Exact asymptotic expressions for local clustering in hyperbolic random graphs

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October 17, 2018

Abstract

Pim: The abstract will be written at the end.

Pim: Below is a list of high and low level to do items. Please add comments using the commands below.
Comment commands:
Markus: \MS
Nikolaos: \NF
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Tobias: \TM

To do: High level list:

- Streamline notations and writing of some sections
- Finish introduction
- Complete bibliography list
- Add additional explanation for the Hyperbolic model.
- Add additional observations based on our main result

Contents

1 Introduction and results

1.1 Motivation

Markus: I would not write a length introduction to complex networks here (because the paper is quite long already and we do not study networks in general in this paper). So, I would remove the first paragraph/list (or incorporate it as a single sentence in the ensuing paragraph). **Pim:** I agree.

Hyperbolic random graphs were suggested as a suitable model for complex networks by Krioukov et al. [14], exhibiting the three main characteristics commonly found in complex networks:

1) Broad degree distribution (power-law/scale-free),

2) Strong clustering (community structure),

3) Small path lengths (small world phenomenon).

Indeed, in their seminal paper they show that these graphs have a power-law degree distribution and exhibit strong clustering, while Bringmann et al. [3] showed that hyperbolic random graphs have doubly logarithmic shortest path lengths. Many other results regarding hyperbolic random graphs have since been established. One important feature of these graphs is that they allow for efficient routing algorithms [4]. Further results include the size of the largest component [2], [9], [More?] and number of cliques and largest clique [1].

The aim of this paper is to study local clustering in hyperbolic random graphs. The first rigorous mathematical analysis of the local clustering coefficient was done by Gugelmann et al. [12]. They show [12, Theorem 2.1] that the local clustering coefficient[1] is concentrated around its expectation which is shown to be $\Theta(1)$ as the number of vertices tends to infinity. In particular this results implies that the local clustering coefficient is asymptotically bounded away from zero. The convergence of this coefficient was not proven. In contrast, the global clustering coefficient was shown [5] to converge in probability to some constant which can be explicitly stated as an integral expression, see Theorem 1.2 in [5]. The authors mention, however, that their analysis needs significant modification to be able to deal with convergence of the local clustering coefficient.

In addition to the clustering coefficients, an important clustering measure that is often studied is the local clustering function $c(k)$. This function computes for any value of k the average of the local clustering coefficient over all vertices of degree k. A general expression of this function for hyperbolic random graphs is given in [14, Equation (59)]. The authors conjecture that as k tends to infinity, $c(k)$ decays as k^{-1}, which they observe (Figure 8 in [14]) in experiments on the infrastructure of the Internet obtained in [6]. However, despite the importance of the local clustering function and these interesting observations, its behavior in hyperbolic random graphs had not been completely determined and the following crucial questions regarding clustering in hyperbolic random graphs remained open so far:

1) Does the local clustering coefficient converge, and if so, what is the limit?

2) What is the limit of the local clustering function and how does it scale with the degree?

In this work we resolve these important open questions. We obtain the asymptotic scaling of $c(k)$ as $k \to \infty$, including the leading constant, as well as an exact expression, in terms of known special functions, of the point-wise limit of the local clustering function as the number of vertices tends to infinity. Interestingly, the scaling of $c(k)$ depends on the exponent of the degree distribution and is only k^{-1} when this exponent exceeds $5/2$. For values less than $5/2$ the scaling is k^{-s} where the value of s depends on the degree distribution exponent. Finally, our analysis allows us to also prove a convergence result for the local clustering coefficient where the limit can again be explicitly expressed in terms of known special functions.

[1] Note that in [12] this is called the global clustering coefficient. However, since this term is more often associated in the literature with the density of triangles compared to the number of paths of length 2, we use the term local clustering coefficient as is done, for instance, in [5].

Pim: I recall that Júlia told me that the results from [13] imply that the local clustering coefficient converges, although they do not mention it in the paper. Should we added a comment related to this?

1.2 Outline of the paper

In the next two subsections 1.3 and 1.4 of the introduction, we give the definitions of the hyperbolic random graph and of the local clustering coefficient and function as used in this paper. After this we present our main results in Section 1.5 and discuss several insights that can be obtained from them and in Section 1.6 show simulations which confirm these results. Section 2 contains a detailed high level outline of the proofs of our main results, split up into different propositions. The proofs of these propositions can be found in the last four sections while Section 3 contains a crucial technical result regarding that will be used through the other sections.

1.3 Hyperbolic random graphs

The hyperbolic plane is an infinite 2-dimensional manifold with constant negative curvature. Many different representations of the hyperbolic plane exist and we refer to [CITE] for more details. We will work with the *native representation* of the hyperbolic plane which takes the elements of $\mathbb{R}^2$ in polar coordinates (r, θ) as the underlying set of points and where distances are determined by the hyperbolic law of cosines.

Let $\alpha > \frac{1}{2}$, $\nu > 0$ and define $R_n = 2\log(n/\nu)$. Then the hyperbolic random graph $G_{\mathbb{H},n}(\alpha, \nu)$ is defined as follows:

1. The vertex set if given by n points $u_1, \dots, u_n$ denoted in polar coordinates (r_i, θ_i), where the angular coordinate θ is chosen uniformly from $(-\pi, \pi]$ while the radial coordinate r is sampled independently according to the cumulative distribution function

$$F_{n,\alpha,\nu}(r) = \begin{cases} 0 & \text{if } r < 0 \\ \frac{\cosh(\alpha r)-1}{\cosh(\alpha R_n)-1} & \text{if } 0 \leq r \leq R_n \\ 1 & \text{if } r > R_n \end{cases} \tag{1}$$

2. Any two vertices $u_i = (r_i, \theta_i)$ and $u_j = (r_j, \theta_j)$ are adjacent if and only if $d_{\mathbb{H}}(u_i, u_j) \leq R_n$, where $d_{\mathbb{H}}$ denotes the distance in the hyperbolic plane, i.e.

$$\cosh(r_i)\cosh(r_j) - \sinh(r_i)\sinh(r_j)\cos(|\theta_i - \theta_j|_{2\pi}) \leq \cosh(R_n),$$

as follows from the hyperbolic cosine rule[CITE?].

Figure 1 shows an example of $G_{\mathbb{H},n}(\alpha, \nu)$.

1.4 Clustering

Clustering measures for networks consider the fraction of triangles (triples of connected vertices) in the network. **Markus:** Do we need the definition of the global clustering coefficient? Also, I thought it was defined as 3 times number of triangles divided by number of length 2 paths and I do not see yet whether this agrees with the definition below. **Pim:** You were right. I have updated the definition. For instance, Let G be a simple graph, with vertex set $V(G)$ and edge set $E(G)$, denote by T_G and Λ_G, respectively, the total number of triangles and the total number of paths of length two in G. Then the *global clustering coefficient* is defined as

$$c_G = \frac{3T_G}{\Lambda_G}.$$

This coefficient measures the fraction of triangles in the network compared to the total possible number of triangles. A different global measure for clustering, *local clustering coefficient* computes

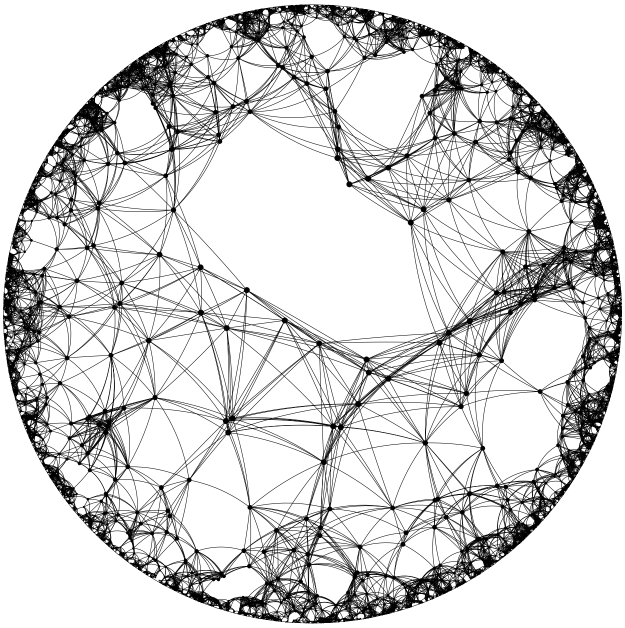


Figure 1: Example of a Hyperbolic Random Graph $G_{\mathbb{H},n}(\alpha,\nu)$ with $\alpha =??$, $\nu =??$ and $n = 5000$. **Pim:** @Tobias: this figure (HyperRGG_N5000L5.eps) is taken from the figures you send me. Could you fill in the value of α and ν you used for the simulation.

the average triangle density per node. To define this function, let $D_G(v)$ denote the degree of node v and $N_G(k)$ the number of nodes with degree k in the graph G. In addition let $T_G(v, v_1, v_2) = \mathbb{1}_{\{(v,v_1)\in E\}}\mathbb{1}_{\{(v,v_2)\in E\}}\mathbb{1}_{\{(v_1,v_2)\in E\}}$ be the indicator that v, v_1 and v_2 form a triangle and write

$$T_G(v) = \sum_{v_1,v_2\in V(G)} T_G(v, v_1, v_2), \tag{2}$$

to denote the number of triangles in which node v participates. Then the *local clustering coefficient* is given by

$$c_G = \frac{1}{|V(G)|}\sum_{v\in V(G)} \frac{T_G(v)}{\binom{D_G(v)}{2}}.$$

However, since the local clustering coefficient assigns just one value to the whole network, representing its triangular structure, it is unable to characterize local structural properties involving triangles. The local clustering function, on the other hand, measures the fraction of triangles to which nodes of a given degree belong, compared to the maximum number of triangles in which they could participate [19, 18]. It therefore characterizes the triangular structure of nodes in the networks, based on their degree, and gives a more detailed look at the overall structure of the network. The formal definition of the *local clustering function* is

$$c_G(k) = \begin{cases} \frac{1}{N_G(k)}\sum_{v\in V(G)} \mathbb{1}_{\{D_G(v)=k\}}\frac{T_G(v)}{\binom{k}{2}} & \text{if } N_G(k) \geq 1 \\ 0 & \text{else.} \end{cases} \tag{3}$$

Remark 1.1 (Notational convention)**.** *In the remainder of this paper we will compare the local clustering function and many other characteristics between several different graph models. To make notation less cluttered we will often use a unique subscript to identify the graph with respect to which the specific property refers instead of the full graph description. For instance, we shall write $c_{\mathbb{H},n}(k)$ to denote $c_{G_{\mathbb{H},n}(\alpha,\nu)}(k)$ and similar, $D_{\mathbb{H},n}(u)$ for $D_{G_{\mathbb{H},n}(\alpha,\nu)}(u)$. For the infinite model we will use the subscript ∞.*

1.5 Main results

We are now ready to state our main results for the clustering coefficient and local clustering function in the hyperbolic random graph. When the degree k is a growing function of n we are able to exactly compute the asymptotic behavior of $c_{\mathbb{H},n}(k)$.

The results are obtained by coupling the hyperbolic random graph to an infinite random graph $G_{\mathcal{P}}(\alpha, \nu)$, which we define in Section 2.1, and show that the limit of the local clustering coefficient and function for the hyperbolic random graph are given by those for the infinite model. To express these expressions we define for any $y \in \mathbb{R}_+$

$$\rho(y, k) = \mathbb{P}\left(\mathrm{Po}\left(\xi_{\alpha,\nu} e^{y/2}\right) = k\right),$$

where $\xi_{\alpha,\nu} = (4\alpha\nu)/(2\alpha - 1)\pi$ and $\mathrm{Po}(\lambda)$ denotes a Poisson random variable with mean λ. In addition we define the (marginal) probability measures on $\mathcal{R} := \mathbb{R}_+ \times \mathbb{R}$ by

$$\eta_y(x', y') = \frac{\alpha\nu e^{-\frac{y}{2}}}{\xi_{\alpha,\nu}\pi} \mathbb{1}_{\left\{|x'| < e^{(y+y')/2}\right\}} e^{-\alpha y'}$$

and let

$$\Delta_{\mathcal{P}}(y) = \iint_{\mathcal{R}^2} T_{\mathcal{P}}(y, x_1, x_2, y_1, y_2) \eta_y(x_1, y_1) \eta_y(x_2, y_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}y_1 \, \mathrm{d}y_2, \tag{4}$$

where

$$T_{\mathcal{P}}(y, x_1, x_2, y_1, y_2) = \mathbb{1}_{\left\{|x_1| \le e^{(y+y_1)/2}\right\}} \mathbb{1}_{\left\{|x_2| \le e^{(y+y_2)/2}\right\}} \mathbb{1}_{\left\{|x_1 - x_2| \le e^{(y_1+y_2)/2}\right\}}.$$

With these expressions we define the limit local clustering coefficient as

$$c_\infty := \alpha \int_0^\infty \Delta_{\mathcal{P}}(y)(1 - \rho(y, 0) - \rho(y, 1)) e^{-\alpha y} \, \mathrm{d}y, \tag{5}$$

while the local clustering function limit is defined as

$$c_\infty(k) = \frac{\int_0^\infty \rho(y, k) \Delta_{\mathcal{P}}(y) e^{-\alpha y} \, \mathrm{d}y}{\int_0^\infty \rho(y, k) e^{-\alpha y} \, \mathrm{d}y}. \tag{6}$$

Finally to express the local clustering coefficient and function in the infinite limit model we need a couple of special functions.

We let $B(a, b)$ denote the beta-function and $B^-(x, a, b)$ the lower incomplete beta-function

$$B^-(x, a, b) = \int_x^1 t^{a-1}(1 - t)^{b-1} \, dt.$$

In addition we write $\Gamma(z)$ for the Gamma function, $\Gamma^+(q, z)$ for the incomplete Gamma function

$$\Gamma^+(q, z) = \int_z^\infty t^{q-1} e^{-t} \, \mathrm{d}t$$

and define $\Gamma^*(q, z) := \Gamma^+(1+q, z) + \Gamma^+(q, z)$. Finally, we write $U(a, b, z)$ for the hypergeometric U-function (also called Tricomi's confluent hypergeometric function), which for $a, b, z \in \mathbb{C}$, $b \notin \mathbb{Z}_{\le 0}$, $\mathrm{Re}(a), \mathrm{Re}(z) > 0$ has the integral representation

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1 + t)^{b-a-1} dt,$$

see [8, p.255 Equation (2)], and let $G^{m,\ell}_{p,q}\left(z \middle| \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\right)$ denote Meijer's G-Function [16], see Appendix A for more details.

1.5.1 Local clustering coefficient

Pim: @All: I would be nice if we could show with simulations how close $c_{\mathbb{H},n}$ actually is to c_∞ for reasonably large n in practice.

Our first result shows that the local clustering coefficient in the Hyperbolic random graph converges in expectation to c_∞.

Theorem 1.1 (Limit for local clustering coefficient in $G_{\mathbb{H},n}(\alpha,\nu)$). *Let $\alpha > \frac{1}{2}$, $\nu > 0$. Then,*

$$\lim_{n\to\infty} \mathbb{E}\left[|c_{\mathbb{H},n} - c_\infty|\right] = 0.$$

Next we establishes the full expression for the clustering coefficient c_∞ in terms of the special function defined above.

Theorem 1.2 (Exact expression for clustering coefficient limit). *Let $\alpha > \frac{1}{2}$, $\nu > 0$. Then, if $\alpha \neq 1$*

$$\begin{aligned}
c_\infty = &\frac{2+4\alpha+13\alpha^2-34\alpha^3-12\alpha^4+24\alpha^5}{16(\alpha-1)^2\alpha(\alpha+1)(2\alpha+1)} + \frac{2^{-1-4\alpha}}{(\alpha-1)^2}\\
&+ \frac{(\alpha-1/2)(B(2\alpha,2\alpha+1)+B^-(1/2;1+2\alpha,-2+2\alpha))}{2(\alpha-1)(3\alpha-1)}\\
&+ \frac{\xi_{\alpha,\nu}^{2\alpha}\Gamma^*(-2\alpha,\xi_{\alpha,\nu})}{4(\alpha-1)} + \frac{\xi_{\alpha,\nu}^{2\alpha+2}\alpha(\alpha-1/2)^2\Gamma^*(-2\alpha-2,\xi_{\alpha,\nu})}{2(\alpha-1)^2}\\
&- \frac{\xi_{\alpha,\nu}^{2\alpha+1}\alpha(2\alpha-1)\Gamma^*(-2\alpha-1,\xi_{\alpha,\nu})}{(\alpha-1)} - \frac{\xi_{\alpha,\nu}^{6\alpha-2}2^{-4\alpha}(3\alpha-1)\Gamma^*(-6\alpha+2,\xi_{\alpha,\nu})}{(\alpha-1)^2}\\
&- \frac{\xi_{\alpha,\nu}^{6\alpha-2}(\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\Gamma^*(-6\alpha+2,\xi_{\alpha,\nu})}{(\alpha-1)}\\
&- \frac{e^{-\xi_{\alpha,\nu}}\Gamma(2\alpha+1)\left(U(2\alpha+1,1-2\alpha,\xi_{\alpha,\nu})+U(2\alpha+1,2-2\alpha,\xi_{\alpha,\nu})\right)}{4(\alpha-1)}\\
&+ \frac{\xi_{\alpha,\nu}^{6\alpha-2}\Gamma(2\alpha+1)\left(G_{2,3}^{3,0}\left(\xi_{\alpha,\nu}\middle|\begin{matrix}1,3-2\alpha\\3-4\alpha,-6\alpha+2,0\end{matrix}\right)+G_{2,3}^{3,0}\left(\xi_{\alpha,\nu}\middle|\begin{matrix}1,3-2\alpha\\3-4\alpha,-6\alpha+3,0\end{matrix}\right)\right)}{4(\alpha-1)},
\end{aligned}$$

while for $\alpha = 1$,

$$\begin{aligned}
c_\infty = &\frac{575-12\pi^2}{576} + \frac{\eta^4(7+\pi^2)\Gamma^*(-4,\eta)}{4}\\
&- \frac{1}{2}\int_0^1 (1-4z+3z^3)\log(1-z)(z+\eta)e^{-\eta/z}\,\mathrm{d}z\\
&- \int_0^1 \mathrm{Li}_2(z)(z^3+\eta z^2)e^{-\eta/z}\,\mathrm{d}z,
\end{aligned}$$

with $\eta = 4\nu/\pi$ and $\mathrm{Li}_2(z) = \sum_{t=1}^\infty z^t/t^2$, the dilogarithm function[2].

1.5.2 Local clustering function

We obtain similar results for the local clustering function. The first states that $c_{\mathbb{H},n}(k)$ is asymptotically equivalent to $c_\infty(k)$ in an L^1 sense.

[2]Note that the integrals in the expression for c_∞ for $\alpha = 1$ exists: for the first one note that $1-4z+3z^2 = (1-z)(1-3z)$, so the integrand can be bounded by $C(1-z)\log(1-z)$ on $[0,1)$ for some constant C, which can be continued continuously to the compact interval $[0,1]$ by noting that the limit for $z \to 1$ is zero, so the integrand is bounded on a bounded domain and hence, this integral is finite; for the second integral note that $\mathrm{Li}_2(z)$ is bounded by $\mathrm{Li}_2(1)$ on $[0,1]$, which is a series with well-known finite limit, so again the integrand is bounded on a bounded domain and hence the second integral is also finite.

Theorem 1.3 (Limit for local clustering function in $G_{\mathbb{H},n}(\alpha,\nu)$)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and $(k_n)_{n\geq 1}$ be any positive sequence, possibly constant, such that $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then,*

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{c_{\mathbb{H},n}(k_n)}{c_\infty(k_n)} - 1\right|\right] = 0.$$

Next we give an exact expression for $c_\infty(k)$ in terms of the special functions defined earlier.

Theorem 1.4 (Exact expression for local clustering function limit)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$. Then, for $\alpha \neq 1$*

$$\begin{aligned} c_\infty(k) &= \frac{\xi_{\alpha,\nu}^{4\alpha-2}\Gamma^+(k-6\alpha+2,\xi_{\alpha,\nu})}{\Gamma^+(k-2\alpha,\xi_{\alpha,\nu})}\left(\frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{(\alpha-\frac{1}{2})B^-\left(\frac{1}{2},2\alpha+1,2\alpha-2\right)}{2\alpha(\alpha-1)}\right) \\ &\quad - \frac{\xi_{\alpha,\nu}^{4\alpha-2}\Gamma(2\alpha+1)}{\Gamma^+(k-2\alpha,\xi_{\alpha,\nu})} G_{2,3}^{3,0}\left(\xi \middle| \begin{matrix} 1, 3-2\alpha \\ 3-4\alpha, -6\alpha+k+2, 0 \end{matrix}\right) \\ &\quad + \frac{1}{8\alpha(\alpha-1)}\left(\frac{\xi_{\alpha,\nu}^{k-2\alpha}e^{-\xi}U(2\alpha+1,1+k-2\alpha,\xi_{\alpha,\nu})}{\Gamma^+(k-2\alpha,\xi_{\alpha,\nu})} - 1\right) \\ &\quad - \frac{(\alpha-1/2)^2\xi_{\alpha,\nu}^2\Gamma^+(k-2\alpha-2,\xi_{\alpha,\nu})}{4(\alpha-1)^2\Gamma^+(k-2\alpha,\xi_{\alpha,\nu})} \end{aligned}$$

while for $\alpha = 1$

$$\begin{aligned} c_\infty(k) &= \frac{9\eta^3}{2k!}\Gamma^+(k-3,\eta) - \frac{\xi_{\alpha,\nu}^4}{k!}\frac{7+\pi^2}{4}\Gamma^+(k-4,\eta) \\ &\quad + \frac{\eta^k}{2k!}\int_0^1 (1-4z+3z^2)\ln(1-z)z^{1-k}e^{-\eta/z}\,\mathrm{d}z \\ &\quad + \frac{\eta^k}{k!}\int_0^1 z^{3-k}\,\mathrm{Li}_2(z)e^{-\eta/z}\,\mathrm{d}z, \end{aligned}$$

with $\eta = 4\nu/\pi$ and $\mathrm{Li}_2(z) = \sum_{t=1}^\infty z^t/t^2$, the dilogarithm function.

From the exact expression for $c_\infty(k)$ we obtain its asymptotic behavior as $k \to \infty$. For this we first define, for any $\frac{1}{2} < \alpha < \frac{3}{4}$,

$$C_\alpha = \frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{\alpha-\frac{1}{2}}{2(\alpha-1)\alpha}B^-(\frac{1}{2},2\alpha+1,2\alpha-2) - \frac{1}{4(\alpha-1)}B(2\alpha,3\alpha-4). \tag{7}$$

Moreover, to simplify the statement we define the scaling function

$$s_\alpha(k_n) = \begin{cases} k^{4\alpha-1} & \frac{1}{2} < \alpha < \frac{3}{4}, \\ \frac{k}{\log(k)} & \alpha = \frac{3}{4}, \\ k & \alpha > \frac{3}{4}. \end{cases} \tag{8}$$

We now have the following result, where the constant C_α emerges when we consider the local clustering function for $\frac{1}{2} < \alpha < \frac{3}{4}$.

Theorem 1.5 (Asymptotic behavior of local clustering function limit)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$. Then,*

$$\lim_{k\to\infty} \frac{c_\infty(k)}{s_\alpha(k)} = \begin{cases} C_\alpha\left(\frac{4\alpha\nu}{\pi(2\alpha-1)}\right)^{4\alpha-2} & \text{if } \frac{1}{2} < \alpha < \frac{3}{4}, \\ \frac{3\nu}{\pi} & \text{if } \alpha = \frac{3}{4}, \\ \frac{8\alpha\nu}{\pi(4\alpha-3)} & \text{if } \alpha \geq \frac{3}{4}, \end{cases}$$

with C_α as defined in (7).

The following is an immediate consequence of Theorem 1.3 and Theorem 1.5.

Corollary 1.6 (Asymptotic behavior of local clustering function in $G_{\mathbb{H},n}(\alpha,\nu)$)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$, $(k_n)_{n\geq 1}$ be any positive sequence such that $k_n \to \infty$ and $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$ and $s_\alpha(k)$ defined as in (8). Then, as $n \to \infty$,*

$$\frac{c_{\mathbb{H},n}(k_n)}{s_\alpha(k_n)} \xrightarrow{L^1} \begin{cases} C_\alpha \left(\frac{4\alpha\nu}{\pi(2\alpha-1)}\right)^{4\alpha-2} & \text{if } \frac{1}{2} < \alpha < \frac{3}{4}, \\ \frac{3\nu}{\pi} & \text{if } \alpha = \frac{3}{4}, \\ \frac{8\alpha\nu}{\pi(4\alpha-3)} & \text{if } \alpha \geq \frac{3}{4}, \end{cases}$$

where $\xrightarrow{L^1}$ denotes converges in expectation.

This result completely characterizes the asymptotic behavior of the local clustering function in Hyperbolic Random Graphs. In particular we observe that the conjectured scaling of k^{-1} from [14] only occurs when $\alpha > 3/4$, or equivalently, when the exponent of the pdf of the degree distribution is larger than $5/2$.

Pim: @All: It would be nice if we could add some simulations showing this scaling in practice, especially the different regimes.

The majority of the paper is dedicated to prove Theorem 1.3, which is based on a sequence of subsequent results for the local clustering function in related random graph models and Theorem 1.5. The exact flow of the argument is explained in Section 2. The proof for Theorem 1.3, using these results, can be found in Section 2.5. Here we also give the proof of Theorem 1.1. The exact result for the local clustering coefficient and function, as well as the proofs of Theorem 1.2 and 1.5 can be found in Section 4.

We end this section with some important observations regarding local clustering in hyperbolic random graphs.

Pim: Populate these paragraphs

Transition in the scaling of local clustering

Maximum scaling for k_n is $n^{\frac{1}{2\alpha+1}}$ All our results for clustering in the hyperbolic random graph are valid for any sequence k_n such that $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Although one would like to have results for any sequence $k_n \leq n$, it turns out that $n^{1/(2\alpha+1)}$ is the optimal scaling for which Theorem 1.1 can be true. To see why this is the case note that by definition of the local clustering function (3) we have that $c_{\mathbb{H},n}(k_n) = 0$ if $N_{\mathbb{H},n}(k_n) = 0$. Hence, it follows by Markov's inequality that for any positive function f

$$\mathbb{E}\left[\left|\frac{c_{\mathbb{H},n}(k_n)}{f(k_n)} - 1\right|\right] \geq \mathbb{P}\left(N_{\mathbb{H},n}(k_n) = 0\right) \geq 1 - \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right].$$

We shall later establish (see Lemma [??] **Pim:** Find the reference for this Lemma) that $\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] = \Theta\left(nk_n^{-(2\alpha+1)}\right)$. Therefore if k_n is such that $k_n = \omega\left(n^{\frac{1}{2\alpha+1}}\right)$ we have

$$\lim_{n\to\infty} nk_n^{-(2\alpha+1)} = \lim_{n\to\infty}\left(n^{-\frac{1}{2\alpha+1}}k_n\right)^{-(2\alpha+1)} = 0$$

and hence

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{c_{\mathbb{H},n}(k_n)}{f(k_n)} - 1\right|\right] \geq \lim_{n\to\infty} 1 - \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] = \lim_{n\to\infty} 1 - \Theta\left(nk_n^{-(2\alpha+1)}\right) = 1 \neq 0,$$

for any positive function f. This implies that we cannot expect a result like that of Theorem 1.1 to hold as soon as $k_n = \omega\left(n^{\frac{1}{2\alpha+1}}\right)$.

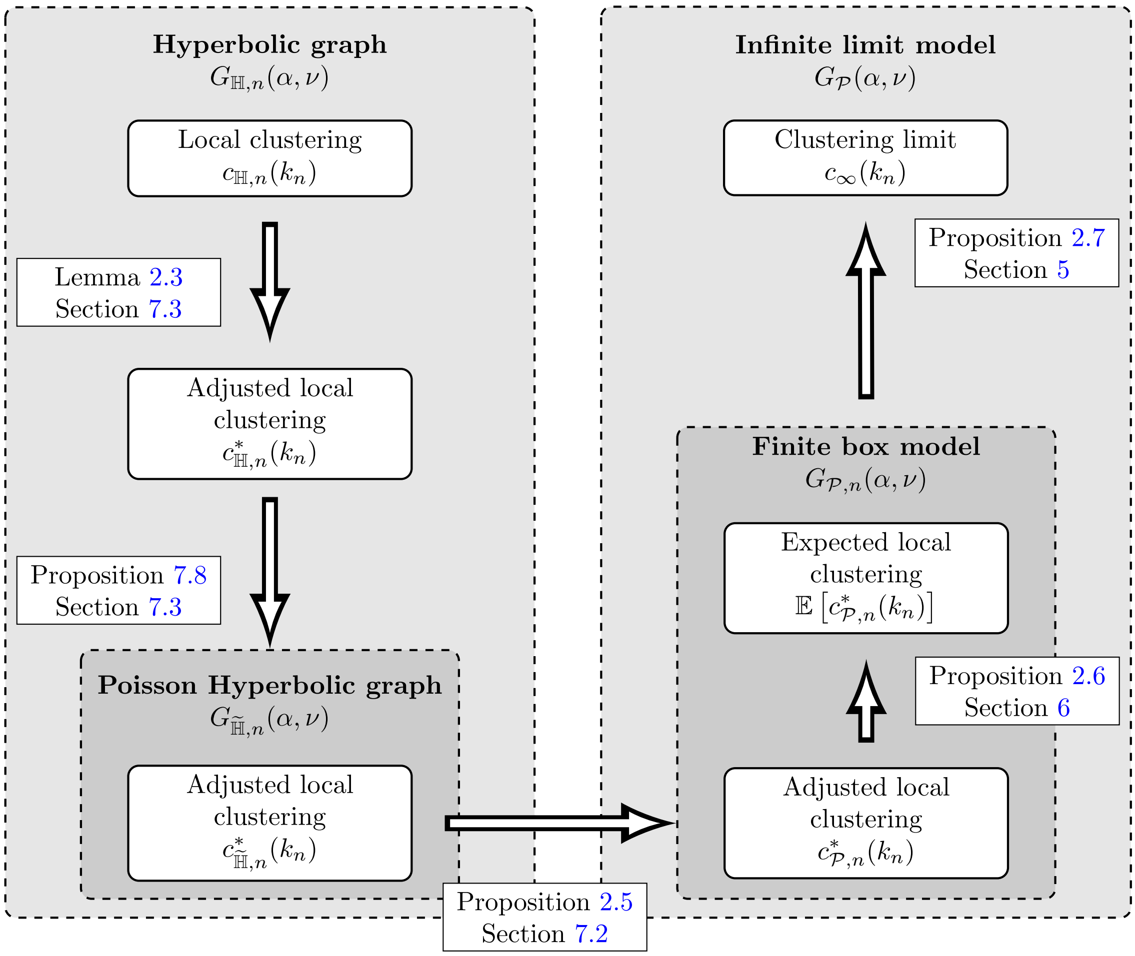


Figure 2: Overview of the proof strategy for Theorem 1.3.

Universality of scaling transition

Geometric inhomogeneous random graphs

1.6 Simulations

2 Overview of the proof strategy for Theorem 1.3

We first give a detailed overview of the structure for the proof of Theorem 1.3. The proof itself can be found in Section 2.5, where we also prove Theorem 1.1. Figure 2 shows a schematic overview of the proof of Theorem 1.3 based on the different propositions described below, plus the sections in which theses propositions are proved.

The main idea behind the proof of Theorem 1.3 is to couple the hyperbolic graph $G_{\mathbb{H},n}(\alpha,\nu)$ with a subgraph of an infinite graph $G_{\mathcal{P}}(\alpha,\nu)$ (see Section 2.1). A similar approach is used in [9] to analyze the size of the largest component in the hyperbolic model and in [13] to analyze explosion times in a more general class of models.. The advantage of the infinite limit model is two-fold. First, it simplifies the actual computations, e.g. by replacing hyperbolic densities with exponential densities and the hyperbolic distance with a computationally easier expression. Secondly, it directly yields the limit as a result rather than having to determine this later from a long expression depending on n.

Very roughly, the proof falls into three parts: justifying the transitioning from the hyperbolic random graph to the infinite limit model, proving concentration results for local clustering and calculating the expected local clustering coefficient and function in the infinite limit model.

In this section, we state the required definitions, auxiliary lemmas and propositions and show how to assemble them to obtain Theorem 1.3.

2.1 Infinite limit model

To analyze the limiting expressions for clustering in the hyperbolic random graph $G_{\mathbb{H},n}(\alpha,\nu)$ we shall use an following infinite graph model which was also considered in [9] to analyze the largest component. Define $\mathcal{R} = \mathbb{R} \times \mathbb{R}_+$ and let $\mathcal{P}_{\alpha,\nu}$ be a Poisson point process on $\mathcal{R}$ with intensity function

$$f_{\alpha,\nu}(x,y) = \frac{\alpha\nu}{\pi} e^{-\alpha y}. \tag{9}$$

We will write $p = (x, y)$ for points in $\mathcal{R}$. Then the *infinite limit model* $G_{\mathcal{P}}(\alpha,\nu)$ is defined to have vertex set $\mathcal{P}$ and edge set such that

$$(p_i, p_j) \in E(G_{\mathcal{P}}(\alpha,\nu)) \iff |x_i - x_j| \le e^{\frac{y_i+y_j}{2}}.$$

For a point $p \in \mathcal{R}$, we will write $\mathcal{B}_{\mathcal{P}}(p)$ to denote the *ball* around p, i.e.

$$\mathcal{B}_{\mathcal{P}}(p) = \{p' \in \mathcal{R} : |x - x'| \le e^{\frac{y+y'}{2}}\}. \tag{10}$$

With this notation we then have that $\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{P}$ denotes the set of neighbors of a vertex $p \in G_{\mathcal{P}}(\alpha,\nu)$. We denote the intensity measure of the Poisson process $\mathcal{P}$ by $\mu_{\alpha,\nu}$, i.e. for every Borel-measurable subset $S \subseteq \mathbb{R} \times \mathbb{R}_+$

$$\mu_{\alpha,\nu}(S) = \int_S f_{\alpha,\nu}(x,y)\, dx\, dy, \tag{11}$$

and for any point $p \in \mathcal{R}$ we let η_p be the corresponding probability measure on $\mathcal{B}_{\mathcal{P}}(p)$

$$\eta_p(S) = \frac{\mu_{\alpha,\nu}(S \cap \mathcal{B}_{\mathcal{P}}(p))}{\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(p))}. \tag{12}$$

In addition we use $\rho(p,k)$ and $\rho(y,k)$ (for a point $p = (x,y) \in \mathcal{R}$) to denote the probability mass function of a Poisson random variable with expectation

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(p)) = \int_0^\infty \int_{-\infty}^\infty \mathbb{1}_{\left\{|x-x'| \le e^{\frac{y+y'}{2}}\right\}} f_{\alpha,\nu}(x',y')\, dx'\, dy' = \frac{4\alpha\nu}{(2\alpha-1)\pi} e^{\frac{y}{2}} := \xi_{\alpha,\nu} e^{\frac{y}{2}}, \tag{13}$$

i.e.

$$\rho(p,k) = \rho(y,k) = \mathbb{P}\left(\mathrm{Po}\left(\xi_{\alpha,\nu} e^{y/2}\right) = k\right).$$

Note that $\rho(y,k)$ denote the probability that a node $(x,y) \in G_{\mathcal{P}}(\alpha,\nu)$ has degree k.

Remark 2.1 (Notations for points). *We will be working extensively with expressions in terms of points of $\mathcal{P}_{\alpha,\nu}$, often relating them back to points in the hyperbolic disc $\mathcal{D}_{\mathcal{R}_n}$. Therefore, in the remainder of this paper we will always use $u = (r,\theta)$ to denote points in $\mathcal{D}_{\mathcal{R}_n}$ in polar coordinates while $p = (x,y)$ will denote points in $\mathcal{R} := \mathbb{R} \times \mathbb{R}_+$ in Cartesian coordinates. In addition, when we write $p' \in \mathbb{R} \times \mathbb{R}_+$ we will denote its Cartesian coordinates by (x',y') and similarly for p_i, u' and u_i, e.g. $p_i = (x_i,y_i)$.*

2.2 Finite box model

Next we define the finite graph $G_{\mathcal{P},n}(\alpha,\nu)$ which we will couple to $G_{\mathbb{H},n}(\alpha,\nu)$. Recall that $R_n = 2\log(n/\nu)$ and that $\mathcal{P}_{\alpha,\nu}$ is a Poisson point process on $\mathcal{R} = \mathbb{R}\times\mathbb{R}_+$, with intensity given by (9). Now consider the finite box $\mathcal{R}_n = (-\frac{\pi}{2}e^{R_n/2}, \frac{\pi}{2}e^{R_n/2}] \times (0, R_n]$ in $\mathcal{R}$. Then the *finite box model* graph $G_{\mathcal{P},n}(\alpha,\nu)$ has vertex set $\mathcal{V}_n := \mathcal{P}_{\alpha,\nu}\cap\mathcal{R}_n$ and edges set such that

$$(p_i,p_j)\in E(G_{\mathcal{P},n}(\alpha,\nu)) \iff |x_i - x_j|_{\pi e^{R_n/2}} \le e^{\frac{y_i+y_j}{2}},$$

where $|x|_{\pi e^{R_n/2}} = \inf_{k\in\mathbb{Z}} |x + k\pi e^{R_n/2}|$. In order words, $G_{\mathcal{P},n}(\alpha,\nu)$ is the subgraph of $G_{\mathcal{P}}(\alpha,\nu)$ induced on $\mathcal{V}_n$, where we have identified the boundaries of $(-\frac{\pi}{2}e^{R_n/2}, \frac{\pi}{2}e^{R_n/2}]$ to exclude boundary effects and to make the model symmetric in the horizontal x-axis.

Similar to $G_{\mathcal{P}}(\alpha,\nu)$, we define for any $p\in\mathcal{R}_n$, its neighborhood ball

$$\mathcal{B}_{\mathcal{P},n}(p) = \left\{p' \in \mathcal{R}_n : |x_i - x_j|_{\pi e^{R_n/2}} \le e^{\frac{y_i+y_j}{2}}\right\}$$

and associated functions $\mu_{\alpha,\nu,n}$ and $\eta_{p,n}$ and others, by appending an additional subscript n. For instance

$$\mu_{\alpha,\nu,n}(S) = \mu_{\alpha,\nu}(S\cap\mathcal{R}_n) \quad \text{and} \quad \eta_{p,n}(S) = \frac{\mu_{\alpha,\nu,n}(S\cap\mathcal{B}_{\mathcal{P},n}(p))}{\mu_{\alpha,\nu,n}(\mathcal{B}_{\mathcal{P},n}(p))} = \frac{\mu_{\alpha,\nu}(S\cap\mathcal{B}_{\mathcal{P},n}(p))}{\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(p))}.$$

2.3 Coupling the hyperbolic model to the finite box model

To couple the hyperbolic graph to $G_{\mathcal{P},n}(\alpha,\nu)$ we consider the original hyperbolic graph on a Poisson distributed number of vertices as an intermediate step. For this we define $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$ to be the hyperbolic random graph where the vertex set is given by $N \stackrel{d}{=} \mathrm{Po}(n)$ points, distributed according to the (α,R_n)-quasi uniform distribution (1) defined in Section 1.3. The following two lemmas from [9] establishes a coupling between this version of the hyperbolic random graph and the finite box model and relate the hyperbolic neighborhoods to the neighborhood balls $\mathcal{B}_{\mathcal{P}}(p)$. The ensuing proposition verifies that this coupling also essentially preserves the local clustering function, i.e. that the difference between the local clustering functions in the hyperbolic model and the finite box graph converges to zero faster than the proposed scalings in the main result, Theorem 1.3. Hence, it justifies the transition to the finite box model.

Lemma 2.1 ([9, Lemma 27]). *Let $\mathcal{V}_{\widetilde{\mathbb{H}},n}$ denote the vertex set of $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$ and $\mathcal{V}_n$ the vertex set of $G_{\mathcal{P},n}(\alpha,\nu)$. Define the map $\Psi : [0,R_n]\times(-\pi,\pi] \to \mathcal{R}_n$ by*

$$\Psi(r,\theta) = \left(\theta\frac{e^{R_n/2}}{2}, R_n - r\right). \tag{14}$$

Then there exists a coupling such that, a.a.s., $\mathcal{V}_n = \Psi(\mathcal{V}_{\widetilde{\mathbb{H}},n})$.

In the remainder of this paper we will write $\mathcal{B}_{\mathbb{H},n}(p)$ to denote the image under Ψ of the ball of hyperbolic radius R_n around the point $\Psi^{-1}(p)$, i.e.

$$\mathcal{B}_{\mathbb{H},n}(p) := \left\{p' := \Psi(u) \,:\, u\in\mathcal{D}_{R_n} \text{ and } d_{\mathbb{H}}(\Psi^{-1}(p),u) \le R_n\right\}.$$

Note that we have that $\mathcal{B}_{\mathbb{H},n}(p)\subseteq\mathcal{R}_n$. In particular, a point $p=(x,y)\in\mathcal{R}$ corresponds to $u := \Psi^{-1}(p) = (2e^{R_n/2}x, R_n - y)$.

Let $u,u'\in\mathcal{D}_R$. The hypebolic law of cosines implies that $d_{\mathbb{H}}(u,u') < R_n$ if and only if their relative angle $\theta(u,u') := |\theta-\theta'|_{2\pi}$ satisfies $\theta(u,u') < \theta_{R_n}(r,r')$, where $\theta_{R_n}(r,r')$ is the solution of the following equation

$$\cosh R_n = \cosh r\cosh r' - \sinh r\sinh r'\cos\theta_{R_n}(r,r').$$

To write this differently, define

$$\Omega(r,r') := \frac{1}{2}e^{R_n/2}\arccos\left(\frac{\cosh r\cosh r' - \cosh R_n}{\sinh r\sinh r'}\right). \tag{15}$$

Then $d_{\mathbb{H}}(u,u') < R_n$ if and only if $\theta(u,u') \le 2e^{-R_n/2}\Omega(r,r')$. Under the coupling map Ψ, this is equivalent to $|x-x'|_{\pi e^{R_n/2}} \le \Omega(r,r')$.

Now consider a point $p \in \mathcal{V}_n$ and write $\Psi^{-1}(p) = (r,\theta)$. Then the triangle inequality implies that for any point $u' \in \mathcal{D}_R$

$$r' \le y = R_n - r \Rightarrow d_H(\Psi^{-1}(p),u') \le R_n. \tag{16}$$

In other words, we have $\mathcal{R}_n([r,R_n]) \subset \mathcal{B}_{\mathbb{H},n}(p)$. The shape of $\mathcal{B}_{\mathbb{H},n}(p)$ at the lower parts of $\mathcal{R}_n$ is described by the following lemma which appears in [9].

Lemma 2.2 ([9, Lemma 28])**.** *There exists a constant $K>0$ such that, for every $\varepsilon>0$ and for R_n sufficiently large, the following holds. For every $r,r' \in [\varepsilon R_n, R_n]$ with $r+r' > R_n$ we have that*

$$e^{\frac{1}{2}(y+y')} - Ke^{\frac{3}{2}(y+y')-R_n} \le \Omega(r,r') \le e^{\frac{1}{2}(y+y')} + Ke^{\frac{3}{2}(y+y')-R_n}, \tag{17}$$

where $y := R_n - r, y' := R - r'$. Moreover:

$$\Omega(r,r') \ge e^{\frac{1}{2}(y+y')} \quad \textit{if} \quad r,r' < R_n - K. \tag{18}$$

This result allows us to couple the neighborhoods in the hyperbolic random graph $G_{\mathbb{H},n}(\alpha,\nu)$ to those in the finite box model $G_{\mathcal{P},n}(\alpha,\nu)$. This coupling is however not exact and hence we need to compute the difference in triangle counts between the two models. For this, and other computations later on, it will be more convenient to consider the following slight modified version of local clustering,

$$c^*_G(k) = \begin{cases} \frac{1}{\mathbb{E}[N_G(k)]}\sum_{v\in V(G)} \mathbb{1}_{\{D_G(v)=k\}}\frac{T_G(v)}{\binom{k}{2}} & \text{if } N_G(k)\ge 1 \\ 0 & \text{else.}\end{cases} \tag{19}$$

Notice that the only difference between $c_G(k)$ and $c^*_G(k)$ is that we replace $N_G(k)$ by its expectation $\mathbb{E}[N_G(k)]$. Following the notational convention, see Remark 1.1, throughout the remainder of this paper we write $c^*_{\widetilde{\mathbb{H}},n}(k)$ and $c^*_{\mathcal{P},n}(k)$ to denote the modified local clustering function in $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$ and $G_{\mathcal{P},n}(\alpha,\nu)$, respectively.

First we show that considering this adjusted version doesn't change anything in terms of the scaling, the proof is given in Section 7.3.

Lemma 2.3. *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and $(k_n)_{n\ge1}$ be any positive sequence, possibly constant, such that $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then, as $n\to\infty$,*

$$\mathbb{E}\left[\left|c_{\mathbb{H},n}(k_n) - c^*_{\mathbb{H},n}(k_n)\right|\right] = o\left(s_\alpha(k_n)\right).$$

We then establish that the adjusted local clustering function in the hyperbolic model $G_{\mathbb{H},n}(\alpha,\nu)$ behaves similar to that in the Poisson version $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$.

Proposition 2.4. *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and $(k_n)_{n\ge1}$ be any increasing sequence such that $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then, as $n\to\infty$,*

$$\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = o\left(s_\alpha(k_n)\right).$$

The next step is to show that the modified clustering isn't influenced by the coupling describe above. The proof can be found in Section 7.2.

Proposition 2.5 (Coupling result for local clustering)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and $(k_n)_{n\geq 1}$ be any increasing sequence such that $k_n = o\left(\min\left\{n^{\frac{1}{2\alpha+1}}, n^{\frac{1}{3}}\right\}\right)$. Then, as $n \to \infty$,*

$$\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\mathcal{P},n}(k_n)\right|\right] = o\left(s_\alpha(k_n)\right).$$

These two results imply that the difference between the local clustering functions in the hyperbolic model and modified version in the finite box graph converges to zero faster than the proposed scaling $s_\alpha(k_n)$ in Theorem 1.3. Hence, to proof this theorem it enough to prove it for $c^*_{\mathcal{P},n}(k)$.

2.4 From the finite to the infinite model

To compute the limit of the adjusted local clustering $c^*_{\mathcal{P},n}(k)$ in the finite graph $G_{\mathcal{P},n}(\alpha,\nu)$ we first prove in Section 6 that it is concentrated around its mean $\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]$.

Proposition 2.6 (Concentration local clustering in $G_{\mathcal{P},n}(\alpha,\nu)$)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and let $(k_n)_{n\geq 1}$ be any non-decreasing sequence satisfying $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then, as $n \to \infty$,*

$$\mathbb{E}\left[\left|c^*_{\mathcal{P},n}(k_n) - \mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right|\right] = o\left(s_\alpha(k_n)\right).$$

With this concentration result, we are left with the task to compute the limit of $\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]$ as $n \to \infty$ and show that it is equivalent to $c_\infty(k_n)$. To accomplish this we move to the infinite limit model $G_{\mathcal{P}}(\alpha,\nu)$. The next result shows that the difference between the expected value of the clustering function $c^*_G(k)$ in $G_{\mathcal{P},n}(\alpha,\nu)$ and $G_{\mathcal{P}}(\alpha,\nu)$ goes to zero faster than the proposed scaling in Theorem 1.3.

Proposition 2.7 (Transition to infinite limit model)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and let $(k_n)_{n\geq 1}$ be any positive sequence tending to infinity and satisfying $k_n = O\left(n^{\frac{1}{2\alpha+1}}\right)$. Then, as $n \to \infty$,*

$$\left|\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right] - c_\infty(k_n)\right| = o\left(s_\alpha(k_n)\right).$$

2.5 Proof of the main results

We are now ready to prove our main results, using the propositions stated in the previous sections. We begin with Theorem 1.3.

