Von Karman disk - Solution

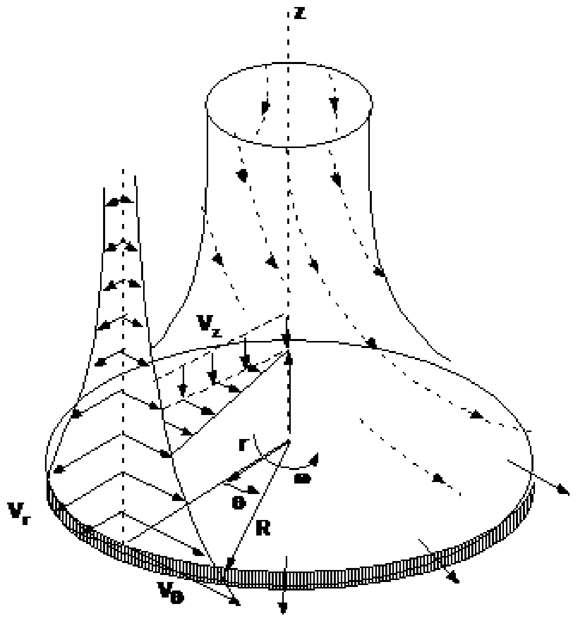

FIGURE 1 –

1. The boundary conditions to be imposed at the disk surface are :

$$u = w = 0, \quad v = \Omega r \tag{1}$$

2.

$$\underbrace{w\frac{\mathrm{d}w}{\mathrm{d}z}}_{\frac{1}{2}\frac{\mathrm{d}w^2}{\mathrm{d}z}} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} + \nu\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} \Rightarrow \frac{p}{\rho} = \nu\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{1}{2}w^2 + F$$

F is a constant since there is no rotation of the fluid far from the disk and p must be independent of r when z is large. Since p is only function of the vertical velocity w it depends only on z.

3.

$$\frac{1}{r}\frac{\partial ru}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial r^2 u/r}{\partial r} + \frac{1}{r}\frac{\partial rv/r}{\partial \theta} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0$$

$$\frac{2u}{r} + r\cancelto{0}{\frac{\partial u/r}{\partial r}} + \cancelto{0}{\frac{\partial v/r}{\partial \theta}} + \cancelto{0}{\frac{v}{r^2}\frac{\partial r}{\partial \theta}} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0$$

4. We proceed as in question 3, keeping in mind that the quantities u/r, v/r, w and p do not depend on r and θ.
5. Replacing the dilatation in the system of equations :

$$\begin{aligned}
&\text{continuity :} \quad \frac{U}{R} + \frac{W}{Z} \sim 0 \Rightarrow U \sim \frac{WR}{Z} \\
&\theta\text{-momentum :} \quad \frac{UV}{R^2} + \frac{WV}{RZ} + \frac{NV}{RZ^2} \sim 0 \Rightarrow W \sim \frac{N}{Z} \\
&\text{no-slip condition :} \quad V \sim OR \\
&r\text{-momentum :} \quad \frac{U^2}{R^2} + \frac{WU}{RZ} + \frac{V^2}{R^2} + \frac{NU}{RZ^2} \sim 0 \Rightarrow \frac{W^2 R}{Z^2} \sim \frac{V^2}{R} \Rightarrow W \sim OZ
\end{aligned} \tag{2}$$

Hence the conditions to be satisfied by the dilatation groups are :

$$U \sim OR,\ V \sim OR,\ W \sim OZ,\ O \sim N/Z^2 \tag{3}$$

6. The velocity vector $\mathbf{u}$ depends on the dimensional quantities r, z, ν and Ω. The general relation among the v-component and the dimensional quantities can be written as :

$$\begin{aligned}
&G(v, r, z, \nu, \Omega) = 0 \\
&G(Vv', Rr', Zz', N\nu', O\Omega') = 0
\end{aligned} \tag{4}$$

In order to eliminate the dependence on scales we can rearrange the variables, as suggested by (3), without loss of generality :

$$\begin{aligned}
G\left(\frac{v'}{\Omega' r'}, z\sqrt{\frac{\Omega'}{\nu'}}, Rr', N\nu', O\Omega'\right) &= 0 \quad \forall R,\ N,\ O \\
G\left(\frac{v'}{\Omega' r'}, z\sqrt{\frac{\Omega'}{\nu'}}\right) &= 0 \Rightarrow v/r = \Omega g(\zeta)
\end{aligned} \tag{5}$$

In a similar way the general relation among the w-velocity component and the dimensional quantities can be written as :

$$\begin{aligned}
&H(w, r, z, \nu, \Omega) = 0 \\
&H(Ww', Rr', Zz', N\nu', O\Omega') = 0
\end{aligned} \tag{6}$$

