Homework #2

TA: Yifan Yu

Due: September 23, 2024 at 6:00 p.m.

Instructions: All your answers must be appropriately justified. Please note that you should

- Submit a zip or pdf file through canvas containing code, plots, and justifications;
- Write your code in a modular, readable way. Code organization will be taken into account for the grading;
- Use the discipline convex programming language CVX to code the problems below. There are several interfaces for CVX in popular programming languages including Python[1], R[2], Julia[3], and MATLAB[4]. Choose your favorite one!

In case you do not have access to canvas, submit your solutions, as a zip or pdf file, to `yyuco@connect.ust.hk`. Late submissions or submissions that are not zip or pdf files won't be considered.

Problem 1 (Linear Regression, 50 points)

Given $n = 50$ pairs of data (x_i, y_i), $i = 1, \ldots, n$. Try to do linear regression: $y = ax + b$.

(a) Write down the problem formulation with respect to the general residual penalty function $\phi(r)$, where $r = ax + b - y$. Please determine the convexity of the problem when the penalty function is specified as $\phi_1(r) = r^2$, $\phi_2(r) = |r|$ and $\phi_3(r) = \log(1 + r^2)$ (with base e).

(b) Solve only the convex problems with CVX using 'dataset1.csv' and display the corresponding estimations. Plot the original data points and all your fitted curves. Suppose the real parameter $a_{\text{true}} = 3, b_{\text{true}} = 4$, and estimation error is defined as $\|[a_{\text{est}}, b_{\text{est}}] - [a_{\text{true}}, b_{\text{true}}]\|_2^2$, which penalty function minimizes the estimation error?

(c) Now we consider the situation where there are additional three outliers in the data. Please use 'dataset2.csv' and then repeat (b).

Problem 2 (50 points)

Smoothing regularization: Given the impulse response function $h(t) = \frac{1}{9}0.9^t(1 - 0.4sin(3t))$, consider a dynamic system with scalar input sequence $\mathbf{u} = [u(0), u(1), \ldots, u(N)]^T$, and scalar output sequence $\mathbf{y} = [y(0), y(1), \ldots, y(N)]^T$, related by convolution:

$$y(t) = \sum_{\tau=0}^{t} h(\tau)u(t-\tau), t = 0, 1, \ldots, N. \tag{1}$$

Use desired output $\mathbf{y}_{\text{des}} = [y_{\text{des}}(0), y_{\text{des}}(1), \ldots, y_{\text{des}}(N)]^T$ in dataset3.csv, where $N = 400$, and try to use cvx to find the following required inputs.

(a) Find the input such that its output should track the desired output sequence $\mathbf{y}_{\text{des}}$. Denote the error as $J_{\text{track}} = \|\mathbf{y} - \mathbf{y}_{\text{des}}\|$. Use different different norms (i.e.,ℓ_1, ℓ_2, and ℓ_∞-norms) in the problem formulation and compare them.

[1] `https://www.cvxpy.org/`
[2] `https://cvxr.rbind.io/`
[3] `https://github.com/jump-dev/Convex.jl`
[4] `http://cvxr.com/cvx/`

- Write down the problem formulations and state what class of problem it is, i.e., LP, QP, etc.
- Use cvx to find the results. Plot and explain them.

(b) Define the magnitude of the input as $J_{\text{mag}} = \|\mathbf{u}\|_2^2$, and the magnitude of the input variations as $J_{\text{var}} = \sum_{t=0}^{N-1}(u(t+1) - u(t))^2$. Use ℓ_2-norm in J_{track}, and the problem is to find the input to minimize the trade-off objective

$$J_{\text{track}} + \delta J_{\text{var}} + \eta J_{\text{mag}}, \tag{2}$$

where $\delta > 0$ and $\eta > 0$.

- State what class of problem it is.
- Choose different δ and η and draw two plots for each set of δ, η:
  - The input $\mathbf{u}^*$ you find.
  - The comparison of the desired output $\mathbf{y}_{\text{des}}$ and the optimized output $\mathbf{H}\mathbf{u}^*$.
- Illustrate and compare the results.