Proof of Theorem 1.3. By applying Proposition 2.7, Proposition 2.7 and Theorem 1.5 we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{c^*_{\mathcal{P},n}(k_n)}{C_{\alpha,\nu}s_\alpha(k_n)} - 1\right|\right] \\
&\quad\leq \frac{\mathbb{E}\left[\left|c^*_{\mathcal{P},n}(k_n) - \mathbb{E}\left[\right] c^*_{\mathcal{P},n}(k_n)\right|\right]}{C_{\alpha,\nu}s_\alpha(k_n)} + \frac{\left|\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right] - c_\infty(k_n)\right|}{C_{\alpha,\nu}s_\alpha(k_n)} + \left|\frac{c_\infty(k_n)}{C_{\alpha,\nu}s_\alpha(k_n)} - 1\right| \\
&\quad= o(1).
\end{aligned}$$

Combining this with Proposition 2.5 yields

$$\mathbb{E}\left[\left|\frac{c^*_{\mathbb{H},n}(k_n)}{C_{\alpha,\nu}s_\alpha(k_n)} - 1\right|\right] \leq \mathbb{E}\left[\left|\frac{c^*_{\mathcal{P},n}(k_n)}{C_{\alpha,\nu}s_\alpha(k_n)} - 1\right|\right] + \frac{\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\mathcal{P},n}(k_n)\right|\right]}{C_{\alpha,\nu}s_\alpha(k_n)} = o(1).$$

and the result then follows by applying Lemma 2.3. □

We can now use this result to prove Theorem 1.1.

Proof of Theorem 1.1. First, let a_n be any sequence such that $a_n \to \infty$. Then

$$\begin{aligned} c_{\mathbb{H},n} &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{D_{\mathbb{H},n}(i)\le a_n\}} \frac{T_{\mathbb{H},n}(i)}{\binom{D_{\mathbb{H},n}(i)}{2}} + \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{D_{\mathbb{H},n}(i)> a_n\}} \frac{T_{\mathbb{H},n}(i)}{\binom{D_{\mathbb{H},n}(i)}{2}} \\ &= \frac{1}{n}\sum_{t=3}^{a_n}\sum_{i=1}^{n} \mathbb{1}_{\{D_{\mathbb{H},n}(i)=t\}} \frac{T_{\mathbb{H},n}(i)}{\binom{t}{2}} + \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{D_{\mathbb{H},n}(i)> a_n\}} \frac{T_{\mathbb{H},n}(i)}{\binom{D_{\mathbb{H},n}(i)}{2}} \\ &= \frac{1}{n}\sum_{t=3}^{a_n} \mathbb{E}\left[N_{\mathbb{H},n}(t)\right] c_{\mathbb{H},n}(t) + \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{D_{\mathbb{H},n}(i)> a_n\}} \frac{T_{\mathbb{H},n}(i)}{\binom{D_{\mathbb{H},n}(i)}{2}}, \end{aligned}$$

where the expectation of last term is bounded from above by $\mathbb{P}\left(D_{\mathbb{H},n} > a_n\right)$. We have a similar result for c_∞. Therefore

$$\begin{aligned} \mathbb{E}\left[\left|c_{\mathbb{H},n} - c_\infty\right|\right] \le \mathbb{E}&\left[\left|\frac{1}{n}\sum_{t=3}^{a_n}\left(\mathbb{E}\left[N_{\mathbb{H},n}(t)\right] c_{\mathbb{H},n}(t) - \mathbb{E}\left[N_{\mathcal{P}}(t)\right] c_\infty(t)\right)\right|\right] \\ &+ \mathbb{P}\left(D_{\mathbb{H},n} > a_n\right) + \mathbb{P}\left(D_{\mathcal{P}} > a_n\right), \end{aligned}$$

and since the last to probabilities are $o\left(1\right)$ it is enough to prove that

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{1}{n}\sum_{t=3}^{a_n}\left(\mathbb{E}\left[N_{\mathbb{H},n}(t)\right] c_{\mathbb{H},n}(t) - \mathbb{E}\left[N_{\mathcal{P}}(t)\right] c_\infty(t)\right)\right|\right] = 0.$$

We write

$$\begin{aligned} \mathbb{E}\left[\left|\frac{1}{n}\sum_{t=3}^{a_n}\left(\mathbb{E}\left[N_{\mathbb{H},n}(t)\right] c_{\mathbb{H},n}(t) - \mathbb{E}\left[N_{\mathcal{P}}(t)\right] c_\infty(t)\right)\right|\right] &\le \frac{1}{n}\sum_{t=3}^{a_n} \mathbb{E}\left[N_{\mathcal{P}}(t)\right] \mathbb{E}\left[\left|c_{\mathbb{H},n}(t) - c_\infty(t)\right|\right] \\ &\quad + \frac{1}{n}\sum_{t=3}^{a_n} \mathbb{E}\left[\left|N_{\mathbb{H},n}(t) - N_{\mathcal{P}}(t)\right|\right] \mathbb{E}\left[c_\infty(t)\right], \end{aligned}$$

and will show that both terms go to zero for some appropriately chosen sequence a_n.

Note that Theorem 1.3 states that for any sequence $k_n = o\left(n^{1/(2\alpha+1)}\right)$

$$\lim_{n\to\infty} \mathbb{E}\left[\left|c_{\mathbb{H},n}(k_n) - c_\infty(k_n)\right|\right] = 0.$$

Let $b_n := \log(n)^{-1} n^{1/(2\alpha+1)}$ and define the sequence k_n by

$$k_n = \arg\max_{3\le t\le b_n} \mathbb{E}\left[\left|c_{\mathbb{H},n}(t) - c_\infty(t)\right|\right].$$

Then $k_n = o\left(n^{1/(2\alpha+1)}\right)$ and by Theorem 1.3

$$\max_{3\le t\le b_n} \mathbb{E}\left[\left|c_{\mathbb{H},n}(t) - c_\infty(t)\right|\right] = \mathbb{E}\left[\left|c_{\mathbb{H},n}(k_n) - c_\infty(k_n)\right|\right] \to 0.$$

Similarly we note that

$$\arg\max_{3\le t\le b_n} \mathbb{E}\left[\left|\frac{N_{\mathbb{H},n}(t)}{N_{\mathcal{P}}(t)} - 1\right|\right] \to 0.$$

Now define

$$a_n := \left\lfloor \min\left\{\left(\max_{3\le t\le b_n} \mathbb{E}\left[\left|c_{\mathbb{H},n}(t) - c_\infty(t)\right|\right]\right)^{-1/2}, \left(\arg\max_{3\le t\le b_n} \mathbb{E}\left[\left|\frac{N_{\mathbb{H},n}(t)}{N_{\mathcal{P}}(t)} - 1\right|\right]\right)^{-1/2}, b_n - 1\right\}\right\rfloor$$

so that $a_n < b_n = o\left(n^{1/(2\alpha+1)}\right)$, $a_n \to \infty$,

$$a_n \max_{3\le t\le a_n} \mathbb{E}\left[\left|c_{\mathbb{H},n}(t) - c_\infty(t)\right|\right] \le a_n \max_{3\le t\le b_n} \mathbb{E}\left[\left|c_{\mathbb{H},n}(t) - c_\infty(t)\right|\right] \to 0,$$

and similarly

$$a_n \arg \max_{3 \le t \le a_n} \mathbb{E}\left[\left|\frac{N_{\mathbb{H},n}(t)}{N_{\mathcal{P}}(t)} - 1\right|\right] \to 0.$$

Then, since $\mathbb{E}[N_{\mathcal{P}}(t)] = \Theta(1)\, n^{-1} t^{-(2\alpha+1)}$

$$\begin{aligned} &\frac{1}{n}\sum_{t=3}^{a_n} \mathbb{E}[N_{\mathcal{P}}(t)]\, \mathbb{E}[|c_{\mathbb{H},n}(t) - c_\infty(t)|] \\ &\qquad \le \Theta(1)\, 3^{-(2\alpha+1)} a_n \max_{3\le t\le a_n} \mathbb{E}[|c_{\mathbb{H},n}(t) - c_\infty(t)|] = o(1), \end{aligned}$$

while $\mathbb{E}[c_\infty(t)] \le 1$ implies that

$$\begin{aligned} &\frac{1}{n}\sum_{t=3}^{a_n} \mathbb{E}[|N_{\mathbb{H},n}(t) - N_{\mathcal{P}}(t)|]\, \mathbb{E}[c_\infty(t)] \\ &\qquad \le \frac{1}{n}\sum_{t=3}^{a_n} \mathbb{E}[N_{\mathcal{P}}(t)]\, \mathbb{E}\left[\left|\frac{N_{\mathbb{H},n}(t)}{N_{\mathcal{P}}(t)} - 1\right|\right] \\ &\qquad \le \Theta(1)\, 3^{-(2\alpha+1)} a_n \max_{3\le t\le a_n} \mathbb{E}\left[\left|\frac{N_{\mathbb{H},n}(t)}{N_{\mathcal{P}}(t)} - 1\right|\right] = o(1). \end{aligned}$$

This finishes the proof. $\square$

3 Concentration of heights for nodes with degree k

Many of the computations in this paper will involve integrals where the integrand contains $\mathbb{P}(\mathrm{Po}(\mu_n(y)) = k)$, where $\mathrm{Po}(\lambda)$ denotes a Poisson random variable with mean λ and $\mu_n(y) \approx e^{y/2}$. For instance, the expected adjusted local clustering function $c_\infty(k)$ in $G_{\mathcal{P}}$ equals

$$\frac{\int_0^\infty \mathbb{P}\left(\mathrm{Po}(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}(0,y)\right)) = k\right) \Delta_{\mathcal{P}}(y) e^{-\alpha}\, \mathrm{d}y}{\int_0^\infty \mathbb{P}\left(\mathrm{Po}(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}(0,y)\right)) = k\right) e^{-\alpha}\, \mathrm{d}y},$$

where $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}(0,y)\right) = \xi_{\alpha,\nu} e^{y/2}$.

In particular, the average degree of a node at height y is approximately Poisson with mean $\xi_{\alpha,\nu} e^{y/2}$ while we are interested in those nodes with degree k_n. Since Poisson random variables are well concentrated around their mean, for our analysis we would therefore like to be able the integration to intervals where $e^{y/2} \approx k_n$. Moreover, we want to be able to do this for all three Poisson models $G_{\widetilde{\mathbb{H}},n}$, $G_{\mathcal{P},n}$ and $G_{\mathcal{P}}$ as well as simultaneously deal with the case where $k_n \to \infty$ and $k_n = \Theta(1)$. In this section we will establish such a result. We start with a concentration lemma for the infinite model (Lemma 3.1) and explain in Remark 3.1 how such a result will be used throughout the paper. To obtain similar results for the other two models we have to first analyze the average degree in these models which we do in Sections 3.2 and 3.3. We conclude this section with a general result that allows us to extend the concentration lemma for the infinite model to the hyperbolic random graph and finite box model.

3.1 Concentration argument for the infinite model

Recall that $\rho(y,k)$ is the probability density function of a Poisson random variable with mean $\mu_{\alpha,\nu}(B_{\mathcal{P}}(p)) = \xi_{\alpha,\nu} e^{\frac{y}{2}}$. For any sequence $\{k_n\}_{n\ge1}$, possibly constant, define

$$\kappa_n := \begin{cases} \log(n) & \text{if } k_n = \Theta(1), \\ \sqrt{k_n \log(k_n)} & \text{else.} \end{cases} \tag{20}$$

and define further, for any $C > 0$

$$\mathcal{K}_C(k_n) = \left\{ p \in \mathbb{R} : \frac{k_n - C\kappa_n}{\xi_{\alpha,\nu}} \vee 0 \le e^{\frac{y}{2}} \le \frac{k_n + C\kappa_n}{\xi_{\alpha,\nu}} \wedge e^{R_n/2} \right\}, \tag{21}$$

Note that if $k_n = \Omega(\log(n))$ then $e^{y/2} = \Theta(k_n)$ whenever $p \in \mathcal{K}_C(k_n)$. The next lemma states that for a large class of functions $h(y)$ to compute the integral

$$\int_{\mathcal{R}_n} \rho(y, k_n) h(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y$$

it is enough to consider integration of $\mathcal{K}_C(k_n)$ instead of $\mathcal{R}_n$.

Lemma 3.1. *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and $\{k_n\}_{n \ge 1}$ be any sequence such that $k_n = o(n^{\frac{1}{2\alpha+1}})$. In addition let $\beta < \alpha$ and $h : \mathbb{R}_+ \mapsto \mathbb{R}$ be a any function such that $h(y) = O\left(e^{\beta y}\right)$ as $y \to \infty$. Then, for any $C > 0$,*

$$\begin{aligned} &\int_{\mathcal{R}_n \setminus \mathcal{K}_C(k_n)} \rho(y, k_n) h(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = \begin{cases} O\left(n^{1-\frac{C}{2}}\right) & \text{if } k_n = \Theta(1) \\ O\left(n \left(k_n^{-(1+C^2)/2} + k_n^{-1/2} e^{-\frac{C\sqrt{k_n \log(k_n)}}{2}}\right)\right) & \text{else.} \end{cases} \end{aligned} \tag{22}$$

as $n \to \infty$.

In particular, if $g_n(y)$ is a function such that $g_n(y) = O\left(n^{-1} k_n^{\beta}\right) \rho(y, k_n)$ for some $s \in \mathbb{R}$. Then for $C > 0$ large enough we have

$$\lim_{n \to \infty} \int_{\mathcal{R}_n \setminus \mathcal{K}_C(k_n)} g_n(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y = 0,$$

or equivalently,

$$\int_{\mathcal{R}_n} g_n(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y = (1 + o(1)) \int_{\mathcal{K}_C(k_n)} g_n(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

Proof. Define

$$\lambda_n^{\pm} = k_n \pm C\kappa_n, \quad a_n^{\pm} = 2 \log\left(\frac{\lambda_n^{\pm}}{\xi_{\alpha,\nu}}\right).$$

Then, since $\rho_y(k_n)$, as a function of y, is strictly increasing on $[0, a_n^-]$ and strictly decreasing on $[a_n^+, \infty)$

$$\begin{aligned} &\int_{\mathcal{R}_n \setminus \mathcal{K}_C(k_n)} h(y) \rho_{k_n}(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = O(1) \int_0^{a_n^-} \int_{-I_n}^{I_n} e^{\beta y} \rho_{k_n}(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y + \int_{a_n^+}^{R_n} \int_{-I_n}^{I_n} e^{\beta y} \rho_{k_n}(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = O(n) \int_0^{a_n^-} \rho_{k_n}(y) e^{-(\alpha-\beta)y} \, \mathrm{d}y + O(n) \int_{a_n^+}^{R_n} \rho_{k_n}(y) e^{-(\alpha-\beta)y} \, \mathrm{d}y \\ &\quad \le O(n) \rho(a_n^-, k_n) \int_0^{a_n^-} e^{-(\alpha-\beta)y} \, \mathrm{d}y + O(n) \rho(y, a_n^+) \int_{a_n^+}^{R_n} e^{-(\alpha-\beta)y} \, \mathrm{d}y. \end{aligned}$$

Note that if $k_n = \Theta(1)$ then, for large enough n, the first integral is zero, it is $O(1)$ when $k_n \to \infty$. Because $\alpha > \beta$ the second integral is always $O(1)$. Therefore

$$\begin{aligned} &\int_{\mathcal{R}_n \setminus \mathcal{K}_C(k_n)} h(y) \rho_{k_n}(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = \begin{cases} O(n) \rho(a_n^+, k_n) & \text{if } k_n = \Theta(1), \\ O(n) \left(\rho(a_n^-, k_n) + \rho(a_n^+, k_n)\right) & \text{else.} \end{cases} \end{aligned} \tag{23}$$

We shall now bound the terms $\rho(a_n^{\pm}, k_n)$, starting with $\rho(a_n^-, k_n)$. Note we only need to bound $\rho(a_n^-, k_n)$ when $k_n \to \infty$, in which case $\kappa_n = \sqrt{k_n \log(k_n)}$. Using that $k! > \sqrt{2\pi} k^{k+1/2} e^{-k}$ we write

$$\begin{aligned}\rho(a_n^-, k_n) &\leq \frac{\mu(a_n^-)^{k_n}}{k_n!} e^{-\mu(a_n^-)} \leq (2\pi)^{-1/2} k_n^{-(k_n+1/2)} e^{-(\mu(a_n^-)-k_n)} \\ &= (2\pi)^{-1/2} k_n^{-1/2} e^{-k_n\left(\frac{\mu(a_n^-)}{k_n} - 1 - \log\left(\frac{\mu(a_n^-)}{k_n}\right)\right)}.\end{aligned}$$

Since

$$\frac{\mu(a_n^-)}{k_n} = \frac{\lambda_n^-}{k_n} = 1 - C\frac{\kappa_n}{k_n} = 1 - C\sqrt{\frac{\log(k_n)}{k_n}},$$

and $-x - \log(1-x) = \Omega(x^2/2)$, we get

$$\begin{aligned}\rho(a_n^-, k_n) &\leq \sqrt{2\pi} k_n^{-1/2} e^{-k_n\left(-C\frac{\kappa_n}{k_n} - \log\left(1 + C\frac{\kappa_n}{k_n}\right)\right)} \\ &\leq (2\pi)^{-1/2} k_n^{-1/2} e^{-\Omega\left(\frac{C^2 \log(k_n)}{2}\right)} = O\left(k_n^{-(1+C^2)/2}\right).\end{aligned} \tag{24}$$

For a_n^+ we again use the bound for $k!$ to get

$$\begin{aligned}\rho(a_n^+, k_n) &\leq \frac{\mu(a_n^+)^{k_n}}{k_n!} e^{-\mu(a_n^+)} \leq (2\pi)^{-1/2} k_n^{-(k_n+1/2)} e^{-(\mu(a_n^+)-k_n)} \\ &= (2\pi)^{-1/2} k_n^{-1/2} e^{-k_n\left(\frac{\mu(a_n^+)}{k_n} - 1 - \log\left(\frac{\mu(a_n^+)}{k_n}\right)\right)}.\end{aligned}$$

Since

$$\frac{\mu(a_n^+)}{k_n} = \frac{\lambda_n^+}{k_n} = 1 + C\frac{\kappa_n}{k_n},$$

with $\kappa_n/k_n = \omega(1)$ as $n \to \infty$, and $x - \log(1+x) = \omega(x/2)$ as $x \to \infty$, we get

$$\rho(a_n^+, k_n) \leq \sqrt{2\pi} k_n^{-1/2} e^{-k_n\left(C\frac{\kappa_n}{k_n} - \log\left(1 + C\frac{\kappa_n}{k_n}\right)\right)} = O\left(k_n^{-1/2} e^{-\frac{C\kappa_n}{2}}\right). \tag{25}$$

Plugging (24) and (25) into (23) yields the result. The second statement immediately follows by our choice of C.

To see that the same holds when we use $\rho_n(y, k_n)$ it satisfies similar bounds as $\rho(y, k_n)$ for $y \in \mathcal{R}_n \setminus \mathcal{K}_\varepsilon(k_n)$ hold for. We first note that $\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(y)) = \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(y))(1 - \phi_n(y))$ where $\phi_n(y) = \Theta\left(e^{-(\alpha - \frac{1}{2})(R_n - y)}\right)$, see Lemma 3.3. In particular we have that $\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(y))$ is increasing in y and hence for $0 \leq y \leq a_n^-$

$$\rho_n(y, k_n) \leq \rho_n(a_n^-, k_n) \leq \rho(a_n^-, k_n) \leq 2k_n^{-\frac{\varepsilon^2}{4}}.$$

It remains to show that the same holds for $a_n^+ \leq y \leq R_n$. For this we define

$$\lambda_n := \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(a_n^+)) = \left(k_n + C\sqrt{k_n \log(k_n)}\right)\left(1 - \phi_n(a_n^+)\right),$$

and note that $\lambda_n \geq k_n$ for large enough n. Hence

$$\begin{aligned}\rho_n(y, k_n) &\leq \rho_n(a_n^+, k_n) = \mathbb{P}\left(\mathrm{Po}(\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(a_n^+))) = k_n\right) \\ &\leq \mathbb{P}\left(|\mathrm{Po}(\lambda_n) - \lambda_n| \geq k_n - \lambda_n\right) \leq 2e^{-\frac{(k_n-\lambda_n)^2}{2(k_n+\lambda_n)}}.\end{aligned}$$

To finish the argument we observe that $k_n + \lambda_n \leq 2k_n$ and $(k_n - \lambda_n)^2 \geq C^2 k_n \log(k_n)$ for n large enough. Therefore

$$\rho_n(y, k_n) \leq 2e^{-\frac{(k_n-\lambda_n)^2}{2(k_n+\lambda_n)}} = O\left(k_n^{-\frac{C^2}{4}}\right)$$

which finishes the proof. $\square$

Remark 3.1 (Concentration argument)**.** *Lemma 3.1 will prove very useful in the remainder of this paper since we often have to deal with integrands of the form $g_n(y)f_{\alpha,\nu}(x,y)$ where $g_n(y) = O\left(n^{-1}k_n^s\right)\rho(y,k_n)$ for some $s \in \mathbb{R}$. In this case the lemma tells us that for a suitable $C > 0$ we only need to integrate over $\mathcal{K}_C(k_n)$. In order words, we may always assume for $g_n(y)$ (for a penalty of $o(1)$) that $k_n - C\kappa_n \le \xi_{\alpha,\nu}e^{y/2} \le k_n + C\kappa_n$. We will refer to this as a* concentration argument*, e.g.* by a concentration argument

$$\int_{\mathcal{R}_n} g_n(y)f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y = (1+o(1))\int_{\mathcal{K}_C(k_n)} \rho(y,k_n)g_n(y)f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

3.2 Average degree in the Hyperbolic graph

Recall that under the coupling between the hyperbolic random graph and the finite box model $|x-x'|_{\pi e^{r_n/2}} \le \Omega(r,r')$, while the coupling lemma (Lemma 2.2) gives that

$$e^{\frac{1}{2}(y+y')} - Ke^{\frac{3}{2}(y+y')-R_n} \le \Omega(r,r') \le e^{\frac{1}{2}(y+y')} + Ke^{\frac{3}{2}(y+y')-R_n},$$

for $y + y' < R_n$. This result enables us to determine the measure of a ball around a given point $p = (0,y)$ which will be fairly useful in our subsequent analysis.

Lemma 3.2. *Let $\varepsilon \in (0,1)$. Then for all $0 \le y \le (1-\varepsilon)R_n$*

$$1 - \phi^{(1)}_{\mathbb{H},n}(y) - \phi^{(2)}_{\mathbb{H},n}(y) \le \frac{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left(0,y\right)\right)}{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)} \le 1 - \phi^{(1)}_{\mathbb{H},n}(y) + \phi^{(2)}_{\mathbb{H},n}(y),$$

where

$$\phi^{(1)}_{\mathbb{H},n}(y) = \frac{2\alpha - 1 - 4\pi}{4\pi}e^{-(\alpha-\frac{1}{2})(R_n-y)} + \left(\alpha - \frac{1}{2}\right)\pi e^{-(\alpha-\frac{1}{2})R_n - y/2},$$

and

$$\phi^{(2)}_{\mathbb{H},n}(y) = +\begin{cases} \frac{(2\alpha-1)K}{3-2\alpha}\left(e^{-(\alpha-\frac{1}{2})(R_n-y)} - e^{-(R_n-y)}\right) & \text{if } 1/2 < \alpha < 3/2, \\ \frac{(2\alpha-1)K}{2}(R_n-y)e^{-(R_n-y)} & \text{if } \alpha = 3/2, \\ \frac{(2\alpha-1)K}{2\alpha-3}\left(e^{-(R_n-y)} - e^{-(\alpha-\frac{1}{2})(R_n-y)}\right) & \text{if } \alpha > 3/2. \end{cases}$$

Proof. We will split the computation of $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left(0,y\right)\right)$ over the case $y' > R_n - y$ and $y' \le R_n - y$ where for the latter we utilize Lemma 2.2. Recall that $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right) = \xi_{\alpha,\nu}e^{y/2}$ where $\xi_{\alpha,\nu} = 4\nu\alpha/(2\alpha-1)\pi$.

By (16), we have $\mathcal{B}_{\mathbb{H},n}\left((0,y)\right) \cap \mathcal{R}_n([R_n-y,R_n]) = \mathcal{R}_n([R_n-y,R_n])$. Thus,

$$\begin{aligned} &\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left((0,y)\right) \cap \mathcal{R}_n([R_n-y,R_n])\right) \\ &\quad= \int_{R_n-y}^{R_n}\int_{I_n} f_{\alpha,\nu}(x',y')\,\mathrm{d}x'\,\mathrm{d}y' = \nu\alpha e^{R_n/2}\left(e^{-\alpha(R_n-y)} - e^{-\alpha R_n}\right) \\ &\quad= \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)\frac{2\alpha-1}{4\pi}\left(e^{-(\alpha-\frac{1}{2})(R_n-y)} - e^{-(\alpha-\frac{1}{2})R_n-y/2}\right) \end{aligned} \tag{26}$$

Next we will establish upper and lower bounds on $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left((0,y)\right) \cap \mathcal{R}_n[(0,R_n-y)]\right)$. Using Lemma 2.2 we have

$$\begin{aligned} \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left((0,y)\right) \cap \mathcal{R}_n[(0,R_n-y)]\right) &\le \frac{2\nu\alpha}{\pi}\int_0^{R_n-y}\left(e^{\frac{y+y'}{2}} + Ke^{\frac{3}{2}(y+y')-R_n}\right)e^{-\alpha y'}\,\mathrm{d}y' \\ &= \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)\left(1 - e^{-(\alpha-\frac{1}{2})(R_n-y)}\right) \\ &\quad+ \frac{2\nu\alpha}{\pi}Ke^{\frac{3y}{2}-R_n}\int_0^{R_n-y} e^{(\frac{3}{2}-\alpha)y'}\,\mathrm{d}y' \end{aligned}$$

The last integral depend on the value of α,

$$\int_0^{R_n-y} e^{(\frac{3}{2}-\alpha)y'}\, \mathrm{d}y' = \begin{cases} \frac{2}{3-2\alpha}\left(e^{(\frac{3}{2}-\alpha)(R_n-y)}-1\right) & \text{if } 1/2<\alpha<3/2, \\ R_n-y & \text{if } \alpha=3/2, \\ \frac{2}{2\alpha-3}\left(1-e^{-(\alpha-\frac{3}{2})(R_n-y)}\right) & \text{if } \alpha>3/2. \end{cases}$$

Therefore we get

$$\begin{aligned} &\frac{2\nu\alpha}{\pi} K e^{\frac{3y}{2}-R_n} \int_0^{R_n-y} e^{(\frac{3}{2}-\alpha)y'}\, \mathrm{d}y' \\ &= \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right) \begin{cases} \frac{(2\alpha-1)K}{3-2\alpha}\left(e^{-(\alpha-\frac{1}{2})(R_n-y)}-e^{-(R_n-y)}\right) & \text{if } 1/2<\alpha<3/2, \\ \frac{(2\alpha-1)K}{2}(R_n-y)e^{-(R_n-y)} & \text{if } \alpha=3/2, \\ \frac{(2\alpha-1)K}{2\alpha-3}\left(e^{-(R_n-y)}-e^{-(\alpha-\frac{1}{2})(R_n-y)}\right) & \text{if } \alpha>3/2. \end{cases} \\ &= \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right) \phi_{\mathbb{H},n}^{(2)}(y) \end{aligned}$$

and hence

$$\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left((0,y)\right)\cap\mathcal{R}_n[(0,R_n-y)]\right) = \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)\left(1-e^{-(\alpha-\frac{1}{2})(R_n-y)}+\phi_{\mathbb{H},n}^{(2)}(y)\right).$$

Combining this with (26) yields the required upper bound.

The lower bound follows by observing that the only difference with the above computations is the change of sign in front of

$$\frac{2\nu\alpha}{\pi} K e^{\frac{3y}{2}-R_n} \int_0^{R_n-y} e^{(\frac{3}{2}-\alpha)y'}\, \mathrm{d}y'.$$

□

3.3 Average degree in the finite box model

For the finite box model $G_{\mathcal{P},n}(\alpha,\nu)$ we obtain a similar result for the average degree.

Lemma 3.3. *For all $p\in\mathcal{R}_n$, such that $y>2\log(\pi/2)$,*

$$\mu_{\alpha,\nu}(B_{\mathcal{P},n}(p)) = \mu_{\alpha,\nu}(B_{\mathcal{P}}(p))\left(1-\phi_n(y)\right)$$

where $\phi_n(y)\geq 0$ is given by

$$\phi_n(y) = \left(\frac{\pi}{2}\right)^{-(2\alpha-1)} e^{-(\alpha-\frac{1}{2})(R_n-y)} - \frac{(2\alpha-1)\pi}{4\alpha}\left(\left(\frac{\pi}{2}\right)^{-2\alpha} e^{-(\alpha-\frac{1}{2})(R_n-y)} - e^{-(\alpha-\frac{1}{2})R_n-\frac{y}{2}}\right).$$

If, on the other hand, $y\leq 2\log(\pi/2)$ then

$$\mu_{\alpha,\nu}(B_{\mathcal{P},n}(p)) = \mu_{\alpha,\nu}(B_{\mathcal{P}}(p))\left(1-e^{-(\alpha-\frac{1}{2})R_n}\right).$$

Proof. First note that since we have identified the boundaries of $[-\frac{\pi}{2}e^{\frac{R_n}{2}}, \frac{\pi}{2}e^{\frac{R_n}{2}}]$ we can assume, without loss of generality, that $p=(0,y)$. We then have that

$$\mathcal{B}_{\mathcal{P},n}\left(p\right) = \left\{p'\in\mathcal{R}_n\,:\,|x'|_n\leq e^{\frac{y+y'}{2}}\right\},$$

whose boundaries given by the equations $x'=\pm e^{\frac{y+y'}{2}}$ intersect the left and right boundaries of $[-\frac{\pi}{2}e^{\frac{R_n}{2}}, \frac{\pi}{2}e^{\frac{R_n}{2}}]$ at height

$$h(y) = R_n + 2\log\left(\frac{\pi}{2}\right) - y.$$

Therefore, if $y \leq 2\log(\pi/2)$ this intersection occurs above the height R_n of the box $\mathcal{R}_n$ while in the other case the full region of the box above $h(y)$ is connected to p.

We will first consider the case where $y > 2\log(\pi/2)$. Recall that $\mu_{\alpha,\nu}(B_{\mathcal{P}}(p)) = \xi_{\alpha,\nu} e^{\frac{y}{2}}$ where $\xi_{\alpha,\nu} = \frac{4\alpha\nu}{(2\alpha-1)\pi}$. Then, after some simple algebra, we have that

$$
\begin{aligned}
\mu_{\alpha,\nu}(B_{\mathcal{P},n}(p)) &= \int_0^{h(y)} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \mathbb{1}_{\left\{|x'| \leq e^{\frac{y+y'}{2}}\right\}} f_{\alpha,\nu}(x', y')\, dx'\, dy' \\
&\quad + \int_{h(y)}^{R_n} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} f_{\alpha,\nu}(x', y')\, dx'\, dy' \\
&= \frac{2\alpha\nu}{\pi} e^{\frac{y}{2}} \int_0^{h(y)} e^{-(\alpha-\frac{1}{2})y'}\, dy' + \alpha\nu e^{\frac{R_n}{2}} \int_{h(y)}^{R_n} e^{-\alpha y'}\, dy' \\
&= \xi_{\alpha,\nu} e^{\frac{y}{2}} \left(1 - \left(\frac{\pi}{2}\right)^{-(2\alpha-1)} e^{-(\alpha-\frac{1}{2})(R_n-y)}\right) \\
&\quad + \nu e^{\frac{R_n}{2}} \left(\left(\frac{\pi}{2}\right)^{-2\alpha} e^{-\alpha(R_n-y)} - e^{-\alpha R_n}\right) \\
&= \mu_{\alpha,\nu}(B_{\mathcal{P}}(p))\,(1 - \phi_n(y)).
\end{aligned}
$$

Since, for all $\alpha > \frac{1}{2}$,

$$
\left(\frac{\pi}{2}\right)^{-(2\alpha-1)} \geq \frac{(2\alpha-1)\pi}{4\alpha} \left(\frac{\pi}{2}\right)^{-2\alpha}
$$

it follows that $\phi_n(y) \geq 0$.

When $y \leq 2\log(\pi/2)$ we have

$$
\begin{aligned}
\mu_{\alpha,\nu}(B_{\mathcal{P},n}(p)) &= \int_0^{R_n} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \mathbb{1}_{\left\{|x'| \leq e^{\frac{y+y'}{2}}\right\}} f_{\alpha,\nu}(x', y')\, dx'\, dy' \\
&= \frac{2\alpha\nu}{\pi} e^{\frac{y}{2}} \int_0^{R_n} e^{-(\alpha-\frac{1}{2})y'}\, dy' \\
&= \mu_{\alpha,\nu}(B_{\mathcal{P}}(p)) \left(1 - e^{-(\alpha-\frac{1}{2})R_n}\right).
\end{aligned}
$$

□

3.4 Concentration argument for Hyperbolic and finite box models

Next we establish a result that allows us to extend the concentration argument to the case where instead of $\rho(y,k)$ we consider the degree distributions $\rho_{\widetilde{\mathbb{H}},n}(y,k)$ and $\rho_n(y,k)$ in $G_{\widetilde{\mathbb{H}},n}$ and $G_{\mathcal{P},n}$, respectively.

Lemma 3.4. *Let $\alpha > \frac{1}{2}, \nu > 0$, k_n be any increasing sequence such that $k_n = o(n^{\frac{1}{2\alpha+1}})$ and $\mathcal{K}_C(k_n)$ defined as in (21) for some $C > 0$. In addition, let $\hat{\mu}_n(y)$ be any function satisfying*

$$
\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}(0,y)\right)(1 - \phi_n^{(1)}(y)) \leq \hat{\mu}_n(y) \leq \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}(0,y)\right)(1 + \phi_n^{(2)}(y))
$$

where $\phi_n^{(i)} : \mathbb{R}_+ \to \mathbb{R}_+$ are such that $\sup_{0 \leq y \leq R_n} \phi_n(y)^{(i)} = o(1)$. Then if we define $\hat{\rho}_n(y,k) = \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k\right)$, we have, for any $\beta < \alpha$,

$$
\int_{\mathcal{R}_n} \hat{\rho}_n(y,k_n) e^{\beta y} f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y = (1 + o(1)) \int_{\mathcal{R}_n} \rho(y,k_n) e^{\beta y} f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y.
$$

Proof. Similar to the proof of Lemma 3.1 we define

$$
\lambda_n^{\pm} = k_n \pm \varepsilon\sqrt{k_n \log(k_n)}, \quad a_n^{\pm} = 2\log\left(\frac{\lambda_n^{\pm}}{\xi_{\alpha,\nu}}\right),
$$

and for simplicity write

$$\mu_1(y) := \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}(y)\right) \quad \text{and} \quad \mu_2(y) = \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(y)\right).$$

Next we notice that for all $y > a_n^+$, $\mu_2(y) \geq \mu_1(a_n^+)(1-\phi_n^{(1)}) = \omega(k_n)$ and hence

$$\begin{aligned}
\hat{\rho}_n(y,k) &\leq \frac{\mu_1(y)^k(1+\phi_n^{(2)}(y))^k}{k!} e^{-\mu_1(y)} e^{-\mu_1(y)\phi_n^{(2)}(y)} \\
&= \rho(y,k) e^{k\log\left(1+\phi_n^{(2)}(y)\right) - \mu_1(y)\phi_n^{(2)}(y)} \\
&= \rho(y,k) O\left(e^{(k-\mu_1(y))\phi_n^{(2)}(y)}\right) \\
&= \rho(y,k) O\left(e^{-C\kappa_n \phi_n^{(2)}(y)}\right) = \rho(y,k) O\left(1\right).
\end{aligned}$$

Similar we have for all $y < a_n^-$,

$$\begin{aligned}
\hat{\rho}_n(y,k) &\leq \frac{\mu_1(y)^k(1-\phi_n^{(1)}(y))^k}{k!} e^{-\mu_1(y)} e^{\mu_1(y)\phi_n^{(1)}(y)} \\
&= \rho(y,k) O\left(e^{(k+\mu_1(y))\phi_n^{(1)}(y)}\right) \\
&= \rho(y,k) O\left(e^{-C\kappa_n \phi_n^{(1)}(y)}\right) = \rho(y,k) O\left(1\right).
\end{aligned}$$

Therefore, by Lemma 3.1, it is enough to show that

$$\int_{a_n^-}^{a_n^+} \hat{\rho}_n(y,k_n) e^{(\beta-\alpha)y} \, \mathrm{d}y = (1+o(1)) \int_{a_n^-}^{a_n^+} \rho_n(y,k_n) e^{(\beta-\alpha)y} \, \mathrm{d}y.$$

For this let us make a change of variables $y \to z$ such that

$$\mu_2(y) = \mu_1(z).$$

and note that since

$$\mu_1^{-1}(y) = 2\log\left(\frac{y}{\xi_{\alpha,\nu}}\right),$$

is strictly monotonic in y, we have

$$z(y) \leq 2\log\left(\frac{\mu_1(y)}{\xi_{\alpha,\nu}}\right) + 2\log\left(1+\phi_n^{(2)}(y)\right) = y + 2\log\left(1+\phi_n^{(2)}(y)\right),$$

and similarly

$$z(y) \geq y + 2\log\left(1+\phi_n^{(2)}(y)\right).$$

Hence, for all $a_n^- \leq y \leq a_n^+$,

$$e^{(\beta-\alpha)y} = (1+o(1)) e^{(\beta-\alpha)z}. \tag{27}$$

For the derivatives we have, for $a_n^- \leq y \leq a_n^+$, $\mu_1'(y) = \mu_1(y)/2$, while $(\phi_n^{(i)})'(y) = O(1)$ and hence

$$\mu_2'(y) = \mu_1'(y)\left(1+o(1)\right).$$

Therefore, with this change of variables, and using

$$\hat{a}_n^{\pm} := z(a_n^{\pm}) = a_n^{\pm}(1+o\left(1\right)),$$

we have

$$\int_{a_n^-}^{a_n^+} \hat{\rho}_n(y,k_n) e^{(\beta-\alpha)y} \, \mathrm{d}y = \int_{a_n^-}^{a_n^+} \mathbb{P}\left(\mathrm{Po}\left(\mu_2(y) = k_n\right)\right) e^{(\beta-\alpha)y} \, \mathrm{d}y$$

$$= (1+o(1)) \int_{\hat{a}_n^-}^{\hat{a}_n^+} \mathbb{P}\left(\mathrm{Po}\left(\mu_1(z) = k_n\right)\right) e^{-\alpha z} \frac{\mu_2'(z)}{\mu_2'(z)}\, dz$$

$$= (1+o(1)) \int_{a_n^-}^{a_n^+} \rho(y, k_n) e^{(\beta-\alpha)z}\, dz.$$

□

Observe that Lemma 3.3 implies that $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left((0,y)\right)\right)$ satisfies the requirements in Lemma 3.4. Lemma 3.2 states that $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}\left(0,y\right)\right)$ satisfies these requirement when $y \le (1-\varepsilon)R_n$. However, since for all $\varepsilon < (2\alpha - 1)/2\alpha$

$$\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left(0,y\right) \cap \mathcal{R}_n[(1-\varepsilon)R_n, R_n]\right) = O\left(e^{\frac{R_n}{2} - \alpha(1-\varepsilon)R_n}\right) = O\left(n^{1-2\alpha(1-\varepsilon)}\right) = o\left(1\right),$$

the conclusion of Lemma 3.4 still holds for this case. We therefore have the following important Corollary.

Corollary 3.5. *Let $\hat{\rho}_n(y,k)$ be any of the two distribution functions $\rho_{\widetilde{\mathbb{H}},n}(y,k)$ and $\rho_n(y,k)$. Then, for any function $g_n(y)$ such that $g_n(y) = O\left(n^{-1}k_n^s\right)\hat{\rho}_n(y,k_n)$, as $n \to \infty$, for some $s \in \mathbb{R}$,*

$$\int_{\mathcal{R}_n} g_n(y) f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y = (1+o(1)) \int_{\mathcal{K}_C(k_n)} g_n(y) f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y,$$

for some $C > 0$ large enough.

In particular we conclude that, similar to the infinite limit model, concentrations arguments as described in Remark 3.1 can be applied in the case of the hyperbolic random graph and finite box model.

4 Clustering in $G_{\mathcal{P}}(\alpha, \nu)$

In this section we will establish the exact expressions for the limit local clustering coefficient and function, c_∞ and $c_\infty(k)$, respectively.

We shall first recall some necessary notations for computing the local clustering function in $G_{\mathcal{P}}$ and introduce some new ones.

Recall that for $p \in \mathbb{R} \times \mathbb{R}_+$,

$$B(p) = \{p' \in \mathbb{R} \times \mathbb{R}_+ : |x - x'| \le e^{\frac{y+y'}{2}}\}.$$

Let $T_{\mathcal{P}}(k)$ denote the number of triangles connected to nodes of degree k in $G_{\mathcal{P}}(\alpha,\nu)$, i.e.

$$T_{\mathcal{P}}(k) = \sum_{p \in \mathcal{P}} \mathbb{1}_{\{D_{\mathcal{P}}(p) = k\}} T(p) = \sum_{p \in \mathcal{P}} \mathbb{1}_{\{D_{\mathcal{P}}(p) = k\}} \sum_{p_1, p_2 \in 2^{\mathcal{P}}} T(p, p_1, p_2),$$

where $2^{\mathcal{P}}$ is the set of distinct pairs in $\mathcal{P} \times \mathcal{P}$ and

$$T(p, p_1, p_2) = \mathbb{1}_{\{p_1 \in B_{\mathcal{P}}(p)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P}}(p)\}} \mathbb{1}_{\{p_1 \in B_{\mathcal{P}}(p_2)\}},$$

is the indicator that $\{p, p_1, p_2\}$ form a triangle. Then the local clustering function is given by

$$c_{\mathcal{P}}(k) = \begin{cases} \frac{T_{\mathcal{P}}(k)}{\binom{k}{2} N_{\mathcal{P}}(k)} & \text{if } N_{\mathcal{P}}(k) \ge 1 \\ 0 & \text{else,} \end{cases} \tag{28}$$

and the adjusted local clustering by

$$c^*_{\mathcal{P}}(k) = \begin{cases} \frac{T_{\mathcal{P}}(k)}{\binom{k}{2} \mathbb{E}[N_{\mathcal{P}}(k)]} & \text{if } N_{\mathcal{P}}(k) \ge 1 \\ 0 & \text{else,} \end{cases}. \tag{29}$$

To analyze the local clustering we define the conditional expected number of triangles of a vertex p with degree k as

$$\Delta_{\mathcal{P}}(p,k) = \mathbb{E}\left[\sum_{(p_1,p_2)\in 2^{\mathcal{P}}} T_{\mathcal{P}}(p,p_1,p_2)\middle| D_{\mathcal{P}}(p) = k\right].$$

To compute this expression, let Z_1, Z_2 be two independent random variables on $B_{\mathcal{P}}(p)$, with (marginal) probability measures $\eta_p = \frac{\mu|_{B_{\mathcal{P}}(p)}}{\mu(B_{\mathcal{P}}(p))}$. Then we define,

$$\Delta_{\mathcal{P}}(p) = \mathbb{P}\left(Z_1 \in \mathcal{B}_{\mathcal{P}}(Z_2)\right) \tag{30}$$

as the probability that two random neighbors of a point $p \in \mathcal{P}$ are adjacent (note that $\mathbb{P} = \mathbb{P}_{\mathcal{P},p}$ denotes the joint probability measure of Z_1 and Z_2).

We shall often abuse notation and write $\Delta_{\mathcal{P}}(y)$ to denote $\Delta_{\mathcal{P}}((0,y))$. Next we note that conditioned on the Poisson process $\mathcal{P}$ having k points in $\mathcal{B}_{\mathcal{P}}(p)$, the points are independent random variables $Z_1, \dots, Z_k$, each with probability measure η_p and hence

$$\Delta_{\mathcal{P}}(p,k) = \binom{k}{2}\Delta_{\mathcal{P}}(p).$$

In addition, since $\mu_{\alpha,\nu}(B_{\mathcal{P}}(p)) = \xi_{\alpha,\nu} e^{\frac{y}{2}}$, with $\xi_{\alpha,\nu} = \frac{4\alpha\nu}{(2\alpha-1)\pi}$, and

$$\Delta_{\mathcal{P}}(p) = \iint_{\mathcal{R}^2} T_{\mathcal{P}}(p,p_1,p_2)\eta_p(x_1,y_1)\eta_p(x_2,y_2)\, dx1\, dx_2\, dy_1\, dy_2,$$

we have that

$$\mathbb{E}\left[T_{\mathcal{P}}(p)\right] = \xi_{\alpha,\nu}^2 \frac{e^y}{2}\Delta_{\mathcal{P}}(p). \tag{31}$$

Remark 4.1 (Notations for the finite graph $G_{\mathcal{P},n}(\alpha,\nu)$)**.** *For the finite graph $G_{\mathcal{P},n}(\alpha,\nu)$ we will use notations similar to those for $G_{\mathcal{P}}(\alpha,\nu)$ where we simply introduce a subscript n. For example,*

$$\Delta_{\mathcal{P},n}(p) = \mathbb{P}_{\mathcal{P},p}(Z_{1,n} \in \mathcal{B}_{\mathcal{P}}\left(Z_{2,n}\right)),$$

where $Z_{1,n}$ and $Z_{2,n}$ are two independent random variables with probability measure $\eta_{p,n}$.

We will start with some preliminary results on the degrees in $G_{\mathcal{P}}(\alpha,\nu)$. The proof of Theorem 1.5 can be found in Section 4.2. The key ingredient for the proof is a result concerning the asymptotic behavior of $\Delta_{\mathcal{P}}(y)$ (Proposition 4.1) whose proof can be found in Section 4.3.

4.1 Degree distribution in $G_{\mathcal{P}}(\alpha,\nu)$

Before we analyze local clustering in the infinite limit model, we first establish some results for its degree distribution. We define the probability distribution for the degree of node $p = (x,y)$ as

$$\rho_p(k) = \mathbb{P}\left(D_{\mathcal{P}}(p) = k\right) = \mathbb{P}\left(\mathrm{Po}(\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}\left(p\right))) = k\right), \tag{32}$$

where $\mathrm{Po}(\lambda)$ denotes a Poisson random variable with mean λ. Recall from equation (**??**) that

$$\mu_{\alpha,\nu}(B(p)) = \xi_{\alpha,\nu} e^{\frac{y}{2}},$$

only depends on the y-coordinate of p. Therefore, we will often write $\rho_y(k)$ instead of $\rho_p(k)$.

Using the above equation and the transformation of variables $z = \xi_{\alpha,\nu} e^{\frac{y}{2}}$, we compute

$$\int_0^\infty \rho(y,k) e^{-\alpha y}\, dy = \frac{1}{\Gamma(k+1)}\int_0^\infty \left(\xi_{\alpha,\nu} e^{\frac{y}{2}}\right)^k e^{-\xi_{\alpha,\nu} e^{\frac{y}{2}}} e^{-\alpha y}\, dy$$

$$
\begin{aligned}
&= \frac{(\xi_{\alpha,\nu})^{2\alpha}}{\Gamma(k+1)} \int_0^\infty \left(\xi_{\alpha,\nu} e^{\frac{y}{2}}\right)^{k-2\alpha} e^{-\xi_{\alpha,\nu} e^{\frac{y}{2}}}\, dy \\
&= \frac{(\xi_{\alpha,\nu})^{2\alpha}}{\Gamma(k+1)} \int_{\xi_{\alpha,\nu}}^\infty z^{k-2\alpha} e^{-z}\, dz \\
&= (\xi_{\alpha,\nu})^{2\alpha} \frac{\Gamma(k-2\alpha)}{\Gamma(k+1)} \left(1 - \frac{\gamma(k-2\alpha, \xi_{\alpha,\nu})}{\Gamma(k-2\alpha)}\right), \qquad (33)
\end{aligned}
$$

where Γ denotes the Gamma-function and γ the incomplete Gamma-function.

