Exact asymptotic expressions for local clustering in a hyperbolic model of complex networks.

Tobias: I prefer not to use the term "hyperbolic random graph". Better : KPKVB random graph or, in the title, "a hyperbolic model of complex networks" **Pim:** I am not really a fan of the term KPKVB random graph so I implemented the second suggestion.

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Abstract

Pim: The abstract will be written at the end.

Contents

1 Introduction and results

1.1 Motivation

Hyperbolic random graphs were suggested as a suitable model for complex networks by Krioukov et al. [18], exhibiting the three main characteristics commonly found in complex networks:

1) Broad degree distribution (power-law/scale-free),

2) Strong clustering (community structure),

3) Small path lengths (small world phenomenon).

Indeed, in their seminal paper they showed that these graphs have a power-law degree distribution and exhibit strong clustering, while Abdullah et al. [1] showed that hyperbolic random graphs have doubly logarithmic shortest path lengths. **Tobias:** In general the proofs that GIRG = HRG are shaky at best, so I do not want to cite anything that relies on that. Of course, in this case Nikoloas & his crew were first anyway. Also, regarding "distances", we probably want to mention graph diameter, studied by Kiwi+Mitsche, Friedrich+Krohmer, Müller + Staps. Many other results regarding hyperbolic random graphs have since been established: for instance the existence of efficient routing algorithms [7], the size of the largest component [6], [12] and the number of cliques and the largest clique [5]. **Pim:** @All: Does anyone have other suggestions for results to add? **Tobias:** Yes, of course! Everything Nikolaos& his crew, and I and my crew, and Kiwi-Mitsche have done on this model. This includes 2nd largest componnent, connectivity, spectral gap, diameter, Hamilton cycles & perfect matchings (with Markus, Nikolaos and Mitsche, in preparation but perhaps submitted around the same time as this one.). Also there is a recent paper on arxiv by Owada+Yogeshwaran on subgraph counts.**Pim:** Great, could you add those to the bib file and mention them here.

The aim of this paper is to study local clustering in hyperbolic random graphs. The first rigorous mathematical analysis of the local clustering coefficient was done by Gugelmann et al. [15]. They show [15, Theorem 2.1] that the local clustering coefficient[1] is concentrated around its expectation which is shown to be $\Theta(1)$ as the number of vertices tends to infinity. In particular this result implies that the local clustering coefficient is asymptotically bounded away from zero. The convergence of this coefficient was however not proven. In contrast, the global clustering coefficient was shown [8] to converge in probability to some constant which can be explicitly stated as an integral expression, see Theorem 1.2 in [8]. The authors mention, however, that their analysis needs significant modification to be able to deal with convergence of the local clustering coefficient.

In addition to the clustering coefficients, an important clustering measure that is often studied is the local clustering function $c(k)$. This function computes for any value of k the average of the local clustering coefficient over all vertices of degree k. A general expression of this function for hyperbolic random graphs is given in [18, Equation (59)]. The authors conjecture that as k tends to infinity, $c(k)$ decays as k^{-1}, which they observe (Figure 8 in [18]) in experiments on the infrastructure of the Internet obtained in [9]. However, despite the importance of the local clustering function and these interesting observations, its behavior in hyperbolic random graphs had not been completely determined and the following crucial questions regarding clustering in hyperbolic random graphs remained open so far:

1) Does the local clustering coefficient converge, and if so, what is the limit?

2) What is the limit of the local clustering function and how does it scale with the degree?

In this work we resolve these important open questions. We obtain the asymptotic scaling of $c(k)$ as $k \to \infty$, including the leading constant, as well as an exact expression, in terms of

[1] Note that in [15] this is called the global clustering coefficient. However, since this term is more often associated in the literature with the density of triangles compared to the number of paths of length 2, we use the term local clustering coefficient as is done, for instance, in [8].

known special functions, of the point-wise limit of the local clustering function as the number of vertices tends to infinity. Interestingly, the scaling of $c(k)$ depends on the exponent of the degree distribution and is only k^{-1} when this exponent exceeds $5/2$. For values less than $5/2$ the scaling is k^{-s} where the value of s depends on the degree distribution exponent. Finally, our analysis allows us to also prove a convergence result for the local clustering coefficient where the limit can again be explicitly expressed in terms of known special functions.

Tobias: I vote to remove this next paragraph. Use of GIRGs is fishy, and also we are not obliged to mention any and all possible alternative approaches. **Pim:** I do not mind if we remove the paragraph. However, the results in Júlias's paper are rigorous so I do not think that is a good reason for removing the paragraph. We remark that it is possible to prove that the local clustering coefficient converges to some limit, using the technical tools in the recent work on explosion times in Geometric Inhomogeneous Random Graphs [17]. However, this problem is not explicitly addressed in that paper, nor is there any reason to expect these tools to give us the exact expressions as we do derive in this work.

1.2 Outline of the paper

Pim: @Tobias and Markus: I haven't update this section yet since it will also depend on your new proofs. Please update it once you have added these to the paper.

In the next two subsections 1.3 and 1.4 of the introduction, we give the definitions of the hyperbolic random graph and of the local clustering coefficient and function as used in this paper. After this we present our main results in Section 1.5 and discuss several insights that can be obtained from them and in Section 1.6 show simulations which confirm these results. Section 2 contains a detailed high level outline of the proofs of our main results, split up into different propositions. The proofs of these propositions can be found in the last four sections. Section 6 contains a crucial technical result that will be used in the other sections. The Appendix contains a small section on Meijer's G-function and some known concentration results we use.

1.3 Hyperbolic random graphs

The hyperbolic plane is an infinite 2-dimensional manifold with constant negative curvature. Many different representations of the hyperbolic plane exist and we refer to [2] [16] [4] for an introduction to hyperbolic geometry. We will work with the *native representation* of the hyperbolic plane which takes the elements of $\mathbb{R}^2$ in polar coordinates (r, θ) as the underlying set of points and where distances are determined by the hyperbolic law of cosines.

Let $\alpha > \frac{1}{2}$, $\nu > 0$ and define $R_n = 2\log(n/\nu)$. Then the hyperbolic random graph $G_{\mathbb{H},n}(\alpha, \nu)$ is defined as follows:

1. The vertex set if given by n points $u_1, \dots, u_n$ denoted in polar coordinates (r_i, θ_i), where the angular coordinate θ is chosen uniformly from $(-\pi, \pi]$ while the radial coordinate r is sampled independently according to the cumulative distribution function

$$F_{n,\alpha,\nu}(r) = \begin{cases} 0 & \text{if } r < 0 \\ \frac{\cosh(\alpha r)-1}{\cosh(\alpha R_n)-1} & \text{if } 0 \le r \le R_n \\ 1 & \text{if } r > R_n \end{cases} \tag{1}$$

2. Any two vertices $u_i = (r_i, \theta_i)$ and $u_j = (r_j, \theta_j)$ are adjacent if and only if $d_{\mathbb{H}}(u_i, u_j) \le R_n$, where $d_{\mathbb{H}}$ denotes the distance in the hyperbolic plane, i.e.

$$\cosh(r_i)\cosh(r_j) - \sinh(r_i)\sinh(r_j)\cos(|\theta_i - \theta_j|_{2\pi}) \le \cosh(R_n),$$

as follows from the hyperbolic law of cosines. **Tobias:** This is only when $r_i + r_j \ge R$! Otherwise the equation does not have a solution as there is no triangle with side lengths r_i, r_j, R. **Pim:** I do not understand this. The hyperbolic law of cosines defines the hyperbolic

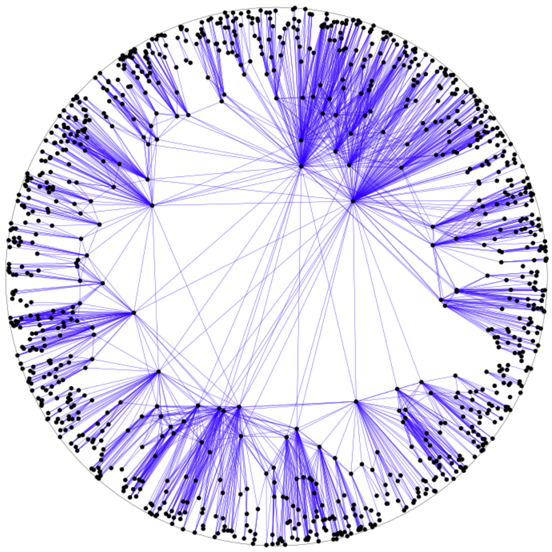

Figure 1: Example of a Hyperbolic Random Graph $G_{\mathbb{H},n}(\alpha,\nu)$ with $\alpha = 0.9$, $\nu = 0.2$ and $n = 5000$.

cosine of the distance between points in terms of hyperbolic cosines and sines of their radii and the cosine of their relative angle:

$$\cosh(d_{\mathbb{H}}(u_i,u_j)) = \cosh(r_i)\cosh(r_j) - \sinh(r_i)\sinh(r_j)\cos(|\theta_i - \theta_j|_{2\pi}).$$

Since cosh is strictly monotonically increasing we have that $\cosh(x) \le \cosh(y) \iff x \le y$ and hence

$$\begin{aligned} & d_{\mathbb{H}}(u_i,u_j) \le R_n \\ \iff & \cosh(d_{\mathbb{H}}(u_i,u_j)) \le \cosh(R_n) \\ \iff & \cosh(r_i)\cosh(r_j) - \sinh(r_i)\sinh(r_j)\cos(|\theta_i - \theta_j|_{2\pi}) \le \cosh(R_n) \end{aligned}$$

The approximation for the hyperbolic balls you and Nicolaos derived in Lemma 28 holds only when $r_i + r_j \ge R$. There were however some sloppy arguments later in Section 2.5, regarding the hyperbolic balls, which I corrected.

Figure 1 shows an example of $G_{\mathbb{H},n}(\alpha,\nu)$.

1.4 Clustering

Clustering measures for networks consider the fraction of triangles (triples of connected vertices) in the network. For instance, for a simple graph G, with vertex set $V(G)$ and edge set $E(G)$, denote by T_G and Λ_G, respectively, the total number of triangles and the total number of paths of length two in G. Then its *global clustering coefficient* is defined as

$$c_G = \frac{3T_G}{\Lambda_G}.$$

Tobias: Do we even need to give this definition? We don't study it, so I would say it suffices to just mention there is another, rival defintion to the one we use. A different concept for clustering is defined for each vertex and by taking averages also for the entire graph as well as for all vertices with a specified degree: the *local clustering coefficient* of a vertex $v \in V(G)$ is a real number between zero and one for the extent to which the neighborhood of v resembles a clique. If the vertex v has $k \ge 2$ neighbors, there are $\binom{k}{2}$ possible edges between them and its local clustering coefficient is the quotient of the number of these pairs that constitute an existing edge in the graph divided by $\binom{k}{2}$. **Tobias:** Here I would add the formula for readability. If the vertex v has no or only one neighbor, its clustering coefficient is set to zero. The (average) local clustering coefficient of the graph is the average over the clustering coefficients of all vertices. The local clustering function maps a natural number k to the average over the clustering coefficients of all vertices with degree k if there are vertices with degree k and zero otherwise.

To introduce the notations for these definitions, let $D_G(v)$ denote the degree of vertex v and $N_G(k)$ the number of vertices with degree k in the graph G. In addition let

$$T_G(v, v_1, v_2) = \mathbb{1}_{\{(v,v_1)\in E\}}\mathbb{1}_{\{(v,v_2)\in E\}}\mathbb{1}_{\{(v_1,v_2)\in E\}}$$

be the indicator that v, v_1 and v_2 form a triangle and write

$$T_G(v) = \sum_{v_1,v_2\in V(G)} T_G(v, v_1, v_2), \tag{2}$$

to denote the number of triangles in which node v participates. Then the *local clustering coefficient* is given by

$$c_G = \frac{1}{|V(G)|}\sum_{v\in V(G)} \frac{T_G(v)}{\binom{D_G(v)}{2}}.$$

Tobias: I would find it natural to define $c(v)$ first and then let $c(G)$ be the average. However, since the local clustering coefficient assigns just one value to the whole network, representing its triangular structure, it is unable to characterize local structural properties involving triangles. **Tobias:** This last sentence seems a bit weird to me, rather abstract. The local clustering function, on the other hand, measures the fraction of triangles to which vertices of a given degree belong, compared to the maximum number of triangles in which they could participate [23, 22]. It describes the triangular structure of vertices in the networks based on their degree and gives a more detailed look at the overall structure of the network. **Tobias:** Similar comment to the previous one. I am not really sure what this means. The formal definition of the *local clustering function* is

$$c_G(k) = \begin{cases} \frac{1}{N_G(k)}\sum_{v\in V(G)} \mathbb{1}_{\{D_G(v)=k\}}\frac{T_G(v)}{\binom{k}{2}} & \text{if } N_G(k) \geq 1 \\ 0 & \text{else.} \end{cases} \tag{3}$$

Tobias: Perhaps we should say that, while it may be more natural to consider $c(k)$ to be undefined when $N_k = 0$, we use this definition for technical convenience. This way $c(k)$ is a plain vanilla random variable and we can compute expectation and so on without issue, and speak about convergence in probability and what not.

Remark 1.1 (Notational convention)**.** *In the remainder of this paper we will compare the local clustering function and many other characteristics between several different graph models. To make notation less cluttered we will often use a unique subscript to identify the graph with respect to which the specific property refers instead of the full graph description. For instance, we shall write $c_{\mathbb{H},n}(k)$ to denote $c_{G_{\mathbb{H},n}(\alpha,\nu)}(k)$ and similar, $D_{\mathbb{H},n}(u)$ for $D_{G_{\mathbb{H},n}(\alpha,\nu)}(u)$. For the infinite model we will use the subscript ∞.*

1.5 Main results

We are now ready to state our main results for the clustering coefficient and local clustering function in the hyperbolic random graph. When the degree k is a growing function of n we are able to exactly compute the asymptotic behavior of $c_{\mathbb{H},n}(k)$. **Tobias:** Seems a little strong. We "exactly compute" the leading terms, I would say.

The results are obtained by coupling the hyperbolic random graph to an infinite random graph $G_{\mathcal{P}}(\alpha, \nu)$, which we define in Section 2.2, and show that the limit of the local clustering coefficient and function for the hyperbolic random graph are given by those for the infinite model.

We let $B(a,b)$ denote the beta-function and $B^-(x,a,b)$ the lower incomplete beta-function. We also write $\Gamma(z)$ for the Gamma function, $\Gamma^+(q,z)$ for the **Tobias:** upper incomplete Gamma function and define $\Gamma^*(q,z) := \Gamma^+(1+q,z)+\Gamma^+(q,z)$. **Tobias:** How useful is this extra definiton? If it is just to make the expression in Thm 1.1 neater, then I dont think we need it. Finally, we write $U(a,b,z)$ for the hypergeometric U-function (also called Tricomi's confluent hypergeometric function), which for $a,b,z\in\mathbb{C}$, $b\notin\mathbb{Z}_{\leq 0}$, $\mathrm{Re}(a),\mathrm{Re}(z)>0$ has the integral representation

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}dt,$$

see [11, p.255 Equation (2)], and let $G^{m,\ell}_{p,q}\left(z\middle|\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right)$ denote Meijer's G-Function [20], see Appendix A for more details. **Tobias:** Should we not maybe also say what the upper incomplete gamma and lower incomplete beta are then?

1.5.1 Local clustering coefficient

Our first result shows that the local clustering coefficient of the hyperbolic random graph converges in expectation to a constant c_∞, which we can explicitly write down in terms of the special functions mentioned above.

Theorem 1.1 (Limit for local clustering coefficient in $G_{\mathbb{H},n}(\alpha,\nu)$)**.** *Let $\alpha>\frac{1}{2}$, $\nu>0$, and $\xi_{\alpha,\nu}=\frac{4\alpha\nu}{\pi(2\alpha-1)}$. Then,*

$$\lim_{n\to\infty}\mathbb{E}\left[|c_{\mathbb{H},n}-c_\infty|\right]=0,$$

Tobias: *As far as I can tell this is just equivalent to $c(G)\xrightarrow{\mathbb{P}}c_\infty$. I would prefer to use this more standard notation and terminology, certainly at least here, where we present our main results.* **Pim:** *Convergence in expectation implies convergence in probability but is a much stronger notion. In addition, for integer valued sequences a_n and b_n we have that $\mathbb{E}\left[|a_n-b_n|\right]\to 0$ implies that $a_n=b_n$ a.a.s.. However, when they are real valued this is no longer true in general.* *where c_∞ is defined for $\alpha\neq 1$ as*

$$\begin{aligned}
c_\infty &= \frac{2+4\alpha+13\alpha^2-34\alpha^3-12\alpha^4+24\alpha^5}{16(\alpha-1)^2\alpha(\alpha+1)(2\alpha+1)}+\frac{2^{-1-4\alpha}}{(\alpha-1)^2}\\
&+\frac{(\alpha-1/2)(B(2\alpha,2\alpha+1)+B^-(1/2;1+2\alpha,-2+2\alpha))}{2(\alpha-1)(3\alpha-1)}\\
&+\frac{\xi_{\alpha,\nu}^{2\alpha}\Gamma^*(-2\alpha,\xi_{\alpha,\nu})}{4(\alpha-1)}+\frac{\xi_{\alpha,\nu}^{2\alpha+2}\alpha(\alpha-1/2)^2\Gamma^*(-2\alpha-2,\xi_{\alpha,\nu})}{2(\alpha-1)^2}\\
&-\frac{\xi_{\alpha,\nu}^{2\alpha+1}\alpha(2\alpha-1)\Gamma^*(-2\alpha-1,\xi_{\alpha,\nu})}{(\alpha-1)}-\frac{\xi_{\alpha,\nu}^{6\alpha-2}2^{-4\alpha}(3\alpha-1)\Gamma^*(-6\alpha+2,\xi_{\alpha,\nu})}{(\alpha-1)^2}\\
&-\frac{\xi_{\alpha,\nu}^{6\alpha-2}(\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\Gamma^*(-6\alpha+2,\xi_{\alpha,\nu})}{(\alpha-1)}\\
&-\frac{e^{-\xi_{\alpha,\nu}}\Gamma(2\alpha+1)\left(U(2\alpha+1,1-2\alpha,\xi_{\alpha,\nu})+U(2\alpha+1,2-2\alpha,\xi_{\alpha,\nu})\right)}{4(\alpha-1)}\\
&+\frac{\xi_{\alpha,\nu}^{6\alpha-2}\Gamma(2\alpha+1)\left(G^{3,0}_{2,3}\left(\xi_{\alpha,\nu}\middle|\begin{matrix}1,3-2\alpha\\3-4\alpha,-6\alpha+2,0\end{matrix}\right)+G^{3,0}_{2,3}\left(\xi_{\alpha,\nu}\middle|\begin{matrix}1,3-2\alpha\\3-4\alpha,-6\alpha+3,0\end{matrix}\right)\right)}{4(\alpha-1)},
\end{aligned}$$

and for $\alpha=1$ as

$$\begin{aligned}
c_\infty &= \frac{575-12\pi^2}{576}+\frac{\eta^4(7+\pi^2)\Gamma^*(-4,\eta)}{4}\\
&-\frac{1}{2}\int_0^1(1-4z+3z^3)\log(1-z)(z+\eta)e^{-\eta/z}\,\mathrm{d}z
\end{aligned}$$

$$-\int_0^1 \mathrm{Li}_2(z)(z^3+\eta z^2)e^{-\eta/z}\,\mathrm{d}z,$$

with $\eta = 4\nu/\pi$ *and* $\mathrm{Li}_2(z) = \sum_{t=1}^{\infty} z^t/t^2$, *the dilogarithm function*[2].

Tobias: Do we want to remark something re: closed form expression for the $\alpha = 1$ case?

1.5.2 Local clustering function

For the local clustering function we also obtain convergence in expectation to a function $c_\infty(k)$ which can be explicitly stated using the special functions.

Theorem 1.2 (Limit for local clustering function in $G_{\mathbb{H},n}(\alpha,\nu)$)**.** *Let* $\alpha > \frac{1}{2}$, $\nu > 0$, $\xi = \frac{4\alpha\nu}{\pi(2\alpha-1)}$ *and* $(k_n)_{n\geq 1}$ *be any positive sequence, possibly constant, such that* $k_n \geq 2$ *and* $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. *Then,*

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{c_{\mathbb{H},n}(k_n)}{c_\infty(k_n)} - 1\right|\right] = 0,$$

where the function $c_\infty(k)$ *is defined for* $\alpha \neq 1$ *as*

$$\begin{aligned}
c_\infty(k) =& \frac{1}{8\alpha(\alpha-1)\Gamma^+(k-2\alpha,\xi)}\left(-\Gamma^+(k-2\alpha,\xi) - 2\frac{\alpha(\alpha-1/2)^2\xi^2\Gamma^+(k-2\alpha-2,\xi)}{(\alpha-1)}\right.\\
&+8\alpha(\alpha-1/2)\xi\Gamma^+(k-2\alpha-1,\xi)\\
&+4\xi^{4\alpha-2}\Gamma^+(k-6\alpha+2,\xi)\left(\frac{2^{-4\alpha}(3\alpha-1)}{(\alpha-1)} + (\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\right)\\
&+\xi^{k-2\alpha}\Gamma(2\alpha+1)e^{-\xi}U(2\alpha+1,1+k-2\alpha,\xi)\\
&\left.-\xi^{4\alpha-2}\Gamma(2\alpha+1)G^{3,0}_{2,3}\left(\left.\begin{matrix}1, 3-2\alpha\\ 3-4\alpha, -6\alpha+k+2, 0\end{matrix}\right|\xi\right)\right)
\end{aligned}$$

and for $\alpha = 1$ *as*

$$\begin{aligned}
c_\infty(k) =& \frac{9\eta^3}{2k!}\Gamma^+(k-3,\eta) - \frac{\xi^4_{\alpha,\nu}}{k!}\frac{7+\pi^2}{4}\Gamma^+(k-4,\eta)\\
&+\frac{\eta^k}{2k!}\int_0^1 (1-4z+3z^2)\ln(1-z)z^{1-k}e^{-\eta/z}\,\mathrm{d}z\\
&+\frac{\eta^k}{k!}\int_0^1 z^{3-k}\,\mathrm{Li}_2(z)e^{-\eta/z}\,\mathrm{d}z,
\end{aligned}$$

with $\eta = 4\nu/\pi$ *and* $\mathrm{Li}_2(z) = \sum_{t=1}^{\infty} z^t/t^2$, *the dilogarithm function.*

Tobias: I would be tempted to split this thm into two. The first states that for k fixed $c(k) \xrightarrow{\mathbb{P}} c_\infty(k)$. Then we say the proof extends to ... the second thm which states that if $1 \ll k_n \ll n^{1/(2\alpha+1)}$ then $c(k_n) = (1+o(1))c_\infty(k_n)$ a.a.s.
In fact, maybe it is good to present a separate – and short – proof of the constant case. The equivalence of the models for constant k can be worked out in a few pages while for the sequence we need 50+ pages of super technical computations. What do you think? **Pim:** I agree with the splitting suggestion. However, as I mentioned before, since $c(k_n)/c_\infty(k_n)$ is real-valued our result does not necessarily imply that $c(k_n)/c_\infty(k_n) = 1$ a.a.s.. I guess this is what you mean with $c(k_n) = (1+o(1))c_\infty(k_n)$ a.a.s. right?

[2]Note that the integrals in the expression for c_∞ for $\alpha = 1$ exists: for the first one note that $1-4z+3z^2 = (1-z)(1-3z)$, so the integrand can be bounded by $C(1-z)\log(1-z)$ on $[0,1)$ for some constant C, which can be continued continuously to the compact interval $[0,1]$ by noting that the limit for $z \to 1$ is zero, so the integrand is bounded on a bounded domain and hence, this integral is finite; for the second integral note that $\mathrm{Li}_2(z)$ is bounded by $\mathrm{Li}_2(1)$ on $[0,1]$, which is a series with well-known finite limit, so again the integrand is bounded on a bounded domain and hence the second integral is also finite.

From the expression for $c_\infty(k)$ we obtain its asymptotic behavior as $k \to \infty$. For this we first define, for any $\frac{1}{2} < \alpha < \frac{3}{4}$,

$$C_\alpha = \frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{\alpha - \frac{1}{2}}{2(\alpha-1)\alpha} B^-(\frac{1}{2}, 2\alpha+1, 2\alpha-2) - \frac{1}{4(\alpha-1)} B(2\alpha, 3\alpha-4). \tag{4}$$

Moreover, to simplify the statement we define the scaling function

$$s_\alpha(k) = \begin{cases} k^{-(4\alpha-2)} & \frac{1}{2} < \alpha < \frac{3}{4}, \\ \log(k)k^{-1} & \alpha = \frac{3}{4}, \\ k^{-1} & \alpha > \frac{3}{4}. \end{cases} \tag{5}$$

Tobias: These exponents are > 0! (see next comment)

We now have the following result, where the constant C_α emerges when we consider the local clustering function for $\frac{1}{2} < \alpha < \frac{3}{4}$.

Theorem 1.3 (Asymptotic behavior of local clustering function limit)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$. Then,*

$$\lim_{k\to\infty} \frac{c_\infty(k)}{s_\alpha(k)} = \begin{cases} C_\alpha \left(\frac{4\alpha\nu}{\pi(2\alpha-1)}\right)^{4\alpha-2} & \text{if } \frac{1}{2} < \alpha < \frac{3}{4}, \\ \frac{6\nu}{\pi} & \text{if } \alpha = \frac{3}{4}, \\ \frac{8\alpha\nu}{\pi(4\alpha-3)} & \text{if } \alpha > \frac{3}{4}, \end{cases}$$

with C_α as defined in (4).

Tobias: There is a typo here. Since you divide by s in this last thm and $s \to \infty$ as the exponents are > 0 and clustering coefficients are $\in [0, 1]$, you would get limiting value zero. **Tobias:** Also, I dont know if this previous result really deserves the status of "theorem". Feels more like a lemma or proposition. The following is an immediate consequence of Theorem 1.2 and Theorem 1.3.

Corollary 1.4 (Asymptotic behavior of local clustering function in $G_{\mathbb{H},n}(\alpha,\nu)$)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$, $(k_n)_{n\geq 1}$ be any positive sequence such that $k_n \to \infty$ and $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$ and $s_\alpha(k)$ defined as in* (5)*. Then, as $n \to \infty$,*

$$\frac{c_{\mathbb{H},n}(k_n)}{s_\alpha(k_n)} \xrightarrow{L^1} \begin{cases} C_\alpha \left(\frac{4\alpha\nu}{\pi(2\alpha-1)}\right)^{4\alpha-2} & \text{if } \frac{1}{2} < \alpha < \frac{3}{4}, \\ \frac{6\nu}{\pi} & \text{if } \alpha = \frac{3}{4}, \\ \frac{8\alpha\nu}{\pi(4\alpha-3)} & \text{if } \alpha > \frac{3}{4}, \end{cases}$$

where $\xrightarrow{L^1}$ denotes convergence in expectation.

Tobias: I had not heard of "convergence in expectation" before this. I have heard of L_1-convergence though. **Markus:** @Pim: Are you sure all constants are correct? I thought for $\alpha = 3/4$ it might be $6\nu/\pi$. But if you are sure, you can remove this comment of course. **Pim:** @Markus: You might be right. I have updated it and we will later check if you were right.

This result completely **Tobias:** "completely" is pretty strong. Note also we seem to stay away from $k = \Theta(n^{1/(2\alpha+1)})$. So unless we change that, there is reason to say we do not to it "completely". characterizes the asymptotic behavior of the local clustering function in Hyperbolic Random Graphs. In particular we observe that the conjectured scaling of k^{-1} from [18] only occurs when $\alpha > 3/4$, or equivalently, when the exponent of the pdf of the degree distribution is larger than $5/2$.

A major part of this paper is dedicated to prove the convergence result in Theorem 1.2, which is based on a sequence of subsequent results for the local clustering function in related random graph models and Theorem 1.3. The exact flow of the argument is explained in Section 2. The proof for the convergence result in Theorem 1.2, using these results, can be found in Section ??. Here we also give the proof of the convergence result in Theorem 1.1. Another major part of this paper is the exact computation of the local clustering coefficient and function limit c_∞ and $c_\infty(k)$, which involves careful computations of several involved integrals. This is done in Section 3, where we also prove Theorem 1.3.

We end this section with some important observations regarding local clustering in hyperbolic random graphs.

Pim: Populate these paragraphs

Uniform convergence Our results for the local clustering function in particular imply uniform convergence of $c_{\mathbb{H},n}(k)$ for all $2 \leq k \leq a_n$ **Tobias:** Why 3? **Pim:** Should have been 2 (corrected) since there are no triangles for node with degree less than 2. where $a_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. To see this let

$$b_n = \arg \max_{3 \leq k \leq a_n} \mathbb{E}\left[\left|\frac{c_{\mathbb{H},n}(k)}{c_\infty(k)} - 1\right|\right].$$

Then $b_n \leq a_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$ and therefore by Theorem 1.2

$$\lim_{n\to\infty} \max_{3 \leq k \leq a_n} \mathbb{E}\left[\left|\frac{c_{\mathbb{H},n}(k)}{c_\infty(k)} - 1\right|\right] = \lim_{n\to\infty} \mathbb{E}\left[\left|\frac{c_{\mathbb{H},n}(b_n)}{c_\infty(b_n)} - 1\right|\right] = 0.$$

The same is true for the result in Corollary 1.4.

Maximum scaling for k_n is $n^{\frac{1}{2\alpha+1}}$ All our results for clustering in the hyperbolic random graph are valid for any sequence k_n such that $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Although one would like to have results for any sequence $k_n \leq n$, it turns out that $n^{1/(2\alpha+1)}$ is the optimal **Tobias:** again seems to strong to me scaling for which Theorem 1.1 can be true. To see why this is the case note that by definition of the local clustering function (3) we have that $c_{\mathbb{H},n}(k_n) = 0$ if $N_{\mathbb{H},n}(k_n) = 0$. Hence, it follows by Markov's inequality that for any positive function f

$$\mathbb{E}\left[\left|\frac{c_{\mathbb{H},n}(k_n)}{f(k_n)} - 1\right|\right] \geq \mathbb{P}\left(N_{\mathbb{H},n}(k_n) = 0\right) \geq 1 - \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right].$$

We shall later establish (see Lemma ??) that $\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] = \Theta\left(nk_n^{-(2\alpha+1)}\right)$. Therefore if k_n is such that $k_n^{-(2\alpha+1)}n$ tends to zero as $n \to \infty$ we have that

$$\lim_{n\to\infty} \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] = 0$$

and hence

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{c_{\mathbb{H},n}(k_n)}{f(k_n)} - 1\right|\right] \geq \lim_{n\to\infty} 1 - \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] = 1 \neq 0,$$

for any positive function f. This implies that we cannot expect a result like that of Theorem 1.1 to hold as soon as k_n is not $o\left(n^{\frac{1}{2\alpha+1}}\right)$.

Transition in the scaling of local clustering **Markus:** @All: If we can, then it would be interesting to remark that there is this phase transition in α which appears when the power-law exponent is $2.5 = \frac{5}{2}$ and it would be cool if we could give a brief intuitive reason for that or mention that this phase transition point has also been observed in other contexts/models etc.

Pim: @All: I agree with Markus. It would be nice to say something on this. Maybe we can leave it out of the first arXived version and in the mean time do some research on this. We could also ask Remco van der Hofstad.

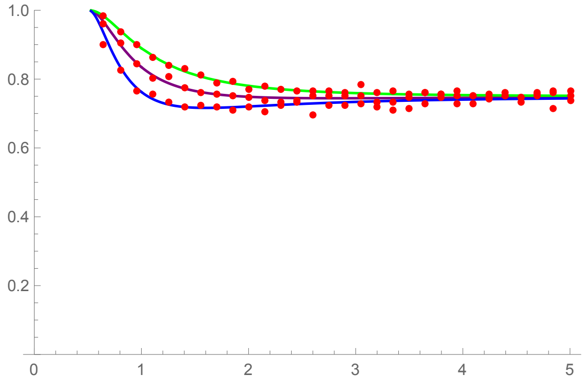

Figure 2: Simulation of $c_{\mathbb{H},n}(k_n)$ for $k_n = 2$, for α varying from 0.5 to 5 on the horizontal axis and for $\nu = \frac{1}{2}$ (blue), $\nu = 1$ (purple), $\nu = 2$ (green); simulations (red dots) with $n = 1000$ and 20 repetitions.

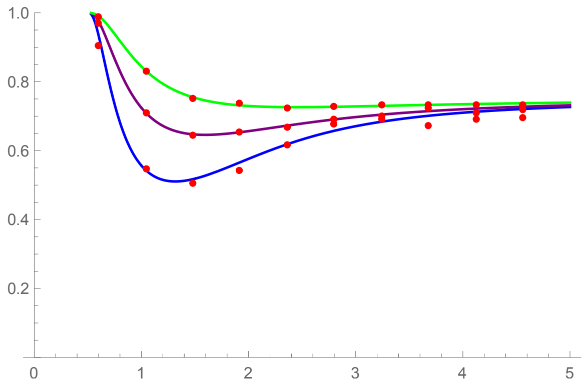

Figure 3: Simulation of $c_{\mathbb{H},n}(k_n)$ for $k_n = 4$, for α varying from 0.5 to 5 on the horizontal axis and for $\nu = \frac{1}{2}$ (blue), $\nu = 1$ (purple), $\nu = 2$ (green); simulations (red dots) with $n = 10000$ and 20 repetitions.

Tobias: **VERY IMPORTANT:** we need to appropriately cite the vdHofstad-vLeeuwaarden-Stegehuis crew. We want to avoid a war. (Just between you and me : I do kind of feel theirs is pretty much non-rigorous though. I am not sure if there is some way to say their work is "non-rigorous" without offending them.)

1.6 Simulations

Tobias: I think the most important simulation to include is for the "overall" clustering coefficient $c(G)$. Can someone please add that?
(The value of k in Figs 2 and 3 seems a bit arbitrary to me btw.)

The simulations of this section (see Figures 2, 3, 4 and 5) appear to be in agreement with the formulas from the result section. The simulations were obtained by generating the hyperbolic random graph for a given choice of the parameters and number of repetitions and then taking the average over all simulation instances.

Tobias: We may want to put the source code in an appendix. In fact RuG now officially requires we deposit the source code in some archive. (gasp.)

2 Overview of the proof strategy for fixed k

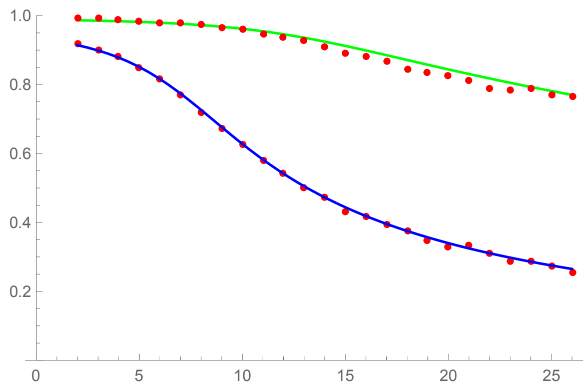

Figure 4: Simulation of $c_{\mathbb{H},n}(k_n)$ for k_n varying from 2 to 26 on the horizontal axis, for $\nu = 2$ and $\alpha = 0.6$ (green) and $\alpha = 0.9$ (blue); simulations (red dots) with $n = 10000$ and 20 repetitions.

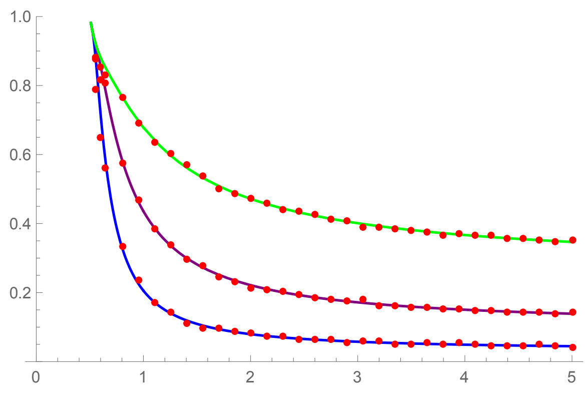

Figure 5: Simulation of $c_{\mathbb{H},n}$ for α varying from 0.5 to 5 on the horizontal axis and for $\nu = \frac{1}{2}$ (blue), $\nu = 1$ (purple), $\nu = 2$ (green); simulations (red dots) with $n = 1000$ and 20 repetitions.

Pim: This section has to be adjusted to include the right references to the new theorems where fixed and diverging k have been separated.

The key result in this paper is Theorem 1.2. In particular, it is the key ingredient in the proofs of Theorem 1.1 and Theorem 1.3. **Tobias:** Sounds weird, since Thm 1.3 is just about asymptotics of certain functions.

Remark 2.1 (Model parameters)**.** *Throughout the remainder of this paper, unless otherwise specified, α and ν will be real numbers satisfying $\alpha > 1/2$ and $\nu > 0$.*

Our approach builds on the work in [12], where tools were developed to analyze the size of the largest component in the hyperbolic model which can be reused here. The main idea is to couple the hyperbolic graph $G_{\mathbb{H},n}(\alpha, \nu)$ with a finite subgraph $G_{\mathcal{P},n}(\alpha, \nu)$ of an infinite graph $G_{\mathcal{P}}(\alpha, \nu)$ whose nodes are given by a Poisson Point Process (see Section 2.2). The infinite graph will then yield the limit expressions for the local clustering coefficient and function. **Tobias:** The Komjathy paper used GIRG, which is fishy. So I say we do not cite that. The advantage of the infinite limit model is two-fold. Firstly, it simplifies the actual computations, e.g. by replacing hyperbolic densities with exponential densities and the hyperbolic distance with a computationally easier expression. Secondly, it directly yields the limit as a result as opposed to some long expression depending on n.

Very roughly, the proof falls into four parts: 1) replacing the randomly sampled nodes in the hyperbolic graph with a Poisson Point Process, 2) justifying the transitioning from this Poisson version of the hyperbolic random graph to the finite model, 3) show that the limit expression correspond to those in the infinite model and finally 4) calculating the expected local clustering coefficient and function in the infinite limit model. **Pim:** @Tobias and Markus: Please include the references to the new theorems for fixed k. This strategy is executed in Section 4 where we prove Theorems [??].

In this section, we define both the infinite and finite model. We then explain how to transition from local clustering in the finite to the infinite model. Finally, we describe the coupling between our finite model and the hyperbolic model and show how this coupling allows us to relate local clustering between them.

We start with some small notational conventions.

2.1 Notations

We adopt standard Landau notations for asymptotic statements. For two functions f and g we write $f(x) = O(g(x))$ as $x \to \infty$ if

$$\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty,$$

while $f(x) = o(g(x))$ as $x \to \infty$ means

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

and $f(x) = \Theta(g(x))$ as $x \to \infty$ whenever

$$\limsup_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| > 0 \quad \text{and} \quad \limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty.$$

Moreover we will write $a \vee b$ for $\max\{a, b\}$ and $a \wedge b$ for $\min\{a, b\}$.

2.2 Infinite limit model

To define the infinite graph $G_{\mathcal{P}}(\alpha, \nu)$ write $\mathcal{R} = \mathbb{R} \times \mathbb{R}_+$ and let $\mathcal{P} = \mathcal{P}_{\alpha,\nu}$ be a Poisson point process on $\mathcal{R}$ with intensity function

$$f_{\alpha,\nu}(x, y) = \frac{\alpha\nu}{\pi} e^{-\alpha y}. \tag{6}$$

We will write $p = (x, y)$ for points in $\mathcal{R}$. In addition we denote the intensity measure of the Poisson process $\mathcal{P}$ by $\mu_{\alpha,\nu}$, i.e. for every Borel-measurable subset $S \subseteq \mathbb{R} \times \mathbb{R}_+$

$$\mu_{\alpha,\nu}(S) = \int_S f_{\alpha,\nu}(x, y)\, dx\, dy, \tag{7}$$

The *infinite limit model* $G_{\mathcal{P}}(\alpha, \nu)$ is defined to have vertex set $\mathcal{P}$ and edge set such that

$$(p_i, p_j) \in E(G_{\mathcal{P}}(\alpha, \nu)) \iff |x_i - x_j| \leq e^{\frac{y_i+y_j}{2}}.$$

For any point $p \in \mathcal{R}$, we write $\mathcal{B}_{\mathcal{P}}(p)$ to denote the *ball* around p, i.e.

$$\mathcal{B}_{\mathcal{P}}(p) = \{p' \in \mathcal{R} : |x - x'| \leq e^{\frac{y+y'}{2}}\}. \tag{8}$$

With this notation we then have that $\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{P}_{\alpha,\nu}$ denotes the set of neighbors of a vertex $p \in G_{\mathcal{P}}(\alpha, \nu)$.

Remark 2.2 (Notations for points)**.** *We will be working extensively with expressions in terms of points of $\mathcal{P}_{\alpha,\nu}$, often relating them back to points in the hyperbolic disc $\mathcal{D}_{\mathcal{R}_n}$. Therefore, in the remainder of this paper we will always use $u = (r, \theta)$ to denote points in $\mathcal{D}_{\mathcal{R}_n}$ in polar coordinates while $p = (x, y)$ will denote points in $\mathcal{R} := \mathbb{R} \times \mathbb{R}_+$ in Cartesian coordinates. In addition, when we write $p' \in \mathbb{R} \times \mathbb{R}_+$ we will denote its Cartesian coordinates by (x', y') and similarly for p_i, u' and u_i, e.g. $p_i = (x_i, y_i)$. In addition, for a point $p = (x, y)$, we will refer to y as the* height *of p.*

2.3 Finite box model

For the definition of the finite graph, recall that $R_n = 2\log(n/\nu)$, write $I_n = \frac{\pi}{2}e^{R_n/2}$ and consider the box $\mathcal{R}_n = (-I_n, I_n] \times (0, R_n]$ in $\mathcal{R}$. Then the *finite box model* graph $G_{\mathcal{P},n}(\alpha, \nu)$ has vertex set $\mathcal{V}_n := \mathcal{P}_{\alpha,\nu} \cap \mathcal{R}_n$ and edges set such that

$$(p_i, p_j) \in E(G_{\mathcal{P},n}(\alpha, \nu)) \iff |x_i - x_j|_{2I_n} \leq e^{\frac{y_i+y_j}{2}},$$

where $|x|_{2I_n} = \inf_{k\in\mathbb{Z}} |x + k2I_n|$. This norm means that the boundaries of $(-I_n, I_n]$ are identified, which is done to exclude boundary effects and make the model invariant under horizontal translation and reflections in vertical lines. The graph $G_{\mathcal{P},n}(\alpha, \nu)$ can thus be seen as a subgraph of $G_{\mathcal{P}}(\alpha, \nu)$ induced on $\mathcal{V}_n$, with some addition edges caused by the identification of the boundaries.

Similar to the infinite graph, for a point $p \in \mathcal{R}_n$ we define the ball $\mathcal{B}_{\mathcal{P},n}(p)$ as

$$\mathcal{B}_{\mathcal{P},n}(p) = \left\{p' \in \mathcal{R}_n : |x_i - x_j|_{2I_n} \leq e^{\frac{y_i+y_j}{2}}\right\}. \tag{9}$$

2.4 Properties of the infinite graph model

Pim: @Tobias: please see if you agree with this setup. The main utility of the infinite graph $G_{\mathcal{P}}(\alpha, \nu)$ is that we can use it to express limits of properties in the finite model, and eventually the hyperbolic random graphs. To this end we add a "typical" point $p_0 := (0, y) \in \mathcal{R}$, with height y exponentially distributed with rate α, and then generate the Poisson Point Process $\mathcal{P}_{\alpha,\nu}$ independently of y. Technically, the vertex set of $G_{\mathcal{P}}(\alpha, \nu)$ then becomes $\mathcal{P}_{\alpha,\nu} \cup p_0$.

A very useful tool to analyze expressions in the both the finite and infinite model is the *Campbell-Mecke Formula* [3, Theorem 3.2], which states that for a Poisson point process $\mathcal{P}$ on a measurable space S with intensity λ and any measurable function $h : S \mapsto \mathbb{R}$,

$$\mathbb{E}\left[\sum_{p\in\mathcal{P}} h(p)\right] = \int_S \mathbb{E}\left[h(x)\right] \mathrm{d}\lambda(x) \tag{10}$$

In essence the results tells us that taking averages of a function over points in a Poisson Points Process simply to integrating the function with respect to the intensity measure. This transformation of summations into integrals is very powerful and we will use this to derive expressions for the degrees and local clustering in the infinite model.

Degrees If we fix a $y \in \mathbb{R}_+$, then it follow from (10) that the expected number of points in $\mathcal{B}_\mathcal{P}((0,y))$ is given by

$$\mathbb{E}\left[\sum_{p\in\mathcal{P}} \mathbb{1}_{\{p\in\mathcal{B}_\mathcal{P}((0,y))\}}\right] = \mu_{\alpha,\nu}(\mathcal{B}_\mathcal{P}((0,y))),$$

which can be computed as,

$$\mu_{\alpha,\nu}(\mathcal{B}_\mathcal{P}((0,y))) = \int_0^\infty \int_{-\infty}^\infty \mathbb{1}_{\{|x'|\le e^{(y+y')/2}\}} f_{\alpha,\nu}(x',y')\,\mathrm{d}x'\,\mathrm{d}y' = \frac{4\alpha\nu}{(2\alpha-1)\pi} e^{\frac{y}{2}} := \xi_{\alpha,\nu} e^{\frac{y}{2}}. \quad (11)$$

We shall write $\rho(p,k)$ and $\rho(y,k)$ (for a point $p=(x,y)\in\mathcal{R}$) to denote the probability mass function of a Poisson random variable with expectation $\mu_{\alpha,\nu}(\mathcal{B}_\mathcal{P}(p))$, i.e.

$$\rho(p,k) = \rho(y,k) = \mathbb{P}\left(\mathrm{Po}\left(\xi_{\alpha,\nu}e^{y/2}\right) = k\right).$$

Let us denote by $D_\mathcal{P}$ the degree of the typical vertex p_0 in $G_\mathcal{P}(\alpha,\nu)$. Then, since the number of points in $\mathcal{B}_\mathcal{P}(p_0)$ is a Poisson random variable with mean $\mu_{\alpha,\nu}(\mathcal{B}_\mathcal{P}(p_0))$, the distribution of $D_\mathcal{P}$ conditioned on the height y is given by $\rho(y,k)$. Therefore, the degree distribution of the typical point is

$$\mathbb{P}(D_\mathcal{P} = k) = \int_0^\infty \rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y,$$

which again can be explicitly computed, see Section 3.1.

Remark 2.3 (Degree distributions in the other models)**.** *Similar to the infinite graph, we will consider the degree of a typical point in a Poisson version of the hyperbolic graph $G_{\widetilde{\mathbb{H}},n}$ and the finite random graph $G_{\mathcal{P},n}$. These will again be distributed as a Poisson random variable with given means and, following the notation above, we will denote these distributions by $\rho_{\widetilde{\mathbb{H}},n}(y,k)$ and $\rho_{\mathcal{P},n}(y,k)$.*

Local clustering We can also define a local clustering coefficient and function in the infinite model. For any $y\in\mathbb{R}_+$ we define the probability density function η_y on $\mathcal{R} := \mathbb{R}\times\mathbb{R}_+$ by

$$\eta_y(x',y') = \frac{\mathbb{1}_{\{p'\in\mathcal{B}_\mathcal{P}((0,y))\}} f_{\alpha,\nu}(x',y')}{\mu_{\alpha,\nu}(\mathcal{B}_\mathcal{P}((0,y)))}. \quad (12)$$

That is η_y is the density $f_{\alpha,\nu}$ restricted to the ball $\mathcal{B}_\mathcal{P}((0,y))$, properly normalized to make it a probability distribution. It can be interpreted as the probability density of a randomly sampled point in $\mathcal{B}_\mathcal{P}((0,y))$, i.e. a randomly sampled neighbor of $(0,y)$ in the infinite model.

We then define

$$\Delta_\mathcal{P}(y) = \iint_{\mathcal{R}^2} T_\mathcal{P}(y,x_1,x_2,y_1,y_2)\eta_y(x_1,y_1)\eta_y(x_2,y_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2, \quad (13)$$

with

$$T_\mathcal{P}(y,x_1,x_2,y_1,y_2) = \mathbb{1}_{\{|x_1|\le e^{(y+y_1)/2}\}}\mathbb{1}_{\{|x_2|\le e^{(y+y_2)/2}\}}\mathbb{1}_{\{|x_1-x_2|\le e^{(y_1+y_2)/2}\}}.$$

Note that $\Delta_\mathcal{P}(y)$ is the probability that two randomly sampled points in $\mathcal{B}_\mathcal{P}((0,y))$ form a triangle with the vertex $(0,y)$.

With these expressions we define the limiting local clustering coefficient as

$$c_\mathcal{P} := \int_0^\infty \Delta_\mathcal{P}(y)(1-\rho(y,0)-\rho(y,1))\alpha e^{-\alpha y}\,\mathrm{d}y, \quad (14)$$

while the limiting local clustering function is defined as

$$c_\mathcal{P}(k) = \frac{\int_0^\infty \rho(y,k)\Delta_\mathcal{P}(y)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}. \quad (15)$$

In Section 3 we will compute both these quantities and show that $c_\mathcal{P} = c_\infty$ and $c_\mathcal{P}(k) = c_\infty(k)$, where c_∞ and $c_\infty(k)$ are given in, respectively, Theorem 1.1 and Theorem 1.2.

2.5 Coupling the hyperbolic model to the finite box model

The final step of our proof strategy is to couple the hyperbolic graph to $G_{\mathcal{P},n}(\alpha,\nu)$. For this we consider the original hyperbolic graph on a Poisson distributed number of vertices as an intermediate step. We define $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$ to be the hyperbolic random graph where the vertex set is given by $N \stackrel{d}{=} \mathrm{Po}(n)$ points, i.i.d. and independent of N, distributed according to the (α, R_n)-quasi uniform distribution (1) defined in Section 1.3.

First we have to show that replacing uniform sampled points with this Poisson Point Process does not influence the properties of the model, so that we can proceed with the latter.

Pim: @Tobias and Markus: I was not sure if you want to state a Lemma here that does this or simply refer to a later section where this is done. So please update this part once you added the proof for fixed k.

The following two lemmas from [12] establish a coupling between the Poisson version of the hyperbolic random graph and the finite box model and relate the hyperbolic neighborhoods to the neighborhood balls $\mathcal{B}_{\mathcal{P}}(p)$.

Lemma 2.1 ([12, Lemma 27])**.** *Let $\mathcal{V}_{\widetilde{\mathbb{H}},n}$ denote the vertex set of $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$ and $\mathcal{V}_n$ the vertex set of $G_{\mathcal{P},n}(\alpha,\nu)$. Define the map $\Psi : [0,R_n]\times(-\pi,\pi] \to \mathcal{R}_n$ by*

$$\Psi(r,\theta) = \left(\theta\frac{e^{R_n/2}}{2}, R_n - r\right). \tag{16}$$

*Then there exists a coupling such that, a.a.s.***Tobias:** *Have we defined a.a.s.? If no, add it before using it.* *, $\mathcal{V}_n = \Psi(\mathcal{V}_{\widetilde{\mathbb{H}},n})$.*

In the remainder of this paper we will write $\mathcal{B}_{\mathbb{H},n}(p)$ to denote the image under Ψ of the ball of hyperbolic radius R_n around the point $\Psi^{-1}(p)$, i.e.

$$\mathcal{B}_{\mathbb{H},n}(p) := \Psi\left[\left\{u \in \mathcal{D}_{R_n} : d_{\mathbb{H}}(\Psi^{-1}(p), u) \le R_n\right\}\right].$$

Note that we have that $\mathcal{B}_{\mathbb{H},n}(p) \subseteq \mathcal{R}_n$. In particular, a point $p = (x,y) \in \mathcal{R}$ corresponds to $u := \Psi^{-1}(p) = (2e^{-R_n/2}x, R_n - y)$.

Let $u, u' \in \mathcal{D}_R$ and write $\theta(u,u') := |\theta - \theta'|_{2\pi}$ for their relative angle. If $r + r' < R_n$, it holds for all $0 \le \phi \le 2\pi$ that

$$\cosh r \cosh r' - \sinh r \sinh r' \cos\phi \le \cosh(R_n),$$

which implies that $d_{\mathbb{H}}(u,u') \le R_n$ and hence u and u' are connected. Now consider a point $p \in \mathcal{V}_n$ and write $\Psi^{-1}(p) = (r,\theta)$. Then it follows that for any point $u' \in \mathcal{D}_R$

$$r' < y = R_n - r \Rightarrow d_{\mathbb{H}}(\Psi^{-1}(p), u') \le R_n. \tag{17}$$

In other words, we have $(-\frac{\pi}{2}e^{R_n/2}, \frac{\pi}{2}e^{R_n/2}] \times [r, R_n] \subset \mathcal{B}_{\mathbb{H},n}(p)$.

When $r + r' > R_n$ the equation

$$\cosh R_n = \cosh r \cosh r' - \sinh r \sinh r' \cos\phi$$

has two solutions of the form $\theta_{R_n}(r,r')$ and $2\pi - \theta_{R_n}(r,r')$ (when $r + r' = R_n$ there is only one solution $\theta_{R_n}(r,r')$). It then follows that $d_{\mathbb{H}}(u,u') \le R_n$ if and only if their relative angle $\theta(u,u')$ satisfies $\theta(u,u') \le \theta_{R_n}(r,r')$. To write this differently, define

$$\Phi(r,r') := \frac{1}{2}e^{R_n/2}\arccos\left(\frac{\cosh r\cosh r' - \cosh R_n}{\sinh r \sinh r'}\right). \tag{18}$$

Pim: @All: Since the choice of Ω here is slightly problematic (we also sometimes use Ω for its asymptotic scaling notation) I have switched $\Omega \mapsto \Phi$.

Then if $r + r' > R_n$ we have that $d_{\mathbb{H}}(u,u') \le R_n$ if and only if $\theta(u,u') \le 2e^{-R_n/2}\Phi(r,r')$. Under the coupling map Ψ, this is equivalent to $|x - x'|_{\pi e^{R_n/2}} \le \Omega(r,r')$. The following lemma, which appears in [12], gives useful bounds on the function $\Phi(r,r')$ in the regime $r + r' > R_n$.

Lemma 2.2 ([12, Lemma 28]). *There exists a constant $K > 0$ such that, for every $\varepsilon > 0$ and for R_n sufficiently large, the following holds. For every $r, r' \in [\varepsilon R_n, R_n]$ with $r + r' > R_n$ we have that*

$$e^{\frac{1}{2}(y+y')} - Ke^{\frac{3}{2}(y+y')-R_n} \leq \Phi(r, r') \leq e^{\frac{1}{2}(y+y')} + Ke^{\frac{3}{2}(y+y')-R_n}, \tag{19}$$

where $y := R_n - r, y' := R - r'$. Moreover:

$$\Phi(r, r') \geq e^{\frac{1}{2}(y+y')} \quad \text{if} \quad r, r' < R_n - K. \tag{20}$$

A key consequence of Lemma 2.2 is that the coupling from Lemma 2.1 preserves edges between points whose heights are small enough, asymptotically.

Lemma 2.3 ([12, Lemma 30]). *On the coupling space of Lemma 2.1 the following holds a.a.s.:*

1. *for any two points $p_i, p_j \in \mathcal{P}_{\alpha,\nu}$ with $y_i, y_j \leq R_n/2$, we have that $(p_i, p_j) \in E(G_{\mathcal{P},n}) \Rightarrow (\Psi^{-1}(p_i), \Psi^{-1}(p_j)) \in E(G_{\widetilde{\mathbb{H}},n})$,*
2. *for any two points $p_i, p_j \in \mathcal{P}_{\alpha,\nu}$ with $y_i, y_j \leq R_n/4$, we have that $(p_i, p_j) \in E(G_{\mathcal{P},n}) \iff (\Psi^{-1}(p_i), \Psi^{-1}(p_j)) \in E(G_{\widetilde{\mathbb{H}},n})$.*

Pim: @Tobias and Markus: Please add additional text to explain the usage of these results in your proof for fixed k.

Pim: @All: Given that the structure of the proof will change, I removed the section where the proofs of the main results were stated.

3 Clustering in $G_{\mathcal{P}}(\alpha, \nu)$

Tobias: @Pim and @Nikolaos : if you have not done so already, can you please check the computations in section 4 very very carefully.

Pim: @All: I haven't touched this section yet.

