

# 04/04/2025: Big O Notation

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CSCI 246: Discrete Structures

Textbook reference: Sec 7.2, Ponomarenko

## Graded Quiz Pickup

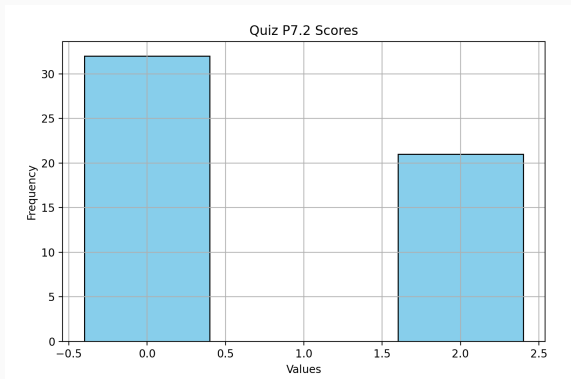
Quizzes are in the front of the room, grouped into four bins (A-G, H-L, M-R, S-Z) by last name. The quizzes are upside down with your last name on the back. Come find yours before, during, or after class. Only turn the quiz over if it's yours.

## Today's Agenda

- Reading and problems quizzes (15 mins)
- Mini-lecture ( $\approx$  15 mins)
- Group exercises ( $\approx$  15 mins)

## **Feedback on Wednesday's Quiz**

# Reading Quiz Scores (Extra Credit)



**Figure 1:** Median Score = 0 points extra credit

**Grading Rubric:** 2 points extra credit for perfect answer.

## Today's quiz

## Problems Quiz (Conditional Prob, Random Variables, Expectation)

1. Let  $(S, P)$  be the sample space with  $S = \{1, 2, \dots, 10\}$  and  $P(x) = \frac{1}{10}$  for all  $x \in S$ . Let  $A$  be the event “is even” and  $B$  be the event “is prime”. Calculate  $P(A | B)$  and  $P(B | A)$ . Show your work.
2. Two cards are drawn at random (without replacement) from a standard deck of 52 cards. Let  $X$  be the rank of the first card and  $Y$  be the rank of the second card. Are  $X$  and  $Y$  independent? Justify your answer with a mathematical argument.

## Reading Quiz (Big O notation)

Consider the sequences  $a_n = 3n + 100$  and  $b_n = n$ . Show that  $a_n = O(b_n)$ . That is, show that there is some  $n_0 \in \mathbb{N}$  and some  $M \in \mathbb{R}$  such that for every  $n \geq n_0$ , we have  $|a_n| \leq M|b_n|$ .

# Thoughts On Big O Notation

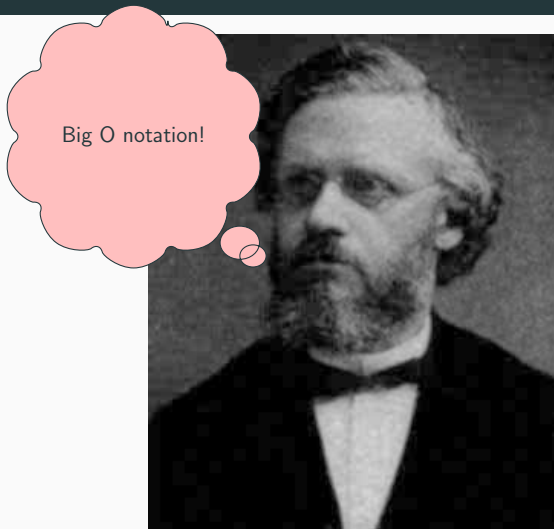
## How to compare the efficiency of algorithms?



**Figure 2:** German mathematician Paul Bachmann



# How to compare the efficiency of algorithms?



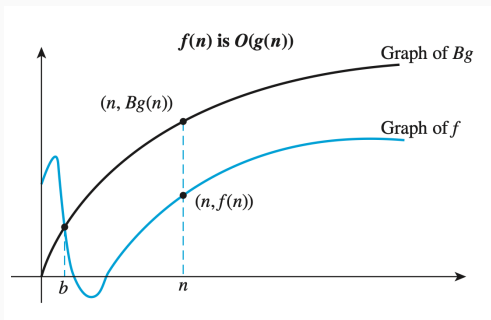
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## Definition

Let  $f$  and  $g$  be real-valued non-negative functions defined on the same set of nonnegative integers.