Thus we deduce that, as $k \to \infty$,

$$
\mathbb{P}\left(D_{\mathcal{P}} = k\right) = \int_0^\infty \rho_y(k) e^{-\alpha y}\, dy \sim (\xi_{\alpha,\nu})^{2\alpha} k^{-(2\alpha+1)}. \qquad (34)
$$

By a similar computation we have the following result, which will be useful later on. For any $\beta > 0$,

$$
\int_0^\infty e^{\beta y} \rho_y(k) e^{-\alpha y}\, dy = (\xi_{\alpha,\nu})^{2(\beta+\alpha)} \frac{\Gamma(k-2(\beta+\alpha))}{\Gamma(k+1)} \left(1 - \frac{\gamma(k-2(\beta+\alpha), \xi_{\alpha,\nu})}{\Gamma(k-2(\beta+\alpha))}\right). \qquad (35)
$$

4.2 Asymptotic behavior of $c_\infty(k)$

Recall that

$$
c_\infty(k) = \frac{\int_0^\infty \rho(y,k) \Delta_{\mathcal{P}}(y) e^{-\alpha y}\ \mathrm{d}y}{\int_0^\infty \rho(y,k) e^{-\alpha y}\ \mathrm{d}y}.
$$

The asymptotic behavior for the denominator follows from (34). Hence, the main term to consider is the numerator

$$
\int_0^\infty \Delta_{\mathcal{P}}(y)\, \rho(y,k) e^{-\alpha y}\, dy,
$$

and in particular the function $\Delta_{\mathcal{P}}(y)$. The following result establishes the asymptotic behavior of the latter.

Proposition 4.1 (Asymptotic behavior of $\Delta_{\mathcal{P}}(y)$)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and C_α as defined in (7). Then there are three, eventually decreasing, functions $\varepsilon_i : \mathbb{R}_+ \mapsto \mathbb{R}$, $i = 1,2,3$, satisfying*

$$
\lim_{y\to\infty} \varepsilon_i(y) = 0,
$$

such that

1. *for $\frac{1}{2} < \alpha < \frac{3}{4}$,*
$$
\Delta_{\mathcal{P}}(y) = e^{-\frac{y}{2}(4\alpha-2)} C_\alpha (1 + \varepsilon_1(y)),
$$
2. *for $\alpha = \frac{3}{4}$,*
$$
\Delta_{\mathcal{P}}(y) = \log\left(\frac{y}{2}\right) e^{-\frac{y}{2}} (1 + \varepsilon_2(y)),
$$
3. *and for $\alpha > \frac{3}{4}$,*
$$
\Delta_{\mathcal{P}}(y) = e^{-\frac{y}{2}} \frac{\alpha - \frac{1}{2}}{\alpha - \frac{3}{4}} (1 + \varepsilon_3(y)).
$$

The proof of this proposition is involved and technical. We therefore postpone it till Section 4.3 and first use it to prove Theorem 1.5.

Proof of Theorem 1.5. **Pim:** Include concentration argument in this proof. We only give the proof for the case $\frac{1}{2} < \alpha < \frac{3}{4}$. The proofs for the other two cases are similar. Let $\varepsilon_1(y)$ be defined as in Proposition 4.1 and define

$$\xi_1(k_n) = \int_0^\infty \varepsilon_1(y) e^{-(4\alpha-2)\frac{y}{2}} \rho_y(k_n) e^{-\alpha y}\, dy$$

Then we have that

$$\int_0^\infty \Delta_{\mathcal{P}}(y)\rho_y(k_n)e^{-\alpha y}\, dy = C_\alpha \int_0^\infty e^{-(4\alpha-2)\frac{y}{2}}\rho_y(k_n)e^{-\alpha y}\, dy + \xi_1(k_n).$$

Using (35) with $\beta = 2\alpha - 1$ yields

$$\int_0^\infty e^{-(4\alpha-2)\frac{y}{2}}\rho_y(k_n)e^{-\alpha y}\, dy = 2\xi^{6\alpha-2}\frac{\Gamma(k_n-(6\alpha-2))}{\Gamma(k_n+1)}\left(1 - \frac{\gamma(k_n-(6\alpha-2),\xi)}{\Gamma(k_n-(6\alpha-2))}\right).$$

Since

$$\lim_{n\to\infty}\frac{\gamma(k_n-2\alpha,\xi)}{\Gamma(k_n-2\alpha)} = 0,$$

and $\Gamma(k_n-t)/\Gamma(k_n+1) \sim k_n^{-(t+1)}$ as $n\to\infty$, it follows that, as $n\to\infty$,

$$c_\infty(k_n) \sim C_\alpha \xi^{4\alpha-2}\frac{k_n^{-(6\alpha-1)}}{k_n^{-(2\alpha-1)}} + \frac{\xi^{-2\alpha}\eta_1(k_n)}{2k_n^{-(2\alpha-1)}} \sim C_\alpha\xi^{4\alpha-2}k_n^{-4\alpha+2} + \frac{\xi^{-2\alpha}\eta_1(k_n)}{2k_n^{-(2\alpha-1)}}.$$

Hence, to prove the result it is enough to show that

$$\lim_{n\to\infty} k_n^{6\alpha-1}\xi_1(k_n) = 0.$$

For this let $0<\varepsilon<1$, define $a_n = 2\log((1-\varepsilon)k_n)$ and split the integral into two pieces

$$\begin{aligned}\xi_1(k_n) &= \int_0^\infty \varepsilon_1(y)e^{-(4\alpha-2)\frac{y}{2}}\rho_y(k_n)e^{-\alpha y}\, dy \\ &= \int_0^{a_n} \varepsilon_1(y)e^{-(4\alpha-2)\frac{y}{2}}\rho_y(k_n)e^{-\alpha y}\, dy \\ &\quad + \int_{a_n}^\infty \varepsilon_1(y)e^{-(4\alpha-2)\frac{y}{2}}\rho_y(k_n)e^{-\alpha y}\, dy.\end{aligned}$$

For the first integral we use that

$$k_n^{6\alpha-1}\rho_{a_n}(k_n) = O\left(\frac{k_n^{6\alpha-1+(1-\varepsilon)k_n}}{k_n!}e^{-(1-\varepsilon)k_n}\right) = o(1),$$

so that

$$\begin{aligned}&\lim_{n\to\infty} k_n^{6\alpha-1}\int_0^{a_n}\varepsilon_1(y)e^{-(4\alpha-2)\frac{y}{2}}\rho_y(k_n)e^{-\alpha y}\, dy \\ &\quad\le \lim_{n\to\infty}\left(\max_{0\le y\le a_n}\varepsilon_1(y)\right)k_n^{6\alpha-1}\rho_{a_n}(k_n)\int_0^\infty e^{-\alpha y}\, dy = 0.\end{aligned}$$

For the second integral we first note that due to (**??**)

$$\int_0^\infty e^{-(4\alpha-2)\frac{y}{2}}\rho_y(k_n)e^{-\alpha y}\, dy \sim k_n^{-(6\alpha-1)}.$$

Therefore, using that $\varepsilon_1(y)$ is eventually decreasing and $\lim_{n\to\infty}\varepsilon_1(a_n) = 0$, we obtain

$$\begin{aligned}&\lim_{n\to\infty} k_n^{6\alpha-1}\int_{a_n}^\infty \varepsilon_1(y)e^{-(4\alpha-2)\frac{y}{2}}\rho_y(k_n)e^{-\alpha y}\, dy \\ &\quad\le \lim_{n\to\infty}\varepsilon_1(a_n)k_n^{6\alpha-1}\int_0^\infty e^{-(4\alpha-2)\frac{y}{2}}\rho_y(k_n)e^{-\alpha y}\, dy = 0,\end{aligned}$$

which finishes the proof. □

4.3 Analyzing $\Delta_{\mathcal{P}}(y)$, an overview.

We derive the following complete asymptotic expansion of $\Delta_{\mathcal{P}}(y)$, from which then the leading terms can be extracted. The following result proves Proposition 4.1, resp. implies it as a corollary after extracting the leading terms.

Proposition 4.2. *1. If $\alpha \neq 1$, then*

$$\begin{aligned}\Delta_{\mathcal{P}}(y) &= -\frac{1}{8(\alpha-1)\alpha} + \frac{(\alpha-1/2)e^{-\frac{1}{2}y}}{\alpha-1} - \frac{(\alpha-1/2)^2e^{-y}}{4(\alpha-1)^2}\\ &\quad + (e^{-\frac{1}{2}y})^{4\alpha-2}\left(\frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{(\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)}{2(\alpha-1)\alpha}\right)\\ &\quad + \frac{(1-e^{-\frac{1}{2}y})^{2\alpha}}{8(\alpha-1)\alpha} - \frac{(e^{-\frac{1}{2}y})^{4\alpha-2}B^-(1-e^{-\frac{1}{2}y};2\alpha,3-4\alpha)}{4(\alpha-1)}\end{aligned}$$

2. If $\alpha = 1$, then

$$\Delta_{\mathcal{P}}(y) = \frac{9}{4}e^{-\frac{1}{2}y} + \frac{1-4e^{-\frac{1}{2}y}+3e^{-y}}{4}\ln(1-e^{-\frac{1}{2}y}) - \frac{7+\pi^2}{8}e^{-y} + \frac{1}{2}e^{-y}\operatorname{Li}_2(e^{-y})$$

where $\operatorname{Li}_2(z) = \int_0^z \frac{\ln(1-t)}{t}dt$ is the dipolylogarithm function.

To derive this result, recall (see (30)) that $\Delta_{\mathcal{P}}(y)$ is the probability that $Z_1 \in \mathcal{B}_{\mathcal{P}}(Z_2)$, where Z_1, Z_2 are two independent random variables on $\mathcal{B}_{\mathcal{P}}(0, y)$ with density

$$\eta_y(y', x') = \frac{f_{\alpha,\nu}(x', y')\mathbb{1}_{\{(x',y' \in B_{\mathcal{P}}(0,y)\}}}{\mu(B_{\mathcal{P}}(0,y))}.$$

For notational convenience we will switch to using y_0 instead of y. Note that the random variables $Z_i = (x_i, y_i)$, for $i = 1, 2$, correspond to first sampling y_i according to the density

$$g(y) := \left(\alpha - \frac{1}{2}\right)e^{-(\alpha-\frac{1}{2})y_1}$$

and then sampling x_i uniformly in $[-e^{\frac{1}{2}(y_0+y_1)}, e^{\frac{1}{2}(y_0+y_1)}]$. With this in mind we define $P(y_0, y_1, y_2)$ to be the probability that $(0, y_0), (x_1, y_1), (x_2, y_2)$ form a triangle where x_1 and x_2 are uniform random variables in, respectively, $[-e^{\frac{1}{2}(y_0+y_1)}, e^{\frac{1}{2}(y_0+y_1)}]$ and $[-e^{\frac{1}{2}(y_0+y_2)}, e^{\frac{1}{2}(y_0+y_2)}]$.

Then we have that

$$\Delta_{\mathcal{P}}(y_0) = (\alpha-1/2)^2\int_0^\infty\int_0^\infty P(y_0, y_1, y_2)e^{-(\alpha-1/2)(y_1+y_2)}\,\mathrm{d}y_2\,\mathrm{d}y_1. \tag{36}$$

4.4 Triangle probability for nodes at given heights

We proceed by first computing $P(y_0, y_1, y_2)$. To compute the integral 36 it will be convenient to use the change of variable $z_i = e^{-y_i/2}$, for $i = 0, 1, 2$. We will write $y_i(z_i)$ to stress the dependence between y_i and z_i. The following result completely characterizes $P(y_0, y_1, y_2)$.

Lemma 4.3.

$$P(y_0(z_0), y_1(z_1), y_2(z_2)) = \begin{cases} 1 & \text{if } z_0 \geq z_1 + z_2, z_0 > z_1 > z_2,\\ 1 - G(z_0, z_1, z_2) & \text{if } z_0 < z_1 + z_2, z_0 > z_1 > z_2,\\ \frac{z_0}{z_1} & \text{if } z_1 \geq z_0 + z_2, z_1 > \max(z_0, z_2),\\ \frac{z_0}{z_1}\left(1 - G(z_1, z_0, z_2)\right) & \text{if } z_1 < z_0 + z_2, z_1 > \max(z_0, z_2),\end{cases}$$

where

$$G(a, b, c) = \frac{1}{4}\left(b^{-1}c + bc^{-1} + a^2b^{-1}c^{-1} + 2 - 2ab^{-1} - 2ac^{-1}\right)$$

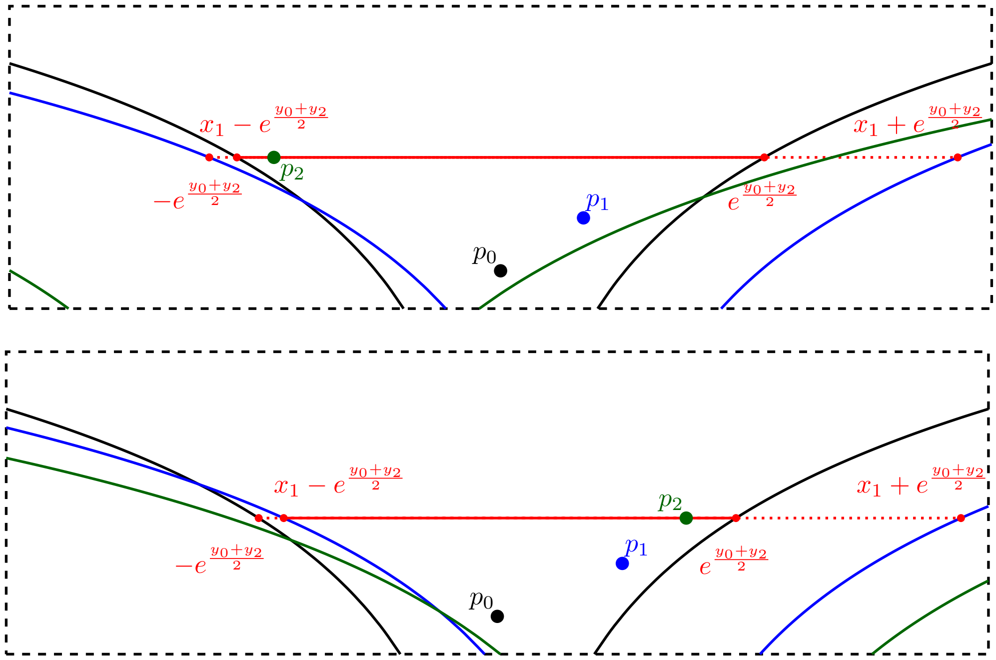


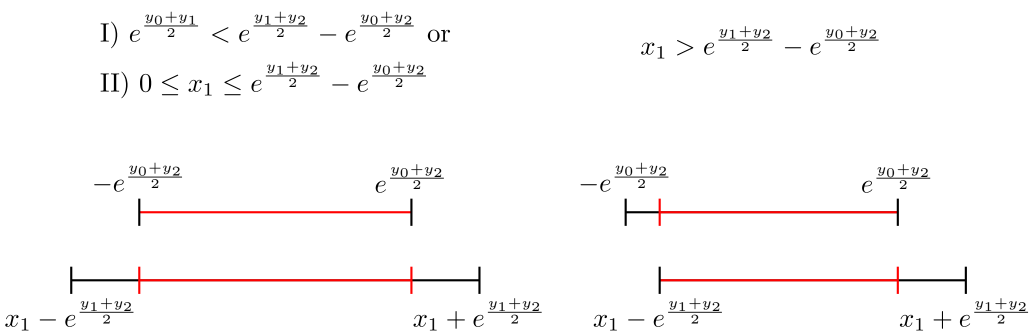
Figure 3: Situation for the intersections of the connection intervals considered in Lemma 4.4, with $y_0 < y_1 < y_2$ fixed and for different cases of $0 \le x_1 \le e^{(y_0+y_1)/2}$. The top figure shows the case where $0 \le x_1 \le e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}$, while the bottom one shows the case $x_1 > e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}$. The solid red line indicates the range for x_2 such that the points p_0, p_1 and p_2 form a triangle. The boundaries of there balls are show in, respectively, black, blue and green.

We split the proof of this lemma into a couple of smaller pieces. We begin with the following lemma.

Lemma 4.4. *Let $z_i = e^{-y_i/2}$, $i = 0, 1, 2$. If $y_0 < y_1 < y_2$ (or equivalently $z_0 > z_1 > z_2$), then*

$$P(y_0(z_0), y_1(z_1), y_2(z_2)) = \begin{cases} 1, & \text{if } z_0 \ge z_1 + z_2, \\ 1 - G(z_0, z_1, z_2), & \text{if } z_0 < z_1 + z_2 \end{cases}$$

Proof. Since the Poisson Point Process is translation invariant in the x-coordinate we can, without loss of generality, take $x_0 = 0$. By definition of the connection rule $P(y_0, y_1, y_2)$ is the probability that $|x_2 - x_1| \le e^{(y_1+y_2)/2}$ (see figure 4). Consider y_0, y_1, y_2 and x_1 fixed. Then we are interested in computing the probability that x_2 falls into the interval $[x_1 - e^{(y_1+y+2)/2}, x_1 + e^{(y_1+y_2)/2}]$ (given



by the bottom interval in the Figure 4), as well as into the interval $[-e^{(y_0+y_2)/2}, e^{(y_0+y_2)/2}]$ (given by the top interval in the Figure 4).

By symmetry, without loss of generality we can consider x_1 uniformly at random from $[0, e^{y_0/2+y_1/2}]$. Since $y_0 < y_1 < y_2$ we have that $e^{(y_1+y_2)/2} > e^{(y_0+y_2)/2}$ and so, when $x_1 \geq 0$, the "right half" of the interval $[-e^{(y_0+y_2)/2}, e^{(y_0+y_2)/2}]$ is always covered by the interval $[x_1 - e^{(y_1+y_2)/2}, x_1 + e^{(y_1+y_2)/2}]$. If $e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} \geq e^{(y_0+y_2)/2}$ then the "left half" is always covered as well. In other words:

$$e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} \geq e^{(y_0+y_2)/2} \Rightarrow P(y_0, y_1, y_2) = 1.$$

Now consider the case where $e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} < e^{(y_0+y_2)/2}$. Then, if $x_1 \in [0, e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}]$ the whole interval $[-e^{(y_0+y_2)/2}, e^{(y_0+y_2)/2}]$ is still covered so that p_0, p_1 and p_2 form a triangle. If, on the other hand $e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2} < x_1 \leq e^{(y_0+y_1)/2}$ then the probability that $|x_2 - x_1| \leq e^{(y_1+y_2)/2}$ equals

$$1 - \frac{x_1 - (e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2})}{2e^{(y_0+y_2)/2}}.$$

Hence, when $e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} < e^{(y_0+y_2)/2}$ we have

$$\begin{aligned}
P(y_0, y_1, y_2) &= \frac{e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}}{e^{(y_0+y_1)/2}} \\
&\quad + \int_{e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}}^{e^{(y_0+y_1)/2}} \left(1 - \frac{x_1 - (e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2})}{2e^{(y_0+y_2)/2}}\right) \cdot \frac{1}{e^{(y_0+y_1)/2}} \, \mathrm{d}x_1 \\
&= 1 - \frac{1}{2e^{y_0+y_1/2+y_2/2}} \int_0^{e^{(y_0+y_1)/2} + e^{(y_0+y_2)/2} - e^{(y_1+y_2)/2}} x_1 \, \mathrm{d}x_1 \\
&= 1 - \frac{\left(e^{(y_0+y_1)/2} + e^{(y_0+y_2)/2} - e^{(y_1+y_2)/2}\right)^2}{4e^{y_0+y_1/2+y_2/2}},
\end{aligned}$$

At this point it is convenient to rewrite everything in terms of $z_i := e^{-y_i/2}$. Note that $y_0 < y_1 < y_2$ if and only if $z_0 > z_1 > z_2$ while the condition $e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} < e^{(y_0+y_2)/2}$ becomes

$$e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} < e^{(y_0+y_2)/2} \Leftrightarrow z_1^{-1} z_2^{-1} < z_0^{-1} z_1^{-1} + z_0^{-1} z_2^{-1} \Leftrightarrow z_0 < z_1 + z_2.$$

We now conclude that

$$P(y_0(z_0), y_1(z_1), y_2(z_2)) = 1 \quad \text{if} \quad z_0 > z_1 > z_2 \text{ and } z_0 \geq z_1 + z_2$$

while for $z_0 > z_1 > z_2$ and $z_0 < z_1 + z_2$

$$\begin{aligned}
P(y_0, y_1, y_2) &= 1 - \frac{z_0^2 z_1 z_2}{4} \cdot \left(z_0^{-1} z_1^{-1} + z_0^{-1} z_2^{-1} - z_1^{-1} z_2^{-1}\right)^2 \\
&= 1 - \frac{1}{4}\left(z_1^{-1} z_2 + z_1 z_2^{-1} + z_0^2 z_1^{-1} z_2^{-1} + 2 - 2z_0 z_1^{-1} - 2z_0 z_2^{-1}\right),
\end{aligned}$$

which finishes the proof. □

The previous lemma covers the case when $y_0 < y_1 < y_2$. We now leverage it to take care of the other cases as well.

Proof of Lemma 4.3. Let $y_i > 0$ and $z_i = e^{-y_i/2}$, $i = 0, 1, 2$. Lemma 4.4 gives the expression for $P(y_0(z_0), y_1(z_1), y_2(z_2))$ in the case $y_0 < y_1 < y_2$, or equivalently $z_0 > z_1 > z_2$, i.e. the first two lines in the claim of Lemma 4.3. To analyze the other cases we shall express $P(y_1, y_0, y_2)$ and $P(y_1, y_2, y_0)$ in terms of $P(y_0, y_1, y_2)$ and z_i. For this we note that we can view $P(y_0, y_1, y_2)$ as a 2-fold integral of the indicator function

$$h(x_0, x_1, x_2) := \mathbb{1}_{\left\{|x_0 - x_1| < e^{(y_0+y_1)/2}, |x_0 - x_2| < e^{(y_0+y_2)/2}, |x_1 - x_2| < e^{(y_1+y_2)/2}\right\}},$$

where x_0 was set to zero, without loss of generality, and the other two x_i are uniform random variables on $[-e^{(y_0)+y_i)/2}, e^{(y_0+y_i)/2}]$. When we consider the probability $P(y_1, y_0, y_2)$, this is the 2-fold integral of $h(x_0, 0, x_2)$ so that

$$\begin{aligned}P(y_1, y_0, y_2) &= \frac{1}{2e^{(y_1+y_0)/2}} \cdot \frac{1}{2e^{(y_1+y_2)/2}} \iint_{\mathbb{R}} h(x_0, 0, x_2)\, \mathrm{d}x_0\, \mathrm{d}x_2 \\ &= \frac{e^{y_0/2}}{e^{y_1/2}} \frac{1}{2e^{(y_0+y_1)/2}} \frac{1}{2e^{(y_0+y_2)/2}} \iint_{\mathbb{R}} h(0, x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\ &= \frac{e^{y_0/2}}{e^{y_1/2}} P(y_0, y_1, y_2) = \frac{z_1}{z_0} P(y_0, y_1, y_2).\end{aligned}$$

Finally we note that $h(x_0, 0, x_2) = h(x_2, 0, x_0)$ from which we conclude that

$$P(y_0, y_1, y_2) = (z_0/z_1)\, P(y_1, y_0, y_2) = (z_0/z_1)\, P(y_1, y_2, y_0). \tag{37}$$

To complete the proof for the other cases we note that since $P(y_0, y_1, y_2)$ is symmetric in y_1 and y_2, we can assume, without loss of generality, that $y_1 < y_2$. Then, there are two more orderings of y_0, y_1, y_2, namely $y_1 < y_0 < y_2$ and $y_1 < y_2 < y_0$, which can be summarized as $y_1 < \min(y_0, y_2)$, or equivalently $z_1 > \max(z_0, z_2)$. For $y_1 < y_0 < y_2$ and $y_1 < y_2 < y_0$ we can apply Lemma 4.4 to obtain $P(y_1, y_0, y_2) = P(y_1, y_2, y_0)$ which happen to agree due to the symmetry in the last two arguments of the expression found in Lemma 4.4. The expression for $P(y_0, y_1, y_2)$ then follows from (37). □

4.5 Computing $\Delta_{\mathcal{P}}(y_0)$

Now that we have established the expression for $P(y_0, y_1, y_2)$ we can proceed with computing $\Delta_{\mathcal{P}}(y_0)$, i.e. proving Proposition 4.2. We start with the following observation.

Lemma 4.5. *The function $\alpha \mapsto \Delta_{\mathcal{P}_{\alpha,\nu}}(y_0)$ is continuous for all $\alpha > \frac{1}{2}$.*

Proof. This follows from the theorem of dominated convergence: Let $\alpha > \frac{1}{2}$ and $(\alpha_n)_{n\in\mathbb{N}}$ a sequence of real numbers converging to α, so we can assume $|\alpha_n - \alpha| < \epsilon := \frac{\alpha - 1/2}{2}$. This means that $-\epsilon < \alpha_n - \alpha < \epsilon$, i.e. $\frac{\alpha-1/2}{2} < \alpha_n - 1/2 < \frac{3\alpha-3/2}{2}$. Define $f_n(y_1, y_2) = P(y_0, y_1, y_2)(\alpha_n - 1/2)^2 e^{-(\alpha_n - 1/2)(y_1+y_2)}$. As the function $x \mapsto x^2$ is increasing in x for $x > 0$ and the function $x \mapsto e^{-(y_1+y_2)x}$ is decreasing in x and $P(y_0, y_1, y_2) \in [0, 1]$, it holds that $|f_n(y_1, y_2)| \leq (\frac{3\alpha-3/2}{2})^2 e^{-(y_1+y_2)\frac{\alpha-1/2}{2}}$ which is integrable over $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ (with integral equals $\frac{(6\alpha-3)^2}{(2\alpha-1)^2}$). Application of the theorem of dominated convergence and using equation (30) yields that $\Delta_{\mathcal{P}_{\alpha_n,\nu}}(y_0) \to \Delta_{\mathcal{P}_{\alpha,\nu}}(y_0)$ which gives the claim as the sequence $(\alpha_n)_n$ was arbitrary. □

Due to this lemma we can first assume $\alpha \notin \{\frac{3}{4}, 1\}$, compute $\Delta_{\mathcal{P}}(y_0)$ and then obtain the values of $\Delta_{\mathcal{P}}(y_0)$ at the remaining two points by taking the corresponding limit in α. This strategy is executed below. It involves the computation of several integrals which are involved and will take up the next 7 pages. The proof is carefully structured, using headers, to aid the reader.

Proof of Proposition 4.2.

When $\alpha \notin \{3/4, 1\}$ Note that when writing $\Delta_{\mathcal{P}}(y_0)$ as in integral as in equation (30), by symmetry in the integration variables y_1 and y_2, we can assume that $y_1 < y_2$ in which case either y_0 or y_1 is the smallest height. This gives half the value of $\Delta_{\mathcal{P}}(y_0)$ and hence

$$\Delta_{\mathcal{P}}(y_0) = 2(I_1(y_0) + I_2(y_0)),$$

where I_1 and I_2 are given by:

$$I_1(y_0) := \int_{0<y_0<y_1<y_2} P(y_0, y_1, y_2) \cdot (\alpha - 1/2)^2 e^{-(\alpha-1/2)(y_1+y_2)}\, \mathrm{d}y_2\, \mathrm{d}y_1$$

$$I_2(y_0) := \int_{0<y_1<y_0,y_2} P(y_0,y_1,y_2)\cdot(\alpha-1/2)^2 e^{-(\alpha-1/2)(y_1+y_2)}\,\mathrm{d}y_2\,\mathrm{d}y_1$$

We proceed with computing each of these two integrals, each of which is split in two parts. The final expressions of those four integrals can be found in (38), (43), (44) and (46).

Computing $I_1(y_0)$ Applying the change of variables $z_i := e^{-y_i/2}$ and Lemma 4.3 gives

$$\begin{aligned}
I_1(y_0) &= 4(\alpha-1/2)^2\cdot\int_{z_0>z_1>z_2>0} P(y_0,y_1(z),y_2(z))z_1^{2\alpha-2}z_2^{2\alpha-2}\,\mathrm{d}z_2\,\mathrm{d}z_1\\
&= 4(\alpha-1/2)^2\cdot\Bigg(\int_{z_0>z_1>z_2>0} 1\cdot z_1^{2\alpha-2}z_2^{2\alpha-2}\,\mathrm{d}z_2\,\mathrm{d}z_1\\
&\quad-\int_{\substack{z_0>z_1>z_2>0,\\ z_0<z_1+z_2}} G(z_0,z_1,z_2)\cdot z_1^{2\alpha-2}z_2^{2\alpha-2}\,\mathrm{d}z_2\,\mathrm{d}z_1\Bigg)\\
&=: 4(\alpha-1/2)^2(I_{11}(y_0)-I_{12}(y_0)). \qquad (38)
\end{aligned}$$

The integral $I_{11}(y_0)$ is easily obtained:

$$\begin{aligned}
I_{11}(y_0) &= \int_0^{z_0}\int_0^{z_1} z_1^{2\alpha-2}z_2^{2\alpha-2}\,\mathrm{d}z_2\,\mathrm{d}z_1\\
&= \int_0^{z_0} z_1^{2\alpha-2}\left[\frac{z_2^{2\alpha-1}}{2\alpha-1}\right]_0^{z_1}\mathrm{d}z_1\\
&= \frac{1}{2\alpha-1}\cdot\int_0^{z_0} z_1^{4\alpha-3}\,\mathrm{d}z_1\\
&= \frac{1}{2(2\alpha-1)^2}\cdot z_0^{4\alpha-2}.
\end{aligned}$$

To deal with I_{12} we note that $G(z_0,z_1,z_2)$ is a linear combination of monomials of the form $z_0^a z_1^b z_2^c$ with $a,b,c\in\{-1,0,1,2\}$ and $a+b+c=0$. Let us consider the integral $J_(a,b,c)(z_0)$

$$J_{a,b,c}(z_0) := z_0^a\int_{\substack{z_0>z_1>z_2>0,\\ z_0<z_1+z_2}} z_1^{b+2\alpha-2}z_2^{c+2\alpha-2}\,\mathrm{d}z_2\,\mathrm{d}z_1. \qquad (39)$$

and note that

$$I_{1,2}(y_0) = \frac{1}{4}(J_{0,-1,1}(z_0)+J_{0,1,-1}(z_0)+J_{2,-1,-1}(z_0)+2J_{0,0,0}(z_0)-2J_{1,-1,0}(z_0)-2J_{1,0,-1}(z_0)). \quad (40)$$

Next we compute $J_{a,b,c}(z_0)$.

$$\begin{aligned}
J_{a,b,c} &= z_0^a\int_{z_0/2}^{z_0}\int_{z_0-z_1}^{z_1} z_1^{b+2\alpha-2}z_2^{c+2\alpha-2}\,\mathrm{d}z_2\,\mathrm{d}z_1 = z_0^a\int_{z_0/2}^{z_0} z_1^{b+2\alpha-2}\left[\frac{z_2^{c+2\alpha-1}}{c+2\alpha-1}\right]_{z_0-z_1}^{z_1}\mathrm{d}z_1\\
&= \frac{z_0^a}{c+2\alpha-1}\cdot\left(\int_{z_0/2}^{z_0} z_1^{b+c+4\alpha-3}\,\mathrm{d}z_1-\int_{z_0/2}^{z_0} z_1^{b+2\alpha-2}(z_0-z_1)^{c+2\alpha-1}\,\mathrm{d}z_1\right)\\
&= \frac{z_0^{a+b+c+4\alpha-2}(1-(1/2)^{b+c+4\alpha-2})}{(c+2\alpha-1)(b+c+4\alpha-2)}-\frac{z_0^{a+b+c+4\alpha-3}}{c+2\alpha-1}\int_{z_0/2}^{z_0}(z_1/z_0)^{b+2\alpha-2}(1-(z_1/z_0))^{c+2\alpha-1}\,\mathrm{d}z_1\\
&= \frac{z_0^{4\alpha-2}(1-(1/2)^{b+c+4\alpha-2})}{(c+2\alpha-1)(b+c+4\alpha-2)}-\frac{z_0^{4\alpha-2}}{c+2\alpha-1}\int_{1/2}^{1}u^{b+2\alpha-2}(1-u)^{c+2\alpha-1}\,\mathrm{d}u\\
&= \frac{z_0^{4\alpha-2}(1-(1/2)^{b+c+4\alpha-2})}{(c+2\alpha-1)(b+c+4\alpha-2)}-\frac{z_0^{4\alpha-2}}{c+2\alpha-1}B^-(1/2;c+2\alpha,b+2\alpha-1),
\end{aligned}$$

where we've used the substitution $u := z_1/z_0$ giving $z_0\,\mathrm{d}u = \mathrm{d}z_1$ in the penultimate line and B^- denotes the (lower) incomplete beta-function. Note that since $c \geq -1$, $-a \in \{0,-1,-2\}$ and by our assumption $\alpha \notin \{\frac{3}{4}\}$, the denominators that occur during the integration are all non-zero.

Plugging this back into (40) gives

$$
\begin{aligned}
I_{1,2}(y_0) &= \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-2})}{32\alpha(\alpha-1/2)} - \frac{z_0^{4\alpha-2}}{8\alpha}B^-(1/2;1+2\alpha,2\alpha-2) \\
&\quad + \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-2})}{32(\alpha-1)(\alpha-1/2)} - \frac{z_0^{4\alpha-2}}{4(2\alpha-2)}B^-(1/2;2\alpha-1,2\alpha) \\
&\quad + \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-4})}{32(\alpha-1)^2} - \frac{z_0^{4\alpha-2}}{4(2\alpha-2)}B^-(1/2;-1+2\alpha,2\alpha-2) \\
&\quad + \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-2})}{16(\alpha-1/2)^2} - \frac{z_0^{4\alpha-2}}{2(2\alpha-1)}B^-(1/2;2\alpha,2\alpha-1) \\
&\quad - \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-3})}{16(\alpha-1/2)(\alpha-3/4)} + \frac{z_0^{4\alpha-2}}{2(2\alpha-1)}B^-(1/2;2\alpha,2\alpha-2) \\
&\quad - \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-3})}{16(\alpha-1)(\alpha-3/4)} + \frac{z_0^{4\alpha-2}}{2(2\alpha-2)}B^-(1/2;-1+2\alpha,2\alpha-1) \\
&= \frac{\left(\frac{3}{64} - \frac{3}{16}2^{-4\alpha} + \alpha(-\frac{41}{128} + \frac{13}{16}2^{-4\alpha}) + \alpha^2(\frac{5}{8} - \frac{3}{4}2^{-4\alpha}) - \frac{15}{32}\alpha^3 + \frac{1}{8}a^4\right) z_0^{4\alpha-2}}{4(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)\alpha} \\
&\quad + \frac{z_0^{4\alpha-2}}{8(\alpha-1)\alpha(2\alpha-1)}(4(\alpha-1)\alpha(B^-(1/2;2\alpha,2\alpha-2) - B^-(1/2;2\alpha,2\alpha-1)) \\
&\quad - (2\alpha-1)\alpha(B^-(1/2;2\alpha-1,2\alpha-2) + B^-(1/2;2\alpha-1,2\alpha) - 2B^-(1/2;2\alpha-1,2\alpha-1)) \\
&\quad - (2\alpha-1)(\alpha-1)B^-(1/2;1+2\alpha,2\alpha-2)) \\
&= \frac{\left(\frac{3}{64} - \frac{3}{16}2^{-4\alpha} + \alpha(-\frac{41}{128} + \frac{13}{16}2^{-4\alpha}) + \alpha^2(\frac{5}{8} - \frac{3}{4}2^{-4\alpha}) - \frac{15}{32}\alpha^3 + \frac{1}{8}a^4\right) z_0^{4\alpha-2}}{4(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)\alpha} \\
&\quad + \frac{z_0^{4\alpha-2}}{8(\alpha-1)\alpha(2\alpha-1)}(4(\alpha-1)\alpha B^-(1/2;2\alpha+1,2\alpha-2) \\
&\quad - (2\alpha-1)\alpha B^-(1/2;2\alpha+1,2\alpha-2) \\
&\quad - (2\alpha-1)(\alpha-1)B^-(1/2;2\alpha+1,2\alpha-2)).
\end{aligned}
$$

For the last step we use the identities

$$B^-(z;a,b) - B^-(z;a,b+1) = B^-(z;a+1,b), \tag{41}$$

$$B^-(z;a,b) + B^-(z;a,b+2) - 2B^-(z;a,b+1) = B^-(z;a+2,b). \tag{42}$$

to obtain

$$
\begin{aligned}
I_{1,2}(y_0) = &\frac{\left(\frac{3}{64} - \frac{3}{16}2^{-4\alpha} + \alpha(-\frac{41}{128} + \frac{13}{16}2^{-4\alpha}) + \alpha^2(\frac{5}{8} - \frac{3}{4}2^{-4\alpha}) - \frac{15}{32}\alpha^3 + \frac{1}{8}a^4\right) z_0^{4\alpha-2}}{4(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)\alpha} \\
&- \frac{z_0^{4\alpha-2}B^-(1/2;2\alpha+1,2\alpha-2)}{8(\alpha-1)\alpha(2\alpha-1)}
\end{aligned}
\tag{43}
$$

Computing $\boldsymbol{I_2(y_0)}$ We will follow a similar strategy as for $I_1(y_0)$. First, using the change of variables $z_i := e^{-y_i/2}$ and Lemma 4.3 we get

$$I_2(y_0) = 4(\alpha-1/2)^2 \cdot \int_{1>z_1>z_2,z_0>0} P(y_0(z_0),y_1(z_1),y_2(z_2))z_1^{2\alpha-2}z_2^{2\alpha-2}\,\mathrm{d}z_2\,\mathrm{d}z_1$$

$$
\begin{aligned}
&= 4(\alpha-1/2)^2 \cdot \Bigg(\int_{1>z_1>z_0, z_2>0} z_0 z_1^{2\alpha-3} z_2^{2\alpha-2} \, \mathrm{d}z_2 \, \mathrm{d}z_1 \\
&\quad - \int_{\substack{1>z_1>z_0, z_2>0 \\ z_1<z_0+z_2}} G(z_1, z_0, z_2) z_0 z_1^{2\alpha-3} z_2^{2\alpha-2} \, \mathrm{d}z_2 \, \mathrm{d}z_1 \Bigg) \\
&=: 4(\alpha-1/2)^2 (I_{21}(y_0) - I_{22}(y_0)).
\end{aligned} \tag{44}
$$

We proceed with the easy integral:

$$
\begin{aligned}
I_{21}(y_0) &= z_0 \int_{1>z_1>\max(z_2, z_0); z_0, z_2>0} z_1^{2\alpha-3} z_2^{2\alpha-2} \, \mathrm{d}z_2 \, \mathrm{d}z_1 = z_0 \int_{z_0}^1 \int_0^{z_1} z_1^{2\alpha-3} z_2^{2\alpha-2} \, \mathrm{d}z_2 \, \mathrm{d}z_1 \\
&= z_0 \int_{z_0}^1 \left[\frac{z_2^{2\alpha-1}}{2\alpha-1} \right]_0^{z_1} z_1^{2\alpha-3} \, \mathrm{d}z_1 = \frac{z_0}{2\alpha-1} \int_{z_0}^1 z_1^{4\alpha-4} \, \mathrm{d}z_1 \\
&= \frac{z_0 - z_0^{4\alpha-2}}{(4\alpha-3)(2\alpha-1)}.
\end{aligned}
$$

We note that the denominators above are non-zero as $\alpha > \frac{1}{2}$ and $\alpha \neq \frac{3}{4}$.

To deal with $I_{22}(y_0)$ we consider the integral function

$$
J'_{a,b,c}(z_0) := z_0^a \int_{\substack{1>z_1>\max(z_0, z_2); z_0, z_2>0 \\ z_1<z_0+z_2}} z_1^{b+2\alpha-2} z_2^{c+2\alpha-2} \, \mathrm{d}z_2 \, \mathrm{d}z_1
$$

and note that

$$
I_{2,2}(y_0) = \frac{1}{4} \left(J'_{0,-1,1}(z_0) + J'_{2,-1,-1}(z_0) + J'_{0,1,-1}(z_0) + 2J'_{1,-1,0}(z_0) - 2J'_{0,0,0}(z_0) - 2J'_{1,0,-1}(z_0) \right) \tag{45}
$$

We know compute $J'_{a,b,c}(z_0)$

$$
\begin{aligned}
J'_{a,b,c}(z_0) &= z_0^a \int_{z_0}^1 \int_{z_1-z_0}^{z_1} z_1^{b+2\alpha-2} z_2^{c+2\alpha-2} \, \mathrm{d}z_2 \, \mathrm{d}z_1 \\
&= z_0^a \int_{z_0}^1 \frac{1}{c+2\alpha-1} z_1^{b+2\alpha-2} (z_1^{c+2\alpha-1} - (z_1-z_0)^{c+2\alpha-1}) \, \mathrm{d}z_1 \\
&= z_0^a \int_{z_0}^1 \frac{1}{c+2\alpha-1} z_1^{b+c+4\alpha-3} \, \mathrm{d}z_1 - z_0^a \int_{z_0}^1 \frac{1}{c+2\alpha-1} z_1^{b+2\alpha-2} (z_1-z_0)^{c+2\alpha-1} \, \mathrm{d}z_1 \\
&= z_0^a \frac{1}{(c+2\alpha-1)(b+c+4\alpha-2)} (1 - z_0^{b+c+4\alpha-2}) \\
&\quad - \frac{z_0^a}{c+2\alpha-1} z_0^{b+c+4\alpha-2} B^-(1-z_0; c+2\alpha, -b-c-4\alpha+2) \\
&= \frac{z_0^a - z_0^{4\alpha-2}}{(c+2\alpha-1)(b+c+4\alpha-2)} - \frac{z_0^{4\alpha-2} B^-(1-z_0; c+2\alpha, -b-c-4\alpha+2)}{c+2\alpha-1}.
\end{aligned}
$$

Here we used that for $x \in \mathbb{R}, y > -1$ (note that as $c \geq -1$, it holds that $c + 2\alpha - 1 > -1$):

$$
\begin{aligned}
\int_{z_0}^1 z_1^x (z_1 - z_0)^y \, \mathrm{d}z_1 &= \int_0^{1-z_0} (s+z_0)^x s^y \, \mathrm{d}s \\
&= z_0^{x+y} \int_0^{1-z_0} ((s/z_0)+1)^x \, (s/z_0)^y \, \mathrm{d}s \\
&= z_0^{x+y+1} \int_0^{1/z_0-1} (t+1)^x t^y \, \mathrm{d}t \\
&= z_0^{x+y+1} \int_0^{1-z_0} u^y (1-u)^{-(x+y+2)} \, \mathrm{d}u
\end{aligned}
$$

$$= z_0^{x+y+1} B^-(1-z_0; y+1, -x-y-1).$$

As $c \geq -1$ and $-a \in \{0,-1,-2\}$ and by our assumption $\alpha \notin \{\frac{3}{4}\}$, the denominators that occur during the computations above are non-zero.

Plugging the expression for $J'_{a,b,c}(z_0)$ back into (45) we get,

$$\begin{aligned}
I_{2,2}(y_0) = & \frac{1-z_0^{4\alpha-2}}{32\alpha(\alpha-1/2)} - \frac{z_0^{4\alpha-2}B^-(1-z_0; 1+2\alpha, -4\alpha+2)}{8\alpha} \\
& + \frac{z_0^2 - z_0^{4\alpha-2}}{32(\alpha-1)^2} - \frac{z_0^{4\alpha-2}B^-(1-z_0; -1+2\alpha, -4\alpha+4)}{8(\alpha-1)} \\
& + \frac{1-z_0^{4\alpha-2}}{32(\alpha-1)(\alpha-1/2)} - \frac{z_0^{4\alpha-2}B^-(1-z_0; -1+2\alpha, -4\alpha+2)}{8(\alpha-1)} \\
& + \frac{z_0 - z_0^{4\alpha-2}}{16(\alpha-1/2)(\alpha-3/4)} - \frac{z_0^{4\alpha-2}B^-(1-z_0; 2\alpha, -4\alpha+3)}{4(\alpha-1/2)} \\
& - \frac{1-z_0^{4\alpha-2}}{16(\alpha-1/2)^2} + \frac{z_0^{4\alpha-2}B^-(1-z_0; 2\alpha, -4\alpha+2)}{4(\alpha-1/2)} \\
& - \frac{z_0 - z_0^{4\alpha-2}}{16(\alpha-1)(\alpha-3/4)} + \frac{z_0^{4\alpha-2}B^-(1-z_0; -1+2\alpha, -4\alpha+3)}{4(\alpha-1)}.
\end{aligned}$$

Using some algebra and the identities (41) and (42) this can be reduced to

$$\begin{aligned}
I_{2,2}(y_0) = & \frac{1}{64\alpha(\alpha-1/2)^2(\alpha-1)} - \frac{(1-z_0)^{2\alpha}}{64\alpha(\alpha-1/2)^2(\alpha-1)} - \frac{z_0}{8(\alpha-1/2)(\alpha-1)(4\alpha-3)} \\
& + \frac{z_0^2}{32(\alpha-1)^2} + \frac{(-6+25\alpha-48\alpha^2+44\alpha^3-16\alpha^4)z_0^{4\alpha-2}}{512\alpha(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)} \\
& + \frac{z_0^{4\alpha-2}B^-(1-z_0; 2\alpha, 3-4\alpha)}{32(\alpha-1)(\alpha-1/2)^2}.
\end{aligned} \tag{46}$$

Combining the results for $I_1(y_0)$ and $I_2(y_0)$ Combining the results for $I_{11}(y_0), I_{12}(y_0), I_{21}(y_0)$ and $I_{22}(y_0)$ we get, after some algebra, an explicit expression for $\Delta_{\mathcal{P}}(y_0)$ as a linear combination of terms of the form z_0^u, $(1-z_0)^u$ and $z_0^u B^-(1-z_0; a, b)$:

$$\begin{aligned}
\Delta_{\mathcal{P}}(y_0) = & 2(I_1+I_2) = 8(\alpha-1/2)^2(I_{1,1} - I_{1,2} + I_{2,1} - I_{2,2}) \\
= & 8(\alpha-1/2)^2 \Bigg(\frac{1}{2(2\alpha-1)^2} z_0^{4\alpha-2} \\
& - \frac{\left(\frac{3}{64} - \frac{3}{16}2^{-4\alpha} + \alpha(-\frac{41}{128} + \frac{13}{16}2^{-4\alpha}) + \alpha^2(\frac{5}{8} - \frac{3}{4}2^{-4\alpha}) - \frac{15}{32}\alpha^3 + \frac{1}{8}a^4\right) z_0^{4\alpha-2}}{4(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)\alpha} \\
& + \frac{z_0^{4\alpha-2}B^-(1/2; 2\alpha+1, 2\alpha-2)}{8(\alpha-1)\alpha(2\alpha-1)} + \frac{z_0 - z_0^{4\alpha-2}}{(4\alpha-3)(2\alpha-1)} \\
& - \frac{1}{64\alpha(\alpha-1/2)^2(\alpha-1)} + \frac{(1-z_0)^{2\alpha}}{64\alpha(\alpha-1/2)^2(\alpha-1)} + \frac{z_0}{8(\alpha-1/2)(\alpha-1)(4\alpha-3)} \\
& - \frac{z_0^2}{32(\alpha-1)^2} - \frac{(-6+25\alpha-48\alpha^2+44\alpha^3-16\alpha^4)z_0^{4\alpha-2}}{512\alpha(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)} \\
& - \frac{z_0^{4\alpha-2}B^-(1-z_0; 2\alpha, 3-4\alpha)}{32(\alpha-1)(\alpha-1/2)^2} \Bigg) \\
= & -\frac{1}{8(\alpha-1)\alpha} + \frac{(\alpha-1/2)z_0}{\alpha-1} - \frac{(\alpha-1/2)^2 z_0^2}{4(\alpha-1)^2} \\
& + z_0^{-2+4\alpha}\left(\frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{(\alpha-1/2)B^-(1/2; 1+2\alpha, -2+2\alpha)}{2(\alpha-1)\alpha} \right)
\end{aligned}$$

$$+\frac{(1-z_0)^{2\alpha}}{8(\alpha-1)\alpha}-\frac{z_0^{4\alpha-2}B^-(1-z_0;2\alpha,3-4\alpha)}{4(\alpha-1)}$$

Observe that the above expression only contains terms of the form $\alpha-1$ in the denominator. The only expression of the form $\alpha-3/4$ is in the lower incomplete beta-function $B^-(1-z_0;2\alpha,3-4\alpha)$ which appears twice in the expression for $\Delta_{\mathcal{P}}(y_0)$.

