Exercise 1

Solution: The Lamb-Oseen vortex

1. Because of symmetry there is no dependence on ϑ. Using this in the continuity equation results in $U_r = 0$, which is necessary to have circular streamlines. There is only U_ϑ, that we will call now v, which is non-zero.

   Using the definition of the rotation in order to determine the irrotational flow field one gets:
   $$\frac{1}{r}\frac{\partial(r\,v)}{\partial r} = 0 \quad \Rightarrow \quad v = \frac{C}{r}.$$

   Evaluation the circulation:
   $$\Gamma = \oint_0^{2\pi} v\, r d\theta \quad \Rightarrow \quad \Gamma = 2\pi C.$$

   We chose to write the velocity defined by a given circulation: $v = \frac{\Gamma}{2\pi r}$.

   Or:
   $$\Gamma = \int \text{rot}(\vec{v}) dA = 0 \ \text{ (if done in a naive way).}$$

   Why? Because the vorticity is concentrated at the origin in a singular point.

2. Continuity is trivially satisfied.

   The momentum equation in radial direction relates the pressure to a known velocity field by
   $$\frac{\partial p}{\partial r} = \varrho \frac{v^2}{r}.$$

   No problem to find a pressure that fulfills the equation.

   The momentum equation in azimutal direction:
   $$0 = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rU)\right),$$

   is also fulfilled. Except at the origin because $v(r = 0) = \infty$.

3. First we non-dimensionalise the initial condition:
   $$V = \frac{\Gamma}{2\pi R}.$$

   Then we non-dimensionalise the momentum equation in the θ-direction, chosing suitable gauges for R and T makes the dimensional group become one.
   $$\frac{\varrho R^2}{T\mu} = 1. \quad \text{for instance} \quad T = \frac{\rho R^2}{\mu}$$

We obtain the non-dimensional system with its boundary conditions:

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (rv)}{\partial r}\right], && (1)\\ v(r,0) &= \frac{1}{r}, && (2)\\ v(0,t) &= 0, && (3)\\ v(\infty,t) &= 0. && (4)\end{aligned}$$

We impose dilatation groups and look for a combination of those that leads to a self-similar system

$$\begin{aligned} r &= r^*\tilde{r} && (5)\\ t &= t^*\tilde{t} && (6)\\ v &= v^*\tilde{v} && (7)\end{aligned}$$

The problem with dilatation groups reads:

$$\begin{aligned}\frac{v^*}{t^*}\frac{\partial \tilde{v}}{\partial \tilde{t}} &= \frac{v^*}{r^{*2}}\frac{\partial}{\partial \tilde{r}}\left[\frac{1}{\tilde{r}}\frac{\partial (\tilde{r}\tilde{v})}{\partial \tilde{r}}\right], && (8)\\ v^*\tilde{v}(r^*\tilde{r},0) &= \frac{1}{r^*}\frac{1}{\tilde{r}}, && (9)\\ v^*v(0,t^*\tilde{t}) &= 0, && (10)\\ v^*(\infty,t^*\tilde{t}) &= 0. && (11)\end{aligned}$$

Using the following relations between the dilatation or streching groups yields a problem that remains frozen. This way we construct a special solution, which remains invariant or self-similar along a certain trajectory in space-time.

$$\begin{aligned} r^{*2} &= t^* && (12)\\ v^* &= \frac{1}{r^*}. && (13)\end{aligned}$$

If we follow a special solution that fulfills the conditions above, which are in some sense particular trajectories, we reduce the number of variables from three to two. This trajectory is described by the self-similar variable, which gives a relation of r and t, on which the problem is self-similar.

We can now eliminate variables, but as long as we guarantee the self-similar relations we are sure to obtain the self-similar solution. Let us find a self-similar solution, the solution is written in the implicit way:

$$F(r,t,v) = 0$$

substituting (12)

$$F(r^*\tilde{r}, r^{*2}\tilde{t}, r^{*-1}\tilde{v}) = 0, \quad \forall r^* > 0$$

in order to eliminate the dependence on r^* let's rearrange the variables as

$$F(r, t/r^2, vr) = 0$$

obtaining using (12)

$$F(r^*\tilde{r}, \tilde{t}/\tilde{r}^2, \tilde{v}\tilde{r}) = 0, \quad \forall r^* > 0$$

which has to be true for whatever r^* therefore:

$$F(\tilde{v}\,\tilde{r}, \tilde{r}^2/\tilde{t}) = 0 \quad \Rightarrow \quad \tilde{v}\,\tilde{r} = f(\tilde{r}^2/\tilde{t}) = f(\eta).$$

We chose another linearly independent combination of the variables in a form that the self-similar variables appear. On the trajectory the solution becomes independent of the free parameters, which can therefore be excluded (r in this case). The last transform is just a step to go from the abstract implicit function F to an explicit (but unknown) function f. Here after we omit the tildes.

Applying the self-similar variable requires a coordinate transform, so the self-similar form could become unnecessarily complicated due to double derivatives. Looking at eq. (1) tells us that r^2 should appear in the nominator[1]. So the self-similar variable is $\eta = r^2/t$.

We proceed in using the newly derived self-similar form and will see that it indeed can verify the Partial Differential Equation (PDE) and simplify it into an Ordinary Differential Equation (ODE).