Then  $f$  is of order at most  $g$ , written  **$f(n)$  is  $O(g(n))$**  ( $f$  of  $n$  is big-O of  $g$  of  $n$ ) if and only if there exist positive real number  $B$  and integer  $b$  such that

$$|f(n)| \leq B|g(n)| \quad \text{for every integer } n \geq b$$

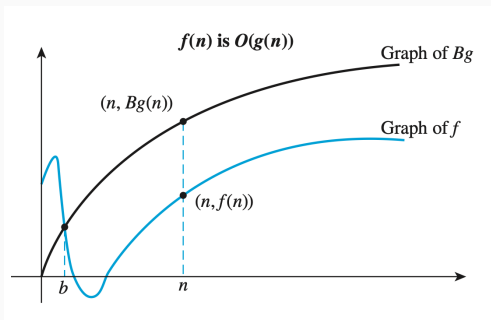


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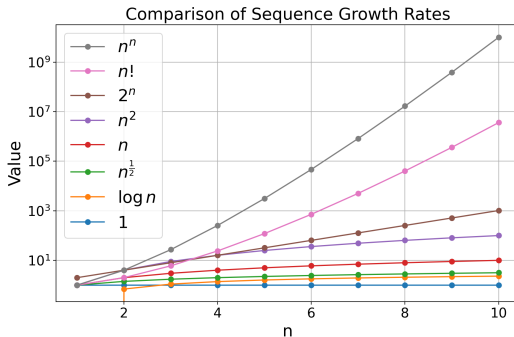
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**Poll.** What's the deal with the highlighted words (especially positive multiple)?

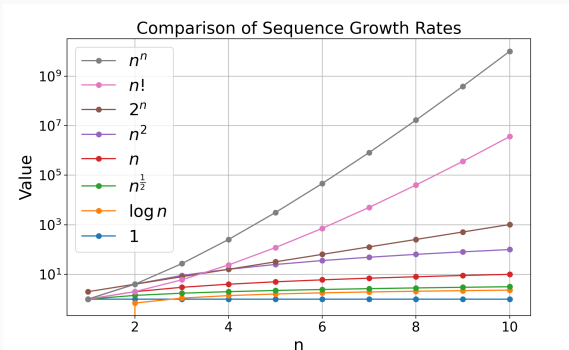


# Classification Theorem



**Poll.** How should we interpret this graph?

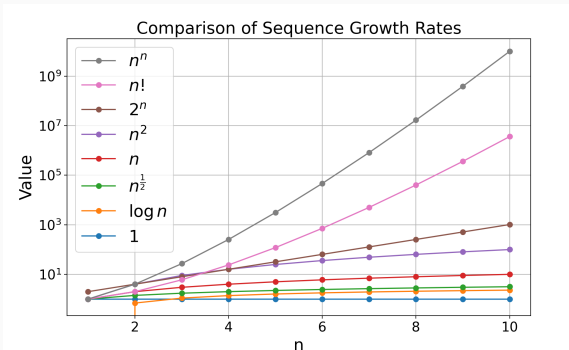
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- A taxonomy of growth rates (or scaling behavior): Any function is big O of everything equal to or above it, but is **not** big O of anything below it.

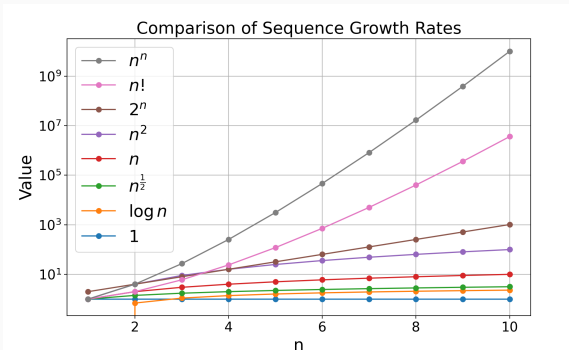
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- **Anti-Example:**  $n$  is not big O of  $n^{1/2}$ . No matter how much you scale up  $n^{1/2}$  by a large constant,  $n$  will eventually get bigger!

# How to check Big O membership?

Example:  $f(n) = 3n + 100$  is  $O(n)$

**Proof.** Take  $n \geq 1$ . Then

$$|3n + 100| \leq |3n + 100n| = 103|n|.$$

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**Proof.** Take  $n \geq 1$ . Then