The case of $\alpha=3/4$

Note that the factor $\alpha-\frac{3}{4}$ does not occur in any denominator of the previously obtained expression. For the lower incomplete beta function, the last argument $3-4\alpha$ is zero for $\alpha=\frac{3}{4}$, however as $z_0<1$ the integration domain of the lower incomplete beta function does not touch the singularity at $t=1$ (note $B^-(1-z_0;2\alpha;3-4\alpha)=\int_0^{1-z_0}t^{2\alpha-1}(1-t)^{2-4\alpha}$). Therefore, the previous expression holds for this case as well.

The case of $\alpha=1$ Pim: Realign formulas to fit them on page.

We want to compute the limit $\lim_{\alpha\to1}\Delta_{\mathcal{P}_{\alpha,\nu}}(y_0(z_0))$. For this, after factoring out $\frac{1}{\alpha-1}$ we compute the first two terms of the Taylor expansion of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$ in α based at $\alpha=1$. The constant term of this Taylor expansion is verified to vanish whereas the linear term gives the expression searched (after cancellation with $\frac{1}{\alpha-1}$). Note that two terms of $P(y_0(z_0))$ contain $(\alpha-1)^2$ in their denominator, so for those two another $\frac{1}{\alpha-1}$ is factored out and the first three terms of the Taylor expansion are computed. Their constant terms cancel and after multiplication with $\frac{1}{\alpha-1}$ their linear and quadratic terms become part of the constant and linear term of the overall expansion.

The first summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$,

$$-\frac{1}{8\alpha}=-\frac{1}{8}+\frac{1}{8}(\alpha-1)+O((\alpha-1)^2)$$

The second summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$,

$$(\alpha-1/2)z_0=\frac{1}{2}z_0+z_0(\alpha-1)+O((\alpha-1)^2)$$

The third summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$, needs a 3-term Taylor expansion after another multiplication with $(\alpha-1)$,

$$-\frac{(\alpha-1/2)^2z_0^2}{4}=-\frac{1}{16}z_0^2-\frac{1}{4}z_0^2(\alpha-1)-\frac{1}{4}z_0^2(\alpha-1)^2+O((\alpha-1)^3)$$

The fourth summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$, needs a 3-term Taylor expansion after another multiplication with $(\alpha-1)$,

$$\begin{aligned}z_0^{-2+4\alpha}\frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha}&=\frac{1}{16}z_0^2+\frac{z_0^2}{4}\left(\frac{1}{8}+\ln\frac{z_0}{2}\right)(\alpha-1)\\&\quad+\frac{z_0^2}{8}\left(4\left(\ln\frac{z_0}{2}\right)^2+\ln\frac{z_0}{2}-\frac{1}{4}\right)(\alpha-1)^2+O((\alpha-1)^3)\end{aligned}$$

as

$$\frac{d}{d\alpha}\left(\left(\frac{z_0}{2}\right)^{-2+4\alpha}2^{-3}\left(3-\frac{1}{\alpha}\right)\right)=\left(\frac{z_0}{2}\right)^{-2+4\alpha}2^{-3}\left(4\ln\left(\frac{z_0}{2}\right)\left(3-\frac{1}{\alpha}\right)+\frac{1}{\alpha^2}\right)$$

and

$$\frac{d^2}{d\alpha^2}\left(\left(\frac{z_0}{2}\right)^{-2+4\alpha}2^{-3}\left(3-\frac{1}{\alpha}\right)\right)=\left(\frac{z_0}{2}\right)^{-2+4\alpha}2^{-3}\left(16\left(\ln\frac{z_0}{2}\right)^2\left(3-\frac{1}{\alpha}\right)+\frac{8\ln(\frac{z_0}{2})}{\alpha^2}-\frac{2}{\alpha^3}\right)$$

The fifth summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$,

$$z_0^{-2+4\alpha}\frac{(\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)}{2\alpha}=\frac{z_0^2}{4}B^-(1/2;3,0)+z_0^2\left(\left(\ln(z_0)+\frac{1}{4}\right)B^-(1/2;3,0)\right.$$
$$\left.+1/2\int_0^{\frac{1}{2}}\ln(t)t^2(1-t)^{-1}+\ln(1-t)t^2(1-t)^{-1}dt\right)(\alpha-1)+O((\alpha-1)^2)$$

as

$$\frac{d}{d\alpha}\left(z_0^{-2+4\alpha}\frac{(\alpha-1/2)}{2\alpha}B^-(1/2;1+2\alpha,-2+2\alpha)\right)=\ln(z_0)4z_0^{-2+4\alpha}\left(\frac{1}{2}-\frac{1}{4\alpha}\right)B^-(1/2;1+2\alpha,-2+2\alpha)$$
$$+z_0^{-2+4\alpha}\frac{1}{4\alpha^2}B^-(1/2;1+2\alpha,-2+2\alpha)+z_0^{-2+4\alpha}\left(\frac{1}{2}-\frac{1}{4\alpha}\right)\int_0^{\frac{1}{2}}\frac{d}{d\alpha}(t^{2\alpha}(1-t)^{2\alpha-3})dt$$

where

$$\int_0^{\frac{1}{2}}\frac{d}{d\alpha}(t^{2\alpha}(1-t)^{2\alpha-3})dt=\int_0^{\frac{1}{2}}\ln(t)2t^{2\alpha}(1-t)^{2\alpha-3}+\ln(1-t)2t^{2\alpha}(1-t)^{2\alpha-3}dt$$

The sixth summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$,

$$\frac{(1-z_0)^{2\alpha}}{8\alpha}=\frac{(1-z_0)^2}{8}+\frac{(1-z_0)^2}{4}(\ln(1-z_0)-1/2)(\alpha-1)+O((\alpha-1)^2)$$

The seventh and last summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$,

$$-\frac{z_0^{4\alpha-2}B^-(1-z_0;2\alpha,3-4\alpha)}{4}=-\frac{z_0^2}{4}B^-(1-z_0;2,-1)-z_0^2\left(\ln(z_0)B^-(1-z_0;2,-1)\right.$$
$$\left.+\int_0^{1-z_0}1/2\ln(t)t(1-t)^{-2}-t\ln(1-t)(1-t)^{-2}dt\right)(\alpha-1)+O((\alpha-1)^2)$$

We see that the constant coefficients cancel:

$$-\frac{1}{8}+\frac{1}{2}z_0-\frac{1}{4}z_0^2+\frac{z_0^2}{32}+\frac{z_0^2}{4}\ln(\frac{z_0}{2})+\frac{z_0^2}{4}B^-(1/2;3,0)+\frac{(1-z_0)^2}{8}-\frac{z_0^2}{4}B^-(1-z_0;2,-1)$$
$$=-\frac{1}{8}+\frac{1}{2}z_0-\frac{1}{4}z_0^2+\frac{z_0^2}{32}+\frac{z_0^2}{4}\ln(z_0)-\frac{z_0^2}{4}\ln 2-\frac{5z_0^2}{32}+\frac{z_0^2}{4}\ln 2+\frac{1}{8}-\frac{z_0}{4}+\frac{z_0^2}{8}+\frac{z_0^2}{4}-\frac{z_0}{4}-\frac{z_0^2}{4}\ln z_0=0$$

The expression for $\alpha=1$ is given by the sum of the linear coefficients:

$$\frac{1}{8}+z_0-\frac{z_0^2}{4}+\frac{z_0^2}{8}(4(\ln\frac{z_0}{2})^2+\ln\frac{z_0}{2}-\frac{1}{4})+\frac{(1-z_0)^2}{4}(\ln(1-z_0)-1/2)$$
$$+z_0^2\left(\left(\ln(z_0)+\frac{1}{4}\right)B^-(1/2;3,0)+1/2\int_0^{\frac{1}{2}}\ln(t)t^2(1-t)^{-1}+\ln(1-t)t^2(1-t)^{-1}dt\right)$$
$$-z_0^2\left(\ln(z_0)B^-(1-z_0;2,-1)+\int_0^{1-z_0}1/2\ln(t)t(1-t)^{-2}-t\ln(1-t)(1-t)^{-2}dt\right)$$
$$=\frac{1}{8}+z_0-\frac{z_0^2}{4}+\frac{z_0^2}{2}(\ln\frac{z_0}{2})^2+\frac{z_0^2}{8}\ln\frac{z_0}{2}-\frac{z_0^2}{32}$$
$$-\frac{5}{8}z_0^2\ln(z_0)+z_0^2\ln(z_0)\ln 2-\frac{5z_0^2}{32}+\frac{z_0^2\ln 2}{4}+z_0^2/2\int_0^{\frac{1}{2}}\ln(t)t^2(1-t)^{-1}+\ln(1-t)t^2(1-t)^{-1}dt$$
$$+\frac{(1-z_0)^2}{4}\ln(1-z_0)-\frac{1}{8}+\frac{z_0}{4}-\frac{z_0^2}{8}$$
$$+z_0^2\ln(z_0)-z_0\ln z_0-z_0^2(\ln z_0)^2-z_0^2\int_0^{1-z_0}1/2\ln(t)t(1-t)^{-2}-t\ln(1-t)(1-t)^{-2}dt$$

$$
\begin{aligned}
&= \frac{5}{4}z_0 - \frac{9}{16}z_0^2 + \frac{z_0^2}{2}(\ln\frac{z_0}{2})^2 + \frac{z_0^2}{8}\ln\frac{z_0}{2} + \frac{(1-z_0)^2}{4}\ln(1-z_0) \\
&\quad + \frac{3}{8}z_0^2\ln(z_0) + z_0^2\ln(z_0)\ln 2 + \frac{z_0^2\ln 2}{4} + z_0^2/2\int_0^{\frac{1}{2}} \ln(t)t^2(1-t)^{-1} + \ln(1-t)t^2(1-t)^{-1}dt \\
&\quad - z_0\ln z_0 - z_0^2(\ln z_0)^2 - z_0^2\int_0^{1-z_0} 1/2\ln(t)t(1-t)^{-2} - t\ln(1-t)(1-t)^{-2}dt \\
&= \frac{5}{4}z_0 - \frac{9}{16}z_0^2 + \frac{z_0^2}{2}(\ln\frac{z_0}{2})^2 + \frac{z_0^2}{8}\ln\frac{z_0}{2} + \frac{(1-z_0)^2}{4}\ln(1-z_0) \\
&\quad + \frac{3}{8}z_0^2\ln(z_0) + z_0^2\ln(z_0)\ln 2 + \frac{z_0^2\ln 2}{4} + z_0^2/2(11/8 - 1/4\ln 2 - 3/2\ln(2)^2 - \mathrm{Li}_2(1/2)) \\
&\quad - z_0\ln z_0 - z_0^2(\ln z_0)^2 + z_0(1 + \frac{1}{2}(2-z_0)\ln(z_0) + \frac{1}{2}z_0\ln(z_0)^2 - \frac{1}{2}(1-z_0)\ln(1-z_0) + \frac{1}{2}z_0\,\mathrm{Li}_2(z_0)) \\
&\quad - z_0^2 - \frac{1}{2}z_0^2\,\mathrm{Li}_2(1) \\
&= \frac{9}{4}z_0 - \frac{25}{16}z_0^2 + \frac{z_0^2}{2}(\ln\frac{z_0}{2})^2 + \frac{z_0^2}{8}\ln\frac{z_0}{2} + \frac{(1-z_0)^2}{4}\ln(1-z_0) \\
&\quad - \frac{1}{8}z_0^2\ln(z_0) + z_0^2\ln(z_0)\ln 2 + \frac{z_0^2\ln 2}{4} + z_0^2/2(11/8 - 1/4\ln 2 - 3/2\ln(2)^2 - \mathrm{Li}_2(1/2) - \mathrm{Li}_2(1) + \mathrm{Li}_2(z_0)) \\
&\quad - \frac{1}{2}z_0^2(\ln z_0)^2 - \frac{1}{2}z_0(1-z_0)\ln(1-z_0)
\end{aligned}
$$

where we used that the first integral is

$$
\begin{aligned}
&[1/2(-2+5t+t^2-\ln(1-t)^2 - 2t\ln t - t^2\ln t - \ln(1-t)(-3+2t+t^2+2\ln t) - 2\,\mathrm{Li}_2(t))]_0^{\frac{1}{2}} \\
&\qquad = 11/8 - 1/4\ln 2 - 3/2\ln(2)^2 - \mathrm{Li}_2(1/2)
\end{aligned}
$$

and the second one is

$$
\begin{aligned}
&[\frac{1}{-1+t}(1 + \frac{1}{2}(1+t)\ln(1-t) + \frac{1}{2}(1-t)\ln(1-t)^2 - \frac{1}{2}t\ln(t) + \frac{1}{2}(1-t)\,\mathrm{Li}_2(1-t))]_0^{1-z_0} \\
&= -\frac{1}{z_0}(1 + \frac{1}{2}(2-z_0)\ln(z_0) + \frac{1}{2}z_0\ln(z_0)^2 - \frac{1}{2}(1-z_0)\ln(1-z_0) + \frac{1}{2}z_0 Li_2(z_0)) + 1 + \frac{1}{2}\,\mathrm{Li}_2(1)
\end{aligned}
$$

By expanding the squares and collecting terms, the last expression can be simplified to

$$
\begin{aligned}
&\frac{9}{4}z_0 + \frac{1-4z_0+3z_0^2}{4}\ln(1-z_0) + z_0^2\left(-7/8 - \frac{\ln(2)^2 + 2\,\mathrm{Li}_2(1/2) + 2\,\mathrm{Li}_2(1)}{4}\right) + \frac{1}{2}z_0^2\,\mathrm{Li}_2(z) \\
=&\frac{9}{4}z_0 + \frac{1-4z_0+3z_0^2}{4}\ln(1-z_0) - \frac{7+\pi^2}{8}z_0^2 + \frac{1}{2}z_0^2\,\mathrm{Li}_2(z)
\end{aligned}
$$

This finishes the computation.

□

4.6 Exact expression for clustering, integrating $\Delta_{\mathcal{P}}(y)$

Pim: This part is still in the original form and has to be converted to match style and notation.

We wish to compute

$$
c_\infty = \alpha\int_0^\infty \Delta_{\mathcal{P}}(y)e^{-\alpha y}\left(1 - e^{-\xi e^{y/2}} - \xi e^{y/2}e^{-\xi e^{y_0/2}}\right)\,\mathrm{d}y,
$$

and

$$
I^{(k)} := \int_0^\infty P(y_0)\alpha e^{-\alpha y_0}\frac{\left(\xi e^{y_0/2}\right)^k}{k!}e^{-\xi e^{y_0/2}}\,\mathrm{d}y_0.
$$

(So $c = I$ and $c(k) = I^{(k)}/p_k$ for $k \geq 2$.)
Note that the integral for I can be written as $J - I^{(0)} - I^{(1)}$ where

$$J := \int_0^\infty P(y_0)\alpha e^{-\alpha y_0}\, \mathrm{d}y_0.$$

Note that, of course, the integrals $I^{(0)}$ and $I^{(1)}$ can be computed even though $c(0)$ and $c(1)$ are technically speaking not well-defined.
It seems helpful to change coordinates to $z_0 := e^{-y_0/2}$. This gives $\mathrm{d}y_0 = -2z_0^{-1}\,\mathrm{d}z_0$, $e^{-\alpha y_0} = z_0^{2\alpha}$. Hence

$$J = 2\alpha \int_0^1 P(y_0(z_0)) \cdot z_0^{2\alpha-1}\,\mathrm{d}z_0,$$

and

$$I^{(k)} = \frac{2\alpha\xi^k}{k!} \cdot \int_0^1 P(y_0(z_0)) \cdot z_0^{2\alpha-(k+1)} e^{-\xi/z_0}\,\mathrm{d}z_0.$$

Given the expression(s) we got for $P(y_0)$, we firstly deal with the case $\alpha \neq 1$.
Now we recall that for $\alpha \neq 1$, $P(y_0(z_0))$ is in fact a linear combination of terms of the form z_0^u, $(1-z_0)^u$ and $z_0^u B^-(1-z_0, v, w)$, so we only need to compute the integrals J and $I^{(k)}$ for these forms.

To work out J we just need to observe that

$$\begin{aligned}
\textstyle\int_0^1 z_0^{u+2\alpha-1} B^-(1-z_0; v, w)\,\mathrm{d}z_0 &= \left[\tfrac{z_0^{u+2\alpha}}{u+2\alpha} B^-(1-z_0; v, w)\right]_0^1 + \tfrac{1}{u+2\alpha}\textstyle\int_0^1 z_0^{u+2\alpha+w-1}(1-z_0)^{v-1}\,\mathrm{d}z_0 \\
&= \tfrac{1}{u+2\alpha} B(u+w+2\alpha, v)
\end{aligned}$$

(using that $\frac{\partial}{\partial z_0} B^-(1-z_0; v, w) = -z_0^{w-1}(1-z_0)^{v-1}$.)

This gives the expression for J (note that it only depends on α but not on ν)

$$\begin{aligned}
J = {} & \frac{2 + 4\alpha + 13\alpha^2 - 34\alpha^3 - 12\alpha^4 + 24\alpha^5}{16(\alpha-1)^2\alpha(\alpha+1)(2\alpha+1)} + \frac{2^{-1-4\alpha}}{(\alpha-1)^2} \\
& + \frac{(\alpha - 1/2)(B(2\alpha, 2\alpha+1) + B^-(1/2; 1+2\alpha, -2+2\alpha))}{2(\alpha-1)(3\alpha-1)}
\end{aligned}$$

To work out $I^{(k)}$ we first point out that

$$\begin{aligned}
\textstyle\int_a^b z_0^{u+2\alpha-(k+1)} e^{-\xi z_0^{-1}}\,\mathrm{d}z_0 &= \xi^{u+2\alpha-k} \textstyle\int_{\xi/b}^{\xi/a} t^{k-1-2\alpha-u} e^{-t}\,\mathrm{d}t \\
&= \xi^{u+2\alpha-k}\left(\Gamma^+(k-2\alpha-u; \xi/b) - \Gamma^+(k-2\alpha-u; \xi/a)\right),
\end{aligned}$$

In particular

$$\int_0^1 z_0^{u+2\alpha-k-1} e^{-\xi z_0^{-1}}\,\mathrm{d}z_0 = \xi^{u+2\alpha-k}\Gamma^+(k-2\alpha-u; \xi)$$

where Γ^+ denotes the (upper) incomplete gamma function, and we've used the subsitution $t = \xi/z_0$ which gives $\mathrm{d}z_0 = -\xi t^{-2}\,\mathrm{d}t$. (And of course it is understood that $\xi/0 = \infty$.)

The hypergeometric U-function (also called Tricomi's confluent hypergeometric function) has the integral representation

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt$$

which holds for $a, b, z \in \mathbb{C}$, $b \notin \mathbb{Z}_{\leq 0}$, $Re(a), Re(z) > 0$, see [?, p.255]. Applying the change of variables $t = \frac{1-s}{s}$ (i.e. $dt = -s^{-2}ds$ and $s = \frac{1}{t+1}$) yields the integral $\int_1^0 e^{-z(-1+\frac{1}{s})}(\frac{1-s}{s})^{a-1}(\frac{1}{s})^{b-a-1}(-s^2)ds = e^z \int_0^1 e^{-z/s}s^{-b}(1-s)^{a-1}ds$, i.e.

$$U(a,b,z) = \frac{e^z}{\Gamma(a)} \int_0^1 s^{-b}(1-s)^{a-1}e^{-z/s}ds$$

Plugging in $a = 2\alpha + 1 > 0$, $b = -2\alpha + k + 1$, $z = \xi > 0$, gives

$$\int_0^1 z_0^{2\alpha-k-1}e^{-\xi/z_0}(1-z_0)^{2\alpha}dz_0 = \Gamma(2\alpha+1)e^{-\xi}U(2\alpha+1, 1+k-2\alpha, \xi)$$

Let $a, c \in \mathbb{R}$, $\xi, b > 0$ positive real numbers. Using the integral definition of the incomplete B-function, the change of variables $s = 1 - t$ gives:

$$\begin{aligned}\int_0^1 z^a e^{-\xi/z} B^-(1-z; b, c)dz &= \int_0^1 z^a e^{-\xi/z} \int_0^{1-z} t^{b-1}(1-t)^{c-1}dtdz \\ &= \int_0^1 z^a e^{-\xi/z} \int_z^1 s^{c-1}(1-s)^{b-1}dsdz\end{aligned}$$

Then changing the order of integration and using the substitution $u = \xi/z$ and recognizing the upper incomplete Γ-function yields

$$\begin{aligned}&= \int_0^1 \int_0^s z^a e^{-\xi/z}dzs^{c-1}(1-s)^{b-1}ds \\ &= \int_0^1 \int_{\xi/s}^\infty \xi^{a+1}u^{-a-2}e^{-u}dus^{c-1}(1-s)^{b-1}ds \\ &= \xi^{a+1} \int_0^1 \Gamma^+(-a-1, \xi/s)s^{c-1}(1-s)^{b-1}ds\end{aligned}$$

Now, the incomplete Γ-function has a representation in terms of Meijer's G-function

$$\Gamma^+(-a-1, \xi/s) = G_{1,2}^{2,0}\left(\begin{matrix}1 \\ -a-1, 0\end{matrix}\middle| \frac{\xi}{s}\right)$$

which holds for any $a \in \mathbb{R}$ and $s > 0$ (see appendix; note that for a fixed second argument, the upper incomplete Γ-function is entire in the first argument, see [?, pp. 899, 1032ff.]). Therefore, we get the integral

$$= \xi^{a+1} \int_0^1 s^{c-1}(1-s)^{b-1}G_{1,2}^{2,0}\left(\begin{matrix}1 \\ -a-1, 0\end{matrix}\middle| \frac{\xi}{s}\right) ds$$

applying the inversion identity for Meijer's G-function (see [?, p. 209, 5.3.1.(9))])

$$= \xi^{a+1} \int_0^1 s^{c-1}(1-s)^{b-1}G_{2,1}^{0,2}\left(\begin{matrix}2+a, 1 \\ 0\end{matrix}\middle| \frac{s}{\xi}\right) ds$$

This is the Euler transform of Meijer's G-function (see [?, p. 214, 5.5.2.(5)]) and (as the conditions $2+1 < 2(0+2)$ and $|\arg(\xi^{-1})| < \frac{\pi}{2}$ (as $\xi > 0$) and $1-c-b < 1-c$ (as $b > 0$) are satisfied) equals

$$= \xi^{a+1}\Gamma(b)G_{3,2}^{0,3}\left(\begin{matrix}1-c, 2+a, 1 \\ 0, 1-c-b\end{matrix}\middle| \xi^{-1}\right)$$

Using again the inversion identity for Meijer's G-function

$$= \xi^{a+1}\Gamma(b)G_{2,3}^{3,0}\left(\begin{matrix}1, b+c \\ c, -1-a, 0\end{matrix}\middle| \xi\right)$$

Plugging in $a = 6\alpha - k - 3$, $b = 2\alpha$, $c = 3 - 4\alpha$ results in

$$= \xi^{6a-k-2}\Gamma(2\alpha)G^{3,0}_{2,3}\left(\begin{matrix} 1, 3-2\alpha \\ 3-4\alpha, -6\alpha+k+2, 0\end{matrix}\middle|\xi\right)$$

This allows us to write:

$$\begin{aligned}
I^{(k)} = \frac{\xi^{2\alpha}}{4k!(\alpha-1)} & \left(-\Gamma^+(k-2\alpha,\xi) - 2\frac{\alpha(\alpha-1/2)^2\xi^2\Gamma^+(k-2\alpha-2,\xi)}{(\alpha-1)} \right. \\
& + 8\alpha(\alpha-1/2)\xi\Gamma^+(k-2\alpha-1,\xi) \\
& + 4\xi^{4\alpha-2}\Gamma^+(k-6\alpha+2,\xi)\left(\frac{2^{-4\alpha}(3\alpha-1)}{(\alpha-1)} + (\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\right) \\
& + \xi^{k-2\alpha}\Gamma(2\alpha+1)e^{-\xi}U(2\alpha+1,1+k-2\alpha,\xi) \\
& \left. - \xi^{4\alpha-2}\Gamma(2\alpha+1)G^{3,0}_{2,3}\left(\begin{matrix} 1, 3-2\alpha \\ 3-4\alpha, -6\alpha+k+2, 0\end{matrix}\middle|\xi\right)\right)
\end{aligned}$$

Finally, we compute

$$p_k := \mathbb{P}(D=k) = \int_0^\infty e^{-\xi e^{y/2}}\frac{(\xi e^{y/2})^k}{k!}\alpha e^{-\alpha y}dy = 2\alpha\int_0^1 e^{-\frac{\xi}{z}}z^{2\alpha-k-1}dz = \frac{2\alpha\xi^{2\alpha}}{k!}\Gamma(k-2\alpha,\xi)$$

and we can write down the average clustering coefficient

$$\begin{aligned}
c &= J - I^{(0)} - I^{(1)} \\
&= \frac{2+4\alpha+13\alpha^2-34\alpha^3-12\alpha^4+24\alpha^5}{16(\alpha-1)^2\alpha(\alpha+1)(2\alpha+1)} + \frac{2^{-1-4\alpha}}{(\alpha-1)^2} \\
&\quad + \frac{(\alpha-1/2)(B(2\alpha,2\alpha+1)+B^-(1/2;1+2\alpha,-2+2\alpha))}{2(\alpha-1)(3\alpha-1)} \\
&\quad - \frac{\xi^{2\alpha}}{4(\alpha-1)}\left(-\Gamma^+(-2\alpha,\xi) - 2\frac{\alpha(\alpha-1/2)^2\xi^2\Gamma^+(-2\alpha-2,\xi)}{(\alpha-1)}\right. \\
&\qquad + 8\alpha(\alpha-1/2)\xi\Gamma^+(-2\alpha-1,\xi) \\
&\qquad + 4\xi^{4\alpha-2}\Gamma^+(-6\alpha+2,\xi)\left(\frac{2^{-4\alpha}(3\alpha-1)}{(\alpha-1)} + (\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\right) \\
&\qquad + \xi^{-2\alpha}\Gamma(2\alpha+1)e^{-\xi}U(2\alpha+1,1-2\alpha,\xi) \\
&\qquad \left. - \xi^{4\alpha-2}\Gamma(2\alpha+1)G^{3,0}_{2,3}\left(\begin{matrix} 1, 3-2\alpha \\ 3-4\alpha, -6\alpha+2, 0\end{matrix}\middle|\xi\right)\right) \\
&\quad - \frac{\xi^{2\alpha}}{4(\alpha-1)}\left(-\Gamma^+(1-2\alpha,\xi) - 2\frac{\alpha(\alpha-1/2)^2\xi^2\Gamma^+(-2\alpha-1,\xi)}{(\alpha-1)}\right. \\
&\qquad + 8\alpha(\alpha-1/2)\xi\Gamma^+(1-2\alpha-1,\xi) \\
&\qquad + 4\xi^{4\alpha-2}\Gamma^+(1-6\alpha+2,\xi)\left(\frac{2^{-4\alpha}(3\alpha-1)}{(\alpha-1)} + (\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\right) \\
&\qquad + \xi^{1-2\alpha}\Gamma(2\alpha+1)e^{-\xi}U(2\alpha+1,2-2\alpha,\xi) \\
&\qquad \left. - \xi^{4\alpha-2}\Gamma(2\alpha+1)G^{3,0}_{2,3}\left(\begin{matrix} 1, 3-2\alpha \\ 3-4\alpha, -6\alpha+3, 0\end{matrix}\middle|\xi\right)\right)
\end{aligned}$$

and

$$c(k) = \frac{I^{(k)}}{p_k} = \frac{1}{8\alpha(\alpha-1)\Gamma^+(k-2\alpha,\xi)}\left(-\Gamma^+(k-2\alpha,\xi) - 2\frac{\alpha(\alpha-1/2)^2\xi^2\Gamma^+(k-2\alpha-2,\xi)}{(\alpha-1)}\right.$$

$$
\begin{aligned}
&+8\alpha(\alpha-1/2)\xi\Gamma^{+}(k-2\alpha-1,\xi)\\
&+4\xi^{4\alpha-2}\Gamma^{+}(k-6\alpha+2,\xi)\left(\frac{2^{-4\alpha}(3\alpha-1)}{(\alpha-1)}+(\alpha-1/2)B^{-}(1/2;1+2\alpha,-2+2\alpha)\right)\\
&+\xi^{k-2\alpha}\Gamma(2\alpha+1)e^{-\xi}U(2\alpha+1,1+k-2\alpha,\xi)\\
&-\xi^{4\alpha-2}\Gamma(2\alpha+1)G^{3,0}_{2,3}\left(\begin{matrix}1,3-2\alpha\\ 3-4\alpha,-6\alpha+k+2,0\end{matrix}\middle|\xi\right)\Bigg)
\end{aligned}
$$

To determine the value of c and $c(k)$ for $\alpha = 1$, it suffices to find the value of J and $I^{(k)}$ at $\alpha = 1$. We can do this by computing those integrals with the expression for $P(y_0)$ that we found for $\alpha = 1$, i.e.

$$
\begin{aligned}
J &= 2\alpha\int_0^1\left(\frac{9}{4}z_0+\frac{1-4z_0+3z_0^2}{4}\ln(1-z_0)-\frac{7+\pi^2}{8}z_0^2+\frac{1}{2}z_0^2\operatorname{Li}_2(z)\right)z_0^{2\alpha-1}dz_0\\
&=\frac{575-12\pi^2}{576}
\end{aligned}
$$

and

$$
\begin{aligned}
I^{(k)} &= \frac{2\alpha\xi^k}{k!}\int_0^1\left(\frac{9}{4}z_0+\frac{1-4z_0+3z_0^2}{4}\ln(1-z_0)-\frac{7+\pi^2}{8}z_0^2+\frac{1}{2}z_0^2\operatorname{Li}_2(z)\right)z_0^{2\alpha-k-1}e^{-\xi/z_0}dz_0\\
&=\frac{2\eta^k}{k!}\int_0^1\left(\frac{9}{4}z_0+\frac{1-4z_0+3z_0^2}{4}\ln(1-z_0)-\frac{7+\pi^2}{8}z_0^2+\frac{1}{2}z_0^2\operatorname{Li}_2(z)\right)z_0^{1-k}e^{-\eta/z_0}dz_0\\
&=\frac{9\eta^k}{2k!}\eta^{3-k}\Gamma^{+}(k-3,\eta)-\frac{\eta^k}{k!}\frac{7+\pi^2}{4}\eta^{4-k}\Gamma^{+}(k-4,\eta)\\
&+\frac{\eta^k}{2k!}\int_0^1(1-4z_0+3z_0^2)\ln(1-z_0)z_0^{1-k}e^{-\eta/z_0}dz_0+\frac{\eta^k}{k!}\int_0^1 z_0^{3-k}\operatorname{Li}_2(z)e^{-\eta/z_0}dz_0\\
&=\frac{9\eta^3}{2k!}\Gamma^{+}(k-3,\eta)-\frac{\eta^4}{k!}\frac{7+\pi^2}{4}\Gamma^{+}(k-4,\eta)\\
&+\frac{\eta^k}{2k!}\int_0^1(1-4z_0+3z_0^2)\ln(1-z_0)z_0^{1-k}e^{-\eta/z_0}dz_0+\frac{\eta^k}{k!}\int_0^1 z_0^{3-k}\operatorname{Li}_2(z)e^{-\eta/z_0}dz_0
\end{aligned}
$$

where $\eta = \frac{4\nu}{\pi}$.

5 From $G_{\mathcal{P},n}(\alpha,\nu)$ to $G_{\mathcal{P}}(\alpha,\nu)$ (Proving Proposition 2.7)

Pim: The text in this section needs a full overhaul.

Our focus will however be on substituting $\rho_n(y,k_n)$ with $\rho(y,k_n)$ in integral expressions. We begin with the following useful lemma.

$\phi_n(y) = \Theta\left(k_n^{2\alpha-1}n^{-(2\alpha-1)}\right)$. This gives the following useful corollary to Lemma 3.3.

Corollary 5.1. *Let $\alpha > \frac{1}{2}$, $0 < \varepsilon < 1$ and k_n be an increasing sequence satisfying $k_n = o\left(n^{\frac{1}{2\alpha-1}}\right)$. Then, for all $p \in \mathcal{K}_\varepsilon(k_n)$,*

$$
\mu_{\alpha,\nu,n}(B_{\mathcal{P},n}(p)) = \mu_{\alpha,\nu}(B_{\mathcal{P}}(p))\left(1-\Theta\left(k_n^{2\alpha-1}n^{-(2\alpha-1)}\right)\right).
$$

With this lemma we can now show that we can replace $\rho_n(y,k_n)$ in integral expressions over $\mathcal{R}_n$ with $\rho(y,k_n)$. This will help make the remaining computations easier.

As a direct consequence of the above lemma we obtain asymptotic result for the number of nodes with degree k_n in $G_{\mathcal{P},n}(\alpha,\nu)$.

Lemma 5.2. *Let k_n be an non-decreasing sequence such that $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then,*

$$\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] = (1+o(1))\alpha n \int_{\mathbb{R}_+} \rho(y,k_n)e^{-\alpha}\ \mathrm{d}y,$$

so that in particular

$$\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] = (1+o\,(1))\alpha n k_n^{-(2\alpha+1)}.$$

Proof. Since

$$\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] = \int_{\mathcal{R}} \rho_n(y,k_n) f_{\alpha,\nu}(x,y)\ \mathrm{d}x\ \mathrm{d}y,$$

and $2I_n/\pi = n/\nu$, the result follows directly from Lemma **??**.

$$\begin{aligned}
\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] &= (1+o\,(1)) \int_{\mathcal{R}} \rho(y,k_n) f_{\alpha,\nu}(x,y)\ \mathrm{d}x\ \mathrm{d}y \\
&= \frac{2\alpha\nu}{\pi} I_n \int_0^{R_n} \rho(y,k_n)e^{-\alpha}\ \mathrm{d}y \\
&= \alpha n \int_0^{R_n} \rho(y,k_n)e^{-\alpha}\ \mathrm{d}y \\
&= (1+o\,(1))\alpha n k_n^{-(2\alpha+1)}.
\end{aligned}$$

□

5.1 Comparing triangles between $G_{\mathcal{P}}(\alpha,\nu)$ and $G_{\mathcal{P},n}(\alpha,\nu)$

We now turn to the task of calculating the expected number of triangles of a node at height y, for both the infinite model and the finite box model. Let $p_0 = (0,y)$ and recall that

$$\mathbb{E}\left[T_{\mathcal{P},n}(p_0)\right] = \frac{1}{2} \iint_{\mathcal{R}_n^2} T_{\mathcal{P},n}(p_0,p_1,p_2) f_{\alpha,\nu}(x_1,y_1) f_{\alpha,\nu}(x_1,y_1)\, dx_1\, dx_2\, dy_1\, dy_2,$$

where

$$T_{\mathcal{P},n}(p_0,p_1,p_2) = \mathbb{1}_{\{p_1 \in B_{\mathcal{P},n}(p)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P},n}(p)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P},n}(p_1)\}}.$$

The difference between the indicator $\mathbb{1}_{\{p_1 \in B_{\mathcal{P},n}(p)\}}$ in the finite box model and $\mathbb{1}_{\{p_1 \in B_{\mathcal{P}}(p)\}}$ is that in $G_{\mathcal{P},n}(\alpha,\nu)$ we identified the boundaries of the interval $[-\frac{\pi}{2}e^{R_n/2}, \frac{\pi}{2}e^{R_n/2}]$. It is clear that for any $0 \le y \le R_n$ we have that $\mathcal{B}_{\mathcal{P},n}(p_0) = \mathcal{B}_{\mathcal{P}}(p_0) \cap \mathcal{R}_n$. However, when we take another point $p' \in \mathcal{B}_{\mathcal{P},n}(p)$ then it could happen that there are points in the intersection $\mathcal{B}_{\mathcal{P},n}(p) \cap \mathcal{B}_{\mathcal{P},n}(p')$ that are not in $\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{B}_{\mathcal{P}}(p')$. Let us denote this region by $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p')$. Then, any $p_2 \in \mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p')$ creates a triangle with p and p' in $G_{\mathcal{P},n}(\alpha,\nu)$ that is not present in $G_{\mathcal{P}}(\alpha,\nu)$.

Define

$$\widetilde{T}_{\mathcal{P},n}(p_0,p_1,p_2) = \mathbb{1}_{\{p_1 \in B_{\mathcal{P},n}(p)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P},n}(p)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P}}(p_1) \cap \mathcal{R}_n\}} \tag{47}$$

Then

$$\sum_{p_1,p_2 \in 2_n^{\mathcal{P}}} T_{\mathcal{P},n}(p_0,p_1,p_2) - \widetilde{T}_{\mathcal{P},n}(p_0,p_1,p_2) = \sum_{p_1,p_2 \in 2_n^{\mathcal{P}}} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathcal{P},n}(p_0)\}} \mathbb{1}_{\{p_2 \in \mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p_0,p_1)\}}$$

Figure 5 shows an example of a configuration where $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p') \neq \emptyset$. We observe that $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p') \neq \emptyset$ because the right boundary of the ball $\mathcal{B}_{\mathcal{P},n}(p')$ exists the right boundary of the box $\mathcal{R}_n$ and then, since we identified the boundaries, continues from the left so that $\mathcal{B}_{\mathcal{P},n}(p')$ covers part of the ball $\mathcal{B}_{\mathcal{P},n}(p)$ which would not be covered in the infinite limit model.

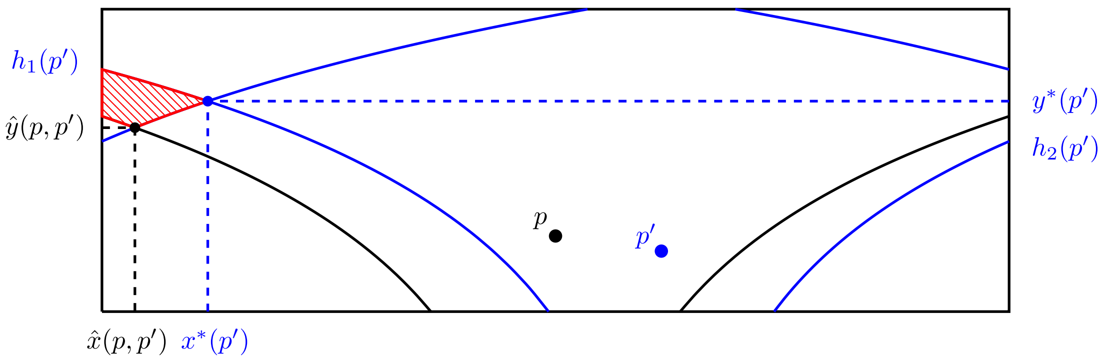


Figure 4: Example configuration of two points p and p' for which $\mathcal{B}_{\mathcal{P},n}(p) \cap \mathcal{B}_{\mathcal{P},n}(p')$ is not a subset of $\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{B}_{\mathcal{P}}(p')$. The red region indicates the area of points belonging to $\mathcal{B}_{\mathcal{P},n}(p) \cap \mathcal{B}_{\mathcal{P},n}(p')$ but not to $\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{B}_{\mathcal{P}}(p')$.

To further analyze this, let us introduce some notation. For any $p = (x, y) \in \mathcal{R}_n$ we will define the left and right boundary functions as, respectfully,

$$b_p^-(z) = \begin{cases} 2\log(x-z) - y & \text{if } -\frac{\pi}{2}e^{R_n/2} \le z \le x - e^{y/2} \\ 2\log\left(\pi e^{R_n/2} + x - z\right) - y & \text{if } x - e^{(y+R_n)/2} + \pi e^{R_n/2} \le z \le \frac{\pi}{2}e^{R_n/2} \\ 0 & \text{else} \end{cases} \tag{48}$$

$$b_p^+(z) = \begin{cases} 2\log(z-x) - y & \text{if } x + e^{y/2} \le z \le \frac{\pi}{2}e^{R_n/2} \\ 2\log\left(\pi e^{R_n/2} + z - x\right) - y & \text{if } -\frac{\pi}{2}e^{R_n/2} \le z \le x + e^{(y+R_n)/2} - \pi e^{R_n/2} \\ 0 & \text{else} \end{cases} \tag{49}$$

Note that these functions describe the boundaries of the ball $\mathcal{B}_{\mathcal{P},n}(p)$. In particular, $p' \in \mathcal{B}_{\mathcal{P},n}(p)$ if and only if $y' \ge \min\left\{b_p^-(x'), b_p^+(x')\right\}$. See Figure 6 for an illustration.

For the remainder of the analysis we restrict ourselves to the case where $x' > 0$. Due to symmetry the situation for $x' < 0$ is the same. There are two important points in the box $\mathcal{R}_n$. These are the intersection between the left boundary of p' and the right boundary of p', as it continues from the left side of the box, and the left boundary of p. We denote by $(x^*(p'), y^*(p'))$ the intersection between the left and right boundary of p' and by $(\hat{x}(p,p'), \hat{y}(p,p'))$ the intersection between the left boundary of p and the right boundary of p' (see also Figure 6). After some simple algebra we obtain the following expressions for the coordinates of these two points

$$\begin{aligned} x^*(p') &= x' - \frac{\pi}{2}e^{R_n/2} \\ y^*(p') &= 2\log\left(\frac{\pi}{2}e^{R_n/2}\right) - y' \\ \hat{x}(p,p') &= \frac{x' - \pi e^{R_n/2}}{1 + e^{(y'-y)/2}} \\ \hat{y}(p,p') &= 2\log\left(\frac{\pi e^{R_n/2} - x'}{e^{y/2} + e^{y'/2}}\right) \end{aligned}$$

The crucial observation is that $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n} = \emptyset$ as long as the point $(x^*(p'), y^*(p'))$ is above the left boundary of p. This happens exactly when $y^*(p') > b_p^-(x^*(p'))$. Therefore the boundary of this event is given by the equation $y^*(p') = b_p^-(x^*(p'))$ which reads

$$2\log\left(\frac{\pi}{2}e^{R_n/2}\right) - y' = 2\log\left(\frac{\pi}{2}e^{R_n/2} - x'\right) - y.$$

Solving this equation gives us the function

$$b_p^*(z) = y - 2\log\left(1 - \frac{z}{\frac{\pi}{2}e^{R_n/2}}\right), \tag{50}$$

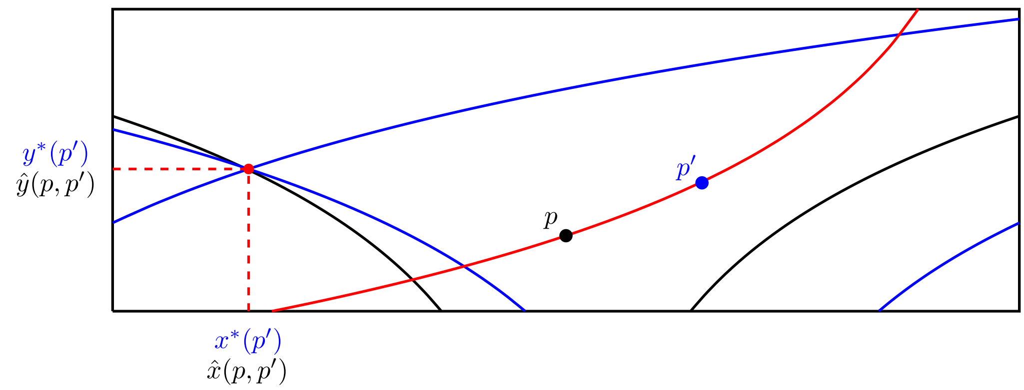


Figure 5: ???.

which is displayed by the red curve in Figure 6. It holds that $y^*(p') > b_p^-(x^*(p'))$ if and only if $y' < b_p^*(x')$ and hence we have that $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n} = \emptyset$ for all $p' \in \mathcal{R}_n$ for which $y' \geq b_p^*(x')$. We also not that when $y' = b_p^*(x')$ the two points $(x^*(p'), y^*(p'))$ and $(\hat{x}(p,p'), \hat{y}(p,p'))$ coincide.

This analysis allows us to compute the expected difference in the number of triangles for $\mathcal{P}$ and $\mathcal{P}_n$, for a node with height y.

Lemma 5.3. *Let $(k_n)_{n\geq 1}$ be any sequence with $k_n \to \infty$ and $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then, for any $0 < \varepsilon < 1$ and $p \in \mathcal{K}_\varepsilon(k_n)$, as $n \to \infty$,*

$$\int_{\mathcal{R}_n} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) f_{\mu,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 = O\left(yn^{-(2\alpha-1)} + n^{-(2\alpha-1)}e^y\right)$$

The proof of the lemma is not difficult but cumbersome, since it involves computing many different integrals. We postpone this proof till the end of this section and proceed with the main goal, proving Proposition 2.7. But first we state a small lemma about the scaling of $s_\alpha(k_n)$ that will be very useful.

Lemma 5.4. *Let $s_\alpha(k_n)$ be as defined in (8). Then for any $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$, as $n \to \infty$,*

$$n^{-(2\alpha-1)} = o\left(s_\alpha(k_n)\right).$$

Proof. First let $\frac{1}{2} < \alpha < \frac{3}{4}$. Then

$$n^{-(2\alpha-1)} s_\alpha(k_n)^{-1} = n^{-(2\alpha-1)} k_n^{4\alpha-2} = o\left(n^{-(2\alpha-1)+\frac{4\alpha-2}{2\alpha+1}}\right) = o\left(n^{-\frac{4\alpha^2-4\alpha+1}{2\alpha+1}}\right) = o\left(1\right),$$

since $4\alpha^2 - 4\alpha + 1 > 0$ for all $\alpha > \frac{1}{2}$. Similarly, for $\alpha \geq \frac{3}{4}$ we have that $4\alpha^2 > 2$ and hence,

$$n^{-(2\alpha-1)} s_\alpha(k_n) = o\left(n^{-(2\alpha-1)} k_n\right) = o\left(n^{-\frac{4\alpha^2-2}{2\alpha+1}}\right) = o\left(1\right).$$

$\square$

Corollary 5.5. *Let $(k_n)_{n\geq 1}$ be any sequence with $k_n \to \infty$ and $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then, for any $0 < \varepsilon < 1$ and $p \in \mathcal{K}_\varepsilon(k_n)$, as $n \to \infty$,*

$$\left|\frac{\mathbb{E}\left[T_{\mathcal{P},n}(p)\right]}{\mathbb{E}\left[T_{\mathcal{P}}(p)\right]} - 1\right| = O\left(s_\alpha(k_n)^{-1} k_n^{-3(2\alpha-1)}\right) = \begin{cases} O\left(k_n^{-(2\alpha-1)}\right) & \text{if } \frac{1}{2} < \alpha < \frac{3}{4}, \\ O\left(\log(k_n)^{-1} k_n^{-\frac{1}{2}}\right) & \text{if } \alpha = \frac{3}{4}, \\ O\left(k_n^{-(6\alpha-4)}\right) & \text{if } \alpha > \frac{3}{4}. \end{cases}$$

Proof of Proposition 2.7. **Pim:** incorporate case where $k_n = \Theta(1)$.