In this section we will establish the exact expressions for the limit local clustering coefficient and function, $c_{\mathcal{P}}$ and $c_{\mathcal{P}}(k)$, respectively. This is done in several steps. Recall that for $p \in \mathbb{R} \times \mathbb{R}_+ =: \mathcal{R}$,

$$\mathcal{B}_{\mathcal{P}}(p) = \{p' \in \mathbb{R} \times \mathbb{R}_+ : |x - x'| \leq e^{\frac{y+y'}{2}}\},$$

denotes the neighborhood ball of p. Furthermore we recall the definition of $\Delta_{\mathcal{P}}$

$$\Delta_{\mathcal{P}}(y) = \iint_{\mathcal{R}^2} T_{\mathcal{P}}(p, p_1, p_2) \eta_y(x_1, y_1) \eta_y(x_2, y_2) \, dx1 \, dx_2 \, dy_1 \, dy_2,$$

where $\eta_y = \frac{\mu_{\alpha,\nu}|_{\mathcal{B}_{\mathcal{P}}((0,y))}}{\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}((0,y)))}$ is the marginal probability density associated with a randomly sampled neighbor of $(0, y)$.

We already noted that $\Delta_{\mathcal{P}}(y)$ denotes the probability that two randomly sampled neighbors of $(0, y)$ are connected. More precisely, let Z_1, Z_2 be two independent random variables on $\mathcal{B}_{\mathcal{P}}((0, y))$, with (marginal) probability measures η_y. Then

$$\Delta_{\mathcal{P}}(p) = \mathbb{P}_{\mathcal{P},p}(Z_1 \in \mathcal{B}_{\mathcal{P}}(Z_2)) \tag{21}$$

where $\mathbb{P}_{\mathcal{P},p}$ denotes the joint probability measure of Z_1 and Z_2. We shall often abuse notation and write $\Delta_{\mathcal{P}}(y)$ to denote $\Delta_{\mathcal{P}}((0, y))$.

We will start with some preliminary results on the degrees in $G_{\mathcal{P}}(\alpha, \nu)$. The proof of Theorem 1.3 can be found in Section 3.2. The key ingredient for the proof is a result concerning the asymptotic behavior of $\Delta_{\mathcal{P}}(y)$ (Proposition 3.1) whose proof can be found in Section 3.3. The final computations for the exact expressions of c_∞ in Theorem 1.1 and $c_\infty(k)$ in Theorem 1.2 can be found in Section 3.6.

3.1 Degree distribution in $G_{\mathcal{P}}(\alpha, \nu)$

Before we analyze local clustering in the infinite limit model, we first establish some results for its degree distribution. **Tobias:** like "randomly sampled", "degree distribution" is a weird concept for infinite graphs. Not sure it even makes sense. Better to speak of "degree of a typical node", i.e. $x = 0$ and y is exponential α, independent of the PPP. We define the probability distribution for the degree of node $p = (x, y)$ **Tobias:** Indeed, for clarity, we ought to add we add the point p to the PPP, and then study its degree. as

$$\rho(p, k) = \mathbb{P}\left(D_{\mathcal{P}}(p) = k\right) = \mathbb{P}\left(\mathrm{Po}(\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(p))) = k\right), \tag{22}$$

where $\mathrm{Po}(\lambda)$ denotes a Poisson random variable with mean λ. Recall from equation (11) that

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(p)) = \xi_{\alpha,\nu} e^{\frac{y}{2}},$$

only depends on the y-coordinate of p. Therefore, we will often write $\rho(y, k)$ instead of $\rho(p, k)$.

Using the above equation and the transformation of variables $z = \xi_{\alpha,\nu} e^{\frac{y}{2}}$ (so $dy = \frac{2}{z} dz$), we compute

$$\begin{aligned}
\int_0^\infty \rho(y,k)\alpha e^{-\alpha y}\, dy &= \frac{1}{k!}\int_0^\infty \left(\xi_{\alpha,\nu} e^{\frac{y}{2}}\right)^k e^{-\xi_{\alpha,\nu} e^{\frac{y}{2}}} \alpha e^{-\alpha y}\, dy \\
&= \frac{\alpha(\xi_{\alpha,\nu})^{2\alpha}}{\Gamma(k+1)} \int_0^\infty \left(\xi_{\alpha,\nu} e^{\frac{y}{2}}\right)^{k-2\alpha} e^{-\xi_{\alpha,\nu} e^{\frac{y}{2}}}\, dy \\
&= \frac{2\alpha(\xi_{\alpha,\nu})^{2\alpha}}{\Gamma(k+1)} \int_{\xi_{\alpha,\nu}}^\infty z^{k-2\alpha-1} e^{-z}\, dz \\
&= 2\alpha(\xi_{\alpha,\nu})^{2\alpha} \frac{\Gamma^+(k-2\alpha, \xi_{\alpha,\nu})}{\Gamma(k+1)},
\end{aligned}$$

where we recall that Γ denotes the Gamma-function and Γ^+ the incomplete Gamma-function. We therefore conclude that

$$\mathbb{P}\left(D_{\mathcal{P}} = k\right) = 2\alpha(\xi_{\alpha,\nu})^{2\alpha} \frac{\Gamma^+(k-2\alpha, \xi_{\alpha,\nu})}{\Gamma(k+1)} \tag{23}$$

Since $\Gamma^+(k - 2\alpha, \xi_{\alpha,\nu})/\Gamma(k - 2\alpha) \sim k^{-(2\alpha+1)}$ as $k \to \infty$ **Tobias:** Typo, bottom should be $\Gamma(k+1)$. **Pim:** @All: Does anyone have a good standard reference for this? **Tobias:** This follows from Stirling's approx, which works also for non-integer values, together with the obvious fact that $\lim_{x\to\infty} \Gamma^x(x, c)/\Gamma(x) = 1$ and some obvious manipulations. We ought to provide reference + explanation though. Ref for the Stirling approx : De Bruijn, "asymptotic methods in analysis", 3rd edition, section 4.5 we deduce that

$$\mathbb{P}\left(D_{\mathcal{P}} = k\right) \sim 2\alpha(\xi_{\alpha,\nu})^{2\alpha} k^{-(2\alpha+1)} \quad \text{as } k \to \infty. \tag{24}$$

By a similar computation we have the following result, which will be useful later on. For any $\beta > 0$, as $k \to \infty$

$$\int_0^\infty e^{-\beta y} \rho(y,k)\alpha e^{-\alpha y}\, dy \sim 2\alpha \left(\xi_{\alpha,\nu}\right)^{2(\beta+\alpha)} k^{-2(\beta+\alpha)-1}. \tag{25}$$

which implies that

$$\frac{\int_0^\infty e^{-\beta y}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y} \sim \xi_{\alpha,\nu}^{2\beta} k^{-2\beta}. \tag{26}$$

Tobias: y is a subscript here, but is an argument just above. Maybe stick with having it as an argument.

3.2 Asymptotic behavior of $c_\infty(k)$

Recall that
$$c_\infty(k) = \frac{\int_0^\infty \rho(y,k)\Delta_{\mathcal{P}}(y)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}.$$
The asymptotic behavior for the denominator follows from (24). Hence, the main term to consider is the numerator
$$\int_0^\infty \Delta_{\mathcal{P}}(y)\,\rho(y,k)\alpha e^{-\alpha y}\,dy,$$
and in particular the function $\Delta_{\mathcal{P}}(y)$. The following result establishes the asymptotic behavior of the latter.

Proposition 3.1 (Asymptotic behavior of $\Delta_{\mathcal{P}}(y)$)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and C_α as defined in (4). Then, as $y \to \infty$,*

1. *for $\frac{1}{2} < \alpha < \frac{3}{4}$,*
$$\Delta_{\mathcal{P}}(y) \sim e^{-\frac{y}{2}(4\alpha-2)}C_\alpha,$$
2. *for $\alpha = \frac{3}{4}$,*
$$\Delta_{\mathcal{P}}(y) \sim \frac{y}{2}e^{-\frac{y}{2}},$$
3. *and for $\alpha > \frac{3}{4}$,*
$$\Delta_{\mathcal{P}}(y) \sim e^{-\frac{y}{2}}\frac{4\alpha-3}{4(\alpha-1)}.$$

Proof. We shall deal with each of the three cases for α separately.

Proof for $1/2 < \alpha < 3/4$ By Proposition 3.2 we get that

$$\begin{aligned} e^{(4\alpha-2)\frac{y}{2}}\Delta_{\mathcal{P}}(y) = {} & \frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{(\alpha-\frac{1}{2})B^-(\frac{1}{2};1+2\alpha,-2+2\alpha)}{2(\alpha-1)\alpha} - \frac{B^-(1-e^{-\frac{y}{2}};2\alpha,3-4\alpha)}{4(\alpha-1)} \\ & + \frac{e^{(4\alpha-2)\frac{y}{2}}}{8(\alpha-1)\alpha}\left((1-e^{-\frac{y}{2}})^{2\alpha}-1\right) + \frac{\alpha-\frac{1}{2}}{\alpha-1}e^{(4\alpha-3)\frac{y}{2}} - \frac{(\alpha-\frac{1}{2})^2}{4(\alpha-1)^2}e^{4(\alpha-1)\frac{y}{2}}. \end{aligned}$$

Because for any $b<1$, $B^-(1-z,a,b)$ converges to $B^-(a,b)<\infty$ as $z \to 0$, we get that as $y \to \infty$, the first line is asymptotically equivalent to

$$,\frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{(\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)}{2(\alpha-1)\alpha} - \frac{B(2\alpha,3-4\alpha)}{4(\alpha-1)} = C_\alpha,$$

as defined in (4). The proof now follows since for $1/2<\alpha<3/4$, the remaining three terms go to zero as $y\to\infty$.

Proof for $\alpha = 3/4$ Similar to the previous case we use Proposition 3.2 to obtain

$$\begin{aligned} \frac{2}{y}e^{\frac{y}{2}}\Delta_{\mathcal{P}}(y) = {} & \frac{2}{y}\frac{B^-(1-e^{-\frac{y}{2}},2\alpha,3-4\alpha)}{4(\alpha-1)} \\ & + \frac{2}{y}\frac{e^{\frac{y}{2}}\left((1-e^{-\frac{y}{2}})^{2\alpha}-1\right)}{8(\alpha-1)\alpha} + \frac{(\alpha-\frac{1}{2})}{(\alpha-1)y} - \frac{(\alpha-\frac{1}{2})^2e^{-\frac{y}{2}}}{4(\alpha-1)^2y} \\ & + \frac{2}{y}\left(\frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{(\alpha-\frac{1}{2})B^-(\frac{1}{2};1+2\alpha,-2+2\alpha)}{2(\alpha-1)\alpha}\right) \end{aligned}$$

First we note that as $y \to \infty$,

$$e^{\frac{y}{2}}\left(\left(1-e^{-\frac{y}{2}}\right)^{2\alpha}-1\right) \sim -2\alpha, \tag{27}$$

which implies that

$$\lim_{y\to\infty} \frac{2}{y}\frac{e^{\frac{y}{2}}\left((1-e^{-\frac{y}{2}})^{2\alpha}-1\right)}{8(\alpha-1)\alpha} = 0.$$

It can now conclude that all terms in $\frac{2}{y}e^{\frac{y}{2}}\Delta_{\mathcal{P}}(y)$ except the first one are $o(1)$ as $y \to \infty$. By writing $z = e^{-\frac{y}{2}}$ we can rewrite this term as

$$\frac{2}{y}\frac{B^{-}(1-e^{-\frac{y}{2}},2\alpha,3-4\alpha)}{4(\alpha-1)} = -\frac{1}{\log(z)}\frac{B^{-}(1-z,2\alpha,3-4\alpha)}{4(\alpha-1)}.$$

Since $B^{-}(1-z,2\alpha,0) \sim -\log(z)$ as $z \to \infty$, see Lemma B.1, it now follows that for $\alpha = 3/4$,

$$\lim_{y\to\infty}\frac{2}{y}\frac{B^{-}(1-e^{-\frac{y}{2}},2\alpha,3-4\alpha)}{4(\alpha-1)} = \lim_{z\to 0} -\frac{1}{\log(z)}\frac{B^{-}(1-z,2\alpha,3-4\alpha)}{4(\alpha-1)} = 1.$$

We therefore conclude that

$$\frac{2}{y}e^{\frac{y}{2}}\Delta_{\mathcal{P}}(y) \sim \frac{y}{2}e^{-\frac{y}{2}},$$

as $y \to \infty$.

Proof for $\alpha > 3/4$ We first deal with the case $\alpha = 1$. Here it follows from Proposition 3.2 that

$$\begin{aligned}
e^{y/2}\Delta_{\mathcal{P}}(y) &= \frac{9}{4} + \frac{e^{y/2}\log(1-e^{-y/2})}{4} \\
&\quad - \log(1-e^{-y/2}) + e^{-y/2}\left(\frac{3}{4}\log(1-e^{-y/2}) - \frac{7+\pi^2}{8} + \frac{1}{2}\mathrm{Li}_2(e^{-y})\right) \\
&= 2 + \left(\frac{e^{y/2}\log(1-e^{-y/2})}{4} + 1\right) \\
&\quad - \log(1-e^{-y/2}) + e^{-y/2}\left(\frac{3}{4}\log(1-e^{-y/2}) - \frac{7+\pi^2}{8} + \frac{1}{2}\mathrm{Li}_2(e^{-y})\right)
\end{aligned}$$

Now we will proof all other cases with $\alpha > 3/4$. For simplicity we write

$$Q_\alpha := \frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{(\alpha-1/2)B^{-}(1/2;1+2\alpha,-2+2\alpha)}{2(\alpha-1)\alpha}.$$

Then, by Proposition 3.2 we get

$$\begin{aligned}
e^{y/2}\Delta_{\mathcal{P}}(y) &= \frac{\alpha-\frac{1}{2}}{\alpha-1} + \frac{e^{\frac{y}{2}}}{8(\alpha-1)\alpha}\left(\left(1-e^{-\frac{y}{2}}\right)^{2\alpha}-1\right) \\
&\quad - e^{-(4\alpha-3)\frac{y}{2}}\frac{B^{-}(1-e^{-\frac{1}{2}y};2\alpha,3-4\alpha)}{4(\alpha-1)} \\
&\quad + e^{-(4\alpha-3)\frac{y}{2}}Q_\alpha + \frac{(\alpha-\frac{1}{2})^2}{4(\alpha-1)^2}e^{-\frac{y}{2}}.
\end{aligned}$$

The first term is constant while the last two terms go to zero as $y \to \infty$. We will therefore focus on the remain two terms. For the first we have, see (27)

$$\frac{e^{\frac{y}{2}}}{8(\alpha-1)\alpha}\left(\left(1-e^{-\frac{y}{2}}\right)^{2\alpha}-1\right) \sim \frac{-2\alpha}{8(\alpha-1)\alpha} = -\frac{1}{4(\alpha-1)},$$

as $y \to \infty$. Finally, writing $z = e^{-\frac{y}{2}}$ we get that

$$e^{-(4\alpha-3)\frac{y}{2}} B^{-}(1 - e^{-\frac{1}{2}y}; 2\alpha, 3 - 4\alpha) = z^{4\alpha-3} B^{-}(1 - z, 2\alpha, 3 - 4\alpha).$$

Therefore it follows, see Lemma B.1, that

$$\lim_{y\to\infty} -e^{-(4\alpha-3)\frac{y}{2}} \frac{B^{-}(1 - e^{-\frac{1}{2}y}; 2\alpha, 3 - 4\alpha)}{4(\alpha - 1)} = \lim_{z\to 0} z^{4\alpha-3} \frac{B^{-}(1 - z, 2\alpha, 3 - 4\alpha)}{4(\alpha - 1)} = \frac{1}{4(\alpha - 1)(4\alpha - 3)}.$$

We conclude that as $y \to \infty$

$$e^{y/2}\Delta_{\mathcal{P}}(y) \sim \frac{\alpha - \frac{1}{2}}{\alpha - 1} - \frac{1}{4(\alpha - 1)} - \frac{1}{4(\alpha - 1)(4\alpha - 3)} = \frac{1 - 3\alpha + 2\alpha^2}{(\alpha - 1)(\alpha - \frac{3}{4})} = \frac{\alpha - \frac{1}{2}}{\alpha - \frac{3}{4}},$$

which finishes the proof. □

The proof of this proposition is involved and technical. We therefore postpone it till Section 3.3 **Tobias:** It is not in section 4.3!!!! IN fact I cannot find it in sec 4 alltogether. Please fix. We do need this worked out explicitly in the paper. and first use it to prove Theorem 1.3.

Proof of Theorem 1.3. We split the proof over the three cases for α.

We will first prove that for any bounded function $\varphi(y)$ satisfying $\lim_{y\to\infty} |\phi(y)| = 0$ and any $\beta > 0$,

$$\int_0^\infty \varphi(y)e^{-\beta y}\rho(y,k)\alpha e^{-\alpha y}\, dy = o\,(1) \int_0^\infty e^{-\beta y}\rho(y,k)\alpha e^{-\alpha y}\, dy. \tag{28}$$

This, in particular, implies that if $\Delta_{\mathcal{P}}(y) \sim Qe^{-\beta y}$ for some constant Q and $\beta > 0$ then

$$\int_0^\infty \Delta_{\mathcal{P}}(y)\rho(y,k)\alpha e^{-\alpha y}\, dy \sim Q \int_0^\infty e^{-\beta y}\rho(y,k)\alpha e^{-\alpha y}\, dy.$$

To establish (28) fix $0 < \delta < 1$ and let $a(k) = 2\log((1-\delta)k/\xi_{\alpha,\nu})$ and recall (25)

$$\int_0^\infty e^{-\beta y}\rho(y,k)\alpha e^{-\alpha y}\, dy \sim 2\alpha\xi_{\alpha,\nu}^{2(\beta+\alpha)} k^{-(\beta+\alpha)-1}$$

Then, since for any $s > 0$

$$k^s\rho(a(k),k) = O\left(\frac{k^{s+(1-\delta)k}}{k!}e^{-(1-\delta)k}\right) = o(1),$$

as $k \to \infty$, it follows that

$$\begin{aligned}
&\lim_{k\to\infty} k^{2(\beta+\alpha)+1} \int_0^{a(k)} \varphi(y)e^{-s\frac{y}{2}}\rho(y,k)\alpha e^{-\alpha y}\, dy \\
&\quad \le \lim_{k\to\infty} \left(\max_{0\le y\le a(k)} \varphi(y)\right) k^{2(\beta+\alpha)+1}\rho(a(k),k) \int_0^\infty \alpha e^{-\alpha y}\, dy = 0,
\end{aligned}$$

where we used that $\rho(y,k)$ is increasing on $[0, a(k)]$. For the other region of integration we use that $a(k) \to \infty$ implies that $\lim_{k\to\infty} \max_{a(k)\le y<\infty} \varphi(a(k)) = 0$. Hence, using (25),

$$\begin{aligned}
&\lim_{k\to\infty} k^{2(\beta+\alpha)+1} \int_{a(k)}^\infty \varphi(y)e^{-\beta y}\rho(y,k)\alpha e^{-\alpha y}\, dy \\
&\quad \le \lim_{k\to\infty} \max_{a(k)\le y<\infty} \varphi(a(k))k^{2(\beta+\alpha)+1} \int_0^\infty e^{-\beta y}\rho(y,k)\alpha e^{-\alpha y}\, dy = 0.
\end{aligned}$$

We now conclude that

$$\lim_{k\to\infty} k^{2(\beta+\alpha)+1} \int_0^\infty \varphi(y)e^{-\beta y}\rho(y,k)\alpha e^{-\alpha y}\, dy = 0,$$

which establishes (28).

Proof when $1/2 < \alpha < 3/4$ By Proposition 3.1 and (28) it is enough to show that

$$\frac{\int_0^\infty e^{-(4\alpha-2)y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \rho_y(k)\alpha e^{-\alpha y}\,\mathrm{d}y} \sim \xi_{\alpha,\nu}^{4\alpha-2}k^{-4\alpha+2}.$$

This follows immediately from (26) with $\beta = 2\alpha - 1$.

Proof when $\alpha = 3/4$ Similar to the previous case Proposition 3.1 and (28) imply that it is enough to show that

$$\frac{\int_0^\infty \frac{y}{2}e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \rho_y(k)\alpha e^{-\alpha y}\,\mathrm{d}y} \sim \xi_{\alpha,\nu}\log(\log(k/\xi_{\alpha,\nu}))k^{-1},$$

This however does not follow immediately from (26) because of the additional logarithmic term. To establish the above scaling we first define

$$a^{\pm}(k) = 2\log\left(\frac{k \pm \sqrt{k\log(k)}}{\xi_{\alpha,\nu}}\right).$$

Since $k \to \infty$, by a concentration of heights argument we have that

$$\int_0^\infty \Delta_{\mathcal{P}}(y)\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y = (1+o(1))\int_{a^-(k)}^{a^+(k)} \Delta_{\mathcal{P}}(y)\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y,$$

as $k \to \infty$.

We will first show that

$$\int_{a^-(k)}^{a^+(k)} \Delta_{\mathcal{P}}(y)\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y \sim \log(k)\int_{a^-(k)}^{a^+(k)} e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y. \tag{29}$$

For this we establish an upper bound for the left hand side

$$\int_{a^-(k)}^{a^+(k)} \frac{y}{2}e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y \le \frac{a^+(k)}{2}\int_{a^-(k)}^{a^+(k)} e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y$$

and similarly, a lower bound

$$\int_{a^-(k)}^{a^+(k)} \frac{y}{2}e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y \ge \frac{a^-(k)}{2}\int_{a^-(k)}^{a^+(k)} e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y$$

Now observe that

$$\frac{a^{\pm}(k)}{2} = \log\left(\frac{k \pm \sqrt{k\log(k)}}{\xi_{\alpha,\nu}}\right) \sim \log(k)$$

and therefore it follows that

$$\limsup_{k\to\infty} \frac{\int_{a^-(k)}^{a^+(k)} \frac{y}{2}e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}{\log(k)\int_{a^-(k)}^{a^+(k)} e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y} \le 1.$$

and

$$\liminf_{k\to\infty} \frac{\int_{a^-(k)}^{a^+(k)} \frac{y}{2}e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}{\log(k)\int_{a^-(k)}^{a^+(k)} e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y} \ge 1.$$

This proofs (29).

Next we note that by (26) with $\beta = 1/2$ we have

$$\frac{\int_0^\infty e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y} \sim \xi_{\alpha,\nu} k^{-1}.$$

Therefore, since

$$\int_0^\infty e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y \sim \int_{a^-(k)}^{a^+(k)} e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y$$

it follows from (29) that

$$\begin{aligned}\frac{\int_0^\infty \Delta_{\mathcal{P}}(y)\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y} &\sim \log(k)\frac{\int_0^\infty e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}\\ &\sim \xi_{\alpha,\nu}\log(k)k^{-1},\end{aligned}$$

which finishes the proof.

Proof when $\alpha > 3/4$ Again, by Proposition 3.1 and (28) it is enough to show that

$$\frac{\int_0^\infty e^{-y/2}\rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \rho_y(k)\alpha e^{-\alpha y}\,\mathrm{d}y} \sim \xi_{\alpha,\nu} k^{-1},$$

which follows from (26) with $\beta = 1/2$. □

3.3 Analyzing $\Delta_{\mathcal{P}}(y)$, an overview.

We derive the following explicit expression for $\Delta_{\mathcal{P}}(y)$, from which then the leading terms can be extracted. The following result proves Proposition 3.1, resp. implies it as a corollary after extracting the leading terms. **Tobias:** Extracting the leading terms needs to be done explicitly in the paper! If it is already in the paper somewhere, add explicit reference.

Proposition 3.2. *1. If $\alpha \neq 1$, then*

$$\begin{aligned}\Delta_{\mathcal{P}}(y) = &-\frac{1}{8(\alpha-1)\alpha} + \frac{(\alpha-1/2)e^{-\frac{1}{2}y}}{\alpha-1} - \frac{(\alpha-1/2)^2e^{-y}}{4(\alpha-1)^2}\\ &+ (e^{-\frac{1}{2}y})^{4\alpha-2}\left(\frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{(\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)}{2(\alpha-1)\alpha}\right)\\ &+ \frac{(1-e^{-\frac{1}{2}y})^{2\alpha}}{8(\alpha-1)\alpha} - \frac{(e^{-\frac{1}{2}y})^{4\alpha-2}B^-(1-e^{-\frac{1}{2}y};2\alpha,3-4\alpha)}{4(\alpha-1)}\end{aligned}$$

2. If $\alpha = 1$, then

$$\Delta_{\mathcal{P}}(y) = \frac{9}{4}e^{-\frac{1}{2}y} + \frac{1-4e^{-\frac{1}{2}y}+3e^{-y}}{4}\ln(1-e^{-\frac{1}{2}y}) - \frac{7+\pi^2}{8}e^{-y} + \frac{1}{2}e^{-y}\operatorname{Li}_2(e^{-y})$$

where $\operatorname{Li}_2(z) = \int_0^z \frac{\ln(1-t)}{t}dt$ is the dipolylogarithm function.

To derive this result, recall (see (13)) that $\Delta_{\mathcal{P}}(y)$ is the probability that $Z_1 \in \mathcal{B}_{\mathcal{P}}(Z_2)$, where Z_1, Z_2 are two independent random variables on $\mathcal{B}_{\mathcal{P}}(0,y)$ with density

$$\eta_y(y',x') = \frac{f_{\alpha,\nu}(x',y')\mathbb{1}_{\{(x',y'\in B_{\mathcal{P}}(0,y)\}}}{\mu(B_{\mathcal{P}}(0,y))}.$$

For notational convenience we will switch to using y_0 instead of y. Note that the random variables $Z_i = (x_i, y_i)$, for $i = 1, 2$, correspond to first sampling y_i according to the density

$$g(y_i) := \left(\alpha - \frac{1}{2}\right) e^{-(\alpha-\frac{1}{2})y_i}$$

and then sampling x_i uniformly in $[-e^{\frac{1}{2}(y+y_i)}, e^{\frac{1}{2}(y+y_i)}]$. With this in mind we define $P(y_0, y_1, y_2)$ to be the probability that $(0, y_0), (x_1, y_1), (x_2, y_2)$ form a triangle where x_1 and x_2 are independent uniform random variables in, respectively, $[-e^{\frac{1}{2}(y_0+y_1)}, e^{\frac{1}{2}(y_0+y_1)}]$ and $[-e^{\frac{1}{2}(y_0+y_2)}, e^{\frac{1}{2}(y_0+y_2)}]$.

Then we have that

$$\Delta_{\mathcal{P}}(y_0) = (\alpha - 1/2)^2 \int_0^\infty \int_0^\infty P(y_0, y_1, y_2) e^{-(\alpha-1/2)(y_1+y_2)} \, \mathrm{d}y_2 \, \mathrm{d}y_1. \tag{30}$$

3.4 Triangle probability for nodes at given heights

We proceed by first computing $P(y_0, y_1, y_2)$. To compute the integral (30) it will be convenient to use the change of variable $z_i = e^{-y_i/2}$, for $i = 0, 1, 2$. We will write $y_i(z_i)$ to stress the dependence between y_i and z_i. The following result completely characterizes $P(y_0, y_1, y_2)$.

Lemma 3.3.

$$P(y_0(z_0), y_1(z_1), y_2(z_2)) = \begin{cases} 1 & \text{if } z_0 \geq z_1 + z_2, z_0 > z_1 > z_2, \\ 1 - G(z_0, z_1, z_2) & \text{if } z_0 < z_1 + z_2, z_0 > z_1 > z_2, \\ \frac{z_0}{z_1} & \text{if } z_1 \geq z_0 + z_2, z_1 > \max(z_0, z_2), \\ \frac{z_0}{z_1}\left(1 - G(z_1, z_0, z_2)\right) & \text{if } z_1 < z_0 + z_2, z_1 > \max(z_0, z_2), \end{cases}$$

where

$$G(a, b, c) = \frac{1}{4}\left(b^{-1}c + bc^{-1} + a^2 b^{-1} c^{-1} + 2 - 2ab^{-1} - 2ac^{-1}\right)$$

We split the proof of this lemma into a couple of smaller pieces. We begin with the following lemma.

Lemma 3.4. *Let $z_i = e^{-y_i/2}$, $i = 0, 1, 2$. If $y_0 < y_1 < y_2$ (or equivalently $z_0 > z_1 > z_2$), then*

$$P(y_0(z_0), y_1(z_1), y_2(z_2)) = \begin{cases} 1, & \text{if } z_0 \geq z_1 + z_2, \\ 1 - G(z_0, z_1, z_2), & \text{if } z_0 < z_1 + z_2 \end{cases}$$

Proof. Since the Poisson Point Process is translation invariant in the x-coordinate we can, without loss of generality, take $x_0 = 0$. **Tobias:** Sentence does not fit with previous discussion, we already said we took $(0, y_0)$ as the central point. By definition of the connection rule $P(y_0, y_1, y_2)$ is the probability that $|x_2 - x_1| \leq e^{(y_1+y_2)/2}$ (see figure 6). Consider y_0, y_1, y_2 and x_1 fixed. Then we are interested in computing the probability that x_2 falls into the interval $[x_1 - e^{(y_1+y_2)/2}, x_1 + e^{(y_1+y_2)/2}]$ (given by the bottom interval in the Figure 6), as well as into the interval $[-e^{(y_0+y_2)/2}, e^{(y_0+y_2)/2}]$ (given by the top interval in the Figure 6).

By symmetry, without loss of generality we can consider x_1 uniformly at random from $[0, e^{y_0/2+y_1/2}]$. Since $y_0 < y_1 < y_2$ we have that $e^{(y_1+y_2)/2} > e^{(y_0+y_2)/2}$ and so, when $x_1 \geq 0$, the "right half" of the interval $[-e^{(y_0+y_2)/2}, e^{(y_0+y_2)/2}]$ is always covered by the interval $[x_1 - e^{(y_1+y_2)/2}, x_1 + e^{(y_1+y_2)/2}]$. If $e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} \geq e^{(y_0+y_2)/2}$ then the "left half" is always covered as well. In other words:

$$e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} \geq e^{(y_0+y_2)/2} \Rightarrow P(y_0, y_1, y_2) = 1.$$

Now consider the case where $e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} < e^{(y_0+y_2)/2}$. Then, if $x_1 \in [0, e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}]$ the whole interval $[-e^{(y_0+y_2)/2}, e^{(y_0+y_2)/2}]$ is still covered so that p_0, p_1 and p_2 form a

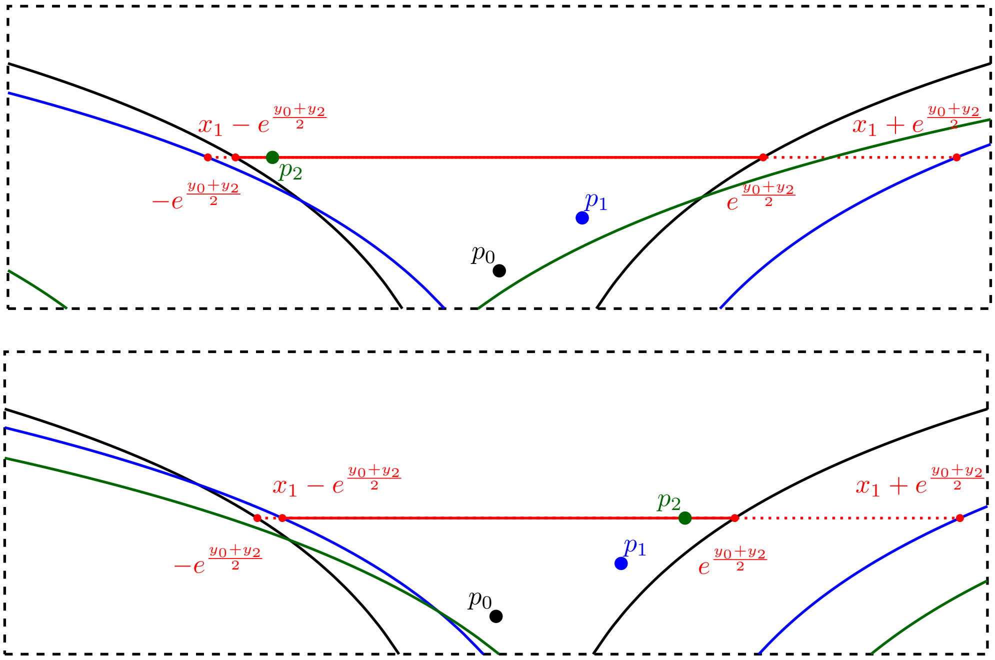

Figure 6: Situation for the intersections of the connection intervals considered in Lemma 3.4, with $y_0 < y_1 < y_2$ fixed and for different cases of $0 \le x_1 \le e^{(y_0+y_1)/2}$. The top figure shows the case where $0 \le x_1 \le e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}$, while the bottom one shows the case $x_1 > e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}$. The solid red line indicates the range for x_2 such that the points p_0, p_1 and p_2 form a triangle. The boundaries of there balls are show in, respectively, black, blue and green.

triangle. If, on the other hand $e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2} < x_1 \le e^{(y_0+y_1)/2}$ then the probability that $|x_2 - x_1| \le e^{(y_1+y_2)/2}$ equals

$$1 - \frac{x_1 - \left(e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}\right)}{2e^{(y_0+y_2)/2}}.$$

Hence, when $e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} < e^{(y_0+y_2)/2}$ we have

$$\begin{aligned}
P(y_0, y_1, y_2) &= \frac{e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}}{e^{(y_0+y_1)/2}} \\
&\quad + \int_{e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}}^{e^{(y_0+y_1)/2}} \left(1 - \frac{x_1 - \left(e^{(y_1+y_2)/2} - e^{(y_0+y_2)/2}\right)}{2e^{(y_0+y_2)/2}}\right) \cdot \frac{1}{e^{(y_0+y_1)/2}} \, dx_1 \\
&= 1 - \frac{1}{2e^{y_0 + y_1/2 + y_2/2}} \int_0^{e^{(y_0+y_1)/2} + e^{(y_0+y_2)/2} - e^{(y_1+y_2)/2}} x_1 \, dx_1 \\
&= 1 - \frac{\left(e^{(y_0+y_1)/2} + e^{(y_0+y_2)/2} - e^{(y_1+y_2)/2}\right)^2}{4e^{y_0 + y_1/2 + y_2/2}},
\end{aligned}$$

At this point it is convenient to rewrite everything in terms of $z_i := e^{-y_i/2}$. Note that $y_0 < y_1 < y_2$ if and only if $z_0 > z_1 > z_2$ while the condition $e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} < e^{(y_0+y_2)/2}$ becomes

$$e^{(y_1+y_2)/2} - e^{(y_0+y_1)/2} < e^{(y_0+y_2)/2} \Leftrightarrow z_1^{-1} z_2^{-1} < z_0^{-1} z_1^{-1} + z_0^{-1} z_2^{-1} \Leftrightarrow z_0 < z_1 + z_2.$$

We now conclude that

$$P(y_0(z_0), y_1(z_1), y_2(z_2)) = 1 \quad \text{if} \quad z_0 > z_1 > z_2 \text{ and } z_0 \ge z_1 + z_2$$

while for $z_0 > z_1 > z_2$ and $z_0 < z_1 + z_2$

$$\begin{aligned} P(y_0, y_1, y_2) &= 1 - \frac{z_0^2 z_1 z_2}{4} \cdot \left(z_0^{-1} z_1^{-1} + z_0^{-1} z_2^{-1} - z_1^{-1} z_2^{-1}\right)^2 \\ &= 1 - \frac{1}{4}\left(z_1^{-1} z_2 + z_1 z_2^{-1} + z_0^2 z_1^{-1} z_2^{-1} + 2 - 2 z_0 z_1^{-1} - 2 z_0 z_2^{-1}\right), \end{aligned}$$

which finishes the proof. □

The previous lemma covers the case when $y_0 < y_1 < y_2$. We now leverage it to take care of the other cases as well.

Proof of Lemma 3.3. Let $y_i > 0$ and $z_i = e^{-y_i/2}$, $i = 0, 1, 2$. Lemma 3.4 gives the expression for $P(y_0(z_0), y_1(z_1), y_2(z_2))$ in the case $y_0 < y_1 < y_2$, or equivalently $z_0 > z_1 > z_2$, i.e. the first two lines in the claim of Lemma 3.3. To analyze the other cases we shall express $P(y_1, y_0, y_2)$ and $P(y_1, y_2, y_0)$ in terms of $P(y_0, y_1, y_2)$ and z_i. For this we note that we can view $P(y_0, y_1, y_2)$ as a 2-fold integral of the indicator function

$$h(x_0, x_1, x_2) := \mathbb{1}_{\left\{|x_0 - x_1| < e^{(y_0+y_1)/2}, |x_0 - x_2| < e^{(y_0+y_2)/2}, |x_1 - x_2| < e^{(y_1+y_2)/2}\right\}},$$

where x_0 was set to zero, without loss of generality, and the other two x_i are uniform random variables on $[-e^{(y_0)+y_i)/2}, e^{(y_0+y_i)/2}]$. When we consider the probability $P(y_1, y_0, y_2)$, this is the 2-fold integral of $h(x_0, 0, x_2)$ so that

$$\begin{aligned} P(y_1, y_0, y_2) &= \frac{1}{2e^{(y_1+y_0)/2}} \cdot \frac{1}{2e^{(y_1+y_2)/2}} \iint_{\mathbb{R}} h(x_0, 0, x_2)\, \mathrm{d}x_0\, \mathrm{d}x_2 \\ &= \frac{e^{y_0/2}}{e^{y_1/2}} \frac{1}{2e^{(y_0+y_1)/2}} \frac{1}{2e^{(y_0+y_2)/2}} \iint_{\mathbb{R}} h(0, x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\ &= \frac{e^{y_0/2}}{e^{y_1/2}} P(y_0, y_1, y_2) = \frac{z_1}{z_0} P(y_0, y_1, y_2). \end{aligned}$$

Finally we note that $h(x_0, 0, x_2) = h(x_2, 0, x_0)$ from which we conclude that

$$P(y_0, y_1, y_2) = (z_0/z_1)\, P(y_1, y_0, y_2) = (z_0/z_1)\, P(y_1, y_2, y_0). \tag{31}$$

To complete the proof for the other cases we note that since $P(y_0, y_1, y_2)$ is symmetric in y_1 and y_2, we can assume, without loss of generality, that $y_1 < y_2$. Then, there are two more orderings of y_0, y_1, y_2, namely $y_1 < y_0 < y_2$ and $y_1 < y_2 < y_0$, which can be summarized as $y_1 < \min(y_0, y_2)$, or equivalently $z_1 > \max(z_0, z_2)$. For $y_1 < y_0 < y_2$ and $y_1 < y_2 < y_0$ we can apply Lemma 3.4 to obtain $P(y_1, y_0, y_2) = P(y_1, y_2, y_0)$ which happen to agree due to the symmetry in the last two arguments of the expression found in Lemma 3.4. The expression for $P(y_0, y_1, y_2)$ then follows from (31). □

3.5 Computing $\Delta_{\mathcal{P}}(y_0)$

Now that we have established the expression for $P(y_0, y_1, y_2)$ we can proceed with computing $\Delta_{\mathcal{P}}(y_0)$, i.e. proving Proposition 3.2. We start with the following observation.

Lemma 3.5. *The function $\alpha \mapsto \Delta_{\mathcal{P}_{\alpha,\nu}}(y_0)$ is continuous for all $\alpha > \frac{1}{2}$.*

Proof. This follows from the theorem of dominated convergence: Let $\alpha > \frac{1}{2}$ and $(\alpha_n)_{n\in\mathbb{N}}$ a sequence of real numbers converging to α, so we can assume $|\alpha_n - \alpha| < \epsilon := \frac{\alpha - 1/2}{2}$. This means that $-\epsilon < \alpha_n - \alpha < \epsilon$, i.e. $\frac{\alpha-1/2}{2} < \alpha_n - 1/2 < \frac{3\alpha - 3/2}{2}$. Define

$$f_n(y_1, y_2) = P(y_0, y_1, y_2)(\alpha_n - 1/2)^2 e^{-(\alpha_n - 1/2)(y_1 + y_2)}.$$

As the function $x \mapsto x^2$ is increasing in x for $x > 0$ and the function $x \mapsto e^{-(y_1+y_2)x}$ is decreasing in x and $P(y_0, y_1, y_2) \in [0, 1]$, it holds that

$$|f_n(y_1, y_2)| \leq \left(\frac{3\alpha - 3/2}{2}\right)^2 e^{-(y_1+y_2)\frac{\alpha-1/2}{2}}$$

which is integrable over $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ (with integral equalling $(6\alpha-3)^2/(2\alpha-1)^2$). Application of the theorem of dominated convergence and using equation (13) yields that $\Delta_{\mathcal{P}_{\alpha_n,\nu}}(y_0) \to \Delta_{\mathcal{P}_{\alpha,\nu}}(y_0)$ which gives the claim as the sequence $(\alpha_n)_n$ was arbitrary. □

Due to this lemma we can first assume $\alpha \notin \{\frac{3}{4}, 1\}$, compute $\Delta_{\mathcal{P}}(y_0)$ and then obtain the values of $\Delta_{\mathcal{P}}(y_0)$ at the remaining two points by taking the corresponding limit in α. This strategy is executed below. It involves the computation of several integrals which are involved and will take up the next 7 pages. The proof is carefully structured, using headers, to aid the reader.

Proof of Proposition 3.2.

When $\boldsymbol{\alpha \notin \{3/4, 1\}}$ Note that when writing $\Delta_{\mathcal{P}}(y_0)$ as in integral as in equation (13), by symmetry in the integration variables y_1 and y_2, we can assume that $y_1 < y_2$ in which case either y_0 or y_1 is the smallest height. This gives half the value of $\Delta_{\mathcal{P}}(y_0)$ and hence

$$\Delta_{\mathcal{P}}(y_0) = 2(I_1(y_0) + I_2(y_0)),$$

where I_1 and I_2 are given by:

$$\begin{aligned}
I_1(y_0) &:= \int_{0<y_0<y_1<y_2} P(y_0, y_1, y_2) \cdot (\alpha - 1/2)^2 e^{-(\alpha-1/2)(y_1+y_2)}\, \mathrm{d}y_2\, \mathrm{d}y_1 \\
I_2(y_0) &:= \int_{0<y_1<y_0,y_2} P(y_0, y_1, y_2) \cdot (\alpha - 1/2)^2 e^{-(\alpha-1/2)(y_1+y_2)}\, \mathrm{d}y_2\, \mathrm{d}y_1
\end{aligned}$$

We proceed with computing each of these two integrals, each of which is split in two parts. The final expressions of those four integrals can be found in (32), (37), (38) and (40).

Computing $\boldsymbol{I_1(y_0)}$ Applying the change of variables $z_i := e^{-y_i/2}$ and Lemma 3.3 gives

$$\begin{aligned}
I_1(y_0) &= 4(\alpha - 1/2)^2 \cdot \int_{z_0>z_1>z_2>0} P(y_0, y_1(z), y_2(z)) z_1^{2\alpha-2} z_2^{2\alpha-2} \mathrm{d}z_2 \mathrm{d}z_1 \\
&= 4(\alpha - 1/2)^2 \cdot \Bigg(\int_{z_0>z_1>z_2>0} 1 \cdot z_1^{2\alpha-2} z_2^{2\alpha-2} \mathrm{d}z_2 \mathrm{d}z_1 \\
&\quad - \int_{\substack{z_0>z_1>z_2>0,\\ z_0<z_1+z_2}} G(z_0, z_1, z_2) \cdot z_1^{2\alpha-2} z_2^{2\alpha-2} \mathrm{d}z_2 \mathrm{d}z_1 \Bigg) \\
&=: 4(\alpha - 1/2)^2 (I_{11}(y_0) - I_{12}(y_0)). \qquad (32)
\end{aligned}$$

The integral $I_{11}(y_0)$ is easily obtained:

$$\begin{aligned}
I_{11}(y_0) &= \int_0^{z_0} \int_0^{z_1} z_1^{2\alpha-2} z_2^{2\alpha-2} \mathrm{d}z_2 \mathrm{d}z_1 \\
&= \int_0^{z_0} z_1^{2\alpha-2} \left[\frac{z_2^{2\alpha-1}}{2\alpha-1}\right]_0^{z_1} \mathrm{d}z_1 \\
&= \frac{1}{2\alpha-1} \cdot \int_0^{z_0} z_1^{4\alpha-3} \mathrm{d}z_1
\end{aligned}$$

$$= \frac{1}{2(2\alpha-1)^2} \cdot z_0^{4\alpha-2}.$$

To deal with I_{12} we note that $G(z_0, z_1, z_2)$ is a linear combination of monomials of the form $z_0^a z_1^b z_2^c$ with $a, b, c \in \{-1, 0, 1, 2\}$ and $a+b+c=0$. Let us consider the integral $J_{(a,b,c)}(z_0)$ defined by

$$J_{a,b,c}(z_0) := z_0^a \int_{\substack{z_0>z_1>z_2>0,\\ z_0<z_1+z_2}} z_1^{b+2\alpha-2} z_2^{c+2\alpha-2} \mathrm{d}z_2 \mathrm{d}z_1. \tag{33}$$

and note that

$$I_{1,2}(y_0) = \frac{1}{4}(J_{0,-1,1}(z_0)+J_{0,1,-1}(z_0)+J_{2,-1,-1}(z_0)+2J_{0,0,0}(z_0)-2J_{1,-1,0}(z_0)-2J_{1,0,-1}(z_0)). \tag{34}$$

Next we compute $J_{a,b,c}(z_0)$.

$$\begin{aligned}
J_{a,b,c} &= z_0^a \int_{z_0/2}^{z_0} \int_{z_0-z_1}^{z_1} z_1^{b+2\alpha-2} z_2^{c+2\alpha-2} \mathrm{d}z_2 \mathrm{d}z_1 = z_0^a \int_{z_0/2}^{z_0} z_1^{b+2\alpha-2} \left[\frac{z_2^{c+2\alpha-1}}{c+2\alpha-1}\right]_{z_0-z_1}^{z_1} \mathrm{d}z_1 \\
&= \frac{z_0^a}{c+2\alpha-1} \cdot \left(\int_{z_0/2}^{z_0} z_1^{b+c+4\alpha-3} \,\mathrm{d}z_1 - \int_{z_0/2}^{z_0} z_1^{b+2\alpha-2} (z_0-z_1)^{c+2\alpha-1} \mathrm{d}z_1 \right) \\
&= \frac{z_0^{a+b+c+4\alpha-2}(1-(1/2)^{b+c+4\alpha-2})}{(c+2\alpha-1)(b+c+4\alpha-2)} \\
&\quad - \frac{z_0^{a+b+c+4\alpha-3}}{c+2\alpha-1} \int_{z_0/2}^{z_0} (z_1/z_0)^{b+2\alpha-2} \left(1-(z_1/z_0)\right)^{c+2\alpha-1} \mathrm{d}z_1 \\
&= \frac{z_0^{4\alpha-2}(1-(1/2)^{b+c+4\alpha-2})}{(c+2\alpha-1)(b+c+4\alpha-2)} - \frac{z_0^{4\alpha-2}}{c+2\alpha-1} \int_{1/2}^{1} u^{b+2\alpha-2}(1-u)^{c+2\alpha-1} \,\mathrm{d}u \\
&= \frac{z_0^{4\alpha-2}(1-(1/2)^{b+c+4\alpha-2})}{(c+2\alpha-1)(b+c+4\alpha-2)} - \frac{z_0^{4\alpha-2}}{c+2\alpha-1} B^-(1/2; c+2\alpha, b+2\alpha-1),
\end{aligned}$$

where we've used the substitution $u := z_1/z_0$ giving $z_0 \mathrm{d}u = \mathrm{d}z_1$ in the penultimate line and B^- denotes the (lower) incomplete beta-function. Note that since $c \geq -1$, $-a \in \{0, -1, -2\}$ and by our assumption $\alpha \notin \{\frac{3}{4}, 1\}$, the denominators that occur during the integration are all non-zero.

Plugging this back into (34) gives

$$\begin{aligned}
I_{1,2}(y_0) &= \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-2})}{32\alpha(\alpha-1/2)} - \frac{z_0^{4\alpha-2}}{8\alpha} B^-(1/2; 1+2\alpha, 2\alpha-2) \\
&\quad + \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-2})}{32(\alpha-1)(\alpha-1/2)} - \frac{z_0^{4\alpha-2}}{4(2\alpha-2)} B^-(1/2; 2\alpha-1, 2\alpha) \\
&\quad + \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-4})}{32(\alpha-1)^2} - \frac{z_0^{4\alpha-2}}{4(2\alpha-2)} B^-(1/2; -1+2\alpha, 2\alpha-2) \\
&\quad + \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-2})}{16(\alpha-1/2)^2} - \frac{z_0^{4\alpha-2}}{2(2\alpha-1)} B^-(1/2; 2\alpha, 2\alpha-1) \\
&\quad - \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-3})}{16(\alpha-1/2)(\alpha-3/4)} + \frac{z_0^{4\alpha-2}}{2(2\alpha-1)} B^-(1/2; 2\alpha, 2\alpha-2) \\
&\quad - \frac{z_0^{4\alpha-2}(1-(1/2)^{4\alpha-3})}{16(\alpha-1)(\alpha-3/4)} + \frac{z_0^{4\alpha-2}}{2(2\alpha-2)} B^-(1/2; -1+2\alpha, 2\alpha-1) \\
&= \frac{\left(\frac{3}{64} - \frac{3}{16}2^{-4\alpha} + \alpha(-\frac{41}{128} + \frac{13}{16}2^{-4\alpha}) + \alpha^2(\frac{5}{8} - \frac{3}{4}2^{-4\alpha}) - \frac{15}{32}\alpha^3 + \frac{1}{8}a^4\right) z_0^{4\alpha-2}}{4(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)\alpha} \\
&\quad + \frac{z_0^{4\alpha-2}}{8(\alpha-1)\alpha(2\alpha-1)} (4(\alpha-1)\alpha(B^-(1/2; 2\alpha, 2\alpha-2) - B^-(1/2; 2\alpha, 2\alpha-1))
\end{aligned}$$

$$
\begin{aligned}
&- (2\alpha-1)\alpha(B^-(1/2;2\alpha-1,2\alpha-2) + B^-(1/2;2\alpha-1,2\alpha) - 2B^-(1/2;2\alpha-1,2\alpha-1))\\
&- (2\alpha-1)(\alpha-1)B^-(1/2;1+2\alpha,2\alpha-2))\\
=\ & \frac{\left(\frac{3}{64} - \frac{3}{16}2^{-4\alpha} + \alpha(-\frac{41}{128} + \frac{13}{16}2^{-4\alpha}) + \alpha^2(\frac{5}{8} - \frac{3}{4}2^{-4\alpha}) - \frac{15}{32}\alpha^3 + \frac{1}{8}a^4\right) z_0^{4\alpha-2}}{4(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)\alpha}\\
&+ \frac{z_0^{4\alpha-2}}{8(\alpha-1)\alpha(2\alpha-1)}(4(\alpha-1)\alpha B^-(1/2;2\alpha+1,2\alpha-2)\\
&- (2\alpha-1)\alpha B^-(1/2;2\alpha+1,2\alpha-2)\\
&- (2\alpha-1)(\alpha-1)B^-(1/2;2\alpha+1,2\alpha-2)).
\end{aligned}
$$

For the last step we use the identities

$$
B^-(z;a,b) - B^-(z;a,b+1) = B^-(z;a+1,b), \tag{35}
$$

$$
B^-(z;a,b) + B^-(z;a,b+2) - 2B^-(z;a,b+1) = B^-(z;a+2,b). \tag{36}
$$

to obtain

$$
\begin{aligned}
I_{1,2}(y_0) = & \frac{\left(\frac{3}{64} - \frac{3}{16}2^{-4\alpha} + \alpha(-\frac{41}{128} + \frac{13}{16}2^{-4\alpha}) + \alpha^2(\frac{5}{8} - \frac{3}{4}2^{-4\alpha}) - \frac{15}{32}\alpha^3 + \frac{1}{8}a^4\right) z_0^{4\alpha-2}}{4(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)\alpha}\\
& - \frac{z_0^{4\alpha-2}B^-(1/2;2\alpha+1,2\alpha-2)}{8(\alpha-1)\alpha(2\alpha-1)}
\end{aligned} \tag{37}
$$

Computing $\boldsymbol{I_2(y_0)}$ We will follow a similar strategy as for $I_1(y_0)$. First, using the change of variables $z_i := e^{-y_i/2}$ we get

$$
\begin{aligned}
I_2(y_0) &= 4(\alpha-1/2)^2 \cdot \int_{1>z_1>z_2,z_0>0} P(y_0(z_0),y_1(z_1),y_2(z_2))z_1^{2\alpha-2}z_2^{2\alpha-2}\mathrm{d}z_2\mathrm{d}z_1\\
&= 4(\alpha-1/2)^2 \cdot \Bigg(\int_{1>z_1>z_0,z_2>0} z_0 z_1^{2\alpha-3}z_2^{2\alpha-2}\mathrm{d}z_2\mathrm{d}z_1\\
&\quad - \int_{\substack{1>z_1>z_0,z_2>0\\ z_1<z_0+z_2}} G(z_1,z_0,z_2)z_0 z_1^{2\alpha-3}z_2^{2\alpha-2}\,\mathrm{d}z_2\mathrm{d}z_1\Bigg)\\
&=: 4(\alpha-1/2)^2(I_{21}(y_0) - I_{22}(y_0)).
\end{aligned} \tag{38}
$$

We proceed with the easy integral:

$$
\begin{aligned}
I_{21}(y_0) &= z_0\int_{1>z_1>\max(z_2,z_0);z_0,z_2>0} z_1^{2\alpha-3}z_2^{2\alpha-2}\mathrm{d}z_2\mathrm{d}z_1 = z_0\int_{z_0}^{1}\int_0^{z_1} z_1^{2\alpha-3}z_2^{2\alpha-2}\mathrm{d}z_2\mathrm{d}z_1\\
&= z_0\int_{z_0}^{1}\left[\frac{z_2^{2\alpha-1}}{2\alpha-1}\right]_0^{z_1} z_1^{2\alpha-3}\mathrm{d}z_1 = \frac{z_0}{2\alpha-1}\int_{z_0}^{1} z_1^{4\alpha-4}\mathrm{d}z_1\\
&= \frac{z_0 - z_0^{4\alpha-2}}{(4\alpha-3)(2\alpha-1)}.
\end{aligned}
$$

We note that the denominators above are non-zero as $\alpha > \frac{1}{2}$ and $\alpha \neq \frac{3}{4}$.

To deal with $I_{22}(y_0)$ we consider the integral function

$$
J'_{a,b,c}(z_0) := z_0^a\int_{\substack{1>z_1>\max(z_0,z_2);z_0,z_2>0\\ z_1<z_0+z_2}} z_1^{b+2\alpha-2}z_2^{c+2\alpha-2}\mathrm{d}z_2\mathrm{d}z_1
$$

and note that

$$
\begin{aligned}
I_{2,2}(y_0) = & \frac{1}{4}\left(J'_{0,-1,1}(z_0) + J'_{2,-1,-1}(z_0) + J'_{0,1,-1}(z_0)\right)\\
& + \frac{1}{2}\left(J'_{1,-1,0}(z_0) - J'_{0,0,0}(z_0) - J'_{1,0,-1}(z_0)\right).
\end{aligned} \tag{39}
$$

We know compute $J'_{a,b,c}(z_0)$

$$
\begin{aligned}
J'_{a,b,c}(z_0) &= z_0^a \int_{z_0}^1 \int_{z_1-z_0}^{z_1} z_1^{b+2\alpha-2} z_2^{c+2\alpha-2} \mathrm{d}z_2 \mathrm{d}z_1 \\
&= z_0^a \int_{z_0}^1 \frac{1}{c+2\alpha-1} z_1^{b+2\alpha-2} (z_1^{c+2\alpha-1} - (z_1-z_0)^{c+2\alpha-1}) \mathrm{d}z_1 \\
&= z_0^a \int_{z_0}^1 \frac{1}{c+2\alpha-1} z_1^{b+c+4\alpha-3} \mathrm{d}z_1 - z_0^a \int_{z_0}^1 \frac{1}{c+2\alpha-1} z_1^{b+2\alpha-2} (z_1-z_0)^{c+2\alpha-1} \mathrm{d}z_1 \\
&= z_0^a \frac{1}{(c+2\alpha-1)(b+c+4\alpha-2)} (1 - z_0^{b+c+4\alpha-2}) \\
&\quad - \frac{z_0^a}{c+2\alpha-1} z_0^{b+c+4\alpha-2} B^-(1-z_0; c+2\alpha, -b-c-4\alpha+2) \\
&= \frac{z_0^a - z_0^{4\alpha-2}}{(c+2\alpha-1)(b+c+4\alpha-2)} - \frac{z_0^{4\alpha-2} B^-(1-z_0; c+2\alpha, -b-c-4\alpha+2)}{c+2\alpha-1}.
\end{aligned}
$$

Here we used that for $x \in \mathbb{R}, y > -1$ (note that as $c \geq -1$, it holds that $c + 2\alpha - 1 > -1$):

$$
\begin{aligned}
\int_{z_0}^1 z_1^x (z_1 - z_0)^y \mathrm{d}z_1 &= \int_0^{1-z_0} (s+z_0)^x s^y \mathrm{d}s \\
&= z_0^{x+y} \int_0^{1-z_0} ((s/z_0)+1)^x \, (s/z_0)^y \mathrm{d}s \\
&= z_0^{x+y+1} \int_0^{1/z_0-1} (t+1)^x t^y \mathrm{d}t \\
&= z_0^{x+y+1} \int_0^{1-z_0} u^y (1-u)^{-(x+y+2)} \mathrm{d}u \\
&= z_0^{x+y+1} B^-(1-z_0; y+1, -x-y-1).
\end{aligned}
$$

As $c \geq -1$ and $-a \in \{0, -1, -2\}$ and by our assumption $\alpha \notin \{\frac{3}{4}\}$, the denominators that occur during the computations above are non-zero.