To transform the derivatives recall:

$$\frac{\partial f(\eta)}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial f(\eta)}{\partial \eta}.$$

Therefore we need at first:

$$\frac{\partial \eta}{\partial t} = -\frac{r^2}{t^2}, \qquad \frac{\partial \eta}{\partial r} = 2\frac{r}{t}.$$

We proceed with the first term in equation 1.

$$\frac{\partial v}{\partial t} = \frac{1}{r}\frac{\partial \eta}{\partial t}\frac{\partial f}{\partial \eta} = -\frac{r}{t^2}f'. \tag{14}$$

The second term on the right side has a double derivative. Starting with the inner term.

$$\frac{\partial (r\,v)}{\partial r} = \frac{\partial \eta}{\partial r}\frac{\partial f}{\partial \eta} = 2\frac{r}{t}f'. \tag{15}$$

Applying the transform to the whole term now:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r\,v)}{\partial r}\right) = \frac{\partial (2f'/t)}{\partial r} = \frac{2}{t}\frac{\partial \eta}{\partial r}f'' = 4\frac{r}{t^2}f''. \tag{16}$$

Inserting the terms from eq. (14) and (16) in the PDE from eq. (1):

$$-\frac{r}{t^2}f' = 4\frac{r}{t^2}f'' \quad \Rightarrow \quad f' + 4f'' = 0. \tag{17}$$

[1]This might seem obscure to you, but try also to derive the self-similar equation using $r/\sqrt{t}$ or t/r^2. These are also an acceptable self-similar variables. What self-similar variable has led you to the simplest self-similar equation?

The result might not always come out that easily but if the self-similar variables are correctly determined the expression should never contain other variables then the self-similar ones.

Well, the general solution to the ODE above is:

$$f = A + Be^{-\eta/4},$$

With the constants A and B. In order to find a non-trivial solution ($A = B = 0$) we verify at first the boundary condition at eq. 3.

$$v(0,t) = 0 = \lim_{r\to 0}(A+B)/r, \quad \Rightarrow \quad A = -B.$$

The division by zero (r in the denominator) is not a problem because by Hopitals rule you can verify that due to the exponential the nominator tends faster to zero than the denominator. We can verify that the other boundary conditions in eq. (2) and (4) are verified as well.

We demonstrate only the first one:

$$\lim_{t\to 0} A(1 - e^{-r^2/t}) = A/r = 1/r \quad \Rightarrow \quad A = 1.$$

Finally gives:

$$v = \frac{1 - \exp\left(-\frac{r^2}{4t}\right)}{r} \quad \Rightarrow \quad V = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{r^2\rho}{4\mu t}\right)\right) \text{(dimensional)}$$

4. We evaluate the vorticity since there is only angular velocity, which changes only in the radial direction. Therefore only the third component of the vorticity vector is non-zero.

   We write:

$$\mathrm{rot}(\vec{v}) \cdot \vec{e}_z = \frac{1}{r}\frac{\partial(rv)}{\partial r} = \frac{1}{r}\left(\frac{2r}{4t}\exp\left(-\frac{r^2}{4t}\right)\right) = \frac{1}{2t}\exp\left(-\frac{r^2}{4t}\right).$$

   We see that for time $t = 0$ the vorticity is concentrated in a Dirac distribution. As time evolves the vorticity diffuses like a Gaussian, just like temperature diffusion in a solid.

Solution: The Rankine vortex

1. It is trivial to assess that the streamlines are circles.

2. For the outer flow field:

$$\mathrm{rot}(\vec{V}) = \frac{1}{r}\frac{\partial}{\partial r}(a^2\Omega) = 0.$$

   The flow field is irrotational.

3. We integrate the pressure field from the momentum equation in the radial direction.

$$\frac{\partial p}{\partial r}|_{out} = \frac{a^4\Omega^2}{r^3}\rho \quad \Rightarrow \quad p_{out} = -\frac{a^4\Omega^2}{2r^2}\rho + P_0.$$

$$\frac{\partial p}{\partial r}|_{in} = r\Omega^2\rho \quad \Rightarrow \quad p_{in} = \Omega\left(\frac{1}{2}r^2 - a^2\right)\rho + P_0.$$

4. Inside:

$$\frac{r^2\Omega^2}{2} + \Omega^2\left(\frac{1}{2}r^2 - a^2\right) + P_0/\rho = \Omega^2(r^2 - a^2) + P_0/\rho \neq \text{constant}.$$

   Outside:

$$\frac{a^4\Omega^2}{2r^2} - \frac{a^4\Omega^2}{2r^2} + P_0/\rho = \text{constant}.$$

   The inner region does not allow to use Bernoulli equation.

5. At the center of the vortex the pressure is (using p_{in} from question 3):

$$P = -a^2\Omega^2\rho + P_0 = V^2\rho + P_0 = P_0 - 9000Pa.$$

6. The air is inviscid compared to water. Therefore we conclude that the moving air does not induce rotation in the water. Only the pressure will cause a hydrostatic rise of the water surface.

$$\rho_{water}gH = 1000\frac{kg}{m^3}10\frac{m}{s^2}H = 9000Pa \quad \Rightarrow \quad H = 0.9m.$$