Recall that

$$\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right] = \frac{\int_{\mathcal{R}_n} \rho_n(y,k_n)\mathbb{E}\left[T_{\mathcal{P},n}(y)\right] f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}.$$

By Lemma 5.2

$$\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] = (1+o(1))\alpha n \int_{\mathbb{R}_+} \rho(y,k_n)e^{-\alpha}\, \mathrm{d}y,$$

and hence it is enough to show that

$$\int_{\mathcal{R}_n} \rho_n(y,k_n)\mathbb{E}\left[T_{\mathcal{P},n}(y)\right] f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y = (1+o(1))\alpha n \int_{\mathbb{R}_+} \rho(y,k_n)\mathbb{E}\left[T_{\mathcal{P}}(y)\right] e^{-\alpha y}\, \mathrm{d}x\, \mathrm{d}y.$$

First we note that

$$|T_{\mathcal{P},n}(p) - T_{\mathcal{P}}(p)| = \sum_{p_1,p_2 \in \mathcal{R}_n} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathcal{P},n}(p)\}}\mathbb{1}_{\{p_2 \in \mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\}} + \sum_{p_1,p_2 \in \mathcal{R}\setminus\mathcal{R}_n} T_{\mathcal{P}}(p,p_1,p_2)$$

and hence

$$\begin{aligned}|\mathbb{E}\left[T_{\mathcal{P},n}(p) - T_{\mathcal{P}}(p)\right]| \le &\int_{\mathcal{R}_n} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) f_{\mu,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 \\ &+ \iint_{\mathcal{R}\setminus\mathcal{R}_n} T_{\mathcal{P}}(p,p_1,p_2) f_{\mu,\nu}(x_1,y_1) f_{\mu,\nu}(x_2,y_2)\, \mathrm{d}x_2\, \mathrm{d}y_2\, \mathrm{d}x_1\, \mathrm{d}y_1.\end{aligned}$$

The first integral was taken care of in Lemma 5.3. For the second integral we have

$$\begin{aligned}&\iint_{\mathcal{R}\setminus\mathcal{R}_n} T_{\mathcal{P}}(p,p_1,p_2) f_{\mu,\nu}(x_1,y_1) f_{\mu,\nu}(x_2,y_2)\, \mathrm{d}x_2\, \mathrm{d}y_2\, \mathrm{d}x_1\, \mathrm{d}y_1 \\ &\quad \le \left(\int_{\mathcal{R}\setminus\mathcal{R}_n} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathcal{P}}(p)\}} f_{\mu,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1\right)^2 \\ &\quad = O\left(\left(e^{y/2}\int_{R_n}^{\infty} e^{-(\alpha-\frac{1}{2})y_1}\, \mathrm{d}y_1\right)^2\right) = O\left(e^y n^{-(2\alpha-1)}\right),\end{aligned}$$

from which we conclude that

$$|\mathbb{E}\left[T_{\mathcal{P},n}(p) - T_{\mathcal{P}}(p)\right]| = O\left(yn^{-(2\alpha-1)} + n^{-(2\alpha-1)}e^y\right) = O\left(n^{-(2\alpha-1)}e^y\right).$$

Next we show that this difference is $o\left(\mathbb{E}\left[T_{\mathcal{P}}(p)\right]\right)$ when $y \in K_\varepsilon(k_n)$. Observe that since $\mathbb{E}\left[T_{\mathcal{P}}(p)\right] = e^y\Delta_{\mathcal{P}}(y)$, Proposition 4.1 implies that $\mathbb{E}\left[T_{\mathcal{P}}(y)\right] = \Theta\left(e^y s_\alpha(e^{y/2})\right)$ as $y \to \infty$. Now consider $y \in K_C(k_n)$. Then $e^{y/2} = \Theta(k_n)$ and hence

$$\frac{|\mathbb{E}\left[T_{\mathcal{P},n}(p) - T_{\mathcal{P}}(p)\right]|}{\mathbb{E}\left[T_{\mathcal{P}}(y)\right]} = O\left(n^{-(2\alpha-1)}s_\alpha(k_n)^{-1}\left(\log(k_n)k^{-2}+1\right)\right) = O\left(\left(n^{2\alpha-1}s_\alpha(k_n)\right)^{-1}\right) = o(1),$$

where the last part follows from Lemma 5.4. It follows that for $y \in \mathcal{K}_\varepsilon(k_n)$ we have

$$\mathbb{E}\left[T_{\mathcal{P},n}(y)\right] = (1+o(1))\mathbb{E}\left[T_{\mathcal{P}}(y)\right],$$

while in general

$$\mathbb{E}\left[T_{\mathcal{P},n}(y)\right] = \mathbb{E}\left[T_{\mathcal{P}}(y)\right]\left(1 + O\left(s_\alpha(e^{y/2})\right)\right) = O\left(e^y s_\alpha(e^{y/2})^2\right).$$

Therefore, by first applying a concentration argument and then Lemma **??**,

$$\int_{\mathcal{R}_n} \rho_n(y,k_n)\mathbb{E}\left[T_{\mathcal{P},n}(y)\right] f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

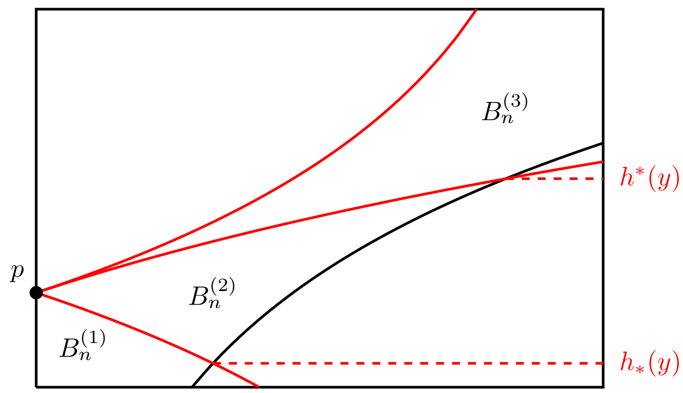


Figure 6: Three different areas $B_n^{(i)}$ used in the proof of Lemma 5.3.

$$
\begin{aligned}
&= (1 + o(1)) \int_{\mathcal{R}_n} \rho_n(y, k_n) \mathbb{E}\left[T_{\mathcal{P}}(y)\right] f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y \\
&= (1 + o(1)) \int_{\mathcal{R}_n} \rho(y, k_n) \mathbb{E}\left[T_{\mathcal{P}}(y)\right] f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y \\
&= (1 + o(1)) \alpha n \int_{\mathbb{R}_+} \rho(y, k_n) \mathbb{E}\left[T_{\mathcal{P}}(y)\right] e^{-\alpha y} \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}
$$

□

5.2 Counting missing triangles

We now come back to computing the expected number of triangles attached to node at height y in $G_{\mathcal{P},n}(\alpha, \nu)$ that are not present in $G_{\mathcal{P}}(\alpha, \nu)$.

Proof of Lemma 5.3. Due to symmetry it is enough to show that

$$
\int_0^{R_n} \int_0^{I_n} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P} \Delta \mathcal{P}_n}(p, p_1)\right) f_{\mu,\nu}(x_1, y_1) \, \mathrm{d}x_1 \, \mathrm{d}y_1 = O\left(y n^{-(2\alpha - 1)} + n^{-(2\alpha-1)} e^y\right) \tag{51}
$$

The proof goes in two stages. First we compute $\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P} \Delta \mathcal{P}_n}(p, p_1)\right)$ by splitting it over three disjoint regimes with respect to p_1, with $x_1 \geq 0$. Then we do the integration with respect to p_1.

Computing $\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P} \Delta \mathcal{P}_n}(p, p_1)\right)$

Pim: I have currently used that $\alpha < 1$, which is the most difficult case. The case $\alpha = 1$ and $\alpha > 1$ will be added.

Define the sets

$$
\begin{aligned}
A_n^{(1)} &= \left\{p_1 \in \mathcal{R}_n \,:\, 0 \leq y_1 \leq y - 2\log(I_n/(I_n - x_1))\right\}, \\
A_n^{(2)} &= \left\{p_1 \in \mathcal{R}_n \,:\, y - 2\log(I_n/(I_n - x_1)) < y_1 \leq y + 2\log\left(1 + \frac{x_1}{I_n}\right)\right\}, \\
A_n^{(3)} &= \left\{p_1 \in \mathcal{R}_n \,:\, y + 2\log\left(1 + \frac{x_1}{I_n}\right) < y_1 \leq y + 2\log\left(\frac{I_n}{I_n - x_1}\right)\right\},
\end{aligned}
$$

and let $B_n^{(i)} = \mathcal{B}_{\mathcal{P},n}(p) \cap A_n^{(i)}$, for $i = 1, 2, 3$, see Figure 7. Here the heights of the two intersections are given by

$$
h_*(y) = y + 2\log\left(\frac{I_n}{I_n + e^y}\right) \tag{52}
$$

$$h^*(y) = y + 2\log\left(\frac{I_n}{I_n - e^y}\right). \tag{53}$$

With these definitions we have that the union $B_n := \bigcup_{i=1}^n B_n^{(i)}$ denotes the area under the red curve in Figure 6 and hence, for all $p_1 \in \mathcal{R}_n \setminus B_n$ with $x_1 \geq 0$ we have that $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1) = \emptyset$. So we only need to consider $p_1 \in B_n$. We shall establish the following result:

$$\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)\right) = \begin{cases} O\left(I_n^{-2\alpha} e^{\alpha y_1}\right) & \text{if } p_1 \in B_n^{(1)} \\ O\left(I_n^{-2\alpha} e^{\alpha y}\right) & \text{if } p_1 \in B_n^{(2)} \cup B_n^{(3)} \end{cases} \tag{54}$$

Depending on which regime p_1 belongs to, the set $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)$ has a different shape. We displayed these shapes in Figure 8 as a visual aid to follow the computations below.

Regime 1: $0 \leq y_1 \leq y - 2\log(I_n/(I_n - x_1))$ In this case the integral over p_2 splits into two parts

$$\mathcal{I}_n^{(1)}(p_1) := \int_{h_2(p_1)}^{y^*(p_1)} \int_{-I_n}^{x_1 + e^{(y_1+y_2)/2} - 2I_n} e^{-\alpha y_2} \, dx_2 \, dy_2$$
$$\mathcal{I}_n^{(2)}(p_1) := \int_{y^*(p_1)}^{h_1(p_1)} \int_{x^*(p_1)}^{x_1 - e^{(y_1+y_2)/2}} e^{-\alpha y_2} \, dx_2 \, dy_2.$$

We first compute $\mathcal{I}_n^{(1)}$.

$$\begin{aligned}
\mathcal{I}_n^{(1)}(p_1) &= \int_{h_2(p_1)}^{y^*(p_1)} \left(x_1 + e^{(y_1+y_2)/2} - I_n\right) e^{-\alpha y_2} \, dx_2 \, dy_2 \\
&\leq e^{y_1/2} \int_{h_2(p_1)}^{y^*(p_1)} e^{-(\alpha - \frac{1}{2})y_2} \, dy_2 \\
&= \frac{2e^{y_1/2}}{2\alpha - 1} \left(e^{-(\alpha-\frac{1}{2})h_2(p_1)} - e^{-(\alpha-\frac{1}{2})y^*(p_1)}\right) \\
&= \frac{2e^{\alpha y_1}}{2\alpha - 1} I_n^{-(2\alpha-1)} \left(\left(1 - \frac{x_1}{I_n}\right)^{-(2\alpha-1)} - 1\right) \\
&= O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\right),
\end{aligned}$$

where we used that $x' \leq e^{(y+y_1)/2} = o(I_n)$ for all $y_1 \leq y$ and $y \in \mathcal{K}_\varepsilon(k_n)$ so that

$$\left(\left(1 - \frac{x_1}{I_n}\right)^{-(2\alpha-1)} - 1\right) = O\left(\frac{x'}{I_n}\right) \quad \text{as } n \to \infty.$$

For $\mathcal{I}_n^{(2)}(p_1)$ we have

$$\begin{aligned}
\mathcal{I}_n^{(2)}(p_1) &= \int_{y^*(p_1)}^{h_1(p_1)} \left(I_n + x_1 - e^{(y_1+y_2)}\right) e^{-\alpha y_2} \, dx_2 \, dy_2 \\
&\leq 2I_n \int_{y^*(p_1)}^{h_1(p_1)} e^{-\alpha y_2} \, dx_2 \, dy_2 \\
&= \frac{2}{\alpha} I_n \left(I_n^{-2\alpha} e^{\alpha y_1} - (I_n + x_1)^{-2\alpha} e^{-\alpha y_1}\right) \\
&= O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\right) = O\left(I_n^{-(2\alpha-1)} e^{\alpha y_1}\right).
\end{aligned}$$

We conclude that for $p_1 \in B_n^{(1)}$:

$$\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)\right) = O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\right),$$

which establishes the first part of (54).

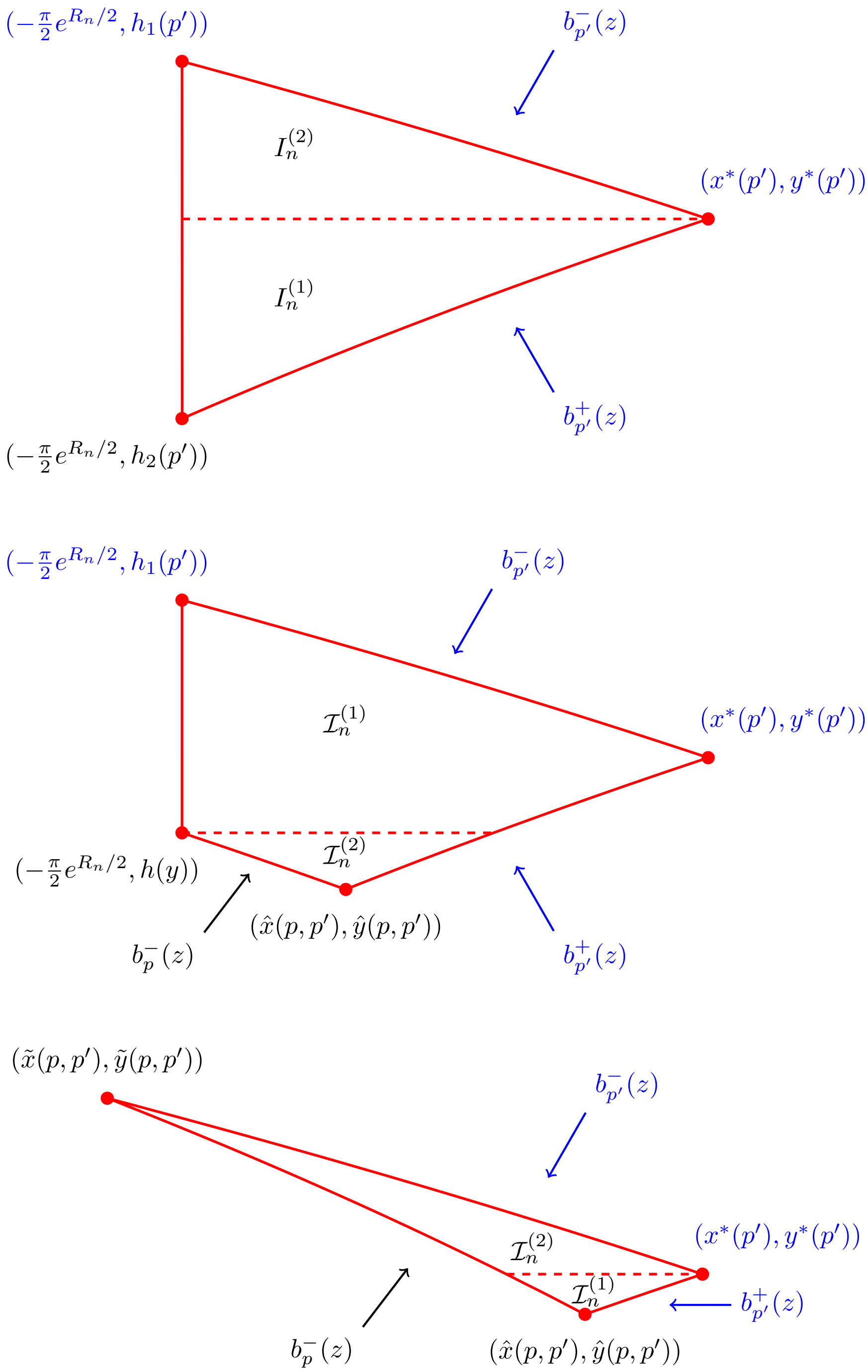


Figure 7: The different shapes of $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)$ depending on the regime to which p_1 belongs. The top figure is for $p_1 \in B_n^{(1)}$, the middle for $p_1 \in B_n^{(2)}$ and the bottom one for $p_1 \in B_n^{(3)}$.

Regime 2: $y - 2\log(I_n/(I_n - x_1)) < y_1 \leq y + 2\log\left(1+\frac{x_1}{I_n}\right)$ Here we split the integration into three parts (see Figure 8). Recall that $x^*(p,p_1) = x_1 - I_n$. Then, for the first part we have

$$\begin{aligned}
\mathcal{I}_n^{(1)}(p,p_1) &\leq \int_{h(y)}^{h_1(p_1)} \int_{-I_n}^{x^*(p,p_1)} f_{\alpha,\nu}(x_2,y_2)\, \mathrm{d}x_2\, \mathrm{d}y_2 \\
&= O\left(x_1\left(e^{-\alpha h(y)} - e^{-\alpha h_1(p_1)}\right)\right) \\
&= O\left(x_1 I_n^{-2\alpha}\left(e^{\alpha y} - e^{\alpha y_1}\left(1+\frac{x_1}{I_n}\right)^{-2\alpha}\right)\right) \\
&= O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\left(\left(1-\frac{x_1}{I_n}\right)^{-2\alpha} - \left(1+\frac{x_1}{I_n}\right)^{-2\alpha}\right)\right) \\
&= O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\right) = O\left(I_n^{-(2\alpha-1)} e^{\alpha y}\right),
\end{aligned}$$

were we used that $y \leq y_1 + 2\log(I_n/(I_n - x_1))$ for $p_1 \in B_n^{(2)}$ for the third line and

$$\left(1-\frac{x_1}{I_n}\right)^{-2\alpha} - \left(1+\frac{x_1}{I_n}\right)^{-2\alpha} = O\left(\frac{x_1}{I_n}\right) = O(1),$$

for the last line.

For the second part we first compute that

$$\begin{aligned}
x_1 + e^{(y_1+y_2)/2} - 2I_n + e^{(y+y_2)/2} &\leq \left(e^{y/2} + e^{y_1/2}\right) e^{y_2/2} \\
&\leq e^{y/2}\left(1 + \frac{I_n}{I_n - e^y}\right) e^{y_2/2} = O\left(e^{(y+y_2)/2}\right),
\end{aligned}$$

since $y \in \mathcal{K}_\varepsilon(k_n)$ and $k_n = o(\sqrt{n})$, so that $e^y = o(n) = o(I_n)$. Then we have

$$\begin{aligned}
\mathcal{I}_n^{(2)} &= \int_{\hat{y}(p,p_1)}^{h(y)} \int_{-e^{(y+y_2)/2}}^{x_1 + e^{(y+y_1)/2} - 2I_n} f_{\alpha,\nu}(x_2,y_2)\, \mathrm{d}x_2\, \mathrm{d}y_2 \\
&= O\left(e^{y/2} \int_{\hat{y}(p,p_1)}^{h(y)} e^{-(\alpha-\frac{1}{2})y_2}\, \mathrm{d}y_2\right) \\
&= O\left(e^{y/2}\left(e^{-(\alpha-\frac{1}{2})\hat{y}(p,p_1)} - e^{-(\alpha-\frac{1}{2})h(y)}\right)\right) \\
&= O\left(e^{y/2}\left(\left(\frac{2I_n - x_1}{e^{y/2}+e^{y_1/2}}\right)^{-(2\alpha-1)} - I_n^{-(2\alpha-1)} e^{(\alpha-\frac{1}{2})y}\right)\right) \\
&= O\left(I_n^{-(2\alpha-1)} e^{\alpha y}\right),
\end{aligned}$$

where for the last line we first used that $(2I_n - x_1)^{-(2\alpha-1)} \leq I_n^{-(2\alpha-1)}$ and then

$$\left(\left(e^{y/2} + e^{y_1/2}\right)^{2\alpha-1} - e^{(\alpha-\frac{1}{2})y}\right) \leq e^{(\alpha-\frac{1}{2})y}\left(\left(1+\sqrt{1+\frac{x_1}{I_n}}\right)^{2\alpha-1} - 1\right) = O\left(e^{(\alpha-\frac{1}{2})y}\right).$$

It then follows that for $p_1 \in B_n^{(2)}$

$$\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) = O\left(I_n^{-(2\alpha-1)} e^{\alpha y}\right).$$

Regime III $p_1 \in B_n^{(3)}$:

$$\begin{aligned}
\mathcal{I}_n^{(1)} &= \int_{y^*}^{\tilde{y}} \int_{-e^{(y+y_2)/2}}^{x_1 - e^{(y_1+y_2)/2}} f_{\alpha,\nu}(x_2, y_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&= O\left(\int_{y^*}^{\tilde{y}} x_1 e^{-\alpha y_2} - \left(e^{y_1/2} - e^{y/2} \right) e^{-(\alpha - \frac{1}{2}) y_2} \, \mathrm{d}y_2 \right) \\
&= O\left(x_1 \int_{y^*}^{\tilde{y}} e^{-\alpha y_2} \, \mathrm{d}y_2 \right).
\end{aligned}$$

Now

$$\begin{aligned}
\int_{y^*}^{\tilde{y}} e^{-\alpha y_2} \, \mathrm{d}y_2 = \frac{1}{\alpha} \left(e^{-\alpha y^*} - e^{-\alpha \tilde{y}} \right) &= \frac{1}{\alpha} \left(I_n^{-2\alpha} e^{\alpha y_1} - \left(\frac{x_1}{e^{y_1/2} - e^{y/2}} \right)^{-2\alpha} \right) \\
&= \frac{I_n^{-2\alpha} e^{\alpha y_1}}{\alpha} \left(1 - \left(1 - e^{(y-y_1)/2} \right)^{2\alpha} \left(\frac{x_1}{I_n} \right)^{-2\alpha} \right) = O\left(I_n^{-2\alpha} e^{\alpha y_1} \right),
\end{aligned}$$

and hence we have

$$\mathcal{I}_n^{(1)} = O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1} \right).$$

For the second integral we have

$$\begin{aligned}
\mathcal{I}_n^{(2)} &= \int_{\hat{y}}^{y^*} \int_{-e^{(y+y_2)/2}}^{e^{(y_1+y_2)/2} + x_1 - 2I_n} f_{\alpha,\nu}(x_2, y_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&= O\left(\int_{\hat{y}}^{y^*} \left(e^{y/2} + e^{y_1/2} \right) e^{-(\alpha - \frac{1}{2}) y_2} \, \mathrm{d}y_2 \right) \\
&= O\left(e^{y_1/2} \int_{\hat{y}}^{y^*} e^{-(\alpha - \frac{1}{2}) y_2} \, \mathrm{d}y_2 \right).
\end{aligned}$$

For the integral we have

$$\begin{aligned}
\int_{\hat{y}}^{y^*} e^{-(\alpha - \frac{1}{2}) y_2} \, \mathrm{d}y_2 &= \frac{2}{2\alpha - 1} \left(e^{-(\alpha - \frac{1}{2}) \hat{y}} - e^{-(\alpha - \frac{1}{2}) y^*} \right) \\
&= \frac{2}{2\alpha - 1} \left(\left(\frac{2I_n - x_1}{e^{y/2} + e^{y_1/2}} \right)^{-(2\alpha - 1)} - I_n^{-(2\alpha - 1)} e^{-(\alpha - \frac{1}{2}) y_1} \right) \\
&= O\left(I_n^{-2\alpha} x_1 e^{-(\alpha - \frac{1}{2}) y_1} \right)
\end{aligned}$$

so that

$$\mathcal{I}_n^{(2)} = O\left(I_n^{-(2\alpha - 1)} e^{(1-\alpha) y_1} \right) = O\left(I_n^{-2\alpha} x_1 e^{\alpha y} \right)$$

and hence for $p_1 \in B_n^{(3)}$

$$\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P} \Delta \mathcal{P}_n}(p, p_1) \right) = O\left(I_n^{-2\alpha} x_1 e^{\alpha y} \right) = O\left(I_n^{-(2\alpha - 1)} e^{\alpha y} \right).$$

Integration over p_1

We now proceed with the second part of the computation leading to (51). Here we will integrate $\mu_{\alpha,\nu}(\mathcal{T}_{\mathcal{P} \Delta \mathcal{P}_n})(p, p_1)$ over the region $B_n := B_n^{(1)} \cup B_n^{(2)} \cup B_n^{(3)}$, see Figure 7. Let us first identify the boundaries of these areas.

The area $B_n^{(1)}$ is bounded from above by the line given by the equation

$$y_1 = y - 2\log\left(\frac{I_n}{I_n - x_1}\right).$$

Solving this for x_1 yields $x_1 = I_n\left(1 - e^{(y_1-y)/2}\right)$ and hence the area $B_n^{(1)}$ is given by

$$B_n^{(1)} = \left\{(x_1, y_1) \,:\, 0 \le y_1 \le y, \quad 0 \le x_1 \le I_n\left(1 - e^{(y_1-y)/2}\right) \wedge e^{(y+y_1)/2}\right\}.$$

In a similar way we have that $B_n^{(2)}$ is bounded from above by line

$$y_1 = y + 2\log\left(\frac{I_n}{I_n + x_1}\right),$$

which yields $x_1 = I_n\left(e^{(y_1-y)/2} - 1\right)$. The lower red boundary is the upper boundary of $B_n^{(2)}$ and hence we have

$$B_n^{(2)} = \left\{(x_1, y_1) \,:\, h_*(y) \le y_1 \le h^*(y),\ I_n\left(1 - e^{(y_1-y)/2}\right) \vee I_n\left(e^{(y_1-y)/2} - 1\right) \le x_1 \le e^{(y+y_1)/2}\right\}.$$

We continue is the same way to obtain for $B_n^{(3)}$

$$B_n^{(3)} = \left\{(x_1, y_1) \,:\, y \le y_1 \le R_n,\ I_n\left(1 - e^{(y-y_1)/2}\right) \le x_1 \le I_n\left(e^{(y_1-y)/2} - 1\right) \wedge e^{(y+y_1)/2} \wedge I_n\right\}.$$

We these characterizations of the areas we now integrate $\mu_{\alpha,\nu}(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n})(p, p_1)$ over B_n, splitting the computations over the three different areas.

$\boldsymbol{p_1 \in B_n^{(1)}}$: We use that $I_n\left(1 - e^{(y_1-y)/2}\right) \wedge e^{(y+y_1)/2} \le I_n\left(1 - e^{(y_1-y)/2}\right)$ so that

$$\begin{aligned}
&\int_{B_n^{(1)}} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)\right) f_{\alpha,\nu}(x_1, y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&\qquad \le \int_0^y \int_0^{I_n(1-e^{(y_1-y)/2})} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)\right) f_{\alpha,\nu}(x_1, y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&\qquad = O\left(I_n^{-2\alpha} \int_0^y \int_0^{e^{(y+y_1)/2}} x_1\, \mathrm{d}x_1\, \mathrm{d}y_1\right) \\
&\qquad = O\left(I_n^{-(2\alpha-1)} \int_0^y \left(1 - e^{(y_1-y)/2}\right)^2 \mathrm{d}y_1\right) \\
&\qquad = O\left(I_n^{-(2\alpha-1)} y\right) = O\left(y n^{-(2\alpha-1)}\right).
\end{aligned}$$

$\boldsymbol{p_1 \in B_n^{(2)}}$: We will show that

$$\mu_{\alpha,\nu}(B_n^{(2)}) = O\left(I_n^{-1} e^{(2-\alpha)y}\right), \tag{55}$$

which together with (54) yields

$$\begin{aligned}
\int_{B_n^{(2)}} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)\right) f_{\alpha,\nu}(x_1, y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 &= O\left(\mu_{\alpha,\nu}(B_n^{(2)}) I_n^{-(2\alpha-1)} e^{\alpha y}\right) \\
&= O\left(I_n^{-2\alpha} e^{2y}\right).
\end{aligned}$$

The integration is split into two parts determined by $I_n\left(1 - e^{(y_1-y)/2}\right) \vee I_n\left(e^{(y_1-y)/2} - 1\right)$:

$$\mu_{\alpha,\nu}(B_n^{(3)}) = \int_{h_*(y)}^y \int_{I_n(1-e^{(y_1-y)/2})}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1, y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1$$

$$+\int_{y}^{h^*(y)}\int_{I_n(e^{(y_1-y)/2}-1)}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1.$$

For the first integral we use that $e^{(y+y_1)/2}-I_n(1-e^{(y_1-y)/2})\le e^{y_1/2}\left(e^{y/2}+e^{-y/2}\right)$ to obtain

$$\begin{aligned}
&\int_{h_*(y)}^{y}\int_{I_n(1-e^{(y_1-y)/2})}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\\
&\quad= O\left(e^{y/2}\int_{h_*(y)}^{y} e^{-(\alpha-\frac{1}{2})y_1}\,\mathrm{d}y_1\right)\\
&\quad= O\left(e^{y/2}\left(e^{-(\alpha-\frac{1}{2})y}-e^{-(\alpha-\frac{1}{2})y}\left(\frac{I_n}{I_n+e^y}\right)^{-(2\alpha-1)}\right)\right)\\
&\quad= O\left(I_n^{-1}e^{(2-\alpha)y}\right).
\end{aligned}$$

For the second integral note that $e^{(y+y_1)/2}-I_n(e^{(y_1-y)/2}-1)\le e^{(y+y_1)/2}$ and hence

$$\begin{aligned}
&\int_{y}^{h^*(y)}\int_{I_n(e^{(y_1-y)/2}-1)}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\\
&\quad= O\left(e^{y/2}\int_{y}^{h^*(y)} e^{-(\alpha-\frac{1}{2})y_1}\,\mathrm{d}y_1\right)\\
&\quad= O\left(e^{y/2}\left(e^{-(\alpha-\frac{1}{2})y}-e^{-(\alpha-\frac{1}{2})y}\left(\frac{I_n}{I_n-e^y}\right)^{-(2\alpha-1)}\right)\right)\\
&\quad= O\left(I_n^{-1}e^{(2-\alpha)y}\right),
\end{aligned}$$

so that (55) follows.

$\boldsymbol{p_1\in B_n^{(3)}}$: For this area we show that

$$\mu_{\alpha,\nu}(B_n^{(3)})=O\left(e^{(1-\alpha)y}\right) \tag{56}$$

so that

$$\begin{aligned}
\int_{B_n^{(3)}}\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right)f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 &= O\left(\mu_{\alpha,\nu}(B_n^{(2)})I_n^{-(2\alpha-1)}e^{\alpha y}\right)\\
&= O\left(I_n^{-(2\alpha-1)}e^{y}\right).
\end{aligned}$$

Here the integral is split into three parts:

$$\begin{aligned}
\mu_{\alpha,\nu}(B_n^{(3)}) &= \int_{y}^{h^*(y)}\int_{I_n(1-e^{(y-y_1)/2})}^{I_n(e^{(y_1-y)/2}-1)} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\\
&\quad+\int_{h^*(y)}^{h(y)}\int_{I_n(1-e^{(y-y_1)/2})}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\\
&\quad+\int_{h(y)}^{R_n}\int_{I_n(1-e^{(y-y_1)/2})}^{I_n} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1.
\end{aligned}$$

Let us first focus on the first integral. Since $I_n(e^{(y_1-y)/2}-1)-I_n(1-e^{(y-y_1)/2})\le I_ne^{(y_1-y)/2}$ we get, using similar arguments as above

$$\int_{y}^{h^*(y)}\int_{I_n(1-e^{(y-y_1)/2})}^{I_n(e^{(y_1-y)/2}-1)} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 = O\left(I_ne^{-y/2}\int_{y}^{h^*(y)} e^{-(\alpha-\frac{1}{2})y_1}\,\mathrm{d}y_1\right)$$

$$= O\left(I_n e^{-\alpha y}\left(1-\left(\frac{I_n}{I_n - e^y}\right)^{-(2\alpha-1)}\right)\right)$$
$$= O\left(e^{(1-\alpha)y}\right).$$

Proceeding to the second integral, we first note that $e^{(y+y_1)/2} - I_n(1-e^{(y-y_1)/2}) = O\left(I_n e^{(y_1-y)/2}\right)$ so that similar calculations as before yield

$$\int_{h^*(y)}^{h(y)} \int_{I_n(1-e^{(y-y_1)/2})}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 = O\left(I_n e^{-y/2} \int_{h^*(y)}^{h(y)} e^{-(\alpha-\frac{1}{2})y_1}\, \mathrm{d}y_1\right) = O\left(e^{(1-\alpha)y}\right).$$

□

6 Concentration for $c_{\mathcal{P},n}(k)$ (Proving Proposition 2.6)

6.1 The main contribution of triangles

Recall the adjusted triangle count function (47)

$$\widetilde{T}_{\mathcal{P},n}(p_0,p_1,p_2) = \mathbb{1}_{\{p_1 \in B_{\mathcal{P},n}(p)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P},n}(p)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P}}(p_1)\cap \mathcal{R}_n\}}.$$

and the concentration set

$$\mathcal{K}_C(k) = \left\{p \in \mathbb{R} : \frac{k - C\sqrt{k\log(k)}}{\xi_{\alpha,\nu}} \le e^{\frac{y}{2}} \le \frac{k + C\sqrt{k\log(k)}}{\xi_{\alpha,\nu}}\right\},$$

We will define the corresponding triangle degree function

$$\widetilde{T}_{\mathcal{P},n}(k,C) = \sum_{p \in \mathcal{P}_n \cap \mathcal{K}_C(k)} \mathbb{1}_{\{D_{\mathcal{P},n}(p)=k\}} \widetilde{T}_{\mathcal{P},n}(p), \tag{57}$$

where

$$\widetilde{T}_{\mathcal{P},n}(p) := \sum_{(p_1,p_2)\in 2^{\mathcal{P}_n}} \widetilde{T}_{\mathcal{P},n}(p,p_1,p_2).$$

By Lemma 5.3 and a concentration argument it follows that for $k_n \to \infty$

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,C)\right] = (1+o(1))\mathbb{E}\left[T_{\mathcal{P}}(k_n)\right],$$

for a appropriately selected $C > 0$. We conclude that the main contribution of triangles of degree k_n is given by $\widetilde{T}_{\mathcal{P},n}(k_n,C)$. Therefore, in order to prove Proposition 2.6 we need to show that $\widetilde{T}_{\mathcal{P},n}(k_n,C)$ is sufficiently concentrated around its mean.

6.2 Proving Proposition 2.6

We start with the concentration result for $\widetilde{T}_{\mathcal{P},n}(k_n,C)$.

Proposition 6.1 (Concentration $\widetilde{T}_{\mathcal{P},n}(k_n,C)$)**.** *Let* $\alpha > \frac{1}{2}$, $0 < \varepsilon < \min\{2\alpha-1, 1\}$ *and let* $(k_n)_{n\ge 1}$ *be any increasing sequence satisfying* $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. *Then, as* $n \to \infty$,

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,C)^2\right] = (1+o(1))\,\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,C)\right]^2.$$

The proof of this proposition is lengthy and we therefore postpone it till Section 6.3. The remaining of this section will be devoted to prove Proposition 2.6. We first show that $\mathbb{E}\left[c_{\mathcal{P},n}(k_n)\right] = (1+o(1))\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]$.

Lemma 6.2. *Let $\nu > 0$, $\alpha > \frac{1}{2}$ and $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then, as $n \to \infty$,*

$$\mathbb{E}\left[\left|c^*_{\mathcal{P},n}(k_n) - c_{\mathcal{P},n}(k_n)\right|\right] = o\left(\mathbb{E}\left[c_{\mathcal{P},n}(k_n)^*\right]\right).$$

Proof. Let $0 < \delta < 1$ and define the following two events

$$A_n = \left\{\left|N_{\mathcal{P},n}(k_n) - \mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]\right| \leq \mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]^{\frac{1+\delta}{2}}\right\}$$
$$B_n = \left\{\left|N_{\mathcal{P},n}(k_n) - \mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]\right| \leq n\right\},$$

so that

$$1 = \mathbb{1}_{\{A_n\}} + \mathbb{1}_{\{A_n^c\}}\mathbb{1}_{\{B_n\}} + \mathbb{1}_{\{B_n^c\}}.$$

Next we note that on the event A_n

$$\left|\frac{\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}{N_{\mathcal{P},n}(k_n)} - 1\right| \leq \frac{\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]^{\frac{1+\delta}{2}}}{\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] + \mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]^{\frac{1+\delta}{2}}} \leq \mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]^{-\frac{1-\delta}{2}},$$

and on the event B_n

$$\left|\frac{\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}{N_{\mathcal{P},n}(k_n)} - 1\right| \leq \frac{n}{\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] + n} \leq 1.$$

Therefore we have

$$\begin{aligned}\mathbb{E}\left[\left|c^*_{\mathcal{P},n}(k_n) - c_{\mathcal{P},n}(k_n)\right|\right] &= \mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\left|\frac{\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}{N_{\mathcal{P},n}(k_n)} - 1\right|\right] \\ &\leq \mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\mathbb{1}_{\{A_n\}}\right]\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]^{-\frac{1-\delta}{2}} \\ &\quad + \mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\mathbb{1}_{\{A_n^c\}}\mathbb{1}_{\{B_n\}}\right] \\ &\quad + \mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\left|\frac{\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}{N_{\mathcal{P},n}(k_n)} - 1\right|\mathbb{1}_{\{B_n^c\}}\right]\end{aligned}$$

Since $\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] \to \infty$, the first term is clearly $o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right)$. For the third term we have

$$\begin{aligned}\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\left|\frac{\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}{N_{\mathcal{P},n}(k_n)} - 1\right|\mathbb{1}_{\{B_n^c\}}\right] &= O\left(\mathbb{P}\left(B_n^c\right)\right) = O\left(\mathbb{P}\left(\left|N_{\mathcal{P},n}(k_n) - \mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]\right| > n\right)\right) \\ &= O\left(e^{-\frac{n^2}{\mathbb{E}\left[N_{k_n}\right]+n}}\right) = O\left(e^{-n}\right) = o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right).\end{aligned}$$

Hence we are left to show that

$$\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\mathbb{1}_{\{A_n^c\}}\mathbb{1}_{\{B_n\}}\right] = o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right). \tag{58}$$

By writing

$$c^*_{\mathcal{P},n}(k_n) = \frac{\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} + \frac{T_{\mathcal{P},n}(k_n,\varepsilon) - \widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}$$

and using Lemma 5.3 we get

$$\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\mathbb{1}_{\{A_n^c\}}\mathbb{1}_{\{B_n\}}\right] \leq \mathbb{E}\left[\frac{(1+o(1))\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}\mathbb{1}_{\{A_n^c\}}\right] + o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right).$$

For the last step we use Hölder's inequality and Proposition 6.1 to get

$$\mathbb{E}\left[\frac{\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}\mathbb{1}_{\{A_n^c\}}\right] \leq \mathbb{E}\left[\left(\frac{\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]}\right)^2\right]^{\frac{1}{2}}\mathbb{P}\left(A_n^c\right)^{\frac{1}{2}}$$

$$= (1+o(1)) \frac{\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right]}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} \mathbb{P}\left(A_n^c\right)^{\frac{1}{2}}$$
$$= O\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right) \mathbb{P}\left(A_n^c\right)^{\frac{1}{2}} = o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right),$$

since $\mathbb{P}(A_n) = o(1)$. This establishes (58) and hence we conclude that

$$\mathbb{E}\left[\left|c^*_{\mathcal{P},n}(k_n) - c_{\mathcal{P},n}(k_n)\right|\right] = o\left(\mathbb{E}\left[c_{\mathcal{P},n}(k_n)^*\right]\right).$$

□

We are now in shape to prove the main result of this section.

Proof of Proposition 2.6. Again, we write

$$c^*_{\mathcal{P},n}(k_n) = \frac{\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} + \frac{\left(T_{\mathcal{P},n}(k_n) - \widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right)}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]},$$

so that by Lemma 5.3,

$$\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right] = \frac{\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right]}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} + o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right).$$

Therefore, by Lemma 6.2,

$$\begin{aligned}\mathbb{E}\left[\left|c_{\mathcal{P},n}(k_n) - \mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right|\right] &\le \mathbb{E}\left[\left|c^*_{\mathcal{P},n}(k_n) - \mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right|\right] + \mathbb{E}\left[\left|c_{\mathcal{P},n}(k_n) - c^*_{\mathcal{P},n}(k_n)\right|\right] \\ &= \frac{\mathbb{E}\left[\left|\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon) - \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right]\right|\right]}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} + o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right). \end{aligned} \tag{59}$$

Next, we use Proposition 6.1 to obtain

$$\begin{aligned}\mathbb{E}\left[\left|\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon) - \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right]\right|\right] &\le \left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)^2\right] - \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right]^2\right)^{\frac{1}{2}} \\ &= o\left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right]\right).\end{aligned}$$

This implies

$$\frac{\mathbb{E}\left[\left|\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon) - \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right]\right|\right]}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} = o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right),$$

which together with (59) finishes the proof. □

6.3 Concentration for main triangle contribution

We now turn to Proposition 6.1. Before we dive into the proof let us first give a high level overview of the strategy and the flow of the arguments.

Recall (see (57)) that for any $C > 0$

$$\widetilde{T}_{\mathcal{P},n}(k,C) = \sum_{p \in \mathcal{P}_n \cap \mathcal{K}_C(k)} \mathbb{1}_{\{D_{\mathcal{P},n}(p) = k\}} \widetilde{T}_{\mathcal{P},n}(p)$$

Then we have

$$\widetilde{T}_{\mathcal{P},n}(k,C)^2 = \sum_{p,p' \in \mathcal{P}_n \cap \mathcal{K}_C(k)} \mathbb{1}_{\{D_{\mathcal{P},n}(p),\, D_{\mathcal{P},n}(p') = k\}} \sum_{(p_1,p_2),(p_1',p_2') \in 2^{\mathcal{P}_n}} T_{\mathcal{P}}(p,p_1,p_2) T_{\mathcal{P}}(p',p_1',p_2'),$$

with $2^{\mathcal{P}_n}$ denoting the distinct pairs in $\mathcal{P}_n$. This expression can be written as the sums of several terms, depending on how $\{p, p_1, p_2\}$ and $\{p', p_1', p_2'\}$ intersect. To this end we define, for $a \in \{0,1\}$ and $b \in \{0,1,2\}$,

$$I_{a,b} = \sum_{\substack{p,p' \in \mathcal{P}_n \cap \mathcal{K}_C(k) \\ |\{p\} \cap \{p'\}| = a}} \mathbb{1}_{\{D_{\mathcal{P},n}(p),\, D_{\mathcal{P},n}(p') = k\}} J_b(p, p'),$$

where

$$J_b(p,p') = \sum_{\substack{(p_1,p_2),(p_1',p_2') \in 2^{\mathcal{P}_n} \\ |\{p_1,p_2\} \cap \{p_1',p_2'\}| = b}} T_{\mathcal{P},n}(p,p_1,p_2) T_{\mathcal{P},n}(p',p_1',p_2').$$

Then we have

$$\widetilde{T}_{\mathcal{P},n}(k,C)^2 = \sum_{a=0}^{1} \sum_{b=0}^{2} I_{a,b}.$$

To prove Proposition 6.1 we will deal with each of the $I_{a,b}$ separately, showing that

$$\mathbb{E}\left[I_{0,0}\right] = (1 + o(1)) \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n)\right]^2 \tag{60}$$

and for all other combinations

$$\mathbb{E}\left[I_{a,b}\right] = o\left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n)\right]^2\right). \tag{61}$$

Note that $J_b(p,p') \le J_0(p,p')$ and, since $I_{1,2} = \widetilde{T}_{\mathcal{P},n}(k_n, C)$, (61) holds for $I_{1,2}$.

Proof of Proposition 6.1. The technical steps of this proof use Lemma B.4 and Lemma B.5, which deal with the joint degree distribution. Let $\mathcal{N}^c_{\mathcal{P},n}(p,p')$ denote the number of nodes not connected to both p and p' an fix $0 < \varepsilon < (2\alpha - 1 \wedge 1)$. To be able to apply both lemmas, we first show that we only need to consider the case where $\mathbb{E}\left[|\mathcal{N}^c_{\mathcal{P},n}(p,p')|\right] \ge k_n^{\varepsilon}$ and $|x - x'| > k_n^{1+\varepsilon}$. To this end define

$$\mathcal{E}_{\varepsilon}(k_n) = \left\{(p,p') \in \mathcal{K}_C(k_n) \times \mathcal{K}_C(k_n) \,:\, \mathbb{E}\left[|\mathcal{N}^c_{\mathcal{P},n}(p,p')|\right] \ge k_n^{\varepsilon} \text{ and } |x - x'| > k_n^{1+\varepsilon}\right\}$$

and let $I^*_{a,b}$ denote the the part of $I_{a,b}$ where $p, p' \in \mathcal{P}_n \cap \mathcal{E}_{\varepsilon}(k_n)$. We first show that

$$\mathbb{E}\left[I_{a,b} - I^*_{a,b}\right] = o\left(\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]^2\right), \tag{62}$$

so that for the remainder of the proof we only need to consider $p, p' \in \mathcal{E}_{\varepsilon}(k_n)$ and hence, we can apply Lemma B.4 and Lemma B.5 to the joint degree distribution inside the integral.

Note that for all $p, p' \in \mathcal{K}_{\varepsilon}(k_n)$ we have that $e^{\frac{y^*}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right) = \Theta\left(k_n^2\right)$, where $y^* = \min\{y, y'\}$. Hence by Lemma B.1 and Lemma B.2 we have for $p, p' \in \mathcal{K}_{\varepsilon}(k_n)$ with $\mathbb{E}\left[|\mathcal{N}_{\mathcal{P},n}(p \Delta p')|\right] \le k_n^{\varepsilon}$ and $|x - x'| > k_n^{1+\varepsilon}$ that either

$$|x - x'| + \left|e^{\frac{y}{2}} - e^{\frac{y'}{2}}\right| = O\left(k_n^{\varepsilon}\right) \quad \text{or} \quad \mathbb{E}\left[|\mathcal{N}_{\mathcal{P},n}(p \Delta p')|\right] = \Omega(k_n).$$

In particular $|x - x'| \le k_n^{1+\varepsilon}$ for all $(p,p') \notin \mathcal{E}_{\varepsilon}(k_n)$. Therefore, we have

$$\begin{aligned}
&\mathbb{E}\left[I_{a,b} - I^*_{a,b}\right] \\
&\le \int_{\mathcal{K}_C(k_n)^2 \setminus \mathcal{E}_{\varepsilon}(k_n)} \rho(p,p',k-i,k-i') \mathbb{E}\left[J_b(p,p')\right] f_{\alpha,\nu}(x,y) f_{\alpha,\nu}(x',y')\, dx'\, dx\, dy'\, dy \\
&\le \int_{\mathcal{K}_C(k_n)^2 \setminus \mathcal{E}_{\varepsilon}(k_n)} \rho(p,p',k-i,k-i') \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right] \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right] f_{\alpha,\nu}(x,y) f_{\alpha,\nu}(x',y')\, dx'\, dx\, dy'\, dy
\end{aligned}$$

$$= O\left(\int_{\mathcal{K}_C(k_n)^2} \mathbb{1}_{\{|x-x'|\le k_n^{1+\varepsilon}\}}\rho_y(k)\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy\right)$$
$$= O\left(k_n^{1+\varepsilon}\binom{k_n}{2}\left(\int_{a_n^-}^{a_n^+}\Delta_{\mathcal{P}}(y')e^{-\alpha y'}\,dy'\right)\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]\right)$$
$$= O\left(k_n^{3+\varepsilon-2\alpha}s_\alpha(k_n)\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]\right)$$
$$= o\left(nk_n^{-(2\alpha-1)}s_\alpha(k_n)\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]\right) = o\left(\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]^2\right),$$

which proves (62). Here we used that $k_n^{2+\varepsilon} = o(n)$ and $\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right] = \Theta\left(nk_n^{-(2\alpha-1)}s_\alpha(k_n)\right)$ for the last line.