Plugging the expression for $J'_{a,b,c}(z_0)$ back into (39) we get,

$$
\begin{aligned}
I_{2,2}(y_0) = &\frac{1 - z_0^{4\alpha-2}}{32\alpha(\alpha-1/2)} - \frac{z_0^{4\alpha-2} B^-(1-z_0; 1+2\alpha, -4\alpha+2)}{8\alpha} \\
&+ \frac{z_0^2 - z_0^{4\alpha-2}}{32(\alpha-1)^2} - \frac{z_0^{4\alpha-2} B^-(1-z_0; -1+2\alpha, -4\alpha+4)}{8(\alpha-1)} \\
&+ \frac{1 - z_0^{4\alpha-2}}{32(\alpha-1)(\alpha-1/2)} - \frac{z_0^{4\alpha-2} B^-(1-z_0; -1+2\alpha, -4\alpha+2)}{8(\alpha-1)} \\
&+ \frac{z_0 - z_0^{4\alpha-2}}{16(\alpha-1/2)(\alpha-3/4)} - \frac{z_0^{4\alpha-2} B^-(1-z_0; 2\alpha, -4\alpha+3)}{4(\alpha-1/2)} \\
&- \frac{1 - z_0^{4\alpha-2}}{16(\alpha-1/2)^2} + \frac{z_0^{4\alpha-2} B^-(1-z_0; 2\alpha, -4\alpha+2)}{4(\alpha-1/2)} \\
&- \frac{z_0 - z_0^{4\alpha-2}}{16(\alpha-1)(\alpha-3/4)} + \frac{z_0^{4\alpha-2} B^-(1-z_0; -1+2\alpha, -4\alpha+3)}{4(\alpha-1)}.
\end{aligned}
$$

Using some algebra and the identities (35) and (36) this can be reduced to

$$\begin{aligned}I_{2,2}(y_0) =& \frac{1}{64\alpha(\alpha-1/2)^2(\alpha-1)} - \frac{(1-z_0)^{2\alpha}}{64\alpha(\alpha-1/2)^2(\alpha-1)} - \frac{z_0}{8(\alpha-1/2)(\alpha-1)(4\alpha-3)}\\ &+ \frac{z_0^2}{32(\alpha-1)^2} + \frac{(-6+25\alpha-48\alpha^2+44\alpha^3-16\alpha^4)z_0^{4\alpha-2}}{512\alpha(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)}\\ &+ \frac{z_0^{4\alpha-2}B^-(1-z_0;2\alpha,3-4\alpha)}{32(\alpha-1)(\alpha-1/2)^2}.\end{aligned} \tag{40}$$

Combining the results for $I_1(y_0)$ and $I_2(y_0)$ Combining the results for $I_{11}(y_0), I_{12}(y_0), I_{21}(y_0)$ and $I_{22}(y_0)$ we get, after some algebra, an explicit expression for $\Delta_{\mathcal{P}}(y_0)$ as a linear combination of terms of the form z_0^u, $(1-z_0)^u$ and $z_0^u B^-(1-z_0;a,b)$:

$$\begin{aligned}\Delta_{\mathcal{P}}(y_0) =& 2(I_1+I_2) = 8(\alpha-1/2)^2(I_{1,1}-I_{1,2}+I_{2,1}-I_{2,2})\\ =& 8(\alpha-1/2)^2\Bigg(\frac{1}{2(2\alpha-1)^2}z_0^{4\alpha-2}\\ &-\frac{\left(\frac{3}{64}-\frac{3}{16}2^{-4\alpha}+\alpha(-\frac{41}{128}+\frac{13}{16}2^{-4\alpha})+\alpha^2(\frac{5}{8}-\frac{3}{4}2^{-4\alpha})-\frac{15}{32}\alpha^3+\frac{1}{8}a^4\right)z_0^{4\alpha-2}}{4(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)\alpha}\\ &+\frac{z_0^{4\alpha-2}B^-(1/2;2\alpha+1,2\alpha-2)}{8(\alpha-1)\alpha(2\alpha-1)} + \frac{z_0-z_0^{4\alpha-2}}{(4\alpha-3)(2\alpha-1)}\\ &-\frac{1}{64\alpha(\alpha-1/2)^2(\alpha-1)} + \frac{(1-z_0)^{2\alpha}}{64\alpha(\alpha-1/2)^2(\alpha-1)} + \frac{z_0}{8(\alpha-1/2)(\alpha-1)(4\alpha-3)}\\ &-\frac{z_0^2}{32(\alpha-1)^2} - \frac{(-6+25\alpha-48\alpha^2+44\alpha^3-16\alpha^4)z_0^{4\alpha-2}}{512\alpha(\alpha-1/2)^2(\alpha-1)^2(\alpha-3/4)}\\ &-\frac{z_0^{4\alpha-2}B^-(1-z_0;2\alpha,3-4\alpha)}{32(\alpha-1)(\alpha-1/2)^2}\Bigg)\\ =& -\frac{1}{8(\alpha-1)\alpha} + \frac{(\alpha-1/2)z_0}{\alpha-1} - \frac{(\alpha-1/2)^2z_0^2}{4(\alpha-1)^2}\\ &+ z_0^{-2+4\alpha}\left(\frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha(\alpha-1)^2} + \frac{(\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)}{2(\alpha-1)\alpha}\right)\\ &+\frac{(1-z_0)^{2\alpha}}{8(\alpha-1)\alpha} - \frac{z_0^{4\alpha-2}B^-(1-z_0;2\alpha,3-4\alpha)}{4(\alpha-1)}\end{aligned}$$

Observe that the above expression only contains terms of the form $\alpha-1$ in the denominator. The only expression of the form $\alpha-3/4$ is in the lower incomplete beta-function $B^-(1-z_0;2\alpha,3-4\alpha)$ which appears twice in the expression for $\Delta_{\mathcal{P}}(y_0)$.

The case of $\alpha = 3/4$

Note that the factor $\alpha-\frac{3}{4}$ does not occur in any denominator of the previously obtained expression. For the lower incomplete beta function, the last argument $3-4\alpha$ is zero for $\alpha=\frac{3}{4}$, however as $z_0<1$ the integration domain of the lower incomplete beta function does not touch the singularity at $t=1$ (note $B^-(1-z_0;2\alpha;3-4\alpha)=\int_0^{1-z_0}t^{2\alpha-1}(1-t)^{2-4\alpha}$). Therefore, the previous expression holds for this case as well.

The case of $\alpha = 1$ We want to compute the limit $\lim_{\alpha\to1}\Delta_{\mathcal{P}_{\alpha,\nu}}(y_0(z_0))$. For this, after factoring out $\frac{1}{\alpha-1}$ in the expression for $\Delta_{\mathcal{P}}(y)$ we obtained for $\alpha\neq1$, we compute the first two terms of the Taylor expansion of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$ in α based at $\alpha=1$. The constant term of this Taylor expansion is verified to vanish whereas the linear term gives the expression searched (after cancellation with $\frac{1}{\alpha-1}$). Note that two terms of $P(y_0(z_0))$ contain $(\alpha-1)^2$ in their denominator, so for those two another $\frac{1}{\alpha-1}$ is factored out and the first three terms of the Taylor expansion are computed. Their constant terms cancel and after multiplication with $\frac{1}{\alpha-1}$ their linear and quadratic terms become part of the constant and linear term of the overall expansion. **Tobias:** I find this rather hard to follow. Maybe it can be made more clear what is meant. Also, We are describing what will happen next. Say that clearly.

The first summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$ is,

$$S_1 = -\frac{1}{8\alpha} = -\frac{1}{8} + \frac{1}{8}(\alpha-1) + O((\alpha-1)^2)$$

The second summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$,

$$S_2 = (\alpha-1/2)z_0 = \frac{1}{2}z_0 + z_0(\alpha-1) + O((\alpha-1)^2)$$

The third summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$, needs **Tobias:** "needs"? Now I am really confused a 3-term Taylor expansion after another multiplication with $(\alpha-1)$,

$$S_3 = -\frac{(\alpha-1/2)^2 z_0^2}{4} = -\frac{1}{16}z_0^2 - \frac{1}{4}z_0^2(\alpha-1) - \frac{1}{4}z_0^2(\alpha-1)^2 + O((\alpha-1)^3)$$

The fourth summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$, needs **Tobias:** again why need? a 3-term Taylor expansion after another multiplication with $(\alpha-1)$,

$$\begin{aligned} S_4 = z_0^{-2+4\alpha}\frac{2^{-4\alpha-1}(3\alpha-1)}{\alpha} &= \frac{1}{16}z_0^2 + \frac{z_0^2}{4}\left(\frac{1}{8} + \ln\frac{z_0}{2}\right)(\alpha-1) \\ &\quad + \frac{z_0^2}{8}\left(4\left(\ln\frac{z_0}{2}\right)^2 + \ln\frac{z_0}{2} - \frac{1}{4}\right)(\alpha-1)^2 + O((\alpha-1)^3) \end{aligned}$$

as

$$\frac{d}{d\alpha}\left(\left(\frac{z_0}{2}\right)^{-2+4\alpha}2^{-3}\left(3-\frac{1}{\alpha}\right)\right) = \left(\frac{z_0}{2}\right)^{-2+4\alpha}2^{-3}\left(4\ln\left(\frac{z_0}{2}\right)\left(3-\frac{1}{\alpha}\right)+\frac{1}{\alpha^2}\right)$$

and

$$\frac{d^2}{d\alpha^2}\left(\left(\frac{z_0}{2}\right)^{-2+4\alpha}2^{-3}\left(3-\frac{1}{\alpha}\right)\right) = \left(\frac{z_0}{2}\right)^{-2+4\alpha}2^{-3}\left(16\left(\ln\frac{z_0}{2}\right)^2\left(3-\frac{1}{\alpha}\right)+\frac{8\ln(\frac{z_0}{2})}{\alpha^2}-\frac{2}{\alpha^3}\right)$$

For the fifth summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$ we get,

$$\begin{aligned} S_5 &= z_0^{-2+4\alpha}\frac{(\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)}{2\alpha} \\ &= \frac{z_0^2}{4}B^-(1/2;3,0) + z_0^2\left(\left(\ln(z_0)+\frac{1}{4}\right)B^-(1/2;3,0)\right. \\ &\quad \left. +1/2\int_0^{\frac{1}{2}}\ln(t)t^2(1-t)^{-1}+\ln(1-t)t^2(1-t)^{-1}dt\right)(\alpha-1)+O((\alpha-1)^2) \end{aligned}$$

as

$$\begin{aligned} &\frac{d}{d\alpha}\left(z_0^{-2+4\alpha}\frac{(\alpha-1/2)}{2\alpha}B^-(1/2;1+2\alpha,-2+2\alpha)\right) \\ &\quad = \ln(z_0)4z_0^{-2+4\alpha}\left(\frac{1}{2}-\frac{1}{4\alpha}\right)B^-(1/2;1+2\alpha,-2+2\alpha) \\ &\quad + z_0^{-2+4\alpha}\frac{1}{4\alpha^2}B^-(1/2;1+2\alpha,-2+2\alpha) + z_0^{-2+4\alpha}\left(\frac{1}{2}-\frac{1}{4\alpha}\right)\int_0^{\frac{1}{2}}\frac{d}{d\alpha}(t^{2\alpha}(1-t)^{2\alpha-3})dt \end{aligned}$$

where

$$\int_0^{\frac{1}{2}} \frac{d}{d\alpha}(t^{2\alpha}(1-t)^{2\alpha-3})dt = \int_0^{\frac{1}{2}} \ln(t)2t^{2\alpha}(1-t)^{2\alpha-3} + \ln(1-t)2t^{2\alpha}(1-t)^{2\alpha-3}dt$$

The sixth summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$,

$$S_6 = \frac{(1-z_0)^{2\alpha}}{8\alpha} = \frac{(1-z_0)^2}{8} + \frac{(1-z_0)^2}{4}(\ln(1-z_0) - 1/2)(\alpha-1) + O((\alpha-1)^2)$$

and finally the seventh and last summand of $\Delta_{\mathcal{P}}(y_0(z_0))(\alpha-1)$ gives,

$$\begin{aligned} S_7 = -\frac{z_0^{4\alpha-2}B^-(1-z_0;2\alpha,3-4\alpha)}{4} &= -\frac{z_0^2}{4}B^-(1-z_0;2,-1) - z_0^2\Big(\ln(z_0)B^-(1-z_0;2,-1) \\ &+ \int_0^{1-z_0} 1/2\ln(t)t(1-t)^{-2} - t\ln(1-t)(1-t)^{-2}dt\Big)(\alpha-1) + O((\alpha-1)^2) \end{aligned}$$

When adding these terms we see that the constant coefficients cancel: **Tobias:** But wait ... some terms were multiplied with $(\alpha-1)$ others with $(\alpha-1)^2$? I am genuinely lost here. What is going on here really needs to explained much better.

$$\begin{aligned} &-\frac{1}{8} + \frac{1}{2}z_0 - \frac{1}{4}z_0^2 + \frac{z_0^2}{32} + \frac{z_0^2}{4}\ln(\frac{z_0}{2}) + \frac{z_0^2}{4}B^-(1/2;3,0) + \frac{(1-z_0)^2}{8} - \frac{z_0^2}{4}B^-(1-z_0;2,-1) \\ &= -\frac{1}{8} + \frac{1}{2}z_0 - \frac{1}{4}z_0^2 + \frac{z_0^2}{32} + \frac{z_0^2}{4}\ln(z_0) - \frac{z_0^2}{4}\ln 2 - \frac{5z_0^2}{32} + \frac{z_0^2}{4}\ln 2 \\ &\quad + \frac{1}{8} - \frac{z_0}{4} + \frac{z_0^2}{8} + \frac{z_0^2}{4} - \frac{z_0}{4} - \frac{z_0^2}{4}\ln z_0 = 0 \end{aligned}$$

The expression for $\alpha = 1$ is given by the sum of the linear coefficients:

$$\begin{aligned} &\frac{1}{8} + z_0 - \frac{z_0^2}{4} + \frac{z_0^2}{8}(4(\ln\frac{z_0}{2})^2 + \ln\frac{z_0}{2} - \frac{1}{4}) + \frac{(1-z_0)^2}{4}(\ln(1-z_0) - 1/2) \\ &\quad + z_0^2\left(\left(\ln(z_0) + \frac{1}{4}\right)B^-(1/2;3,0) + 1/2\int_0^{\frac{1}{2}} \ln(t)t^2(1-t)^{-1} + \ln(1-t)t^2(1-t)^{-1}dt\right) \\ &\quad - z_0^2\left(\ln(z_0)B^-(1-z_0;2,-1) + \int_0^{1-z_0} 1/2\ln(t)t(1-t)^{-2} - t\ln(1-t)(1-t)^{-2}dt\right) \\ &= \frac{1}{8} + z_0 - \frac{z_0^2}{4} + \frac{z_0^2}{2}(\ln\frac{z_0}{2})^2 + \frac{z_0^2}{8}\ln\frac{z_0}{2} - \frac{z_0^2}{32} \\ &\quad - \frac{5}{8}z_0^2\ln(z_0) + z_0^2\ln(z_0)\ln 2 - \frac{5z_0^2}{32} + \frac{z_0^2\ln 2}{4} \\ &\quad + z_0^2/2\int_0^{\frac{1}{2}} \ln(t)t^2(1-t)^{-1} + \ln(1-t)t^2(1-t)^{-1}\,\mathrm{dt} \\ &\quad + \frac{(1-z_0)^2}{4}\ln(1-z_0) - \frac{1}{8} + \frac{z_0}{4} - \frac{z_0^2}{8} \\ &\quad + z_0^2\ln(z_0) - z_0\ln z_0 - z_0^2(\ln z_0)^2 - z_0^2\int_0^{1-z_0} 1/2\ln(t)t(1-t)^{-2} - t\ln(1-t)(1-t)^{-2}dt \\ &= \frac{5}{4}z_0 - \frac{9}{16}z_0^2 + \frac{z_0^2}{2}(\ln\frac{z_0}{2})^2 + \frac{z_0^2}{8}\ln\frac{z_0}{2} + \frac{(1-z_0)^2}{4}\ln(1-z_0) \\ &\quad + \frac{3}{8}z_0^2\ln(z_0) + z_0^2\ln(z_0)\ln 2 + \frac{z_0^2\ln 2}{4} + z_0^2/2\int_0^{\frac{1}{2}} \ln(t)t^2(1-t)^{-1} + \ln(1-t)t^2(1-t)^{-1}dt \\ &\quad - z_0\ln z_0 - z_0^2(\ln z_0)^2 - z_0^2\int_0^{1-z_0} 1/2\ln(t)t(1-t)^{-2} - t\ln(1-t)(1-t)^{-2}dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{5}{4}z_0 - \frac{9}{16}z_0^2 + \frac{z_0^2}{2}(\ln\frac{z_0}{2})^2 + \frac{z_0^2}{8}\ln\frac{z_0}{2} + \frac{(1-z_0)^2}{4}\ln(1-z_0) \\
&\quad + \frac{3}{8}z_0^2\ln(z_0) + z_0^2\ln(z_0)\ln 2 + \frac{z_0^2\ln 2}{4} + z_0^2/2(11/8 - 1/4\ln 2 - 3/2\ln(2)^2 - \mathrm{Li}_2(1/2)) \\
&\quad - z_0\ln z_0 - z_0^2(\ln z_0)^2 + z_0(1 + \frac{1}{2}(2-z_0)\ln(z_0) + \frac{1}{2}z_0\ln(z_0)^2 - \frac{1}{2}(1-z_0)\ln(1-z_0) \\
&\quad + \frac{1}{2}z_0\,\mathrm{Li}_2(z_0)) - z_0^2 - \frac{1}{2}z_0^2\,\mathrm{Li}_2(1) \\
&= \frac{9}{4}z_0 - \frac{25}{16}z_0^2 + \frac{z_0^2}{2}(\ln\frac{z_0}{2})^2 + \frac{z_0^2}{8}\ln\frac{z_0}{2} + \frac{(1-z_0)^2}{4}\ln(1-z_0) \\
&\quad - \frac{1}{8}z_0^2\ln(z_0) + z_0^2\ln(z_0)\ln 2 + \frac{z_0^2\ln 2}{4} + z_0^2/2(11/8 - 1/4\ln 2 - 3/2\ln(2)^2 \\
&\quad - \mathrm{Li}_2(1/2) - \mathrm{Li}_2(1) + \mathrm{Li}_2(z_0)) - \frac{1}{2}z_0^2(\ln z_0)^2 - \frac{1}{2}z_0(1-z_0)\ln(1-z_0)
\end{aligned}
$$

where we used that

$$
z_0^2/2\int_0^{\frac{1}{2}} \ln(t)t^2(1-t)^{-1} + \ln(1-t)t^2(1-t)^{-1}\,\mathrm{d}t = 11/8 - 1/4\ln 2 - 3/2\ln(2)^2 - \mathrm{Li}_2(1/2),
$$

and

$$
\begin{aligned}
&z_0^2\int_0^{1-z_0} 1/2\ln(t)t(1-t)^{-2} - t\ln(1-t)(1-t)^{-2}dt \\
&= -\frac{1}{z_0}(1 + \frac{1}{2}(2-z_0)\ln(z_0) + \frac{1}{2}z_0\ln(z_0)^2 - \frac{1}{2}(1-z_0)\ln(1-z_0) + \frac{1}{2}z_0 Li_2(z_0)) + 1 + \frac{1}{2}\,\mathrm{Li}_2(1).
\end{aligned}
$$

By expanding the squares and collecting terms, the last expression can be simplified to

$$
\begin{aligned}
&\frac{9}{4}z_0 + \frac{1-4z_0+3z_0^2}{4}\ln(1-z_0) + z_0^2\left(-7/8 - \frac{\ln(2)^2 + 2\,\mathrm{Li}_2(1/2) + 2\,\mathrm{Li}_2(1)}{4}\right) + \frac{1}{2}z_0^2\,\mathrm{Li}_2(z) \\
=&\frac{9}{4}z_0 + \frac{1-4z_0+3z_0^2}{4}\ln(1-z_0) - \frac{7+\pi^2}{8}z_0^2 + \frac{1}{2}z_0^2\,\mathrm{Li}_2(z)
\end{aligned}
$$

which finishes the computation.

□

3.6 Exact expressions for the local clustering coefficient and function, integrating $\Delta_{\mathcal{P}}(y)$

Now that we have an expression for $\Delta_{\mathcal{P}}(y)$ we can compute the limit local clustering coefficient $c_{\mathcal{P}}$ and the limit local clustering function $c_{\mathcal{P}}(k)$ and prove that they equal c_∞ and $c_\infty(k)$ given in, respectively, Theorem 1.1 and Theorem 1.2.

First we recall that $\rho(y,k) = (\xi_{\alpha,\nu}e^{y/2})^k/k!\,e^{-\xi_{\alpha,\nu}e^{y/2}}$ and

$$
\int_0^\infty \rho(y,k)\alpha e^{-\alpha y}\,\mathrm{d}y = \frac{2\alpha\xi^{2\alpha}}{k!}\Gamma^+(k-2\alpha,\xi) := \rho(k).
$$

We also define

$$
I^{(k)} := \int_0^\infty \Delta_{\mathcal{P}}(y)\alpha e^{-\alpha y}\frac{\left(\xi e^{y/2}\right)^k}{k!}e^{-\xi e^{y/2}}\,\mathrm{d}y
$$

and

$$
J := \int_0^\infty \Delta_{\mathcal{P}}(y)\alpha e^{-\alpha y}\,\mathrm{d}y.
$$

Then

$$c_{\mathcal{P}} = J - I^{(1)} - I^{(2)} \quad \text{and} \quad c_{\mathcal{P}}(k) = \frac{I^{(k)}}{\rho(k)}$$

and hence we first need to compute J and $I^{(k)}$.

For this it will be helpful to change coordinates to $z := e^{-y/2}$. This yields

$$J = 2\alpha \int_0^1 \Delta_{\mathcal{P}}(y) z^{2\alpha-1}\,\mathrm{d}z,$$

and

$$I^{(k)} = \frac{2\alpha\xi_{\alpha,\nu}^k}{k!} \cdot \int_0^1 P(y(z)) \cdot z^{2\alpha-(k+1)} e^{-\xi_{\alpha,\nu} z^{-1}}\,\mathrm{d}z.$$

TM P should be Δ?

We first deal with the case $\alpha \neq 1$. We observe from Proposition 3.2 that for $\alpha \neq 1$, $\Delta_{\mathcal{P}}(y(z))$ is in fact a linear combination of terms of the form z^u, $(1-z)^u$ and $z_0^u B^-(1-z_0, v, w)$.

To compute J we observe that, by partial integration,

$$\begin{aligned}\int_0^1 z^{u+2\alpha-1} B^-(1-z;v,w)\,\mathrm{d}z &= \left[\frac{z^{u+2\alpha}}{u+2\alpha} B^-(1-z;v,w)\right]_0^1 + \frac{1}{u+2\alpha}\int_0^1 z^{u+2\alpha+w-1}(1-z)^{v-1}\,\mathrm{d}z \\ &= \frac{1}{u+2\alpha} B(u+w+2\alpha, v)\end{aligned}$$

where we used that $\frac{\partial}{\partial z} B^-(1-z;v,w) = -z^{w-1}(1-z)^{v-1}$. This takes care of the two integrands involving the Beta function in $\Delta_{\mathcal{P}}(y)$. The other integrals are easily computed and yield the following expression for J (note that it only depends on α but not on ν)

$$\begin{aligned}J = &\frac{2+4\alpha+13\alpha^2-34\alpha^3-12\alpha^4+24\alpha^5}{16(\alpha-1)^2\alpha(\alpha+1)(2\alpha+1)} + \frac{2^{-1-4\alpha}}{(\alpha-1)^2} \\ &+ \frac{(\alpha-1/2)(B(2\alpha, 2\alpha+1) + B^-(1/2; 1+2\alpha, -2+2\alpha))}{2(\alpha-1)(3\alpha-1)}\end{aligned}$$

We proceed to work out $I^{(k)}$. For this we will compute the integrals involving terms in $\Delta_{\mathcal{P}}(y(z))$ of the form z^u, $(1-z)^u$ and $B(1-z,v,w)$ separately. We first point out that for any $0 \leq a < b \leq 1$

$$\begin{aligned}\int_a^b z^{u+2\alpha-(k+1)} e^{-\xi_{\alpha,\nu} z^{-1}}\,\mathrm{d}z &= \xi_{\alpha,\nu}^{u+2\alpha-k} \int_{\xi_{\alpha,\nu}/b}^{\xi_{\alpha,\nu}/a} t^{k-1-2\alpha-u} e^{-t}\,\mathrm{d}t \\ &= \xi_{\alpha,\nu}^{u+2\alpha-k}\left(\Gamma^+(k-2\alpha-u;\xi_{\alpha,\nu}/b) - \Gamma^+(k-2\alpha-u;\xi_{\alpha,\nu}/a)\right),\end{aligned}$$

In particular

$$\int_0^1 z^{u+2\alpha-k-1} e^{-\xi_{\alpha,\nu} z^{-1}}\,\mathrm{d}z = \xi_{\alpha,\nu}^{u+2\alpha-k}\Gamma^+(k-2\alpha-u;\xi_{\alpha,\nu}) \tag{41}$$

where Γ^+ denotes the (upper) incomplete gamma function, and we've used the subsitution $t = \xi_{\alpha,\nu}/z$ which gives $\mathrm{d}z = -\xi_{\alpha,\nu} t^{-2}\,\mathrm{d}t$. (And of course it is understood that $\xi_{\alpha,\nu}/0 = \infty$). This takes care of the integrals of all terms in $\Delta_{\mathcal{P}}(y(z))$ of the form z^u.

Next we will consider the integrals over the terms in $\Delta_{\mathcal{P}}(y(z))$ of the form $(1-z)^u$. For this we need the hypergeometric U-function (also called Tricomi's confluent hypergeometric function), which has the integral representation

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt} t^{a-1}(1+t)^{b-a-1}\,\mathrm{d}t$$

which holds for $a, b, z \in \mathbb{C}$, $b \notin \mathbb{Z}_{\leq 0}$, $Re(a), Re(z) > 0$, see [11, p.255]. Applying the change of variables $t = \frac{1-s}{s}$ (i.e. $\mathrm{d}t = -s^{-2}\,\mathrm{d}s$ and $s = \frac{1}{t+1}$) yields

$$U(a,b,z) = \frac{e^z}{\Gamma(a)}\int_0^1 s^{-b}(1-s)^{a-1} e^{-z/s} ds$$

Plugging in $a = 2\alpha + 1 > 0$, $b = -2\alpha + k + 1$, $z = \xi_{\alpha,\nu} > 0$, then gives

$$\int_0^1 z_0^{2\alpha-k-1} e^{-\xi_{\alpha,\nu}/z_0} (1-z_0)^{2\alpha} dz_0 = \Gamma(2\alpha+1) e^{-\xi_{\alpha,\nu}} U(2\alpha+1, 1+k-2\alpha, \xi_{\alpha,\nu}) \tag{42}$$

Finally we need to deal with the terms in $\Delta_{\mathcal{P}}(y(z))$ that involve the incomplete Beta-function. Let $a, c \in \mathbb{R}$, $\xi_{\alpha,\nu}, b > 0$ positive real numbers. Using the integral definition of the incomplete B-function, the change of variables $s = 1 - t$ gives:

$$\begin{aligned}\int_0^1 z^a e^{-\xi_{\alpha,\nu}/z} B^-(1-z; b, c)\,\mathrm{d}z &= \int_0^1 z^a e^{-\xi_{\alpha,\nu}/z} \int_0^{1-z} t^{b-1}(1-t)^{c-1}\,\mathrm{d}t\,\mathrm{d}z \\ &= \int_0^1 z^a e^{-\xi_{\alpha,\nu}/z} \int_z^1 s^{c-1}(1-s)^{b-1}\,\mathrm{d}s\,\mathrm{d}z\end{aligned}$$

Then changing the order of integration and using the substitution $u = \xi_{\alpha,\nu}/z$ and recognizing the upper incomplete Γ-function yields

$$\begin{aligned}&\int_0^1 z^a e^{-\xi_{\alpha,\nu}/z} \int_z^1 s^{c-1}(1-s)^{b-1}\,\mathrm{d}s\,\mathrm{d}z \\ &\quad= \int_0^1 \int_0^s z^a e^{-\xi_{\alpha,\nu}/z}\,\mathrm{d}z\, s^{c-1}(1-s)^{b-1}\,\mathrm{d}s \\ &\quad= \int_0^1 \int_{\xi_{\alpha,\nu}/s}^{\infty} \xi_{\alpha,\nu}^{a+1} u^{-a-2} e^{-u}\,\mathrm{d}u\, s^{c-1}(1-s)^{b-1}\,\mathrm{d}s \\ &\quad= \xi_{\alpha,\nu}^{a+1} \int_0^1 \Gamma^+(-a-1, \xi_{\alpha,\nu}/s) s^{c-1}(1-s)^{b-1}\,\mathrm{d}s.\end{aligned} \tag{43}$$

To compute this last integral we make use of the fact that the incomplete Γ-function has a representation in terms of Meijer's G-function (see Lemma A.1 in Appendix A)

$$\Gamma^+(-a-1, \xi_{\alpha,\nu}/s) = G^{2,0}_{1,2}\left(\begin{matrix}1 \\ -a-1, 0\end{matrix}\middle| \frac{\xi_{\alpha,\nu}}{s}\right),$$

which holds for any $a \in \mathbb{R}$ and $s > 0$ (that for a fixed second argument, the upper incomplete Γ-function is entire in the first argument, see [14, pp. 899, 1032ff.]). **Pim:** @Markus: What does entire mean? We can now evaluate the integral in (43) using several identities for Meijer's G-function. First, inserting the expression for the incomplete Gamma-function into (43) gives

$$\xi_{\alpha,\nu}^{a+1} \int_0^1 s^{c-1}(1-s)^{b-1} G^{2,0}_{1,2}\left(\begin{matrix}1 \\ -a-1, 0\end{matrix}\middle| \frac{\xi_{\alpha,\nu}}{s}\right) \mathrm{d}s$$

Next we apply the inversion identity for Meijer's G-function (see [11, p. 209, 5.3.1.(9))]) to get

$$\xi_{\alpha,\nu}^{a+1} \int_0^1 s^{c-1}(1-s)^{b-1} G^{0,2}_{2,1}\left(\begin{matrix}2+a, 1 \\ 0\end{matrix}\middle| \frac{s}{\xi_{\alpha,\nu}}\right) \mathrm{d}s$$

This expression is actually the Euler transform of Meijer's G-function (see [11, p. 214, 5.5.2.(5)]) and (as the conditions $2 + 1 < 2(0 + 2)$ and $|\arg(\xi_{\alpha,\nu}^{-1})| < \frac{\pi}{2}$ (as $\xi_{\alpha,\nu} > 0$) and $1 - c - b < 1 - c$ (as $b > 0$) are satisfied) it equals

$$\xi_{\alpha,\nu}^{a+1} \Gamma(b) G^{0,3}_{3,2}\left(\begin{matrix}1-c, 2+a, 1 \\ 0, 1-c-b\end{matrix}\middle| \xi_{\alpha,\nu}^{-1}\right)$$

Using again the inversion identity for Meijer's G-function we now get

$$\xi_{\alpha,\nu}^{a+1} \Gamma(b) G^{3,0}_{2,3}\left(\begin{matrix}1, b+c \\ c, -1-a, 0\end{matrix}\middle| \xi_{\alpha,\nu}\right)$$

Finally, plugging in $a = 6\alpha - k - 3$, $b = 2\alpha$, $c = 3 - 4\alpha$ we obtain

$$\int_0^1 z^a e^{-\xi_{\alpha,\nu}/z} B^-(1-z;b,c)\,\mathrm{d}z = \xi_{\alpha,\nu}^{6a-k-2}\Gamma(2\alpha) G^{3,0}_{2,3}\left(\begin{matrix}1, 3-2\alpha \\ 3-4\alpha, -6\alpha+k+2, 0\end{matrix}\middle|\xi_{\alpha,\nu}\right) \tag{44}$$

Using equation (41), (42) and (44) we get

$$\begin{aligned}
I^{(k)} = \frac{\xi_{\alpha,\nu}^{2\alpha}}{4k!(\alpha-1)}\Bigg(&-\Gamma^+(k-2\alpha,\xi_{\alpha,\nu}) - 2\frac{\alpha(\alpha-1/2)^2\xi_{\alpha,\nu}^2\Gamma^+(k-2\alpha-2,\xi_{\alpha,\nu})}{(\alpha-1)} \\
&+8\alpha(\alpha-1/2)\xi_{\alpha,\nu}\Gamma^+(k-2\alpha-1,\xi_{\alpha,\nu}) \\
&+4\xi_{\alpha,\nu}^{4\alpha-2}\Gamma^+(k-6\alpha+2,\xi_{\alpha,\nu})\left(\frac{2^{-4\alpha}(3\alpha-1)}{(\alpha-1)} + (\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\right) \\
&+\xi_{\alpha,\nu}^{k-2\alpha}\Gamma(2\alpha+1)e^{-\xi_{\alpha,\nu}}U(2\alpha+1,1+k-2\alpha,\xi_{\alpha,\nu}) \\
&-\xi_{\alpha,\nu}^{4\alpha-2}\Gamma(2\alpha+1)G^{3,0}_{2,3}\left(\begin{matrix}1, 3-2\alpha \\ 3-4\alpha, -6\alpha+k+2, 0\end{matrix}\middle|\xi_{\alpha,\nu}\right)\Bigg)
\end{aligned}$$

With the expressions for J and $I^{(k)}$ and using $\Gamma^*(q,z) = \Gamma^+(q+1,z) + \Gamma^+(q,z)$ we now obtain, after some algebra, the expression for $c_{\mathcal{P}}$

$$\begin{aligned}
c_{\mathcal{P}} &= J - I^{(0)} - I^{(1)} \\
&= \frac{2+4\alpha+13\alpha^2-34\alpha^3-12\alpha^4+24\alpha^5}{16(\alpha-1)^2\alpha(\alpha+1)(2\alpha+1)} + \frac{2^{-1-4\alpha}}{(\alpha-1)^2} \\
&\quad+\frac{(\alpha-1/2)(B(2\alpha,2\alpha+1)+B^-(1/2;1+2\alpha,-2+2\alpha))}{2(\alpha-1)(3\alpha-1)} \\
&\quad-\frac{\xi_{\alpha,\nu}^{2\alpha}}{4(\alpha-1)}\Bigg(-\Gamma^+(-2\alpha,\xi_{\alpha,\nu}) - 2\frac{\alpha(\alpha-1/2)^2\xi_{\alpha,\nu}^2\Gamma^+(-2\alpha-2,\xi_{\alpha,\nu})}{(\alpha-1)} \\
&\quad+8\alpha(\alpha-1/2)\xi_{\alpha,\nu}\Gamma^+(-2\alpha-1,\xi_{\alpha,\nu}) \\
&\quad+4\xi_{\alpha,\nu}^{4\alpha-2}\Gamma^+(-6\alpha+2,\xi_{\alpha,\nu})\left(\frac{2^{-4\alpha}(3\alpha-1)}{(\alpha-1)} + (\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\right) \\
&\quad+\xi_{\alpha,\nu}^{-2\alpha}\Gamma(2\alpha+1)e^{-\xi_{\alpha,\nu}}U(2\alpha+1,1-2\alpha,\xi_{\alpha,\nu}) \\
&\quad-\xi_{\alpha,\nu}^{4\alpha-2}\Gamma(2\alpha+1)G^{3,0}_{2,3}\left(\begin{matrix}1, 3-2\alpha \\ 3-4\alpha, -6\alpha+2, 0\end{matrix}\middle|\xi_{\alpha,\nu}\right)\Bigg) \\
&\quad-\frac{\xi_{\alpha,\nu}^{2\alpha}}{4(\alpha-1)}\Bigg(-\Gamma^+(1-2\alpha,\xi_{\alpha,\nu}) - 2\frac{\alpha(\alpha-1/2)^2\xi_{\alpha,\nu}^2\Gamma^+(-2\alpha-1,\xi_{\alpha,\nu})}{(\alpha-1)} \\
&\quad+8\alpha(\alpha-1/2)\xi_{\alpha,\nu}\Gamma^+(1-2\alpha-1,\xi_{\alpha,\nu}) \\
&\quad+4\xi_{\alpha,\nu}^{4\alpha-2}\Gamma^+(1-6\alpha+2,\xi_{\alpha,\nu})\left(\frac{2^{-4\alpha}(3\alpha-1)}{(\alpha-1)} + (\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\right) \\
&\quad+\xi_{\alpha,\nu}^{1-2\alpha}\Gamma(2\alpha+1)e^{-\xi_{\alpha,\nu}}U(2\alpha+1,2-2\alpha,\xi_{\alpha,\nu}) \\
&\quad-\xi_{\alpha,\nu}^{4\alpha-2}\Gamma(2\alpha+1)G^{3,0}_{2,3}\left(\begin{matrix}1, 3-2\alpha \\ 3-4\alpha, -6\alpha+3, 0\end{matrix}\middle|\xi_{\alpha,\nu}\right)\Bigg) \\
&= \frac{2+4\alpha+13\alpha^2-34\alpha^3-12\alpha^4+24\alpha^5}{16(\alpha-1)^2\alpha(\alpha+1)(2\alpha+1)} + \frac{2^{-1-4\alpha}}{(\alpha-1)^2} \\
&\quad+\frac{(\alpha-1/2)(B(2\alpha,2\alpha+1)+B^-(1/2;1+2\alpha,-2+2\alpha))}{2(\alpha-1)(3\alpha-1)} \\
&\quad+\frac{\xi_{\alpha,\nu}^{2\alpha}\Gamma^*(-2\alpha,\xi_{\alpha,\nu})}{4(\alpha-1)} + \frac{\xi_{\alpha,\nu}^{2\alpha+2}\alpha(\alpha-1/2)^2\Gamma^*(-2\alpha-2,\xi_{\alpha,\nu})}{2(\alpha-1)^2}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\xi_{\alpha,\nu}^{2\alpha+1}\alpha(2\alpha-1)\Gamma^*(-2\alpha-1,\xi_{\alpha,\nu})}{(\alpha-1)} - \frac{\xi_{\alpha,\nu}^{6\alpha-2}2^{-4\alpha}(3\alpha-1)\Gamma^*(-6\alpha+2,\xi_{\alpha,\nu})}{(\alpha-1)^2} \\
&- \frac{\xi_{\alpha,\nu}^{6\alpha-2}(\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\Gamma^*(-6\alpha+2,\xi_{\alpha,\nu})}{(\alpha-1)} \\
&- \frac{e^{-\xi_{\alpha,\nu}}\Gamma(2\alpha+1)\left(U(2\alpha+1,1-2\alpha,\xi_{\alpha,\nu})+U(2\alpha+1,2-2\alpha,\xi_{\alpha,\nu})\right)}{4(\alpha-1)} \\
&+ \frac{\xi_{\alpha,\nu}^{6\alpha-2}\Gamma(2\alpha+1)\left(G_{2,3}^{3,0}\left(\xi_{\alpha,\nu}\middle|\begin{matrix}1,3-2\alpha\\3-4\alpha,-6\alpha+2,0\end{matrix}\right)+G_{2,3}^{3,0}\left(\xi_{\alpha,\nu}\middle|\begin{matrix}1,3-2\alpha\\3-4\alpha,-6\alpha+3,0\end{matrix}\right)\right)}{4(\alpha-1)}.
\end{aligned}
$$

and note that this equals c_∞ in Theorem 1.1 when $\alpha \neq 1$.

Similarly, we get

$$
\begin{aligned}
c_{\mathcal{P}}(k) &= \frac{I^{(k)}}{\rho(k)} \\
&= \frac{1}{8\alpha(\alpha-1)\Gamma^+(k-2\alpha,\xi)}\Bigg(-\Gamma^+(k-2\alpha,\xi) - 2\frac{\alpha(\alpha-1/2)^2\xi^2\Gamma^+(k-2\alpha-2,\xi)}{(\alpha-1)} \\
&\quad +8\alpha(\alpha-1/2)\xi\Gamma^+(k-2\alpha-1,\xi) \\
&\quad +4\xi^{4\alpha-2}\Gamma^+(k-6\alpha+2,\xi)\left(\frac{2^{-4\alpha}(3\alpha-1)}{(\alpha-1)} + (\alpha-1/2)B^-(1/2;1+2\alpha,-2+2\alpha)\right) \\
&\quad +\xi^{k-2\alpha}\Gamma(2\alpha+1)e^{-\xi}U(2\alpha+1,1+k-2\alpha,\xi) \\
&\quad -\xi^{4\alpha-2}\Gamma(2\alpha+1)G_{2,3}^{3,0}\left(\begin{matrix}1,3-2\alpha\\3-4\alpha,-6\alpha+k+2,0\end{matrix}\middle|\xi\right)\Bigg),
\end{aligned}
$$

which equals $c_\infty(k)$ in Theorem 1.2 when $\alpha \neq 1$.

Finally, to determine the value of $c_{\mathcal{P}}$ and $c_{\mathcal{P}}(k)$ for $\alpha = 1$, it suffices to find the value of J and $I^{(k)}$ at $\alpha = 1$. We can do this by computing the integrals with the expression for $\Delta_{\mathcal{P}}(y)$ that we found for $\alpha = 1$, i.e.

$$
\begin{aligned}
J &= 2\alpha\int_0^1\left(\frac{9}{4}z + \frac{1-4z+3z^2}{4}\ln(1-z) - \frac{7+\pi^2}{8}z^2 + \frac{1}{2}z^2\operatorname{Li}_2(z)\right)z^{2\alpha-1}\,\mathrm{d}z \\
&= \frac{575-12\pi^2}{576}
\end{aligned}
$$

and

$$
\begin{aligned}
I^{(k)} &= \frac{2\alpha\xi_{\alpha,\nu}^k}{k!}\int_0^1\left(\frac{9}{4}z + \frac{1-4z+3z^2}{4}\ln(1-z) - \frac{7+\pi^2}{8}z^2 + \frac{1}{2}z^2\operatorname{Li}_2(z)\right)z^{2\alpha-k-1}e^{-\xi_{\alpha,\nu}/z}\,\mathrm{d}z \\
&= \frac{2\eta^k}{k!}\int_0^1\left(\frac{9}{4}z + \frac{1-4z+3z^2}{4}\ln(1-z) - \frac{7+\pi^2}{8}z^2 + \frac{1}{2}z^2\operatorname{Li}_2(z)\right)z^{1-k}e^{-\eta/z}\,\mathrm{d}z \\
&= \frac{9\eta^k}{2k!}\eta^{3-k}\Gamma^+(k-3,\eta) - \frac{\eta^k}{k!}\frac{7+\pi^2}{4}\eta^{4-k}\Gamma^+(k-4,\eta) \\
&\quad + \frac{\eta^k}{2k!}\int_0^1(1-4z+3z^2)\ln(1-z)z^{1-k}e^{-\eta/z}dz + \frac{\eta^k}{k!}\int_0^1 z^{3-k}\operatorname{Li}_2(z)e^{-\eta/z}\,\mathrm{d}z \\
&= \frac{9\eta^3}{2k!}\Gamma^+(k-3,\eta) - \frac{\eta^4}{k!}\frac{7+\pi^2}{4}\Gamma^+(k-4,\eta) \\
&\quad + \frac{\eta^k}{2k!}\int_0^1(1-4z+3z^2)\ln(1-z)z^{1-k}e^{-\eta/z}dz + \frac{\eta^k}{k!}\int_0^1 z^{3-k}\operatorname{Li}_2(z)e^{-\eta/z}\,\mathrm{d}z
\end{aligned}
$$

where $\eta = \frac{4\nu}{\pi}$ and and $\operatorname{Li}_2(z) = \sum_{t=1}^\infty z^t/t^2$, the dilogarithm function. Plugging this into the definitions for $c_{\mathcal{P}}$ and $c_{\mathcal{P}}(k)$ yields the expressions c_∞ in Theorem 1.1 and $c_\infty(k)$ in Theorem 1.2, respectively.

4 Proofs of the main results for fixed k

5 Overview of the proof strategy for $k \to \infty$

Pim: References need to be included once Section 1 is updated.

The proofs for Theorems [??] follow the same strategy as outlined in Section 2. However, the fact that $k = k_n \to \infty$ as $n \to \infty$, introduces additional technical challenges. For example, the coupling we use becomes less exact so that we can no longer use Lemma 2.3 to conclude that triangle counts in $G_{\widetilde{\mathbb{H}},n}$ and $G_{\mathcal{P},n}$ are asymptotically equivalent. In this section we explain the challenges with each step and give a detailed overview of the structure for the proof of Theorem [??] using intermediate results for each of the steps. Since we are ultimately interested in recovering the scaling of $c_{\mathbb{H},n}(k_n)$, which Theorem [??] claims is $s_\alpha(k_n)$, we need to show that each step only introduces error terms that are of smaller order, i.e. that are $o\left(s_\alpha(k_n)\right)$. We end this section with the proof of Theorem [??], based on the intermediate results.

Remark 5.1 (Diverging k_n)**.** *Throughout the remainder of this paper $\{k_n\}_{n\geq 1}$ will always denote a sequence of integers satisfying $k_n \to \infty$ and $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$, as $n \to \infty$.*

We start with introducing a slightly modified version of the local clustering function, which will be convenient for computations later,

$$c^*_G(k) = \frac{1}{\mathbb{E}\left[N_G(k)\right]} \sum_{v \in V(G)} \mathbb{1}_{\{D_G(v)=k\}} \frac{T_G(v)}{\binom{k}{2}}. \tag{45}$$

Notice that the only difference between $c_G(k)$ and $c^*_G(k)$ is that we replace $N_G(k)$ by its expectation $\mathbb{E}\left[N_G(k)\right]$. The advantage is that now, the only randomness is in triangle counting. In addition, note that since $\mathbb{E}\left[N_G(k)\right] > 0$ a case distinction for N_k is no longer needed for $c^*_G(k)$. It is however still relevant since we are eventually interested in $c_G(k)$. Following the notational convention, see Remark 1.1, throughout the remainder of this paper we write $c^*_{\widetilde{\mathbb{H}},n}(k)$ and $c^*_{\mathcal{P},n}(k)$ to denote the modified local clustering function in $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$ and $G_{\mathcal{P},n}(\alpha,\nu)$, respectively.

Figure 7 shows a schematic overview of the proof of Theorem [??] based on the different propositions described below, plus the sections in which theses propositions are proved. Observe that the order in which the intermediate results are proved is reversed with respect to the natural order of reasoning. This does not create any circular logic, since each intermediate result is independent of the others. We choose this order because results proved in the later stages are helpful to deal with error terms coming up in proofs at earlier stages and hence help streamline those proofs.

5.1 Adjusted clustering and Poisson nodes in hyperbolic graphs

Recall that the first step for the fixed k case was to show that the transition from the hyperbolic random graph $G_{\mathbb{H},n}$ to the Poisson version $G_{\widetilde{\mathbb{H}},n}$ did not influence clustering. Here we first make a transition from the local clustering function $c_{\mathbb{H},n}(k)$ to the adjusted version $c^*_{\mathbb{H},n}(k)$. The following lemma justifies working with this modified version. The proof uses a concentration result for $\mathbb{N}_{\mathbb{H},n}(k_n)$ and full details can be found in Section 9.3.

Lemma 5.1. *As $n \to \infty$,*

$$\mathbb{E}\left[\left|c_{\mathbb{H},n}(k_n) - c^*_{\mathbb{H},n}(k_n)\right|\right] = o\left(s_\alpha(k_n)\right).$$

We then establish that the modified local clustering function in the hyperbolic model $G_{\mathbb{H},n}(\alpha,\nu)$ behaves similarly to that in the Poisson version $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$. This is based on a standard coupling between a Binomial Point Process and Poisson Point Process.

Proposition 5.2. *As $n \to \infty$,*

$$\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = o\left(s_\alpha(k_n)\right).$$

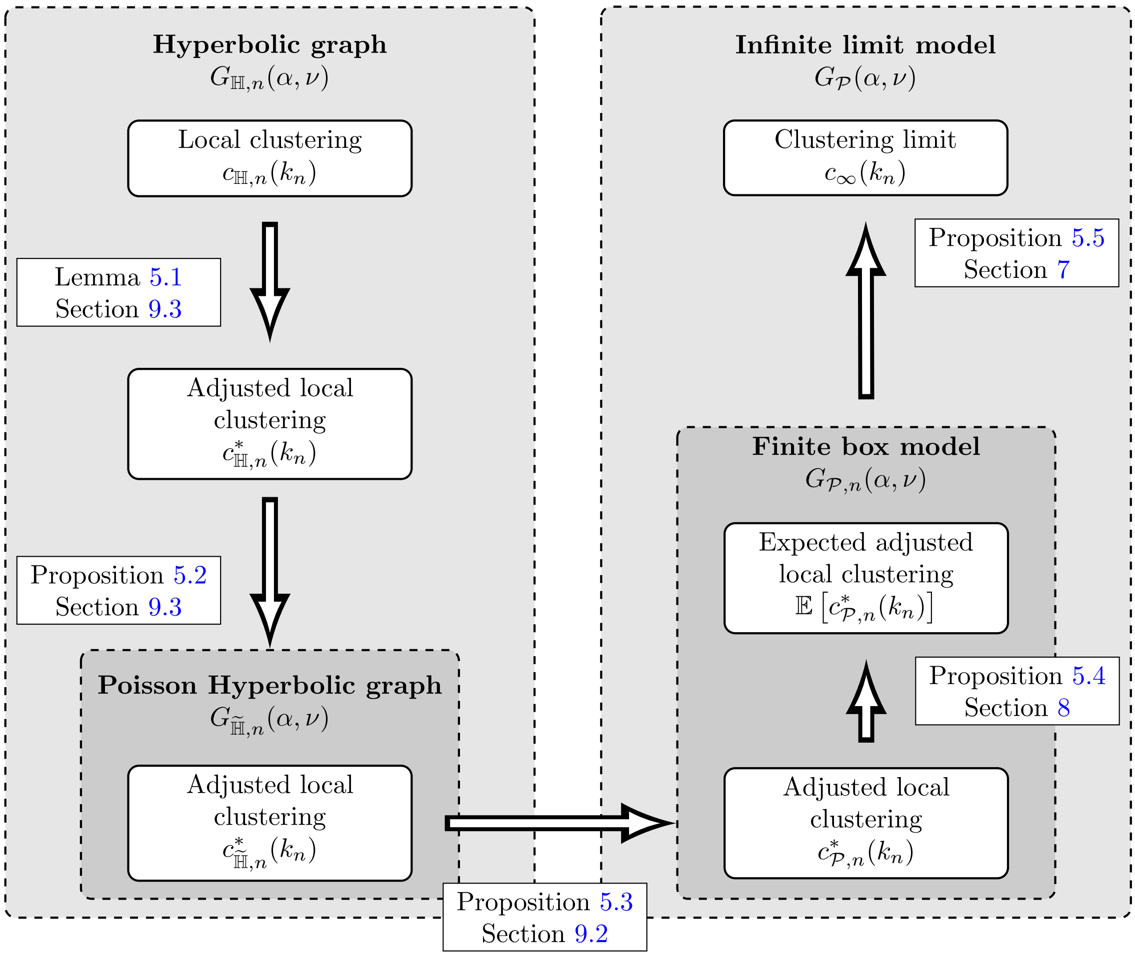

Figure 7: Overview of the proof strategy for Theorem 1.2.

5.2 Coupling of local clustering between $G_{\widetilde{\mathbb{H}},n}$ and $G_{\mathcal{P},n}$

The next step is to show that the modified clustering is preserved under the coupling described in Section 2.5. The proof can be found in Section 9.2. This is one of the key technical challenges we face.

To understand why, recall that the degree k of a node is related to its height y, roughly speaking, by $k \approx \xi_{\alpha,\nu} e^{y/2}$. Therefore, when k is fixed we have that the heights of nodes with that degree are also fixed, in particular $y < R_n/4$ for large enough n. In addition, the main contribution of triangles would also come from nodes with heights $y' < R_n/4$. This allowed us to use Lemma 2.3 and conclude that the triangles present in the graph $G_{\widetilde{\mathbb{H}},n}$ where exactly those present in $G_{\mathcal{P},n}$ and therefore the local clustering function was the same in both models. When $k_n \to \infty$ this is no longer true in general. For instance, suppose $k_n = n^{\frac{1-\varepsilon}{2\alpha+1}}$, for some small $0 < \varepsilon < 1$. Then the relation $k_n \approx \xi_{\alpha,\nu} e^{y_n/2}$ implies that $y_n \approx \frac{2(1-\varepsilon)}{2\alpha+1}\log(n) - 2\log(\xi_{\alpha,\nu})$. Since $R_n/4 = \frac{1}{2}\log(n) - \frac{1}{2}\log(\nu)$ we get that $R_n/4 = o(y_n)$ for all $\alpha > (3-4\varepsilon)/2$ and hence $y_n > R_n/4$ for large enough n, violating the conditions of Lemma 2.3. However, by carefully analyzing the difference between the adjusted local clustering function in both models we can still make the same conclusion. This is summarized in the following proposition whose proof is found in Section 9.2.

Proposition 5.3 (Coupling result for local clustering)**.** *As $n \to \infty$,*

$$\mathbb{E}\left[\left|c^*_{\widetilde{\mathbb{H}},n}(k_n) - c^*_{\mathcal{P},n}(k_n)\right|\right] = o\left(s_\alpha(k_n)\right).$$

Tobias: Maybe we could replace these three with a statement on $|c_{\mathbb{H},n}(k_n) - c^*_{\mathcal{P},n}(k_n)|$, at least at this point of the paper. This is only the high level description and that is all we need for the "final proof".**Pim:** I would vote for keeping them split, since this allows us to clearly point to the main technical challenge we have to overcome to obtain the final result. These three results together imply that the difference between the local clustering function of the hyperbolic random graph and the modified local clustering function of the finite box graph converges to zero faster than the proposed scaling $s_\alpha(k_n)$ in Theorem [??]. Hence, to prove this theorem it is enough to prove it for $c^*_{\mathcal{P},n}(k)$.

5.3 From the finite to the infinite model

To compute the limit of the modified local clustering function $c^*_{\mathcal{P},n}(k)$ in the finite graph $G_{\mathcal{P},n}(\alpha,\nu)$ we first prove in Section 8 that it is concentrated around its expectation $\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]$.

Tobias: "concentration" is a loaded term in probability. I am not sure this use of the word will not be counterintuitive to many. **Pim:** I am not sure. Concentration generally refers to how much a random variable deviates from its expectation. This is exactly what this proposition tells us. I would therefore vote for keeping this terminology, although I would have no problem with replacing it with an other term if someone has a good suggestion.

Proposition 5.4 (Concentration for local clustering function in $G_{\mathcal{P},n}(\alpha,\nu)$)**.** *As $n \to \infty$,*

$$\mathbb{E}\left[\left|c^*_{\mathcal{P},n}(k_n) - \mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right|\right] = o\left(s_\alpha(k_n)\right).$$

This result is another one of the technical challenges we face when considering $k_n \to \infty$. For the proof, we first identify the specific range of heights that give the main contribution to the triangle count, showing that the triangles coming from nodes with heights outside this range is of smaller order. Then we prove a concentration result for the main term, by carefully analyzing the joint neighborhoods of two nodes whose heights fall into the identified range. The full details are found in Section 8.

Assuming this concentration result, we are left with the task to compute the limit of $\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]$ as $n \to \infty$ and show that it is equivalent to $c_\infty(k_n)$. To accomplish this we move to the infinite limit model $G_{\mathcal{P}}(\alpha,\nu)$ and show that the difference between the expected value of the clustering function $c^*_G(k)$ in $G_{\mathcal{P},n}(\alpha,\nu)$ and $G_{\mathcal{P}}(\alpha,\nu)$ goes to zero faster than the proposed scaling in Theorem 1.2.