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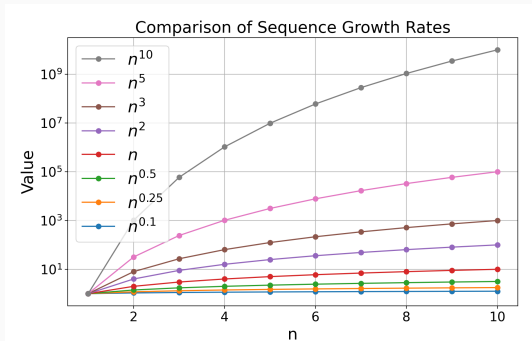
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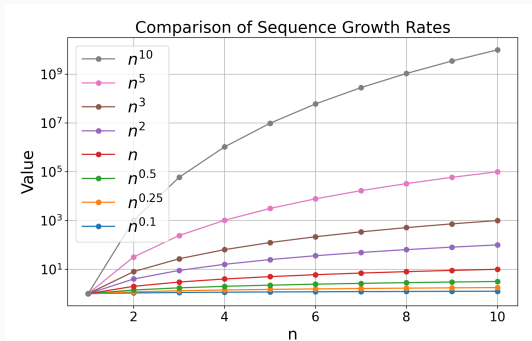
**Remark.** ...and also big O of still higher-order terms. E.g. for  $n \geq 1$ ,

$$14|n^2| \leq 14|n^3| \leq 14|n^4| \leq \dots$$

# Classification Theorem: Zooming In On Polynomials

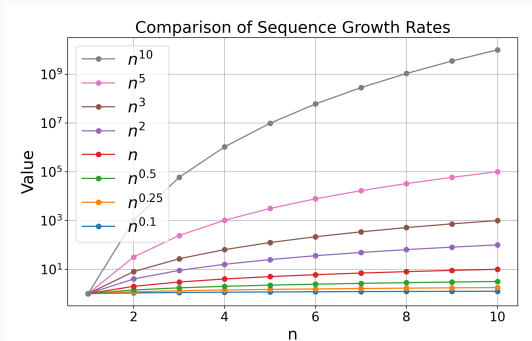


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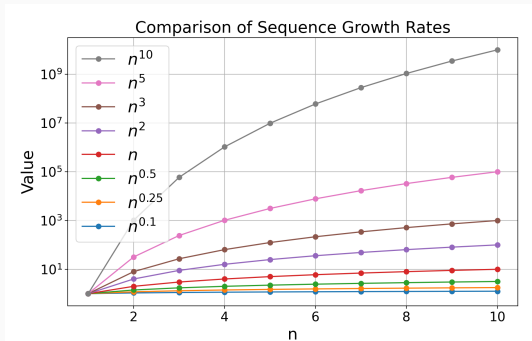
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## Theorem

For any function  $f$  and positive real numbers  $u$  and  $v$  with  $u < v$ ,

$$f \text{ is } O(x^u) \implies f \text{ is } O(x^v)$$

## A major problem with Big O statements

**Poll.** Suppose a person is analyzing the efficiency of algorithms, and finds that  $f$  is  $O(x^5)$  and  $g$  is  $O(x^4)$ . Because  $4 < 5$ , this person concludes that  $g$  has a better algorithmic efficiency than  $f$ . Is this person's conclusion correct?

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**Solution.** No. For instance, take  $f(x) = x^2$  and  $g(x) = x^3$ . Then  
f is  $O(x^2), O(x^3), O(x^4), O(x^5), \dots$   
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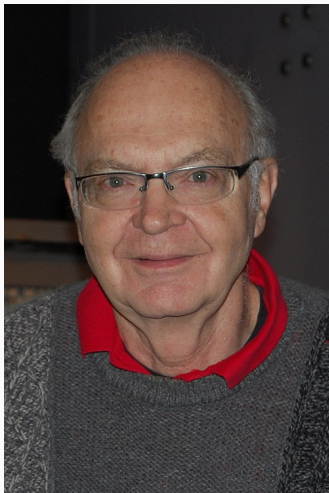
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**Remark.** The problem is that the upper bounds for Big O can be needlessly large!

We should interpret “ $g$  is  $O(x^4)$ ” as: “ $g$  is of order **at most**  $x^4$ ”.

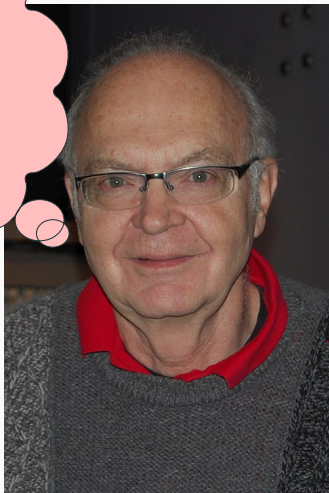
# How to compare the efficiency of algorithms?



**Figure 3:** American computer scientist and mathematician Donald Knuth

# How to correct the efficiency of algorithms?

Big Theta notation!