We will now proceed to establish (60) and (61). We split these up into separate subsections because each term requires different bounding techniques.

I) $I^*_{0,0}$: Let $i = |\{p', p_1, p_2, p_1', p_2'\} \cap B_{\mathcal{P},n}(p)|$ and $j = |\{p' \cap B_{\mathcal{P},n}(p)\}|$ and let i', j' be defined, similarly, by interchanging the primed and non-primed variables. Then, by Lemma B.4 and Lemma B.5

$$\rho_(p, p', k_n - i, k_n - i') = (1 + o(1))\rho(p, p', k_n - j, k_n - j') = (1 + o(1))\rho_n(p, k_n)\rho_n(p', k_n)$$

which now no longer depends on the other four points p_1, p_2, p_1', p_2'. Hence, using the Mecke formula, we get

$$\mathbb{E}\left[I^*_{0,0}\right] = \frac{1+o(1)}{4}\int_{\mathcal{E}_\varepsilon(k_n)} \rho_n(p,k_n)\rho_n(p',k_n)\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy,$$

where the factor $\frac{1}{4}$ is because we consider distinct pairs (p_1, p_2) and (p_1', p_2').

Next, by Lemma 5.3 and Proposition ?? we have for $y \in K_C(k_n)$,

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right] = (1+o(1))\mathbb{E}\left[\widetilde{T}_{\mathcal{P}}(p)\right] = (1+o(1))k_n^2\Delta_{\mathcal{P}}(y)$$

and similar result holds for p'. Hence

$$\frac{1+o(1)}{4}\int_{\mathcal{E}_\varepsilon(k_n)} \rho_{p,p'}(k-j,k-j')\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy$$
$$= (1+o(1))\binom{k_n}{2}^2\int_{\mathcal{E}_\varepsilon(k_n)} \rho_{p,p'}(k-j,k-j')\Delta_{\mathcal{P}}(y)\Delta_{\mathcal{P}}(y')f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy$$
$$= (1+o(1))\left(\binom{k_n}{2}\int_{a_n^-}^{a_n^+}\int_{I_n}\rho_{y,n}(k_n)\Delta_{\mathcal{P}}(y)f_{\alpha,\nu}(x,y)\,dx\,dy\right)^2$$
$$= (1+o(1))\left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right]\right)^2,$$

which proves (60).

II) $I^*_{0,1}$: Without loss of generality we will assume that $p_1 = p_1'$. Similar to the previous computations, we let $i = |\{p', p_1, p_2, p_2'\} \cap B_{\mathcal{P},n}(p)|$ and $j = |\{p'\} \cap B_{\mathcal{P},n}(p)|$ and let i', j' be defined, similarly, by interchanging the primed and non-primed variables. Then, if we define

$$T^{(0,1)}_{\mathcal{P},n}(p,p') = \sum_{(p_1,p_2)\in 2^{\mathcal{P}_n}}\sum_{p_2'\in\mathcal{P}_n}\widetilde{T}_{\mathcal{P},n}(p,p_1,p_2)\widetilde{T}_{\mathcal{P},n}(p',p_1,p_2'),$$

we have

$$\mathbb{E}\left[I^*_{0,1}\right] = \frac{1+o(1)}{2}\int_{\mathcal{E}_\varepsilon(k_n)} \rho_n(p,k)\rho_n(p',k)\mathbb{E}\left[T^{(0,1)}_{\mathcal{P},n}(p,p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy,$$

where we again used Lemma B.4 and Lemma B.5. We will show that

$$\mathbb{E}\left[T^{(0,1)}_{\mathcal{P},n}(p,p')\right] = o\left(k_n^4 s_\alpha(k_n)^2\right),$$

from which (61) follows since $k_n^4 s_\alpha(k_n)^2 = O\left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right]\right)$ on $\mathcal{K}_C(k_n) \times \mathcal{K}_C(k_n)$.

First we consider the contribution coming from $y_1 > 4\log(k_n)$. Since the integration of $T_{\mathcal{P}}(p,p_1,p_2)T_{\mathcal{P}}(p',p_1,p_2')$ over x_1, x_2 and x_2' is bounded by $O\left(e^y e^{\frac{y'}{2}} e^{\frac{y_1+y_2+y_2'}{2}}\right)$ it follows that contribution to $\mathbb{E}\left[T^{(0,1)}_{\mathcal{P},n}(p,p')\right]$ is bounded by

$$\begin{aligned}
&O\left(e^y e^{\frac{y'}{2}} \int_{4\log(k_n)}^{a_n^+} e^{-(\alpha-\frac{1}{2})y_1}\, dy_1\right)\\
&\qquad\qquad = O\left(k_n^3 \int_{4\log(k_n)}^{a_n^+} e^{-(\alpha-\frac{1}{2})y_1}\, dy_1\right)\\
&\qquad\qquad = O\left(k_n^{3-(4\alpha-2)}\right) = o\left(k_n^4 s_\alpha(k_n)^2\right).
\end{aligned}$$

To deal with the case where $y_1 \leq 4\log(k_n)$ we define $b_n = 2\varepsilon\log(k_n))$ and will consider different cases for $\mathbb{E}\left[T^*_{\mathcal{P},n}(p,p')\right]$, depending on whether $y_2 \leq b_n$ or $y_2 > b_n$ and similar for y_2'.

When $y_1 \leq 4\log(k_n)$ and $y_2 > b_n$, the contribution to $\mathbb{E}\left[T^{(0,1)}_{\mathcal{P},n}(p,p')\right]$ is bounded by

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right] O\left(e^{\frac{y'}{2}} \int_{b_n}^{a_n^+} e^{-(\alpha-\frac{1}{2})y_2}\, dy_2\right) = O\left(k_n^{1-\varepsilon}\right)\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right] = o\left(k_n^4 s_\alpha(k_n)^2\right).$$

Due to the symmetry in p_2 and p_2' the same results holds for the cases where $y_2 > b_n$.

Finally, when $y_1 \leq 4\log(k_n)$ and both $y_2, y_2' \leq b_n$ we have that

$$|x_2 - x_2'| \leq |x_1 - x_2| + |x_1 - x_2'| \leq e^{\frac{y_1}{2}}\left(e^{\frac{y_2}{2}} + e^{\frac{y_2'}{2}}\right) \leq 2k_n^{2+\varepsilon}$$

whenever $T_{\mathcal{P}}(p,p_1,p_2)T_{\mathcal{P}}(p',p_1,p_2') > 0$ while both $|x - x_2|, |x' - x_2'| = O\left(k_n^{1+\varepsilon}\right)$. Hence it follows that

$$|x - x'| \leq |x - x_2| + |x_2 - x_2'| + |x_2' - x'| = O\left(k_n^{2+\varepsilon}\right).$$

Next, by integrating only over x_2' and y_2' we get the contribution to $\mathbb{E}\left[T^{(0,1)}_{\mathcal{P},n}(p,p')\right]$ for this regime is bounded by

$$O\left(e^{\frac{y'}{2}}\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\right) = O\left(k_n\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\right) = o\left(k_n^4 s_\alpha(k_n)^2\right).$$

The proofs for the other two cases $I^*_{0,2}$ and $I^*_{0,2}$ follows similar arguments.

□

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7 Equivalence for local clustering in hyperbolic and Poissonized random graph

In this section we establish the equivalence between $c^*_{\mathbb{H},n}(k)$ and $c^*_{\mathcal{P},n}(k)$ as expressed in Proposition 2.5, using the coupling procedure explained in Section 2.3.

Recall that $\mathcal{P}_{\alpha,\nu}$ denotes a Poisson process on $\mathbb{R}\times\mathbb{R}_+$, with intensity $f_{\alpha,\nu}(x,y)$, $I_n = \left(-\frac{\pi}{2}e^{R_n/2}, \frac{\pi}{2}e^{R_n/2}\right)$, $\mathcal{R}_n = I_n \times (0, R_n]$ and $\mathcal{V}_n = \mathcal{P}_{\alpha,\nu} \cap \mathcal{R}_n$. In addition we define for any interval $I \subseteq \mathbb{R}_+$, $\mathcal{R}_n(I) := I_n \times I$ and denote by $\mathcal{B}_{\mathcal{P}}(p)$ the *ball*

$$\mathcal{B}_{\mathcal{P}}(p) = \left\{p' \in \mathcal{V}_n : |x - x'|_{\pi e^{R_n/2}} < e^{\frac{y+y'}{2}}\right\}.$$

Note that when $p \in \mathcal{V}_n$ then $\mathcal{B}_{\mathcal{P}}(p)$ denotes its neighborhood in the graph $G_{\mathcal{P},n}$. Note that the above definition implies that for all $y \in [0, R_n]$ we have

$$\mathcal{R}_n([R_n - y - 2\ln(\pi/2), R_n]) \subseteq \mathcal{B}_{\mathcal{P}}((0,y)) \tag{63}$$

- this is a fact which we are going to use several times in our analysis.

For any Borel-measurable subset $S \subseteq \mathbb{R} \times \mathbb{R}_+$, we let

$$\mu_{\alpha,\nu}(S) = \int_S f_{\alpha,\nu}(x,y)\,dx\,dy = \frac{\nu\alpha}{\pi}\int_S e^{-\alpha y}dy.$$

Thus, the number of points of $\mathcal{P}_{\alpha,\nu}$ inside S is distributed as $\mathrm{Po}(\mu_{\alpha,\nu,}(S))$.

Finally, we remind the reader that $\mathcal{B}_{\mathbb{H},n}(p)$ denotes the image under Ψ of the ball of hyperbolic radius R_n around the point $\Psi^{-1}(p)$ and that under the coupling between the hyperbolic random graph and the finite box model, described in Section 2.3, two point p and p' are connected if and only if

$$|x - x'|_{\pi e^{r_n/2}} \le \Omega(R_n - y, R_n - y'),$$

where the function Ω can be approximated, for $y + y' < R_n$, using Lemma 2.2 by

$$e^{\frac{1}{2}(y+y')} - Ke^{\frac{3}{2}(y+y')-R_n} \le \Omega(r,r') \le e^{\frac{1}{2}(y+y')} + Ke^{\frac{3}{2}(y+y')-R_n}.$$

To prove Proposition 2.5 we calculate the error in two steps. First we show in Section 7.2 that

$$\lim_{n\to\infty} s_\alpha(k_n)\mathbb{E}\left[\left|c^*_{\widetilde{\mathbb{H}},n}(k_n) - c^*_{\mathcal{P},n}(k_n)\right|\right] = 0,$$

Then, in Section 7.3, we prove Lemma 2.3 and prove that

$$\lim_{n\to\infty} s_\alpha(k_n)\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = 0.$$

Together these results yield Proposition 2.5.

7.1 Some results on the hyperbolic geometric graph

We start with some basic results for the hyperbolic random geometric graph. Observe that Lemma 2.2 implies the following.

Corollary 7.1.

$$\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{R}_n([K, R_n]) \subseteq \mathcal{B}_{\mathbb{H},n}(p) \cap \mathcal{R}_n(K, R_n).$$

Furthermore, Lemma 2.2 enables us to determine the measure of a ball around a given point $p = (0, y)$ - this is will be fairly useful in our subsequent analysis.

Lemma 7.2. *Let $\alpha > 1/2$, $\nu > 0$ and $\{k_n\}_{n\ge1}$ be a sequence such that $k_n = o\left(n^{1/(2\alpha+1)}\right)$. Then*

$$\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right] = \Theta(1)\,nk_n^{-(2\alpha+1)}, \tag{64}$$

and

$$\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] = \Theta(1)\,nk_n^{-(2\alpha+1)}. \tag{65}$$

Proof. Recall that

$$\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] = \int_{\mathcal{R}_n} \rho_{\widetilde{\mathbb{H}},n}(y,k_n) f_{\alpha,\nu}(x,y) \, \mathrm{d}x \, \mathrm{d}y.$$

Then by a concentration argument

$$\begin{aligned}\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] &= (1+o(1)) \int_{\mathcal{R}_n} \rho(y,k_n) f_{\alpha,\nu}(x,y) \, \mathrm{d}x \, \mathrm{d}y \\ &= (1+o(1)) n \mathbb{E}\left[N_{\mathcal{P}}(k_n)\right] = \Theta(1)\, n k_n^{-(2\alpha+1)}.\end{aligned}$$

The proof for $\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]$ is similar. ∎

Let $p \in \mathcal{R}_n$. Then we can see that the curve $x' = e^{\frac{1}{2}(y+y')}$ with $x' \geq 0$ meets the right boundary of $\mathcal{R}$, that is, the line $x' = \frac{\pi}{2} e^{R_n/2}$ at $y' = R_n - y + 2\ln\frac{\pi}{2}$. Hence, any point $p' \in \mathcal{R}([R_n - y + 2\ln\frac{\pi}{2}, R_n])$ is included in $\mathcal{B}_{\mathcal{P}}(p)$. In other words,

$$\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{R}([R_n - y + 2\ln\frac{\pi}{2}, R_n]) = \mathcal{R}([R_n - y + 2\ln\frac{\pi}{2}, R_n]).$$

This together with (16) implies that

$$(\mathcal{B}_{\mathbb{H},n}(p) \triangle \mathcal{B}_{\mathcal{P}}(p)) \cap \mathcal{R}([R_n - y + 2\ln\frac{\pi}{2}, R_n]) = \emptyset. \tag{66}$$

Lemma 7.3. *For any $y' \in (0, R_n - y)$ we have*

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}((0,y')) \triangle \mathcal{B}_{\mathcal{P}}((0,y'))) = \Theta(1) \cdot \begin{cases} e^{(1/2-\alpha)R_n + \alpha y'}, & \text{if } \alpha < 3/2 \\ (R_n - y) e^{3y'/2 - R_n}, & \text{if } \alpha = 3/2 \\ e^{3y'/2 - R_n}, & \text{if } \alpha > 3/2 \end{cases}.$$

Proof. Let $r_n := R_n - y_n$. Lemma 2.2 implies that for such a p_n, if a point p' belongs to $\mathcal{B}_{\mathbb{H},n}(p_n) \triangle \mathcal{B}_{\mathcal{P}}(p_n) \cap \mathcal{R}([0, r_n])$ then

$$|x_n - x'| = \Theta(1) \cdot e^{\frac{3}{2}(y_n + y') - R_n}.$$

Now, if $p' \in [r_n, r_n + 2\ln\frac{\pi}{2})]$ and also $p' \in \mathcal{B}_{\mathbb{H},n}(p_n) \triangle \mathcal{B}_{\mathcal{P}}(p_n)$, then

$$|x_n - x'| = \frac{\pi}{2} e^{R_n/2} - e^{\frac{1}{2}(y_n + y')}.$$

Finally, (66) implies that no point in $\mathcal{R}([r_n + 2\ln\frac{\pi}{2}, R_n])$ belongs to $\mathcal{B}_{\mathbb{H},n}(p_n) \triangle \mathcal{B}_{\mathcal{P}}(p_n)$. We first compute the expected number of points in $p' \in \mathcal{B}_{\mathbb{H},n}(p_n) \triangle \mathcal{B}_{\mathcal{P}}(p_n)$ that have $R_n - y' \leq r_n$. The result depends on the value of α, yielding the following three cases

$$\begin{aligned}\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(p_n) \triangle \mathcal{B}_{\mathcal{P}}(p_n) \cap \mathcal{R}([0, r_n])) &= \Theta(1) \cdot e^{3y_n/2 - R_n} \int_0^{r_n} e^{(3/2-\alpha)y} \, dy \\ &= \Theta(1) \cdot \begin{cases} e^{(1/2-\alpha)R_n + \alpha y_n}, & \text{if } \alpha < 3/2 \\ r_n e^{3y_n/2 - R_n}, & \text{if } \alpha = 3/2 \\ e^{3y_n/2 - R_n}, & \text{if } \alpha > 3/2 \end{cases}.\end{aligned}$$

Next we compute the number of remaining points in $\mathcal{B}_{\mathbb{H},n}(p_n) \triangle \mathcal{B}_{\mathcal{P}}(p_n)$,

$$\begin{aligned}\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(p_n) \triangle \mathcal{B}_{\mathcal{P}}(p_n) \cap \mathcal{R}([r_n, R_n])) &= \frac{\nu\alpha}{\pi} \int_{r_n}^{r_n + 2\ln\frac{\pi}{2}} \left(\frac{\pi}{2} e^{R_n/2} - e^{\frac{1}{2}(y_n + y)}\right) e^{-\alpha y} \, dy \\ &= O(1) \cdot e^{R_n/2} \int_{r_n}^{r_n + 2\ln\frac{\pi}{2}} e^{-\alpha y} \, dy = O(1) \cdot e^{R_n/2} e^{-\alpha r_n} \\ &= O(1) \cdot e^{(1/2-\alpha)R_n + \alpha y_n}.\end{aligned}$$

Now note that for any $\alpha > 3/2$, we have

$$((1/2-\alpha)R_n + \alpha y_n) - (3y_n/2 - R_n) \to -\infty,$$

since

$$((1/2-\alpha)R_n + \alpha y_n) - (3y_n/2 - R_n) = (3/2-\alpha)R_n - (3/2-\alpha)y_n \quad = (3/2-\alpha)(R_n - y_n) \to -\infty.$$

For $\alpha = 3/2$, these two quantities are equal. From these observations, we deduce that

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(p_n) \triangle \mathcal{B}_{\mathcal{P}}(p_n)) = \Theta(1) \cdot \begin{cases} e^{(1/2-\alpha)R_n+\alpha y_n}, & \text{if } \alpha < 3/2 \\ r_n e^{3y_n/2-R_n}, & \text{if } \alpha = 3/2 \\ e^{3y_n/2-R_n}, & \text{if } \alpha > 3/2 \end{cases}.$$

□

7.2 Equivalence clustering $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$ and $G_{\mathcal{P},n}(\alpha,\nu)$

We are going to show the following lemma:

Lemma 7.4. *Let k_n be an increasing sequence of positive integers such that $k_n = O(n^{\frac{1}{2\alpha+1}})$. The following hold:*

1. *If $1/2 < \alpha \leq 3/4$, then*
$$\lim_{n\to\infty} k_n^{4\alpha-2} \cdot \mathbb{E}\left[|c^*_{\mathbb{H}}(k_n) - c^*_{\mathcal{P}}(k_n)|\right] = 0.$$
2. *If $\alpha = 3/4$, then*
$$\lim_{n\to\infty} \frac{k_n}{\log k_n} \cdot \mathbb{E}\left[|c^*_{\mathbb{H}}(k_n) - c^*_{\mathcal{P}}(k_n)|\right] = 0.$$
3. *If $3/4 < \alpha$, then*
$$\lim_{n\to\infty} k_n \cdot \mathbb{E}\left[|c^*_{\mathbb{H}}(k_n) - c^*_{\mathcal{P}}(k_n)|\right] = 0.$$

Of course Case 2 is implied by Case 1 as $k_n^{4\alpha-2} = k_n$ when $\alpha = 3/4$. So it suffices to prove 1 and 3. We will set $I_n = \frac{\pi}{2}e^{R_n/2}$, and therefore $\mathcal{R}_n = (-I_n, I_n] \times [0, R_n]$.

Recall the definition of $\mathcal{K}_C(k_n)$

$$\mathcal{K}_C(k_n) = \left\{p \in \mathbb{R} : \frac{k_n - C\kappa_n}{\xi_{\alpha,\nu}} \vee 0 \leq e^{\frac{y}{2}} \leq \frac{k_n + C\kappa_n}{\xi_{\alpha,\nu}} \wedge e^{R_n/2}\right\},$$

with $C > 0$ and

$$\kappa_n := \begin{cases} \log(n) & \text{if } k_n = \Theta(1), \\ \sqrt{k_n \log(k_n)} & \text{else.} \end{cases}$$

The following lemma will be frequently used in the proof of Proposition 2.5

Lemma 7.5. *Let $s, t, r \in \mathbb{R}$ be fixed and let $\lambda_y = \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(0,y))$. Then*

$$\int_{\mathcal{K}_C(k_n)} e^{ys} e^{-\lambda_y} \frac{\lambda_y^{k_n-r}}{(k_n-t)!} e^{-\alpha y}\, \mathrm{d}y = O(1)\, \frac{\Gamma(k_n - (2\alpha + r - 2s))}{\Gamma(k_n - t + 1)} = O(1)\, k_n^{-2\alpha+t-r-1+2s}.$$

Proof. To evaluate this integral we perform a change of variable setting $z = \lambda_y$. We have $dz = \frac{1}{2}\lambda_y dy$ and therefore

$$\int_{I_\varepsilon(k_n)} e^{ys} \cdot e^{-\lambda_y} \cdot \lambda_y^{k_n-r} \cdot \lambda_y^{-2\alpha} dy = O(1) \cdot \int_{k_n^{1-\varepsilon}}^{k_n^{1+\varepsilon}} e^{-z} \cdot z^{k_n-1-r-2\alpha+2s} dz. \tag{67}$$

Since $k_n \to \infty$, then as $n \to \infty$ we have

$$\int_{k_n^{1-\varepsilon}}^{k_n^{1+\varepsilon}} e^{-z} \cdot z^{k_n-(2\alpha+1+r-2s)} dz / \Gamma(k_n - (2\alpha + r - 2s)) \to 1. \tag{68}$$

The claim follows from Stirling's formula. □

Proof of Lemma 7.4. Using the Campbell-Mecke formula (**??**), we have

$$\mathbb{E}\left[|c^*_{\mathbb{H}}(k_n) - c^*_{\mathcal{P}}(k_n)|\right] = \binom{k_n}{2}^{-1} \mathbb{E}\left[\left|\sum_{p\in\mathcal{P}} \frac{\mathbb{1}_{\{D_{\mathbb{H}}(p)=k_n\}}}{\mathbb{E}\left[N_{\mathbb{H}}(k_n)\right]} \Delta_{\mathbb{H}}(p) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}(p)=k_n\}}}{\mathbb{E}\left[N_{\mathcal{P}}(k_n)\right]} \Delta_{\mathcal{P}}(p)\right|\right]$$

$$= \binom{k_n}{2}^{-1} \int_{-I_n}^{I_n} \int_0^{R_n} \mathbb{E}\left[\left|\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\mathbb{E}\left[N_{\mathbb{H}}(k_n)\right]} \Delta_{\mathbb{H}}((0,y)) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}((0,y))=k_n\}}}{\mathbb{E}\left[N_{\mathcal{P}}(k_n)\right]} \Delta_{\mathcal{P}}((0,y))\right|\right] e^{-\alpha y} dy dx.$$

We abbreviate $\mathbb{E}\left[N_{\mathbb{H}}(k_n)\right]$ and $\mathbb{E}\left[N_{\mathcal{P}}(k_n)\right]$ by $\overline{n}_{\mathbb{H}}(k_n)$ and $\overline{n}_{\mathcal{P}}(k_n)$, respectively. Since

$$\begin{aligned}
\mathbb{E}\left[\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\mathbb{E}\left[N_{\mathbb{H}}(k_n)\right]} \Delta_{\mathbb{H}}((0,y))\right] &\leq \binom{k_n}{2} \rho_{\widetilde{\mathbb{H}},n}(y,k_n) \overline{n}_{\mathbb{H}}(k_n)^{-1} \\
&= \binom{k_n}{2} \rho_{\widetilde{\mathbb{H}},n}(y,k_n) \Theta\left(\overline{n}_{\mathcal{P}}(k_n)^{-1}\right) \\
&= \Theta\left(n^{-1} k_n^{2\alpha+3}\right) \rho_{\widetilde{\mathbb{H}},n}(y,k_n)
\end{aligned}$$

and similar for the other term, it follows that

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\mathbb{E}\left[N_{\mathbb{H}}(k_n)\right]} \Delta_{\mathbb{H}}((0,y)) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}((0,y))=k_n\}}}{\mathbb{E}\left[N_{\mathcal{P}}(k_n)\right]} \Delta_{\mathcal{P}}((0,y))\right|\right] \\
&\qquad \leq \Theta\left(n^{-1} k_n^{2\alpha+3}\right) \left(\rho_{\widetilde{\mathbb{H}},n}(k,n) + \rho_n(y,k_n)\right).
\end{aligned}$$

Therefore, by a concentration argument, it is enough to consider the integral

$$\binom{k_n}{2}^{-1} \int_{\mathcal{K}_C(k_n)} \mathbb{E}\left[\left|\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\mathbb{E}\left[N_{\mathbb{H}}(k_n)\right]} \Delta_{\mathbb{H}}((0,y)) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}((0,y))=k_n\}}}{\mathbb{E}\left[N_{\mathcal{P}}(k_n)\right]} \Delta_{\mathcal{P}}((0,y))\right|\right] e^{-\alpha y} \, \mathrm{d}y \, \mathrm{d}x. \tag{69}$$

We will first expand the integrand. We write $D_{\mathbb{H}}(y, k_n; \mathcal{P})$ for the indicator which is equal to 1 if and only if $\mathcal{B}_{\mathbb{H},n}((0,y))$ contains k_n points from $\mathcal{P} \setminus \{(0,y)\}$ and define $D_{\mathcal{P}}(y, k_n; \mathcal{P})$ analogously for the ball $\mathcal{B}_{\mathcal{P}}((0,y))$. Again the Campbel-Mecke formula (**??**) yields

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\overline{n}_{\mathbb{H}}(k_n)} \Delta_{\mathbb{H}}((0,y)) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}((0,y))=k_n\}}}{\overline{n}_{\mathcal{P}}(k_n)} \Delta_{\mathcal{P}}((0,y))\right|\right] \leq \\
&\mathbb{E}\left[\sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}}^{\neq} \left| D_{\mathbb{H}}(y, k_n - 2; \mathcal{P} \setminus \{(0,y), p_1, p_2\}) \frac{\Delta_{\mathbb{H}}((0,y), p_1, p_2)}{\overline{n}_{\mathbb{H}}(k_n)} \right.\right. \\
&\qquad\qquad \left.\left. - D_{\mathcal{P}}(y, k_n - 2; \mathcal{P} \setminus \{(0,y), p_1, p_2\}) \frac{\Delta_{\mathcal{P}}((0,y), p_1, p_2)}{\overline{n}_{\mathcal{P}}(k_n)} \right|\right],
\end{aligned}$$

where the sum ranges over all distinct pairs of points in $\mathcal{P} \setminus \{(0,y)\}$. In what follows, we will set $\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p') = \mathcal{B}_{\mathbb{H},n}(p') \triangle \mathcal{B}_{\mathcal{P}}(p')$ and $\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}(p') = \mathcal{B}_{\mathbb{H},n}(p') \cap \mathcal{B}_{\mathcal{P}}(p')$ to denote the symmetric difference between the two neighborhoods and their intersection, respectively.

We will now bound the sum that is inside the expectation. Note that each summand is the absolute value of the difference between two quantities that are either equal to 0 or of order $\overline{n}_{\mathbb{H}}(k_n)^{-1}$ $(\overline{n}_{\mathcal{P}}(k_n)^{-1})$. We will split these summands into 6 classes. All but the last one are combinations of $p_1, p_2 \in \mathcal{P} \setminus \{(0,y)\}$ for which only one of the two terms of this difference is non-zero.

1. both p_1 and p_2 have $y(p_1), y(p_2) < (1-\varepsilon)R_n \wedge (R_n - y)$ and
   (a) p_1 is in $\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))$ but $p_2 \in \mathcal{B}_{\mathbb{H},n}(p_1) \setminus \mathcal{B}_{\mathcal{P}}(p_1)$ and $\mathcal{B}_{\mathbb{H},n}((0,y))$ contains exactly $k_n - 2$ or $k_n - 1$ other points (depending on whether $p_2 \in \mathcal{B}_{\mathbb{H},n}((0,y))$ or not).
   (b) p_1 is in $\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))$ but $p_2 \in \mathcal{B}_{\mathcal{P}}(p_1) \setminus \mathcal{B}_{\mathbb{H},n}(p_1)$ and $\mathcal{B}_{\mathcal{P}}((0,y))$ contains exactly $k_n - 2$ or $k_n - 1$ other points (depending on whether $p_2 \in \mathcal{B}_{\mathcal{P}}((0,y))$ or not).
2. the above cases but with $y(p_1) \geq (1-\varepsilon)R_n \wedge (R_n - y)$ - in this we demand that $\mathcal{B}_{\mathbb{H},n}((0,y))$ contains $k_n - 1$ other points and p_1 thus adjacent to at most k_n other points therein (accounts for the choices of p_2).
3. $y(p_1) \geq K$ and $p_1 \in \mathcal{B}_{\mathbb{H},n}((0,y)) \setminus \mathcal{B}_{\mathcal{P}}((0,y))$ and $p_2 \in \mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))$ - here we use Corollary 7.1 which implies that if $p_1 \in \mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))$ and $y(p_1) \geq K$, then in fact $p_1 \in \mathcal{B}_{\mathbb{H},n}((0,y)) \setminus \mathcal{B}_{\mathcal{P}}((0,y))$.
4. $y(p_1) < K$ and $p_1 \in \mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))$ and $p_2 \in \mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))$.
5. p_1 and p_2 are such that $\Delta_{\mathbb{H}}((0,y), p_1, p_2) = \Delta_{\mathcal{P}}((0,y), p_1, p_2) = 1$.

We bound this sum by the following expression:

$$
\begin{aligned}
&\sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y\}}^{\neq} \left| D_{\mathbb{H}}(y,k_n;\mathcal{P})\frac{\Delta_{\mathbb{H}}((0,y),p_1,p_2)}{\overline{n}_{\mathbb{H}}(k_n)} - D_{\mathcal{P}}(y,k_n;\mathcal{P})\frac{\Delta_{\mathcal{P}}((0,y),p_1,p_2)}{\overline{n}_{\mathcal{P}}(k_n)}\right| \\
&\leq \sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\} \\ y(p_1),y(p_2)<(1-\varepsilon)R_n\wedge(R_n-y)}} \frac{\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\}}}{\overline{n}_{\mathbb{H}}(k_n)} \times \\
&\qquad\qquad (D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\}) + D_{\mathbb{H}}(y,k_n-1;\mathcal{P}\setminus\{p_1,p_2\})) \\
&+ \overline{n}_{\mathcal{P}}(k_n)^{-1} \times \\
&\sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\},\ y(p_1),y(p_2)<(1-\varepsilon)R_n\wedge(R_n-y)} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\}}\times \\
&\qquad\qquad (D_{\mathcal{P}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\}) + D_{\mathcal{P}}(y,k_n-1;\mathcal{P}\setminus\{p_1,p_2\})) \\
&+ \overline{n}_{\mathbb{H}}(k_n)^{-1} \sum_{p_1\ \in\mathcal{P}\setminus\{(0,y)\},\ y(p_1)\geq(1-\varepsilon)R_n\wedge(R_n-y)} k_n\cdot D_{\mathbb{H}}(y,k_n-1;\mathcal{P}\setminus\{p_1\}) \\
&+ \overline{n}_{\mathcal{P}}(k_n)^{-1} \sum_{p_1\ \in\mathcal{P}\setminus\{(0,y)\},\ y(p_1)\geq(1-\varepsilon)R_n\wedge(R_n-y)} k_n\cdot D_{\mathcal{P}}(y,k_n-1;\mathcal{P}\setminus\{p_1\}) \\
&+ 2\overline{n}_{\mathbb{H}}(k_n)^{-1}\times \\
&\sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\},\ y(p_1)\geq K} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}((0,y))\setminus\mathcal{B}_{\mathcal{P}}((0,y))\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))\}}\cdot D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\}) \\
&+ (\overline{n}_{\mathbb{H}}(k_n)^{-1} + \overline{n}_{\mathcal{P}}(k_n)^{-1})\cdot \sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\},\ y(p_1)<K} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))\}} \\
&+ \sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}}^{\neq} \mathbb{1}_{\{p_1,p_2\in\mathcal{B}_{\mathbb{H},n}((0,y))\cup\mathcal{B}_{\mathcal{P}}((0,y))\}}\cdot D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\cdot\left|\overline{n}_{\mathbb{H}}(k_n)^{-1} - \overline{n}_{\mathcal{P}}(k_n)^{-1}\right|.
\end{aligned}
$$

In the following sections we will give upper bounds on the expected values of each one of these partial sums.

First and second term For the first summand, we use the Camplell-Mecke formula (**??**):

$$\mathbb{E}\left[\sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\},\ y(p_1),y(p_2)\le(1-\varepsilon)R_n\wedge(R_n-y)} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(p_1)\triangle\mathcal{B}_{\mathcal{P}}(p_1)\}}\times\right.$$
$$\left.(D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{(0,y),p_1,p_2\})+D_{\mathcal{P}}(y,k_n-1;\mathcal{P}\setminus\{(0,y),p_1,p_2\}))\right]$$
$$=\int_{-I_n}^{I_n}\int_0^{(1-\varepsilon)R_n\wedge(R_n-y)}\int_{-I_n}^{I_n}\int_0^{(1-\varepsilon)R_n\wedge(R_n-y)}\mathbb{1}_{\{(x',y')\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))\}}\cdot\mathbb{1}_{\{(x'',y'')\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((x',y'))\}}\times$$
$$\mathbb{P}\left(D_{\mathbb{H}}((0,y))=k_n-2;\mathcal{P}\setminus\{(0,y),(x',y'),(x'',y'')\}\right)e^{-\alpha y'}\cdot e^{-\alpha y''}dy''dx''dy'dx'. \tag{70}$$

In particular, we have

$$\mathbb{P}\left(D_{\mathbb{H}}((0,y))=k_n-2;\mathcal{P}\setminus\{(0,y),(x',y'),(x'',y'')\}\right)=\rho(y,k_n-2),$$

and

$$\mathbb{P}\left(D_{\mathbb{H}}((0,y))=k_n-1;\mathcal{P}\setminus\{(0,y),(x',y'),(x'',y'')\}\right)=\rho(y,k_n-1).$$

Thus, we can extract these terms out of the quadruple integral.

In particular, recall that $\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}((0,y)))=\lambda_y$ and moreover $\lambda_y=O(1)\cdot e^{y/2}$. Hence,

$$\rho(y,k_n-2)=e^{-\lambda_y}\frac{\lambda_y^{k_n-2}}{(k_n-2)!}=e^{-\lambda_y}\lambda_y^{k_n-2}\frac{1}{\Gamma(k_n-1)}. \tag{71}$$

Lemma 2.2 implies that for $y'\le R_n-y$, we have that if $(x',y')\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))$, then $|x'|<(1+K)e^{y/2+y'/2}$, where $K>0$ is as in Lemma 2.2. For $y>R_n-y$, we will simply take $|x'|<I_n$.

Lemma 2.2 also implies that if $(x'',y'')\in\mathcal{B}_{\mathbb{H},n}((x',y'))\triangle\mathcal{B}_{\mathcal{P}}((x',y'))$, then x'' ranges in an interval of length at most $Ke^{3y'/2+3y''/2-R_n}$. Furthermore, such a point satisfies $y''<R_n-y'+2\ln(\pi/2)$, for otherwise it would belong to $\mathcal{B}_{\mathbb{H},n}((x',y'))\cap\mathcal{B}_{\mathcal{P}}((x',y'))$.

These two observations yield

$$\int_{-I_n}^{I_n}\int_0^{(1-\varepsilon)R_n\wedge(R_n-y)}\int_{-I_n}^{I_n}\int_0^{(1-\varepsilon)R_n\wedge(R_n-y)}\mathbb{1}_{\{(x',y')\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))\}}\cdot\mathbb{1}_{\{(x'',y'')\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((x',y'))\}}\times$$
$$e^{-\alpha y'}\cdot e^{-\alpha y''}dy''dx''dy'dx$$
$$\le\int_{-I_n}^{I_n}\int_0^{(1-\varepsilon)R_n\wedge(R_n-y)}\int_{-I_n}^{I_n}\int_0^{R_n-y'+2\ln(\pi/2)}$$
$$\mathbb{1}_{\{(x',y')\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))\}}\cdot\mathbb{1}_{\{(x'',y'')\in\mathcal{B}_{\mathbb{H},n}((x',y'))\triangle\mathcal{B}_{\mathcal{P}}((x',y'))\}}\cdot e^{-\alpha y'}\cdot e^{-\alpha y''}dy''dx''dy'dx$$

This integral becomes:

$$\int_{-I_n}^{I_n}\int_0^{(1-\varepsilon)R_n\wedge(R_n-y)}\int_{-I_n}^{I_n}\int_0^{R_n-y'+2\ln(\pi/2)} \mathbb{1}_{\{(x',y')\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))\}}\cdot\mathbb{1}_{\{(x'',y'')\in\mathcal{B}_{\mathbb{H},n}((x',y'))\triangle\mathcal{B}_{\mathcal{P}}((x',y'))\}}\times$$
$$e^{-\alpha y'}\cdot e^{-\alpha y''}dy''dx''dy'dx$$
$$= O(1)\cdot e^{y/2}\int_0^{(1-\varepsilon)R_n\wedge(R_n-y)} e^{y'/2}\int_0^{R_n-y'+2\ln(\pi/2)} e^{3y'/2+3y''/2-R_n}\cdot e^{-\alpha y''}dy''\cdot e^{-\alpha y'}dy'$$
$$\leq O(1)\cdot e^{y/2-R_n}\int_0^{R_n-y} e^{2y'}\int_0^{R_n-y'+2\ln(\pi/2)} e^{3/2(R_n-y')-\alpha y''}dy''\cdot e^{-\alpha y'}dy'$$
$$= O(1)\cdot e^{y/2-R_n}\int_0^{R_n-y} e^{2y'}\cdot e^{3/2(R_n-y')}\cdot e^{-\alpha y'}dy'$$
$$= O(1)\cdot e^{y/2+(1/2-\alpha)R_n}\int_0^{R_n-y} e^{y'/2-\alpha y'}dy'$$
$$= O(1)\cdot e^{y/2+(1/2-\alpha)R_n}.$$

The contribution of this term to (69) is

$$\binom{k_n}{2}^{-1}\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot e^{(1/2-\alpha)R_n}\cdot\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{y/2}\left(\rho(y,k_n-2)+\rho(y,k_n-1)\right)e^{-\alpha y}dydx$$
$$\overset{(71)}{=} O(1)\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot e^{(1/2-\alpha)R_n}\cdot I_n\cdot\int_{I_\varepsilon(k_n)} e^{y/2}\cdot e^{-\lambda_y}\left(\frac{\lambda_y^{k_n-2}}{k_n!}+\frac{\lambda_y^{k_n-1}}{(k_n+1)!}\right)e^{-\alpha y}dydx$$
$$\overset{I_n=O(n)}{=} O(1)\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot e^{(1/2-\alpha)R_n}\cdot n\cdot\int_{I_\varepsilon(k_n)} e^{-\lambda_y}\left(\frac{\lambda_y^{k_n-2}}{k_n!}+\frac{\lambda_y^{k_n-1}}{(k_n+1)!}\right)e^{-\alpha y}dy.$$

We will show that the latter integral can be approximated by the ratio of Gamma functions. As this approximation will be applied several times in the sequel, we will state it is a more general form.

Therefore,

$$\binom{k_n}{2}^{-1}\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot e^{(1/2-\alpha)R_n}\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{y/2}\left(\rho(y,k_n-2)+\rho(y,k_n-1)\right)e^{-\alpha y}dydx$$
$$= O(1)\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot e^{(1/2-\alpha)R_n}\cdot n\cdot\left(\frac{\Gamma(k_n-(2\alpha+1))}{\Gamma(k_n+1)}+\frac{\Gamma(k_n-(2\alpha+3))}{\Gamma(k_n+2)}\right)$$
$$= O(1)\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot e^{(1/2-\alpha)R_n}\cdot n\cdot k_n^{-(2\alpha+2)}.$$

But $\overline{n}_{\mathbb{H}}(k_n)=\Theta(1)\cdot n\cdot k_n^{2\alpha+1}$. Therefore,

$$\binom{k_n}{2}^{-1}\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot e^{(1/2-\alpha)R_n}\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{y/2}\mathbb{P}\left(D_{\mathbb{H}}((0,y))=k_n-2\right)e^{-\alpha y}dydx$$
$$O(1)\cdot e^{(1/2-\alpha)R_n}\cdot k_n^{-1}.$$

Therefore, for $1/2<\alpha<3/4$ we have

$$k_n^{4\alpha-2}\cdot\binom{k_n}{2}^{-1}\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot e^{(1/2-\alpha)R_n}\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{y/2}\left(\rho(y,k_n-2)+\rho(y,k_n-1)\right)e^{-\alpha y}dydx=$$
$$= O(1)\cdot k_n^{4\alpha-3}\cdot e^{(1/2-\alpha)R_n}\to 0,\ \text{as } n\to\infty.$$

Similarly, for $\alpha>3/4$, we have

$$k_n\cdot\binom{k_n}{2}^{-1}\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot e^{(1/2-\alpha)R_n}\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{y/2}\left(\rho(y,k_n-2)+\rho(y,k_n-1)\right)e^{-\alpha y}dydx=$$
$$= O(1)\cdot e^{(1/2-\alpha)R_n}\to 0,\ \text{as } n\to\infty.$$

The calculations for the second term are almost identical and we omit them.

The third term The expected value of the third term can be bounded as follows:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{p_1\ \in\mathcal{P}\setminus\{(0,y)\},\ y(p_1)\geq(1-\varepsilon)R_n\wedge(R_n-y)} k_n\cdot D_{\mathbb{H}}(y,k_n-1;\mathcal{P}\setminus\{p_1\})\right]\\
&= k_n\cdot\int_{-I_n}^{I_n}\int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} D_{\mathbb{H}}(y,k_n-1;\mathcal{P}\setminus\{p_1\})e^{-\alpha y}dydx\\
&= k_n\cdot\rho(y,k_n-1)\cdot\int_{-I_n}^{I_n}\int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} e^{-\alpha y}dydx\\
&= O(1)\cdot k_n\cdot\rho(y,k_n-1)\cdot I_n\cdot e^{-\alpha((1-\varepsilon)R_n\wedge(R_n-y))}\\
&= O(1)\cdot k_n\cdot\rho(y,k_n-1)\cdot e^{(1/2-\alpha)R_n+\alpha\cdot(\varepsilon R_n\vee y)}.
\end{aligned}$$

Now, if $y>\varepsilon R_n$, then $e^{(1/2-\alpha)R_n+\alpha\cdot(\varepsilon R_n\vee y)}=e^{(1/2-\alpha)R_n+\alpha y}<e^{y/2}$, as $y<R_n$. If $y\leq\varepsilon R_n$, then $e^{(1/2-\alpha)R_n+\alpha\cdot(\varepsilon R_n\vee y)}=e^{(1/2-\alpha)R_n+\alpha\varepsilon R_n}<e^{y/2}$, provided that $\varepsilon=\varepsilon(\alpha)$ is small enough and n is sufficiently large. Therefore,

$$\mathbb{E}\left[\sum_{p_1\ \in\mathcal{P}\setminus\{(0,y)\},\ y(p_1)\geq(1-\varepsilon)R_n\wedge(R_n-y)} k_n\cdot D_{\mathbb{H}}(y,k_n-1;\mathcal{P}\setminus\{p_1\})\right]= O(1)\cdot k_n\cdot\rho(y,k_n-1)\cdot e^{y/2}.$$

Recall also that

$$\rho(y,k_n-1)=e^{-\lambda_y}\frac{\lambda_y^{k_n-1}}{(k_n-1)!}.$$

The contribution of this term to (69) is:

$$\begin{aligned}
&\binom{k_n}{2}^{-1}\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot k_n\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{y/2}\cdot e^{-\lambda_y}\frac{\lambda_y^{k_n-1}}{(k_n-1)!}e^{-\alpha y}dydx\\
&= O(1)\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{y/2}\cdot e^{-\lambda_y}\frac{\lambda_y^{k_n-1}}{k_n!}e^{-\alpha y}dydx\\
&\overset{I_n=O(n)}{=} O(1)\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot n\cdot\int_{I_\varepsilon(k_n)} e^{y/2}\cdot e^{-\lambda_y}\frac{\lambda_y^{k_n-1}}{k_n!}e^{-\alpha y}dy.
\end{aligned}$$

By Claim 7.5 we have

$$\int_{I_\varepsilon(k_n)} e^{y/2}\cdot e^{-\lambda_y}\frac{\lambda_y^{k_n-1}}{k_n!}e^{-\alpha y}dy=O(1)\cdot\frac{\Gamma(k_n-2\alpha)}{\Gamma(k_n+1)}=O(1)\cdot\frac{1}{k_n^{2\alpha+1}}.$$

Substituting this into the above expression we obtain:

$$\begin{aligned}
&\binom{k_n}{2}^{-1}\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot k_n\cdot\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{y/2}\cdot e^{-\lambda_y}\frac{\lambda_y^{k_n-1}}{(k_n-1)!}e^{-\alpha y}dydx\\
&= O(1)\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot n\cdot\frac{1}{k_n^{2\alpha+1}}.
\end{aligned}$$

But $\overline{n}_{\mathbb{H}}(k_n)=O(1)\cdot n\cdot k_n^{-(2\alpha+1)}$. We thus conclude that

$$\begin{aligned}
&\binom{k_n}{2}^{-1}\cdot\overline{n}_{\mathbb{H}}(k_n)^{-1}\cdot k_n\cdot e^{(1/2-\alpha)R_n+\alpha\cdot(\varepsilon R_n\vee y)}\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{-\lambda_y}\frac{\lambda_y^{k_n-1}}{(k_n-1)!}e^{-\alpha y}dydx\\
&= O(1)\cdot e^{(1/2-\alpha)R_n+\alpha\cdot(\varepsilon R_n\vee y)}\cdot\frac{1}{k_n}.
\end{aligned}$$

Thus, for $1/2 < \alpha \leq 3/4$, we have

$$\begin{aligned} & k_n^{4\alpha-3} \cdot \binom{k_n}{2}^{-1} \cdot \overline{n}_{\mathbb{H}}(k_n)^{-1} \cdot k_n \cdot e^{(1/2-\alpha)R_n+\varepsilon R_n} \int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} e^{-\lambda_y} \frac{\lambda_y^{k_n-1}}{(k_n-1)!} e^{-\alpha y} dy dx \\ & = O(1) \cdot k_n^{4\alpha-3} \cdot e^{(1/2-\alpha)R_n+\varepsilon R_n} \to 0, \end{aligned} \tag{72}$$

provided that $\varepsilon = \varepsilon(\alpha) > 0$ is small enough. Similarly, for $\alpha > 3/4$ we get

$$\begin{aligned} & k_n \cdot \binom{k_n}{2}^{-1} \cdot \overline{n}_{\mathbb{H}}(k_n)^{-1} \cdot k_n \cdot e^{(1/2-\alpha)R_n+\varepsilon R_n} \int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} e^{-\lambda_y} \frac{\lambda_y^{k_n-1}}{(k_n-1)!} e^{-\alpha y} dy dx \\ & = O(1) \cdot e^{(1/2-\alpha)R_n+\varepsilon R_n} \to 0, \end{aligned} \tag{73}$$

for $\varepsilon = \varepsilon(\alpha) > 0$ sufficiently small.

