Description of matrix operator for finite-volume form of advection-diffusion equation

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We seek to descritize an operator of the form:

$$M[f] = \frac{1}{J}\frac{\partial}{\partial x}(J\Gamma) = \frac{1}{J}\frac{\partial}{\partial x}\left[JAf + JD\frac{\partial f}{\partial x}\right] \tag{1}$$

using a finite-volume method with piecewise constant elements. Each element has a average/constant value of f and x given by f_i and x_i, respectively, for $1 \leq i \leq N$. The fluxes $\Gamma_{i+1/2}$ and $\Gamma_{i-1/2}$ are defined on the faces $x_{i+1/2}$ and $x_{i-1/2}$. The Jacobian of the (arbitrary) coordinate x is $J(x)$. Using the average of the adjacent cells to represent the value of f at the cell boundary, we can descritize Eq. 1 as:

$$\begin{aligned}
M[f_i] \approx & \frac{1}{J_i}\frac{1}{x_{i+1/2}-x_{i-1/2}}\left[(JA)_{i+1/2}\frac{f_{i+1}+f_i}{2} - (JA)_{i-1/2}\frac{f_i+f_{i-1}}{2}\right. \\
& \left. + (JD)_{i+1/2}\frac{f_{i+1}-f_i}{v_{i+1}-v_i} - (JD)_{i-1/2}\frac{f_i-f_{i-1}}{v_i-v_{i-1}}\right] & (2) \\
= & a_{i-1}f_{i-1} + b_i f_i + c_{i+1}f_{i+1}. & (3)
\end{aligned}$$

The tridiagonal coefficients for interior points are given by:

$$a_{i-1} = \frac{1}{J_i\left(x_{i+1/2}-x_{i-1/2}\right)}\left[-\frac{1}{2}(JA)_{i-1/2} + \frac{(JD)_{i-1/2}}{x_i - x_{i-1}}\right] \tag{4}$$

$$b_i = \frac{1}{J_i\left(x_{i+1/2}-x_{i-1/2}\right)}\left[\frac{1}{2}(JA)_{i+1/2} - \frac{1}{2}(JA)_{i-1/2} - \frac{(JD)_{i+1/2}}{x_{i+1}-x_i} - \frac{(JD)_{i-1/2}}{x_i - x_{i-1}}\right] \tag{5}$$

$$c_{i+1} = \frac{1}{J_i\left(x_{i+1/2} - x_{i-1/2}\right)}\left[\frac{1}{2}(JA)_{i+1/2} + \frac{(JD)_{i+1/2}}{x_{i+1} - x_i}\right] \tag{6}$$

For Dirichlet boundary conditions, these are replaced by unity on the diagonal element, and zeros elsewhere. For flux-specified boundary conditions at $i = 1$:

$$M[f_1] \approx \frac{1}{J_1}\frac{1}{x_{3/2} - x_{1/2}}\left[(JA)_{3/2}\frac{f_2 + f_1}{2} + (JD)_{3/2}\frac{f_2 - f_1}{v_2 - v_1}\right] \tag{7}$$

and

$$b_1 = \frac{1}{J_1\left(x_{3/2} - x_{1/2}\right)}\left[\frac{1}{2}(JA)_{3/2} - \frac{(JD)_{3/2}}{x_2 - x_1}\right] \tag{8}$$

$$c_2 = \frac{1}{J_1\left(x_{3/2} - x_{1/2}\right)}\left[\frac{1}{2}(JA)_{3/2} + \frac{(JD)_{3/2}}{x_2 - x_1}\right], \tag{9}$$

while the incoming flux $\Gamma_{1/2} = \Gamma\left(x_{1/2}\right)$ is given and the following must be added to the source:

$$\Delta S_1 = \frac{J_{1/2}\Gamma_{1/2}}{J_1\left(x_{3/2} - x_{1/2}\right)}. \tag{10}$$

For flux conditions on the right side, we have similarly:

$$M[f_N] \approx \frac{1}{J_N}\frac{1}{x_{N+1/2} - x_{N-1/2}}\left[-(JA)_{N-1/2}\frac{f_N + f_{N-1}}{2} - (JD)_{N-1/2}\frac{f_N - f_{N-1}}{v_N - v_{N-1}}\right] \tag{11}$$

$$a_{N-1} = \frac{1}{J_N\left(x_{N+1/2} - x_{N-1/2}\right)}\left[-\frac{1}{2}(JA)_{N-1/2} + \frac{(JD)_{N-1/2}}{x_N - x_{N-1}}\right] \tag{12}$$

$$b_N = \frac{1}{J_N\left(x_{N+1/2} - x_{N-1/2}\right)}\left[-\frac{1}{2}(JA)_{N-1/2} - \frac{(JD)_{N-1/2}}{x_N - x_{N-1}}\right] \tag{13}$$

$$\Delta S_N = \frac{-J_{N+1/2}\Gamma_{N+1/2}}{J_N\left(x_{N+1/2} - x_{N-1/2}\right)}. \tag{14}$$