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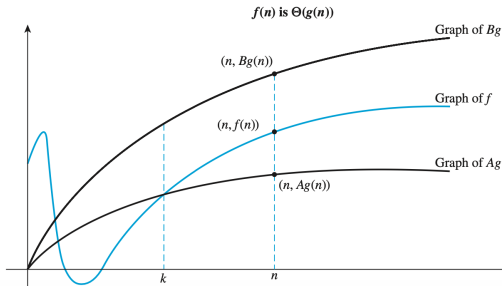
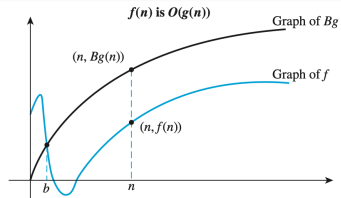
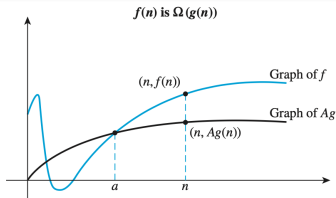
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**Big O**      $f$  is  $O(g)$       $f$  is of order **at most**  $g$      The values of  $f$  are eventually **less** than those of a positive multiple of  $g$ .

**Big Omega**      $f$  is  $\Omega(g)$       $f$  is of order **at least**  $g$      The values of  $f$  are eventually **greater** than those of a positive multiple of  $g$ .

**Big Theta**      $f$  is  $\Theta(g)$       $f$  is of order  $g$      The values of  $f$  are eventually **less** than those of a positive multiple of  $g$  and **greater** than those of a positive multiple of  $g$ .

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### Theorem: On Polynomial Orders

If  $m$  is any integer with  $m \geq 0$  and  $c_0, c_1, c_2, \dots, c_m$  are real numbers with  $c_m \geq 0$ , then

$$c_m x^m + c_{m-1} x^{m-1} + \dots + c_1 x + c_0 \text{ is } \Theta(x^m).$$



## **Group exercises**

aaron.loomis: 15  
adam.wyszynski: 13  
alexander.goetz: 2  
alexander.knutson: 3  
anthony.mann: 17  
blake.leone: 19  
bridger.voss: 19  
caitlin.hermanson: 15  
cameron.wittrock: 4  
carsten.brooks: 7  
carver.wambold: 8  
colter.huber: 16  
conner.reed1: 12  
connor.mizner: 6  
connor.yetter: 21  
derek.price4: 14  
devon.maurer: 2  
emmeri.grooms: 4  
erik.moore3: 3  
ethan.johnson18: 8  
evan.barth: 1

evan.schoening: 9  
griffin.short: 14  
jack.fry: 10  
jacob.ketola: 11  
jacob.ruiz1: 17  
jacob.shepherd1: 4  
jada.zorn: 13  
jakob.kominsky: 12  
james.brubaker: 10  
jeremiah.mackey: 18  
jett.girard: 21  
john.fotheringham: 5  
jonas.zeiler: 1  
joseph.mergenthaler: 9  
joseph.triem: 12  
julia.larsen: 10  
justice.mosso: 16  
kaden.price: 13  
lucas.jones6: 15  
luka.derry: 17  
luke.donaldson1: 7

lynsey.read: 14  
mason.barnocky: 11  
matthew.nagel: 1  
micaylyn.parker: 6  
michael.oswald: 20  
nolan.scott1: 19  
owen.obrien: 11  
pendleton.johnston: 2  
peter.buckley1: 6  
reid.pickert: 18  
ryan.barrett2: 3  
samuel.hemmen: 20  
samuel.mosier: 9  
samuel.rollins: 20  
sarah.periolat: 8  
timothy.true: 16  
tristan.nogacki: 5  
tyler.broesel: 18  
william.elder1: 21  
yebin.wallace: 7  
zeke.baumann: 5

## Exercises.

1. Let  $a_n = 1,000,000n + 3,000,000$ . Prove that  $a_n = O(n)$ .
2. Let  $a_n = 5 + \frac{1}{n} + \frac{1}{n+1}$ . Prove that  $a_n = O(1)$ .
3. Let  $a_n = n^2 + n + 1 + \frac{1}{n} + \sin n$ . Prove that  $a_n = O(n^2)$ .
4. Let  $a_n = 3n^2 + 7$ . Prove that  $a_n = \Theta(n^2)$ .
5. Prove Theorem 7.9 from the Ponomarenko reading.

**Theorem 7.9 (Ponomarenko).** Let  $a_n$  be a sequence, and  $b_n, c_n$  be test sequences. If  $a_n = O(b_n)$  and  $b_n = O(c_n)$ , then  $a_n = O(c_n)$ .