The fourth term We will give an upper bound on the expectation of

$$\overline{n}_{\mathbb{H}}(k_n)^{-1} \cdot \sum_{p_1,p_2 \in \mathcal{P}\setminus\{(0,y)\}} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}((0,y)) \setminus \mathcal{B}_{\mathcal{P}}((0,y))\}} \mathbb{1}_{\{p_2 \in \mathcal{B}_{\mathbb{H},n}((0,y)) \cap \mathcal{B}_{\mathbb{H},n}(p_1)\}} \cdot D_{\mathbb{H}}(y, k_n-2; \mathcal{P}\setminus\{p_1,p_2\}).$$

Let us set $p = (0, y)$. Recall that $\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p) \cap \mathcal{R}([R_n - y + 2\log\left(\frac{\pi}{2}\right), R_n]) = \emptyset$. Thus, the summand in the above sum is equal to 0, when $y(p_1) > R_n - y + 2\log(\pi/2)$.

Recall that we have

$$\mathbb{E}\left[D_{\mathbb{H}}(y, k_n - 2; \mathcal{P} \setminus \{p_1, p_2\})\right] = O(1) \cdot \rho(y, k_n - 2). \tag{74}$$

We are going to define an extended ball around p that contains both $\mathcal{B}_{\mathbb{H},n}(p)$ and $\mathcal{B}_{\mathcal{P}}(p)$ and we are going to use as an upper approximation on $\mathcal{B}_{\mathbb{H},n}(p)$. For $K > 0$ as in Lemma 2.2, we define:

$$\check{\mathcal{B}}_{\mathbb{H},n}(p) := \{p' : y' < R_n - y,\ |x'| < (1+K)e^{\frac{1}{2}(y+y')}\}. \tag{75}$$

Observe that,

$$\mathcal{B}_{\mathbb{H},n}(p) \cap \mathcal{R}([0, r(p))) \subseteq \check{\mathcal{B}}_{\mathbb{H},n}(p)$$

and

$$\mathcal{B}_{\mathbb{H},n}(p) \cap \mathcal{R}([r(p), R_n]) = \mathcal{R}([r(p), R_n]).$$

We thus conclude that

$$\mathcal{B}_{\mathbb{H},n}(p) \subseteq \check{\mathcal{B}}_{\mathbb{H},n}(p) \cup \mathcal{R}([r(p), R_n]). \tag{76}$$

Hence, if we set

$$h_y(p_1, \mathcal{P}) := \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p) \setminus \mathcal{B}_{\mathcal{P}}(p)\}} \cdot \left(\mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(p_1) \cap \check{\mathcal{B}}_{\mathbb{H},n}(p)\right) + \mu_{\alpha,\nu}\left(\mathcal{R}([r(p), R_n])\right)\right),$$

then

$$\begin{aligned} & \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p) \setminus \mathcal{B}_{\mathcal{P}}(p)\}} \cdot \mathbb{E}\left[\left(\sum_{p_2 \in \mathcal{P}\setminus\{p,p_1\}} \mathbb{1}_{\{p_2 \in \mathcal{B}_{\mathbb{H},n}(p) \cap \mathcal{B}_{\mathcal{P}}(p_1)\}}\right) \cdot D_{\mathbb{H}}(y, k_n - 2; \mathcal{P} \setminus \{p_1, p_2\})\right] \\ & = O(1) \cdot \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p) \setminus \mathcal{B}_{\mathcal{P}}(p)\}} \cdot \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(p) \cap \mathcal{B}_{\mathbb{H},n}(p_1)) \cdot \rho(y, k_n - 2) \\ & \leq O(1) \cdot h_y(p_1, \mathcal{P}) \cdot \rho(y, k_n - 2). \end{aligned}$$

To calculate the expectation of the above function we need to approximate the intersection of the two balls $\check{\mathcal{B}}_{\mathbb{H},n}(p)$ and $\check{\mathcal{B}}_{\mathbb{H},n}(p_1)$, where $p_1 = (x_1, y_1)$. Let us assume without loss of generality that $x_1 > 0$. The right boundary of $\check{\mathcal{B}}_{\mathbb{H},n}(p)$ is given by the equation $x = x(y_1) = (1+K)e^{\frac{1}{2}(y+y_1)}$ whereas the left boundary of $\check{\mathcal{B}}_{\mathbb{H},n}(p_1)$ is given by the curve $x = x(y_1) = x_1 - (1+K)e^{\frac{1}{2}(y+y_1)}$.

The equation that determines the intersecting point of the two curves is

$$x_1 - (1+K)e^{(\hat{y}+y_1)/2} = (1+K)e^{(\hat{y}+y)/2},$$

where $\hat{y}$ is the y-coordinate of the intersecting point. We can solve the above for $\hat{y}$

$$x_1 = (1+K)e^{\hat{y}/2}\left(e^{y/2}+e^{y_1/2}\right).$$

But since $p_1 = (x_1, y_1) \in \mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p)$, we also have $x_1 > e^{\frac{y+y_1}{2}}$. Therefore,

$$e^{\hat{y}/2} > \frac{1}{1+K}\ \frac{e^{\frac{y+y_1}{2}}}{e^{y/2}+e^{y_1/2}} \geq \frac{1}{2(1+K)}\ \frac{e^{\frac{y_1+y}{2}}}{e^{\max\{y,y_1\}/2}} > \frac{1}{2(1+K)}\ e^{\min\{y,y_1\}/2}.$$

The above yields

$$\hat{y} > \min\{y, y_1\} - 2\log(2(1+K)) := c(y_1, y). \tag{77}$$

which, in turn, implies the following

$$p \in \check{\mathcal{B}}_{\mathbb{H},n}((0,y)) \cap \check{\mathcal{B}}_{\mathbb{H},n}(p_1) \Rightarrow y(p) \geq c(y_1, y). \tag{78}$$

We thus conclude that

$$\mathcal{B}_{\mathbb{H},n}(p_1) \cap \mathcal{B}_{\mathbb{H},n}(p) \subseteq \check{\mathcal{B}}_{\mathbb{H},n}(p) \cap \mathcal{R}([c(y_1,y), R_n]) \bigcup \mathcal{R}([R_n - y, R_n]),$$

which in turn implies that

$$\mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(p_1) \cap \mathcal{B}_{\mathbb{H},n}(p)\right) \leq \mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(p) \cap \mathcal{R}([c(y_1,y), R_n]\right) + \mu_{\alpha,\nu}(\mathcal{R}([R_n - y, R_n])).$$

Therefore,

$$h_y(p_1, \mathcal{P}) \leq \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p)\setminus\mathcal{B}_{\mathcal{P}}(p)\}}\mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(p) \cap \mathcal{R}([c(y_1,y), R_n])\right) \tag{79}$$

$$+ \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p)\setminus\mathcal{B}_{\mathcal{P}}(p)\}}\mu_{\alpha,\nu}\left(\mathcal{R}([R_n - y, R_n])\right). \tag{80}$$

Now, (**??**) gives

$$\mathbb{E}\left[\left(\sum_{p_1\in\mathcal{P}} h_y(p_1, \mathcal{P}\setminus\{p_1\})\right)\right] = \frac{\nu\alpha}{\pi}\int_{\mathcal{R}_n} \mathbb{E}\left[h_y((x_1,y_1), \mathcal{P}\setminus\{(x_1,y_1)\})\right] e^{-\alpha y_1}\, dx_1\, dy_1.$$

Recall that $(\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))) \cap \mathcal{R}([R_n - y + 2\log\left(\frac{\pi}{2}\right), R_n]) = \emptyset$. We will calculate the measure of each one of the two summands. The first one is:

$$\begin{aligned}\mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}((0,y)) \cap \mathcal{R}([c(y_1,y), R_n])\right) &\leq (1+K)\frac{\nu\alpha}{\pi}\cdot e^{y/2}\int_{c(y_1,y)}^{R_n} e^{-(\alpha-\frac{1}{2})y'}\, dy' \\ &= O\left(e^{\frac{y}{2}-(\alpha-\frac{1}{2})\min\{y,y_1\}}\right).\end{aligned}$$

The second summand is:

$$\mu_{\alpha,\nu}\left(\mathcal{R}([R_n - y, R_n])\right) = \frac{\nu\alpha}{\pi}\int_{R_n-y}^{R_n} \pi e^{\frac{R_n}{2}} e^{-\alpha y'}\, dy' = O\left(e^{\frac{R_n}{2}} e^{-\alpha(R_n - y)}\right) = O\left(e^{\alpha y - (\alpha-\frac{1}{2})R_n}\right).$$

Using these,

$$\int_{\mathcal{R}_n([0, R_n - y_n + 2\ln\frac{\pi}{2}])} \mathbb{E}\left[h_y((x_1,y_1), \mathcal{P}\setminus\{(x_1,y_1)\})\right] e^{-\alpha y_1}\, dx_1\, dy_1 =$$

$$O(1)\cdot\left(\int_{\mathcal{R}_n([0, R_n - y + 2\ln\frac{\pi}{2}])} \mathbb{1}_{\{(x_1,y_1)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p)\}} e^{\frac{y}{2}-(\alpha-\frac{1}{2})\min\{y,y_1\}-\alpha y_1}\, dx_1\, dy_1\right.$$

$$+\int_{\mathcal{R}_n([0,R_n-y+2\ln\frac{\pi}{2}])} \mathbb{1}_{\{(x,y)\in\mathcal{B}_{\mathbb{H},n}((0,y))\}} e^{\alpha y-(\alpha-\frac{1}{2})R_n-\alpha y_1}\,dx_1\,dy_1\Bigg). \tag{81}$$

Now, Lemma 2.2 implies that for any $y\in[0,R_n-y_n+2\ln\frac{\pi}{2}]$, we have

$$\int \mathbb{1}_{\{(x_1,y_1)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}\,dx_1 \le Ke^{\frac{3}{2}(y_1+y)-R_n}.$$

Therefore, the first integral in (81) is

$$\begin{aligned}
&\int_{\mathcal{R}([0,R_n-y+2\ln\frac{\pi}{2}])} \mathbb{E}\left[h_y((x_1,y_1),\mathcal{P}\setminus\{(x_1,y_1)\})\right] e^{-\alpha y_1}\,dx_1\,dy_1\\
&\quad= O(1)\cdot e^{2y-R_n}\int_0^{R_n-y+2\ln\frac{\pi}{2}} e^{\frac{3y_1}{2}-(\alpha-\frac{1}{2})\min\{y_1,y\}-\alpha y_1}\,dy_1\\
&\quad= O(1)\cdot e^{2y-R_n}\left(\int_0^y e^{\frac{3y_1}{2}-(2\alpha-\frac{1}{2})y_1}\,dy_1+e^{-(\alpha-\frac{1}{2})y}\int_y^{R_n-y+2\ln\frac{\pi}{2}} e^{(\frac{3}{2}-\alpha)y_1}\,dy_1\right)\\
&\quad= O(1)\cdot\begin{cases} e^{(4-2\alpha)y-R_n}, & \text{if } \alpha<2\\ R_n\cdot e^{2y-R_n}, & \text{if } \alpha\ge 2\end{cases} + \begin{cases} e^{-(\alpha-\frac{1}{2})R_n+y}, & \text{if } \alpha<3/2\\ R_n\cdot e^{2(2-\alpha)y-R_n}, & \text{if } \alpha\ge 3/2\end{cases}.
\end{aligned}$$

Similarly, the second integral in (81) is

$$\begin{aligned}
&\int_{\mathcal{R}([0,R_n-y+2\ln\frac{\pi}{2}])} \mathbb{1}_{\{(x_1,y_1)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}} e^{\alpha y-(\alpha-\frac{1}{2})R_n-\alpha y_1}\,dx_1\,dy_1\\
&\quad= e^{\frac{3y}{2}-R_n+\alpha y-(\alpha-\frac{1}{2})R_n}\cdot\int_0^{R_n-y+2\ln\frac{\pi}{2}} e^{\frac{3y_1}{2}-\alpha y_1}\,dy_1\\
&\quad= O(1)\cdot\begin{cases} e^{\frac{3y}{2}-R_n+\alpha y-(\alpha-\frac{1}{2})R_n+(\frac{3}{2}-\alpha)(R_n-y)}, & \text{if } \alpha<3/2\\ R_n\cdot e^{(\frac{3}{2}+\alpha)y-(\alpha+\frac{1}{2})R_n}, & \text{if } \alpha\ge 3/2\end{cases}\\
&\quad= O(1)\cdot\begin{cases} e^{-(2\alpha-1)R_n+2\alpha y}, & \text{if } \alpha<3/2\\ R_n\cdot e^{(\frac{3}{2}+\alpha)y-(\alpha+\frac{1}{2})R_n}, & \text{if } \alpha\ge 3/2\end{cases}.
\end{aligned}$$

We thus conclude that

$$\mathbb{E}\left[\left(\sum_{p_1\in\mathcal{P}\setminus\{p\}} h_y(p_1,\mathcal{P}\setminus\{p_1\})\right)\right] = O(1)\cdot(\Lambda_1+\Lambda_2+\Lambda_3), \tag{82}$$

where

$$\begin{aligned}
\Lambda_1=\Lambda_1(y) &= \begin{cases} e^{(4-2\alpha)y-R_n}, & \text{if } \alpha<2\\ R_n\cdot e^{2y-R_n}, & \text{if } \alpha\ge 2\end{cases},\\
\Lambda_2=\Lambda_2(y) &= \begin{cases} e^{-(\alpha-\frac{1}{2})R_n+y}, & \text{if } \alpha<3/2\\ R_n\cdot e^{2(2-\alpha)y-R_n}, & \text{if } \alpha\ge 3/2\end{cases}\\
&\overset{2-\alpha\le 1/2}{\le} \begin{cases} e^{-(\alpha-\frac{1}{2})R_n+y}, & \text{if } \alpha<3/2\\ R_n\cdot e^{y-R_n}, & \text{if } \alpha\ge 3/2\end{cases}\\
\text{and } \Lambda_3=\Lambda_3(y) &= \begin{cases} e^{-(2\alpha-1)R_n+2\alpha y}, & \text{if } \alpha<3/2\\ R_n\cdot e^{(\frac{3}{2}+\alpha)y-(\alpha+\frac{1}{2})R_n}, & \text{if } \alpha\ge 3/2\end{cases}.
\end{aligned}$$

Substituting this into (69) we now need to calculate:

$$\binom{k_n}{2}^{-1} \cdot \overline{n}_{\mathbb{H}}(k_n)^{-1} \cdot \int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} \mathbb{E}\left[\left(\sum_{p_1 \in \mathcal{P}} h_y(p_1, \mathcal{P} \setminus \{p_1\})\right)\right] \cdot \rho(y, k_n - 2) \cdot e^{-\alpha y} dy.$$

Firstly, note that as $\overline{n}_{\mathbb{H}}(k_n) = \Theta(1) \cdot n \cdot k_n^{-(2\alpha+1)}$, we have

$$\binom{k_n}{2}^{-1} \cdot \overline{n}_{\mathbb{H}}(k_n)^{-1} = O(1) \cdot \frac{k_n^{2\alpha-1}}{n}.$$

Also, $\mathbb{E}\left[\left(\sum_{p_1 \in \mathcal{P}} h_y(p_1, \mathcal{P} \setminus \{p_1\})\right)\right]$ is given as the sum of Λ_1, Λ_2 and Λ_3 (cf. (82)). We need to integrate these expressions together with $e^{-\lambda_y} \frac{\lambda_y^{k_n-2}}{(k_n-2)!}$. For this, we will use again Claim 7.5.

Using $n = \nu e^{R_n/2}$ as well as Claim 7.5, we deduce

$$M_1 := \int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} \Lambda_1(y) \rho(y, k_n - 2) e^{-\alpha y} dy = O(1) \cdot \begin{cases} \frac{k_n^{7-6\alpha}}{n}, & \text{if } \alpha < 2 \\ R_n^2 \cdot \frac{k_n^{3-2\alpha}}{n}, & \text{if } \alpha \geq 2 \end{cases}.$$

$$\begin{aligned} M_2 := \int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} \Lambda_2(y) \rho(y, k_n - 2) e^{-\alpha y} dy &= O(1) \cdot \begin{cases} e^{-(\alpha-1)R_n} k_n^{-2\alpha+1}, & \text{if } \alpha < 3/2 \\ R_n \cdot \frac{k_n^{7-6\alpha}}{n}, & \text{if } \alpha \geq 3/2 \end{cases} \\ &= \begin{cases} \frac{k_n^{1-2\alpha}}{n^{2(\alpha-1)}}, & \text{if } \alpha < 3/2 \\ R_n \cdot \frac{k_n^{7-6\alpha}}{n}, & \text{if } \alpha \geq 3/2 \end{cases} \end{aligned}$$

and finally

$$\begin{aligned} M_3 := \int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} \Lambda_3(y) \rho(y, k_n - 2) e^{-\alpha y} dy &= O(1) \cdot \begin{cases} e^{-(2\alpha-3/2)R_n} k_n^{2\alpha-1}, & \text{if } \alpha < 3/2 \\ R_n \cdot n \cdot e^{-(\alpha+\frac{1}{2})R_n} k_n^2, & \text{if } \alpha \geq 3/2 \end{cases} \\ &= O(1) \cdot \begin{cases} \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}}, & \text{if } \alpha < 3/2 \\ R_n \cdot \frac{k_n^2}{n^{2\alpha}}, & \text{if } \alpha \geq 3/2 \end{cases}. \end{aligned}$$

Setting

$$J_3 = \frac{k_n^{2\alpha-1}}{n} \cdot (M_1 + M_2 + M_3)$$

we deduce that

$$\begin{aligned} &\binom{k_n}{2}^{-1} \cdot \overline{n}_{\mathbb{H}}(k_n)^{-1} \cdot \int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} \mathbb{E}\left[\left(\sum_{p_1 \in \mathcal{P} \setminus \{(0,y)\}} h_y(p_1, \mathcal{P} \setminus \{p_1\})\right)\right] \cdot \rho(y, k_n - 2) \cdot e^{-\alpha y} dy \\ &= O(1) \cdot J_3 \end{aligned}$$

Now, we will consider the two cases according to the value of α. Assume first that $1/2 < \alpha \leq 3/4$. In this case, we want to show that

$$\lim_{n \to \infty} k_n^{4\alpha-2} \cdot J_3 = 0. \tag{83}$$

Using the above expression for J_3, we have

$$k_n^{4\alpha-2} \cdot J_3 = O(1) \cdot \frac{k_n^{6\alpha-3}}{n} \cdot \left(\frac{k_n^{7-6\alpha}}{n} + \frac{k_n^{2(1-\alpha)}}{n^{2(\alpha-1)}} + \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}}.\right)$$

We wish to show that each one of the above three terms is $o(1)$ for $k_n = O(n^{\frac{1}{2\alpha+1}})$. For the first one we have

$$\frac{k_n^{6\alpha-3}}{n} \cdot \frac{k_n^{7-6\alpha}}{n} = \frac{k_n^4}{n^2} = O(1) \cdot n^{\frac{4}{2\alpha+1}-2} \overset{\alpha>1/2}{=} o(1).$$

The second one yields:

$$\frac{k_n^{6\alpha-3}}{n} \cdot \frac{k_n^{-2\alpha+1}}{n^{2(\alpha-1)}} = \frac{k_n^{4\alpha-2}}{n^{2\alpha-1}} = O(1)\frac{n^{\frac{4\alpha-2}{2\alpha+1}}}{n^{2\alpha-1}}.$$

We need to show that $\frac{4\alpha-2}{2\alpha+1} < 2\alpha - 1$. Indeed, rearranging this yields, $4\alpha - 2 < 4\alpha^2 - 1$, which is equivalent to $0 < 4\alpha^2 - 4\alpha + 1 = (2\alpha - 1)^2$. This holds for all $\alpha > 1/2$.

Finally, the third one yields:

$$\frac{k_n^{6\alpha-3}}{n} \cdot \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}} = \frac{k_n^{8\alpha-4}}{n^{2(2\alpha-1)}} = \frac{k_n^{4(2\alpha-1)}}{n^{2(2\alpha-1)}}.$$

But $k_n^4 \leq O(1) \cdot n^{\frac{4}{2\alpha+1}} = o(n^2)$, as $2\alpha + 1 > 2$.

For $\alpha > 3/4$, we would like to show that

$$\lim_{n\to\infty} k_n \cdot J_3 = 0. \tag{84}$$

Firstly, if $3/4 < \alpha < 3/2$ we have,

$$k_n \cdot J_3 = O(1) \cdot \frac{k_n^{2\alpha}}{n} \cdot \left(\frac{k_n^{7-6\alpha}}{n} + \frac{k_n^{-2\alpha+1}}{n^{2(\alpha-1)}} + \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}}.\right)$$

As above we will deal with the three term of this. For the first one we have

$$\frac{k_n^{2\alpha}}{n} \cdot \frac{k_n^{7-6\alpha}}{n} = \frac{k_n^{7-4\alpha}}{n^2} \overset{\alpha>3/4}{<} \frac{k_n^4}{n^2} = o(1).$$

The second one yields:

$$\frac{k_n^{2\alpha}}{n} \cdot \frac{k_n^{-2\alpha+1}}{n^{2(\alpha-1)}} = \frac{k_n}{n^{2\alpha-1}} \overset{\alpha>3/4}{<} \frac{k_n}{n^{1/2}} = O(1)\frac{n^{\frac{1}{2\alpha+1}}}{n^{1/2}} = o(1).$$

Finally, the third one yields:

$$\frac{k_n^{2\alpha}}{n} \cdot \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}} = \frac{k_n^{4\alpha-1}}{n^{2(2\alpha-1)}} = O(1)\frac{n^{\frac{4\alpha-1}{2\alpha+1}}}{n^{2(2\alpha-1)}}.$$

We need to show that $\frac{4\alpha-1}{2\alpha+1} < 2(2\alpha - 1)$, which is equivalent to $8\alpha^2 - 4\alpha - 1 > 0$; this is indeed the case for any $\alpha \geq 3/4$.

For $3/2 \leq \alpha < 2$, it is only M_2 and M_3 that change values. In particular, for any $\alpha \geq 3/2$ we have

$$\frac{k_n}{n} \cdot M_2 = O(1) \cdot R_n \cdot \frac{k_n^{2\alpha}}{n} \cdot \frac{k_n^{7-6\alpha}}{n} = o(1),$$

as above. Also,

$$\frac{k_n}{n} \cdot M_3 = O(1) \cdot R_n \cdot \frac{k_n}{n} \cdot \frac{k_n^2}{n^{2\alpha}} = R_n \cdot \frac{k_n^3}{n^{2\alpha+1}} = o(1),$$

since $k_n = o(n^{1/2})$ (and, therefore, $k_n^3 = o(n^{3/2})$) but $2\alpha + 1 > 2$.

If $\alpha \geq 2$ too, then M_1 changes value and we have

$$\frac{k_n}{n} \cdot M_1 = O(1) \cdot R_n^2 \cdot \frac{k_n}{n} \cdot \frac{k_n^{3-2\alpha}}{n} = \frac{k_n^{4-2\alpha}}{n^2} = o(1),$$

since $\alpha \geq 2$.

The fifth term Now, we will give an upper bound on the term

$$\mathbb{E}\left[\sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\},\ y(p_1)<K} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))\}}\right].$$

Using the Campbell-Mecke formula (**??**), we write

$$\mathbb{E}\left[\sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\},\ y(p_1)<K} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))\}}\right] =$$
$$\leq \int_{-I_n}^{I_n}\int_0^K\int_{-I_n}^{I_n}\int_0^{R_n} \mathbb{1}_{\{(x_1,y_1)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}\mathbb{1}_{\{(x_2,y_2)\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))\}}e^{-\alpha y_2}e^{-\alpha y_1}dx_2dy_2dx_1dy_1$$
$$\leq \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}((0,y)))\cdot\int_{-I_n}^{I_n}\int_0^K \mathbb{1}_{\{(x_1,y_1)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}e^{-\alpha y_1}dx_1dy_1.$$

By (**??**), the first factor is

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}((0,y))) = O(1)\cdot e^{y/2}.$$

We bound the second factor using Lemma 2.2. In particular, (17) implies that if $(x_1,y_1)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))$, then

$$|x_1 - e^{(y+y_1)/2}| \leq e^{(y+y_1)/2}\cdot Ke^{y+y_1-R_n} \overset{y_1\leq K}{=} O(1)e^{(y+y_1)/2}\cdot e^{y-R_n}.$$

Therefore,

$$\int_{-I_n}^{I_n}\int_0^K \mathbb{1}_{\{(x_1,y_1)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}e^{-\alpha y_1}dx_1dy_1 = O(1)\cdot e^{y-R_n}\cdot\int_0^K e^{(y+y_1)/2}e^{-\alpha y_1}dy_1 = O(1)\cdot e^{3y/2-R_n}.$$

Therefore,

$$\mathbb{E}\left[\sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\},\ y(p_1)<K} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}((0,y))\cap\mathcal{B}_{\mathcal{P}}((0,y))\}}\right] = O(1)\cdot e^{2y-R_n}.$$

Now, we integrate this over y:

$$\begin{aligned}
e^{-R_n}\int_{-I_n}^{I_n}\int_{I_\varepsilon(k_n)} e^{2y-\alpha y}dydx &= O(1)\cdot n\cdot e^{-R_n}\int_{I_\varepsilon(k_n)} e^{2y-\alpha y}dy \\
&\overset{n=\nu e^{R_n/2}}{=} O(1)\cdot n^{-1}\int_{I_\varepsilon(k_n)} e^{2y-\alpha y}dy \\
&= O(1)\cdot n^{-1}\cdot\begin{cases} k_n^{2(2-\alpha)(1+\varepsilon)}, & \text{if } \alpha<2 \\ \log k_n, & \text{if } \alpha=2 \\ 1, & \text{if } \alpha>2 \end{cases}.
\end{aligned}$$

We will multiply the above by

$$\binom{k_n}{2}^{-1}(\overline{n}_{\mathbb{H}}(k_n)^{-1}+\overline{n}_{\mathcal{P}}(k_n)^{-1}) = O(1)\cdot n^{-1}\cdot k_n^{-2+2\alpha+1} = O(1)\cdot n^{-1}\cdot k_n^{2\alpha-1}.$$

Assume first that $1/2<\alpha\leq 3/4$. In this case, we will consider

$$k_n^{4\alpha-2}\cdot n^{-2}\cdot k_n^{2\alpha-1+4-2\alpha} = n^{-2}\cdot k_n^{4\alpha+1}.$$

But, $\alpha\leq 3/4$, we have $4\alpha+1\leq 4$ and $k_n = o(n^{1/2})$, whereby $k_n^{4\alpha+1} = o(n^2)$.

Now, suppose that $3/4 < \alpha < 2$. Here, we will consider

$$k_n \cdot n^{-2} \cdot k_n^{2\alpha-1+4-2\alpha} = \frac{k_n^2}{n^2} = o(1).$$

When $\alpha \geq 2$, we will bound $\log k_n$ and 1 by k_n and we will consider

$$k_n \cdot n^{-2} \cdot k_n^{2\alpha-1+1} = n^{-2} k_n^{2\alpha+1}.$$

But $k_n = O(1) \cdot n^{\frac{1}{2\alpha+1}}$, whereby $k_n^{2\alpha+1} = O(n)$ and the above term is therefore $o(1)$.

The sixth term For the last term it suffices to bound

$$\mathbb{E}\left[\sum_{p_1,p_2 \in \mathcal{P}\setminus\{(0,y)\}} \mathbb{1}_{\{p_1,p_2 \in \mathcal{B}_{\mathbb{H},n}((0,y)) \cup \mathcal{B}_{\mathcal{P}}((0,y))\}} \cdot D_{\mathbb{H}}(y, k_n - 2; \mathcal{P} \setminus \{(0,y), p_1, p_2\})\right] \cdot \left|\overline{n}_{\mathbb{H}}(k_n)^{-1} - \overline{n}_{\mathcal{P}}(k_n)^{-1}\right|.$$

For the first term of this, the Camplell-Mecke formula (??) yields

$$\mathbb{E}\left[\sum_{p_1,p_2 \in \mathcal{P}\setminus\{(0,y)\}} \mathbb{1}_{\{p_1,p_2 \in \mathcal{B}_{\mathbb{H},n}((0,y)) \cup \mathcal{B}_{\mathcal{P}}((0,y))\}} \cdot D_{\mathbb{H}}(y, k_n - 2; \mathcal{P} \setminus \{(0,y), p_1, p_2\})\right] =$$
$$O(1) \cdot e^y \cdot \rho(y, k_n - 2).$$

Recall that by Lemma ?? $\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}((0,y))) = \lambda_y = O(1) \cdot e^{y/2}$. Hence,

$$\rho(y, k_n - 2) = e^{-\lambda_y} \frac{\lambda_y^{k_n-2}}{(k_n-2)!} = e^{-\lambda_y} \lambda_y^{k_n-2} \frac{1}{\Gamma(k_n-1)}.$$

This part of (69) gives:

$$\int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} e^y \cdot \rho(y, k_n - 2) \cdot e^{-\alpha y} dy dx = O(1) \cdot I_n \cdot \frac{1}{\Gamma(k_n - 1)} \int_{I_\varepsilon(k_n)} \lambda_y^2 \cdot e^{-\lambda_y} \cdot \lambda_y^{k_n-2} \cdot \lambda_y^{2\alpha} dy.$$

We evaluate this integral using Proposition ??

$$\begin{aligned}\int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} e^y \cdot \rho(y, k_n - 2) \cdot e^{-\alpha y} dy dx &= O(1) \cdot I_n \cdot \frac{\Gamma(k_n - 2\alpha)}{\Gamma(k_n - 1)} \\ &= O(n) \cdot k_n^{-2\alpha+1} = O(1) \cdot n \cdot k_n^{-2\alpha+1}.\end{aligned}$$

Recall also that by (??) we have

$$\overline{n}_{\mathbb{H}}(k_n), \overline{n}_{\mathcal{P}}(k_n) = \Theta(1) \cdot n \cdot k_n^{-(2\alpha+1)}.$$

Thus, we can express the above double integral as follows:

$$I := \int_{-I_n}^{I_n} \int_{I_\varepsilon(k_n)} e^y \cdot \rho(y, k_n - 2) \cdot e^{-\alpha y} dy dx = O(1) \cdot \overline{n}_{\mathbb{H}}(k_n) \cdot k_n^2.$$

Therefore, (69) yields

$$\begin{aligned}\binom{k_n}{2}^{-1} \cdot I \cdot \left|\overline{n}_{\mathbb{H}}(k_n)^{-1} - \overline{n}_{\mathcal{P}}(k_n)^{-1}\right| &= O(1) \cdot \overline{n}_{\mathbb{H}}(k_n) \cdot \left|\overline{n}_{\mathbb{H}}(k_n)^{-1} - \overline{n}_{\mathcal{P}}(k_n)^{-1}\right| \\ &= O(1) \cdot \left|1 - \frac{\overline{n}_{\mathbb{H}}(k_n)}{\overline{n}_{\mathcal{P}}(k_n)}\right|.\end{aligned}$$

We shall estimate this difference through the following lemma.

Lemma 7.6. *We have*

$$|\overline{n}_{\mathbb{H}}(k_n) - \overline{n}_{\mathcal{P}}(k_n)| = O(1) \cdot \begin{cases} R_n^2 \cdot n^{2(1-\alpha)} & \text{if } 1/2 < \alpha \leq 3/2 \\ n^{-1} & \text{if } \alpha > 3/2 \end{cases}.$$

Proof. We will bound this difference as follows

$$|\overline{n}_{\mathbb{H}}(k_n) - \overline{n}_{\mathcal{P}}(k_n)| \leq \mathbb{E}\left[\sum_{p \in \mathcal{P}} \mathbb{1}_{\{\mathcal{B}_{\mathbb{H} \triangle \mathcal{P}}(p) \cap \mathcal{P} \neq \emptyset\}}\right].$$

We will calculate this expectation using the Camplell-Mecke formula (**??**):

$$\mathbb{E}\left[\sum_{p \in \mathcal{P}} \mathbb{1}_{\{\mathcal{B}_{\mathbb{H} \triangle \mathcal{P}}(p) \cap \mathcal{P} \neq \emptyset\}}\right] \int_{-I_n}^{I_n} \int_0^{R_n} \mathbb{E}\left[\mathbb{1}_{\{\mathcal{B}_{\mathbb{H} \triangle \mathcal{P}}(p) \cap \mathcal{P} \setminus \{(x,y)\} \neq \emptyset\}}; \mathcal{P} \setminus \{(x,y)\}\right] e^{-\alpha y} dy dx$$
$$= \int_{-I_n}^{I_n} \int_0^{R_n} \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H} \triangle \mathcal{P}}((0,y))) \cdot e^{-\alpha y} dy dx$$
$$= \int_{-I_n}^{I_n} \int_0^{(1-\varepsilon)R_n} \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H} \triangle \mathcal{P}}((0,y))) \cdot e^{-\alpha y} dy dx + \int_{-I_n}^{I_n} \int_{(1-\varepsilon)R_n}^{R_n} \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H} \triangle \mathcal{P}}((0,y))) \cdot e^{-\alpha y} dy dx.$$

Using Lemma 7.3, for $\alpha \leq 3/2$ we have

$$\int_{-I_n}^{I_n} \int_0^{(1-\varepsilon)R_n} \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H} \triangle \mathcal{P}}((0,y))) \cdot e^{-\alpha y} dy dx =$$
$$\Theta(R_n) \cdot e^{(1/2-\alpha)R_n} \cdot 2I_n \int_0^{(1-\varepsilon)R_n} e^{\alpha y - \alpha y} dy \overset{I_n = \Theta(1) \cdot e^{R_n/2}}{=} \Theta(R_n^2) \cdot e^{(1-\alpha)R_n}$$
$$= \Theta(R_n^2) \cdot n^{2(1-\alpha)}.$$

For $\alpha > 3/2$, the corresponding estimate in Lemma 7.3 implies that

$$\int_{-I_n}^{I_n} \int_0^{(1-\varepsilon)R_n} \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H} \triangle \mathcal{P}}((0,y))) \cdot e^{-\alpha y} dy dx =$$
$$\Theta(1) \cdot e^{-R_n} \cdot 2I_n \int_0^{(1-\varepsilon)R_n} e^{3y/2 - \alpha y} dy \overset{I_n = \Theta(1) \cdot e^{R_n/2}}{=} \Theta(1) \cdot e^{-R_n/2}$$
$$= \Theta(1) \cdot n^{-1}.$$

□

Hence, since $\overline{n}_{\mathcal{P}}(k_n) = \Theta(1) \cdot n \cdot k_n^{-(2\alpha+1)}$, we deduce that for $1/2 < \alpha \leq 3/2$

$$\left|1 - \frac{\overline{n}_{\mathbb{H}}(k_n)}{\overline{n}_{\mathcal{P}}(k_n)}\right| = O(R_n^2) \cdot k_n^{2\alpha+1} \cdot n^{1-2\alpha}.$$

For $1/2 < \alpha < 3/4$, we then have

$$k_n^{4\alpha-2} \cdot I = O(R_n^2) \cdot k_n^{6\alpha-1} \cdot n^{1-2\alpha}.$$

But recall that $k_n = O(1) \cdot n^{\frac{1}{2\alpha+1}}$. Therefore,

$$k_n^{4\alpha-2} \cdot I = O(R_n^2) \cdot n^{\frac{6\alpha-1}{2\alpha+1} - 2\alpha}.$$

But $\frac{6\alpha-1}{2\alpha+1} - 2\alpha = \frac{4\alpha+1-4\alpha^2}{2\alpha+1}$. An elementary calculation reveals that the numerator of this fraction is negative for $\alpha > 1/2$, whereby

$$k_n^{4\alpha-2} \cdot \binom{k_n}{2}^{-1} \cdot I \to 0, \text{as } n \to \infty.$$

This also works for the case $\alpha = 3/4$, with $k_n/\log k_n$ instead of $k_n^{4\alpha-2} = k_n$.

Suppose now that $\alpha > 3/4$. In this case, we examine the product:

$$k_n \cdot \binom{k_n}{2}^{-1} \cdot I = O(R_n^2) \cdot k_n^{2\alpha+2} \cdot n^{-2\alpha} = O(R_n^2) \cdot n^{\frac{2\alpha+2}{2\alpha+1}-2\alpha}.$$

But $\frac{2\alpha+2}{2\alpha+1} - 2\alpha = 1 + \frac{1}{2\alpha+1} - 2\alpha = \frac{1}{2\alpha+1} - (2\alpha - 1) = \frac{1-4\alpha^2+1}{2\alpha+1} = 2\frac{1-2\alpha^2}{2\alpha+1}$. This is negative for any $\alpha > 1/\sqrt{2}$. But as $3/4 > 1/\sqrt{2}$, we deduce that the exponent of n is negative as therefore

$$k_n \cdot \binom{k_n}{2}^{-1} \cdot I \to 0, \text{ as } n \to \infty.$$

□

7.3 Coupling $G_{\mathbb{H},n}$ to $G_{\widetilde{\mathbb{H}},n}$

The main result of this section is the following

$$\lim_{n\to\infty} s_\alpha(k_n)\mathbb{E}\left[\left|c_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = 0.$$

We start by proving Lemma 2.3

Proof of Lemma 2.3. Note that by Proposition 7.8 and Proposition Let $0 < \delta < 1$ and define the event

$$A_n = \left\{ |N_{\mathbb{H},n}(k_n) - \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]| \le \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{\frac{1+\delta}{2}} \right\}.$$

Since $N_{\mathbb{H},n}(k_n) = \sum_{i=1}^n \mathbb{1}_{\{D_{\mathbb{H}}(i)=k_n\}}$ it follows from Lemma C.1, with $c = \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{-\frac{1-\delta}{2}}$, that

$$\mathbb{P}(A_n) \ge 1 - O\left(e^{-\frac{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^\delta}{2}}\right) = 1 - O\left(e^{-\frac{n^\delta k_n^{-\delta(2\alpha+1)}}{2}}\right), \tag{85}$$

where the last part is due to Lemma 7.7.

On the event A_n

$$\left|\frac{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]}{N_{\mathbb{H},n}(k_n)} - 1\right| \le \frac{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{\frac{1+\delta}{2}}}{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] + \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{\frac{1+\delta}{2}}} \le \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{-\frac{1-\delta}{2}}.$$

Therefore we have

$$\begin{aligned}
\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c_{\mathbb{H},n}(k_n)\right|\right] &\le \mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c_{\mathbb{H},n}(k_n)\right| \mathbb{1}_{\{A_n\}}\right] + O\left(1 - \mathbb{P}(A_n)\right) \\
&= \mathbb{E}\left[c^*_{\mathbb{H},n}(k_n)\left|\frac{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]}{N_{\mathbb{H},n}(k_n)} - 1\right| \mathbb{1}_{\{A_n\}}\right] + O\left(e^{-\frac{n^\delta k_n^{-\delta(2\alpha+1)}}{2}}\right) \\
&\le \mathbb{E}\left[c^*_{\mathbb{H},n}(k_n)\right]\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{-\frac{1-\delta}{2}} + O\left(e^{-\frac{n^\delta k_n^{-\delta(2\alpha+1)}}{2}}\right).
\end{aligned}$$

Since $\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] = \Theta\left(nk_n^{-(2\alpha+1)}\right) \to \infty$, the first term is clearly $o\left(\mathbb{E}\left[c^*_{\mathbb{H},n}(k_n)\right]\right)$. This term is $o\left(s_\alpha(k_n)\right)$ finishes the proof. □

Next we shall prove Proposition 2.5. First note that by combining Proposition 2.7 and Proposition **??** we have that

$$\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right] = \Theta\left(s_\alpha(k_n)\right) \tag{86}$$

To achieve the results we consider the standard coupling between the binomial and Poisson process. That is, we take a sequence of i.i.d. random elements $z_1, z_2, \ldots$ uniformly on the hyperbolic disk of radius R_n, i.e. according to the distribution (1). Then the original hyperbolic random graph consists of the first n points and the poissonized version of the first $N \stackrel{d}{=} Po(n)$ many points (N is a Poisson random variable with mean n). Under this coupling $N_{\mathbb{H},n}(k) = \sum_{j=1}^{n} \mathbb{1}_{\{D_{\mathbb{H}}(z_j)=k\}}$ denotes the number of degree k vertices in the original Hyperbolic random graph model with n vertices and $N_{\widetilde{\mathbb{H}},n}(k) = \sum_{j=1}^{N} \mathbb{1}_{\{D_{\widetilde{\mathbb{H}}}(z_j)=k\}}$ denotes the number of degree k vertices in the Poisson version of the Hyperbolic random graph.

Lemma 7.7. *Let $\{k_n\}_{n\geq 1}$ be sequence of natural numbers with $0 \leq k_n \leq n-1$ and $k_n = o(n^{\frac{1}{2\alpha+1}})$. Then*

$$\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right) = o\left(nk^{-(2\alpha+1)}\right).$$

Proof. We use the Chernoff concentration result for a Poisson random (97), with probability $n^{-c^2/2}$ the Poisson random variable N with expectation n is contained in the interval $[n - c\sqrt{n\log n}, n + c\sqrt{n\log n}]$. We proceed by bounding the effect on the number of degree k_n vertices of adding or removing $c\sqrt{n\log n}$ many vertices to $G_{\mathbb{H},n}(\alpha, \nu$ and from $G_{\widetilde{\mathbb{H}},n}(\alpha, \nu)$, respectively.

Define the events

$$A_n^{(1)} := \{N \in [n, n + c\sqrt{n\log n}]\}$$
$$A_n^{(2)} := \{N \in [n - c\sqrt{n\log n}, n)\}$$

and let $A_n = A_n^{(1)} \cup A_n^{(2)}$. Then,

$$\begin{aligned}\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] &\leq \mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right| \Big| A_n\right] + O\left(n^{1-c^2/2}\right)\\ &= \mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right| \Big| A_n\right] + o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right)\end{aligned}$$

by choosing c large enough, e.g. $c > \sqrt{2}$. What is left to show is that for any $c > 0$

$$\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right| \Big| A_n\right] = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right).$$

Let $V_{\mathbb{H},n}(k_n)$ be the set of degree k_n vertices in the binomial graph $G_{\mathbb{H},n}$ and $V_{\widetilde{\mathbb{H}},n}(k_n)$ be the set of degree k_n vertices in the Poisson model $G_{\widetilde{\mathbb{H}},n}$. Then

$$|N_k^{(n)} - N_k^{(Po(n))}| = |V_{\mathbb{H},n}(k_n)\Delta V_{\widetilde{\mathbb{H}},n}(k_n)| = |V_{\mathbb{H},n}(k_n)\backslash V_{\widetilde{\mathbb{H}},n}(k_n)| + |V_{\widetilde{\mathbb{H}},n}(k_n)\backslash V_{\mathbb{H},n}(k_n)|,$$

where $A\Delta B$ denotes the symmetric difference between two sets A and B.

We first consider the case $N \in [n, n + c\sqrt{n\log n}]$, i.e. event $A_n^{(1)}$. For $z \in V_{\widetilde{\mathbb{H}},n}(k_n)\backslash V_{\mathbb{H},n}(k_n)$, z has degree k in the Poisson graph, but not in the binomial graph; so as $N \geq n$, either z or one of its k neighbors must have been removed during the transition from the Poisson graph to the binomial graph. On the event A_n, at most $c\sqrt{n\log n}$ many vertices are removed. The probability of hitting a degree k vertex or one of its neighbors is $\leq \frac{k+1}{N} \leq \frac{k+1}{n}$. Therefore, by the union bound the probability that a particular degree k vertex of the Poisson graph is removed is upper bounded by $c\sqrt{n\log n}\frac{k+1}{n}$. Hence, the expected number of degree k_n vertices that disappear in the transition from the Poisson graph to the binomial graph is bounded by

$$\mathbb{E}\left[|V_{\widetilde{\mathbb{H}},n}(k_n)\backslash V_{\mathbb{H},n}(k_n)\Big| A_n^{(1)}\right] \leq \mathbb{E}[N_{\widetilde{\mathbb{H}},n}(k_n)]c\sqrt{n\log n}\frac{k_n+1}{n} = o(\mathbb{E}[N_k^{(Po(n))}]),$$

where the last line follows since for $\alpha > 1/2$,

$$k_n\sqrt{\frac{\log(n)}{n}} = o\left(n^{\frac{1}{2\alpha+1}}\sqrt{\frac{\log(n)}{n}}\right) = o\left(n^{-\frac{2\alpha-1}{4\alpha+2}}\sqrt{\log(n)}\right) = o\left(1\right).$$

For $z \in V_{\mathbb{H},n}(k_n)\backslash V_{\widetilde{\mathbb{H}},n}(k_n)$, z is a degree k_n vertex in the binomial graph, but must have degree $k_n + \ell$ in the Poisson graph (where $1 \leq \ell \leq c\sqrt{n \log n}$). By linearity of expectation the expected number of degree $k_n + \ell$ vertices of the Poisson graph which turn into degree k_n vertices of the binomial graph is equal to the expected number of degree $k_n + \ell$ vertices in the Poisson graph times the probability that a degree $k_n + \ell$ vertex turns into a degree k_n vertex in the transition back, from the Poisson graph to the binomial graph. The probability of choosing uniformly a set of ℓ neighbors of a degree $k_n + \ell$ vertex of the Poisson graph is given by $\frac{k_n+\ell}{N} \cdots \frac{k+1}{N-\ell+1}$. Now, using $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right) = o\left(c\sqrt{n \log n}\right)$ for $\alpha > \frac{1}{2}$, $\ell \leq c\sqrt{n \log n}$ and $N - \ell + 1 \geq n$, this probability is bounded from above by $(c+1)^\ell (\frac{\sqrt{n \log n}}{n})^\ell = ((c+1)\sqrt{\frac{\log n}{n}})^\ell$ which is upper bounded by $(\frac{1}{2})^\ell$ for n large enough, i.e. $n \geq n_0$. Therefore, using the geometric series, we conclude

$$\begin{aligned}
\mathbb{E}\left[|V_{\mathbb{H},n}(k_n)\backslash V_{\widetilde{\mathbb{H}},n}(k_n)|\,\Big|\, A_n^{(1)}\right] &\leq \sum_{\ell=1}^{\sqrt{n \log n}} \mathbb{E}[N_{k_n+\ell}^{(Po(n))}]((c+1)\sqrt{\frac{\log n}{n}})^\ell \\
&\leq \sum_{\ell=1}^{\sqrt{n \log n}} \Theta\left(n(k_n+\ell)^{-2\alpha-1}\right)((c+1)\sqrt{\frac{\log n}{n}})^\ell \\
&= O\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right) \sum_{\ell=1}^{\sqrt{n \log n}} ((c+1)\sqrt{\frac{\log n}{n}})^\ell = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right),
\end{aligned}$$

and hence

$$\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\,\Big|\, A_n^{(1)}\right] = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right).$$

The case $N \in [n - c\sqrt{n \log n}, n)$ (event $A_n^{(2)}$) works similarly. As $N < n$, a vertex $z \in V_{\widetilde{\mathbb{H}},n}(k_n)\backslash V_{\mathbb{H},n}(k_n)$ with degree k_n in the Poisson graph must have a strictly larger degree in the binomial graph, i.e. in the transition from the Poisson graph to the binomial graph, a vertex must have been dropped in the neighborhood of z. By the union bound, this can be upper bounded by the number of additional vertices (of the binomial graph) times the probability that a random point falls into the neighborhood of a degree k vertex. We obtain

$$\mathbb{E}\left[|V_{\widetilde{\mathbb{H}},n}(k_n)\backslash V_{\mathbb{H},n}(k_n)|\,\Big|\, A_n^{(2)}\right] = O\left(\sqrt{n \log n}\frac{k}{n}\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right) = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right)$$

A vertex $z \in V_{\mathbb{H},n}(k_n)\backslash V_{\widetilde{\mathbb{H}},n}(k_n)$ could be one of the additional vertices in the binomial graph or it is a degree $k_n - \ell$ vertex of the Poisson graph which receives exactly ℓ new vertices in its neighborhood in the transition from the Poisson graph to the binomial graph. The probability that one of the additional vertices of the binomial graph (compared to the smaller Poisson graph) has degree k_n has the asymptotic order $k_n^{-(2\alpha+1)}$ (as can be seen by considering the alternative coupling between the binomial and the Poisson process, where instead of taking $z_1, \ldots, z_N$ for the Poisson process, we take the points $z_n, z_{n-1}, \ldots, z_{n-N+1}$ (resp. points with index larger than n after we hit z_1): for this graph, we have that the expected number of degree k_n vertices is $\Theta\left(nk^{-2\alpha-1}\right)$, so the probability that a vertex chosen uniformly from the Poisson graph has degree k is $\Theta(k^{-2\alpha-1})$). Therefore, the expected number of additional points with degree k_n is $O\left(\sqrt{n \log n}k_n^{-2\alpha-1}\right) = o\left(nk_n^{-2\alpha-1}\right) = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right)$. The expected number of degree $k_n - \ell$ vertices of the Poisson graph which receive exactly ℓ new vertices can be bounded in a sum resp. series similarly as done for $z \in V_{\mathbb{H},n}(k_n)\backslash V_{\widetilde{\mathbb{H}},n}(k_n)$ in the case $N \geq n$. We therefore conclude that

$$\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\,\Big|\, A_n^{(2)}\right] = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right),$$

which finishes the proof. □

Proposition 7.8.

$$\lim_{n\to\infty} s_\alpha(k_n)\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = 0.$$

Proof. **Pim:** Proof taken from Markus notes (slightly edited). Should probably be improved. We are looking at the modified clustering coefficient, where we divide by the expected number of degree k_n vertices. As the expected numbers of degree k_n vertices in the Poisson and binomial graph are asymptotically equivalent, it is therefore sufficient to consider the sum of the clustering coefficients of all vertices of degree k. Given again the standard coupling between the binomial and Poisson process, we denote by $V_{\mathbb{H},n}(k_n)$ the set of degree k_n vertices in the binomial graph and by $V_{\widetilde{\mathbb{H}},n}(k_n)$ the set of degree k_n vertices in the Poisson graph. If a vertex is contained in both sets, it must have the same degree in both the Poisson and binomial graph, and given the nature of the coupling, the neighbourhoods are therefore the same and hence also their clustering coefficients agree.