In order to eliminate the dependence on scales we can rearrange the variables, as suggested by (3), without loss of generality :

$$\begin{aligned}
H\left(\frac{w'}{\Omega' z'}, z\sqrt{\frac{\Omega'}{\nu'}}, Rr', N\nu', O\Omega'\right) &= 0 \quad \forall R,\ N,\ O \\
H\left(\frac{w'}{\Omega' z'}, z\sqrt{\frac{\Omega'}{\nu'}}\right) &= 0 \Rightarrow w = \Omega z \bar{h}(\zeta) = \sqrt{\Omega\nu}\,\underbrace{\zeta\bar{h}(\zeta)}_{h(\zeta)}
\end{aligned} \tag{7}$$

The u-component and pressure are obtained by continuity and vertical momentum :

$$\begin{aligned} &\frac{2u}{r} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0 \;\Rightarrow\; u/r = -\frac{1}{2}\Omega h' \\ &\frac{p}{\rho} = \nu\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{1}{2}w^2 \;\Rightarrow\; \frac{p}{\rho} = \nu h' - \frac{1}{2}h^2 \end{aligned} \tag{8}$$

The corresponding self-similar system (r and θ momentum together with boundary conditions) is :

$$\begin{aligned} &\frac{1}{4}h'^2 - \frac{1}{2}hh'' - g^2 = -\frac{1}{2}h''' \\ &-gh' + g'h = g'' \\ &h = h' = 0,\; g = 1, \quad \text{at} \;\; \zeta = 0 \\ &h' \to 0,\; g \to 0, \quad \text{at} \;\; \zeta \to \infty \end{aligned} \tag{9}$$

7. The tangential stress is :

$$\sigma_{z\phi} = \mu\frac{\partial v}{\partial \zeta}\bigg|_{z=0} = \mu\Omega r\,\frac{\partial g}{\partial z}\bigg|_{z=0} = \mu\Omega r\,\frac{\partial g}{\partial \zeta}\bigg|_{\zeta=0}\frac{\partial \zeta}{\partial z}\bigg|_{z=0} = \mu\frac{\Omega}{\delta}rg'(0) = \rho\sqrt{\nu}\Omega^{3/2}rg'(0) \tag{10}$$

Hence the torque exerted by the fluid on both sides is given by :

$$T = 2\int_0^{2\pi}\int_0^D \sigma_{z\phi}r^2\mathrm{d}r\mathrm{d}\phi = 4\pi\int_0^D \sigma_{z\phi}r^2\mathrm{d}r = \pi D^4\rho\nu^{1/2}\Omega^{3/2}g'(0) \tag{11}$$

8. Far from the disk there is an axial motion towards the disk, $h(\zeta \to \infty) < 0$, while u and w go to zero. This happens because a circular motion of the fluid near the disk is not possible, because there is no imposed radial pressure gradient (p depends only on z) to provide centripetal force, and the fluid near the disk therefore spirals outwards. This outward radial motion near the disk must be balanced by a negative axial velocity flux toward the disk in order to satisfy mass conservation, see figure 2

Stokes disk - Solution

1. The boundary conditions to be imposed at the disk surface are :

$$u = 0 = w = 0, \quad v = \dot{\phi} = -\phi_0 \omega \sin(\omega t) \tag{12}$$

2. Let scale the velocity with the characteristic velocity $\phi_0 \omega R$. The unsteady, convective and viscous terms scale as :

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &\sim \phi_0 \\ \mathbf{v} \cdot \nabla \mathbf{v} &\sim \phi_0^2 \\ \nu \Delta \mathbf{v} &\sim \phi_0 \end{aligned} \tag{13}$$

Hence, in the limit of small ϕ_0

$$\begin{aligned} \mathbf{v} \cdot \nabla \mathbf{v} &\ll \frac{\partial \mathbf{v}}{\partial t} \\ \mathbf{v} \cdot \nabla \mathbf{v} &\ll \nu \Delta \mathbf{v} \end{aligned} \tag{14}$$

3. By considering an axisymmetric solution of the type $u = w = 0$ and $v = r\Omega(z,t)$ the continuity equation is trivially satisfied and the momentum equations in the vertical and radial direction reduces to $\partial p/\partial z = 0$ and $\partial p/\partial r = 0$. Hence the pressure is uniform in the flow domain. The azimuthal momentum equation for axisymmetric flow and $\phi_0 \ll 1$ is :

$$\frac{\partial v}{\partial t} = \nu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv) \right) + \frac{\partial^2 v}{\partial z^2} \right),$$

by imposing $v = r\Omega(z,t)$ the first term on the right hand side is nill and the equation reads :

$$\frac{\partial \Omega}{\partial t} = \nu \frac{\partial^2 \Omega}{\partial z^2} \tag{15}$$