Proposition 5.5 (Transition to the infinite limit model)**.** *As $n \to \infty$,*

$$\left|\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right] - c_{\mathcal{P}}(k_n)\right| = o\left(s_\alpha(k_n)\right).$$

Tobias: This is only for $c(k)$. Should we not also mean c? **Pim:** I do not think so. We can prove everything for c once we have the result for $c(k_n)$.

Recall that for the finite box model the left and right boundaries of $\mathcal{R}_n$ where identified, so that graph $G_{\mathcal{P},n}$ contains some additional edge with respect to the induced subgraph of $G_{\mathcal{P}}$ on $\mathcal{R}_n$. The prove of Proposition 5.5 therefore relies on analyzing the number of triangles coming from these additional edges and showing that their contribution to the local clustering function are of negligible order, see Section 7.

5.4 Proof of the main results

We are now ready to prove Theorem [??], using the propositions stated in the previous sections. As mentioned before, we focus here on the convergence statements. The computation of the exact expressions is done in Section 3. We begin with Theorem 1.2.

Proof of Theorem 1.2. First of all, due to cancellation of equal terms we can rewrite

$$c_{\mathbb{H},n}(k_n) - c_\infty(k_n) = c_{\mathbb{H},n}(k_n) - c^*_{\mathbb{H},n}(k_n) + c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n) + c^*_{\widetilde{\mathbb{H}},n}(k_n) - c^*_{\mathcal{P},n}(k_n)$$

$$+ c^*_{\mathcal{P},n}(k_n) - \mathbb{E}c^*_{\mathcal{P},n}(k_n) + \mathbb{E}c^*_{\mathcal{P},n}(k_n) - c_\infty(k_n)$$

Then, we take absolute values and apply the triangle inequality. By monotonicity of expectation, we can apply it to both sides and obtain

$$\begin{aligned}\mathbb{E}[|c_{\mathbb{H},n}(k_n) - c_\infty(k_n)|] \leq & \mathbb{E}[|c_{\mathbb{H},n}(k_n) - c^*_{\mathbb{H},n}(k_n)|] + \mathbb{E}[|c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)|] \\ & + \mathbb{E}[|c^*_{\widetilde{\mathbb{H}},n}(k_n) - c^*_{\mathcal{P},n}(k_n)|] + \mathbb{E}[|c^*_{\mathcal{P},n}(k_n) - \mathbb{E}c^*_{\mathcal{P},n}(k_n)|] \\ & + \mathbb{E}[|\mathbb{E}c^*_{\mathcal{P},n}(k_n) - c_\infty(k_n)|]\end{aligned}$$

At this point, the lemmas and propositions presented above in this section can be applied in order to show that all summands are $o\left(s_\alpha(k_n)\right)$: Lemma 5.1 for the transition to the modified clustering function in the first term, Proposition 5.2 for the Poissonization in the disk in the second term, Proposition 5.3 for the coupling from the disk to the finite box model in the third term, Proposition 5.4 for the concentration in the fourth term and finally Proposition 5.5 for the transition to the infinite limit model where we also used that $c_{\mathcal{P}}(k_n) = c_\infty(k_n)$ as obtained at the end of Section 3. **Tobias:** Maybe we can have a more clear reference? Now, at this place in the paper this proof feels a bit like cheating since there are almost no details given. **Pim:** I completely agree. I think that once the results for fixed k have been included we can include a specific statement at the end of Section 3 we can refer to. All of this together yields that:

$$\mathbb{E}[|c_{\mathbb{H},n}(k_n) - c_\infty(k_n)|] = o\left(s_\alpha(k_n)\right) = o\left(c_\infty(k_n)\right)$$

Tobias: Last equality seems to rely on Thm 1.3, which is not yet proven at this point. **Pim:** We can fix this by proving the scaling of $c_\infty(k)$ in Section 3. i.e. the statement of the theorem. □

6 Concentration of heights for vertices with degree k

Many of the computations in this paper will involve integrals whose integrand contains the function $\mathbb{P}\left(\mathrm{Po}(\mu_n(y)) = k\right)$, where $\mathrm{Po}(\lambda)$ denotes a Poisson random variable with expectation λ and $\mu_n(y) \approx e^{y/2}$. For instance, the expected modified local clustering function $c_\infty(k)$ in $G_{\mathcal{P}}$ equals

$$\frac{\int_0^\infty \mathbb{P}\left(\mathrm{Po}(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)) = k\right)\Delta_{\mathcal{P}}(y)\alpha e^{-\alpha y}\,\mathrm{d}y}{\int_0^\infty \mathbb{P}\left(\mathrm{Po}(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)) = k\right)\alpha e^{-\alpha y}\,\mathrm{d}y},$$

where $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right) = \xi_{\alpha,\nu}e^{y/2}$ and $\Delta_{\mathcal{P}}(y)$ is defined by (13).

In particular, we already established that the degree of a node at height y in the infinite model is Poisson with mean $\xi_{\alpha,\nu}e^{y/2}$. Since we are mostly interested in those nodes with degree k_n and Poisson random variables are well concentrated around their mean, we would therefore like to be able the restrict integration to intervals where $e^{y/2} \approx k_n$. Moreover, we want to be able to do this for all three Poisson models $G_{\widetilde{\mathbb{H}},n}$, $G_{\mathcal{P},n}$ and $G_{\mathcal{P}}$ as well as simultaneously deal with the case where $k_n \to \infty$ and $k_n = \Theta\left(1\right)$. In this section we will establish such a result. We start with a concentration lemma for the infinite model (Lemma 6.1) and explain in Remark 6.1 how such a result will be used throughout the paper. To obtain similar results for the other two models we have to first analyze the average degree in these models which we do in Sections 6.2 and 6.3. We conclude this section with a general result that allows us to extend the concentration lemma for the infinite model to the hyperbolic random graph and finite box model.

6.1 Concentration of heights argument for the infinite model

Recall that $\rho(y,k)$ is the probability density function of a Poisson random variable with expectation $\mu_{\alpha,\nu}(B_{\mathcal{P}}(y)) = \xi_{\alpha,\nu}e^{\frac{y}{2}}$. We will consider two different types of sequences $\{k_n\}_{n\geq 1}$, those that are asymptotically bounded and those that diverge. We define

$$\kappa_n := \begin{cases} \log(n) & \text{if } k_n = \Theta\left(1\right), \\ \sqrt{k_n \log(k_n)} & \text{if } k_n = \omega(1). \end{cases} \tag{46}$$

Tobias: This is not a good definition. There are sequences that neither stay bounded nor tend to infinity, e.q. odd values equal to 2 even values equal to $\log n$. How do you decide for a given k_n which of the two cases you pick. To solve this we could either restrict to sequences that are either a) constant or b) tending to infinity; or we could make a cut-off at some specific, slow growing function such as $\log\log\log n$. **Pim:** You are right, I overlooked this fact. I would opt to go with the setting where we consider only sequences that are either asymptotically bounded or tend to infinity. After the new proves are added we can see where we specify this. and define further, for any $C > 0$

$$\mathcal{K}_C(k_n) = \left\{ y \in \mathbb{R}_+ : \frac{k_n - C\kappa_n}{\xi_{\alpha,\nu}} \le e^{\frac{y}{2}} \le \frac{k_n + C\kappa_n}{\xi_{\alpha,\nu}} \right\}, \tag{47}$$

Note that if $k_n = \Omega(\log(n))$ then $e^{y/2} = \Theta(k_n)$ whenever $y \in \mathcal{K}_C(k_n)$. The next lemma states that for a large class of functions $h(y)$, to compute the integral

$$\int_{\mathbb{R}_+} \rho(y, k_n) h(y) e^{-\alpha y} \, \mathrm{d}y$$

Tobias: This part is supposed to be for the infinite model, yet you integrate over the finite box $\mathcal{R}_n$. Is this really meant? I would have expected we just integrate wrt. y and replace $f_{\alpha,\nu}(x, y)$ by $\alpha e^{-\alpha y}$, the exponential density.**Pim:** I understand this confusion and agree that your suggestion makes more sense here. However, I choose this setup to minimize notations later. I am for changing the setup to what you are suggesting but I will have to spend some time first on checking how that will effect the definitions in later sections. it is enough to consider integration over $\mathcal{K}_C(k_n)$ instead of $\mathbb{R}_+$. More precisely, for any $\ell_n = (1 + o(1))k_n$ it is enough to consider $\mathcal{K}_C(\ell_n)$. The flexibility of using $(1 + o(1))k_n$ instead of k_n will prove useful in later sections.

Lemma 6.1. *Let $\alpha > \frac{1}{2}$, $\nu > 0$, $\{k_n\}_{n\ge 1}$ be any positive sequence such that $k_n = o(n^{\frac{1}{2\alpha+1}})$ and let $\ell_n = k_n(1 + \epsilon_n)$, with $\epsilon_n \to 0$. In addition let $\beta < \alpha$ and $h : \mathbb{R}_+ \to \mathbb{R}$ be a any function such that $h(y) = g(y)e^{\beta y}$ with $|g(y)|$ uniformly bounded on $\mathbb{R}_+$. Then, if we define for $C > 0$*

$$\lambda_n^{\pm} = (\ell_n \pm C\kappa_n) \wedge \xi_{\alpha,\nu}, \quad a_n^{\pm} = 2\log\left(\frac{\lambda_n^{\pm}}{\xi_{\alpha,\nu}}\right),$$

Tobias: *If k is constant, then $\ell^- < 0$ and hence a^- is undefined! Need to change the definitions.* **Pim:** *You are right. I changed it by taking the minimum with $\xi_{\alpha,\nu}$.* *we have*

$$\int_{\mathbb{R}_+ \setminus \mathcal{K}_C(k_n)} \rho(y, k_n) h(y) \alpha e^{-\alpha y} \, \mathrm{d}y = \begin{cases} O\left(\log(n)^{k_n} n^{-\frac{C}{2}}\right) & \text{if } k_n = \Theta(1) \\ O\left(k_n^{-(1+C^2)/2}\right) & \text{if } k_n = \omega(1), \end{cases} \tag{48}$$

Tobias: *As before, this is not a proper dichotomy. (unless we assume more about the sequences k_n, i.e. either constant or tending to infinity* *as $n \to \infty$.*

In particular, if $h_n(y)$ is a function such that $h_n(y) = O\left(n^{-1}k_n^s e^{\beta y}\right) \rho(y, k_n)$ for some $s \in \mathbb{R}$ and $\beta < \alpha$ as $n \to \infty$. Then for $C > 0$ large enough we have

$$\lim_{n\to\infty} \int_{\mathcal{R}_n \setminus (\mathbb{R} \times \mathcal{K}_C(\ell_n))} h_n(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y = 0,$$

or equivalently,

$$\int_{\mathcal{R}_n \setminus (\mathbb{R} \times \mathcal{K}_C(\ell_n))} h_n(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y = (1 + o(1)) \int_{\mathcal{K}_C(\ell_n)} h_n(y) f_{\alpha,\nu}(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

Proof. Observe that in this case

$$\kappa_n := \begin{cases} \log(n) & \text{if } k_n = \Theta(1), \\ \sqrt{\ell_n \log(\ell_n)} & \text{else.} \end{cases}$$

Now consider $\rho(y, k_n)$ as a function of y. Then, since

$$\frac{\partial \rho}{\partial y} = \left(k_n y^{-1} - 1\right) \rho(y, k_n),$$

it follows that $\rho(y, k_n)$ attains its maximum at $y = k_n$. Moreover we see that the derivative is strictly positive on $[0, k_n)$ and strictly negative on (k_n, ∞). Therefore, since $a_n^- < k_n$ and $a_n^+ > k_n$, $\rho(y, k_n)$, as a function of y, is strictly increasing on $[0, a_n^-]$ and strictly decreasing on $[a_n^+, \infty)$. Therefore, by our assumption on $h(y)$,

$$\begin{aligned}
&\int_{\mathbb{R}_+ \setminus [a_n^-, a_n^+]} h(y)\rho(y, k_n)\alpha e^{-\alpha y} \, \mathrm{d}y \\
&\quad = O(1) \int_0^{a_n^-} e^{\beta y} \rho(y, k_n)\alpha e^{-\alpha y} \, \mathrm{d}y + O(1) \int_{a_n^+}^{\infty} e^{\beta y} \rho(y, k_n)\alpha e^{-\alpha y} x \, \mathrm{d}y \\
&\quad = O(1) \int_0^{a_n^-} \rho(y, k_n) e^{-(\alpha-\beta)y} \, \mathrm{d}y + O(1) \int_{a_n^+}^{\infty} \rho(y, k_n) e^{-(\alpha-\beta)y} \, \mathrm{d}y \\
&\quad \leq O(1)\, \rho(a_n^-, k_n) \int_0^{a_n^-} e^{-(\alpha-\beta)y} \, \mathrm{d}y + O(1)\, \rho(a_n^+, k_n) \int_{a_n^+}^{\infty} e^{-(\alpha-\beta)y} \, \mathrm{d}y.
\end{aligned}$$

Note that if $k_n = \Theta(1)$ then, for large enough n, the first integral is zero. We conclude that

$$\int_{\mathbb{R}_+ \setminus [a_n^-, a_n^+]} h(y)\rho(y, k_n)\alpha e^{-\alpha y} \, \mathrm{d}y = \begin{cases} O(1)\, \rho(a_n^+, k_n) & \text{if } k_n = \Theta(1), \\ O(1)\left(\rho(a_n^-, k_n) + \rho(a_n^+, k_n)\right) & \text{if } k_n = \omega(1). \end{cases} \tag{49}$$

We shall now bound the terms $\rho(a_n^{\pm}, k_n)$, starting with $\rho(a_n^+, k_n)$. Using that $k! > \sqrt{2\pi} k^{k+1/2} e^{-k}$ **Tobias:** Reference! we write

$$\begin{aligned}
\rho(a_n^+, k_n) &= \frac{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(a_n^+\right)\right)^{k_n}}{k_n!} e^{-\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(a_n^+\right)\right)} \\
&\leq (2\pi)^{-1/2} k_n^{-1/2} \left(\frac{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(a_n^+\right)\right)}{k_n}\right)^{k_n} e^{-\left(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(a_n^+\right)\right) - k_n\right)} \\
&= (2\pi)^{-1/2} k_n^{-1/2} e^{-k_n\left(\frac{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(a_n^+\right)\right)}{k_n} - 1 - \log\left(\frac{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(a_n^+\right)\right)}{k_n}\right)\right)}.
\end{aligned}$$

Let us first consider the case $k_n \to \infty$, in which case $\kappa_n = \sqrt{\ell_n \log(\ell_n)}$. Since

$$\frac{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(a_n^+\right)\right)}{k_n} = \frac{\lambda_n^+}{k_n} = 1 + \epsilon_n + C\frac{\kappa_n}{k_n} = 1 + \epsilon_n + C\sqrt{\frac{(1+\epsilon_n)\log((1+\epsilon_n)k_n)}{k_n}},$$

and $x - \log(1+x) \sim x^2/2$ as $x \to 0$, we get

$$\begin{aligned}
\rho(a_n^+, k_n) &\leq \sqrt{2\pi} k_n^{-1/2} e^{-k_n\left(\epsilon_n + C\frac{\kappa_n}{k_n} - \log\left(1 + \epsilon_n + C\frac{\kappa_n}{k_n}\right)\right)} \\
&\sim (2\pi)^{-1/2} k_n^{-1/2} e^{-\frac{k_n(\epsilon_n + C\kappa_n/k_n)^2}{2}} \\
&= O\left(k_n^{-(1+C^2)/2}\right),
\end{aligned} \tag{50}$$

where for the last line we used that

$$-k_n \frac{\left(\epsilon_n + C\kappa_n/k_n\right)^2}{2} = -\frac{C^2}{2}\log(k_n) + \Theta(1).$$

Let use now consider the case $k_n = \Theta(1)$. Then $\kappa_n = \log(n)$,

$$\frac{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(a_n^+\right)\right)}{k_n} = \frac{\lambda_n^+}{k_n} = 1 + \epsilon_n + C\frac{\log(n)}{k_n},$$

and hence

$$\rho(a_n^-, k_n) \leq O\left(\log(n)^{k_n} n^{-C}\right). \tag{51}$$

Note that for $\rho(a_n^-, k_n)$ we only need to consider the case $k_n \to \infty$. A similar analysis as for a_n^- yields

$$\rho(a_n^-, k_n) \leq \Theta(1)\, k_n^{-1/2} e^{-\frac{k_n(\epsilon_n - C\kappa_n/k_n)^2}{2}} = O\left(k_n^{-(1+C^2)/2}\right). \tag{52}$$

Plugging (52), (50) and (51) into (49) yields the result. The second statement immediately follows from the first by choosing a large enough C and observing that

$$\int_{\mathcal{R}_n \setminus (\mathbb{R} \times \mathcal{K}_C(\ell_n))} h_n(y) f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y \leq n \int_{\mathbb{R}_+ \setminus [a_n^-, a_n^+]} h_n(y) \alpha e^{-\alpha y}\, \mathrm{d}y.$$

□

Remark 6.1 (Concentration of heights argument)**.** *Lemma 6.1 will prove very useful in the remainder of this paper since we often have to deal with integrands of the form $g_n(y) f_{\alpha,\nu}(x,y)$ where $g_n(y) = O\left(n^{-1} k_n^s\right) e^{\beta y} \rho(y, k_n)$ for some $s \in \mathbb{R}$ and $\beta < \alpha$. In this case the lemma tells us that for a suitable $C > 0$ we only need to integrate over $\mathcal{K}_C(k_n)$. In order words, we may always assume for $g_n(y)$ (for a penalty of $o(1)$) that $k_n - C\kappa_n \leq \xi_{\alpha,\nu} e^{y/2} \leq k_n + C\kappa_n$. We will refer to this as a* concentration of heights argument*, e.g.* by a concentration of heights argument

$$\int_{\mathcal{R}_n} g_n(y) \rho(y, k_n) f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y = (1 + o(1)) n \int_{\mathcal{K}_C(k_n)} g_n(y) \rho(y, k_n) \alpha e^{-\alpha y}\, \mathrm{d}y.$$

Pim: *@All:It might be a good idea to add one more example of how the concentration argument is used in the paper. I someone finds a good example we use later on, please add it here.*

We will later establish similar results for the cases where we consider the hyperbolic and finite box model, i.e. when we have $\rho_{\mathbb{H},n}(y,k)$ or $\rho_n(y,k)$ instead of $\rho(y,k)$. For this the following, slightly more general version of Lemma 6.1 will be important.

Lemma 6.2. *Let $\alpha > \frac{1}{2}, \nu > 0$, k_n be any positive sequence such that $k_n = o(n^{\frac{1}{2\alpha+1}})$, $\ell_n = (1+\epsilon_n)k_n$, with $\epsilon_n \to 0$ and let $\mathcal{K}_{C,n}(\ell_n)$ be defined as in (56). In addition, define $\hat{\rho}_n(y,k) = \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k\right)$, where $\hat{\mu}_n(y)$ satisfies,*

$$\hat{\mu}_n(y) = (1 + \phi_n(y)) \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}(0,y)\right),$$

where $\phi_n(y)$ is a continuous differentiable function such that for some $0 < \varepsilon < 1$

$$\sup_{0 \leq y \leq (1-\varepsilon)R_n} |\phi_n(y)| = 0.$$

Then for any sequence of continuous functions $h_n(y) : \mathbb{R}_+ \to \mathbb{R}$ such that, uniformly on $(0, R_n]$, $h_n(y) = O\left(k_n^s e^{\beta y} \hat{\rho}_n(y, k_n)\right)$, as $n \to \infty$, **Tobias:** *Why not adjust h so that we incorporate $\hat{\rho}$ in line below. Then both items look more similar.* **Pim:** *The reason for the bound on $h_n(y)$ in the first statement is that we want to use this for cases where $h_n(y) \leq k_n^s \hat{\rho}_n(y, k_n)$ (see for instance the proof of Proposition 5.5. for some $s < 2\alpha(2\alpha+1)$ and $\beta < \alpha$, we have,*

$$\int_{\mathcal{R}_n} h_n(y) f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y = (1 + o(1)) \int_{\mathcal{K}_{C,n}(\ell_n)} h_n(y) f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y,$$

for some $C > 0$ large enough.

Proof. Similarly to the proof of Lemma 6.1 we define

$$\lambda_n^\pm = (\ell_n \pm C\kappa_n) \wedge \xi_{\alpha,\nu}, \quad \text{and} \quad a_n^\pm = 2\log\left(\frac{\lambda_n^\pm}{\xi_{\alpha,\nu}}\right).$$

In addition we fix $\varepsilon^* > 0$ such that $\varepsilon < \min\{1 - \frac{s}{2\alpha(2\alpha+1)}, \varepsilon, 1\}$ and. Then since $h_n(y) = \hat{\rho}_n(y,k_n) O\left(k_n^s\right)$ we have

$$k_n^s \int_{(1-\varepsilon^*)R_n}^{R_n} \hat{\rho}_n(y,k_n) e^{-\alpha y} \, \mathrm{d}y = O\left(1\right) \hat{\rho}_n((1-\varepsilon^*)R_n, k_n) k_n^s e^{-\alpha(1-\varepsilon^*)R_n}$$

$$= O\left(\hat{\rho}_n((1-\varepsilon^*)R_n, k_n) k_n^s n^{-2\alpha(1-\varepsilon^*)}\right)$$

$$= o\left(\hat{\rho}_n((1-\varepsilon^*)R_n, k_n) n^{\frac{s}{2\alpha+1} - 2\alpha(1-\varepsilon^*)}\right) = o\left(1\right),$$

where the last step follows by our choice of ε^*. Hence it is enough to prove both statement on $\mathcal{R}_n((1-\varepsilon)R_n, R_n)$ instead of $\mathcal{R}_n$. **Pim:** @All: This is where we use that $s < 2\alpha(2\alpha+1)$. I think this can be removed by using a more careful bound but I am not sure this is needed.

Since we already showed that

$$\lim_{n\to\infty} k_n^s \int_{(1-\varepsilon)R_n}^{R_n} \hat{\rho}_n(y,k_n) e^{(\beta-\alpha)y} \, \mathrm{d}y = 0,$$

by our assumption on $h_n(y)$, it is enough to show that for sufficiently large $C > 0$,

$$\lim_{n\to\infty} k_n^s \int_0^{a_n^-} \hat{\rho}_n(y,k_n) e^{(\beta-\alpha)y} \, \mathrm{d}y = 0, \tag{53}$$

and

$$\lim_{n\to\infty} k_n^s \int_{a_n^+}^{(1-\varepsilon)R_n} \hat{\rho}_n(y,k_n) e^{(\beta-\alpha)y} \, \mathrm{d}y = 0. \tag{54}$$

For simplicity we write $\mu(y) := \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(y\right)\right)$.. Now fix some $0 < \delta < 1$ and let n be large enough such that

1. $\sup_{0 < y \le (1-\varepsilon)R_n} |\phi_n(y)| < \delta$,
2. $(\ell_n + C\kappa_n)(1-\delta) > k_n$ and
3. $(\ell_n - C\kappa_n)(1+\delta) < k_n$

Next, recall that the function $\lambda \mapsto \mathbb{P}\left(\mathrm{Po}(\lambda) = k\right)$ is monotonic increasing on $[0,k]$ and monotonic decreasing on $[k,\infty)$. Then since for n large enough we have

$$\hat{\mu}_n(y) = \mu(y)(1+\phi_n(y)) \ge \mu(y)(1-\delta) \ge \mu(a_n^+)(1-\delta) = (\ell_n + C\kappa_n)(1-\delta) > k_n,$$

it follows that

$$\hat{\rho}_n(y,k_n) = \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k_n\right) \le \mathbb{P}\left(\mathrm{Po}(\mu(y)(1-\delta)) = k_n\right),$$

for all $a_n^+ \le y \le (1-\varepsilon)R_n$. By making the change of variables $z = \mu^{-1}(\mu(y)(1-\delta)) = y + 2\log(1-\delta)$ we then get

$$k_n^s \int_{a_n^+}^{(1-\varepsilon)R_n} \hat{\rho}_n(y,k_n) e^{(\beta-\alpha)y} \, \mathrm{d}y \le k_n^s \int_{a_n^+}^{(1-\varepsilon)R_n} \mathbb{P}\left(\mathrm{Po}(\mu(y)(1-\delta)) = k_n\right) e^{(\beta-\alpha)y} \, \mathrm{d}y$$

$$= k_n^s (1-\delta)^{\beta-\alpha} \int_{a_n^+ + 2\log(1-\delta)}^{(1-\varepsilon)R_n + 2\log(1-\delta)} \mathbb{P}\left(\mathrm{Po}(\mu(z)) = k_n\right) e^{(\beta-\alpha)z} \, \mathrm{d}z$$

$$\le k_n^s (1-\delta)^{\beta-\alpha} \int_{a_n^+}^{\infty} \rho(z,k_n) e^{(\beta-\alpha)z} \, \mathrm{d}z.$$

Since Lemma 6.1 implies that for large enough C,

$$\lim_{n\to\infty} k_n^s \int_{a_n^+}^{\infty} \rho(z,k_n) e^{(\beta-\alpha)z} \, \mathrm{d}z = 0,$$

we have proven (61).

The proof of (60) follows the same line of reasoning. This time we use that for $0 \le y \le a_n^-$,

$$\hat{\mu}_n(y) \le \mu(y)(1+\delta) \ge \mu(a_n^-)(1+\delta) = (\ell_n - C\kappa_n)(1+\delta) < k_n,$$

so that

$$\hat{\rho}_n(y, k_n) = \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k_n\right) \le \mathbb{P}\left(\mathrm{Po}(\mu(y)(1+\delta)) = k_n\right).$$

Making a similar change of variables $z = y + 2\log(1+\delta)$ we then get

$$\begin{aligned} k_n^s \int_0^{a_n^-} \hat{\rho}_n(y, k_n) e^{(\beta-\alpha)y} \, \mathrm{d}y &= k_n^s \int_0^{a_n^-} \mathbb{P}\left(\mathrm{Po}(\mu(y)(1-\delta)) = k_n\right) e^{(\beta-\alpha)y} \, \mathrm{d}y \\ &\le k_n^s (1+\delta)^{\beta-\alpha} \int_0^{a_n^- + 2\log(1+\delta)} \rho(z, k_n) e^{(\beta-\alpha)z} \, \mathrm{d}z, \end{aligned}$$

and (60) follows by another application of Lemma 6.1. □

6.2 Expected number of points in balls in the hyperbolic random graph

Tobias: the term "average degree" does not fit very well with what goes on in this section. We are considering the expected number of points in a ball, but not yet computing the average degree of the graph. **Pim:** I agree and have changed the header. Recall that under the coupling between the hyperbolic random graph and the finite box model, for two points p, p' with $y + y' < R_n$, $p' \in \mathcal{B}_{\mathbb{H},n}(p)$ exactly when $|x - x'|_{\pi e^{R_n/2}} \le \Phi(r, r')$. In this setting, the coupling lemma (Lemma 2.2) gives that

$$e^{\frac{1}{2}(y+y')} - K e^{\frac{3}{2}(y+y') - R_n} \le \Phi(r, r') \le e^{\frac{1}{2}(y+y')} + K e^{\frac{3}{2}(y+y') - R_n},$$

for some constant K. This result enables us to determine the measure of a ball around a given point $p = (0, y)$ which will be fairly useful in our subsequent analysis. Recall that the hyperbolic ball $\mathcal{B}_{\mathbb{H},n}(p)$ is a subset of $\mathcal{R}_n$ and not of the hyperbolic disc $\mathcal{D}_{R_n}$, i.e. the balls $\mathcal{B}_{\mathbb{H},n}(p)$ "live" in the finite box and not the hyperbolic disc.

Lemma 6.3. *Let $\varepsilon \in (0,1)$. Then for all $0 \le y \le (1-\varepsilon)R_n$*

$$1 - \phi_{\mathbb{H},n}^{(1)}(y) - \phi_{\mathbb{H},n}^{(2)}(y) \le \frac{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}(0,y)\right)}{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}(0,y)\right)} \le 1 - \phi_{\mathbb{H},n}^{(1)}(y) + \phi_{\mathbb{H},n}^{(2)}(y),$$

where

$$\phi_{\mathbb{H},n}^{(1)}(y) = \frac{2\alpha - 1 - 4\pi}{4\pi} e^{-(\alpha - \frac{1}{2})(R_n - y)} + \left(\alpha - \frac{1}{2}\right) \pi e^{-(\alpha - \frac{1}{2})R_n - y/2},$$

and

$$\phi_{\mathbb{H},n}^{(2)}(y) = + \begin{cases} \frac{(2\alpha-1)K}{3-2\alpha}\left(e^{-(\alpha-\frac{1}{2})(R_n-y)} - e^{-(R_n-y)}\right) & \text{if } 1/2 < \alpha < 3/2, \\ \frac{(2\alpha-1)K}{2}(R_n - y)e^{-(R_n-y)} & \text{if } \alpha = 3/2, \\ \frac{(2\alpha-1)K}{2\alpha-3}\left(e^{-(R_n-y)} - e^{-(\alpha-\frac{1}{2})(R_n-y)}\right) & \text{if } \alpha > 3/2, \end{cases}$$

with K being the constant coming from the approximation of Φ in Lemma 2.2.

Proof. We perform the computation of $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}(0,y)\right)$ by splitting the integration with respect to the height y' into the cases $y' > R_n - y$ and $y' \le R_n - y$, where for the latter we utilize Lemma 2.2. Recall that $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}(0,y)\right) = \xi_{\alpha,\nu} e^{y/2}$ where $\xi_{\alpha,\nu} = \frac{4\alpha\nu}{\pi(2\alpha-1)}$.

By (17), we have $\mathcal{B}_{\mathbb{H},n}((0,y)) \cap \mathcal{R}_n([R_n - y, R_n]) = \mathcal{R}_n([R_n - y, R_n])$. Thus,

$$\begin{aligned} &\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}((0,y)) \cap \mathcal{R}_n([R_n - y, R_n])\right) \\ &\quad = \int_{R_n - y}^{R_n} \int_{I_n} f_{\alpha,\nu}(x', y') \, \mathrm{d}x' \, \mathrm{d}y' = \nu\alpha e^{R_n/2}\left(e^{-\alpha(R_n - y)} - e^{-\alpha R_n}\right) \end{aligned}$$

$$= \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)\frac{2\alpha-1}{4\pi}\left(e^{-(\alpha-\frac{1}{2})(R_n-y)} - e^{-(\alpha-\frac{1}{2})R_n-y/2}\right) \tag{55}$$

Next we will establish upper and lower bounds on $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left((0,y)\right)\cap\mathcal{R}_n[(0,R_n-y)]\right)$. Using Lemma 2.2 we have

$$\begin{aligned}\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left((0,y)\right)\cap\mathcal{R}_n[(0,R_n-y)]\right) &\leq \frac{2\nu\alpha}{\pi}\int_0^{R_n-y}\left(e^{\frac{y+y'}{2}}+Ke^{\frac{3}{2}(y+y')-R_n}\right)e^{-\alpha y'}\,\mathrm{d}y'\\ &= \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)\left(1-e^{-(\alpha-\frac{1}{2})(R_n-y)}\right)\\ &\quad+\frac{2\nu\alpha}{\pi}Ke^{\frac{3y}{2}-R_n}\int_0^{R_n-y}e^{(\frac{3}{2}-\alpha)y'}\,\mathrm{d}y'\end{aligned}$$

The last integral depends on the value of α,

$$\int_0^{R_n-y}e^{(\frac{3}{2}-\alpha)y'}\,\mathrm{d}y' = \begin{cases}\frac{2}{3-2\alpha}\left(e^{(\frac{3}{2}-\alpha)(R_n-y)}-1\right) & \text{if } 1/2<\alpha<3/2,\\ R_n-y & \text{if } \alpha=3/2,\\ \frac{2}{2\alpha-3}\left(1-e^{-(\alpha-\frac{3}{2})(R_n-y)}\right) & \text{if } \alpha>3/2.\end{cases}$$

Therefore we get

$$\begin{aligned}&\frac{2\nu\alpha}{\pi}Ke^{\frac{3y}{2}-R_n}\int_0^{R_n-y}e^{(\frac{3}{2}-\alpha)y'}\,\mathrm{d}y'\\ &\quad= \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)\begin{cases}\frac{(2\alpha-1)K}{3-2\alpha}\left(e^{-(\alpha-\frac{1}{2})(R_n-y)}-e^{-(R_n-y)}\right) & \text{if } 1/2<\alpha<3/2,\\ \frac{(2\alpha-1)K}{2}(R_n-y)e^{-(R_n-y)} & \text{if } \alpha=3/2,\\ \frac{(2\alpha-1)K}{2\alpha-3}\left(e^{-(R_n-y)}-e^{-(\alpha-\frac{1}{2})(R_n-y)}\right) & \text{if } \alpha>3/2.\end{cases}\\ &\quad= \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)\phi^{(2)}_{\mathbb{H},n}(y)\end{aligned}$$

and hence

$$\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left((0,y)\right)\cap\mathcal{R}_n[(0,R_n-y)]\right) = \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(0,y\right)\right)\left(1-e^{-(\alpha-\frac{1}{2})(R_n-y)}+\phi^{(2)}_{\mathbb{H},n}(y)\right).$$

Combining this with (55) yields the required upper bound.

The lower bound follows by observing that the only difference with the above computations is the change of sign in front of

$$\frac{2\nu\alpha}{\pi}Ke^{\frac{3y}{2}-R_n}\int_0^{R_n-y}e^{(\frac{3}{2}-\alpha)y'}\,\mathrm{d}y'.$$

□

6.3 Expected number of points in balls in the finite box model

For the finite box model $G_{\mathcal{P},n}(\alpha,\nu)$ we obtain a similar result for expected size of the balls.

Lemma 6.4. *For all $p\in\mathcal{R}_n$, such that $y>2\log(\pi/2)$,*

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}\left(p\right)) = \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}\left(p\right))\left(1-\phi_n(y)\right)$$

Tobias: *where did the curly B's go?* **Pim:** *Typo. Fixed where $\phi_n(y)\geq 0$ is given by*

$$\phi_n(y) = \left(\frac{\pi}{2}\right)^{-(2\alpha-1)}e^{-(\alpha-\frac{1}{2})(R_n-y)} - \frac{(2\alpha-1)\pi}{4\alpha}\left(\left(\frac{\pi}{2}\right)^{-2\alpha}e^{-(\alpha-\frac{1}{2})(R_n-y)}-e^{-(\alpha-\frac{1}{2})R_n-\frac{y}{2}}\right).$$

On the other hand, if $y\leq 2\log(\pi/2)$ then

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}\left(p\right)) = \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}\left(p\right))\left(1-e^{-(\alpha-\frac{1}{2})R_n}\right).$$

Proof. First note that since we have identified the boundaries of $[-\frac{\pi}{2}e^{\frac{R_n}{2}}, \frac{\pi}{2}e^{\frac{R_n}{2}}]$ we can assume, without loss of generality, that $p = (0, y)$. We then have that the boundaries of $\mathcal{B}_{\mathcal{P},n}(p)$ are given by the equations $x' = \pm e^{\frac{y+y'}{2}}$, which intersect the left and right boundaries of $[-\frac{\pi}{2}e^{\frac{R_n}{2}}, \frac{\pi}{2}e^{\frac{R_n}{2}}]$ at height

$$h(y) = R_n + 2\log\left(\frac{\pi}{2}\right) - y.$$

Therefore, if $y \le 2\log(\pi/2)$ this intersection occurs above the height R_n of the box $\mathcal{R}_n$ while in the other case the full region of the box above $h(y)$ is connected to p.

We will first consider the case where $y > 2\log(\pi/2)$. Recall that $\mu_{\alpha,\nu}(B_{\mathcal{P}}(p)) = \xi_{\alpha,\nu}e^{\frac{y}{2}}$ where $\xi_{\alpha,\nu} = \frac{4\alpha\nu}{(2\alpha-1)\pi}$. Then, after some simple algebra, we have that

$$\begin{aligned}
\mu_{\alpha,\nu}(B_{\mathcal{P},n}(p)) &= \int_0^{h(y)} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \mathbb{1}_{\left\{|x'|\le e^{\frac{y+y'}{2}}\right\}} f_{\alpha,\nu}(x', y')\, dx'\, dy' \\
&\quad + \int_{h(y)}^{R_n} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} f_{\alpha,\nu}(x', y')\, dx'\, dy' \\
&= \frac{2\alpha\nu}{\pi} e^{\frac{y}{2}} \int_0^{h(y)} e^{-(\alpha-\frac{1}{2})y'}\, dy' + \alpha\nu e^{\frac{R_n}{2}} \int_{h(y)}^{R_n} e^{-\alpha y'}\, dy' \\
&= \xi_{\alpha,\nu} e^{\frac{y}{2}} \left(1 - \left(\frac{\pi}{2}\right)^{-(2\alpha-1)} e^{-(\alpha-\frac{1}{2})(R_n-y)}\right) \\
&\quad + \nu e^{\frac{R_n}{2}} \left(\left(\frac{\pi}{2}\right)^{-2\alpha} e^{-\alpha(R_n-y)} - e^{-\alpha R_n}\right) \\
&= \mu_{\alpha,\nu}(B_{\mathcal{P}}(p))\left(1 - \phi_n(y)\right).
\end{aligned}$$

Since, for all $\alpha > \frac{1}{2}$,

$$\left(\frac{\pi}{2}\right)^{-(2\alpha-1)} \ge \frac{(2\alpha-1)\pi}{4\alpha}\left(\frac{\pi}{2}\right)^{-2\alpha}$$

it follows that $\phi_n(y) \ge 0$.

When $y \le 2\log(\pi/2)$ we have

$$\begin{aligned}
\mu_{\alpha,\nu}(B_{\mathcal{P},n}(p)) &= \int_0^{R_n} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \mathbb{1}_{\left\{|x'|\le e^{\frac{y+y'}{2}}\right\}} f_{\alpha,\nu}(x', y')\, dx'\, dy' \\
&= \frac{2\alpha\nu}{\pi} e^{\frac{y}{2}} \int_0^{R_n} e^{-(\alpha-\frac{1}{2})y'}\, dy' \\
&= \mu_{\alpha,\nu}(B_{\mathcal{P}}(p))\left(1 - e^{-(\alpha-\frac{1}{2})R_n}\right).
\end{aligned}$$

□

6.4 Concentration of heights argument for hyperbolic and finite box model

With the results on the number of points in the balls in the two models $G_{\widetilde{\mathbb{H}},n}$ and $G_{\mathcal{P},n}$, we can now establish a result that allows us to extend the concentration of heights argument to the case where instead of $\rho(y,k)$ we consider the functions $\rho_{\widetilde{\mathbb{H}},n}(y,k)$ and $\rho_n(y,k)$. The lemma is stated in a more general form, to make it applicable to slightly other cases later in the paper. To understand the general conditions presented in the statement of the lemma we write $\mu_n(y) = \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(y))$ and $\mu(y) = \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(y))$ and recall the result from Lemma 6.4

$$\mu_n(y) = \mu(y)(1 - \phi_n(y)).$$

For the error function $\phi_n(y)$ we note that for any $0 < \varepsilon < 1$, as $n \to \infty$,

$$\sup_{0 \le y \le (1-\varepsilon)R_n} \phi_n(y) = o(1),$$

while

$$\frac{\partial \phi_n(y)}{\partial y} = \Theta(\phi_n(y)).$$

These are exactly the crucial properties of the error function that allow for a concentration argument. Moreover, if the integrand also contains a function $g_n(y)$ that is reasonably nice, we can replace the $\rho_n(y, k_n)$ in the integrand by $\rho(y, k_n)$. In particular, the triangle counting function $\Delta_{\mathcal{P}}(y)$ will satisfy these conditions, see Section 7. We will slightly abuse notation and write

$$\mathcal{K}_{C,n}(\kappa_n) := (-I_n, I_n] \times ((0, R_n] \cap \mathcal{K}_C(\kappa_n)), \tag{56}$$

with $\mathcal{K}_C$ defined in (47).

Lemma 6.5. *Let $\alpha > \frac{1}{2}, \nu > 0$, k_n be any positive sequence such that $k_n = o(n^{\frac{1}{2\alpha+1}})$, $\ell_n = (1+\epsilon_n)k_n$, with $\epsilon_n \to 0$ and let $\mathcal{K}_{C,n}(\ell_n)$ be defined as in (56). In addition, define $\hat{\rho}_n(y,k) = \mathbb{P}(\mathrm{Po}(\hat{\mu}_n(y)) = k)$, where $\hat{\mu}_n(y)$ is any differentiable function satisfying,*

$$\hat{\mu}_n(y) = (1 + o(1))\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(0, y)),$$

as $n \to \infty$ uniformly on $(0, (1-\varepsilon)R_n]$, for some $0 < \varepsilon < 1$. Then the following holds for some $C > 0$ large enough:

1. *for any sequence of continuous functions $h_n(y) : \mathbb{R}_+ \to \mathbb{R}$ such that, uniformly on $(0, R_n]$, $h_n(y) = O\left(k_n^s e^{\beta y} \hat{\rho}_n(y, k_n)\right)$* **Tobias:** *Why not adjust h so that we incorporate $\hat{\rho}$ in line below. Then both items look more similar.* **Pim:** *The reason for the bound on $h_n(y)$ in the first statement is that we want to use this for cases where $h_n(y) \le k_n^s \hat{\rho}_n(y, k_n)$ (see for instance the proof of Proposition 5.5. for some $s < 2\alpha(2\alpha+1)$ and $\beta < \alpha$, we have,*

$$\int_{\mathcal{R}_n} h_n(y) f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y = (1 + o(1)) \int_{\mathcal{K}_{C,n}(\ell_n)} h_n(y) f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

2. *for any continuous and uniformly bounded function $h : \mathbb{R}_+ \to \mathbb{R}$,*

$$\int_{\mathcal{R}_n} h(y)\hat{\rho}_n(y, k_n) f_{\alpha,\nu}(x,y)\, \mathrm{d}x\, \mathrm{d}y = (1 + o(1))n \int_0^\infty \rho(y, k_n) h(y) \alpha e^{-\alpha y}\, \mathrm{d}y.$$

Proof. Similarly to the proof of Lemma 6.1 we define

$$\lambda_n^\pm = (\ell_n \pm C\kappa_n) \wedge \kappa_n, \quad \text{and} \quad a_n^\pm = 2\log\left(\frac{\lambda_n^\pm}{\xi_{\alpha,\nu}}\right).$$

In addition we fix $\varepsilon > 0$ such that $\varepsilon < \min\{1 - \frac{s}{2\alpha(2\alpha+1)}, 1\}$. Then since $h_n(y) = \hat{\rho}_n(y, k_n) O(k_n^s)$ we have

$$\begin{aligned}
k_n^s \int_{(1-\varepsilon)R_n}^{R_n} \hat{\rho}_n(y, k_n) e^{-\alpha y}\, \mathrm{d}y &= O(1)\, \hat{\rho}_n((1-\varepsilon)R_n, k_n) k_n^s e^{-\alpha(1-\varepsilon)R_n} \\
&= O\left(\hat{\rho}_n((1-\varepsilon)R_n, k_n) k_n^s n^{-2\alpha(1-\varepsilon)}\right) \\
&= o\left(\hat{\rho}_n((1-\varepsilon)R_n, k_n) n^{\frac{s}{2\alpha+1} - 2\alpha(1-\varepsilon)}\right) = o(1),
\end{aligned}$$

where the last step follows by our choice of ε. Hence it is enough to prove both statement on $\mathcal{R}_n((1-\varepsilon)R_n, R_n)$ instead of $\mathcal{R}_n$. **Pim:** @All: This is where we use that $s < 2\alpha(2\alpha+1)$. I think this can be removed by using a more careful bound but I am not sure this is needed.

For simplicity we write $\mu(y) := \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(y))$. For both statements we shall make a change of variables $y \to z$ such that

$$\hat{\mu}_n(y) = \mu(z), \tag{57}$$

and therefore

$$\mu^{-1}(y) = 2\log\left(\frac{y}{\xi_{\alpha,\nu}}\right),$$

Observe that since $\mu^{-1}(y)$ is strictly monotonic in y, we have, by our assumptions on $\hat{\mu}_n$, that

$$z(y) \leq 2\log\left(\frac{\mu(y)}{\xi_{\alpha,\nu}}\right) + 2\log\left(1+\phi_n^{(2)}(y)\right) = y + 2\log\left(1+\phi_n^{(2)}(y)\right), \tag{58}$$

and similarly

$$z(y) \geq y + 2\log\left(1-\phi_n^{(1)}(y)\right) \tag{59}$$

Tobias: Please elaborate! Why would $\hat{\mu}$ be diff'ble anyway? **Pim:** I have added the diff'ble assumption to the lemma statement Then by assumption $(\phi_n^{(i)})'(z) = \Theta\left(\phi_n^{(i)}(z)\right)$ and hence

$$\hat{\mu}_n'(z) = \mu'(z)\left(1+o(1)\right).$$

Proof of statement 1. Recall that we have already showed that

$$\lim_{n\to\infty} k_n^s \int_{(1-\varepsilon)R_n}^{R_n} \hat{\rho}_n(y,k_n)e^{(\beta-\alpha)y}\, \mathrm{d}y = 0.$$

Therefore, by our assumption on $h_n(y)$, it is enough to show that for sufficiently large $C > 0$,

$$\lim_{n\to\infty} k_n^s \int_0^{a_n^-} \hat{\rho}_n(y,k_n)e^{(\beta-\alpha)y}\, \mathrm{d}y = 0, \tag{60}$$

and

$$\lim_{n\to\infty} k_n^s \int_{a_n^+}^{(1-\varepsilon)R_n} \hat{\rho}_n(y,k_n)e^{(\beta-\alpha)y}\, \mathrm{d}y = 0. \tag{61}$$

For simplicity we write $\mu(y) := \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(y))$. Now fix some $0 < \delta < 1$ and let n be large enough such that

1. $\sup_{0<y\leq(1-\varepsilon)R_n} \phi_n^{(1)}(y) < \delta$,
2. $\sup_{0<y\leq(1-\varepsilon)R_n} \phi_n^{(2)}(y) < \delta$,
3. $(\ell_n + C\kappa_n)(1-\delta) > k_n$ and
4. $(\ell_n - C\kappa_n)(1+\delta) < k_n$

Next, recall that the function $\lambda \mapsto \mathbb{P}(\mathrm{Po}(\lambda) = k)$ is monotonic increasing on $[0,k]$ and monotonic decreasing on $[k,\infty)$. Then, since for n large enough we have that

$$\hat{\mu}_n(y) \geq \mu(y)(1-\phi_n^{(1)}(y)) \geq \mu(y)(1-\delta) \geq \mu(a_n^+)(1-\delta) = (\ell_n + C\kappa_n)(1-\delta) > k_n,$$

it follows that

$$\hat{\rho}_n(y,k_n) = \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k_n\right) \leq \mathbb{P}\left(\mathrm{Po}(\mu(y)(1-\delta)) = k_n\right),$$

for all $a_n^+ \leq y \leq (1-\varepsilon)R_n$. By making the change of variables $z = \mu^{-1}(\mu(y)(1-\delta)) = y + 2\log(1-\delta)$ we then get

$$k_n^s \int_{a_n^+}^{(1-\varepsilon)R_n} \hat{\rho}_n(y,k_n)e^{(\beta-\alpha)y}\, \mathrm{d}y \leq k_n^s \int_{a_n^+}^{(1-\varepsilon)R_n} \mathbb{P}\left(\mathrm{Po}(\mu(y)(1-\delta)) = k_n\right) e^{(\beta-\alpha)y}\, \mathrm{d}y$$

$$= k_n^s(1-\delta)^{\beta-\alpha}\int_{a_n^+ + 2\log(1-\delta)}^{(1-\varepsilon)R_n + 2\log(1-\delta)} \mathbb{P}\left(\mathrm{Po}(\mu(z)) = k_n\right) e^{(\beta-\alpha)z}\, \mathrm{d}z$$
$$\leq k_n^s(1-\delta)^{\beta-\alpha}\int_{a_n^+}^{\infty} \rho(z,k_n) e^{(\beta-\alpha)z}\, \mathrm{d}z.$$

Since Lemma 6.1 implies that for large enough C,

$$\lim_{n\to\infty} k_n^s \int_{a_n^+}^{\infty} \rho(z,k_n)e^{(\beta-\alpha)z}\,\mathrm{d}z = 0,$$

we have proven (61).

The proof of (60) follows the same line of reasoning. This time we use that for $0 \leq y \leq a_n^-$,

$$\hat{\mu}_n(y) \leq \mu(y)(1+\phi_n^{(2)}(y)) \geq \mu(y)(1+\delta) \geq \mu(a_n^-)(1+\delta) = (\ell_n - C\kappa_n)(1+\delta) < k_n,$$

so that

$$\hat{\rho}_n(y,k_n) = \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k_n\right) \leq \mathbb{P}\left(\mathrm{Po}(\mu(y)(1+\delta)) = k_n\right).$$

Making a similar change of variables $z = y + 2\log(1+\delta)$ we then get

$$k_n^s \int_0^{a_n^-} \hat{\rho}_n(y,k_n)e^{(\beta-\alpha)y}\,\mathrm{d}y = k_n^s \int_0^{a_n^-} \mathbb{P}\left(\mathrm{Po}(\mu(y)(1-\delta)) = k_n\right) e^{(\beta-\alpha)y}\,\mathrm{d}y$$
$$\leq k_n^s(1+\delta)^{\beta-\alpha}\int_0^{a_n^- + 2\log(1+\delta)} \rho(z,k_n)e^{(\beta-\alpha)z}\,\mathrm{d}z,$$

and (60) follows by another application of Lemma 6.1.

Proof of statement 2. The proof of the second statement follows the same line of reasoning as above. First note that by Lemma 6.1 and the first statement, it is enough to show that

$$\int_{a_n^-}^{a_n^+} \rho_n(y,k_n)h(y)e^{-\alpha y}\,\mathrm{d}y = (1+o(1))\int_{a_n^-}^{a_n^+} \rho(z,k_n)h(z)e^{-\alpha z}\,\mathrm{d}z.$$

For this we observe that (58) and (59) imply that there exists a $b_n \to 0$ such that $\mu^{-1}(\hat{\mu}_n(y)) = y + b_n$, uniformly for $0 \leq y \leq (1-\varepsilon)R_n$. Hence, if we define $g_n(z) = h(z+b_n)$, then $g_n(z)$ is uniformly bounded and converges pointwise to $h(z)$. **Tobias:** I don't quite follow. Do we assume h is continuous? **Pim:** Yes, we probably should. I have added this to the lemma statement.

Using the change of variable $z = \mu^{-1}(\hat{\mu}_n(y))$ and define $\hat{a}_n^\pm = \hat{\mu}_n^{-1}(\mu(a_n^\pm))$,

$$\int_{a_n^-}^{a_n^+} \rho(z,k_n)h(z)e^{-\alpha z}\,\mathrm{d}z$$
$$= \int_{a_n^-}^{a_n^+} \mathbb{P}\left(\mathrm{Po}(\mu(z)) = k_n\right) h(z)e^{-\alpha z}\,\mathrm{d}z$$
$$= \int_{\hat{a}_n^-}^{\hat{a}_n^+} \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k_n\right) h(\mu^{-1}(\hat{\mu}_n(y)))e^{-\alpha\mu^{-1}(\hat{\mu}_n(y))}\frac{\hat{\mu}_n'(y)}{\mu'(\mu^{-1}(\hat{\mu}_n(y)))}\,\mathrm{d}y,$$

where the fraction is the last line follows from the chain rule and the fact that $(\mu^{-1})'(t) = (\mu'(\mu^{-1}(t)))^{-1}$.

Now since $\hat{\mu}_n(y) = (1+o(1))\mu(y)1$, uniformly on $[0,(1-\varepsilon)R_n]$ we have that $\hat{\mu}_n'(y) = (1+o(1))\mu'(y)$ as $n\to\infty$, uniformly on $(0,(1-\varepsilon)R_n)$. In particular this holds uniformly on $[a_n^-, a_n^+]$. Next we note that $\mu'(y) = \mu(y)/2$ from which it follows that

$$\frac{\hat{\mu}_n'(y)}{\mu'(\mu^{-1}(\hat{\mu}_n(y)))} = \frac{2\hat{\mu}_n'(y)}{\hat{\mu}_n(y)} = \frac{(1+o(1))2\mu'(y)}{(1+o(1))\mu(y)} = (1+o(1))$$

$$\mu^{-1}(\hat{\mu}_n(y)) = 2\log(\hat{\mu}(y)/\xi_{\alpha,\nu}) = 2\log(\mu(y)/\xi_{\alpha,\nu}) + 2\log(1+o(1)) = y + o(1).$$

$$\begin{aligned}
&\int_{a_n^-}^{a_n^+} \rho(z,k_n)h(z)e^{-\alpha z}\,\mathrm{d}z \\
&\qquad = (1+o(1))e^{-\alpha b_n}\int_{\hat{a}_n^-}^{\hat{a}_n^+} \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k_n\right) h(y+b_n)e^{-\alpha y}\,\mathrm{d}y \\
&\qquad = (1+o(1))\int_{\hat{a}_n^-}^{\hat{a}_n^+} \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k_n\right) h(y+b_n)e^{-\alpha y}\,\mathrm{d}y
\end{aligned}$$

Therefore, with the change of variables (57) we get

$$\begin{aligned}
\int_{a_n^-}^{a_n^+} \rho_n(y,k_n)h(y)e^{-\alpha y}\,\mathrm{d}y &= \int_{a_n^-}^{a_n^+} \mathbb{P}\left(\mathrm{Po}(\hat{\mu}_n(y)) = k_n\right) h(y)e^{-\alpha y}\,\mathrm{d}y \\
&= (1+o(1))\int_{a_n^-}^{a_n^+} \mathbb{P}\left(\mathrm{Po}(\mu(z))\right) g_n(z)e^{-\alpha z}\frac{\hat{\mu}_n'(z)}{\mu'(z)}\,\mathrm{d}z \\
&= (1+o(1))\int_{a_n^-}^{a_n^+} \mathbb{P}\left(\mathrm{Po}(\mu(z))\right) g_n(z)e^{-\alpha z}\,\mathrm{d}z \\
&= (1+o(1))\int_{a_n^-}^{a_n^+} \rho(z,k_n)h(z)e^{-\alpha z}\,\mathrm{d}z.
\end{aligned}$$

Tobias: again, please elaborate 2nd eq. (in the paper) where the last line follow by dominated convergence. □

Observe that Lemma 6.4 implies that $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathbb{H},n}\left((0,y)\right)\right)$ satisfies the requirements in Lemma 6.5 while Lemma 6.3 states that this holds for $\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}\left(0,y\right)\right)$. We therefore have the following important Corollary.

Corollary 6.6. *The statements of Lemma 6.5 holds for the two distribution functions* $\rho_{\widetilde{\mathbb{H}},n}(y,k)$ *and* $\rho_n(y,k)$.

In particular we conclude that, similarly to the infinite limit model, concentration of heights arguments as described in Remark 6.1 can be applied in the case of the hyperbolic random graphs and the finite box model.

7 From $G_{\mathcal{P},n}(\alpha,\nu)$ to $G_{\mathcal{P}}(\alpha,\nu)$ (Proving Proposition 5.5)

In this section we shall relate the clustering in the finite box model $G_{\mathcal{P},n}$ to that of the infinite model. The main goal is to prove Proposition 5.5 which states that

$$\mathbb{E}\left[\left|c^*_{\mathcal{P},n}(k_n) - \mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right|\right] = o\left(s_\alpha(k_n)\right).$$

Recall that $G_{\mathcal{P},n}$ is obtained by restricting the Poisson Point Process $\mathcal{P}_{\alpha,\nu}$ to the box $\mathcal{R}_n = (-I_n, I_n] \times (0, R_n]$, with $I_n = \frac{\pi}{2}e^{R_n/2}$ (we write $\mathcal{P}_n$ for this process), and connecting two points $p_1, p_2 \in \mathcal{R}_n$ if $|x_1 - x_2|_{\pi e^{R_n/2}} \le e^{(y_1+y_2)/2}$. We recall that by definition of the norm $|.|_{\pi e^{R_n/2}}$ the left and right boundaries of $\mathcal{R}_n$ are identified. See Section 2.3 for more details. **Tobias:** This last bit of the sentence is already contained in the fact we us $|.|_{\pi e^{R_n/2}}$. To me it does not add anything, except possibly confusion. **Pim:** I have updated this and just added a reminder of this.