The difference of the sum of the clustering coefficients therefore comes from all the clustering coefficients of the symmetric difference $V_{\mathbb{H},n}(k_n)\Delta V_{\widetilde{\mathbb{H}},n}(k_n)$. This symmetric difference is again a Poisson process, whose expected number of points is $\mathbb{E}|N_k^{(n)} - N_k^{(Po(n))}| = o(\mathbb{E}N_k^{(Po(n))})$. Using the Palm-Mecke formula, we have that

$$\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] \le \frac{\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\right]}{\overline{n}_{\mathbb{H}}(k_n)} \mathbb{E}\left[c^*_{\mathbb{H},n}(k_n)\right] = o\left(1\right)\mathbb{E}\left[c^*_{\mathbb{H},n}(k_n)\right],$$

where the last line follows from Lemma 7.7. The result now follows by applying Proposition 2.5, 2.7 and Proposition **??**. ☐

A Meijer's G-function

Recall that $\Gamma(z)$ denotes the Gamma function. Let p, q, m, ℓ be four integers satisfying $0 \le m \le q$ and $0 \le \ell \le p$ and consider two sequences $\mathbf{a}_p = \{a_1, \dots, a_p\}$ and $\mathbf{b}_q = \{b_1, \dots, b_q\}$ of reals such that $a_i - b_j$ is not a positive integer for all $1 \le i \le p$ and $1 \le j \le q$ and $a_i - a_j$ is not an integer for all distinct indices $1 \le i, j \le p$. Then, with $\imath$ denoting the complex unit, Meijer's G-Function [16] is defined as

$$G_{p,q}^{m,\ell}\left(z\middle|\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right) = \frac{1}{2\pi\imath}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - t)\prod_{j=1}^\ell \Gamma(1 - a_j + t)}{\prod_{j=m+1}^q \Gamma(1 - b_j + t)\prod_{j=\ell+1}^p \Gamma_n(a_j - t)}\, z^t \,\mathrm{d}t, \tag{87}$$

where the path L is an upward oriented loop contour which separates the poles of the function $\prod_{j=1}^m \Gamma(b_j - t)$ from those of $\prod_{j=1}^n \Gamma(1 - a_j + t)$ and begins and ends at $+\infty$ or $-\infty$.

The Meijer's G-Function is of a very general nature and has relation to many known special functions such as the Gamma function and the generalized hypergeometric function. For more details, such as many identities for $G_{p,q}^{m,\ell}\left(z\middle|\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right)$ see [11, 15].

For our purpose we need the following identity which follows from an Mellin transform operation.

Lemma A.1. *For any $a \in \mathbb{R}$ and $\xi, s > 0$,*

$$\Gamma^+(-a-1, \xi/s) = G_{1,2}^{2,0}\left(\frac{\xi}{s}\middle|\begin{matrix}1\\ -a-1, 0\end{matrix}\right)$$

Proof. Let $x > 0$ and $q \in \mathbb{R}$ and note that as the Γ-function is the Mellin transform of e^{-x}, by the inverse Mellin transform formula, we have $e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(p)x^{-p}dp$ for $c > 0$ (see [7, p.196]). Applying the change of variable $p(r) = q - r$ yields $e^{-x} = \frac{1}{2\pi i}\int_{c+q-i\infty}^{c+q+i\infty}\Gamma(q-r)x^{r-q}dr$, then multiplying both sides with $-x^{q-1}$ gives $-x^{q-1}e^{-x} = -\frac{1}{2\pi i}\int_{c+q-i\infty}^{c+q+i\infty}\Gamma(q-r)x^{r-1}dr$. Now, integrating both sides gives $\int_x^\infty t^{q-1}e^{-t}dt = \frac{1}{2\pi i}\int_{c+q-i\infty}^{c+q+i\infty}\frac{\Gamma(q-r)}{-r}x^r dr$. On the left-hand side is the incomplete gamma function and on the right-hand side with using $-r = \frac{\Gamma(1-r)}{\Gamma(-r)}$ is the Meijer

G-function, i.e. $\Gamma^+(q,x) = G^{2,0}_{1,2}\left(\begin{matrix}1\\ q,0\end{matrix}\middle| x\right)$. The claim follows by plugging in $q=-a-1$ and $x=\frac{\xi}{s}$. $\square$

Properties of $G_{\mathbb{H},n}(\alpha,\nu)$

Pim: @Markus: is the lemma below only needed to prove the second lemma? If so than we can remove it because the statements of the second lemma have already been established in other parts of the paper.

Lemma A.2 (Intensity measure of the intersection of two neighbourhood balls)**.** *Consider the Poisson hyperbolic random graph conditioned on having two vertices with degree k at heights $y_1, y_2 \in [y_{k,-}, y_{k,+}]$ and horizontal distance d. Then the expected number of vertices in the intersection of their neighbourhood balls in the infinite limit model is $\Theta(k^{2\alpha}d^{1-2\alpha})$.*

Proof. Denote the two vertices by v and w. If the neighbourhood balls of v and w intersect at the minimal height y_h, then $d = e^{\frac{y_h+y_1}{2}} + e^{\frac{y_h+y_2}{2}}$.

Due to our assumption $y_1, y_2 \in [y_{k,-}, y_{k,+}]$, this condition implies $d \le 2e^{\frac{y_h+y_{k,+}}{2}}$. This implies $y_h \ge 2\ln\frac{d}{2} - y_{k,+}$. (In a similar fashion, we infer that $y_h \le 2\ln\frac{d}{2} - y_{k,-}$.)

Therefore, using $y_1, y_2 \in [y_{k,-}, y_{k,+}]$ and $e^{\frac{y_1}{2}}, e^{\frac{y_2}{2}} \sim k$, $e^{\frac{y_h}{2}} \sim \frac{d}{k}$:

$$\begin{aligned}\mu_3 &= \int_{y_h}^{\infty} (e^{\frac{y_1+y}{2}} + e^{\frac{y_2+y}{2}} - d)\alpha e^{-\alpha y}dy = \Theta(k)\frac{\alpha}{\alpha-\frac{1}{2}}e^{(\frac{1}{2}-\alpha)y_h} + de^{-\alpha y_h}\\ &= \Theta(k(\frac{d}{k})^{1-2\alpha} + d(\frac{d}{k})^{-2\alpha}) = \Theta(k^{2\alpha}d^{1-2\alpha})\end{aligned}$$

$\square$

The following proposition establishes several properties on the number of vertices with degrees k_n in the Poisson hyperbolic graph $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$.

Proposition A.3. *Let $\alpha > \frac{1}{2}$, $\nu > 0$, consider the Poisson hyperbolic random geometric graph $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$ and let $N_{\widetilde{\mathbb{H}},n}(k_n)$ be the number of degree k_n vertices.*

1. *If $2 \le k_n \le n-1$ such that $k_n \to \infty$ as $n \to \infty$, then $\mathbb{E}[N_{\widetilde{\mathbb{H}},n}(k_n)] \sim 2\alpha\xi^{2\alpha}nk_n^{-(2\alpha+1)}$ as $n \to \infty$.*
2. *In particular if $k_n \gg n^{\frac{1}{2\alpha+1}}$, then $N_{\widetilde{\mathbb{H}},n}(k_n) = 0$ w.h.p.*
3. *Furthermore, if $k_n \ll n^{\frac{1}{2\alpha+1}}$, then $\frac{\mathbb{E}[N_{\widetilde{\mathbb{H}},n}(k_n)^2]}{\mathbb{E}[N_{\widetilde{\mathbb{H}},n}(k_n)]^2} \to 1$, so in particular in this case w.h.p.*

$$N_{\widetilde{\mathbb{H}},n}(k_n) = (1+o(1))\mathbb{E}[N_{\widetilde{\mathbb{H}},n}(k_n)]$$

and $N_{\widetilde{\mathbb{H}},n}(k_n) > 0$ w.h.p.

Proof. **Pim:** The results from this lemma are already stated elsewhere but I kept the lemma to not break any references. It will be cleaned next round. $\square$

B Properties of $G_{\mathcal{P},n}(\alpha,\nu)$

B.1 Neighborhoods in $G_{\mathcal{P},n}(\alpha,\nu)$

Next we consider the number of disjoint neighbors for two points p and p' in $G_{\mathcal{P},n}(\alpha,\nu)$, which we denote by $\mathcal{N}_{\mathcal{P},n}(p\Delta p')$. We will distinguish between the cases where the distance between their

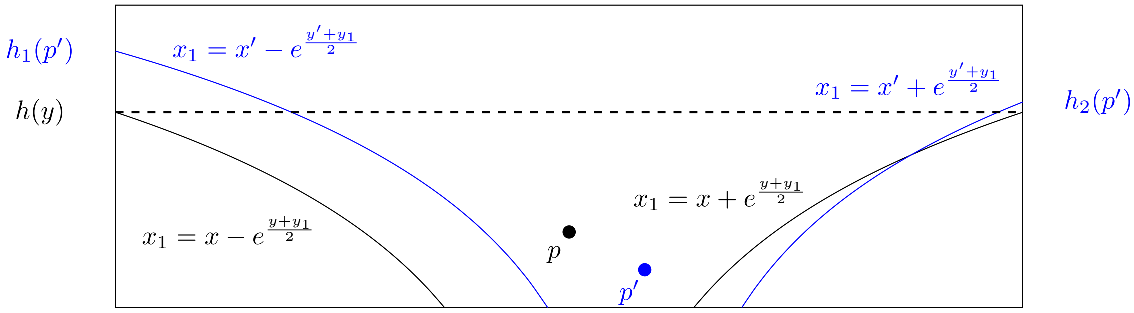

Figure 8: Schematic representation of the neighborhoods of p and p' in $G_{\mathcal{P},n}(\alpha,\nu)$ when $|x-x'| \leq e^{\frac{y+y'}{2}}$ used for the proof of Lemma B.3.

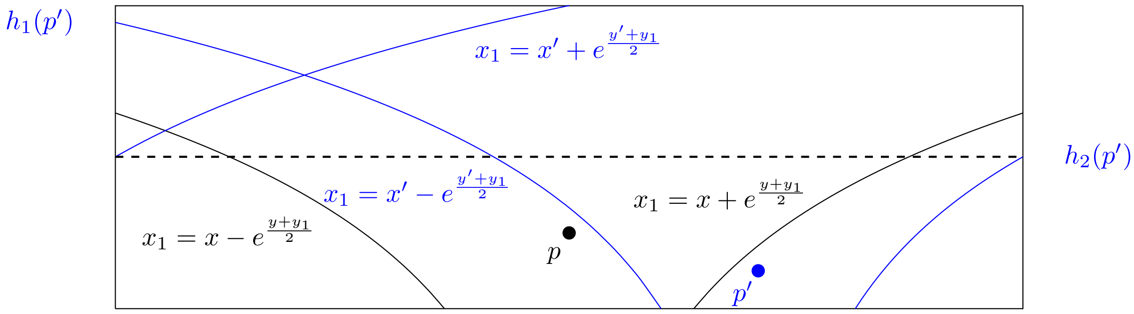

Figure 9: Schematic representation of the neighborhoods of p and p' in $G_{\mathcal{P},n}(\alpha,\nu)$ when $e^{\frac{y+y'}{2}} < |x-x'| \leq e^{\frac{y^*}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right)$ used for the proof of Lemma B.3.

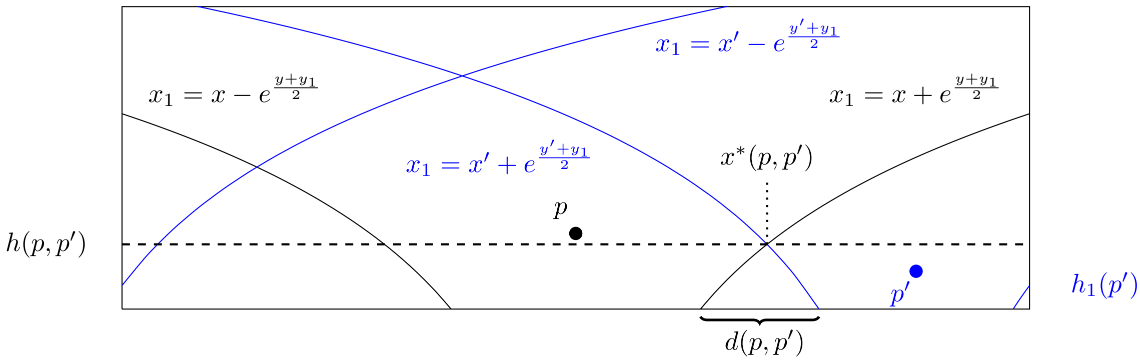

Figure 10: Schematic representation of the neighborhoods of p and p' in $G_{\mathcal{P},n}(\alpha,\nu)$ when $|x-x'| > e^{\frac{y^*}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right)$ used for the proof of Lemma B.2.

x-coordinates is small and large. Due to symmetry and the fact that we have identified the left and right boundaries of the box $\mathcal{R}_n$, we can, without loss of generality, assume that $p = (0, y)$ and $p' = (x', y')$ with $x' > 0$ and $y' \leq y$. To understand the computation it is helpful to have a picture of the different situations. Figure 9 and Figure 10 show two different situations for small distance in the x-coordinates, in which case the number of disjoint neighbors is small. The case where this distance is large, and the number of disjoint neighbors is expected to be large, is show in Figure 11. There several different quantities that are important. The are the heights $h_1(p')$ and $h_2(p')$ where, respectively, the left and right boundaries of the ball $B_{\mathcal{P},n}(p')$ go outside the box $\mathcal{R}_n$. Note that when $x = 0$ then these height are the same and we denote this by $h(y)$. We also need to know the coordinates $h(p,p')$ and $x^*(p,p')$ of the intersection of the right boundary of the neighborhoods of p with the left boundary of that of p'. Finally we will denote by $d(p,p')$ the distance between the lower right boundary of $B_{\mathcal{P},n}(p)$ and the lower left of $B_{\mathcal{P},n}(p')$, which is positive only when the bottom parts of both neighborhoods do not intersect, compare Figures 9 and 11. The full expressions of all these functions are given below for further reference.

$$h(y) = R_n - y + 2\log\left(\frac{\pi}{2}\right) \tag{88}$$

$$h_1(p') = 2\log\left(x' + \frac{\pi}{2}e^{\frac{R_n}{2}}\right) - y' \tag{89}$$

$$h_2(p') = 2\log\left(\frac{\pi}{2}e^{\frac{R_n}{2}} - x'\right) - y' \tag{90}$$

$$h((p,p') = 2\log\left(\frac{|x-x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}\right) \tag{91}$$

$$x^*(p,p') = \frac{xe^{\frac{y'}{2}} + x'e^{\frac{y}{2}}}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}, \tag{92}$$

$$d(p,p') = |x-x'| - e^{\frac{y_*}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right) > 0, \tag{93}$$

We start with the result for points whose x-coordinates are close, which is when $d(p,p') < 0$.

Lemma B.1. *Let $p, p' \in \mathcal{R}_n$ and let $y^* = \min\{y, y'\}$. Then whenever $|x-x'| \le e^{\frac{y^*}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right)$,*

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right] = \Omega\left(|x-x'| + \left|e^{\frac{y}{2}} - e^{\frac{y'}{2}}\right|\right),$$

as $n \to \infty$.

Proof. In order to proof the result we will consider the area in between the two left boundaries of the balls $B_{\mathcal{P},n}(p)$ and $B_{\mathcal{P},n}(p')$ up to the height $h^* := \min\{h(y), h_2(p')\}$. We get

$$\begin{aligned}
\mathbb{E}\left[\mathcal{P}_n \cap (B_{\mathcal{P},n}(p)\Delta B_{\mathcal{P},n}(p'))\right] &\ge \int_0^{h^*}\int_{x-e^{\frac{y+y_1}{2}}}^{x'-e^{\frac{y'+y_1}{2}}} f_{\alpha,\nu}(x_1,y_1)\,dx_1\,dy_1 \\
&= \frac{\alpha\nu}{\pi}|x'-x|\int_0^{h^*} e^{-\alpha y_1}\,dy_1 + \frac{\alpha\nu}{\pi}\left|e^{\frac{y}{2}} - e^{\frac{y'}{2}}\right|\int_0^{h^*} e^{-(\alpha-\frac{1}{2})y_1}\,dy_1 \\
&= \Theta\left(|x-x'| + \left|e^{\frac{y}{2}} - e^{\frac{y'}{2}}\right|\right),
\end{aligned}$$

where we used that h^* $\square$

Now we will consider the case where $|x-x'| > e^{\frac{y^*}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right)$

Lemma B.2. *Let $p, p' \in \mathcal{R}_n$ and let $y^* = \min\{y, y'\}$. Then, whenever $|x-x'| > e^{\frac{y^*}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right)$,*

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right] \ge \left(\mu_{\alpha,\nu,n}(B_{\mathcal{P},n}(p)) + \mu_{\alpha,\nu,n}(B_{\mathcal{P},n}(p'))\right)\left(1 - \Theta\left(e^{-(\alpha-\frac{1}{2})y^*} - e^{-(\alpha-\frac{1}{2})R_n}\right)\right)$$

as $n \to \infty$.

Proof. We will prove the results by using that

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right] \ge \int_0^{h(p,p')}\int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \mathbb{P}\left(p_1 \in B_{\mathcal{P},n}(p) \cup B_{\mathcal{P},n}(p')\right) f_{\alpha,\nu}(x_1,y_1)\,dx_1\,dy_1,$$

and computing the integral on the right. Before we proceed we show that the neighborhoods of p and p' below $h(p,p')$ are disjoint. This is clearly true when $h(p,p') \le h_2(p')$ so suppose that

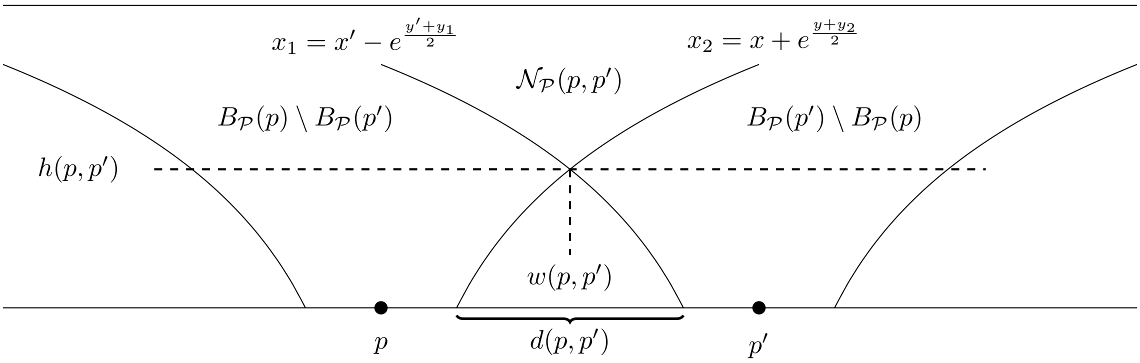

Figure 11: Schematic representation of the neighborhoods of p and p' in $G_{\mathcal{P},n}$ used for the proof of Lemma B.3.

$h(p,p') > h_2(p')$. Then, because we identified the right and left boundaries of the box $\mathcal{R}_n$ the right boundary of $B_{\mathcal{P},n}(p')$ continues from the left boundary of the box and is described by the equation

$$x_1 = x' + e^{\frac{y'+y_1}{2}} - \pi e^{\frac{R_n}{2}}.$$

Now, let x'_{right} and x_{left} denote the x-coordinate of the intersection of the line $h(p,p')$ with, respectively, the right boundary of $B_{\mathcal{P},n}(p')$ and the left boundary of $B_{\mathcal{P},n}(p)$. Then

$$\begin{aligned}
x'_{\text{right}} &= x' + e^{\frac{y'+h(p,p')}{2}} - \pi e^{\frac{R_n}{2}} \\
&= x' + e^{\frac{h(p,p')}{2}} \left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right) - e\frac{y + h(p,p')}{2} - \pi e^{\frac{R_n}{2}} \\
&= x' + |x - x'| - e\frac{y + h(p,p')}{2} - \pi e^{\frac{R_n}{2}} \\
&= x - e\frac{y + h(p,p')}{2} + 2|x - x'| - \pi e^{\frac{R_n}{2}} \\
&\leq x - e\frac{y + h(p,p')}{2} = x_{\text{right}},
\end{aligned}$$

and hence the neighborhoods of p and p' below $h(p,p')$ are disjoint. It then follows that

$$\begin{aligned}
\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p,p')\right] &\geq \int_0^{h(p,p')} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \mathbb{P}\left(p_1 \in B_{\mathcal{P},n}(p) \cup B_{\mathcal{P},n}(p')\right) f_{\alpha,\nu}(x_1, y_1)\, dx_1\, dy_1 \\
&= \int_0^{h(p,p')} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \left(\mathbb{P}\left(p_1 \in B_{\mathcal{P},n}(p)\right) + \mathbb{P}\left(p_1 \in B_{\mathcal{P},n}(p')\right)\right) f_{\alpha,\nu}(x_1, y_1)\, dx_1\, dy_1 \\
&= \left(\mu_{\alpha,\nu,n}(B_{\mathcal{P},n}(p)) + \mu_{\alpha,\nu,n}(B_{\mathcal{P},n}(p'))\right) \left(1 - \frac{2\alpha\nu}{\pi} \int_{h(p,p')}^{R_n} e^{-(\alpha - \frac{1}{2})y_1}\, dy_1\right),
\end{aligned}$$

from which the result follows since

$$\int_{h(p,p')}^{R_n} e^{-(\alpha-\frac{1}{2})y_1}\, dy_1 = \Theta\left(e^{-(\alpha-\frac{1}{2})y^*} - e^{-(\alpha-\frac{1}{2})R_n}\right).$$

□

Denote by $\mathcal{N}_{\mathcal{P}}(p,p')$ the number if common neighbors between two nodes p and p' in $G_{\mathcal{P}}$.

Lemma B.3. *Let $p, p' \in \mathcal{R}_n$ and let $y_* = \min\{y, y'\}$. Then, whenever $|x - x'| > e^{\frac{y_*}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right)$,*

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P}}(p, p')\right] = \frac{\nu\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right)}{2\alpha - 1}\left(\left(\frac{|x - x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}\right)^{-(2\alpha-1)} + \Theta\left(e^{-(\alpha - \frac{1}{2})R_n}\right)\right)$$

Proof. Assume, without loss of generality, that $y \geq y'$ and $x \leq x'$ and consider the boundaries of the "balls" $B_{\mathcal{P},n}(p)$ and $B_{\mathcal{P},n}(p')$ as drawn in Figure 12. The left boundary of $B_{\mathcal{P},n}(p)$ intersects the right boundary of $B_{\mathcal{P},n}(p')$ if an only if

$$d(p, p') := |x - x'| - e^{\frac{y_*}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right) > 0,$$

which is the condition we imposed in the statement of the lemma. The x-coordinate of the intersection is given by

$$w(p, p') := \frac{xe^{\frac{y'}{2}} + x'e^{\frac{y}{2}}}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}},$$

while the y-coordinate equals

$$h((p, p') := 2\log\left(\frac{|x - x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}\right).$$

Therefore (see Figure 12)

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P}}(p, p')\right] = \int_{h(p,p')}^{\infty}\int_{x - e^{\frac{y+y_1}{2}}}^{w(p,p')} f_{\alpha,\nu}(x_1, y_1)\, dx_1\, dy_1 \tag{94}$$

$$+ \int_{h(p,p')}^{\infty}\int_{w(p,p')}^{x + e^{\frac{y+y_2}{2}}} f_{\alpha,\nu}(x_2, y_2)\, dx_2\, dy_2 \tag{95}$$

We first compute (94).

$$\begin{aligned}
&\int_{h(p,p')}^{\infty}\int_{x - e^{\frac{y+y_1}{2}}}^{w(p,p')} f_{\alpha,\nu}(x_1, y_1)\, dx_1\, dy_1 \\
&\quad = \frac{\alpha\nu}{\pi}\int_{h(p,p')}^{\infty}\left(e^{\frac{y'+y_1}{2}} - (x' - w(p, p'))\right)e^{-\alpha y_1}\, dy_1 \\
&\quad = \frac{\alpha\nu}{\pi}e^{\frac{y'}{2}}\int_{h(p,p')}^{\infty}\left(e^{-(\alpha-\frac{1}{2})y_1} - \frac{|x - x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}e^{-\alpha y_1}\right)dy_1 \\
&\quad = \frac{\alpha\nu}{\pi(\alpha - \frac{1}{2})}e^{\frac{y'}{2}}\left(\frac{|x - x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}\right)^{-(2\alpha-1)} + \frac{\nu}{\pi}e^{\frac{y'}{2}}\frac{|x - x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}\left(\frac{|x - x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}\right)^{-2\alpha} \\
&\quad = \frac{\nu}{2\alpha - 1}e^{\frac{y'}{2}}\left(\frac{|x - x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}\right)^{-(2\alpha-1)}.
\end{aligned}$$

In a similar way we get that (95) equals

$$\frac{\nu}{2\alpha - 1}e^{\frac{y}{2}}\left(\frac{|x - x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}\right)^{-(2\alpha-1)}.$$

Therefore we conclude that

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P}}(p, p')\right] = \frac{\nu\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right)}{2\alpha - 1}\left(\frac{|x - x'|}{e^{\frac{y}{2}} + e^{\frac{y'}{2}}}\right)^{-(2\alpha-1)}.$$

□

B.2 Joint distributions in $G_{\mathcal{P},n}(\alpha, \nu)$

To ease notations we introduce the following short-hand notation for the conditional joint degree distribution

$$\varphi_{p,p'}(k, k') := \mathbb{P}\left(D_{\mathcal{P},n}(p) = k, D_{\mathcal{P},n}(p') = k'\right).$$

Lemma B.4. *Let $\alpha > \frac{1}{2}$, $\nu > 0$, $k_n \to \infty$ and fix $\varepsilon > 0$. Then for any fixed $i, j, i', j' \in \mathbb{Z}$ and $p, p' \in \mathcal{K}_\varepsilon(k_n)$ such that $\mathbb{E}\left[|\mathcal{N}_{\mathcal{P},n}(p, p')|\right] \geq k_n^\varepsilon$,*

$$\varphi_{p,p'}(k_n + i, k_n + i') = (1 + o(1))\varphi_{p,p'}(k_n + j, k_n + j') \pm e^{-\Omega(k_n^\varepsilon)}.$$

Proof. Define

$$\begin{aligned}
X_n &= \left|\mathcal{N}_{\mathcal{P},n}(p) \setminus \mathcal{N}_{\mathcal{P},n}(p')\right|, \\
Y_n &= \left|\mathcal{N}_{\mathcal{P},n}(p) \cap \mathcal{N}_{\mathcal{P},n}(p')\right|, \\
Z_n &= \left|\mathcal{N}_{\mathcal{P},n}(p') \setminus \mathcal{N}_{\mathcal{P},n}(p)\right|.
\end{aligned}$$

Then it follows that X_n, Y_n and Z_n are independent Poisson random variables satisfying $\mathbb{E}[X_n] + \mathbb{E}[Y_n] = \mu_n(B_{\mathcal{P},n}(p))$ and $\mathbb{E}[Z_n] + \mathbb{E}[Y_n] = \mu_n(B_{\mathcal{P},n}(p'))$ while

$$\begin{aligned}
\varphi(p, p', k_n + i, k_n + i') &= \mathbb{P}\left(X_n + Y_n = k + i, Z_n + Y_n = k + i'\right) \\
&= \sum_{\ell=0}^{\infty} \mathbb{P}\left(Y_n = \ell\right) \mathbb{P}\left(X_n = k + i - \ell, Z_n = k + i' - \ell\right) \\
&= \sum_{\ell=0}^{\infty} \mathbb{P}\left(Y_n = \ell\right) \mathbb{P}\left(X_n = k + i - \ell\right) \mathbb{P}\left(Z_n = k + i' - \ell\right).
\end{aligned}$$

Next define $\delta_n = k_n^{-\frac{1-\varepsilon}{2}}$, let n be large enough such that $0 < \delta_n < 1$ and note that by a Chernoff bound,

$$\mathbb{P}\left(|X_n - \mathbb{E}[X_n]| > \delta_n \mathbb{E}[X_n]\right) = O\left(e^{-\frac{\delta_n^2}{4(1+\delta_n)}\mathbb{E}[X_n]}\right),$$

and similar for Y_n and Z_n. Finally, we define

$$\begin{aligned}
L_X &= \{\ell : (1 - \delta_n)\mathbb{E}[X_n] \leq k + i - \ell \leq (1 + \delta_n)\mathbb{E}[X_n]\} \\
L_y &= \{\ell : (1 - \delta_n)\mathbb{E}[Y_n] \leq \ell \leq (1 + \delta_n)\mathbb{E}[Y_n]\} \\
L_Z &= \{\ell : (1 - \delta_n)\mathbb{E}[Z_n] \leq k + i' - \ell \leq (1 + \delta_n)\mathbb{E}[Z_n]\}
\end{aligned}$$

We will now make distinguish between the cases $\mathbb{E}[Y_n] \leq k_n/2$ and $\mathbb{E}[Y_n] > k_n/n$.

Let us first assume that $\mathbb{E}[Y_n] \leq k_n/2$. Then, since $p, p' \in \mathcal{K}_\varepsilon(k_n)$, $\mu_{\alpha,\nu,n}(B_{\mathcal{P},n}(p)) = \Theta(e^{\frac{y}{2}}) = \Theta(k_n)$ by Lemma 3.3. Therefore, $\mathbb{E}[X_n] \geq \Omega(k_n)$ and hence

$$\mathbb{P}\left(|X_n - \mathbb{E}[X_n]| > \delta_n \mathbb{E}[X_n]\right) = O\left(e^{-\frac{\delta_n^2}{4(1+\delta_n)}\mathbb{E}[X_n]}\right) = e^{-\Omega(k_n^\varepsilon)}.$$

In particular, this implies

$$\begin{aligned}
&\sum_{\ell \notin L_X} \mathbb{P}\left(Y_n = \ell\right) \mathbb{P}\left(X_n = k + i - \ell\right) \mathbb{P}\left(Z_n = k + i' - \ell\right) \\
&\qquad = O\left(\mathbb{P}\left(|X_n - \mathbb{E}[X_n]| > \delta_n \mathbb{E}[X_n]\right)\right) = e^{-\Omega(k_n^\varepsilon)},
\end{aligned}$$

Finally we note that, for $\ell \in L_X$, we have

$$\frac{\mathbb{P}\left(X_n = k + i - \ell\right)}{\mathbb{P}\left(X_n = k + j - \ell\right)} = \mathbb{E}[X_n]^{i-j} \frac{(k + j - \ell)!}{(k + i - \ell)!} \leq (1 + \delta_n)^{2|i-j|}. \tag{96}$$

and observe that we have similar results for Z_n. Therefore,

$$\begin{aligned}
&\sum_{\ell\in L_X\cap L_Z} \mathbb{P}\left(Y_n=\ell\right)\mathbb{P}\left(X_n=k+i-\ell\right)\mathbb{P}\left(Z_n=k+i'-\ell\right)\\
&\qquad\le (1+\delta_n)^{2(j-i)+2(j'-i')}\sum_{\ell\in L_x\cap L_Z} \mathbb{P}\left(Y_n=\ell\right)\mathbb{P}\left(X_n=k+j-\ell\right)\mathbb{P}\left(Z_n=k+j'-\ell\right)\\
&\qquad= (1+o(1))(1+\delta_n)^{2|i-j|+2|i'-j'|}\mathbb{P}\left(D_p=k_n+j, D_{p'}=k_n+j'\right)\\
&\qquad= (1+o(1))\mathbb{P}\left(D_p=k_n+j, D_{p'}=k_n+j'\right)
\end{aligned}$$

and hence

$$\mathbb{P}\left(D_p=k_n+i, D_{p'}=k_n+i'\right)=(1+o(1))\mathbb{P}\left(D_p=k_n+j, D_{p'}=k_n+j'\right)+e^{-\Omega(k_n^\varepsilon)}.$$

Now assume that $\mathbb{E}\left[Y_n\right]>k_n/2$. Then, since $\mathbb{E}\left[|\mathcal{N}_{\mathcal{P},n}(p,p')|\right]\ge k_n^\varepsilon$ it follows that $\mathbb{E}\left[X_n\right]=\Omega(k_n^\varepsilon)$ or $\mathbb{E}\left[Z_n\right]=\Omega(k_n^\varepsilon)$. Without loss of generality we assume that $\mathbb{E}\left[X_n\right]=\Omega(k_n^\varepsilon)$. Similar to (96) we have for Y_n

$$\frac{\mathbb{P}\left(Y_n=\ell\right)}{\mathbb{P}\left(Y_n=\ell+j'-i'\right)}\le(1+\delta_n)^{2(|i'-j'|)}.$$

Using similar computations as above we then have

$$\begin{aligned}
&\mathbb{P}\left(D_p=k_n+i, D_{p'}=k_n+i'\right)\\
&\qquad=\sum_{\ell\in L_X\cap L_Y}\mathbb{P}\left(Y_n=\ell\right)\mathbb{P}\left(X_n=k+i-\ell\right)\mathbb{P}\left(Z_n=k+i'-\ell\right)+e^{-\Omega(k_n^\varepsilon)}\\
&\qquad=(1+o(1))\mathbb{P}\left(D_p=k_n+j, D_{p'}=k_n+j'\right)+e^{-\Omega(k_n^\varepsilon)}.
\end{aligned}$$

□

Lemma B.5. *Let $\alpha>\frac{1}{2}$, $\nu>0$ and fix $\varepsilon>0$. Then for any $p,p'\in\mathcal{K}_\varepsilon(k)$ such that $|x-x'|\ge k^{1+\varepsilon}$, as $n\to\infty$,*

$$\begin{aligned}
&\mathbb{P}\left(D_{\mathcal{P},n}(p)=k-|B_{\mathcal{P},n}(p)\cap\{p'\}|, D_{\mathcal{P},n}(p')=k-|B_{\mathcal{P},n}(p')\cap\{p\}|\right)\\
&\qquad=(1+o(1))\mathbb{P}\left(D_{\mathcal{P},n}(p)=k\right)\mathbb{P}\left(D_{\mathcal{P},n}(p')=k\right)+k^{-\omega(1)}.
\end{aligned}$$

Proof. This is a direct consequence of Lemma B.4 and Lemma C.3. □

C Some results for random variables

We start with the following concentration result which follows from [10, Theorem 4], together with the note directly after it.

Lemma C.1. *Let X_n be a sum of n, possibly dependent, indicators and $c>0$. Then*

$$\mathbb{P}\left(|X_n-\mathbb{E}\left[X_n\right]|>c\mathbb{E}\left[X_n\right]\right)\le 2e^{-\frac{c\mathbb{E}[X_n]}{2}}.$$

Let $H(x)=x\log(x)-x+1$. Then by a Chernoff bound, see for instance [17, Lemma 1.2],

$$\begin{aligned}
\mathbb{P}\left(\mathrm{Po}(\lambda)\ge k\right)&\le e^{-\lambda H(k/\lambda)}\quad\text{for all } k\ge\lambda\\
\mathbb{P}\left(\mathrm{Po}(\lambda)\le k\right)&\le e^{-\lambda H(k/\lambda)}\quad\text{for all } k\le\lambda.
\end{aligned}$$

Note that $H(x)\le(x-1)^2/2$ for all $0\le x\le 1$. Therefore,

$$\mathbb{P}\left(|\mathrm{Po}(\lambda)-\lambda|\ge x\right)\ge 1-e^{-\lambda H(1-x/\lambda)}-e^{-\lambda H(1+x/\lambda)}$$

$\mathbb{P}(X > k) \le e^{-\mu H(\frac{k}{\mu})}$ and for $k < \mu$, $\mathbb{P}(X < k) \le e^{-\mu H(\frac{k}{\mu})}$, where $H(x) = x\ln x - x + 1$, [?]), it follows that $\mathbb{P}(N \in [n - c\sqrt{n\log n}, n + c\sqrt{n\log n}]) \ge 1 - e^{-nH(\frac{n-c\sqrt{n\log n}}{n})} - e^{-nH(\frac{n+c\sqrt{n\log n}}{n})} \ge 1 - 2e^{-n\frac{c^2 n\log n}{n^2}} = 1 - 2e^{-c^2\log n} = 1 - 2n^{-c^2}$ (where we have used that $H(x) = (x-1)^2$ for x close to 1)

By a Chernoff bound we have

$$\mathbb{P}\left(|\mathrm{Po}(\lambda) - \lambda| \ge x\right) \le 2e^{-\frac{x^2}{2(\lambda+x)}}. \tag{97}$$

In particular, if $\lambda_n \to \infty$, then, for any $0 < \varepsilon < 1$,

$$\mathbb{P}\left(|\mathrm{Po}(\lambda_n) - \lambda_n| \ge \lambda_n^{\frac{1+\varepsilon}{2}}\right) \le 2e^{-\frac{\lambda_n^{\varepsilon}}{2\left(1+\lambda_n^{-(1-\varepsilon)/2}\right)}} = O\left(e^{-\lambda_n^{\varepsilon}}\right).$$

Pim: The results below still need to be converted to match style and notation.

Lemma C.2. *Let k_n be a sequence of natural numbers, $\mu_n^{(1)}, \mu_n^{(2)}$ sequences of positive real numbers and $X_n = \mathrm{Po}(\mu_n^{(1)})$ be a Poison random variable with mean $\mu_n^{(1)}$. In addition, Let $h_n = o\left(\sqrt{\frac{k_n}{\log k_n}}\right)$, $c > 0$ and assume that $k_n - \mu_n^{(2)} = \mu_n^{(1)} + a_n$ with $a_n \in [-c\sqrt{k_n\log k_n}, c\sqrt{k_n\log k_n}]$ and $\mu_n^{(1)} = \Theta(k_n)$. Then, as $n \to \infty$,*

$$\mathbb{P}\left(X_n = k_n - \mu_n^{(3)}\right) \sim \mathbb{P}\left(X_n = k_n - \mu_n^{(2)} + h_n\right).$$

Proof. Firstly, we upper bound the ratio:

$$\begin{aligned}\frac{\mathbb{P}(Po(\mu_1) = k - \mu_3)}{\mathbb{P}(Po(\mu_1) = k - \mu_3 + h)} &= \mu_1^{-h}\frac{(k-\mu_3+h)!}{(k-\mu_3)!} \le \mu_1^{-h}(k-\mu_3+h)^h \\ &= \mu_1^{-h}(\mu_1+g+h)^h = (1+\frac{g+h}{\mu_1})^h \le e^{\frac{(g+h)h}{\mu_1}}\end{aligned}$$

which tends to 1 by the assumptions on g, h and μ_1.

Similarly, the ratio can be lower bounded by 1:

$$\begin{aligned}\frac{\mathbb{P}(Po(\mu_1) = k - \mu_3)}{\mathbb{P}(Po(\mu_1) = k - \mu_3 + h)} &= \mu_1^{h}\frac{(k-\mu_3+h)!}{(k-\mu_3)!} \ge \mu_1^{-h}(k-\mu_3)^h \\ &= \mu_1^{-h}(\mu_1+g)^h = (1+\frac{g}{\mu_1})^h \ge 1\end{aligned}$$

□

Lemma C.3 (Near independence of Poisson variables)**.** *Let X_1, X_2, X_3 be independent Poisson random variables with expectations $\lambda_1, \lambda_2, \lambda_3$ (where everything depends on n, i.e. formally a sequence of such random variables). Let $k = k_n \in \mathbb{N}_0$ be a sequence of natural numbers. Let $C > 0$.*

If $\lambda_3 \le k^{1-\epsilon}$ (for any $\epsilon > 0$) and $\lambda_1, \lambda_2 = \Theta(k)$, then (as $n \to \infty$)

$$\mathbb{P}(X_1 + X_3 = X_2 + X_3 = k) = (1 + o(1))\mathbb{P}(X_1 + X_3 = k)\mathbb{P}(X_2 + X_3 = k) + O(k^{-C})$$

Proof. Let $C > 0$ and $c = \sqrt{C}$. From $\lambda_3 \le k^{1-\epsilon}$ it follows by the Chernov bound for Poisson random variables that $\mathbb{P}(|X_3 - \lambda_3| \ge c\sqrt{k^{1-\epsilon}\log k}) \le k^{-c^2} = k^{-C}$. Note that $h := c\sqrt{k^{1-\epsilon}\log k} = o(\sqrt{\frac{k}{\log k}})$.

Conditioning on the value of X_3 and then separating the values according to the event A that $X_3 \in [\lambda_3 - h, \lambda_3 + h]$ gives

$$\mathbb{P}(X_1 + X_3 = X_2 + X_3 = k) = \sum_{k_3 \in [\lambda_3 - h, \lambda_3 + h]} \mathbb{P}(X_1 = X_2 = k - k_3)\mathbb{P}(X_3 = k_3)$$

$$+\sum_{k_3\notin[\lambda_3-h,\lambda_3+h]}\mathbb{P}(X_1=X_2=k-k_3)\mathbb{P}(X_3=k_3)$$
$$=\sum_{k_3\in[\lambda_3-h,\lambda_3+h]}\mathbb{P}(X_1=k-k_3)\mathbb{P}(X_2=k-k_3)\mathbb{P}(X_3=k_3)+k^{-C}$$

where we have used the independence of X_1 and X_2 in the last line. Now applying lemma E2 please replace with proper reference) and then summarizing all the terms results in

$$=\sum_{k_3\in[\lambda_3-h,\lambda_3+h]}(1+o(1))\mathbb{P}(X_1=k-\lambda_3)\mathbb{P}(X_2=k-\lambda_3)\mathbb{P}(X_3=k_3)+k^{-C}$$
$$=(1+o(1))\mathbb{P}(X_1=k-\lambda_3)\mathbb{P}(X_2=k-\lambda_3)+k^{-C}$$

On the other hand, again by conditioning on the event A,

$$\begin{aligned}\mathbb{P}(X_1+X_3=k)&=\sum_{k_3\in[\lambda_3-h,\lambda_3+h]}\mathbb{P}(X_1=k-k_3)\mathbb{P}(X_3=k_3)+\mathbb{P}(X_1+X_3=k|A^c)\mathbb{P}(A^c)\\&=(1+o(1))\mathbb{P}(X_1=k-\lambda_3)\mathbb{P}(A)+k^{-C}\\&=(1+o(1))\mathbb{P}(X_1=k-\lambda_3)+k^{-C}\end{aligned}$$

and likewise

$$\mathbb{P}(X_2+X_3=k)=(1+o(1))\mathbb{P}(X_2=k-\lambda_3)+k^{-C}$$

Therefore, their product has the form

$$\begin{aligned}&\mathbb{P}(X_1+X_3=k)\mathbb{P}(X_2+X_3=k)\\&=(1+o(1))\mathbb{P}(X_1=k-\lambda_3)\mathbb{P}(X_2=k-\lambda_3)+O(k^{-C})\end{aligned}$$

We conclude that both sides in the claim agree. □

References

[1] Thomas Bläsius, Tobias Friedrich, and Anton Krohmer. Cliques in hyperbolic random graphs. *Algorithmica*, 80(8):2324–2344, 2018.

[2] Fountoulakis Nikolaos Bode, Michel and Tobias Müller. Law of large numbers for the largest component in a hyperbolic model of complex networks. *The Annals of Applied Probability*, 28(1):607–650, 2018.

[3] Karl Bringmann, Ralph Keusch, and Johannes Lengler. Average distance in a general class of scale-free networks with underlying geometry. *arXiv*, 2016.

[4] Karl Bringmann, Ralph Keusch, Johannes Lengler, Yannic Maus, and Anisur Rahaman Molla. Greedy routing and the algorithmic small-world phenomenon. In *Proceedings of the ACM Symposium on Principles of Distributed Computing*, pages 371–380. ACM, 2017.

[5] Elisabetta Candellero and Nikolaos Fountoulakis. Clustering and the hyperbolic geometry of complex networks. *Internet Mathematics*, 12(1-2):2–53, 2016.

[6] Kimberly Claffy, Young Hyun, Ken Keys, Marina Fomenkov, and Dmitri Krioukov. Internet mapping: from art to science. In *Conference For Homeland Security, 2009. CATCH'09. Cybersecurity Applications & Technology*, pages 205–211. IEEE, 2009.

[7] Brian Davies. *Integral transforms and their applications*, volume 41. Springer Science & Business Media, 2012.

[8] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, Francesco G Tricomi, et al. Higher transcendental functions, vol. 1, 1953.

[9] Nikolaos Fountoulakis and Tobias Müller. Law of large numbers for the largest component in a hyperbolic model of complex networks. *The Annals of Applied Probability*, 28(1):607–650, 2018.

[10] David Freedman. Another note on the borel–cantelli lemma and the strong law, with the poisson approximation as a by-product. *The Annals of Probability*, pages 910–925, 1973.

[11] Izrail Solomonovich Gradshteyn and Iosif Moiseevich Ryzhik. *Table of Integrals, Series, and Products*. Elsevier, 2015.

[12] Luca Gugelmann, Konstantinos Panagiotou, and Ueli Peter. Random hyperbolic graphs: degree sequence and clustering. In *International Colloquium on Automata, Languages, and Programming*, pages 573–585. Springer, 2012.

[13] Júlia Komjáthy and Bas Lodewijks. Explosion in weighted hyperbolic random graphs and geometric inhomogeneous random graphs. *arXiv*, 2018.

[14] Dmitri Krioukov, Fragkiskos Papadopoulos, Maksim Kitsak, Amin Vahdat, and Marián Boguná. Hyperbolic geometry of complex networks. *Physical Review E*, 82(3):036106, 2010.

[15] Yudell L Luke. *Mathematical functions and their approximations*. Academic Press, 2014.

[16] CS Meijer. On the G-function. I-VIII. In *Nederl. Akad. Wetensch., Proc.*, volume 49, 1946.

[17] Mathew Penrose. *Random geometric graphs*. Number 5. Oxford university press, 2003.

[18] M Ángeles Serrano and Marian Boguna. Clustering in complex networks. i. general formalism. *Physical Review E*, 74(5):056114, 2006.

[19] Alexei Vázquez, Romualdo Pastor-Satorras, and Alessandro Vespignani. Large-scale topological and dynamical properties of the internet. *Physical Review E*, 65(6):066130, 2002.