etc.), some of which were discussed in the text.
- The study of how to combine and change propositions under logical connectives to form more complex propositions is called **propositional logic**.

Group work!

Students are randomly assigned into groups of 3 on the next slide.

Each group gets $\frac{1}{2}$ of a white board.

If the $\frac{1}{2}$ white board is inconvenient, feel free to write on a window!

Group 1: timothy.true,conner.reed1,connor.mizner
Group 2: jacob.ruiz1,evan.barth,evan.schoening
Group 3: matthew.nagel,connor.graville,adam.wyszynski
Group 4: lynsey.read,connor.yetter,ryan.barrett2
Group 5: caitlin.hermanson,james.brubaker,peter.buckley1
Group 6: derek.price4,alexander.goetz,jacob.ketola
Group 7: tristan.nogacki,jeremiah.mackey,michael.oswald
Group 8: nicholas.harrington1,aaron.loomis,joseph.windmann
Group 9: samuel.rollins,zeke.baumann,samuel.hemmen
Group 10: erik.moore3,colter.huber,devon.maurer
Group 11: jonas.zeiler,luke.donaldson1,carver.wambold
Group 12: jett.girard,carsten.brooks,justice.mosso
Group 13: luka.derry,nolan.scott1,owen.obrien
Group 14: anthony.mann,samuel.mosier,blake.leone
Group 15: yebin.wallace,peyton.trigg,emmeri.grooms
Group 16: julia.larsen,tyler.broesel,sarah.periolat
Group 17: bridger.voss,jack.fry,micaylyn.parker
Group 18: jacob.shepherd1,ethan.johnson18,joseph.triem
Group 19: cameron.wittrock,lucas.jones6,jada.zorn
Group 20: reid.pickert,delaney.rubb,alexander.knutson
Group 21: griffin.short,jakob.kominsky,john.fotheringham
Group 22: mason.barnocky,william.elder1,kaden.price
Group 23: pendleton.johnston,joseph.mergenthaler

Group exercises

1. It is a common mistake to confuse the following two statements (i) If A, then B and (ii) If B, then A. Find two conditions A and B such that statement (i) is true but statement (ii) is false. Then find two conditions A and B such that both statements are true.
2. Two propositions are considered *equivalent* if they have the same truth table values. Show that the biconditional $A \iff B$ is equivalent to $(A \implies B)\,\texttt{and}\,(B \implies A)$.
3. Consider these two statements: (i) If A, then B, (ii) If (not B), then (not A). Under what circumstances are these statements true? When are they false? Explain whether these statements are identical or not. [Note: (ii) is called the **contrapositive** of (i).]
4. (Challenge problem, from philosopher Norman Swartz.) Is the following statement true or false, and why? *A's-being-a-necessary-condition-for-B is both a necessary and sufficient condition for B's-being-a-sufficient-condition-for-A.*

Question 1: Solution

$A \implies B$ but $B \not\Rightarrow A$:
A = I lived in Los Angeles
B = I lived in California.

$A \iff B$:
A = Valentine's Day is this month
B = This month is February.

Question 2: Solution

The truth table for the "and" operator (also written $\wedge$) is given by

Original propositions		New propositions
X	Y	$X \wedge Y$
T	T	T
T	F	F
F	T	F
F	F	F

Now we apply the $\wedge$ operator to the results of the $\implies$ and $\impliedby$ operators.

Orig. props.		New props.		
A	B	$\overbrace{A \implies B}^{X}$	$\overbrace{B \implies A}^{Y}$	$\overbrace{(A \implies B) \wedge (B \implies A)}^{X \wedge Y}$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that $(A \implies B) \wedge (B \implies A)$ gives the same results as $A \iff B$ as on Slide 3.

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T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that $(A \implies B) \wedge (B \implies A)$ gives the same results as $A \iff B$ as on Slide 3.

So what? This guides us towards how we'll prove $A \iff B$ in Sec. 5.

Question 3: Solution

Recall from Slide 3 that the truth table for the $\implies$ operator is given by

X	Y	$\overbrace{X \implies Y}^{\text{If X, then Y}}$
T	T	T
T	F	F
F	T	T
F	F	T

Now we apply the $\implies$ operator to the results of the "`not`" operator (also written $\neg$).

Orig. props.		**New props.**		
A	B	$\overbrace{\neg A}^{Y}$	$\overbrace{\neg B}^{X}$	$\overbrace{\neg B \implies \neg A)}^{X \implies Y}$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Note that $\neg B \implies \neg A$ gives the same results as $A \implies B$ as on Slide 3.

Question 3: Solution

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X	Y	$\overbrace{X \implies Y}^{\text{If X, then Y}}$
T	T	T
T	F	F
F	T	T
F	F	T

Now we apply the $\implies$ operator to the results of the "`not`" operator (also written $\neg$).

Orig. props.		**New props.**		
A	B	$\overbrace{\neg A}^{Y}$	$\overbrace{\neg B}^{X}$	$\overbrace{\neg B \implies \neg A)}^{X \implies Y}$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Note that $\neg B \implies \neg A$ gives the same results as $A \implies B$ as on Slide 3.

Remark: We've shown that a proposition is logically equivalent to its contrapositive. So what? Sometimes it's easier to verify the contrapositive version.

Question 4: Solution

The simplest way to see this is as follows:

- A's-being-a-necessary-condition-for-B can be expressed as $B \implies A$.
- B's-being-a-sufficient-condition-for-A can be expressed as $B \implies A$.
- In other words, both propositions are the same: $B \implies A$. And a proposition is always necessary and sufficient for itself.