Remark 7.1 (Diverging k_n). *Throughout this section $\{k_n\}_{n\geq 1}$ will denote an sequence of integers satisfying $k_n \to \infty$ and $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$, as $n \to \infty$.*

In this section we shall compute the asymptotic difference between triangle counts in the finite and infinite model. First we recall the definition of $\mathcal{K}_C(k_n)$

$$\mathcal{K}_C(k_n) = \left\{ p \in \mathcal{R} : \frac{k_n - C\sqrt{k_n \log(k_n)}}{\xi_{\alpha,\nu}} \leq e^{\frac{y}{2}} \leq \frac{k_n + C\sqrt{k_n \log(k_n)}}{\xi_{\alpha,\nu}} \right\},$$

Tobias: Typo : $p \in \mathbb{R}$. Do you mean $\mathcal{R}_n$? Or maybe $\mathbb{R} \times \mathbb{R}^+$? Still not sure that max and min in defn add anything. **Pim:** It was a typo and is corrected. I have removed the max and min. In addition we note that the function $\phi_n(y)$ in Lemma 6.4 satisfies

$$\phi_n(y) = \Theta\left(e^{-(\alpha-\frac{1}{2})(R_n - y)}\right),$$

as $R_n - y \to \infty$. Since $e^{y/2} = \Theta(k_n) = o\left(e^{R_n}\right)$ as $n \to \infty$, uniformly on $\mathcal{K}_C$, it follows that $\phi_n(y) = \Theta\left(k_n^{2\alpha-1} n^{-(2\alpha-1)}\right)$, uniformly on $\mathcal{K}_C$. This yields the following useful corollary to Lemma 6.4.

Corollary 7.1. *Let $\alpha > \frac{1}{2}$, $C > 0$. Then, for all $p \in \mathcal{K}_C(k_n)$,*

$$\mu_{\alpha,\nu,n}(B_{\mathcal{P},n}(p)) = \mu_{\alpha,\nu}(B_{\mathcal{P}}(p))\left(1 - \Theta\left(k_n^{2\alpha-1} n^{-(2\alpha-1)}\right)\right).$$

Tobias: *I don't follow. Spell it out.* **Pim:** *I added additional explanation above. Please see if this solves the confusion.*

7.1 Comparing triangles between $G_{\mathcal{P}}(\alpha,\nu)$ and $G_{\mathcal{P},n}(\alpha,\nu)$

We now turn to the task of calculating the expected number of triangles of a node at height y, for both the infinite model and the finite box model. Recall that we added a typical point $p_0 = (0, y)$, with exponentially distributed height, to the Poisson Point Process and that

$$\mathbb{E}\left[T_{\mathcal{P},n}(p_0)\right] = \frac{1}{2}\iint_{\mathcal{R}_n^2} T_{\mathcal{P},n}(p_0,p_1,p_2) f_{\alpha,\nu}(x_1,y_1) f_{\alpha,\nu}(x_1,y_1)\, dx_1\, dx_2\, dy_1\, dy_2,$$

where

$$T_{\mathcal{P},n}(p_0,p_1,p_2) = \mathbb{1}_{\{p_1 \in B_{\mathcal{P},n}(p_0)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P},n}(p_0)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P},n}(p_1)\}}.$$

Tobias: p is supposed to be p_0 I assume. **Pim:** Indeed. It has been corrected. The difference between the indicator $\mathbb{1}_{\{p_1 \in B_{\mathcal{P},n}(p)\}}$ in the finite box model and $\mathbb{1}_{\{p_1 \in B_{\mathcal{P}}(p)\}}$ is that in $G_{\mathcal{P},n}(\alpha,\nu)$ we identified the boundaries of the interval $[-\frac{\pi}{2}e^{R_n/2}, \frac{\pi}{2}e^{R_n/2}]$ and we stop at height $y = R_n$. It is clear that for any $0 \leq y \leq R_n$ we have that $\mathcal{B}_{\mathcal{P},n}(p_0) = \mathcal{B}_{\mathcal{P}}(p_0) \cap \mathcal{R}_n$. However, when we take another point $p' \in \mathcal{B}_{\mathcal{P},n}(p)$ then it could happen that there are points in the intersection $\mathcal{B}_{\mathcal{P},n}(p) \cap \mathcal{B}_{\mathcal{P},n}(p')$ that are not in $\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{B}_{\mathcal{P}}(p')$. Let us denote this region by $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p')$. Then, any $p_2 \in \mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p')$ creates a triangle with p and p' in $G_{\mathcal{P},n}(\alpha,\nu)$ that is not present in $G_{\mathcal{P}}(\alpha,\nu)$.

Define

$$\widetilde{T}_{\mathcal{P},n}(p_0,p_1,p_2) = \mathbb{1}_{\{p_1 \in B_{\mathcal{P},n}(p_0)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P},n}(p_0)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P}}(p_1) \cap \mathcal{R}_n\}} \tag{62}$$

Then

$$\sum_{p_1,p_2 \in \mathcal{P}_n}^{\neq} T_{\mathcal{P},n}(p_0,p_1,p_2) - \widetilde{T}_{\mathcal{P},n}(p_0,p_1,p_2) = \sum_{p_1,p_2 \in \mathcal{P}_n}^{\neq} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathcal{P},n}(p_0)\}} \mathbb{1}_{\{p_2 \in \mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p_0,p_1)\}},$$

Tobias: Note you cannot really force/assume $p_0 = (0, y)$ is in your PPP $\mathcal{P}_n$. (The summation over $\mathcal{P}_n \setminus (0, y)$ suggests this is what you do.) For taking the expectation of sort of thing, Palm

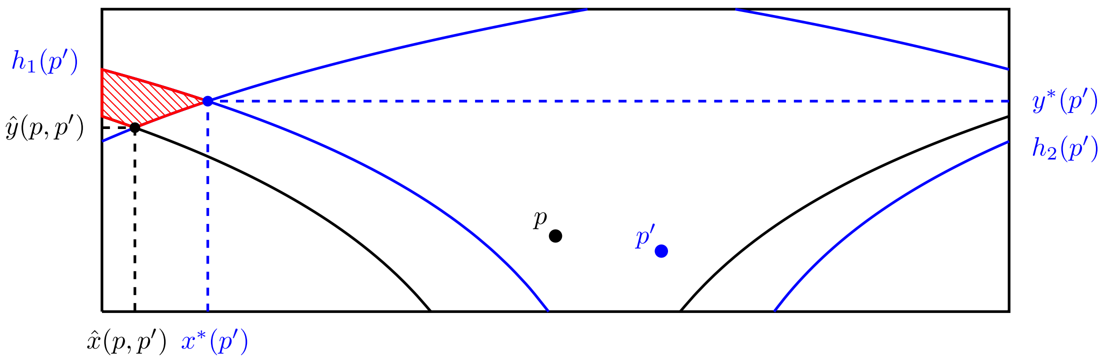

Figure 8: Example configuration of two points p and p' for which $\mathcal{B}_{\mathcal{P},n}(p) \cap \mathcal{B}_{\mathcal{P},n}(p')$ is not a subset of $\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{B}_{\mathcal{P}}(p')$. The red region indicates the area of points belonging to $\mathcal{B}_{\mathcal{P},n}(p) \cap \mathcal{B}_{\mathcal{P},n}(p')$ but not to $\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{B}_{\mathcal{P}}(p')$.

theory / Mecke are used typically. At this point though you can again still just say we add p_0 to the PPP. **Pim:** I changed the text at the beginning of this section to mention this. Since we now consider the point process $\mathcal{P}_n \cup p_0$ the above summation is over $\mathcal{P}_n$. where the sums are over all distinct pairs $(p_1, p_2) \in \mathcal{P}_n$.

Figure 8 shows an example of a configuration where $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p') \neq \emptyset$. We observe that $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p') \neq \emptyset$ because the right boundary of the ball $\mathcal{B}_{\mathcal{P},n}(p')$ exists the right boundary of the box $\mathcal{R}_n$ and then, since we identified the boundaries, continues from the left so that $\mathcal{B}_{\mathcal{P},n}(p')$ covers part of the ball $\mathcal{B}_{\mathcal{P},n}(p)$ which would not be covered in the infinite limit model.

To further analyze this, let us introduce some notation. For any $p = (x, y) \in \mathcal{R}_n$ we will define the left and right boundary functions as, respectively,

$$b_p^-(z) = \begin{cases} 2\log(x - z) - y & \text{if } -\frac{\pi}{2}e^{R_n/2} \leq z \leq x - e^{y/2} \\ 2\log\left(\pi e^{R_n/2} + x - z\right) - y & \text{if } x - e^{(y+R_n)/2} + \pi e^{R_n/2} \leq z \leq \frac{\pi}{2}e^{R_n/2} \\ 0 & \text{else} \end{cases} \tag{63}$$

$$b_p^+(z) = \begin{cases} 2\log(z - x) - y & \text{if } x + e^{y/2} \leq z \leq \frac{\pi}{2}e^{R_n/2} \\ 2\log\left(\pi e^{R_n/2} + z - x\right) - y & \text{if } -\frac{\pi}{2}e^{R_n/2} \leq z \leq x + e^{(y+R_n)/2} - \pi e^{R_n/2} \\ 0 & \text{else} \end{cases} \tag{64}$$

Note that these functions describe the boundaries of the ball $\mathcal{B}_{\mathcal{P},n}(p)$. In particular, $p' = x', y') \in \mathcal{B}_{\mathcal{P},n}(p)$ if and only if $y' \geq \min\left\{b_p^-(x'), b_p^+(x')\right\}$.

Since we have identified the left and right boundary of $\mathcal{R}_n$ we can assume, without loss of generality that $x = 0$. Due to symmetry it is then enough to restrict the analysis to the case where $x' > 0$. **Tobias:** does this not depend on p? **Pim:** I do not think so, because we can always take $p = (0, y)$ due to the invariance in the x-direction. I have updated the text to better reflect this. There are two important points in the box $\mathcal{R}_n$. These are the intersection between the left boundary of p' and the right boundary of p', as it continues from the left side of the box, and the left boundary of p. We denote by $(x^*(p'), y^*(p'))$ the intersection between the left and right boundary of p' and by $(\hat{x}(p,p'), \hat{y}(p,p'))$ the intersection between the left boundary of p and the right boundary of p', see Figure 8.

Let us derive the expressions for the coordinates of these two points, starting with $(x^*(p'), y^*(p'))$. The x-coordinate $x^*(p')$ is the solution to the equation $b_{p'}^+(z) = b_{p'}^-(z)$ for $-\frac{\pi}{2}e^{R_n/2} \leq z \leq x + e^{(y+R_n)/2} - \pi e^{R_n/2}$. This equation becomes

$$2\log\left(\pi e^{R_n/2} + z - x'\right) - y' = 2\log(x' - z) - y',$$

whose solution is $x^*(p') := x' - \frac{\pi}{2}e^{R_n/2}$. Plugging this into either the left or right hand side of the above equation yields the y-coordinate $y^*(p') = 2\log\left(\frac{\pi}{2}e^{R_n/2}\right) - y'$. In a similar way,

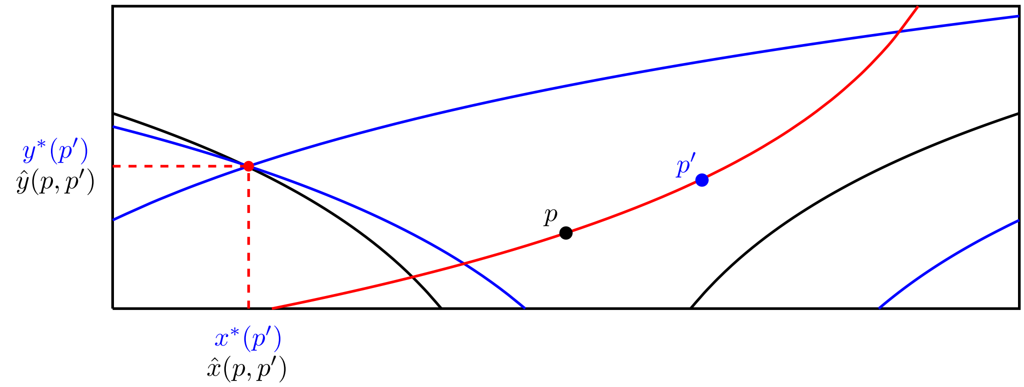

Figure 9: Example for a given p of the boundary function $x' \mapsto b_p^*(x')$, given by the red curve, which determines whether $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n} = \emptyset$. We see that when $y' = b_p^*(x')$ then $(\hat{x}(p,p'), \hat{y}(p,p')) = (x^*(p'), y^*(\prime))$.

the x-coordinate $\hat{x}(p,p')$ is the solution to the equation $b_{p'}^+(z) = b_p^-(z)$ for $-\frac{\pi}{2}e^{R_n/2} \leq z \leq x + e^{(y+R_n)/2} - \pi e^{R_n/2}$, i.e.

$$2\log\left(\pi e^{R_n/2} + z - x'\right) - y' = 2\log\left(x - z\right) - y.$$

This solution is $\frac{x' - \pi e^{R_n/2}}{1 + e^{(y'-y)/2}}$ and again $\hat{y}(p,p')$ is obtained by plugging the solution into either left or right hand side of the equation, yielding $\hat{y}(p,p') = 2\log\left(\frac{\pi e^{R_n/2} - x'}{e^{y/2} + e^{y'/2}}\right)$.

To summarize we have:

$$\begin{aligned} x^*(p') &= x' - \frac{\pi}{2}e^{R_n/2} \\ y^*(p') &= 2\log\left(\frac{\pi}{2}e^{R_n/2}\right) - y' \\ \hat{x}(p,p') &= \frac{x' - \pi e^{R_n/2}}{1 + e^{(y'-y)/2}} \\ \hat{y}(p,p') &= 2\log\left(\frac{\pi e^{R_n/2} - x'}{e^{y/2} + e^{y'/2}}\right) \end{aligned}$$

The crucial observation is that $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n} = \emptyset$ as long as the point $(x^*(p'), y^*(p'))$ is above the left boundary of p. This happens exactly when $y^*(p') > b_p^-(x^*(p'))$. Therefore the boundary of this event is given by the equation $y^*(p') = b_p^-(x^*(p'))$ which reads

$$2\log\left(\frac{\pi}{2}e^{R_n/2}\right) - y' = 2\log\left(\frac{\pi}{2}e^{R_n/2} - x'\right) - y.$$

Solving this equation gives us the function

$$b_p^*(z) = y - 2\log\left(1 - \frac{z}{\frac{\pi}{2}e^{R_n/2}}\right), \tag{65}$$

which is displayed by the red curve in Figure 9. It holds that $y^*(p') > b_p^-(x^*(p'))$ if and only if $y' < b_p^*(x')$ and hence we have that $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n} = \emptyset$ for all $p' \in \mathcal{R}_n$ for which $y' \geq b_p^*(x')$. We also not that when $y' = b_p^*(x')$ the two points $(x^*(p'), y^*(p'))$ and $(\hat{x}(p,p'), \hat{y}(p,p'))$ coincide.

This analysis allows us to compute the expected difference in the number of triangles for $\mathcal{P}$ and $\mathcal{P}_n$, for a node with height y.

Lemma 7.2. *Let $(k_n)_{n\geq 1}$ be any sequence such that $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then, for some $C > 0$ and $p \in \mathcal{K}_C(k_n)$, as $n \to \infty$,*

$$\int_{\mathcal{R}_n} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)\right) f_{\alpha,\nu}(x_1, y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 = O\left(yn^{-(2\alpha-1)} + n^{-(2\alpha-1)}e^y\right)$$

The proof of the lemma is not difficult but cumbersome, since it involves computing many different integrals. We postpone this proof till the end of this section and proceed with the main goal, proving Proposition 5.5. But first we state a small lemma about the scaling of $s_\alpha(k_n)$ that will be very useful.

Lemma 7.3. *Let $s_\alpha(k_n)$ be as defined in (5). Then for any $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$, as $n \to \infty$,*

$$n^{-(2\alpha-1)} = o\left(s_\alpha(k_n)\right).$$

Proof. First let $\frac{1}{2} < \alpha < \frac{3}{4}$. Then

$$n^{-(2\alpha-1)}s_\alpha(k_n)^{-1} = n^{-(2\alpha-1)}k_n^{4\alpha-2} = o\left(n^{-(2\alpha-1)+\frac{4\alpha-2}{2\alpha+1}}\right) = o\left(n^{-\frac{4\alpha^2-4\alpha+1}{2\alpha+1}}\right) = o\left(1\right),$$

since $4\alpha^2 - 4\alpha + 1 > 0$ for all $\alpha > \frac{1}{2}$. Similarly, for $\alpha \geq \frac{3}{4}$ we have that $4\alpha^2 > 2$ and hence,

$$n^{-(2\alpha-1)}s_\alpha(k_n) = o\left(n^{-(2\alpha-1)}k_n\right) = o\left(n^{-\frac{4\alpha^2-2}{2\alpha+1}}\right) = o\left(1\right).$$

□

Proof of Proposition 5.5. Recall that

$$\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right] = \frac{\int_{\mathcal{R}_n} \mathbb{E}\left[\mathbb{1}_{\{D_{\mathcal{P},n}(y)=k_n\}}T_{\mathcal{P},n}(y)\right] f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]},$$

and the definition of $\mathcal{K}_{C,n}(\kappa_n)$

$$\mathcal{K}_{C,n}(\kappa_n) = \left\{(x,y) \in \mathcal{R}_n \,:\, \frac{k_n - C\kappa_n}{\xi_{\alpha,\nu}} \leq e^{\frac{y}{2}} \leq \frac{k_n + C\kappa_n}{\xi_{\alpha,\nu}}\right\},$$

where for $k_n \to \infty$ we defined $\kappa_n := \sqrt{k_n \log(k_n)}$.

Since

$$\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] = \int_{\mathcal{R}_n} \rho_n(y, k_n) f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

the Corollary 6.6, using the second statement of Lemma 6.5, implies

$$\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] = (1 + o(1))\alpha n \int_0^\infty \rho(y, k_n)e^{-\alpha}\,\mathrm{d}y = (1 + o\left(1\right))n\mathbb{E}\left[N_{\mathcal{P}}(k_n)\right].$$

Moreover, since $\mathbb{E}\left[\mathbb{1}_{\{D_{\mathbb{H},n}(y)=k_n\}}T_{\mathcal{P},n}(y)\right] \leq \rho_n(y,k_n)k_n^2$ a concentration of heights argument (cf. first statement of Lemma 6.5) yields

$$\begin{aligned}
&\int_{\mathcal{R}_n} \mathbb{E}\left[\mathbb{1}_{\{D_{\mathbb{H},n}(y)=k_n\}}T_{\mathcal{P},n}(y)\right] f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = (1 + o\left(1\right)) \int_{\mathcal{K}_{C,n}(k_n)} \rho_n(y,k_n)\mathbb{E}\left[T_{\mathcal{P},n}(y)\right] f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = (1 + o\left(1\right))\alpha n \int_{a_n^-}^{a_n^+} \rho_n(y,k_n)\mathbb{E}\left[T_{\mathcal{P},n}(y)\right]e^{-\alpha}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

with $a_n^\pm = 2\log\left(\frac{(k_n \pm C\kappa_n)\wedge \xi_{\alpha,\nu}}{\xi_{\alpha,\nu}}\right)$. Next we note that for $a_n^- \le y \le a_n^+$,

$$\mathbb{E}\left[T_{\mathcal{P}}(y)\right] = \frac{\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P}}\left(y\right)\right)^2}{2}\Delta_{\mathcal{P}}(y) = (1+o\left(1\right))\binom{k_n}{2}\Delta_{\mathcal{P}}(y) \tag{66}$$

and hence it now suffices to show that

$$\binom{k_n}{2}^{-1}\int_{a_n^-}^{a_n^+} \rho_n(y,k_n)\mathbb{E}\left[T_{\mathcal{P},n}(y)\right]e^{-\alpha y}\,\mathrm{d}y = (1+o\left(1\right))\int_{a_n^-}^{a_n^+}\rho(y,k_n)\Delta_{\mathcal{P}}(y)e^{-\alpha y}\,\mathrm{d}y.$$

We do this in two stages. First we prove that

$$\int_{a_n^-}^{a_n^+}\rho_n(y,k_n)\Delta_{\mathcal{P}}(y)e^{-\alpha y}\,\mathrm{d}y = (1+o\left(1\right))\int_{a_n^-}^{a_n^+}\rho(y,k_n)\Delta_{\mathcal{P}}(y)e^{-\alpha y}\,\mathrm{d}y. \tag{67}$$

Then we show that

$$\binom{k_n}{2}^{-1}\int_{a_n^-}^{a_n^+}\rho_n(y,k_n)\left|\mathbb{E}\left[T_{\mathcal{P},n}(y) - T_{\mathcal{P}}(y)\right]\right|e^{-\alpha y}\,\mathrm{d}y = o\left(1\right)\int_{a_n^-}^{a_n^+}\rho(y,k_n)\Delta_{\mathcal{P}}(y)e^{-\alpha y}\,\mathrm{d}y. \tag{68}$$

Since $h(y) := \Delta_{\mathcal{P}}(y)$ is uniformly bounded (67) follows directly from the second statement in Lemma 6.5.

For the error term (68) we write

$$|T_{\mathcal{P},n}(p) - T_{\mathcal{P}}(p)| = \sum_{p_1,p_2\in\mathcal{R}_n}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathcal{P},n}(p)\}}\mathbb{1}_{\{p_2\in\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\}} + \sum_{p_1,p_2\in\mathcal{R}\setminus\mathcal{R}_n}T_{\mathcal{P}}(p,p_1,p_2)$$

so that by the Campbell-Mecke formula (10)

$$\begin{aligned}\left|\mathbb{E}\left[T_{\mathcal{P},n}(p) - T_{\mathcal{P}}(p)\right]\right| \le &\int_{\mathcal{R}_n}\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right)f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\\ &+\int_{\mathcal{R}\setminus\mathcal{R}_n}\int_{\mathcal{R}\setminus\mathcal{R}_n}T_{\mathcal{P}}(p,p_1,p_2)f_{\alpha,\nu}(x_1,y_1)f_{\alpha,\nu}(x_2,y_2)\,\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_1\,\mathrm{d}y_1.\end{aligned}$$

Tobias: in the 1st integral we could have restricted to the ball of p, and possibly saved a lot. Does that not help? **Pim:** the first integral is taken care of by another Lemma so it does not matter I think. The first integral was taken care of in Lemma 7.2. For the second integral we have

$$\begin{aligned}&\iint_{\mathcal{R}\setminus\mathcal{R}_n}T_{\mathcal{P}}(p,p_1,p_2)f_{\alpha,\nu}(x_1,y_1)f_{\alpha,\nu}(x_2,y_2)\,\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_1\,\mathrm{d}y_1\\ &\quad\le\left(\int_{\mathcal{R}\setminus\mathcal{R}_n}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathcal{P}}(p)\}}f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\right)^2 = O\left(\left(e^{y/2}\int_{R_n}^\infty e^{-(\alpha-\frac{1}{2})y_1}\,\mathrm{d}y_1\right)^2\right)\\ &\quad= O\left(e^y e^{-(2\alpha-1)R_n}\right) = O\left(e^y n^{-(4\alpha-2)}\right),\end{aligned}$$

from which we conclude that

$$\left|\mathbb{E}\left[T_{\mathcal{P},n}(p) - T_{\mathcal{P}}(p)\right]\right| = O\left(yn^{-(2\alpha-1)} + n^{-(4\alpha-2)}e^y\right) = O\left(n^{-(4\alpha-2)}e^y\right). \tag{69}$$

Therefore, since $e^y = O\left(1\right)k_n^2$ on $\mathcal{K}_C(\kappa_n)$,

$$\begin{aligned}&\binom{k_n}{2}^{-1}\int_{a_n^-}^{a_n^+}\rho(y,k_n)\left|\mathbb{E}\left[T_{\mathcal{P},n}(p) - T_{\mathcal{P}}(p)\right]\right|e^{-\alpha y}\,\mathrm{d}y\\ &\quad= O\left(1\right)n^{-(4\alpha-2)}\int_0^\infty\rho(y,k_n)e^{-\alpha y}\,\mathrm{d}y\end{aligned}$$

$$= O(1)\, n^{-(4\alpha-2)} k_n^{-(2\alpha+1)} = o\left(s_\alpha(k_n) k_n^{-(2\alpha+1)}\right),$$

where the last part follows from Lemma 7.3. **Tobias:** Also justify penultimate step! **Pim:** What do you mean? To finish the argument we observe that

$$\int_{a_n^-}^{a_n^+} \rho(y, k_n)\Delta_{\mathcal{P}}(y) e^{-\alpha}\, \mathrm{d}y = \Theta(1)\, \mathbb{E}\left[N_{\mathcal{P}}(k_n)\right] c_\infty(k_n) = \Theta\left(s_\alpha(k_n) k_n^{-(2\alpha+1)}\right).$$

□

From the proof of Proposition 5.5 we can extract the following corollary, which will be useful later on in Section 8.

Corollary 7.4. *Uniformly for all $y \in \mathcal{K}_C(\kappa_n)$, as $n \to \infty$*

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(y)\right] = (1 + o(1))\mathbb{E}\left[T_{\mathcal{P}}(y)\right],$$

where

$$\widetilde{T}_{\mathcal{P},n}(y) = \sum_{(p_1,p_2)\in\mathcal{P}\setminus p}^{\neq} \mathbb{1}_{\{p_1\in B_{\mathcal{P},n}(y)\}} \mathbb{1}_{\{p_2\in B_{\mathcal{P},n}(y)\}} \mathbb{1}_{\{p_2\in B_{\mathcal{P}}(p_1)\cap\mathcal{R}_n\}}.$$

In particular,

$$\binom{k_n}{2}^{-1} \int_{\mathcal{K}_C(k_n)} \rho_n(y, k_n)\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(y)\right] f_{\alpha,\nu}(x, y)\, \mathrm{d}x\, \mathrm{d}y = (1+o(1)) \int_{\mathcal{R}_n} \rho(y, k_n)\Delta_{\mathcal{P}}(y) f_{\alpha,\nu}(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

Proof. First we note that equation (69) implies that for $y \in \mathcal{K}_C(\kappa_n)$

$$\mathbb{E}\left[\left|\widetilde{T}_{\mathcal{P},n}(y) - T_{\mathcal{P}}(y)\right|\right] = O\left(n^{-(4\alpha-2)} k_n^2\right).$$

Next, equation 66 together with Proposition 3.1 yield

$$\mathbb{E}\left[T_{\mathcal{P}}(y)\right] = (1 + o(1))\binom{k_n}{2}\Delta_{\mathcal{P}}(y) = \Theta\left(k_n^2 s_\alpha(k_n)\right),$$

on $\mathcal{K}_C(\kappa_n)$, with $s_\alpha(k)$ defined by (5). By Lemma 7.3 we now have that for all $y \in \mathcal{K}_C(\kappa_n)$

$$n^{-(4\alpha-2)} k_n^2 = o\left(k_n^2 s_\alpha(k_n)\right) = o\left(\mathbb{E}\left[T_{\mathcal{P}}(y)\right]\right)$$

and hence

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(y)\right] = \mathbb{E}\left[T_{\mathcal{P}}(y)\right] + O\left(\mathbb{E}\left[\left|\widetilde{T}_{\mathcal{P},n}(y) - T_{\mathcal{P}}(y)\right|\right]\right) = (1 + o(1))\mathbb{E}\left[T_{\mathcal{P}}(y)\right].$$

Using this we derive the second statement as follows

$$\begin{aligned}
&\binom{k_n}{2}^{-1} \int_{\mathcal{K}_C(k_n)} \rho_n(y, k_n)\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(y)\right] f_{\alpha,\nu}(x, y)\, \mathrm{d}x\, \mathrm{d}y \\
&\quad = (1 + o(1))\binom{k_n}{2}^{-1} \int_{\mathcal{K}_C(k_n)} \rho_n(y, k_n)\mathbb{E}\left[T_{\mathcal{P}}(y)\right] f_{\alpha,\nu}(x, y)\, \mathrm{d}x\, \mathrm{d}y \\
&\quad = (1 + o(1)) \int_{\mathcal{K}_C(k_n)} \rho_n(y, k_n)\Delta_{\mathcal{P}}(y) f_{\alpha,\nu}(x, y)\, \mathrm{d}x\, \mathrm{d}y \\
&\quad = (1 + o(1)) \int_{\mathcal{R}_n} \rho_n(y, k_n)\Delta_{\mathcal{P}}(y) f_{\alpha,\nu}(x, y)\, \mathrm{d}x\, \mathrm{d}y
\end{aligned}$$

where the third line is due to equation 66 and the last line follows by a concentration of heights argument. □

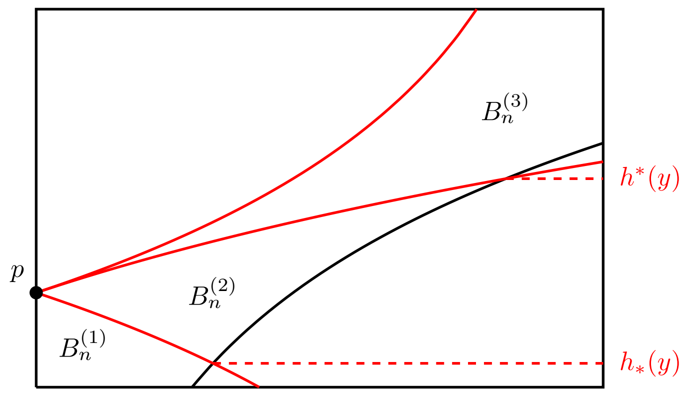

Figure 10: Three different areas $B_n^{(i)}$ used in the proof of Lemma 7.2.

7.2 Counting missing triangles

We now come back to computing the expected number of triangles attached to node at height y in $G_{\mathcal{P},n}(\alpha,\nu)$ that are not present in $G_{\mathcal{P}}(\alpha,\nu)$.

Proof of Lemma 7.2. Due to symmetry it is enough to show that

$$\int_0^{R_n}\int_0^{I_n} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) f_{\mu,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 = O\left(yn^{-(2\alpha-1)} + n^{-(2\alpha-1)}e^y\right) \tag{70}$$

The proof goes in two stages. First we compute $\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right)$ by splitting it over three disjoint regimes with respect to p_1, with $x_1 \geq 0$. Then we do the integration with respect to p_1.

Computing $\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right)$

Recall that $I_n = \frac{\pi}{2}e^{R_n/2}$ and define the sets

$$\begin{aligned}
A_n^{(1)} &= \{p_1 \in \mathcal{R}_n \,:\, 0 \leq y_1 \leq y - 2\log(I_n/(I_n - x_1))\},\\
A_n^{(2)} &= \left\{p_1 \in \mathcal{R}_n \,:\, y - 2\log(I_n/(I_n - x_1)) < y_1 \leq y + 2\log\left(1 + \frac{x_1}{I_n}\right)\right\},\\
A_n^{(3)} &= \left\{p_1 \in \mathcal{R}_n \,:\, y + 2\log\left(1 + \frac{x_1}{I_n}\right) < y_1 \leq y + 2\log\left(\frac{I_n}{I_n - x_1}\right)\right\},
\end{aligned}$$

and let $B_n^{(i)} = \mathcal{B}_{\mathcal{P},n}(p) \cap A_n^{(i)}$, for $i = 1,2,3$, see Figure 10. Here the heights of the two intersections are given by

$$h_*(y) = y + 2\log\left(\frac{I_n}{I_n + e^y}\right) \tag{71}$$

$$h^*(y) = y + 2\log\left(\frac{I_n}{I_n - e^y}\right). \tag{72}$$

With these definitions we have that the union $B_n := \bigcup_{i=1}^n B_n^{(i)}$ denotes the area under the red curve in Figure 9 and hence, for all $p_1 \in \mathcal{R}_n \setminus B_n$ with $x_1 \geq 0$ we have that $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1) = \emptyset$. So we only need to consider $p_1 \in B_n$. We shall establish the following result:

$$\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) = \begin{cases} O\left(I_n^{-2\alpha}e^{\alpha y_1}\right) & \text{if } p_1 \in B_n^{(1)}\\ O\left(I_n^{-2\alpha}e^{\alpha y}\right) & \text{if } p_1 \in B_n^{(2)} \cup B_n^{(3)}\end{cases} \tag{73}$$

Depending on which regime p_1 belongs to, the set $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)$ has a different shape. We displayed these shapes in Figure 11 as a visual aid to follow the computations below.

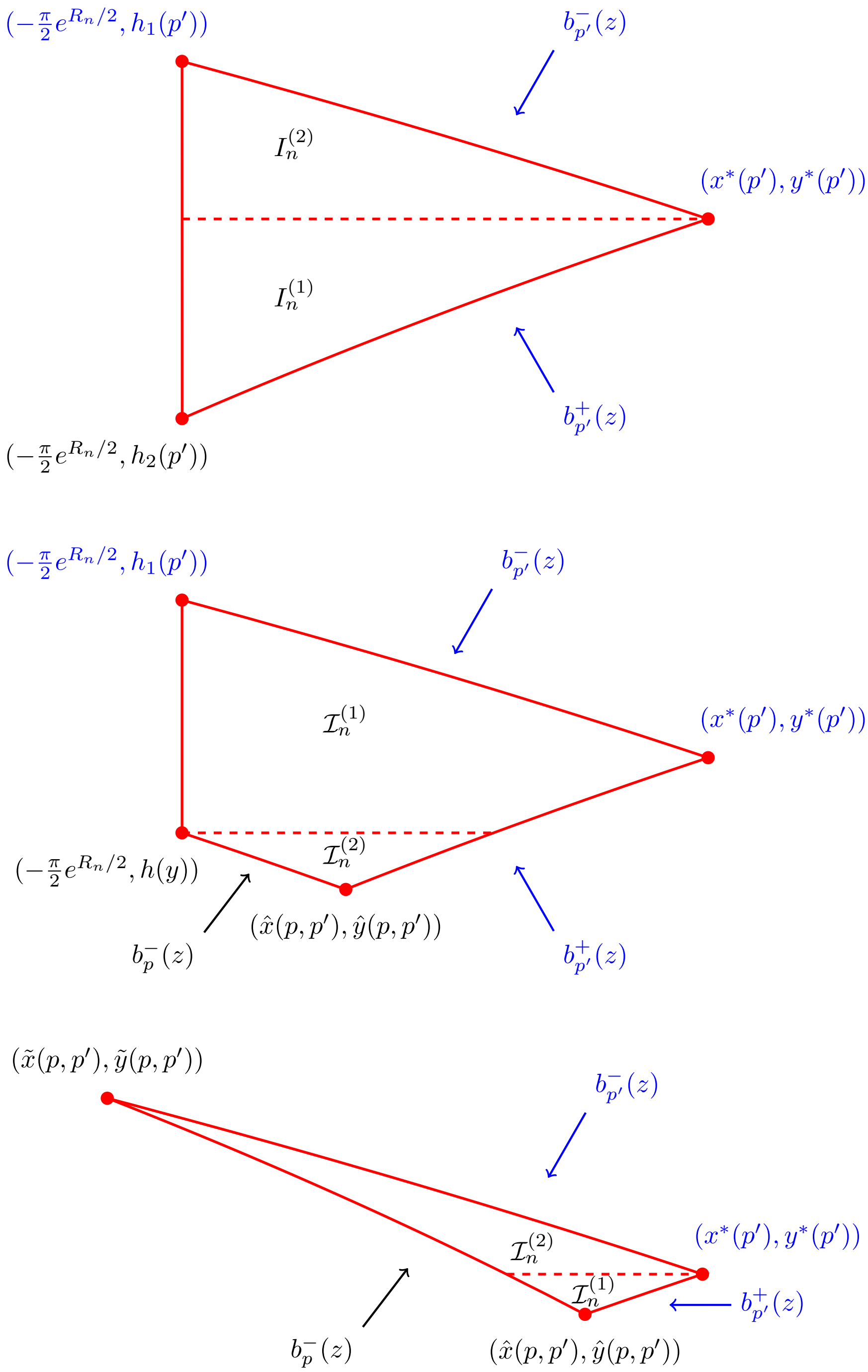

Figure 11: The different shapes of $\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)$ depending on the regime to which p_1 belongs. The top figure is for $p_1 \in B^{(1)}_n$, the middle for $p_1 \in B^{(2)}_n$ and the bottom one for $p_1 \in B^{(3)}_n$.

Regime 1: $0 \leq y_1 \leq y - 2\log(I_n/(I_n - x_1))$ In this case the integral over p_2 splits into two parts

$$\mathcal{I}_n^{(1)}(p_1) := \int_{h_2(p_1)}^{y^*(p_1)} \int_{-I_n}^{x_1+e^{(y_1+y_2)/2}-2I_n} e^{-\alpha y_2} \, \mathrm{d}x_2 \, \mathrm{d}y_2$$
$$\mathcal{I}_n^{(2)}(p_1) := \int_{y^*(p_1)}^{h_1(p_1)} \int_{x^*(p_1)}^{x_1-e^{(y_1+y_2)/2}} e^{-\alpha y_2} \, \mathrm{d}x_2 \, \mathrm{d}y_2.$$

We first compute $\mathcal{I}_n^{(1)}$.

$$\begin{aligned}
\mathcal{I}_n^{(1)}(p_1) &= \int_{h_2(p_1)}^{y^*(p_1)} \left(x_1 + e^{(y_1+y_2)/2} - I_n\right) e^{-\alpha y_2} \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&\leq e^{y_1/2} \int_{h_2(p_1)}^{y^*(p_1)} e^{-(\alpha - \frac{1}{2})y_2} \, \mathrm{d}y_2 \\
&= \frac{2e^{y_1/2}}{2\alpha - 1} \left(e^{-(\alpha-\frac{1}{2})h_2(p_1)} - e^{-(\alpha-\frac{1}{2})y^*(p_1)}\right) \\
&= \frac{2e^{\alpha y_1}}{2\alpha - 1} I_n^{-(2\alpha-1)} \left(\left(1 - \frac{x_1}{I_n}\right)^{-(2\alpha-1)} - 1\right) \\
&= O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\right),
\end{aligned}$$

where we used that $x' \leq e^{(y+y_1)/2} = o\left(I_n\right)$ for all $y_1 \leq y$ and $y \in \mathcal{K}_C(k_n)$ so that

$$\left(\left(1 - \frac{x_1}{I_n}\right)^{-(2\alpha-1)} - 1\right) = O\left(\frac{x'}{I_n}\right) \quad \text{as } n \to \infty.$$

For $\mathcal{I}_n^{(2)}(p_1)$ we have

$$\begin{aligned}
\mathcal{I}_n^{(2)}(p_1) &= \int_{y^*(p_1)}^{h_1(p_1)} \left(I_n + x_1 - e^{(y_1+y_2)}\right) e^{-\alpha y_2} \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&\leq 2I_n \int_{y^*(p_1)}^{h_1(p_1)} e^{-\alpha y_2} \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&= \frac{2}{\alpha} I_n \left(I_n^{-2\alpha} e^{\alpha y_1} - \left(I_n + x_1\right)^{-2\alpha} e^{-\alpha y_1}\right) \\
&= O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\right) = O\left(I_n^{-(2\alpha-1)} e^{\alpha y_1}\right).
\end{aligned}$$

We conclude that for $p_1 \in B_n^{(1)}$:

$$\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)\right) = O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\right),$$

which establishes the first part of (73).

Regime 2: $y - 2\log(I_n/(I_n - x_1)) < y_1 \leq y + 2\log\left(1 + \frac{x_1}{I_n}\right)$ Here we split the integration into two parts (see Figure 11). Recall that $x^*(p, p_1) = x_1 - I_n$. Then, for the first part we have

$$\begin{aligned}
\mathcal{I}_n^{(1)}(p, p_1) &\leq \int_{h(y)}^{h_1(p_1)} \int_{-I_n}^{x^*(p,p_1)} f_{\alpha,\nu}(x_2, y_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&= O\left(x_1\left(e^{-\alpha h(y)} - e^{-\alpha h_1(p_1)}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= O\left(x_1 I_n^{-2\alpha}\left(e^{\alpha y} - e^{\alpha y_1}\left(1+\frac{x_1}{I_n}\right)^{-2\alpha}\right)\right)\\
&= O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\left(\left(1-\frac{x_1}{I_n}\right)^{-2\alpha} - \left(1+\frac{x_1}{I_n}\right)^{-2\alpha}\right)\right)\\
&= O\left(I_n^{-2\alpha} x_1 e^{\alpha y_1}\right) = O\left(I_n^{-(2\alpha-1)} e^{\alpha y}\right),
\end{aligned}
$$

were we used that $y \le y_1 + 2\log(I_n/(I_n - x_1))$ for $p_1 \in B_n^{(2)}$ for the third line and

$$
\left(1-\frac{x_1}{I_n}\right)^{-2\alpha} - \left(1+\frac{x_1}{I_n}\right)^{-2\alpha} = O\left(\frac{x_1}{I_n}\right) = O\left(1\right),
$$

for the last line.

For the second part we first compute that

$$
\begin{aligned}
x_1 + e^{(y_1+y_2)/2} - 2I_n + e^{(y+y_2)/2} &\le \left(e^{y/2} + e^{y_1/2}\right) e^{y_2/2}\\
&\le e^{y/2}\left(1 + \frac{I_n}{I_n - e^y}\right) e^{y_2/2} = O\left(e^{(y+y_2)/2}\right),
\end{aligned}
$$

since $y \in \mathcal{K}_C(k_n)$ and $k_n = o\left(\sqrt{n}\right)$, so that $e^y = o\left(n\right) = o\left(I_n\right)$. Then we have

$$
\begin{aligned}
\mathcal{I}_n^{(2)} &= \int_{\hat{y}(p,p_1)}^{h(y)} \int_{-e^{(y+y_2)/2}}^{x_1 + e^{(y+y_1)/2} - 2I_n} f_{\alpha,\nu}(x_2, y_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2\\
&= O\left(e^{y/2} \int_{\hat{y}(p,p_1)}^{h(y)} e^{-(\alpha-\frac{1}{2})y_2} \, \mathrm{d}y_2\right)\\
&= O\left(e^{y/2}\left(e^{-(\alpha-\frac{1}{2})\hat{y}(p,p_1)} - e^{-(\alpha-\frac{1}{2})h(y)}\right)\right)\\
&= O\left(e^{y/2}\left(\left(\frac{2I_n - x_1}{e^{y/2} + e^{y_1/2}}\right)^{-(2\alpha-1)} - I_n^{-(2\alpha-1)} e^{(\alpha-\frac{1}{2})y}\right)\right)\\
&= O\left(I_n^{-(2\alpha-1)} e^{\alpha y}\right),
\end{aligned}
$$

where for the last line we first used that $(2I_n - x_1)^{-(2\alpha-1)} \le I_n^{-(2\alpha-1)}$ and then

$$
\left(\left(e^{y/2} + e^{y_1/2}\right)^{2\alpha-1} - e^{(\alpha-\frac{1}{2})y}\right) \le e^{(\alpha-\frac{1}{2})y}\left(\left(1 + \sqrt{1 + \frac{x_1}{I_n}}\right)^{2\alpha-1} - 1\right) = O\left(e^{(\alpha-\frac{1}{2})y}\right).
$$

It then follows that for $p_1 \in B_n^{(2)}$

$$
\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p, p_1)\right) = O\left(I_n^{-(2\alpha-1)} e^{\alpha y}\right).
$$

Regime III $p_1 \in B_n^{(3)}$:

$$
\begin{aligned}
\mathcal{I}_n^{(1)} &= \int_{y^*}^{\tilde{y}} \int_{-e^{(y+y_2)/2}}^{x_1 - e^{(y_1+y_2)/2}} f_{\alpha,\nu}(x_2, y_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2\\
&= O\left(\int_{y^*}^{\tilde{y}} x_1 e^{-\alpha y_2} - \left(e^{y_1/2} - e^{y/2}\right) e^{-(\alpha-\frac{1}{2})y_2} \, \mathrm{d}y_2\right)\\
&= O\left(x_1 \int_{y^*}^{\tilde{y}} e^{-\alpha y_2} \, \mathrm{d}y_2\right).
\end{aligned}
$$

Now

$$\int_{y^*}^{\tilde{y}} e^{-\alpha y_2}\,\mathrm{d}y_2 = \frac{1}{\alpha}\left(e^{-\alpha y^*} - e^{-\alpha\tilde{y}}\right) = \frac{1}{\alpha}\left(I_n^{-2\alpha}e^{\alpha y_1} - \left(\frac{x_1}{e^{y_1/2} - e^{y/2}}\right)^{-2\alpha}\right)$$
$$= \frac{I_n^{-2\alpha}e^{\alpha y_1}}{\alpha}\left(1 - \left(1 - e^{(y-y_1)/2}\right)^{2\alpha}\left(\frac{x_1}{I_n}\right)^{-2\alpha}\right) = O\left(I_n^{-2\alpha}e^{\alpha y_1}\right),$$

and hence we have

$$\mathcal{I}_n^{(1)} = O\left(I_n^{-2\alpha}x_1 e^{\alpha y_1}\right).$$

For the second integral we have

$$\begin{aligned}
\mathcal{I}_n^{(2)} &= \int_{\hat{y}}^{y^*}\int_{-e^{(y+y_2)/2}}^{e^{(y_1+y_2)/2}+x_1-2I_n} f_{\alpha,\nu}(x_2,y_2)\,\mathrm{d}x_2\,\mathrm{d}y_2 \\
&= O\left(\int_{\hat{y}}^{y^*}\left(e^{y/2}+e^{y_1/2}\right)e^{-(\alpha-\frac{1}{2})y_2}\,\mathrm{d}y_2\right) \\
&= O\left(e^{y_1/2}\int_{\hat{y}}^{y^*} e^{-(\alpha-\frac{1}{2})y_2}\,\mathrm{d}y_2\right).
\end{aligned}$$

For the integral we have

$$\begin{aligned}
\int_{\hat{y}}^{y^*} e^{-(\alpha-\frac{1}{2})y_2}\,\mathrm{d}y_2 &= \frac{2}{2\alpha-1}\left(e^{-(\alpha-\frac{1}{2})\hat{y}} - e^{-(\alpha-\frac{1}{2})y^*}\right) \\
&= \frac{2}{2\alpha-1}\left(\left(\frac{2I_n - x_1}{e^{y/2}+e^{y_1/2}}\right)^{-(2\alpha-1)} - I_n^{-(2\alpha-1)}e^{-(\alpha-\frac{1}{2})y_1}\right) \\
&= O\left(I_n^{-2\alpha}x_1 e^{-(\alpha-\frac{1}{2})y_1}\right)
\end{aligned}$$

so that

$$\mathcal{I}_n^{(2)} = O\left(I_n^{-(2\alpha-1)}e^{(1-\alpha)y_1}\right) = O\left(I_n^{-2\alpha}x_1 e^{\alpha y}\right)$$

and hence for $p_1 \in B_n^{(3)}$

$$\mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) = O\left(I_n^{-2\alpha}x_1 e^{\alpha y}\right) = O\left(I_n^{-(2\alpha-1)}e^{\alpha y}\right).$$

Integration over p_1

We now proceed with the second part of the computation leading to (70). Here we will integrate $\mu_{\alpha,\nu}(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n})(p,p_1)$ over the region $B_n := B_n^{(1)} \cup B_n^{(2)} \cup B_n^{(3)}$, see Figure 10. Let us first identify the boundaries of these areas.

The area $B_n^{(1)}$ is bounded from above by the line given by the equation

$$y_1 = y - 2\log\left(\frac{I_n}{I_n - x_1}\right).$$

Solving this for x_1 yields $x_1 = I_n\left(1 - e^{(y_1-y)/2}\right)$ and hence the area $B_n^{(1)}$ is given by

$$B_n^{(1)} = \left\{(x_1,y_1)\,:\, 0 \le y_1 \le y, \quad 0 \le x_1 \le I_n\left(1 - e^{(y_1-y)/2}\right) \wedge e^{(y+y_1)/2}\right\}.$$

In a similar way we have that $B_n^{(2)}$ is bounded from above by line

$$y_1 = y + 2\log\left(\frac{I_n}{I_n + x_1}\right),$$

which yields $x_1 = I_n\left(e^{(y_1-y)/2}-1\right)$. The lower red boundary is the upper boundary of $B_n^{(2)}$ and hence we have

$$B_n^{(2)} = \left\{(x_1,y_1) \,:\, h_*(y) \le y_1 \le h^*(y),\; I_n\left(1-e^{(y_1-y)/2}\right) \vee I_n\left(e^{(y_1-y)/2}-1\right) \le x_1 \le e^{(y+y_1)/2}\right\}.$$

We continue is the same way to obtain for $B_n^{(3)}$

$$B_n^{(3)} = \left\{(x_1,y_1) \,:\, y \le y_1 \le R_n,\; I_n\left(1-e^{(y-y_1)/2}\right) \le x_1 \le I_n\left(e^{(y_1-y)/2}-1\right) \wedge e^{(y+y_1)/2} \wedge I_n\right\}.$$

We these characterizations of the areas we now integrate $\mu_{\alpha,\nu}(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n})(p,p_1)$ over B_n, splitting the computations over the three different areas.

$\boldsymbol{p_1 \in B_n^{(1)}}$: We use that $I_n\left(1-e^{(y_1-y)/2}\right) \wedge e^{(y+y_1)/2} \le I_n\left(1-e^{(y_1-y)/2}\right)$ so that

$$\begin{aligned}
&\int_{B_n^{(1)}} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
&\qquad \le \int_0^y \int_0^{I_n(1-e^{(y_1-y)/2})} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
&\qquad = O\left(I_n^{-2\alpha}\int_0^y \int_0^{e^{(y+y_1)/2}} x_1\,\mathrm{d}x_1\,\mathrm{d}y_1\right) \\
&\qquad = O\left(I_n^{-(2\alpha-1)}\int_0^y \left(1-e^{(y_1-y)/2}\right)^2 \mathrm{d}y_1\right) \\
&\qquad = O\left(I_n^{-(2\alpha-1)}y\right) = O\left(yn^{-(2\alpha-1)}\right).
\end{aligned}$$

$\boldsymbol{p_1 \in B_n^{(2)}}$: We will show that

$$\mu_{\alpha,\nu}(B_n^{(2)}) = O\left(I_n^{-1}e^{(2-\alpha)y}\right), \tag{74}$$

which together with (73) yields

$$\begin{aligned}
\int_{B_n^{(2)}} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 &= O\left(\mu_{\alpha,\nu}(B_n^{(2)})I_n^{-(2\alpha-1)}e^{\alpha y}\right) \\
&= O\left(I_n^{-2\alpha}e^{2y}\right).
\end{aligned}$$

The integration is split into two parts determined by $I_n\left(1-e^{(y_1-y)/2}\right) \vee I_n\left(e^{(y_1-y)/2}-1\right)$:

$$\begin{aligned}
\mu_{\alpha,\nu}(B_n^{(3)}) = &\int_{h_*(y)}^y \int_{I_n(1-e^{(y_1-y)/2})}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
&+ \int_y^{h^*(y)} \int_{I_n(e^{(y_1-y)/2}-1)}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1.
\end{aligned}$$

For the first integral we use that $e^{(y+y_1)/2} - I_n(1-e^{(y_1-y)/2}) \le e^{y_1/2}\left(e^{y/2}+e^{-y/2}\right)$ to obtain

$$\begin{aligned}
&\int_{h_*(y)}^y \int_{I_n(1-e^{(y_1-y)/2})}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
&\qquad = O\left(e^{y/2}\int_{h_*(y)}^y e^{-(\alpha-\frac{1}{2})y_1}\,\mathrm{d}y_1\right) \\
&\qquad = O\left(e^{y/2}\left(e^{-(\alpha-\frac{1}{2})y} - e^{-(\alpha-\frac{1}{2})y}\left(\frac{I_n}{I_n+e^y}\right)^{-(2\alpha-1)}\right)\right)
\end{aligned}$$

$$= O\left(I_n^{-1} e^{(2-\alpha)y}\right).$$

For the second integral note that $e^{(y+y_1)/2} - I_n(e^{(y_1-y)/2} - 1) \le e^{(y+y_1)/2}$ and hence

$$\begin{aligned}
&\int_y^{h^*(y)} \int_{I_n(e^{(y_1-y)/2}-1)}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&= O\left(e^{y/2} \int_y^{h^*(y)} e^{-(\alpha-\frac{1}{2})y_1}\, \mathrm{d}y_1\right) \\
&= O\left(e^{y/2}\left(e^{-(\alpha-\frac{1}{2})y} - e^{-(\alpha-\frac{1}{2})y}\left(\frac{I_n}{I_n - e^y}\right)^{-(2\alpha-1)}\right)\right) \\
&= O\left(I_n^{-1} e^{(2-\alpha)y}\right),
\end{aligned}$$

so that (74) follows.

$\boldsymbol{p_1 \in B_n^{(3)}}$: For this area we show that

$$\mu_{\alpha,\nu}(B_n^{(3)}) = O\left(e^{(1-\alpha)y}\right) \tag{75}$$

so that

$$\begin{aligned}
\int_{B_n^{(3)}} \mu_{\alpha,\nu}\left(\mathcal{T}_{\mathcal{P}\Delta\mathcal{P}_n}(p,p_1)\right) f_{\alpha,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 &= O\left(\mu_{\alpha,\nu}(B_n^{(2)}) I_n^{-(2\alpha-1)} e^{\alpha y}\right) \\
&= O\left(I_n^{-(2\alpha-1)} e^{y}\right).
\end{aligned}$$

Here the integral is split into three parts:

$$\begin{aligned}
\mu_{\alpha,\nu}(B_n^{(3)}) = &\int_y^{h^*(y)} \int_{I_n(1-e^{(y-y_1)/2})}^{I_n(e^{(y_1-y)/2}-1)} f_{\alpha,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&+ \int_{h^*(y)}^{h(y)} \int_{I_n(1-e^{(y-y_1)/2})}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&+ \int_{h(y)}^{R_n} \int_{I_n(1-e^{(y-y_1)/2})}^{I_n} f_{\alpha,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1.
\end{aligned}$$

Let us first focus on the first integral. Since $I_n(e^{(y_1-y)/2} - 1) - I_n(1 - e^{(y-y_1)/2}) \le I_n e^{(y_1-y)/2}$ we get, using similar arguments as above

$$\begin{aligned}
\int_y^{h^*(y)} \int_{I_n(1-e^{(y-y_1)/2})}^{I_n(e^{(y_1-y)/2}-1)} f_{\alpha,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 &= O\left(I_n e^{-y/2} \int_y^{h^*(y)} e^{-(\alpha-\frac{1}{2})y_1}\, \mathrm{d}y_1\right) \\
&= O\left(I_n e^{-\alpha y}\left(1 - \left(\frac{I_n}{I_n - e^y}\right)^{-(2\alpha-1)}\right)\right) \\
&= O\left(e^{(1-\alpha)y}\right).
\end{aligned}$$

Proceeding to the second integral, we first note that $e^{(y+y_1)/2} - I_n(1 - e^{(y-y_1)/2}) = O\left(I_n e^{(y_1-y)/2}\right)$ so that similar calculations as before yield

$$\int_{h^*(y)}^{h(y)} \int_{I_n(1-e^{(y-y_1)/2})}^{e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 = O\left(I_n e^{-y/2} \int_{h^*(y)}^{h(y)} e^{-(\alpha-\frac{1}{2})y_1}\, \mathrm{d}y_1\right) = O\left(e^{(1-\alpha)y}\right).$$

□

8 Concentration for $c_{\mathcal{P},n}(k)$ (Proving Proposition 5.4)

In this section we establish a concentration result for the local clustering function $c^*_{\mathcal{P},n}(k)$ in the finite model $G_{\mathcal{P},n}$. Similar to the previous section we will focus on those nodes $p = (x, y) \in \mathcal{K}_C(k_n)$.

8.1 The main contribution of triangles

Recall the adjusted triangle count function (62)

$$\widetilde{T}_{\mathcal{P},n}(p_0, p_1, p_2) = \mathbb{1}_{\{p_1 \in B_{\mathcal{P},n}(p)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P},n}(p)\}} \mathbb{1}_{\{p_2 \in B_{\mathcal{P}}(p_1) \cap \mathcal{R}_n\}}.$$

and the concentration set

$$\mathcal{K}_C(k_n) = \left\{ p \in \mathbb{R} : \frac{k_n - C\kappa_n}{\xi_{\alpha,\nu}} \vee 0 \leq e^{\frac{y}{2}} \leq \frac{k_n + C\kappa_n}{\xi_{\alpha,\nu}} \wedge e^{R_n/2} \right\},$$

with

$$\kappa_n := \begin{cases} \log(n) & \text{if } k_n = \Theta(1), \\ \sqrt{k_n \log(k_n)} & \text{else.} \end{cases}$$

We will define the corresponding triangle degree function

$$\widetilde{T}_{\mathcal{P},n}(k, C) = \sum_{p \in \mathcal{P}_n \cap \mathcal{K}_C(k)} \mathbb{1}_{\{D_{\mathcal{P},n}(p) = k\}} \widetilde{T}_{\mathcal{P},n}(p), \tag{76}$$

where

$$\widetilde{T}_{\mathcal{P},n}(p) := \sum_{(p_1,p_2) \in \mathcal{P} \setminus (0,y)}^{\neq} \widetilde{T}_{\mathcal{P},n}(p, p_1, p_2),$$

where the sum is over all distinct pairs $(p_1, p_2) \in \mathcal{P} \setminus \{(0, y)\}$.