4. By substituting the modal expansion $\Omega = \hat{\Omega}(z) e^{i\omega t} + c.c.$ in the governing equation for $\Omega(z,t)$ we get :

$$\hat{\Omega}'' - i \frac{\omega}{\nu} \hat{\Omega}, \tag{16}$$

where the $'$ denotes the derivative with respect to the vertical direction z. This equation has the general solution :

$$\hat{\Omega}(z) = A e^{-(1+i)z\sqrt{\omega/(2\nu)}} + B e^{(1+i)z\sqrt{\omega/(2\nu)}}. \tag{17}$$

Thus the general solution for $\Omega(z,t)$ is :

$$\Omega(z,t) = A e^{-(1+i)z\sqrt{\omega/(2\nu)}} e^{i\omega t} + B e^{(1+i)z\sqrt{\omega/(2\nu)}} e^{i\omega t} + c.c., \tag{18}$$

where the complex constants A and B are determined by imposing the boundary conditions. In particular, B is nill to preclude the angular velocity to diverge at $z = \infty$. At $z = 0$ the no-split boundary condition yields $A = i\phi_0\omega$, and the solution is :

$$\Omega(z,t) = -\omega \phi_0 e^{-z\sqrt{\omega/(2\nu)}} \sin(\omega t - z\sqrt{\omega/(2\nu)}) \tag{19}$$

5. By using the definition of the Stokes layer $\delta = \sqrt{2\nu/\omega}$ the angular velocity reads :

$$\Omega(z,t) = -\omega\phi_0 e^{-z/\delta}\sin(\omega t - z/\delta), \tag{20}$$

hence δ is the penetration depth of the rotational motion in the vertical direction.

6. The instantaneous viscous torque M_z is :

$$M_z = 2\int_0^R r 2\pi r \mu \left.\frac{\partial v}{\partial z}\right|_{z=0} \mathrm{d}r = \omega\phi_0\pi\sqrt{\omega\rho\mu}R^4\cos(\omega t - \pi/4) \tag{21}$$

7. The instantaneous rotational angle and the instantaneous viscous torque are not synchronized since a phase lag of $\pi/4$ is present.

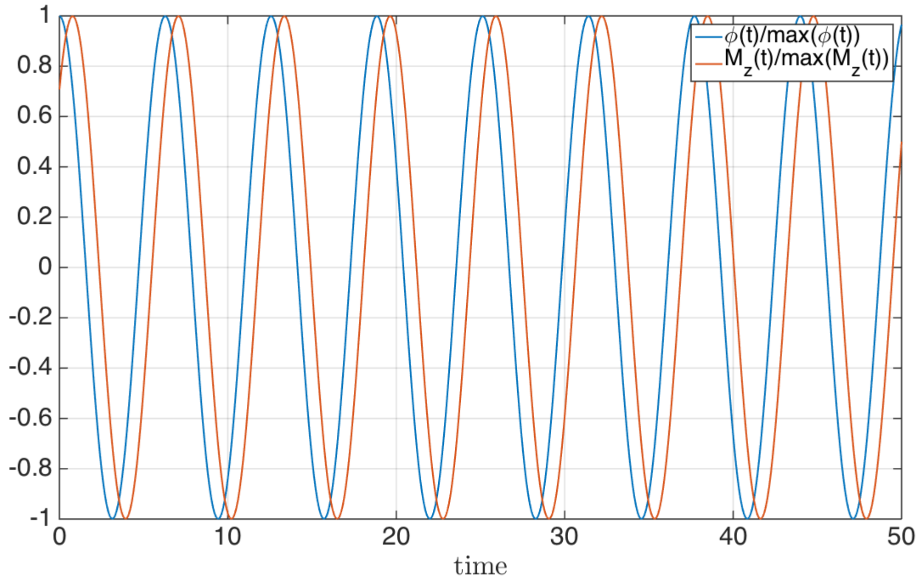

FIGURE 2 –

8. The energy dissipation averaged over one period is :

$$\overline{\dot{E}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} M_z\dot{\phi}\mathrm{d}t = \frac{\omega}{2\pi}\omega\phi_0\pi\sqrt{\omega\rho\mu}R^4(-\phi_0\omega)\underbrace{\int_0^{2\pi/\omega}\cos(\omega t - \pi/4)\sin(\omega t)\mathrm{d}t}_{\pi/(\omega\sqrt{2})} = \\ -\frac{\pi}{2\sqrt{2}}\omega^2\phi_0^2\sqrt{\omega\rho\mu}R^4 \tag{22}$$