Next we note that the second result from Corollary 7.4 states that

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n, C)\right] = (1 + o(1))\mathbb{E}\left[T_{\mathcal{P}}(k_n)\right], \tag{77}$$

so that we can conclude that the main contribution of triangles of degree k_n is given by $\widetilde{T}_{\mathcal{P},n}(k_n, C)$. Therefore, in order to prove Proposition 5.4 it suffices to show that $\widetilde{T}_{\mathcal{P},n}(k_n, C)$ is sufficiently concentrated around its mean. This is done in the following proposition.

Proposition 8.1 (Concentration $\widetilde{T}_{\mathcal{P},n}(k_n, C)$)**.** *Let $\alpha > \frac{1}{2}$, $\nu > 0$ and let $(k_n)_{n \geq 1}$ be any positive sequence satisfying $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$. Then for any $C > 0$, as $n \to \infty$,*

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n, C)^2\right] = (1 + o(1))\,\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n, C)\right]^2.$$

We postpone the proof of this proposition postpone till Section 8.3 and first use it to prove Proposition 5.4.

Proof of Proposition 5.4. Again, we write

$$c^*_{\mathcal{P},n}(k_n) = \frac{\widetilde{T}_{\mathcal{P},n}(k_n, C)}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} + \frac{\left(T_{\mathcal{P},n}(k_n) - \widetilde{T}_{\mathcal{P},n}(k_n, C)\right)}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]},$$

so that by (77),

$$\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right] = \frac{\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n, C)\right]}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} + o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right). \tag{78}$$

Next, we use Proposition 8.1 to obtain

$$\mathbb{E}\left[\left|\widetilde{T}_{\mathcal{P},n}(k_n,C) - \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,C)\right]\right|\right] \le \left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,C)^2\right] - \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,C)\right]^2\right)^{\frac{1}{2}}$$
$$= o\left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,C)\right]\right).$$

This implies

$$\frac{\mathbb{E}\left[\left|\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon) - \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,\varepsilon)\right]\right|\right]}{\binom{k_n}{2}\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} = o\left(\mathbb{E}\left[c^*_{\mathcal{P},n}(k_n)\right]\right),$$

which together with (78) yields the required result. □

8.2 Joint neighborhoods and degrees in $G_{\mathcal{P},n}(\alpha,\nu)$

To prove Proposition 8.1 we need to understand the joint degree distribution in $G_{\mathcal{P},n}$ and subsequently the joint neighborhoods of two points $p, p' \in \mathcal{R}_n$. We perform the analysis in this section. The two main results, which will be the crucial technical ingredients for the proof of Proposition 8.1, are Lemma 8.5 and Lemma 8.6.

Neighborhoods

We start with analyzing joint neighborhoods in $G_{\mathcal{P},n}$. Let $p, p' \in \mathcal{R}_n$. Then we denote by $\mathcal{N}_{\mathcal{P},n}(p\Delta p')$ the number of disjoint neighbors of p and p' in $G_{\mathcal{P},n}$, i.e. those points that belong to either $\mathcal{B}_{\mathcal{P},n}(p)$ or $\mathcal{B}_{\mathcal{P},n}(p')$ but not to their intersection. In addition we denote by $\mathcal{N}_{\mathcal{P},n}(p,p')$ the number of joint neighbors of p and p'. We shall establish lower bounds on the expect number of these disjoint neighbors $\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right]$ as well as an asymptotic expression for the expected number of joint neighbors. For this we will distinguish between the cases where the distance between the x-coordinates of p and p' is small or large. Figure 12-14 show the different situations that occur.

We start by analyzing the shape of the neighborhoods. Due to symmetry and the fact that we have identified the left and right boundaries of the box $\mathcal{R}_n$, we can, without loss of generality, assume that $p = (0, y)$ and $p' = (x', y')$ with $x' > 0$ and $y' \le y$. To understand the computation it is helpful to have a picture of the different situations. Figure 12 and Figure 13 show two different situations for small distance in the x-coordinates, in which case the number of disjoint neighbors is small. The case where this distance is large, and the number of disjoint neighbors is expected to be large, is show in Figure 14. There are several different quantities that are important. The first are the heights $h_1(p')$ and $h_2(p')$ where, respectively, the left and right boundaries of the ball $\mathcal{B}_{\mathcal{P},n}(p')$ go outside the box $\mathcal{R}_n$. Note that when $x = 0$ then these height are the same and we denote this by $h(y)$. We also need to know the coordinates $h(p,p')$ and $x^*(p,p')$ of the intersection of the right boundary of the neighborhood of p with the left boundary of the neighborhood of p'. Finally we will denote by $d(p,p')$ the distance between the lower right boundary of $\mathcal{B}_{\mathcal{P},n}(p)$ and the lower left of $\mathcal{B}_{\mathcal{P},n}(p')$, which is positive only when the bottom parts of both neighborhoods do not intersect, compare Figures 12 and 14. The full expressions of all these functions are given below for further reference.

$$h(y) = R_n - y + 2\log\left(\frac{\pi}{2}\right) \tag{79}$$

$$h_1(p') = 2\log\left(x' + \frac{\pi}{2}e^{\frac{R_n}{2}}\right) - y' \tag{80}$$

$$h_2(p') = 2\log\left(\frac{\pi}{2}e^{\frac{R_n}{2}} - x'\right) - y' \tag{81}$$

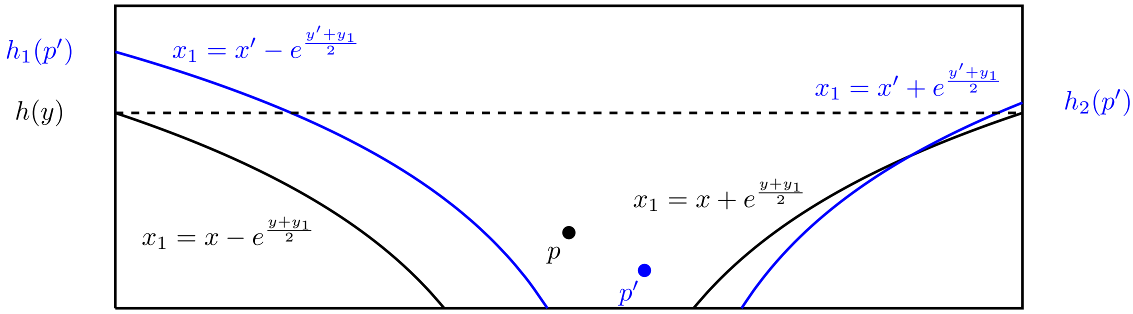

Figure 12: Schematic representation of the neighborhoods of p and p' in $G_{\mathcal{P},n}(\alpha,\nu)$ when $|x-x'| \leq e^{\frac{y+y'}{2}}$ used for the proof of Lemma 8.2 and Lemma 8.4.

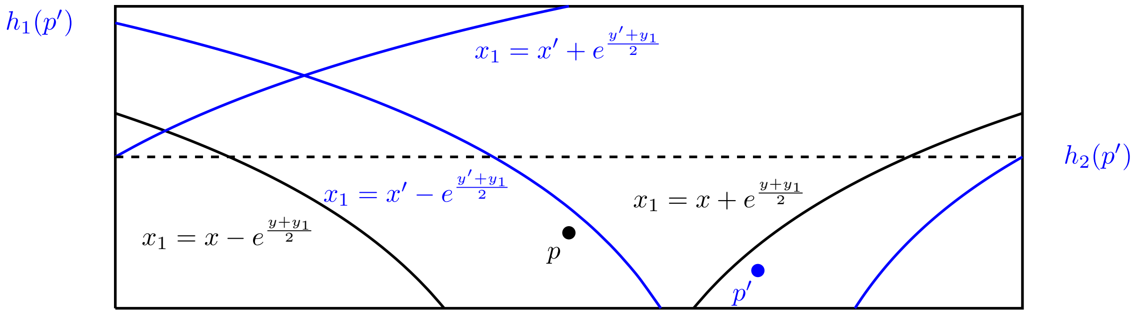

Figure 13: Schematic representation of the neighborhoods of p and p' in $G_{\mathcal{P},n}(\alpha,\nu)$ when $e^{\frac{y+y'}{2}} < |x-x'| \leq e^{\frac{y}{2}} + e^{\frac{y'}{2}}$ used for the proof of Lemma 8.2 and Lemma 8.4.

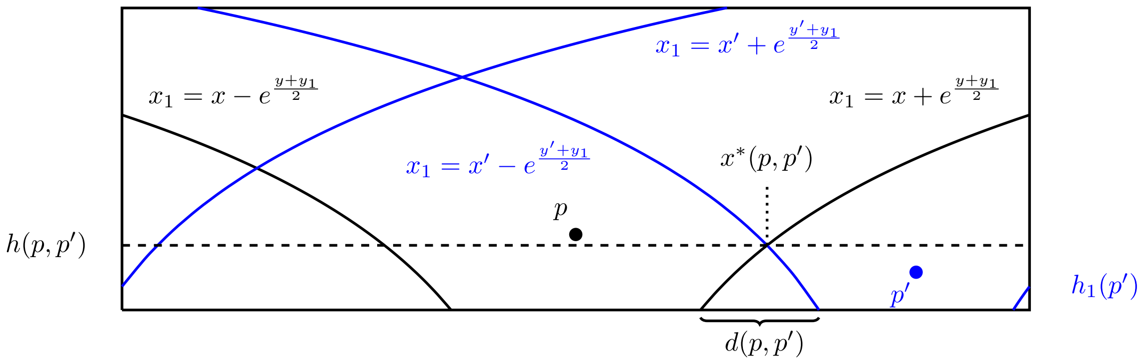

Figure 14: Schematic representation of the neighborhoods of p and p' in $G_{\mathcal{P},n}(\alpha,\nu)$ when $|x-x'| > e^{\frac{y}{2}} + e^{\frac{y'}{2}}$ used for the proof of Lemma 8.3.

$$h((p,p') = 2\log\left(\frac{|x-x'|}{e^{\frac{y}{2}}+e^{\frac{y'}{2}}}\right) \tag{82}$$

$$x^*(p,p') = \frac{xe^{\frac{y'}{2}}+x'e^{\frac{y}{2}}}{e^{\frac{y}{2}}+e^{\frac{y'}{2}}}, \tag{83}$$

$$d(p,p') = |x-x'| - \left(e^{\frac{y}{2}}+e^{\frac{y'}{2}}\right). \tag{84}$$

We start with the result for points whose x-coordinates are close, which is when $d(p,p') < 0$.

Lemma 8.2. *Let $p, p' \in \mathcal{R}_n$. Then whenever $|x-x'| \le e^{\frac{y}{2}}+e^{\frac{y'}{2}}$,*

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right] \ge \frac{\nu}{\pi}|x'-x|\left(1-\left(\frac{2}{\pi}\right)^{2\alpha}e^{-\alpha(R_n-y^*)}\right) + \xi_{\alpha,\nu}\left|e^{\frac{y}{2}}-e^{\frac{y'}{2}}\right|\left(1-\left(\frac{2}{\pi}\right)^{2\alpha-1}e^{-(\alpha-\frac{1}{2})(R_n-y^*)}\right).$$

Proof. In order to proof the result we will consider the area in between the two left boundaries of the balls $\mathcal{B}_{\mathcal{P},n}(p)$ and $\mathcal{B}_{\mathcal{P},n}(p')$ up to the height $h^* := \min\{h(y), h_2(p')\}$, see Figure 12 and Figure 13 for reference. Note that here we do not need to consider different cases depending on whether $|x-x'| \le e^{(y+y')/2}$ or $|x-x'| > e^{(y+y')/2}$.

By the Campbell-Mecke formula we get

$$\begin{aligned}
\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right] &= \mu_{\alpha,\nu}\left(B_{\mathcal{P},n}(p)\Delta B_{\mathcal{P},n}(p')\right) \\
&\ge \int_0^{h^*}\int_{x-e^{\frac{y+y_1}{2}}}^{x'-e^{\frac{y'+y_1}{2}}} f_{\alpha,\nu}(x_1,y_1)\,dx_1\,dy_1 \\
&= \frac{\alpha\nu}{\pi}|x'-x|\int_0^{h^*} e^{-\alpha y_1}dy_1 + \frac{\alpha\nu}{\pi}\left|e^{\frac{y}{2}}-e^{\frac{y'}{2}}\right|\int_0^{h^*} e^{-(\alpha-\frac{1}{2})y_1}\,dy_1 \\
&= \frac{\nu}{\pi}|x'-x|\left(1-\left(\frac{2}{\pi}\right)^{2\alpha}e^{-\alpha(R_n-y^*)}\right) \\
&\quad + \xi_{\alpha,\nu}\left|e^{\frac{y}{2}}-e^{\frac{y'}{2}}\right|\left(1-\left(\frac{2}{\pi}\right)^{2\alpha-1}e^{-(\alpha-\frac{1}{2})(R_n-y^*)}\right).
\end{aligned}$$

□

Now we will consider the case where $|x-x'| > e^{\frac{y}{2}}+e^{\frac{y'}{2}}$

Lemma 8.3. *Let $p, p' \in \mathcal{R}_n$. Then, whenever $|x-x'| > e^{\frac{y}{2}}+e^{\frac{y'}{2}}$,*

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right] \ge \left(\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(p)) + \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(p'))\right)\left(1-\phi_n(p,p')\right).$$

$$\phi_n(p,p') = \xi_{\alpha,\nu}\left(\left(\frac{e^{y/2}+e^{y'/2}}{|x-x'|}\right)^{2\alpha-1} - e^{-(\alpha-\frac{1}{2})R_n}\right)$$

Proof. We will prove the results by using that

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right] \ge \int_0^{h(p,p')}\int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathcal{P},n}(p)\cup\mathcal{B}_{\mathcal{P},n}(p')\}} f_{\alpha,\nu}(x_1,y_1)\,dx_1\,dy_1,$$

and computing the integral on the right. We refer to Figure 14 for further clarification.

Before we proceed we show that the neighborhoods of p and p' below $h(p,p')$ are disjoint. This is clearly true when $h(p,p') \le h_2(p')$ so suppose that $h(p,p') > h_2(p')$. Then, because we identified the right and left boundaries of the box $\mathcal{R}_n$ the right boundary of $\mathcal{B}_{\mathcal{P},n}(p')$ continues from the left boundary of the box and is described by the equation

$$x_1 = x' + e^{\frac{y'+y_1}{2}} - \pi e^{\frac{R_n}{2}}.$$

Now, let x'_{right} and x_{left} denote the x-coordinate of the intersection of the line $h(p,p')$ with, respectively, the right boundary of $\mathcal{B}_{\mathcal{P},n}(p')$ and the left boundary of $\mathcal{B}_{\mathcal{P},n}(p)$. Then

$$\begin{aligned}
x'_{\text{right}} &= x' + e^{\frac{y'+h(p,p')}{2}} - \pi e^{\frac{R_n}{2}} \\
&= x' + e^{\frac{h(p,p')}{2}}\left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right) - e\frac{y+h(p,p')}{2} - \pi e^{\frac{R_n}{2}} \\
&= x' + |x-x'| - e\frac{y+h(p,p')}{2} - \pi e^{\frac{R_n}{2}} \\
&= x - e\frac{y+h(p,p')}{2} + 2|x-x'| - \pi e^{\frac{R_n}{2}} \\
&\le x - e\frac{y+h(p,p')}{2} = x_{\text{right}},
\end{aligned}$$

and hence the neighborhoods of p and p' below $h(p,p')$ are disjoint. It then follows that

$$\begin{aligned}
\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p,p')\right] &\ge \int_0^{h(p,p')} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathcal{P},n}(p) \cup \mathcal{B}_{\mathcal{P},n}(p')\}} f_{\alpha,\nu}(x_1,y_1)\, dx_1\, dy_1 \\
&= \int_0^{h(p,p')} \int_{-\frac{\pi}{2}e^{\frac{R_n}{2}}}^{\frac{\pi}{2}e^{\frac{R_n}{2}}} \left(\mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathcal{P},n}(p)\}} + \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathcal{P},n}(p')\}}\right) f_{\alpha,\nu}(x_1,y_1)\, dx_1\, dy_1 \\
&= \left(\mu_{\alpha,\nu,n}(\mathcal{B}_{\mathcal{P},n}(p)) + \mu_{\alpha,\nu,n}(\mathcal{B}_{\mathcal{P},n}(p'))\right)\left(1 - \frac{2\alpha\nu}{\pi}\int_{h(p,p')}^{R_n} e^{-(\alpha-\frac{1}{2})y_1}\, dy_1\right),
\end{aligned}$$

from which the result follows since

$$\begin{aligned}
\frac{2\alpha\nu}{\pi}\int_{h(p,p')}^{R_n} e^{-(\alpha-\frac{1}{2})y_1}\, dy_1 &= \xi_{\alpha,\nu}\left(e^{-(\alpha-\frac{1}{2})h(p,p')} - e^{-(\alpha-\frac{1}{2})R_n}\right) \\
&= \xi_{\alpha,\nu}\left(\left(\frac{e^{y/2}+e^{y'/2}}{|x-x'|}\right)^{2\alpha-1} - e^{-(\alpha-\frac{1}{2})R_n}\right).
\end{aligned}$$

□

Next we consider the number of common neighbors between two nodes p and p' in $G_{\mathcal{P}}$, which we denote by $\mathcal{N}_{\mathcal{P}}(p,p')$.

Lemma 8.4. *Let $p, p' \in \mathcal{R}_n$. Then, whenever $|x-x'| > \left(e^{\frac{y}{2}} + e^{\frac{y'}{2}}\right)$,*

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p,p')\right] = \left(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p)\right) + \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p')\right)\right)\phi_n(p,p'),$$

where

$$\phi_n(p,p') = \frac{1+2\alpha}{4\alpha}\left(\frac{|x-x'|}{e^{\frac{y}{2}}+e^{\frac{y'}{2}}}\right)^{-(2\alpha-1)} + \frac{2\alpha-1}{4\alpha}|x'-x|e^{-\alpha R_n} - e^{-(\alpha-\frac{1}{2})R_n}.$$

Proof. Assume, without loss of generality, that $y \ge y'$ and $x \le x'$ and consider the boundaries of the balls $\mathcal{B}_{\mathcal{P},n}(p)$ and $\mathcal{B}_{\mathcal{P},n}(p')$ as drawn in Figure 14. The left boundary of $\mathcal{B}_{\mathcal{P},n}(p)$ intersects the right boundary of $\mathcal{B}_{\mathcal{P},n}(p')$ if an only if $d(p,p') > 0$. We observe from the definition of $d(p,p')$, (84), that this is exactly the condition we imposed in the statement of the lemma. The y-coordinate of the intersection is then given by

$$h((p,p') := 2\log\left(\frac{|x-x'|}{e^{\frac{y}{2}}+e^{\frac{y'}{2}}}\right).$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p,p')\right] &= \int_{h(p,p')}^{R_n}\int_{x'-e^{(y'+y_1)/2}}^{x+e^{(y+y_1)/2}} f_{\alpha,\nu}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
&= \left(e^{y/2}+e^{y'/2}\right)\frac{\alpha\nu}{\pi}\int_{h(p,p')}^{R_n} e^{-(\alpha-\frac{1}{2})y_1}\,\mathrm{d}y_1 \\
&\quad -(x'-x)\frac{\alpha\nu}{\pi}\int_{h(p,p')}^{R_n} e^{-\alpha y_1}\,\mathrm{d}y_1.
\end{aligned}$$

For the first term we get

$$\begin{aligned}
&\left(e^{y/2}+e^{y'/2}\right)\frac{\alpha\nu}{\pi}\int_{h(p,p')}^{R_n} e^{-(\alpha-\frac{1}{2})y_1}\,\mathrm{d}y_1 \\
&\quad= \xi_{\alpha,\nu}\left(e^{y/2}+e^{y'/2}\right)\left(e^{-(\alpha-\frac{1}{2})h(p,p')}-e^{-(\alpha-\frac{1}{2})R_n}\right) \\
&\quad= \xi_{\alpha,\nu}\left(e^{y/2}+e^{y'/2}\right)\left(\left(\frac{|x'-x|}{e^{y/2}+e^{y'/2}}\right)^{-(2\alpha-1)}-e^{-(\alpha-\frac{1}{2})R_n}\right).
\end{aligned}$$

For the other term we compute

$$\begin{aligned}
&(x'-x)\frac{\alpha\nu}{\pi}\int_{h(p,p')}^{R_n} e^{-\alpha y_1}\,\mathrm{d}y_1 \\
&\quad= (x'-x)\frac{\nu}{\pi}\left(e^{-\alpha(p,p')}-e^{-\alpha R_n}\right) \\
&\quad= \frac{\nu}{\pi}\left(e^{y/2}+e^{y'/2}\right)\left(\left(\frac{|x'-x|}{e^{y/2}+e^{y'/2}}\right)^{-(2\alpha-1)}-(x'-x)e^{-\alpha R_n}\right) \\
&\quad= \xi_{\alpha,\nu}\left(e^{y/2}+e^{y'/2}\right)\left(\frac{2\alpha-1}{4\alpha}\left(\frac{|x'-x|}{e^{y/2}+e^{y'/2}}\right)^{-(2\alpha-1)}-\frac{2\alpha-1}{4\alpha}|x'-x|e^{-\alpha R_n}\right).
\end{aligned}$$

Combining these two results and noticing that $\xi_{\alpha,\nu}e^{y/2} = \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p)\right)$ and similar for p' finishes the proof. □

Degrees

We now turn to the joint degree distribution of nodes in $G_{\mathcal{P},n}$. To ease notations we introduce the following short-hand notation for the conditional joint degree distribution

$$\rho_n(p,p',k,k') := \mathbb{P}\left(D_{\mathcal{P},n}(p)=k, D_{\mathcal{P},n}(p')=k'\right).$$

We first establish an almost independence result for integrals over the joint degree distribution $\rho_n(p,p',k,k')$ for the case where p and p' are sufficiently separated. For this we note that when $|x-x'| \gg e^{y/2}+e^{y'/2}$ it follows from Lemma 8.3

$$\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right] = \left(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p)\right)+\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p')\right)\right)(1+o(1)),$$

for $p, p' \in \mathcal{K}_C(k_n) \times \mathcal{K}_C(k_n)$. This implies that the joint neighborhoods are almost independent and hence their degrees must be as well. To make this more precise, let $0 < \varepsilon < \min\{(2\alpha-1)^{-1}, 1\}$ and define the following two sets

$$\mathcal{E}(k_n) = \left\{(p, p') \in \mathcal{K}_C(k_n) \times \mathcal{K}_C(k_n) \,:\, |x - x'| > \left(e^{y/2} + e^{y'/2}\right) \log(n)\right\} \tag{85}$$

$$\mathcal{E}_\varepsilon(k_n) = \left\{(p, p') \in \mathcal{K}_C(k_n) \times \mathcal{K}_C(k_n) \,:\, \mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p, p')\right] \geq k_n^\varepsilon \text{ and } |x - x'| > k_n^{1+\varepsilon}\right\}. \tag{86}$$

The next result shows that for integrals over either $\mathcal{E}(k_n)$, when $k_n = \Theta(1)$, or $\mathcal{E}_\varepsilon(k_n)$, when $k_n \to \infty$, the joint degree distribution satisfies $\rho_n(p, p', k_n, k_n) = (1 + o(1))\rho(p, k_n)\rho(p', k_n)$.

Lemma 8.5. *Let $k_n \to \infty$ be such that $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right)$ let $h : \mathbb{R}_+ \to \mathbb{R}$ be a uniformly bounded function and set*

$$\mathcal{E}_n := \begin{cases} \mathcal{E}(k_n) & \text{if } k_n = \Theta(1) \\ \mathcal{E}_\varepsilon(k_n) & \text{if } k_n \to \infty. \end{cases}$$

Then, as $n \to \infty$,

$$\begin{aligned}
&\int_{\mathcal{E}_n} \rho_n(p, p', k_n, k_n) h(y) h(y') f_{\alpha,\nu}(x, y) f_{\alpha,\nu}(x', y') \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}x' \, \mathrm{d}y' \\
&\qquad = (1 + o(1)) \int_{\mathcal{E}_n} \rho(y, k_n) \rho(y', k_n) h(y) h(y') f_{\alpha,\nu}(x, y) f_{\alpha,\nu}(x', y') \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}x' \, \mathrm{d}y'.
\end{aligned}$$

Proof. Recall that $\mathrm{Po}(\lambda)$ denotes a Poisson random variable with mean λ. Now define the random variables

$$\begin{aligned}
X_1(p, p') &:= \mathrm{Po}\left(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p) \setminus \mathcal{B}_{\mathcal{P},n}(p')\right)\right), \\
X_2(p, p') &:= \mathrm{Po}\left(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p') \setminus \mathcal{B}_{\mathcal{P},n}(p)\right)\right), \\
X_3(p, p') &:= \mathrm{Po}\left(\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p) \cup \mathcal{B}_{\mathcal{P},n}(p')\right)\right),
\end{aligned}$$

and note that

$$\begin{aligned}
\rho_n(p, p', k_n, k_n) &= \mathbb{P}\left(X_1(p, p') + X_3(p, p') = k_n, X_2(p, p') + X_3(p, p') = k_n\right) \\
&= \sum_{t=0}^{\infty} \mathbb{P}\left(X_3(p, p') = t\right) \mathbb{P}\left(X_1(p, p') = k_n - t\right) \mathbb{P}\left(X_2(p, p') = k_n - t\right).
\end{aligned}$$

We proceed with the case $k_n \to \infty$.

Case $k_n \to \infty$:

Define $\delta_n = k_n^{-\frac{(1-\varepsilon)\varepsilon}{2}}$. Then, since $\mathbb{E}\left[X_3(p, p')\right] = \mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p, p')\right] \geq k_n^\varepsilon$ for $(p, p') \in \mathcal{E}_\varepsilon(k_n)$, a Chernoff bound (c.f. (138)) implies that

$$\mathbb{P}\left(\left|X_3(p, p') - \mathbb{E}\left[X_i(p, p')\right]\right| > \delta_n \mathbb{E}\left[X_3(p, p')\right]\right) = O\left(e^{-k_n^\varepsilon}\right).$$

If we now define

$$A_n(p, p') = \left\{t : (1 - \delta_n)\mathbb{E}\left[X_3(p, p')\right] \leq t \leq (1 + \delta_n)\mathbb{E}\left[X_3(p, p')\right]\right\},$$

then it follows that

$$\begin{aligned}
&\sum_{t > \lfloor (1+\delta_n)\mathbb{E}[X_3(p,p')] \rfloor} \mathbb{P}\left(X_3(p, p') = t\right) \mathbb{P}\left(X_1(p, p') = k_n - t\right) \mathbb{P}\left(X_2(p, p') = k_n - t\right) \\
&\qquad \leq \mathbb{P}\left(X_3(p, p') > (1 + \delta_n)\mathbb{E}\left[X_3(p, p')\right]\right) \\
&\qquad \leq \mathbb{P}\left(\left|X_3(p, p') - \mathbb{E}\left[X_3(p, p')\right]\right| > \delta_n \mathbb{E}\left[X_3(p, p')\right]\right) \\
&\qquad = O\left(e^{-k_n^\varepsilon}\right),
\end{aligned}$$

and similar for the sum over all $t < (1-\delta_n)\mathbb{E}[X_3(p,p')]$. Hence we conclude that

$$\rho_n(p,p',k_n,k_n)$$
$$= \sum_{t\in A_n(p,p')} \mathbb{P}(X_3(p,p')=t)\,\mathbb{P}(X_1(p,p')=k_n-t)\,\mathbb{P}(X_2(p,p')=k_n-t) + O\left(e^{-k_n^{\varepsilon}}\right) \tag{87}$$

The idea for the remainder of the proof is to first remove the dependence on p and p' in the summation over t so that

$$\rho_n(p,p',k_n,k_n) = (1+o(1))\mathbb{P}(X_1(p,p')=k_n-t_n)\,\mathbb{P}(X_2(p,p')=k_n-t_n) + O\left(e^{-k_n^{\varepsilon}}\right),$$

where $t_n = o(k_n)$. Then, since $\mathbb{E}[X_1(p,p')] = \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(p)\setminus\mathcal{B}_{\mathcal{P},n}(p')) = (1+o(1))\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(p))$, by Lemma 8.3, and similar for X_2, the proof will follow by applying a concentration argument twice.

To execute this plan we note that $\mathbb{E}[X_3(p,p')] = \mathbb{E}[\mathcal{N}_{\mathcal{P},n}(p,p')]$. Now recall the error function $\phi_n(p,p')$ from Lemma 8.4. Then, for $p,p'\in\mathcal{K}_C(k_n)$ we have that

$$\begin{aligned}
\phi_n(p,p') &\le \frac{1+2\alpha}{4\alpha}\left(\frac{k_n^{1+\varepsilon}}{e^{y/2}+e^{y'/2}}\right)^{-(2\alpha-1)} - \left(1-\frac{(2\alpha-1)\pi}{4\alpha}\right)e^{-(\alpha-\frac{1}{2})R_n}\\
&\le \frac{1+2\alpha}{4\alpha}\left(\frac{k_n^{1+\varepsilon}}{e^{y/2}+e^{y'/2}}\right)^{-(2\alpha-1)} = \Theta\left(k_n^{-\varepsilon(2\alpha-1)}\right),
\end{aligned}$$

where we used that $|x-x'|\le \pi e^{R_n/2}$. In a similar fashion we get

$$\begin{aligned}
\phi_n(p,p') &\ge \frac{1+2\alpha}{4\alpha}\left(\frac{\pi e^{R_n/2}}{e^{y/2}+e^{y'/2}}\right)^{-(2\alpha-1)} + \frac{2\alpha-1}{4\alpha}k_n^{1+\varepsilon}e^{-\alpha R_n} - e^{-(\alpha-\frac{1}{2})R_n}\\
&\ge \frac{1+2\alpha}{4\alpha}\left(\frac{\pi e^{R_n/2}}{e^{y/2}+e^{y'/2}}\right)^{-(2\alpha-1)} - e^{-(\alpha-\frac{1}{2})R_n} = \Theta\left(k_n^{2\alpha-1}n^{-(2\alpha-1)}\right).
\end{aligned}$$

Therefore, since $\mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P}}(p)), \mu_{\alpha,\nu}(\mathcal{B}_{\mathcal{P},n}(p')) = \Theta(k_n)$ for $p,p'\in\mathcal{K}_C(k_n)$, there exist two n-dependent constants

$$b_n^- = \Theta\left(k_n^{2\alpha}n^{-(2\alpha-1)}\right) \quad\text{and}\quad b_n^+ = \Theta\left(k_n^{1-\varepsilon(2\alpha-1)}\right)$$

such that

$$A_n(p,p') \subseteq \left\{t\,:\,(1-\delta_n)b_n^- \le t \le (1+\delta_n)b_n^+\right\} := B_n.$$

If we now define

$$\begin{aligned}
t_n^- &= \arg\min_{t\in B_n} \mathbb{P}(X_1(p,p')=k_n-t)\,\mathbb{P}(X_2(p,p')=k_n-t)\\
t_n^+ &= \arg\max_{t\in B_n} \mathbb{P}(X_1(p,p')=k_n-t)\,\mathbb{P}(X_2(p,p')=k_n-t),
\end{aligned}$$

then $t_n^\pm = o(k_n)$ and we get

$$\begin{aligned}
&\sum_{t\in A_n(p,p')} \mathbb{P}(X_3(p,p')=t)\,\mathbb{P}(X_1(p,p')=k_n-t)\,\mathbb{P}(X_2(p,p')=k_n-t)\\
&\quad\le \mathbb{P}\left(X_1(p,p')=k_n-t_n^+\right)\mathbb{P}\left(X_2(p,p')=k_n-t_n^+\right)\sum_{t\in A_n(p,p')}\mathbb{P}(X_3(p,p')=t)\\
&\quad= \mathbb{P}\left(X_1(p,p')=k_n-t_n^+\right)\mathbb{P}\left(X_2(p,p')=k_n-t_n^+\right)(1+o(1)).
\end{aligned} \tag{88}$$

and similarly

$$\sum_{t\in A_n(p,p')} \mathbb{P}(X_3(p,p')=t)\,\mathbb{P}(X_1(p,p')=k_n-t)\,\mathbb{P}(X_2(p,p')=k_n-t)$$

$$\geq \mathbb{P}\left(X_1(p,p') = k_n - t_n^-\right)\mathbb{P}\left(X_2(p,p') = k_n - t_n^-\right)(1+o(1)). \tag{89}$$

Next, it follows from Lemma 8.3 that

$$\mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p)\right)(1-\phi_n(p,p')) \leq \mathbb{E}\left[X_1(p,p')\right] \leq \mu_{\alpha,\nu}\left(\mathcal{B}_{\mathcal{P},n}(p)\right),$$

where

$$\sup_{p,p' \in \mathcal{E}_\varepsilon(k_n)} \phi_n(p,p') = O\left(k_n^{-\varepsilon(2\alpha-1)}\right).$$

This implies that on $\mathcal{E}_\varepsilon(k_n)$, $\mu_n^{(1)}(p) := \mathbb{E}\left[X_1(p,p')\right]$ satisfies the assumptions of Lemma 6.5. The same conclusion holds for $\mu_n^{(2)}(p')$. Therefore, if we write $\hat{\rho}_n^{(i)}(p,k) := \mathbb{P}\left(X_i(p,p') = k - t_n^{\pm}\right)$, the second statement of the Lemma 6.5 (applied twice) yields

$$\begin{aligned}
&\int_{\mathcal{E}_\varepsilon(k_n)} \hat{\rho}_n^{(1)}(p,k_n)\hat{\rho}_n^{(2)}(p',k_n)h(y)h(y')f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y' \\
&= (1+o(1))\int_{\mathcal{E}_\varepsilon(k_n)} \rho(y,k_n)\rho(y',k_n)h(y)h(y')f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y'.
\end{aligned}$$

Together with the approximation (87) and the bounds (88) and (89) we get

$$\begin{aligned}
&\int_{\mathcal{E}_\varepsilon(k_n)} \rho_n(p,p',k_n,k_n)h(y)h(y')f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y' \\
&= (1+o(1))\int_{\mathcal{E}_\varepsilon(k_n)} \rho(y,k_n)\rho(y',k_n)h(y)h(y')f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad + O(1)\,e^{-k_n^\varepsilon}\int_{\mathcal{E}_\varepsilon(k_n)} f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y',
\end{aligned}$$

and the result follows since the last term is of smaller order than the first.

□

Next we show that when $k_n \to \infty$ and the expected number of disjoint neighbors of p and p' in $\mathcal{K}_C(k_n)$ grows only slightly with k_n then any fixed shift in the joint degree distribution does not effect its asymptotic behavior.

Pim: @All: The proof of this lemma follows almost the exact same steps as that of the previous one. Is there any way we could merge them?

Lemma 8.6. *Let $\alpha > \frac{1}{2}$, $\nu > 0$, $k_n \to \infty$ and fix $\varepsilon > 0$. Then for any fixed $i, j, i', j' \in \mathbb{Z}$ and $p, p' \in \mathcal{K}_C(k_n)$ such that $\mathbb{E}\left[\mathcal{N}_{\mathcal{P},n}(p\Delta p')\right] \geq k_n^\varepsilon$,*

$$\rho_n(p,p',k_n+i,k_n+i') = (1+o(1))\rho_n(p,p',k_n+j,k_n+j') \pm e^{-\Omega(k_n^\varepsilon)}.$$

Proof. Define

$$\begin{aligned}
X_n &= \left|\mathcal{B}_{\mathcal{P},n}(p) \setminus \mathcal{B}_{\mathcal{P},n}(p')\right|, \\
Y_n &= \left|\mathcal{B}_{\mathcal{P},n}(p) \cap \mathcal{B}_{\mathcal{P},n}(p')\right|, \\
Z_n &= \left|\mathcal{B}_{\mathcal{P},n}(p') \setminus \mathcal{B}_{\mathcal{P},n}(p)\right|.
\end{aligned}$$

Then it follows that X_n, Y_n and Z_n are independent Poisson random variables satisfying $\mathbb{E}\left[X_n\right] + \mathbb{E}\left[Y_n\right] = \mu_{\alpha,\nu}\left(B_{\mathcal{P},n}(p)\right)$ and $\mathbb{E}\left[Z_n\right] + \mathbb{E}\left[Y_n\right] = \mu_{\alpha,\nu}\left(B_{\mathcal{P},n}(p')\right)$ while

$$\begin{aligned}
\rho_n(p,p',k_n+i,k_n+i') &= \mathbb{P}\left(X_n + Y_n = k+i, Z_n + Y_n = k+i'\right) \\
&= \sum_{\ell=0}^{\infty} \mathbb{P}\left(Y_n = \ell\right)\mathbb{P}\left(X_n = k+i-\ell, Z_n = k+i'-\ell\right)
\end{aligned}$$

$$= \sum_{\ell=0}^{\infty} \mathbb{P}\left(Y_n = \ell\right) \mathbb{P}\left(X_n = k + i - \ell\right) \mathbb{P}\left(Z_n = k + i' - \ell\right).$$

Next define $\delta_n = k_n^{-\frac{1-\varepsilon}{2}}$, let n be large enough such that $0 < \delta_n < 1$ and note that by a Chernoff bound,

$$\mathbb{P}\left(\left|X_n - \mathbb{E}\left[X_n\right]\right| > \delta_n \mathbb{E}\left[X_n\right]\right) = O\left(e^{-\frac{\delta_n^2}{4(1+\delta_n)}\mathbb{E}[X_n]}\right),$$

and similar for Y_n and Z_n. Finally, we define

$$\begin{aligned} L_X(k_n) &= \{\ell : (1-\delta_n)\mathbb{E}\left[X_n\right] \le k + i - \ell \le (1+\delta_n)\mathbb{E}\left[X_n\right]\} \\ L_y(k_n) &= \{\ell : (1-\delta_n)\mathbb{E}\left[Y_n\right] \le \ell \le (1+\delta_n)\mathbb{E}\left[Y_n\right]\} \\ L_Z(k_n) &= \{\ell : (1-\delta_n)\mathbb{E}\left[Z_n\right] \le k + i' - \ell \le (1+\delta_n)\mathbb{E}\left[Z_n\right]\} \end{aligned}$$

We will now make distinguish between the cases $\mathbb{E}\left[Y_n\right] \le k_n/2$ and $\mathbb{E}\left[Y_n\right] > k_n/2$.

Let us first assume that $\mathbb{E}\left[Y_n\right] \le k_n/2$. Then, since $p \in \mathcal{K}_\varepsilon(k_n)$ and $\mu_{\alpha,\nu}(B_{\mathcal{P},n}(p)) = \Theta\left(e^{y/2}\right) = \Theta\left(k_n\right)$ it follows that $\mathbb{E}\left[X_n\right] = \Omega(k_n)$ and hence

$$\mathbb{P}\left(\left|X_n - \mathbb{E}\left[X_n\right]\right| > \delta_n \mathbb{E}\left[X_n\right]\right) = O\left(e^{-\frac{\delta_n^2}{4(1+\delta_n)}\mathbb{E}[X_n]}\right) = e^{-\Omega(k_n^{(1+\varepsilon)/2})} = e^{-\Omega(k_n^\varepsilon)}.$$

In particular, this implies

$$\begin{aligned} &\sum_{\ell \notin L_X} \mathbb{P}\left(Y_n = \ell\right) \mathbb{P}\left(X_n = k + i - \ell\right) \mathbb{P}\left(Z_n = k + i' - \ell\right) \\ &\qquad = O\left(\mathbb{P}\left(\left|X_n - \mathbb{E}\left[X_n\right]\right| > \delta_n \mathbb{E}\left[X_n\right]\right)\right) = e^{-\Omega(k_n^{(1+\varepsilon)/2})}. \end{aligned}$$

Finally we note that, for $\ell \in L_X$, we have

$$\frac{\mathbb{P}\left(X_n = k + i - \ell\right)}{\mathbb{P}\left(X_n = k + j - \ell\right)} = \mathbb{E}\left[X_n\right]^{i-j} \frac{(k+j-\ell)!}{(k+i-\ell)!} \le (1+\delta_n)^{2|i-j|}. \tag{90}$$

and observe that we have similar results for Z_n. Therefore,

$$\begin{aligned} &\sum_{\ell \in L_X \cap L_Z} \mathbb{P}\left(Y_n = \ell\right) \mathbb{P}\left(X_n = k + i - \ell\right) \mathbb{P}\left(Z_n = k + i' - \ell\right) \\ &\quad \le (1+\delta_n)^{2(j-i)+2(j'-i')} \sum_{\ell \in L_x \cap L_Z} \mathbb{P}\left(Y_n = \ell\right) \mathbb{P}\left(X_n = k + j - \ell\right) \mathbb{P}\left(Z_n = k + j' - \ell\right) \\ &\quad = (1+o(1))(1+\delta_n)^{2|i-j|+2|i'-j'|} \mathbb{P}\left(D_p = k_n + j, D_{p'} = k_n + j'\right) \\ &\quad = (1+o(1)) \mathbb{P}\left(D_p = k_n + j, D_{p'} = k_n + j'\right) \end{aligned}$$

and hence

$$\rho_n(p, p', k+i, k'+i') = (1+o(1))\rho_n(p, p', k+j, k'+j') + e^{-\Omega(k_n^\varepsilon)}.$$

Now assume that $\mathbb{E}\left[Y_n\right] > k_n/2$. Then, since $\mathbb{E}\left[|\mathcal{N}_{\mathcal{P},n}(p\Delta p')|\right] \ge k_n^\varepsilon$ it follows that $\mathbb{E}\left[X_n\right] = \Omega(k_n^\varepsilon)$ or $\mathbb{E}\left[Z_n\right] = \Omega(k_n^\varepsilon)$. Without loss of generality we assume that $\mathbb{E}\left[X_n\right] = \Omega(k_n^\varepsilon)$. Similar to (90) we have for Y_n

$$\frac{\mathbb{P}\left(Y_n = \ell\right)}{\mathbb{P}\left(Y_n = \ell + j' - i'\right)} \le (1+\delta_n)^{2(|i'-j'|)}.$$

Using similar computations as above we then have

$$\begin{aligned} &\rho_n(p, p', k+i, k'+i') \\ &\quad = \sum_{\ell \in L_X \cap L_Y} \mathbb{P}\left(Y_n = \ell\right) \mathbb{P}\left(X_n = k + i - \ell\right) \mathbb{P}\left(Z_n = k + i' - \ell\right) + e^{-\Omega(k_n^{(1+\varepsilon)/2})} \\ &\quad = (1+o(1))\rho_n(p, p', k+j, k'+j') + e^{-\Omega(k_n^{(1+\varepsilon)/2})}. \end{aligned}$$

$\square$

8.3 Concentration for main triangle contribution

We now turn to Proposition 8.1. Before we dive into the proof let us first give a high level overview of the strategy and the flow of the arguments.

Recall (see (76)) that for any $C > 0$

$$\widetilde{T}_{\mathcal{P},n}(k, C) = \sum_{p \in \mathcal{P}_n \cap \mathcal{K}_C(k)} \mathbb{1}_{\{D_{\mathcal{P},n}(p)=k\}} \widetilde{T}_{\mathcal{P},n}(p)$$

Then we have

$$\widetilde{T}_{\mathcal{P},n}(k, C)^2 = \sum_{p,p' \in \mathcal{P}_n \cap \mathcal{K}_C(k)} \mathbb{1}_{\{D_{\mathcal{P},n}(p),\, D_{\mathcal{P},n}(p')=k\}} \sum_{(p_1,p_2),(p_1',p_2') \in \mathcal{P}_n \setminus (0,y)}^{\neq} \widetilde{T}_{\mathcal{P}}(p, p_1, p_2) \widetilde{T}_{\mathcal{P}}(p', p_1', p_2'),$$

This expression can be written as the sums of several terms, depending on how $\{p, p_1, p_2\}$ and $\{p', p_1', p_2'\}$ intersect. To this end we define, for $a \in \{0, 1\}$ and $b \in \{0, 1, 2\}$,

$$I_{a,b} = \sum_{\substack{p,p' \in \mathcal{P}_n \cap \mathcal{K}_C(k) \\ |\{p\} \cap \{p'\}| = a}} \mathbb{1}_{\{D_{\mathcal{P},n}(p),\, D_{\mathcal{P},n}(p')=k\}} J_b(p, p'),$$

where

$$J_b(p, p') = \sum_{\substack{p_1,p_2,p_1',p_2' \in \mathcal{P}_n \\ |\{p_1,p_2\} \cap \{p_1',p_2'\}| = b}}^{\neq} T_{\mathcal{P},n}(p, p_1, p_2) T_{\mathcal{P},n}(p', p_1', p_2'),$$

with the summation being over all two distinct pairs (p_1, p_2) and (p_1', p_2'). Then we have

$$\widetilde{T}_{\mathcal{P},n}(k, C)^2 = \sum_{a=0}^{1} \sum_{b=0}^{2} I_{a,b}.$$

To prove Proposition 8.1 we will deal with each of the $I_{a,b}$ separately, showing that

$$\mathbb{E}[I_{0,0}] = (1 + o(1)) \mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n)\right]^2 \tag{91}$$

and for all other combinations

$$\mathbb{E}[I_{a,b}] = o\left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n)\right]^2\right). \tag{92}$$

Note that $J_b(p, p') \le J_0(p, p')$ and, since $I_{1,2} = \widetilde{T}_{\mathcal{P},n}(k_n, C)$, (92) holds for $I_{1,2}$.

To prove (67) and (68) we have to consider the cases $k_n = \Theta(1)$ and $k_n \to \infty$ separately. However, each case follows a similar strategy. First we define a set $\mathcal{E} \subseteq \mathcal{K}_C(k_n) \times \mathcal{K}_C(k_n)$ such that outside this set the contributions of $\mathbb{E}[I_{a,b}]$ are negligible while on this set the joint degree distribution factorizes. These will be the sets defined by (85) and (86). In particular, for these sets we have that $a = 1$ is not possible and hence we only need to consider the three different cases for b. For $a = 0 = b$, the factorization of the degree distributions will enable us to prove (67). For $b = 1, 2$ we derive sufficient bounds for $\mathbb{E}[J_b(p, p')]$ in terms of k_n for (p, p') in the set $\mathcal{E}$.

Proof of Proposition 8.1. Throughout this proof we set $i = |\{p', p_1, p_2, p_1', p_2'\} \cap \mathcal{B}_{\mathcal{P},n}(p)|$, $j = |\{p'\} \cap \mathcal{B}_{\mathcal{P},n}(p)|$ and define i', j' in a similar way by interchanging the primed and non-primed variables. In addition, we write $D_{\mathcal{P},n}(p, p', k, \ell)$ to denote the indicator that $|\mathcal{B}_{\mathcal{P},n}(p) \cap (\mathcal{P}_n \setminus \{p, p', p_1, p_2, p_1', p_2'\})| = k$ and $|\mathcal{B}_{\mathcal{P},n}(p') \cap (\mathcal{P}_n \setminus \{p, p', p_1, p_2, p_1', p_2'\})| = \ell$. Then,

$$\mathbb{E}\left[\mathbb{1}_{\{D_{\mathcal{P},n}(p)=k_n, D_{\mathcal{P},n}(p')=k_n\}} J_b(p, p')\right]$$

$$= \mathbb{E}\left[\sum_{\substack{p_1,p_2,p_1',p_2' \in \mathcal{P}_n \\ |\{p_1,p_2\}\cap\{p_1',p_2'\}|=b}}^{\neq} D_{\mathcal{P},n}(p,p',k_n-i,k_n-i')\,\widetilde{T}_{\mathcal{P},n}(p,p_1,p_2)\widetilde{T}_{\mathcal{P},n}(p',p_1',p_2')\right],$$

where the sum is over all distinct pairs (p_1,p_2) and (p_1',p_2'). We also that

$$\mathbb{E}\left[T_{\mathcal{P}}(k_n)\right] = \Theta\left(nk_n^{-(2\alpha-1)}s_\alpha(k_n)\right).$$

Recall the definition of $\mathcal{E}_\varepsilon(k_n)$

$$\mathcal{E}_\varepsilon(k_n) = \left\{(p,p') \in \mathcal{K}_C(k_n)\times\mathcal{K}_C(k_n) \,:\, \mathbb{E}\left[\left|\mathcal{N}_{\mathcal{P},n}^c(p,p')\right|\right] \ge k_n^\varepsilon \text{ and } |x-x'| > k_n^{1+\varepsilon}\right\}$$

let $\mathcal{E}_\varepsilon(k_n)^c$ denote its complement and let $I_{a,b}^*$ denote the the part of $I_{a,b}$ where $p,p' \in \mathcal{P}_n \cap \mathcal{E}_\varepsilon(k_n)$. We first show that

$$\mathbb{E}\left[I_{a,b}-I_{a,b}^*\right] = o\left(\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]^2\right), \tag{93}$$

so that for the remainder of the proof we only need to consider $p,p' \in \mathcal{E}_\varepsilon(k_n)$ and hence, we can apply Lemma 8.5. For this we note that by Lemma 8.2 and Lemma 8.3 we have for $p,p' \in \mathcal{K}_\varepsilon(k_n)$ that $\mathbb{E}\left[|\mathcal{N}_{\mathcal{P},n}(p\Delta p')|\right] \le k_n^\varepsilon$ implies that eventually and $|x-x'| \le k_n^{1+\varepsilon}$. In particular $|x-x'| \le k_n^{1+\varepsilon}$ for all $(p,p') \in \mathcal{E}_\varepsilon(k_n)^c$. Therefore, we have

$$\begin{aligned}
&\mathbb{E}\left[I_{a,b}-I_{a,b}^*\right] \\
&\le \int_{\mathcal{K}_C(k_n)^2\setminus\mathcal{E}_\varepsilon(k_n)} \rho(p,p',k-i,k-i')\mathbb{E}\left[J_b(p,p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy \\
&\le \int_{\mathcal{K}_C(k_n)^2\setminus\mathcal{E}_\varepsilon(k_n)} \rho(p,p',k-i,k-i')\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy \\
&= O\left(\int_{\mathcal{K}_C(k_n)^2} \mathbb{1}_{\left\{|x-x'|\le k_n^{1+\varepsilon}\right\}}\rho_y(k)\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy\right) \\
&= O\left(k_n^{1+\varepsilon}\binom{k_n}{2}\left(\int_{a_n^-}^{a_n^+}\Delta_{\mathcal{P}}(y')e^{-\alpha y'}\,dy'\right)\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]\right) \\
&= O\left(k_n^{3+\varepsilon-2\alpha}s_\alpha(k_n)\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]\right) \\
&= o\left(nk_n^{-(2\alpha-1)}s_\alpha(k_n)\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]\right) = o\left(\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right]^2\right),
\end{aligned}$$

which proves (93). Here we used that $k_n^{2+\varepsilon} = o(n)$ and $\mathbb{E}\left[T_{\mathcal{P},n}(k_n)\right] = \Theta\left(nk_n^{-(2\alpha-1)}s_\alpha(k_n)\right)$ for the last line.

We will now proceed to establish (91) and (92). We start with $I_{0,0}^*$

By Lemma 8.6

$$\rho_(p,p',k_n-i,k_n-i') = (1+o(1))\rho(p,p',k_n-j,k_n-j') = (1+o(1))\rho_n(p,p',k_n,k_n)$$

which now no longer depends on the other four points p_1,p_2,p_1',p_2'. Hence, using the Campbell-Mecke formula, we get

$$\mathbb{E}\left[I_{0,0}^*\right] = (1+o(1))\int_{\mathcal{E}_\varepsilon(k_n)} \rho_n(p,p',k_n,k_n)\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy,$$

Next, by Corollary 7.4 we have for $y \in K_C(k_n)$,

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right] = (1+o(1))\mathbb{E}\left[\widetilde{T}_{\mathcal{P}}(p)\right] = (1+o(1))\binom{k_n}{2}\Delta_{\mathcal{P}}(y)$$

and similar for p'. Hence, by Lemma 8.5

$$\begin{aligned}
&(1+o(1))\int_{\mathcal{E}_\varepsilon(k_n)} \rho_n(p,p',k_n,k_n)\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy \\
&\quad = (1+o(1))\binom{k_n}{2}^2 \int_{\mathcal{E}_\varepsilon(k_n)} \rho(y,k_n)\rho(y',k_n)\Delta_{\mathcal{P}}(y)\Delta_{\mathcal{P}}(y')f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy \\
&\quad = (1+o(1))\left(\binom{k_n}{2}\int_{a_n^-}^{a_n^+}\int_{I_n} \rho(y,k_n)\Delta_{\mathcal{P}}(y)f_{\alpha,\nu}(x,y)\,dx\,dy\right)^2 \\
&\quad = (1+o(1))\left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(k_n,C)\right]\right)^2,
\end{aligned}$$

which proves (91).

Next we consider $I^*_{0,1}$. The proofs for the other two cases $I^*_{1,1}$ and $I^*_{0,2}$ follow using similar arguments and hence we omit them here.

Without loss of generality we will assume that $p_1 = p'_1$. Notice that now $i = |\{p', p_1, p_2, p'_2\} \cap B_{\mathcal{P},n}(p)|$ and $j = |\{p'\} \cap B_{\mathcal{P},n}(p)|$ and i', j' are defined, similarly, by interchanging the primed and non-primed variables. Then, if we define

$$T^{(0,1)}_{\mathcal{P},n}(p,p') = \sum_{(p_1,p_2)\in 2^{\mathcal{P}_n}} \sum_{p'_2\in\mathcal{P}_n} \widetilde{T}_{\mathcal{P},n}(p,p_1,p_2)\widetilde{T}_{\mathcal{P},n}(p',p_1,p'_2),$$

we have

$$\mathbb{E}\left[I^*_{0,1}\right] = (1+o(1))\int_{\mathcal{E}_\varepsilon(k_n)} \rho_n(p,p',k_n,k_n)\mathbb{E}\left[T^{(0,1)}_{\mathcal{P},n}(p,p')\right] f_{\alpha,\nu}(x,y)f_{\alpha,\nu}(x',y')\,dx'\,dx\,dy'\,dy,$$

where we again used Lemma 8.6. We will show that

$$\mathbb{E}\left[T^{(0,1)}_{\mathcal{P},n}(p,p')\right] = o\left(k_n^4 s_\alpha(k_n)^2\right),$$

from which (92) follows since $k_n^4 s_\alpha(k_n)^2 = O\left(\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p')\right]\right)$ on $\mathcal{K}_C(k_n)\times\mathcal{K}_C(k_n)$.

First we consider the contribution coming from $y_1 > 4\log(k_n)$. Since the integration of $T_{\mathcal{P}}(p,p_1,p_2)T_{\mathcal{P}}(p',p_1,p'_2)$ over x_1, x_2 and x'_2 is bounded by $O\left(e^y e^{\frac{y'}{2}} e^{\frac{y_1+y_2+y'_2}{2}}\right)$ it follows that contribution to $\mathbb{E}\left[T^{(0,1)}_{\mathcal{P},n}(p,p')\right]$ is bounded by

$$\begin{aligned}
O\left(e^y e^{\frac{y'}{2}}\int_{4\log(k_n)}^{a_n^+} e^{-(\alpha-\frac{1}{2})y_1}\,dy_1\right) &= O\left(k_n^3\int_{4\log(k_n)}^{a_n^+} e^{-(\alpha-\frac{1}{2})y_1}\,dy_1\right) \\
&= O\left(k_n^{3-(4\alpha-2)}\right) = o\left(k_n^4 s_\alpha(k_n)^2\right).
\end{aligned}$$

To deal with the case where $y_1 \le 4\log(k_n)$ we define $b_n = 2\varepsilon\log(k_n))$ and will consider different cases for $\mathbb{E}\left[T^*_{\mathcal{P},n}(p,p')\right]$, depending on whether $y_2 \le b_n$ or $y_2 > b_n$ and similar for y'_2.

When $y_1 \le 4\log(k_n)$ and $y_2 > b_n$, the contribution to $\mathbb{E}\left[T^{(0,1)}_{\mathcal{P},n}(p,p')\right]$ is bounded by

$$\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right] O\left(e^{\frac{y'}{2}}\int_{b_n}^{a_n^+} e^{-(\alpha-\frac{1}{2})y_2}\,dy_2\right) = O\left(k_n^{1-\varepsilon}\right)\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right] = o\left(k_n^4 s_\alpha(k_n)^2\right).$$

Due to the symmetry in p_2 and p'_2 the same results holds for the cases where $y_2 > b_n$.

Finally, when $y_1 \le 4\log(k_n)$ and both $y_2, y'_2 \le b_n$ we have that

$$|x_2 - x'_2| \le |x_1 - x_2| + |x_1 - x'_2| \le e^{\frac{y_1}{2}}\left(e^{\frac{y_2}{2}} + e^{\frac{y'_2}{2}}\right) \le 2k_n^{2+\varepsilon}$$

whenever $T_{\mathcal{P}}(p,p_1,p_2)T_{\mathcal{P}}(p',p_1,p_2') > 0$ while both $|x-x_2|, |x'-x_2'| = O\left(k_n^{1+\varepsilon}\right)$. Hence it follows that

$$|x-x'| \le |x-x_2| + |x_2-x_2'| + |x_2'-x'| = O\left(k_n^{2+\varepsilon}\right).$$

Next, by integrating only over x_2' and y_2' we get the contribution to $\mathbb{E}\left[T_{\mathcal{P},n}^{(0,1)}(p,p')\right]$ for this regime is bounded by

$$O\left(e^{\frac{y'}{2}}\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\right) = O\left(k_n\mathbb{E}\left[\widetilde{T}_{\mathcal{P},n}(p)\right]\right) = o\left(k_n^4 s_\alpha(k_n)^2\right).$$

□

9 Equivalence for local clustering in hyperbolic and Poissonized random graph

In this section we establish the equivalence between $c^*_{\mathbb{H},n}(k)$ and $c^*_{\mathcal{P},n}(k)$ as expressed in Proposition 5.3, using the coupling procedure explained in Section 2.5.

Recall that $\mathcal{P}_{\alpha,\nu}$ denotes a Poisson process on $\mathbb{R}\times\mathbb{R}_+$, with intensity $f_{\alpha,\nu}(x,y)$, $I_n = \left(-\frac{\pi}{2}e^{R_n/2}, \frac{\pi}{2}e^{R_n/2}\right)$, $\mathcal{R}_n = I_n \times (0,R_n]$ and $\mathcal{V}_n = \mathcal{P}_{\alpha,\nu}\cap\mathcal{R}_n$. In addition we define for any interval $I \subseteq \mathbb{R}_+$, $\mathcal{R}_n(I) := I_n \times I$ and denote by $\mathcal{B}_{\mathcal{P},n}(p)$ the *ball*

$$\mathcal{B}_{\mathcal{P},n}(p) = \left\{p' \in \mathcal{V}_n : |x-x'|_{\pi e^{R_n/2}} < e^{\frac{y+y'}{2}}\right\}.$$

Note that when $p \in \mathcal{V}_n$ then $\mathcal{B}_{\mathcal{P},n}(p)$ denotes its neighborhood in the graph $G_{\mathcal{P},n}$. Note that the above definition implies that for all $y \in [0,R_n]$ we have

$$\mathcal{R}_n([R_n - y - 2\ln(\pi/2), R_n]) \subseteq \mathcal{B}_{\mathcal{P},n}((0,y)) \tag{94}$$

- this is a fact which we are going to use several times in our analysis. **Pim:** @All: Maybe we should either add figure to illustrate this or we could refer to a previous figure.

For any Borel-measurable subset $S \subseteq \mathbb{R}\times\mathbb{R}_+$, we let

$$\mu_{\alpha,\nu}(S) = \int_S f_{\alpha,\nu}(x,y)\,dx\,dy = \frac{\nu\alpha}{\pi}\int_S e^{-\alpha y}dy.$$

Thus, the number of points of $\mathcal{P}_{\alpha,\nu}$ inside S is distributed as $\mathrm{Po}(\mu_{\alpha,\nu,}(S))$.

Finally, we recall the map Ψ from (16)

$$\Psi(r,\theta) = \left(\theta\frac{e^{R_n/2}}{2}, R_n - r\right),$$

and remind the reader that $\mathcal{B}_{\mathbb{H},n}(p)$ denotes the image under Ψ of the ball of hyperbolic radius R_n around the point $\Psi^{-1}(p)$ and that under the coupling between the hyperbolic random graph and the finite box model, described in Section 2.5, two points p and p' are connected if and only if

$$|x-x'|_{\pi e^{r_n/2}} \le \Omega(R_n - y, R_n - y'),$$

where the function Ω can be approximated, for $y + y' < R_n$, using Lemma 2.2 by

$$e^{\frac{1}{2}(y+y')} - Ke^{\frac{3}{2}(y+y')-R_n} \le \Omega(R_n-y,R_n-y') \le e^{\frac{1}{2}(y+y')} + Ke^{\frac{3}{2}(y+y')-R_n}.$$

9.1 Some results on the hyperbolic geometric graph

We start with some basic results for the hyperbolic random geometric graph. Observe that (20) from Lemma 2.2 implies the following.

Corollary 9.1.

$$\mathcal{B}_{\mathcal{P}}(p)\cap\mathcal{R}_n([K,R_n]) \subseteq \mathcal{B}_{\mathbb{H},n}(p)\cap\mathcal{R}_n(K,R_n).$$

Furthermore, Lemma 2.2 enables us to determine the measure of a ball around a given point $p = (0, y)$ - this is will be fairly useful in our subsequent analysis.

Let $p \in \mathcal{R}_n$. Then we can see that the curve $x' = e^{\frac{1}{2}(y+y')}$ with $x' \geq 0$ meets the right boundary of $\mathcal{R}_N$, that is, the line $x' = \frac{\pi}{2}e^{R_n/2}$ at $y' = R_n - y + 2\ln\frac{\pi}{2}$. Hence, any point $p' \in \mathcal{R}_n([R_n - y + 2\ln\frac{\pi}{2}, R_n])$ is included in $\mathcal{B}_{\mathcal{P}}(p)$. In other words,

$$\mathcal{B}_{\mathcal{P}}(p) \cap \mathcal{R}_n([R_n - y + 2\ln\frac{\pi}{2}, R_n]) = \mathcal{R}_n([R_n - y + 2\ln\frac{\pi}{2}, R_n]).$$

This together with (17) implies that

$$(\mathcal{B}_{\mathbb{H},n}(p) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p)) \cap \mathcal{R}([R_n - y + 2\ln\frac{\pi}{2}, R_n]) = \emptyset, \tag{95}$$

where $A \bigtriangleup B$ denotes the symmetric difference. We can now compute the expected number of points in $\mathcal{B}_{\mathbb{H},n}(p) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p)$, i.e. those that belong are a neighbor of p in only one of the two models.

Lemma 9.2. *Let $0 \leq y_n < R_n$ be such that $R_n - y_n \to \infty$ and write $p_n = (x_n, y_n)$. Then we have, as $n \to \infty$,*

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(p_n) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p_n)) = \Theta(1) \cdot \begin{cases} e^{(1/2-\alpha)R_n + \alpha y_n}, & \text{if } \alpha < 3/2 \\ (R_n - y_n)e^{3y/2 - R_n}, & \text{if } \alpha = 3/2 \\ e^{3y_n/2 - R_n}, & \text{if } \alpha > 3/2 \end{cases}.$$

Proof. Let $r_n := R_n - y$. Lemma 2.2 implies that for such a p_n, if a point p belongs to $\mathcal{B}_{\mathbb{H},n}(p_n) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p_n) \cap \mathcal{R}([0, r_n])$ then

$$|x_n - x| = \Theta(1) \cdot e^{\frac{3}{2}(y_n + y) - R_n}.$$

Now, if $p \in [r_n, r_n + 2\ln\frac{\pi}{2})]$ and also $p \in \mathcal{B}_{\mathbb{H},n}(p_n) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p_n)$, then

$$|x_n - x| = \frac{\pi}{2}e^{R_n/2} - e^{\frac{1}{2}(y_n + y)}.$$

Finally, (95) implies that no point in $\mathcal{R}([r_n + 2\ln\frac{\pi}{2}, R_n])$ belongs to $\mathcal{B}_{\mathbb{H},n}(p_n) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p_n)$. We first compute the expected number of points in $p \in \mathcal{B}_{\mathbb{H},n}(p_n) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p_n)$ that have $R_n - y \leq r_n$. The result depends on the value of α, yielding the following three cases

$$\begin{aligned} \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(p_n) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p_n) \cap \mathcal{R}([0, r_n])) &= \Theta(1) \cdot e^{3y_n/2 - R_n} \int_0^{r_n} e^{(3/2-\alpha)y}\, dy \\ &= \Theta(1) \cdot \begin{cases} e^{(1/2-\alpha)R_n + \alpha y_n}, & \text{if } \alpha < 3/2 \\ (R_n - y_n)e^{3y_n/2 - R_n}, & \text{if } \alpha = 3/2 \\ e^{3y_n/2 - R_n}, & \text{if } \alpha > 3/2 \end{cases}. \end{aligned}$$

Next we compute the number of remaining points in $\mathcal{B}_{\mathbb{H},n}(p_n) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p_n)$,

$$\begin{aligned} \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(p_n) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p_n) \cap \mathcal{R}([r_n, R_n])) &= \frac{\nu\alpha}{\pi} \int_{r_n}^{r_n + 2\ln\frac{\pi}{2}} \left(\frac{\pi}{2}e^{R_n/2} - e^{\frac{1}{2}(y_n + y)}\right) e^{-\alpha y}\, dy \\ &= O(1) \cdot e^{R_n/2} \int_{r_n}^{r_n + 2\ln\frac{\pi}{2}} e^{-\alpha y}\, dy = O(1) \cdot e^{R_n/2} e^{-\alpha r_n} \\ &= O(1) \cdot e^{(1/2-\alpha)R_n + \alpha y_n}. \end{aligned}$$

Now note that for any $\alpha > 3/2$, we have

$$((1/2 - \alpha)R_n + \alpha y_n) - (3y_n/2 - R_n) = (3/2 - \alpha)(R_n - y_n) \to -\infty,$$

by our assumption on y_n. For $\alpha = 3/2$, these two quantities are equal. From these observations, we deduce that

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(p_n) \bigtriangleup \mathcal{B}_{\mathcal{P}}(p_n)) = \Theta(1) \cdot \begin{cases} e^{(1/2-\alpha)R_n + \alpha y_n}, & \text{if } \alpha < 3/2 \\ r_n e^{3y_n/2 - R_n}, & \text{if } \alpha = 3/2 \\ e^{3y_n/2 - R_n}, & \text{if } \alpha > 3/2 \end{cases}.$$

$\square$

9.2 Equivalence clustering $G_{\widetilde{\mathbb{H}},n}(\alpha,\nu)$ and $G_{\mathcal{P},n}(\alpha,\nu)$

Here we prove Proposition 5.3. We first establish a few results regarding the number of nodes of degree k_n in both the **Tobias:** Poissonized? hyperbolic random graph $G_{\widetilde{\mathbb{H}},n}$ and the finite box model $G_{\mathcal{P},n}$.

Lemma 9.3. *Let $\alpha > 1/2$, $\nu > 0$ and $\{k_n\}_{n\geq 1}$ be a sequence such that $k_n = O\left(n^{1/(2\alpha+1)}\right)$. Then*

$$\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right] = \Theta(1)\, n k_n^{-(2\alpha+1)}, \tag{96}$$

and

$$\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] = \Theta(1)\, n k_n^{-(2\alpha+1)}. \tag{97}$$

Moreover,

$$\lim_{n\to\infty}\left|\frac{\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]}{\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right]} - 1\right| = 0. \tag{98}$$

Proof. Recall that

$$\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] = \int_{\mathcal{R}_n} \rho_{\widetilde{\mathbb{H}},n}(y,k_n) f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Then by Lemma 6.5 and a concentration argument **Tobias:** Please provide details for the "concentration argumnent".

$$\begin{aligned}\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] &= (1+o(1))\int_{\mathcal{R}_n} \rho(y,k_n) f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= (1+o(1))n\int_0^{R_n} \rho(y,k_n) f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \Theta(1)\, n k_n^{-(2\alpha+1)}.\end{aligned}$$

Similarly,

$$\mathbb{E}\left[N_{\mathcal{P},n}(k_n)\right] = (1+o(1))\int_{\mathcal{R}_n} \rho(y,k_n) f_{\alpha,\nu}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

From which both (97) and (98) follow. □

Recall that Proposition 5.3 states

$$\lim_{n\to\infty} s_\alpha(k_n)^{-1}\,\mathbb{E}\left[\left|c^*_{\widetilde{\mathbb{H}},n}(k_n) - c^*_{\mathcal{P},n}(k_n)\right|\right] = 0.$$

Since for $\alpha > 3/4$, $s_{3/4}(k_n) = \log(k_n)^{-1} s_\alpha(k_n) = o\left(s_\alpha(k_n)\right)$ it suffices to prove the following two cases:

1. if $1/2 < \alpha \leq 3/4$, then

$$\lim_{n\to\infty} k_n^{4\alpha-2}\cdot\mathbb{E}\left[\left|c^*_{\widetilde{\mathbb{H}}}(k_n) - c^*_{\mathcal{P}}(k_n)\right|\right] = 0,$$

2. if $3/4 < \alpha$, then

$$\lim_{n\to\infty} k_n\cdot\mathbb{E}\left[\left|c^*_{\widetilde{\mathbb{H}}}(k_n) - c^*_{\mathcal{P}}(k_n)\right|\right] = 0.$$

Recall the definition of $\mathcal{K}_C(k_n)$

$$\mathcal{K}_C(k_n) = \left\{p\in\mathbb{R} : \frac{k_n - C\kappa_n}{\xi_{\alpha,\nu}} \vee 0 \leq e^{\frac{y}{2}} \leq \frac{k_n + C\kappa_n}{\xi_{\alpha,\nu}} \wedge e^{R_n/2}\right\},$$

with $C > 0$ and

$$\kappa_n := \begin{cases}\log(n) & \text{if } k_n = \Theta(1), \\ \sqrt{k_n\log(k_n)} & \text{else.}\end{cases}$$

The following lemma will be frequently used in the proof of Proposition 5.3.

Lemma 9.4. *Let $t, r \in \mathbb{R}$ be fixed and let $\hat{\rho}(y,k)$ be any of the three probability functions $\rho_{\widetilde{\mathbb{H}},n}(y,k), \rho_{\mathcal{P},n}(y,k)$ or $\rho(y,k)$. Then for any sequence k_n of positive integers with $k_n = O\left(n^{\frac{1}{2\alpha+1}}\right)$ and $C > 0$ large enough,*

$$\int_{\mathcal{K}_C} e^{ty} \hat{\rho}_n(y, k_n - r) e^{-\alpha y} \, dy = O(1) \, n k_n^{-2\alpha-1+2t}$$

as $n \to \infty$.

Proof. Note that on $\mathcal{K}_C(k_n)$ we have that $e^{ty} = \Theta\left(k_n^{2t}\right)$. Hence, by a concentration argument

$$\begin{aligned}\int_{\mathcal{K}_C} e^{ty} \hat{\rho}_n(y, k_n - r) e^{-\alpha y} \, dy &= \Theta\left(k_n^{2t}\right) \int_{\mathcal{K}_C} \hat{\rho}_n(y, k_n - r) e^{-\alpha y} \, dy \\ &= O\left(k_n^{2t}\right) n \mathbb{E}\left[N_{\mathcal{P}}(k_n)\right] = O(1) \, n k_n^{-2\alpha-1+2t}.\end{aligned}$$

□

Proof of Proposition 5.3. To keep notations concise we abbreviate $\mathbb{E}\left[N_{\widetilde{\mathbb{H}}}(k_n)\right]$ and $\mathbb{E}\left[N_{\mathcal{P}}(k_n)\right]$ by $\overline{n}_{\widetilde{\mathbb{H}}}(k_n)$ and $\overline{n}_{\mathcal{P}}(k_n)$, respectively. We will also suppress the subscripts n in most expression regarding the graphs $G_{\widetilde{\mathbb{H}},n}$ and $G_{\mathcal{P},n}$. Then we have

$$\begin{aligned}\mathbb{E}\left[\left|c^*_{\widetilde{\mathbb{H}}}(k_n) - c^*_{\mathcal{P}}(k_n)\right|\right] &= \binom{k_n}{2}^{-1} \mathbb{E}\left[\left|\sum_{p \in \mathcal{P}} \frac{\mathbb{1}_{\{D_{\widetilde{\mathbb{H}}}(p)=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)} \Delta_{\widetilde{\mathbb{H}}}(p) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}(p)=k_n\}}}{\overline{n}_{\mathcal{P}}(k_n)} \Delta_{\mathcal{P}}(p)\right|\right] \\ &\leq \binom{k_n}{2}^{-1} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \mathbb{E}\left[\left|\sum_{p \in \mathcal{P}} \mathbb{1}_{\left\{D_{\widetilde{\mathbb{H}}}((0,y))=k_n\right\}} \Delta_{\widetilde{\mathbb{H}}}(p) - \mathbb{1}_{\{D_{\mathcal{P}}(p)=k_n\}} \Delta_{\mathcal{P}}(p)\right|\right] \\ &\quad + \binom{k_n}{2}^{-1} \left|\frac{1}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)} - \frac{1}{\overline{n}_{\mathcal{P}}(k_n)}\right| \mathbb{E}\left[\sum_{p \in \mathcal{P}} \mathbb{1}_{\{D_{\mathcal{P}}(p)=k_n\}} \Delta_{\mathcal{P}}(p)\right]\end{aligned}$$

The last term can be rewritten as

$$\left|1 - \frac{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}{\overline{n}_{\mathcal{P}}(k_n)}\right| \mathbb{E}\left[c^*_{\mathcal{P}}(k_n)\right] = \left|1 - \frac{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}{\overline{n}_{\mathcal{P}}(k_n)}\right| c_\infty(k_n)(1 + o(1)),$$

where we used Proposition 5.5 (See Section 7). The first term in this product converges to zero by Lemma 9.3 while the second term scales as $s_\alpha(k_n)$ by Theorem 1.3. Hence

$$\left|1 - \frac{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}{\overline{n}_{\mathcal{P}}(k_n)}\right| \mathbb{E}\left[c^*_{\mathcal{P}}(k_n)\right] = o\left(s_\alpha(k_n)\right),$$

and therefore we are left to analyze the other term. By the Campbell-Mecke formula (10) we have that

$$\begin{aligned}&\mathbb{E}\left[\left|\sum_{p \in \mathcal{P}} \frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)} \Delta_{\mathbb{H}}(p) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}(p)=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)} \Delta_{\mathcal{P}}(p)\right|\right] \\ &= \int_{\mathcal{R}_n} \mathbb{E}\left[\left|\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)} \Delta_{\mathbb{H}}((0,y)) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)} \Delta_{\mathcal{P}}((0,y))\right|\right] f_{\alpha,\nu}(x,y) \, dy \, dx.\end{aligned}$$

Since

$$\begin{aligned}\mathbb{E}\left[\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)} \Delta_{\mathbb{H}}((0,y))\right] &\leq \binom{k_n}{2} \rho_{\widetilde{\mathbb{H}}}(y, k_n) \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \\ &= \binom{k_n}{2} \rho_{\widetilde{\mathbb{H}}}(y, k_n) \Theta\left(\overline{n}_{\mathcal{P}}(k_n)^{-1}\right)\end{aligned}$$

$$= \Theta\left(n^{-1}k_n^{2\alpha+3}\right)\rho_{\widetilde{\mathbb{H}},n}(y,k_n)$$

and similar for the other term, it follows that

$$\mathbb{E}\left[\left|\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathbb{H}}((0,y)) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathcal{P}}((0,y))\right|\right]$$
$$\leq \Theta\left(n^{-1}k_n^{2\alpha+3}\right)\left(\rho_{\widetilde{\mathbb{H}},n}(k,n) + \rho_n(y,k_n)\right).$$

Therefore, by a concentration argument (c.f. Corollary 6.6), it is enough to consider the integral

$$\int_{\mathcal{K}_C(k_n)} \mathbb{E}\left[\left|\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathbb{H}}((0,y)) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathcal{P}}((0,y))\right|\right] e^{-\alpha y}\, dy\, dx, \tag{99}$$

where we also used that $f_{\alpha,\nu}(x,y)$ is simply a constant multiple of the function $e^{-\alpha y}$. We shall proceed by expanding the integrand and analyzing the individual terms. With a slight abuse of notation we shall write y instead of $(0,y)$ in expression such as $\mathcal{B}_{\mathbb{H},n}(y)$. In addition we write $D_{\mathbb{H}}(y,k_n;\mathcal{P})$ for the indicator which is equal to 1 if and only if $\mathcal{B}_{\mathbb{H},n}((0,y))$ contains k_n points from $\mathcal{P}\setminus\{(0,y)\}$. We define $D_{\mathcal{P}}(y,k_n;\mathcal{P})$ analogously for the ball $\mathcal{B}_{\mathcal{P}}((0,y))$.

We need to split the integrand over several terms and then analyze each of these separately. Applying the Campbel-Mecke formula (10) yields

$$\mathbb{E}\left[\left|\frac{\mathbb{1}_{\{D_{\mathbb{H}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathbb{H}}((0,y)) - \frac{\mathbb{1}_{\{D_{\mathcal{P}}((0,y))=k_n\}}}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathcal{P}}((0,y))\right|\right] \leq$$
$$\mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{P}\setminus\{(0,y)\}}^{\neq}\left|\frac{D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathbb{H}}(y,p_1,p_2)\right.\right.$$
$$\left.\left. - \frac{D_{\mathcal{P}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathcal{P}}(y,p_1,p_2)\right|\right],$$

where the sum ranges over all distinct pairs of points in $\mathcal{P}\setminus\{(0,y)\}$. In what follows, we will set $\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p') = \mathcal{B}_{\mathbb{H},n}(p') \triangle \mathcal{B}_{\mathcal{P}}(p')$ and $\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}(p') = \mathcal{B}_{\mathbb{H},n}(p') \cap \mathcal{B}_{\mathcal{P}}(p')$. We will now bound the sum that is inside the expectation. Note that each summand is the absolute value of the difference between two quantities that are either equal to 0 or of order $\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}$ $(\overline{n}_{\mathcal{P}}(k_n)^{-1})$. We will split these summands into 5 classes which are all combinations of $p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}$ for which only one of the two terms of this difference is non-zero.

1. both p_1 and p_2 have $y_1,y_2 < (1-\varepsilon)R_n \wedge (R_n - y)$ and
   (a) p_1 is in $\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}(y)$ but $p_2 \in \mathcal{B}_{\mathbb{H},n}(p_1)\setminus\mathcal{B}_{\mathcal{P}}(p_1)$ and $\mathcal{B}_{\mathbb{H},n}(y)$ contains exactly k_n-2 or k_n-1 other points (depending on whether $p_2\in\mathcal{B}_{\mathbb{H},n}(y)$ or not).
   (b) p_1 is in $\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}(y)$ but $p_2 \in \mathcal{B}_{\mathcal{P}}(p_1)\setminus\mathcal{B}_{\mathbb{H},n}(p_1)$ and $\mathcal{B}_{\mathcal{P}}(y)$ contains exactly k_n-2 or k_n-1 other points (depending on whether $p_2\in\mathcal{B}_{\mathcal{P}}(y)$ or not).
2. the above cases but with $y_1 \geq (1-\varepsilon)R_n \wedge (R_n - y)$.
3. $y_1 \geq K$ and $p_1 \in \mathcal{B}_{\mathbb{H},n}(y)\setminus\mathcal{B}_{\mathcal{P}}(y)$ and $p_2\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}(y)$ - here we use Corollary 9.1 which implies that if $p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)$ and $y_1\geq K$, then in fact $p_1\in\mathcal{B}_{\mathbb{H},n}(y)\setminus\mathcal{B}_{\mathcal{P}}(y)$.
4. $y_1 < K$ and $p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)$ and $p_2\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}(y)$.

We bound this sum by the following expression:

$$\sum_{(p_1,p_2)\in\mathcal{P}\setminus\{(0,y)\}}^{\neq}\left|\frac{D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathbb{H}}(y,p_1,p_2)\right.$$

$$-\frac{D_{\mathcal{P}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})}{\overline{n}_{\widetilde{\mathbb{H}}}(k_n)}\Delta_{\mathcal{P}}(y,p_1,p_2)\Bigg|$$

$$\leq \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y_1,y_2<(1-\varepsilon)R_n\wedge(R_n-y)}}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\}}D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\}) \tag{100}$$

$$\leq \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y_1,y_2<(1-\varepsilon)R_n\wedge(R_n-y)}}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\}}D_{\mathcal{P}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\}) \tag{101}$$

$$+\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y_1\geq(1-\varepsilon)R_n\wedge(R_n-y)}}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}(y)\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\}) \tag{102}$$

$$+\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y_1\geq(1-\varepsilon)R_n\wedge(R_n-y)}}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}(y)\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}D_{\mathcal{P}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\}) \tag{103}$$

$$+2\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y(p_1)\geq K}}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\setminus\mathcal{B}_{\mathcal{P}}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\cap\mathcal{B}_{\mathcal{P}}(y)\}}D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\}) \tag{104}$$

$$+(\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}+\overline{n}_{\mathcal{P}}(k_n)^{-1})\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y(p_1)<K}}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\cap\mathcal{B}_{\mathcal{P}}(y)\}} \tag{105}$$

In the following sections we will give upper bounds on the expected values of each one of these partial sums.

The sums (100) and (101) We will analyze (100). The analysis of the other sum (101) is similar. Note first that for any two points p_1,p_2 the following holds: $p_1\in\mathcal{B}_{\mathbb{H},n}(y)$ and $p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)$, then $p_2\in\mathcal{B}_{\mathbb{H},n}(y)$ and $p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_2)\cap\mathcal{B}_{\mathbb{H},n}(y)$. Using this symmetry, it suffices to consider distinct pairs $(p_1,p_2)\in\mathcal{P}\setminus\{(0,y)\}$ with $0\leq y_2\leq y_1\leq R-y$. Let $\mathcal{D}$ denote the set of these pairs.

We are going to consider several sub-cases and, thereby, split the domain $\mathcal{D}$ into the corresponding sub-domains. Let $\omega=\omega(n)\to\infty$ as $n\to\infty$ be a slowly growing function and set $y_\omega:=y+\omega$. We let

$$\begin{aligned}\mathcal{D}_1&=\{(p_1,p_2)\in\mathcal{D}\ :\ y\leq y_1\leq R_n/2,\ y_\omega\leq y_2\leq y_1\},\\ \mathcal{D}_2&=\{(p_1,p_2)\in\mathcal{D}\ :\ y_1\leq R_n/2,\ y_2\leq y_\omega\}\text{ and}\\ \mathcal{D}_3&=\{(p_1,p_2)\in\mathcal{D}\ :R_n/2<y_1\leq R_n-y,\ y_2\leq y_1\}.\end{aligned}$$

Note that $\mathcal{D}\subseteq\mathcal{D}_1\cup\mathcal{D}_2\cup\mathcal{D}_3$. Hence, we can write

$$\begin{aligned}&\mathbb{E}\left[\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y_1,y_2\leq(1-\varepsilon)R_n\wedge(R_n-y)}}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right]\\ &=\sum_{i=1}^{3}\mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_i}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right].\end{aligned} \tag{106}$$

We bound each one of the above three summands as follows:

$$\begin{aligned}&\mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_1}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right]\\ &\leq\mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_1}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\}}D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right]:=\mathcal{I}_n^{(1)}(y),\end{aligned} \tag{107}$$

$$\begin{aligned}
&\mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_2} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}\, D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right]\\
&\mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_2} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\}}\, D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right] := \mathcal{I}_n^{(2)}(y)
\end{aligned} \tag{108}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_3} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}\, D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right]\\
&\le \mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_3} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\}}\, D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right] := \mathcal{I}_n^{(3)}.
\end{aligned} \tag{109}$$

We will bound each term using the Campbell-Mecke formula (10) and show for $i = 1,2,3$ that for $1/2 < \alpha < 3/4$

$$\lim_{n\to\infty} k_n^{4\alpha-2}\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(i)}e^{-\alpha}\,\mathrm{d}y = 0, \tag{110}$$

and for $\alpha \ge 3/4$

$$\lim_{n\to\infty} k_n\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(i)}e^{-\alpha}\,\mathrm{d}y = 0. \tag{111}$$

For the first term (107), we note that

$$\mathbb{P}\left(D_{\mathbb{H}}(y) = k_n-2;\mathcal{P}\setminus\{p_1,p_2\}\right) = \rho_{\widetilde{\mathbb{H}}}(y,k_n-2).$$

and hence $\mathcal{I}_n^{(1)}(y)$ becomes

$$\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)\int_{-I_n}^{I_n}\int_y^{R_n/2}\int_{-I_n}^{I_n}\int_{y_\omega}^{y_1} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}(y)\}}\,\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\}}e^{-\alpha(y_1+y_2)}\,\mathrm{d}y_2\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}x_1. \tag{112}$$

Next, Lemma 2.2 implies that for $y' \le R_n - y$, we have that if $(x',y') \in \mathcal{B}_{\mathbb{H},n}(y)$, then $|x'| < (1+K)e^{y/2+y'/2}$, where $K > 0$ is as in Lemma 2.2. Using these observations, we obtain:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{p_1,p_2\in\mathcal{D}_1} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right]\\
&\qquad = \rho_{\widetilde{\mathbb{H}}}(y,k_n-2)e^y\int_y^{R_n/2} e^{y_1/2}\int_{y_\omega}^{y_1} e^{y_2/2}e^{-\alpha y_2}\cdot e^{-\alpha y_1}dy_2dy_1.
\end{aligned}$$

Now, the double integral becomes

$$\begin{aligned}
&\int_y^{R_n/2} e^{y_1/2}\int_{y_\omega}^{y_1} e^{y_2/2}e^{-\alpha y_2}\cdot e^{-\alpha y_1}dy_2dy_1 =\\
&O(1)\cdot\int_y^{R_n/2} e^{y_1/2-\alpha y_1}\cdot e^{(1/2-\alpha)y_\omega}dy_1\\
&= O(1)\cdot e^{(1/2-\alpha)y_\omega}\cdot\int_y^{R_n/2} e^{y_1/2-\alpha y_1}dy_1\\
&= O(1)\cdot e^{(1/2-\alpha)y_\omega+(1/2-\alpha)y}\\
&\ll e^{(1-2\alpha)y},
\end{aligned} \tag{113}$$

since $y_\omega = y + \omega$ and $\omega \to \infty$. We then deduce that

$$\mathbb{E}\left[\sum_{p_1,p_2\in\mathcal{D}_1} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\cap\mathcal{P}}((0,y))\}} \cdot \mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\}} \cdot D_{\mathbb{H}}(y, k_n-2; \mathcal{P}\setminus\{p_1,p_2\})\right] \\ \ll \rho_{\widetilde{\mathbb{H}}}(y,k_n-2)e^{(1-2\alpha)y}. \tag{114}$$

We now integrate this with respect to y and determine its contribution to (99) is

$$\binom{k_n}{2}^{-1} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \rho_{\widetilde{\mathbb{H}}}(y,k_n-2)e^{(1-2\alpha)y}e^{-\alpha y}\,\mathrm{d}y\,\mathrm{d}x \\ = O\left(k^{2\alpha-1}k_n^{-6\alpha+1}\right) = O\left(k_n^{-4\alpha}\right),$$

where we used Lemma 9.4 with $s = 1-2\alpha$.

Since $k_n^{-4\alpha} = o\left(\min\{k_n^{-4\alpha+2}, k_n^{-1}\}\right)$ for all $\alpha > 1/2$ we deduce that for $1/2 < \alpha < 3/4$

$$\lim_{n\to\infty} k_n^{4\alpha-2}\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(1)}(y)e^{-\alpha y}\,\mathrm{d}y = 0,$$

while for $\alpha \geq 3/4$

$$\lim_{n\to\infty} k_n\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(1)}(y)e^{-\alpha y}\,\mathrm{d}y = 0.$$

We will now bound the term in (108). Using similar observations as for the previous term we get that $\mathcal{I}_n^{(2)}(y)$ equals

$$\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)\int_{-I_n}^{I_n}\int_0^{R_n/2}\int_{-I_n}^{I_n}\int_0^{y_\omega} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\,\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}e^{-\alpha(y_1+y_2)}\,\mathrm{d}y_2 d\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}x_1.$$

Now, Lemma 2.2 implies that for $y_2 \leq R_n - y_1$, we have that if $(x_2,y_2)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}\left((x_1,y_1)\right)$, then x_2 lies in an interval of length $Ke^{3y/2+3y'/2-R_n}$, where $K>0$ is again the constant in Lemma 2.2. Using these observations we obtain:

$$\mathcal{I}_n^{(2)}(y) = \rho_{\widetilde{\mathbb{H}}}(y,k_n-2)e^{y/2}\int_0^{R_n/2} e^{y_1/2+3y_1/2}\int_0^{y_\omega} e^{3y_2/2-R_n}e^{-\alpha y_2}\cdot e^{-\alpha y_1}\,\mathrm{d}y_2\,\mathrm{d}y_1. \tag{115}$$

Now the latter integral is

$$e^{-R_n}\left(\int_0^{R_n/2} e^{(2-\alpha)y_1}\,\mathrm{d}y_1\right)\left(\int_0^{y} e^{(3/2-\alpha)y_2}\,\mathrm{d}y_2\right) \\ = O(1)\,e^{-R_n}\left(\begin{cases} e^{(1-\alpha/2)R_n} & \text{if } \frac{1}{2}<\alpha<2 \\ R_n & \text{if } \alpha\geq 2\end{cases}\right)\left(\begin{cases} e^{(3/2-\alpha)y} & \text{if } \frac{1}{2}<\alpha<\frac{3}{2} \\ y & \text{if } \alpha\geq\frac{3}{2}\end{cases}\right) \\ = O(1)\begin{cases} e^{-\frac{\alpha}{2}R_n}e^{(3/2-\alpha)y} & \text{if } \frac{1}{2}<\alpha<\frac{3}{2} \\ ye^{\frac{\alpha}{2}R_n} & \text{if } \frac{3}{2}\leq\alpha<2 \\ yR_ne^{-R_n} & \text{if } \alpha\geq 2.\end{cases}$$

Since $y \leq R_n = O(\log(n))$ we conclude that

$$\mathcal{I}_n^{(2)}(y) = O(1)\,\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)\begin{cases} n^{-\alpha}e^{(3/2-\alpha)y} & \text{if } \frac{1}{2}<\alpha<\frac{3}{2} \\ n^{-\alpha}\log(n) & \text{if } \frac{3}{2}\leq\alpha<2 \\ n^{-2}\log(n)^2 & \text{if } \alpha\geq 2.\end{cases}$$

We proceed with the integration of this with respect to y and determine its contribution to (99) is

$$
\begin{aligned}
&O(1)\binom{k_n}{2}^{-2}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)}\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)e^{-\alpha y}\,\mathrm{d}y\cdot\begin{cases} n^{-\alpha}k_n^{3-2\alpha} & \text{if } \frac{1}{2}<\alpha<\frac{3}{2}\\ n^{-\alpha}\log(n) & \text{if } \frac{3}{2}\leq\alpha<2\\ n^{-2}\log(n)^2 & \text{if } \alpha\geq 2,\end{cases}\\
&=O(1)\cdot\begin{cases} n^{-\alpha}k_n^{1-2\alpha} & \text{if } \frac{1}{2}<\alpha<\frac{3}{2}\\ n^{-\alpha}\log(n)k_n^{-2} & \text{if } \frac{3}{2}\leq\alpha<2\\ n^{-2}\log(n)^2k_n^{-2} & \text{if } \alpha\geq 2.\end{cases}
\end{aligned}
$$

Now for $1/2<\alpha<3/4$ it holds that $4\alpha^2-\alpha+1>0$. Hence since $k_n=O\left(n^{\frac{1}{2\alpha+1}}\right)$, we have

$$
k_n^{4\alpha-2}n^{-\alpha}k_n^{1-2\alpha}=n^{-\alpha}k_n^{2\alpha-1}=O\left(n^{-\alpha+\frac{2\alpha-1}{2\alpha+1}}\right)=O\left(k_n^{-\frac{4\alpha^2-\alpha+1}{2\alpha+1}}\right)=o(1),
$$

from which we deduce that

$$
\lim_{n\to\infty}k_n^{4\alpha-2}\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)}\mathcal{I}_n^{(2)}(y)e^{-\alpha y}\,\mathrm{d}y=0.
$$

For $\alpha\geq 3/4$ we have that both $n^{-\alpha}\log(n)k_n^{-1}$ and $n^{-2}\log(n)^2k_n^{-1}$ converge to zero as $n\to\infty$ and hence in this case

$$
\lim_{n\to\infty}k_n\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)}\mathcal{I}_n^{(2)}(y)e^{-\alpha y}\,\mathrm{d}y=0.
$$

We will now consider the term in (109). Recall that $\mathcal{D}_3$ consists of all pairs $(p_1,p_2)\in\mathcal{D}$ such that $R_n/2<y_1\leq(1-\varepsilon)R_n\wedge(R_n-y)$ and $y_2\leq y_\omega$ with the property that $p_1\in\mathcal{B}_{\mathbb{H},n}(y)$ and $p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)$. So, in particular, $p_2\in(\mathcal{B}_{\mathbb{H},n}(p_1)\cup\mathcal{B}_{\mathcal{P}}(p_1))\cap\mathcal{B}_{\mathbb{H},n}(y)$.

We will consider this intersection more closely. We use Lemma 2.2 to define a ball around p_1 that contains both $\mathcal{B}_{\mathbb{H},n}(p_1)$ and $\mathcal{B}_{\mathcal{P}}(p_1)$. For $K>0$, we define, for any point $p_1=(x_1,y_1)\in\mathbb{R}\times\mathbb{R}_+$,

$$
\check{\mathcal{B}}_{\mathbb{H},n}(p_1):=\{(x',y')\ :\ y'<R_n-y_1,\ |x_1-x'|<(1+K)e^{\frac{1}{2}(y_1+y')}\}. \tag{116}
$$

It is an implication of Lemma 2.2 that

$$
(\mathcal{B}_{\mathbb{H},n}(p_1)\cup\mathcal{B}_{\mathcal{P}}(p_1))\cap\mathcal{R}([0,R_n-y_1])\subseteq\check{\mathcal{B}}_{\mathbb{H},n}(p_1)
$$

Therefore, any point $p_2=(x_2,y_2)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)$ with $y_2\leq R-y_1$ must belong to $\check{\mathcal{B}}_{\mathbb{H},n}(p_1)\cap\check{\mathcal{B}}_{\mathbb{H},n}((0,y))$.

We will use this in order to derive a lower bound on y_2 as a function of x_1,y_1. Let us suppose without loss of generality that $x_1<0$. The left boundary of $\check{\mathcal{B}}_{\mathbb{H},n}((0,y))$ is given by the equation $x'=(1+K)e^{\frac{1}{2}(y+y')}$ whereas the right boundary of $\check{\mathcal{B}}_{\mathbb{H},n}(p_1)$ is given by the curve having equation $x'=x_1+(1+K)e^{\frac{1}{2}(y_1+y')}$. The equation that determines the intersection point $(\hat{x},\hat{y})$ of these curves is

$$
x_1+(1+K)e^{(y_1+\hat{y})/2}=(1+K)e^{(y+\hat{y})/2}.
$$

We can solve the above for $\hat{y}$

$$
|x_1|=(1+K)e^{\hat{y}/2}\left(e^{y_1/2}+e^{y/2}\right).
$$

But $y_1>R_n/2$ and $y<(1+\varepsilon)R_n/(2\alpha+1)$. So if ε is small enough depending on α, we have

$$
|x_1|=(1+K)e^{\hat{y}/2}\left(e^{y_1/2}+e^{y/2}\right)=(1+K+o(1))e^{\hat{y}/2+y_1/2}.
$$

Let c_K denote the multiplicative term $1+K+o(1)$, which appears in the above. The above yields

$$\hat{y} = \Big(2\log(|x_1|e^{-y_1/2}) - \log c_K\Big) \vee 0 := c(x_1, y_1). \tag{117}$$

In particular, note that $\hat{y} = 0$ if and only if $|x_1| \leq c_K e^{y_1/2}$. Moreover, since $p_1 \in \mathcal{B}_{\mathbb{H},n}(y)$ and $x_1 \leq R_n - y$, we also have that $|x_1| \leq e^{(y+y_1)/2}(1+o(1))$. This upper bound on $|x_1|$ together with (124), imply that for n sufficiently large, we have $\hat{y} \leq y$. This observation will be used below, where we integrate over y_2, thus ensuring that the integrals are non-zero.

We conclude that

$$p' \in \check{\mathcal{B}}_{\mathbb{H},n}(y) \cap \check{\mathcal{B}}_{\mathbb{H},n}((x_1,y_1)) \Rightarrow y' \geq c(x_1,y_1),$$

Therefore we have

$$\mathbb{1}_{\{p_2 \in \mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap \mathcal{B}_{\mathbb{H},n}(y)\}} \leq \mathbb{1}_{\left\{y_2 \geq c(x_1,y_1), p_2 \in \check{\mathcal{B}}_{\mathbb{H},n}((0,y))\right\}}. \tag{118}$$

If we integrate this over x_2, y_2 we get

$$\begin{aligned}
&\int_{-I_n}^{I_n}\int_0^{y_\omega} \mathbb{1}_{\{p_2 \in \mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap \mathcal{B}_{\mathbb{H},n}(y)\}} e^{-\alpha y_2} dy_2 dx_2 \\
&\leq \int_{-I_n}^{I_n}\int_0^{y_\omega} \mathbb{1}_{\left\{y_2 \geq c(x_1,y_1), p_2 \in \check{\mathcal{B}}_{\mathbb{H},n}(y)\right\}} e^{-\alpha y_2} dy_2 dx_2 \leq (1+K)\cdot e^{y/2}\int_{c(x_1,y_1)}^{y_\omega} e^{y_2/2-\alpha y_2} dy_2 \\
&= O(1)\cdot e^{y/2+(1/2-\alpha)c(x_1,y_1)}.
\end{aligned}$$

Note also that

$$\mathbb{E}\left[D_{\mathbb{H}}(y, k_n-2; \mathcal{P}\setminus\{p_1,p_2\})\right] = \rho_{\widetilde{\mathbb{H}}}(y, k_n-2),$$

uniformly over all $(p_1,p_2) \in \mathcal{D}_3$.

So the Campbell-Mecke formula yields that $\mathcal{I}_n^{(3)}(y)$ equals:

$$\begin{aligned}
&O(1)\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)\, e^{y/2}\int_{-I_n}^{I_n}\int_{R_n/2}^{(R_n-y)\wedge(1-\varepsilon)R_n} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(y)\}} e^{(1/2-\alpha)c(x_1,y_1)-\alpha y_1} dy_1 dx_1 \\
&= O(1)\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)\, e^{y/2}\int_{-I_n}^{I_n}\int_{R_n/2}^{(R_n-y)\wedge(1-\varepsilon)R_n} \mathbb{1}_{\left\{p_1 \in \check{\mathcal{B}}_{\mathbb{H},n}(y)\right\}} e^{(1/2-\alpha)c(x_1,y_1)-\alpha y_1} dy_1 dx_1.
\end{aligned} \tag{119}$$

Due to the symmetry of $\check{\mathcal{B}}_{\mathbb{H},n}(y)$, the integration over x_1 is:

$$O(1)\cdot e^{y/2}\cdot \int_0^{(1+K)e^{y/2+y_1/2}} e^{c(x_1,y_1)(1/2-\alpha)} dx_1$$

We will split this integral into two parts according to the value of $c(x_1,y_1)$:

$$\int_0^{(1+K)e^{y/2+y_1/2}} e^{c(x_1,y_1)(1/2-\alpha)} dx_1 = \int_{c_K e^{y_1/2}}^{(1+K)e^{y/2+y_1/2}} e^{c(x_1,y_1)(1/2-\alpha)} dx_1 + \int_0^{c_K e^{y_1/2}} dx_1.$$

The first integral becomes:

$$\begin{aligned}
&\int_{c_K e^{y_1/2}}^{(1+K)e^{y/2+y_1/2}} e^{c(x_1,y_1)(1/2-\alpha)} dx_1 = \int_{c_K e^{y_1/2}}^{(1+K)e^{y/2+y_1/2}} e^{c(x_1,y_1)/2(1-2\alpha)} dx_1 \\
&= O(1)\cdot \int_{c_K e^{y_1/2}}^{(1+K)e^{y/2+y_1/2}} x_1^{1-2\alpha} e^{-\frac{y_1}{2}(1-2\alpha)} dx_1 \\
&= O(1)\cdot e^{-y_1/2+\alpha y_1}\cdot e^{\frac{(y+y_1)}{2}2(1-\alpha)} \\
&= O(1)\cdot e^{y_1/2+y(1-\alpha)}.
\end{aligned}$$

The second integral trivially gives:

$$\int_0^{c_K e^{y_1/2}} dx_1 = O(1) \cdot e^{y_1/2} = O(1) \cdot e^{y_1/2 + y(1-\alpha)}.$$

We conclude that

$$e^{y/2} \cdot \int_0^{(1+K)e^{y/2+y_1/2}} e^{c(x_1,y_1)(1/2-\alpha)} dx_1 = O(1) \cdot e^{y_1/2+y(3/2-\alpha)}.$$

Now, we integrate this with respect to y_1 and get

$$e^{y(3/2-\alpha)} \int_{R_n/2}^{R_n - y} e^{(1/2-\alpha)y_1} dy_1 = O(1) \cdot e^{y(3/2-\alpha)} e^{(1/2-\alpha)R_n/2} = O(1) \cdot n^{1/2-\alpha} \cdot e^{y(3/2-\alpha)},$$

from which we deduce

$$\mathcal{I}_n^{(3)}(y) = O(1) \cdot n^{1/2-\alpha} e^{y(3/2-\alpha)} \, \rho_{\widetilde{\mathbb{H}}}(y, k_n - 2). \tag{120}$$

We now apply Lemma 9.4 and get

$$\begin{aligned}
&\binom{k_n}{2}^{-2} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(3)}(y) e^{-\alpha y} \, \mathrm{d}y \\
&\quad = O\left(1\right) k_n^{2\alpha-1} n^{-1} \int_{\mathcal{K}_C(k_n)} e^{(3/2-\alpha)y} \rho_{\widetilde{\mathbb{H}}}(y, k_n - 2) e^{-\alpha y} \, \mathrm{d}y \\
&\quad = O\left(1\right) k_n^{1-2\alpha} n^{-(\alpha - 1/2)}.
\end{aligned}$$

Since for $\alpha > 1/2$, $k_n = O\left(n^{\frac{1}{2\alpha+1}}\right) = o\left(n^{1/2}\right)$ we have that $k_n^{4\alpha-2} k_n^{1-2\alpha} n^{-(\alpha-1/2)} = o\left(1\right)$ and hence for $1/2 < \alpha < 3/4$.

$$\lim_{n\to\infty} k_n^{4\alpha-2} \binom{k_n}{2}^{-2} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(3)}(y) e^{-\alpha y} \, \mathrm{d}x \, \mathrm{d}y = 0,$$

For $\alpha \geq 3/4$ we observe that $2\alpha^2 + 2\alpha - 2 < 0$. Hence, since

$$k_n n^{-(\alpha-1/2)} k_n^{1-2\alpha} = O\left(n^{-(\alpha-1/2)} n^{\frac{2-2\alpha}{2\alpha+1}}\right) = O\left(n^{-\frac{2\alpha^2+2\alpha-2}{2\alpha+1}}\right) = o\left(1\right).$$

we get for $\alpha \geq 3/4$

$$\lim_{n\to\infty} k_n \binom{k_n}{2}^{-2} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(3)}(y) e^{-\alpha y} \, \mathrm{d}x \, \mathrm{d}y = 0.$$

The sums (102) **and** (103) Again, we will only consider (102) since the analysis for the other term is similar. Recall that in this case, we consider pairs (p_1, p_2), with $p_1 = (x_1, y_1)$ satisfying $y_1 \geq (R_n - y) \wedge (1-\varepsilon)R_n$, and $p_1 \in \mathcal{B}_{\mathbb{H},n}\left(y\right)$, $p_2 \in \mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}\left(p_1\right) \cap \mathcal{B}_{\mathbb{H},n}\left(y\right)$. We split this into three sub-domains: i. $y_2 \geq R_n - y$; ii. $R_n - y_1 \leq y_2 \leq R_n - y$ and iii. $y_2 < R_n - y_1$. Similar to the analysis above we define

$$\begin{aligned}
\mathcal{D}_1 &:= \{(p_1,p_2) \ : \ p_1, p_2 \ \in \mathcal{P} \setminus \{(0,y)\}, \ y_1 \geq (1-\varepsilon)R_n \wedge (R_n - y), \ R_n - y \leq y_2 \leq R_n\} \\
\mathcal{D}_2 &:= \{(p_1,p_2) \ : \ p_1, p_2 \ \in \mathcal{P} \setminus \{(0,y)\}, \ y_1 \geq (1-\varepsilon)R_n \wedge (R_n - y), \ R_n - y_1 \leq y_2 \leq R_n - y\} \\
\mathcal{D}_3 &:= \{(p_1,p_2) \ : \ p_1, p_2 \ \in \mathcal{P} \setminus \{(0,y)\}, \ y_1 \geq (1-\varepsilon)R_n \wedge (R_n - y), \ y_2 \leq R_n - y_1\}
\end{aligned}$$

and write, for $i = 1, 2, 3$,

$$\mathcal{I}_n^{(i)}(y) := \mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_i} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}} \cdot \mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}} \cdot D_{\mathbb{H}}(y, k_n-2; \mathcal{P}\setminus\{p_1,p_2\})\right].$$

In the first case, note that $y_1 + y_2 \geq 2(R_n - y) > R_n$, since $2y < 2(1+\varepsilon)\frac{R_n}{2\alpha+1} < R_n$. Thus, $p_2 \in \mathcal{B}_{\mathbb{H},n}(p_1)$. Furthermore, $y_2 > R_n - y_1 + 2\ln(\pi/2)$, which implies that $p_2 \in \mathcal{B}_{\mathcal{P}}(p_1)$ too. Hence, the contribution from these pairs is zero.

The Campbell-Mecke formula yields that:

$$\begin{aligned}\mathcal{I}_n^{(1)}(y) = O(1) &\int_{-I_n}^{I_n}\int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\times \\ &\int_{-I_n}^{I_n}\int_{R_n-y}^{R_n} \mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)\cdot e^{-\alpha(y_2+y_1)}\,\mathrm{d}y_2\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}x_1.\end{aligned}$$

We proceed to bound the integral:

$$\begin{aligned}&\int_{-I_n}^{I_n}\int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\int_{-I_n}^{I_n}\int_{R_n-y}^{R_n} \mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}e^{-\alpha(y_1+y_2)}\,\mathrm{d}y_2\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}x_1 \\ &\leq \int_{-I_n}^{I_n}\int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n}\int_{-I_n}^{I_n}\int_{R_n-y}^{R_n} e^{-\alpha(y_1+y_2)}\,\mathrm{d}y_2\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}x_1 \\ &= \left(\int_{-I_n}^{I_n}\int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} e^{-\alpha y_1}\,\mathrm{d}y_1\,\mathrm{d}x_1\right)\left(\int_{-I_n}^{I_n}\int_{R_n-y}^{R_n} e^{-\alpha y_2}\,\mathrm{d}y_2\,\mathrm{d}x_2\right).\end{aligned}$$

We evaluate

$$\int_{-I_n}^{I_n}\int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} e^{-\alpha y_1}dy_1dx_1 = O(1)\cdot n\cdot e^{-\alpha R_n+((\varepsilon R_n)\vee y))\alpha} = O(1)\cdot n\cdot e^{-\alpha R_n+\alpha y+\alpha\varepsilon R_n}$$

and

$$\int_{-I_n}^{I_n}\int_{R_n-y}^{R_n} e^{-\alpha y_2}dy_2dx_2 = O(1)\cdot n\cdot e^{-\alpha R_n+\alpha y}.$$

Also, $n\cdot e^{-\alpha R_n} = O(1)\cdot e^{(1/2-\alpha)R_n}$, whereby we deduce that

$$\begin{aligned}&\int_{\mathcal{D}_1} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}e^{-\alpha(y_1+y_2)}dy_2dx_2dy_1dx_1 \\ &\quad = O(1)\cdot e^{(1-2\alpha)R_n+2\alpha y+\alpha\varepsilon R_n} = O(1)\cdot n^{2(1-2\alpha)+2\alpha\varepsilon}\cdot e^{2\alpha y}.\end{aligned}$$

With these computations we obtain

$$\begin{aligned}&\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)}\mathcal{I}_n^{(1)}(y)e^{-\alpha y}\,\mathrm{d}x\,\mathrm{d}y \\ &= O(1)n^{(2(1-2\alpha)+2\alpha\varepsilon}\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)} e^{2\alpha y}\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)e^{-\alpha y}\,\mathrm{d}y\,\mathrm{d}x \\ &= O(1)n^{(2(1-2\alpha)+2\alpha\varepsilon}\,k_n^{-2}\,\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\,n\,k_n^{2\alpha-1} = O(1)\,n^{(2(1-2\alpha)+2\alpha\varepsilon}k_n^{4\alpha-2}.\end{aligned}$$

Thus, for $1/2 < \alpha < 3/4$, we have

$$k_n^{4\alpha-2}\,n^{2(1-2\alpha)+2\alpha\varepsilon}\,k_n^{4\alpha-2} = n^{2\alpha\varepsilon}\cdot\left(\frac{k_n^2}{n}\right)^{2(2\alpha-1)} = o(1),$$

provided that $\varepsilon = \varepsilon(\alpha) > 0$ is small enough, and hence for such ε

$$\lim_{n\to\infty} k_n^{4\alpha-2} \binom{k_n}{2}^{-1} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(1)}(y) e^{-\alpha y} \, \mathrm{d}x \, \mathrm{d}y = 0.$$

When $\alpha > 3/4$ we have $2\alpha - 1 > 1/2$ and we get

$$k_n \cdot n^{2(1-2\alpha)+2\alpha\varepsilon} \cdot k_n^{4\alpha-2} \ll k_n \cdot n^{-1/2+2\alpha\varepsilon} \cdot k_n^{4\alpha-2} \cdot n^{1-2\alpha} = o(1),$$

provided that ε is small enough, depending on α, so that

$$\lim_{n\to\infty} k_n \binom{k_n}{2}^{-1} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(1)}(y) e^{-\alpha y} \, \mathrm{d}x \, \mathrm{d}y = 0.$$

We now consider the second sub-domain $\mathcal{D}_2$. The Campbell-Mecke formula yields that:

$$\begin{aligned}
\mathcal{I}_n^{(2)}(y) &= \mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_2} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}} \mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}} D_{\mathbb{H}}(y, k_n-2; \mathcal{P}\setminus\{p_1\}) \right] \\
&= O(1)\rho_{\widetilde{\mathbb{H}}}(y, k_n-2) \cdot \int_{-I_n}^{I_n} \int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}} \times \\
&\quad \int_{-I_n}^{I_n} \int_{R_n-y_1}^{R_n-y} \mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}} e^{-\alpha(y_1+y_2)} \, \mathrm{d}y_2 \, \mathrm{d}x_2 \, \mathrm{d}y_1 \, \mathrm{d}x_1.
\end{aligned}$$

We bound the integral as follows:

$$\begin{aligned}
&\int_{-I_n}^{I_n} \int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}} \int_{-I_n}^{I_n} \int_{R_n-y_1}^{R_n-y} \mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}} e^{-\alpha(y_1+y_2)} \, \mathrm{d}y_2 \, \mathrm{d}x_2 \, \mathrm{d}y_1 \, \mathrm{d}x_1 \\
&\leq \int_{-I_n}^{I_n} \int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}} \int_{-I_n}^{I_n} \int_{R_n-y_1}^{R_n-y} \mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\}} e^{-\alpha(y_1+y_2)} \, \mathrm{d}y_2 \, \mathrm{d}x_2 \, \mathrm{d}y_1 \, \mathrm{d}x_1.
\end{aligned}$$

Now, by Lemma **??** **Pim:** @Nikolaos: Which lemma are you referring to here?

$$\begin{aligned}
&\int_{-I_n}^{I_n} \int_{R_n-y_1}^{R_n-y} \mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\}} \cdot e^{-\alpha y_2} dy_2 dx_2 = O(1) \cdot e^{y/2} \int_{R_n-y_1}^{R_n-y} e^{(1/2-\alpha)y_2} dy_2 \\
&= O(1) \cdot e^{y/2+(1/2-\alpha)(R_n-y_1)}.
\end{aligned}$$

We then integrate with respect to y_1:

$$\begin{aligned}
O(1) \cdot e^{y/2} \cdot &\int_{-I_n}^{I_n} \int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}} e^{(1/2-\alpha)(R_n-y_1)} e^{-\alpha y_1} dy_1 dx_1 \\
&\leq O(1) \cdot e^{y/2+(1/2-\alpha)R_n} \cdot \int_{-I_n}^{I_n} \int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n} e^{(\alpha-1/2)y_1} e^{-\alpha y_1} dy_1 dx_1 \\
&= O(1) \cdot e^{y/2+(1-\alpha)R_n-((1-\varepsilon)R_n\wedge(R_n-y))/2} \\
&= O(1) \cdot e^{y/2+(1/2-\alpha)R_n+((\varepsilon R_n)\vee y)/2} \\
&= O(1) \cdot e^{y+(1/2-\alpha)R_n+\varepsilon R_n} = O(1) \cdot n^{1-2\alpha+\varepsilon} \cdot e^y.
\end{aligned}$$

Therefore, the contribution of this term to (99) is:

$$\binom{k_n}{2}^{-1} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(2)}(y) e^{-\alpha y} \, \mathrm{d}x \, \mathrm{d}y$$

$$
\begin{aligned}
&= O\left(n^{1-2\alpha+\varepsilon}\right)\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)}\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)e^{y}e^{-\alpha y}\,\mathrm{d}x\,\mathrm{d}y\\
&= O\left(1\right)n^{1-2\alpha+\varepsilon},
\end{aligned}
$$

where we used Lemma 9.4 with $s=1$.

For $1/2<\alpha<3/4$, we have

$$
k_n^{4\alpha-2}\cdot n^{1-2\alpha+\varepsilon}=n^{\varepsilon}\left(\frac{k_n^2}{n}\right)^{2\alpha-1}=o(1),
$$

provided that $\varepsilon=\varepsilon(\alpha)>0$ is small enough yielding

$$
\lim_{n\to\infty}k_n^{4\alpha-2}\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)}\mathcal{I}_n^{(2)}(y)e^{-\alpha y}\,\mathrm{d}x\,\mathrm{d}y=0.
$$

Similarly, for $\alpha>3/4$ we have $2\alpha-1>1/2$ and we get

$$
k_n\cdot n^{1-2\alpha+\varepsilon}\ll n^{-1/2+\varepsilon}\cdot k_n=o(1),
$$

provided that ε is small enough, so that

$$
\lim_{n\to\infty}k_n\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)}\mathcal{I}_n^{(2)}(y)e^{-\alpha y}\,\mathrm{d}x\,\mathrm{d}y=0.
$$

For the third sub-domain $\mathcal{D}_3$ we shall use (118) which states that if $p_2=(x_2,y_2)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)$ and $y_2\le R_n-y_1$, then $y_2\ge c(x_1,y_1)$, where $c(x_1,y_1)=\left(2\log(|x_1|e^{-y_1/2})-\log c_K\right)\vee 0$ (cf. (124)). Moreover, $p_2\in\check{\mathcal{B}}_{\mathbb{H},n}(p_1)$.

Again, we will use the Campbell-Mecke formula:

$$
\begin{aligned}
\mathcal{I}_n^{(3)}(y)&=\mathbb{E}\left[\sum_{(p_1,p_2)\in\mathcal{D}_3}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}\cdot D_{\mathbb{H}}(y,k_n-2;\mathcal{P}\setminus\{p_1,p_2\})\right]\\
&=O(1)\rho_{\widetilde{\mathbb{H}}}(y,k_n-2)\int_{-I_n}^{I_n}\int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\times\\
&\qquad\int_{-I_n}^{I_n}\int_0^{R_n-y_1}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}e^{-\alpha(y_1+y_2)}dy_2dx_2dy_1dx_1
\end{aligned}
$$

The inner integral with respect to $p_2:=(x_2,y_2)$ is

$$
\begin{aligned}
&\int_{-I_n}^{I_n}\int_0^{R_n-y_1}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}e^{-\alpha y_2}dy_2dx_2\\
&\qquad\le\int_{-I_n}^{I_n}\int_0^{R_n-y_1}\mathbb{1}_{\left\{y_2\ge c(x_1,y_1),p_2\in\check{\mathcal{B}}_{\mathbb{H},n}((0,y))\right\}}e^{-\alpha y_2}dy_2dx_2\\
&\qquad=O(1)e^{y/2}\int_{c(x_1,y_1)}^{R_n-y_1}e^{y_2/2-\alpha y_2}dy_2\\
&\qquad=O(1)e^{y/2+(1/2-\alpha)c(x_1,y_1)}.
\end{aligned}
$$

Thus, we get

$$
\int_{-I_n}^{I_n}\int_{(1-\varepsilon)R_n\wedge(R_n-y)}^{R_n}\mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}(y)\}}\int_{-I_n}^{I_n}\int_0^{R_n-y_1}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p_1)\cap\mathcal{B}_{\mathbb{H},n}(y)\}}\times
$$

$$e^{-\alpha(y_1+y_2)} dy_2 dx_2 dy_1 dx_1$$
$$\leq O(1) \int_{-I_n}^{I_n} \int_{(1-\varepsilon)R_n \wedge (R_n - y)}^{R_n} e^{y/2+(1/2-\alpha)c(x_1,y_1)} e^{-\alpha y_1} dy_1 dx_1.$$

Due to symmetry, to bound the integral it is enough to integrate this with respect to x_1 from 0 to I_n. We will split this integral into two parts according to the value of $c(x_1, y_1)$:

$$\int_0^{I_n} e^{c(x_1,y_1)(1/2-\alpha)} dx_1 = \int_{c_K e^{y_1/2}}^{I_n} e^{c(x_1,y_1)(1/2-\alpha)} dx_1 + \int_0^{c_K e^{y_1/2}} dx_1.$$

The first integral becomes:

$$\begin{aligned}
\int_{c_K e^{y_1/2}}^{I_n} e^{c(x_1,y_1)(1/2-\alpha)} dx_1 &= O(1) \cdot \int_{c_K e^{y_1/2}}^{I_n} x_1^{1-2\alpha} e^{-\frac{y_1}{2}(1-2\alpha)} dx_1 \\
&= \begin{cases} O(R_n) \cdot e^{-y_1/2+\alpha y_1} \cdot e^{\frac{R_n}{2} 2(1-\alpha)} & \text{if } \alpha \leq 1 \\ O(1) \cdot e^{-y_1/2+\alpha y_1 + 2(1-\alpha) y_1/2} & \text{if } \alpha > 1 \end{cases} \\
&= \begin{cases} O(R_n) \cdot e^{(\alpha-1/2)y_1} \cdot n^{2(1-\alpha)} & \text{if } \alpha \leq 1 \\ O(1) \cdot e^{y_1/2} & \text{if } \alpha > 1 \end{cases}.
\end{aligned}$$

The second integral trivially gives:

$$\int_0^{c_K e^{y_1/2}} dx_1 = O(1) \cdot e^{y_1/2}.$$

Putting these two together we conclude that

$$e^{y/2} \cdot \int_0^{I_n} e^{c(x_1,y_1)(1/2-\alpha)} dx_1 = O(1) \cdot e^{y_1/2 + y(3/2-\alpha)}.$$

Now, we integrate these with respect to y_1:

$$n^{2(1-\alpha)} \cdot \int_{(1-\varepsilon)R_n \wedge (R_n - y)}^{R_n} e^{(\alpha-1/2)y_1 - \alpha y_1} dy_1 = O(1) \cdot n^{2(1-\alpha)} \cdot e^{-R_n/2 + \varepsilon R_n/2 + y/2} \tag{121}$$
$$= O(1) \cdot n^{1-2\alpha+\varepsilon} \cdot e^{y/2}. \tag{122}$$

Therefore, we conclude that

$$\mathcal{I}_n^{(3)}(y) = O(R_n)\, n^{1-2\alpha+\varepsilon(2\alpha-1)}\, e^{y/2} \rho_{\widetilde{\mathbb{H}}}(y, k_n - 2)$$

and hence, using again Lemma 9.4,

$$\begin{aligned}
&\binom{k_n}{2}^{-1} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(3)}(y) e^{-\alpha y} \,\mathrm{d}x \,\mathrm{d}y \\
&\quad = O(R_n)\, n^{1-2\alpha+\varepsilon(2\alpha-1)} k_n^{-2} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} e^{y/2} \rho_{\widetilde{\mathbb{H}}}(y, k_n - 2) e^{-\alpha y} \,\mathrm{d}x \,\mathrm{d}y \\
&\quad = O(R_n)\, n^{1-2\alpha+\varepsilon(2\alpha-1)}.
\end{aligned}$$

It follows that for $\varepsilon = \varepsilon(\alpha)$ small enough

$$k_n^{4\alpha-2} R_n n^{1-2\alpha+\varepsilon(2\alpha-1)} = R_n n^{\varepsilon(2\alpha-1)} \left(\frac{k_n^2}{n}\right)^{2\alpha-1} = o(1)$$

and hence for $\alpha > 1/2$,

$$\lim_{n\to\infty} k_n^{4\alpha-2} \binom{k_n}{2}^{-1} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(3)}(y) e^{-\alpha y} \, \mathrm{d}x \, \mathrm{d}y = 0.$$

Since $4\alpha - 2 \geq 1$ when $\alpha \geq 3/4$ it immediately follows that

$$\lim_{n\to\infty} k_n \binom{k_n}{2}^{-1} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(3)}(y) e^{-\alpha y} \, \mathrm{d}x \, \mathrm{d}y = 0.$$

The sum of (104) We will give an upper bound for

$$\mathbb{E}\left[\sum_{\substack{p_1, p_2 \in \mathcal{P}\setminus\{(0,y)\} \\ y(p_1) \geq K}} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(y)\setminus \mathcal{B}_{\mathcal{P}}(y)\}} \mathbb{1}_{\{p_2 \in \mathcal{B}_{\mathbb{H},n}(y) \cap \mathcal{B}_{\mathcal{P}}(y)\}} \, D_{\mathbb{H}}(y, k_n - 2; \mathcal{P} \setminus \{p_1, p_2\}) \right].$$

Let us set $p = (0, y)$. Recall that $\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y) \cap \mathcal{R}([R_n - y + 2\log\left(\frac{\pi}{2}\right), R_n]) = \emptyset$. Thus, the summand in the above sum is equal to 0, when $y_1 > R_n - y + 2\log(\pi/2)$.

Recall the definition of the extended ball $\check{\mathcal{B}}_{\mathbb{H},n}(p)$ around p (116) that contains both $\mathcal{B}_{\mathbb{H},n}(p)$ and $\mathcal{B}_{\mathcal{P}}(p)$

$$\check{\mathcal{B}}_{\mathbb{H},n}(p) := \{p' : y' < R_n - y, \ |x'| < (1+K)e^{\frac{1}{2}(y+y')}\},$$

and that we have $\mathbb{E}\left[D_{\mathbb{H}}(y, k_n - 2; \mathcal{P} \setminus \{p_1, p_2\})\right] = \rho_{\widetilde{\mathbb{H}}}(y, k_n - 2)$.

Observe that,

$$\mathcal{B}_{\mathbb{H},n}(p) \cap \mathcal{R}([0, r(p))) \subseteq \check{\mathcal{B}}_{\mathbb{H},n}(p)$$

and

$$\mathcal{B}_{\mathbb{H},n}(p) \cap \mathcal{R}([r(p), R_n]) = \mathcal{R}([r(p), R_n]).$$

We thus conclude that

$$\mathcal{B}_{\mathbb{H},n}(p) \subseteq \check{\mathcal{B}}_{\mathbb{H},n}(p) \cup \mathcal{R}([r(p), R_n]). \tag{123}$$

Hence, if we set

$$h_y(p_1, \mathcal{P}) := \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p)\setminus\mathcal{B}_{\mathcal{P}}(p)\}} \cdot \left(\mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(p_1) \cap \check{\mathcal{B}}_{\mathbb{H},n}(p)\right) + \mu_{\alpha,\nu}\left(\mathcal{R}([R_n - y, R_n])\right)\right),$$

then

$$\begin{aligned}
&\mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p)\setminus\mathcal{B}_{\mathcal{P}}(p)\}} \cdot \mathbb{E}\left[\left(\sum_{p_2 \in \mathcal{P}\setminus\{p, p_1\}} \mathbb{1}_{\{p_2 \in \mathcal{B}_{\mathbb{H},n}(p)\cap\mathcal{B}_{\mathcal{P}}(p_1)\}}\right) \cdot D_{\mathbb{H}}(y, k_n - 2; \mathcal{P} \setminus \{p_1, p_2\})\right] \\
&= O(1) \cdot \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p)\setminus\mathcal{B}_{\mathcal{P}}(p)\}} \cdot \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(p) \cap \mathcal{B}_{\mathbb{H},n}(p_1)) \rho_{\widetilde{\mathbb{H}}}(y, k_n - 2) \\
&\leq O(1) \cdot h_y(p_1, \mathcal{P}) \rho_{\widetilde{\mathbb{H}}}(y, k_n - 2).
\end{aligned}$$

To calculate the expectation of the above function we need to approximate the intersection of the two balls $\check{\mathcal{B}}_{\mathbb{H},n}(p)$ and $\check{\mathcal{B}}_{\mathbb{H},n}(p_1)$, where $p_1 = (x_1, y_1)$. Let us assume without loss of generality that $x_1 > 0$. The right boundary of $\check{\mathcal{B}}_{\mathbb{H},n}(p)$ is given by the equation $x = x(y_1) = (1+K)e^{\frac{1}{2}(y+y_1)}$ whereas the left boundary of $\check{\mathcal{B}}_{\mathbb{H},n}(p_1)$ is given by the curve $x = x(y_1) = x_1 - (1+K)e^{\frac{1}{2}(y+y_1)}$.

The equation that determines the intersecting point of the two curves is

$$x_1 - (1+K)e^{(\hat{y}+y_1)/2} = (1+K)e^{(\hat{y}+y)/2},$$

where $\hat{y}$ is the y-coordinate of the intersecting point. We can solve the above for $\hat{y}$

$$x_1 = (1+K)e^{\hat{y}/2}\left(e^{y/2} + e^{y_1/2}\right).$$

But since $p_1 = (x_1, y_1) \in \mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p)$, we also have $x_1 > e^{\frac{y+y_1}{2}}$. Therefore,

$$e^{\hat{y}/2} > \frac{1}{1+K} \frac{e^{\frac{y+y_1}{2}}}{e^{y/2}+e^{y_1/2}} \geq \frac{1}{2(1+K)} \frac{e^{\frac{y_1+y}{2}}}{e^{\max\{y,y_1\}/2}} > \frac{1}{2(1+K)} e^{\min\{y,y_1\}/2}.$$

The above yields

$$\hat{y} > \min\{y, y_1\} - 2\log(2(1+K)) := c(y_1, y). \tag{124}$$

which, in turn, implies the following

$$p \in \check{\mathcal{B}}_{\mathbb{H},n}((0,y)) \cap \check{\mathcal{B}}_{\mathbb{H},n}(p_1) \Rightarrow y(p) \geq c(y_1, y). \tag{125}$$

We thus conclude that

$$\mathcal{B}_{\mathbb{H},n}(p_1) \cap \mathcal{B}_{\mathbb{H},n}(p) \subseteq \left(\check{\mathcal{B}}_{\mathbb{H},n}(p) \cap \mathcal{R}([c(y_1,y), R_n])\right) \cup \mathcal{R}([R_n - y, R_n]),$$

which in turn implies that

$$\mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(p_1) \cap \mathcal{B}_{\mathbb{H},n}(p)\right) \leq \mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(p) \cap \mathcal{R}([c(y_1,y), R_n]\right) + \mu_{\alpha,\nu}(\mathcal{R}([R_n - y, R_n])).$$

Therefore,

$$\begin{aligned} h_y(p_1, \mathcal{P}) \leq\ & \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p)\setminus\mathcal{B}_{\mathcal{P}}(p)\}} \mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(p) \cap \mathcal{R}([c(y_1,y), R_n])\right) \\ & + \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p)\setminus\mathcal{B}_{\mathcal{P}}(p)\}} \mu_{\alpha,\nu}\left(\mathcal{R}([R_n - y, R_n])\right). \end{aligned}$$

Now, the Campbell-Mecke formula (10) gives

$$\begin{aligned} & \mathbb{E}\left[\sum_{\substack{p_1,p_2 \in \mathcal{P}\setminus\{(0,y)\} \\ y(p_1)\geq K}} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(y)\setminus\mathcal{B}_{\mathcal{P}}(y)\}} \mathbb{1}_{\{p_2 \in \mathcal{B}_{\mathbb{H},n}(y)\cap\mathcal{B}_{\mathcal{P}}(y)\}} D_{\mathbb{H}}(y, k_n - 2; \mathcal{P}\setminus\{p_1,p_2\})\right] \\ & \leq \mathbb{E}\left[\left(\sum_{p_1\in\mathcal{P}} h_y(p_1, \mathcal{P}\setminus\{p_1\})\right)\right] \\ & = \frac{\nu\alpha}{\pi}\int_{\mathcal{R}_n} \mathbb{E}\left[h_y(p_1, \mathcal{P}\setminus\{p_1\})\right] e^{-\alpha y_1}\, \mathrm{d}x_1\, \mathrm{d}y_1 \\ & \leq \frac{\nu\alpha}{\pi}\int_{\mathcal{R}_n} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p)\setminus\mathcal{B}_{\mathcal{P}}(p)\}} \mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(p) \cap \mathcal{R}([c(y_1,y), R_n])\right) e^{-\alpha y_1}\, \mathrm{d}x_1\, \mathrm{d}y_1 \qquad (126) \\ & \quad + \frac{\nu\alpha}{\pi}\int_{\mathcal{R}_n} \mathbb{1}_{\{p_1 \in \mathcal{B}_{\mathbb{H},n}(p)\setminus\mathcal{B}_{\mathcal{P}}(p)\}} \mu_{\alpha,\nu}\left(\mathcal{R}([R_n - y, R_n])\right) e^{-\alpha y_1}\, \mathrm{d}x_1\, \mathrm{d}y_1. \qquad (127) \end{aligned}$$

Recall that $(\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))) \cap \mathcal{R}([R_n - y + 2\log\left(\frac{\pi}{2}\right), R_n]) = \emptyset$. We will first calculate the measures $\mu_{\alpha,\nu}$ appearing in (126) and (127). The first one is:

$$\begin{aligned} \mu_{\alpha,\nu}\left(\check{\mathcal{B}}_{\mathbb{H},n}(y) \cap \mathcal{R}([c(y_1,y), R_n])\right) & \leq (1+K)\frac{\nu\alpha}{\pi} \cdot e^{y/2} \int_{c(y_1,y)}^{R_n} e^{-(\alpha-\frac{1}{2})y'}\, dy' \\ & = O\left(e^{\frac{y}{2} - (\alpha-\frac{1}{2})\min\{y,y_1\}}\right). \end{aligned}$$

The second term is:

$$\mu_{\alpha,\nu}\left(\mathcal{R}([R_n - y, R_n])\right) = \frac{\nu\alpha}{\pi}\int_{R_n - y}^{R_n} \pi e^{\frac{R_n}{2}} e^{-\alpha y'}\, dy' = O\left(e^{\frac{R_n}{2}} e^{-\alpha(R_n - y)}\right) = O\left(e^{\alpha y - (\alpha-\frac{1}{2})R_n}\right).$$

Using these, we get

$$\int_{\mathcal{R}_n([0, R_n - y_n + 2\ln\frac{\pi}{2}])} \mathbb{E}\left[h_y(p_1, \mathcal{P}\setminus\{p_1\})\right] e^{-\alpha y_1}\, dx_1\, dy_1$$

$$= O(1) \int_{\mathcal{R}_n([0,R_n-y+2\ln\frac{\pi}{2}])} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(p)\}} e^{\frac{y}{2}-(\alpha-\frac{1}{2})\min\{y,y_1\}-\alpha y_1}\, dx_1\, dy_1 \tag{128}$$

$$+ O(1) \int_{\mathcal{R}_n([0,R_n-y+2\ln\frac{\pi}{2}])} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H},n}((0,y))\}} e^{\alpha y-(\alpha-\frac{1}{2})R_n-\alpha y_1}\, dx_1\, dy_1. \tag{129}$$

Now, Lemma 2.2 implies that for any $y \in [0, R_n - y_n + 2\ln\frac{\pi}{2}]$, we have

$$\int_{-I_n}^{I_n} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)\}}\, dx_1 \leq 2Ke^{\frac{3}{2}(y_1+y)-R_n}.$$

Therefore, (128) is

$$\begin{aligned}
&O(1)\cdot e^{2y-R_n} \int_0^{R_n-y+2\ln\frac{\pi}{2}} e^{\frac{3y_1}{2}-(\alpha-\frac{1}{2})\min\{y_1,y\}-\alpha y_1}\, dy_1 \\
&= O(1)\cdot e^{2y-R_n}\left(\int_0^y e^{\frac{3y_1}{2}-(2\alpha-\frac{1}{2})y_1}\, dy_1 + e^{-(\alpha-\frac{1}{2})y}\int_y^{R_n-y+2\ln\frac{\pi}{2}} e^{(\frac{3}{2}-\alpha)y_1}\, dy_1\right) \\
&= O(1)\left(\begin{cases} e^{(4-2\alpha)y-R_n}, & \text{if } \alpha < 2 \\ R_n\cdot e^{2y-R_n}, & \text{if } \alpha \geq 2 \end{cases} + \begin{cases} e^{-(\alpha-\frac{1}{2})R_n+y}, & \text{if } \alpha < 3/2 \\ R_n\cdot e^{2(2-\alpha)y-R_n}, & \text{if } \alpha \geq 3/2 \end{cases}\right).
\end{aligned}$$

Similarly, for (129) we have

$$\begin{aligned}
&\int_{\mathcal{R}([0,R_n-y+2\ln\frac{\pi}{2}])} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}} e^{\alpha y-(\alpha-\frac{1}{2})R_n-\alpha y_1}\, dx_1\, dy_1 \\
&\quad= e^{\frac{3y}{2}-R_n+\alpha y-(\alpha-\frac{1}{2})R_n}\cdot\int_0^{R_n-y+2\ln\frac{\pi}{2}} e^{\frac{3y_1}{2}-\alpha y_1}\, dy_1 \\
&\quad= O(1)\cdot\begin{cases} e^{\frac{3y}{2}-R_n+\alpha y-(\alpha-\frac{1}{2})R_n+(\frac{3}{2}-\alpha)(R_n-y)}, & \text{if } \alpha < 3/2 \\ R_n\cdot e^{(\frac{3}{2}+\alpha)y-(\alpha+\frac{1}{2})R_n}, & \text{if } \alpha \geq 3/2 \end{cases} \\
&\quad= O(1)\cdot\begin{cases} e^{-(2\alpha-1)R_n+2\alpha y}, & \text{if } \alpha < 3/2 \\ R_n\cdot e^{(\frac{3}{2}+\alpha)y-(\alpha+\frac{1}{2})R_n}, & \text{if } \alpha \geq 3/2 \end{cases}.
\end{aligned}$$

We thus conclude, using $2(2-\alpha)y \leq y$ for $\alpha > 3/2$, that

$$\mathbb{E}\left[\left(\sum_{p_1\in\mathcal{P}\setminus\{p\}} h_y(p_1, \mathcal{P}\setminus\{p_1\})\right)\right] \leq O(1)\cdot\left(\mathcal{I}_n^{(1)}(y) + \mathcal{I}_n^{(2)}(y) + \mathcal{I}_n^{(3)}(y)\right), \tag{130}$$

where

$$\begin{aligned}
\mathcal{I}_n^{(1)}(y) &= \begin{cases} e^{(4-2\alpha)y-R_n}, & \text{if } \alpha < 2 \\ R_n\cdot e^{2y-R_n}, & \text{if } \alpha \geq 2 \end{cases}, \\
\mathcal{I}_n^{(2)}(y) &= \begin{cases} e^{-(\alpha-\frac{1}{2})R_n+y}, & \text{if } \alpha < 3/2 \\ R_n\cdot e^{y-R_n}, & \text{if } \alpha \geq 3/2 \end{cases} \\
\mathcal{I}_n^{(3)}(y) &= \begin{cases} e^{-(2\alpha-1)R_n+2\alpha y}, & \text{if } \alpha < 3/2 \\ R_n\cdot e^{(\frac{3}{2}+\alpha)y-(\alpha+\frac{1}{2})R_n}, & \text{if } \alpha \geq 3/2 \end{cases}.
\end{aligned}$$

We now need to calculate:

$$\binom{k_n}{2}^{-1}\cdot\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\int_{\mathcal{K}_C(k_n)} \mathbb{E}\left[\left(\sum_{p_1\in\mathcal{P}} h_y(p_1, \mathcal{P}\setminus\{p_1\})\right)\right]\cdot\rho_{\widetilde{\mathbb{H}}}(y, k_n-2)e^{-\alpha y}\, \mathrm{d}y\, \mathrm{d}x.$$

Firstly, note that as $\overline{n}_{\widetilde{\mathbb{H}}}(k_n) = \Theta(1) \cdot n \cdot k_n^{-(2\alpha+1)}$, we have

$$\binom{k_n}{2}^{-1} \cdot \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} = O(1) \cdot \frac{k_n^{2\alpha-1}}{n}.$$

Also, $\mathbb{E}\left[\left(\sum_{p_1 \in \mathcal{P}} h_y(p_1, \mathcal{P} \setminus \{p_1\})\right)\right]$ is given as the sum of $\mathcal{I}_n^{(1)}(y), \mathcal{I}_n^{(2}(y)$ and $\mathcal{I}_n^{(3)}(y)$ (cf. (130)). Setting

$$J_3 = \frac{k_n^{2\alpha-1}}{n} \cdot (M_1 + M_2 + M_3)$$

with

$$M_i = \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(i)}(y) \rho_{\widetilde{\mathbb{H}}}(y, k_n - 1) e^{-\alpha y} \, \mathrm{d}y \, \mathrm{d}x$$

it follows that

$$\binom{k_n}{2}^{-1} \overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1} \int_{\mathcal{K}_C(k_n)} \mathbb{E}\left[\left(\sum_{p_1 \in \mathcal{P} \setminus \{(0,y)\}} h_y(p_1, \mathcal{P} \setminus \{p_1\})\right)\right] \rho_{\widetilde{\mathbb{H}}}(y, k_n - 1) e^{-\alpha y} \, \mathrm{d}y \, \mathrm{d}x$$
$$= O(1) \cdot J_3$$

Computing each of the integral separately we obtain, using Lemma 9.4 and the fact that $n = \nu e^{R_n/2}$,

$$M_1 := \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(1)}(y) \rho_{\widetilde{\mathbb{H}}}(y, k_n - 1) e^{-\alpha y} \, \mathrm{d}y \, \mathrm{d}x = O(1) \cdot \begin{cases} \frac{k_n^{7-6\alpha}}{n}, & \text{if } \alpha < 2 \\ R_n^2 \frac{k_n^{3-2\alpha}}{n}, & \text{if } \alpha \geq 2 \end{cases}.$$

$$\begin{aligned} M_2 := \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(2)}(y) \rho_{\widetilde{\mathbb{H}}}(y, k_n - 1) e^{-\alpha y} dy &= O(1) \cdot \begin{cases} e^{-(\alpha-1)R_n} k_n^{-2\alpha+1}, & \text{if } \alpha < 3/2 \\ R_n \frac{k_n^{7-6\alpha}}{n}, & \text{if } \alpha \geq 3/2 \end{cases} \\ &= \begin{cases} \frac{k_n^{1-2\alpha}}{n^{2(\alpha-1)}}, & \text{if } \alpha < 3/2 \\ R_n \frac{k_n^{7-6\alpha}}{n}, & \text{if } \alpha \geq 3/2 \end{cases} \end{aligned}$$

and finally

$$\begin{aligned} M_3 := \int_{\mathcal{K}_C(k_n)} \mathcal{I}_n^{(3)}(y) \rho_{\widetilde{\mathbb{H}}}(y, k_n - 1) e^{-\alpha y} dy &= O(1) \cdot \begin{cases} e^{-(2\alpha-3/2)R_n} k_n^{2\alpha-1}, & \text{if } \alpha < 3/2 \\ R_n e^{-\alpha R_n} k_n^2, & \text{if } \alpha \geq 3/2 \end{cases} \\ &= O(1) \cdot \begin{cases} \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}}, & \text{if } \alpha < 3/2 \\ R_n \cdot \frac{k_n^2}{n^{2\alpha}}, & \text{if } \alpha \geq 3/2 \end{cases}. \end{aligned}$$

Now, we will consider the two cases according to the value of α. Assume first that $1/2 < \alpha < 3/4$. In this case, we want to show that

$$\lim_{n \to \infty} k_n^{4\alpha-2} \cdot J_3 = 0. \tag{131}$$

Using the above expression for J_3, we have

$$k_n^{4\alpha-2} \cdot J_3 = O(1) \cdot \frac{k_n^{6\alpha-3}}{n} \cdot \left(\frac{k_n^{7-6\alpha}}{n} + \frac{k_n^{2(1-\alpha)}}{n^{2(\alpha-1)}} + \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}}.\right)$$

We wish to show that each one of the above three terms is $o(1)$ for $k_n = O(n^{\frac{1}{2\alpha+1}})$. For the first one we have

$$\frac{k_n^{6\alpha-3}}{n} \cdot \frac{k_n^{7-6\alpha}}{n} = \frac{k_n^4}{n^2} = O(1) \cdot n^{\frac{4}{2\alpha+1}-2} = o(1),$$

since $4 < 4\alpha + 2$ when $1/2 < \alpha$. The second one yields:

$$\frac{k_n^{6\alpha-3}}{n} \cdot \frac{k_n^{-2\alpha+1}}{n^{2(\alpha-1)}} = \frac{k_n^{4\alpha-2}}{n^{2\alpha-1}} = O(1)\frac{n^{\frac{4\alpha-2}{2\alpha+1}}}{n^{2\alpha-1}}.$$

We need to show that $\frac{4\alpha-2}{2\alpha+1} < 2\alpha - 1$. Indeed, rearranging this yields, $4\alpha - 2 < 4\alpha^2 - 1$, which is equivalent to $0 < 4\alpha^2 - 4\alpha + 1 = (2\alpha - 1)^2$. This holds for all $\alpha > 1/2$.

Finally, the third one yields:

$$\frac{k_n^{6\alpha-3}}{n} \cdot \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}} = \frac{k_n^{8\alpha-4}}{n^{2(2\alpha-1)}} = \frac{k_n^{4(2\alpha-1)}}{n^{2(2\alpha-1)}}.$$

But $k_n^4 \leq O(1) \cdot n^{\frac{4}{2\alpha+1}} = o(n^2)$, as $2\alpha + 1 > 2$.

For $\alpha \geq 3/4$, we would like to show that

$$\lim_{n\to\infty} k_n \cdot J_3 = 0. \tag{132}$$

Firstly, if $3/4 < \alpha < 3/2$ we have,

$$k_n \cdot J_3 = O(1) \cdot \frac{k_n^{2\alpha}}{n} \cdot \left(\frac{k_n^{7-6\alpha}}{n} + \frac{k_n^{-2\alpha+1}}{n^{2(\alpha-1)}} + \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}}.\right)$$

As above we will deal with the three term of this. For the first one we have

$$\frac{k_n^{2\alpha}}{n}\frac{k_n^{7-6\alpha}}{n} = \frac{k_n^{7-4\alpha}}{n^2} \leq \frac{k_n^4}{n^2} = o(1).$$

The second one yields:

$$\frac{k_n^{2\alpha}}{n}\frac{k_n^{-2\alpha+1}}{n^{2(\alpha-1)}} = \frac{k_n}{n^{2\alpha-1}} \leq \frac{k_n}{n^{1/2}} = O(1)\frac{n^{\frac{1}{2\alpha+1}}}{n^{1/2}} = o(1).$$

Finally, the third one yields:

$$\frac{k_n^{2\alpha}}{n} \cdot \frac{k_n^{2\alpha-1}}{n^{4\alpha-3}} = \frac{k_n^{4\alpha-1}}{n^{2(2\alpha-1)}} = O(1)\frac{n^{\frac{4\alpha-1}{2\alpha+1}}}{n^{2(2\alpha-1)}}.$$

We need to show that $\frac{4\alpha-1}{2\alpha+1} < 2(2\alpha - 1)$, which is equivalent to $8\alpha^2 - 4\alpha - 1 > 0$; this is indeed the case for any $\alpha \geq 3/4$.

For $3/2 \leq \alpha < 2$, it is only M_2 and M_3 that change values. In particular, for any $\alpha \geq 3/2$ we have

$$\frac{k_n}{n} \cdot M_2 = O(1) \cdot R_n \cdot \frac{k_n^{2\alpha}}{n} \cdot \frac{k_n^{7-6\alpha}}{n} = o(1),$$

as above. Also,

$$\frac{k_n}{n} \cdot M_3 = O(1) \cdot R_n \cdot \frac{k_n}{n} \cdot \frac{k_n^2}{n^{2\alpha}} = R_n \cdot \frac{k_n^3}{n^{2\alpha+1}} = o(1),$$

since $k_n = o(n^{1/2})$ (and, therefore, $k_n^3 = o(n^{3/2})$) but $2\alpha + 1 > 2$.

If $\alpha \geq 2$ too, then M_1 changes value and we have

$$\frac{k_n}{n} \cdot M_1 = O(1) \cdot R_n^2 \cdot \frac{k_n}{n} \cdot \frac{k_n^{3-2\alpha}}{n} = \frac{k_n^{4-2\alpha}}{n^2} = o(1),$$

since $\alpha \geq 2$.

The sum of (105) Now, we will first give an upper bound on the term

$$\mathbb{E}\left[\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y_1<K}} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\cap\mathcal{B}_{\mathcal{P}}(y)\}}\right].$$

Using the Campbell-Mecke formula (10), we write

$$\begin{aligned}
&\mathbb{E}\left[\sum_{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\},\ y_1<K} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\cap\mathcal{B}_{\mathcal{P}}(y)\}}\right] = \\
&\le \int_{-I_n}^{I_n}\int_0^K\int_{-I_n}^{I_n}\int_0^{R_n} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\cap\mathcal{B}_{\mathcal{P}}(y)\}} e^{-\alpha y_2}e^{-\alpha y_1}dx_2dy_2dx_1dy_1 \\
&\le \mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(y))\cdot\int_{-I_n}^{I_n}\int_0^K \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)\}}e^{-\alpha y_1}dx_1dy_1.
\end{aligned}$$

By Lemma 6.3 and a concentration argument, the first factor is

$$\mu_{\alpha,\nu}(\mathcal{B}_{\mathbb{H},n}(y)) = O(1)e^{y/2}.$$

We bound the second factor using Lemma 2.2. In particular, (19) implies that if $(x_1,y_1)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))$, then because $y_1<K$

$$|x_1 - e^{(y+y_1)/2}| \le e^{(y+y_1)/2}\cdot Ke^{y+y_1-R_n} = O(1)e^{(y+y_1)/2}\cdot e^{y-R_n}.$$

Therefore,

$$\int_{-I_n}^{I_n}\int_0^K \mathbb{1}_{\{(x_1,y_1)\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}((0,y))\}}e^{-\alpha y_1}dx_1dy_1 = O(1)\cdot e^{y-R_n}\cdot\int_0^K e^{(y+y_1)/2}e^{-\alpha y_1}dy_1 = O(1)\cdot e^{3y/2-R_n},$$

and hence

$$\mathbb{E}\left[\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y_1<K}} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\cap\mathcal{B}_{\mathcal{P}}(y)\}}\right] = O(1)\cdot e^{2y-R_n}.$$

Now, we integrate this over y:

$$\begin{aligned}
e^{-R_n}\int_{\mathcal{K}_C(k_n)} e^{2y-\alpha y}dydx &= O(1)ne^{-R_n}\int_{I_\varepsilon(k_n)} e^{2y-\alpha y}dy \\
&= O(1)\cdot n^{-1}\int_{I_\varepsilon(k_n)} e^{2y-\alpha y}dy \\
&= O(1)\cdot n^{-1}\cdot\begin{cases} k_n^{2(2-\alpha)(1+\varepsilon)}, & \text{if } \alpha<2 \\ \log k_n, & \text{if } \alpha=2 \\ 1, & \text{if } \alpha>2 \end{cases}.
\end{aligned}$$

We deduce that

$$\binom{k_n}{2}^{-1}\overline{n}_{\widetilde{\mathbb{H}}}(k_n)^{-1}\mathbb{E}\left[\sum_{\substack{p_1,p_2\in\mathcal{P}\setminus\{(0,y)\}\\ y_1<K}} \mathbb{1}_{\{p_1\in\mathcal{B}_{\mathbb{H}\triangle\mathcal{P}}(y)\}}\mathbb{1}_{\{p_2\in\mathcal{B}_{\mathbb{H},n}(y)\cap\mathcal{B}_{\mathcal{P}}(y)\}}\right] = O(1)\cdot n^{-1}\cdot k_n^{2\alpha-1}.$$

To finish the argument assume first that $1/2<\alpha\le 3/4$. In this case, we will consider

$$k_n^{4\alpha-2}\cdot n^{-2}\cdot k_n^{2\alpha-1+4-2\alpha} = n^{-2}\cdot k_n^{4\alpha+1}.$$

But, $\alpha \leq 3/4$, we have $4\alpha + 1 \leq 4$ and $k_n = o(n^{1/2})$, whereby $k_n^{4\alpha+1} = o(n^2)$.

Now, suppose that $3/4 < \alpha < 2$. Here, we will consider

$$k_n \cdot n^{-2} \cdot k_n^{2\alpha-1+4-2\alpha} = \frac{k_n^2}{n^2} = o(1).$$

When $\alpha \geq 2$, we will bound $\log k_n$ and 1 by k_n and we will consider

$$k_n \cdot n^{-2} \cdot k_n^{2\alpha-1+1} = n^{-2} k_n^{2\alpha+1}.$$

But $k_n = O(1) \cdot n^{\frac{1}{2\alpha+1}}$, whereby $k_n^{2\alpha+1} = O(n)$ and the above term is therefore $o(1)$. □

9.3 Coupling $G_{\mathbb{H},n}$ to $G_{\widetilde{\mathbb{H}},n}$

Now that we have established the equivalence of the local clustering function between the Poisson hyperbolic graph $G_{\widetilde{\mathbb{H}},n}$ and $G_{\mathcal{P},n}$ the final step is to relate local clustering in $G_{\widetilde{\mathbb{H}},n}$ to the original hyperbolic random graph $G_{\mathbb{H},n}$. As mentioned in Section 2.5, this is done by moving from $c_{\mathbb{H},n}(k)$ to the adjusted local clustering $c^*_{\mathbb{H},n}(k)$ (Lemma 5.1) and then to $c^*_{\widetilde{\mathbb{H}},n}(k)$ (Proposition 5.2). We prove Proposition 5.2 first and will end this section with the proof of Lemma 5.1.

To achieve the results we consider the standard coupling between the binomial and Poisson process. That is, we take a sequence of i.i.d. random elements $z_1, z_2, \dots$ uniformly on the hyperbolic disk of radius R_n, i.e. according to the distribution (1). Then the original hyperbolic random graph consists of the first n points and the poissonized version of the first $N \overset{d}{=} \mathrm{Po}(n)$ many points (N is a Poisson random variable with mean n). Under this coupling $N_{\mathbb{H},n}(k) = \sum_{j=1}^{n} \mathbb{1}_{\{D_{\mathbb{H}}(z_j)=k\}}$ denotes the number of degree k vertices in the original hyperbolic random graph model with n vertices and $N_{\widetilde{\mathbb{H}},n}(k) = \sum_{j=1}^{N} \mathbb{1}_{\{D_{\widetilde{\mathbb{H}}}(z_j)=k\}}$ denotes the (random) number of degree k vertices in the Poisson version of the hyperbolic random graph.

We start with a result that relates the number of nodes with degree k_n in both models.

Lemma 9.5. *Let $\{k_n\}_{n\geq 1}$ be sequence of natural numbers with $0 \leq k_n \leq n-1$ and $k_n = o(n^{\frac{1}{2\alpha+1}})$. Then*

$$\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right) = o\left(nk^{-(2\alpha+1)}\right),$$

and in particular

$$\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] = \Theta\left(k_n^{-(2\alpha+1)} n\right).$$

Tobias: I thought it might be a good idea to show somewhere in the paper that the result of Gugelmann et al. that $N_k = (1 + o(1))p_k n$ a.a.s., extends for all $k \ll n^{1/(2\alpha+1)}$ – where p_k the expression in terms on Γ^+, ξ etc. Gugelmann et al. had show it only until some small power of n. Adding this just means we have to compute the leading constants for the expectation $\mathbb{E}N_k$, and bound the variance, in addition to what we do here. Somehow I had expected we would include that. In fact Markus had prepared a short write up of the variance part of the argument. Should not be too hard to add the rest, no?
We already have the degree sequence for the infinite model somewhere in the paper.

Proof. The second result follows directly from the first and Lemma 9.3. Therefore we will only need to prove the first statement of the lemma.

We use the Chernoff concentration result for a Poisson random (138), which states that with probability $n^{-C^2/2}$ the Poisson random variable N with expectation n is contained in the interval $[n - C\sqrt{n \log n}, n + C\sqrt{n \log n}]$. We proceed by bounding the effect on the number of degree k_n vertices by adding or removing $C\sqrt{n \log n}$ many vertices to $G_{\mathbb{H},n}(\alpha, \nu$ and from $G_{\widetilde{\mathbb{H}},n}(\alpha, \nu)$, respectively.

Define the events

$$A_n^{\pm} := \{N \in [n, n \pm C\sqrt{n \log n}]\},$$

Tobias: weird notation and let $A_n = A_n^- \cup A_n^+$. Then,

$$\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] \leq \mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\middle| A_n\right] + O\left(n^{1-C^2/2}\right)$$
$$= \mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\middle| A_n\right] + o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right)$$

by choosing C large enough, e.g. $C > \sqrt{2}$. What is left to show is that for any $C > 0$

$$\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\middle| A_n\right] = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right).$$

Let $V_{\mathbb{H},n}(k_n)$ be the set of degree k_n vertices in the binomial graph $G_{\mathbb{H},n}$ and $V_{\widetilde{\mathbb{H}},n}(k_n)$ be the set of degree k_n vertices in the Poisson graph $G_{\widetilde{\mathbb{H}},n}$. Then

$$\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right| = |V_{\mathbb{H},n}(k_n)\Delta V_{\widetilde{\mathbb{H}},n}(k_n)|,$$

where $A\Delta B$ denotes the symmetric difference between two sets A and B.

We first consider the case $N \in [n, n+C\sqrt{n\log n}]$, i.e. the event A_n^+. For $z \in V_{\widetilde{\mathbb{H}},n}(k_n)\backslash V_{\mathbb{H},n}(k_n)$, z has degree k_n in the Poisson graph, but not in the binomial graph; **Tobias:** I dont particularly like the name "binomial graph". Maybe just use "standard KPKVB graph" or "fixed number of node graph" so as $N \geq n$, either z or one of its k_n neighbors must have been removed during the transition from the Poisson graph to the binomial graph. On the event A_n, at most $C\sqrt{n\log n}$ many vertices are removed. The probability of hitting a degree k_n vertex or one of its neighbors is at most $\frac{k_n+1}{N} \leq \frac{k_n+1}{n}$. Therefore, by the union bound, the probability that a particular degree k_n vertex of the Poisson graph is removed is upper bounded by $C\sqrt{n\log n}\frac{k_n+1}{n}$. Hence, the expected number of degree k_n vertices that disappear in the transition from the Poisson graph to the binomial graph is bounded by

$$\mathbb{E}\left[|V_{\widetilde{\mathbb{H}},n}(k_n) \setminus V_{\mathbb{H},n}(k_n)\middle| A_n^{(1)}\right] \leq \mathbb{E}[N_{\widetilde{\mathbb{H}},n}(k_n)]C\sqrt{n\log n}\frac{k_n+1}{n} = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right),$$

Tobias: absolute bar missing above where the last line follows since for $\alpha > 1/2$,

$$k_n\sqrt{\frac{\log(n)}{n}} = o\left(n^{\frac{1}{2\alpha+1}}\sqrt{\frac{\log(n)}{n}}\right) = o\left(n^{-\frac{2\alpha-1}{4\alpha+2}}\sqrt{\log(n)}\right) = o(1).$$

For $z \in V_{\mathbb{H},n}(k_n)\backslash V_{\widetilde{\mathbb{H}},n}(k_n)$, z is a degree k_n vertex in the binomial graph, but must have degree $k_n + \ell$ in the Poisson graph (where $1 \leq \ell \leq c\sqrt{n\log n}$). By linearity of expectation the expected number of degree $k_n + \ell$ vertices of the Poisson graph which turn into degree k_n vertices of the binomial graph is equal to the expected number of degree $k_n + \ell$ vertices in the Poisson graph times the probability that a degree $k_n + \ell$ vertex turns into a degree k_n vertex in the transition back, from the Poisson graph to the binomial graph. The probability of choosing uniformly a set of ℓ neighbors of a degree $k_n + \ell$ vertex of the Poisson graph is given by $\frac{k_n+\ell}{N}\cdots\frac{k_n+1}{N-\ell+1}$. **Tobias:** what exactly is the chance experiment here? You are doing ordered, which is not what I would have expected. Maybe that can spelled out. Now, using $k_n = o\left(n^{\frac{1}{2\alpha+1}}\right) = o\left(C\sqrt{n\log n}\right)$ for $\alpha > \frac{1}{2}$, $\ell \leq C\sqrt{n\log n}$ and $N - \ell + 1 \geq n$, this probability is bounded from above by $(C+1)^\ell(\frac{\sqrt{n\log n}}{n})^\ell = ((C+1)\sqrt{\frac{\log n}{n}})^\ell$ which is upper bounded by $(\frac{1}{2})^\ell$ for n large enough, i.e. $n \geq n_0$. Therefore, using the geometric series, we conclude

$$\mathbb{E}\left[|V_{\mathbb{H},n}(k_n) \setminus V_{\widetilde{\mathbb{H}},n}(k_n)|\middle| A_n^+\right] \leq \sum_{\ell=1}^{\sqrt{n\log n}} \mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n+\ell)\right]\left((c+1)\sqrt{\frac{\log n}{n}}\right)^\ell$$
$$\leq \sum_{\ell=1}^{\sqrt{n\log n}} \Theta\left(n(k_n+\ell)^{-2\alpha-1}\right)\left((c+1)\sqrt{\frac{\log n}{n}}\right)^\ell$$

$$= O\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right)\sum_{\ell=1}^{\sqrt{n\log n}}\left((c+1)\sqrt{\frac{\log n}{n}}\right)^{\ell} = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right),$$

and hence

$$\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\middle| A_n^+\right] = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right).$$

The case $N \in [n - C\sqrt{n\log n}, n)$ (event A_n^-) follows by similar arguments. As $N < n$, a vertex $z \in V_{\widetilde{\mathbb{H}},n}(k_n) \setminus V_{\mathbb{H},n}(k_n)$ with degree k_n in the Poisson graph must have a strictly larger degree in the binomial graph, i.e. in the transition from the Poisson graph to the binomial graph, a vertex must have been dropped in the neighborhood of z. By the union bound, this can be upper bounded by the number of additional vertices (of the binomial graph) times the probability that a random point falls into the neighborhood of a degree k_n vertex. We obtain

$$\mathbb{E}\left[|V_{\widetilde{\mathbb{H}},n}(k_n)\backslash V_{\mathbb{H},n}(k_n)|\middle| A_n^{(2)}\right] = O\left(\sqrt{n\log n}\frac{k_n}{n}\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right) = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right)$$

Tobias: Not so fast! You seem assume that dropping into the ball around a **degree** k vertex has order k/n. That has to be argued. It is of course only true if the height of the particular vertex is appropriate. Please correct / provide the relevant details. I guess we may have to appeal to "concentration of heights" here. A vertex $z \in V_{\mathbb{H},n}(k_n) \setminus V_{\widetilde{\mathbb{H}},n}(k_n)$ could be one of the additional vertices in the binomial graph or it is a degree $k_n - \ell$ vertex of the Poisson graph which receives exactly ℓ new vertices in its neighborhood in the transition from the Poisson graph to the binomial graph. The probability that one of the additional vertices of the binomial graph (compared to the smaller Poisson graph) has degree k_n is asymptotically of the order $k_n^{-(2\alpha+1)}$ (as can be seen by considering the alternative coupling between the binomial and the Poisson process, where instead of taking $z_1, \ldots, z_N$ for the Poisson process, we take the points $z_n, z_{n-1}, \ldots, z_{n-N+1}$ (resp. points with index larger than n after we hit z_1): **Tobias:** I don't quite follow the construction here. for this graph, we have that the expected number of degree k_n vertices is $\Theta\left(nk^{-2\alpha-1}\right)$, so the probability that a vertex chosen uniformly from the Poisson graph has degree k is $\Theta(k^{-2\alpha-1})$). Therefore, the expected number of additional points with degree k_n is $O\left(\sqrt{n\log n}k_n^{-2\alpha-1}\right) = o\left(nk_n^{-2\alpha-1}\right) = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right)$. The expected number of degree $k_n - \ell$ vertices of the Poisson graph which receive exactly ℓ new vertices can be bounded in a sum resp. series similarly as done for $z \in V_{\mathbb{H},n}(k_n) \setminus V_{\widetilde{\mathbb{H}},n}(k_n)$ in the case $N \geq n$. We therefore conclude that

$$\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\middle| A_n^-\right] = o\left(\mathbb{E}\left[N_{\widetilde{\mathbb{H}},n}(k_n)\right]\right),$$

which finishes the proof. □

We note that this result together with Lemma 9.3 implies Lemma ??. Next we prove Proposition 5.2, which states

$$\lim_{n\to\infty} s_\alpha(k_n)\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = 0.$$

Proof of Proposition 5.2. First we note that Proposition 5.3, Proposition 5.4, Proposition 5.5 and Theorem 1.3 together imply that

$$\mathbb{E}\left[c^*_{\widetilde{\mathbb{H}},n}(k_n)\right] = (1 + o(1))s_\alpha(k_n)$$

Therefore it suffices to show that

$$\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = o\left(\mathbb{E}\left[c^*_{\widetilde{\mathbb{H}},n}(k_n)\right]\right).$$

For this we observe that we are looking at the modified clustering coefficient, where we divide by the expected number of degree k_n vertices. As the expected numbers of degree k_n vertices in $G_{\widetilde{\mathbb{H}},n}$ and $G_{\mathbb{H},n}$ are asymptotically equivalent (see Lemma 9.5), it is therefore sufficient to consider the sum of the clustering coefficients of all vertices of degree k_n. Given again the standard coupling between the binomial and Poisson process (as used in the proof of Lemma 9.5), we denote by $V_{\mathbb{H},n}(k_n)$ the set of degree k_n vertices in $G_{\mathbb{H},n}$ and by $V_{\widetilde{\mathbb{H}},n}(k_n)$ the set of degree k_n vertices in the graph $G_{\widetilde{\mathbb{H}},n}$. If a vertex is contained in both sets, it must have the same degree in both the Poisson and binomial graph, and given the nature of the coupling, the neighbourhoods are therefore the same and hence also their clustering coefficients agree.

The difference of the sum of the clustering coefficients therefore comes from all the clustering coefficients of the symmetric difference $V_{\mathbb{H},n}(k_n)\Delta V_{\widetilde{\mathbb{H}},n}(k_n)$. This symmetric difference is again a Poisson process, whose expected number of points is $\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] = o\left(\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]\right)$ by Lemma 9.5. Therefore we have that

$$\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c^*_{\widetilde{\mathbb{H}},n}(k_n)\right|\right] \leq \frac{\mathbb{E}\left[\left|N_{\mathbb{H},n}(k_n) - N_{\widetilde{\mathbb{H}},n}(k_n)\right|\right]}{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]}\mathbb{E}\left[c^*_{\widetilde{\mathbb{H}},n}(k_n)\right] = o\left(1\right)\mathbb{E}\left[c^*_{\widetilde{\mathbb{H}},n}(k_n)\right],$$

which finishes the proof. □

We end this section with the proof of Lemma 5.1, whose statement is

$$\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c_{\mathbb{H},n}(k_n)\right|\right] = o\left(s_\alpha(k_n)\right).$$

Proof of Lemma 5.1. Let $0 < \delta < 1$ and define the event

$$A_n = \left\{\left|N_{\mathbb{H},n}(k_n) - \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]\right| \leq \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{\frac{1+\delta}{2}}\right\}.$$

Since $N_{\mathbb{H},n}(k_n) = \sum_{i=1}^n \mathbb{1}_{\{D_{\mathbb{H}}(i)=k_n\}}$ it follows from Lemma C.1, with $c = \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{-\frac{1-\delta}{2}}$, that

$$\mathbb{P}\left(A_n\right) \geq 1 - O\left(e^{-\frac{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^\delta}{2}}\right) = 1 - O\left(e^{-\frac{n^\delta k_n^{-\delta(2\alpha+1)}}{2}}\right), \tag{133}$$

where the last part is due to Lemma 9.5.

On the event A_n

$$\left|\frac{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]}{N_{\mathbb{H},n}(k_n)} - 1\right| \leq \frac{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{\frac{1+\delta}{2}}}{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right] + \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{\frac{1+\delta}{2}}} \leq \mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{-\frac{1-\delta}{2}}.$$

Therefore we have

$$\begin{aligned}
\mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c_{\mathbb{H},n}(k_n)\right|\right] &\leq \mathbb{E}\left[\left|c^*_{\mathbb{H},n}(k_n) - c_{\mathbb{H},n}(k_n)\right| \mathbb{1}_{\{A_n\}}\right] + O\left(1 - \mathbb{P}\left(A_n\right)\right) \\
&= \mathbb{E}\left[c^*_{\mathbb{H},n}(k_n)\left|\frac{\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]}{N_{\mathbb{H},n}(k_n)} - 1\right| \mathbb{1}_{\{A_n\}}\right] + O\left(e^{-\frac{n^\delta k_n^{-\delta(2\alpha+1)}}{2}}\right) \\
&\leq \mathbb{E}\left[c^*_{\mathbb{H},n}(k_n)\right]\mathbb{E}\left[N_{\mathbb{H},n}(k_n)\right]^{-\frac{1-\delta}{2}} + O\left(e^{-\frac{n^\delta k_n^{-\delta(2\alpha+1)}}{2}}\right).
\end{aligned}$$

The second term is clearly $o\left(s_\alpha(k_n)\right)$. The first term is clearly $o\left(\mathbb{E}\left[c^*_{\mathbb{H},n}(k_n)\right]\right)$ which is $o\left(s_\alpha(k_n)\right)$ by Proposition 5.2. □

Tobias: Maybe a "conclusion and further work" section here, before the references and appendix?
Also, appendix usually comes after references in my experience.

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A Meijer's G-function

Recall that $\Gamma(z)$ denotes the Gamma function. Let p, q, m, ℓ be four integers satisfying $0 \leq m \leq q$ and $0 \leq \ell \leq p$ and consider two sequences $\mathbf{a}_p = \{a_1, \ldots, a_p\}$ and $\mathbf{b}_q = \{b_1, \ldots, b_q\}$ of reals such that $a_i - b_j$ is not a positive integer for all $1 \leq i \leq p$ and $1 \leq j \leq q$ and $a_i - a_j$ is not an integer for all distinct indices $1 \leq i, j \leq p$. Then, with ι denoting the complex unit, Meijer's G-Function [20] is defined as

$$G_{p,q}^{m,\ell}\left(z \middle| \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\right) = \frac{1}{2\pi\iota} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - t) \prod_{j=1}^\ell \Gamma(1 - a_j + t)}{\prod_{j=m+1}^q \Gamma(1 - b_j + t) \prod_{j=\ell+1}^p \Gamma_n(a_j - t)} z^t \, \mathrm{d}t, \tag{134}$$

where the path L is an upward oriented loop contour which separates the poles of the function $\prod_{j=1}^m \Gamma(b_j - t)$ from those of $\prod_{j=1}^n \Gamma(1 - a_j + t)$ and begins and ends at $+\infty$ or $-\infty$.

The Meijer's G-Function is of a very general nature and has relation to many known special functions such as the Gamma function and the generalized hypergeometric function. For more details, such as many identities for $G_{p,q}^{m,\ell}\left(z \middle| \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\right)$ see [14, 19].

For our purpose we need the following identity which follows from an Mellin transform operation.

Lemma A.1. *For any $a \in \mathbb{R}$ and $\xi, s > 0$,*

$$\Gamma^+(-a - 1, \xi/s) = G_{1,2}^{2,0}\left(\frac{\xi}{s} \middle| \begin{matrix} 1 \\ -a - 1, 0 \end{matrix}\right)$$

Proof. Let $x > 0$ and $q \in \mathbb{R}$ and note that as the Γ-function is the Mellin transform of e^{-x}, by the inverse Mellin transform formula, we have $e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(p) x^{-p} dp$ for $c > 0$ (see [10, p.196]). Applying the change of variable $p(r) = q - r$ yields $e^{-x} = \frac{1}{2\pi i} \int_{c+q-i\infty}^{c+q+i\infty} \Gamma(q - r) x^{r-q} dr$, then multiplying both sides with $-x^{q-1}$ gives $-x^{q-1} e^{-x} = -\frac{1}{2\pi i} \int_{c+q-i\infty}^{c+q+i\infty} \Gamma(q - r) x^{r-1} dr$. Now, integrating both sides gives $\int_x^\infty t^{q-1} e^{-t} dt = \frac{1}{2\pi i} \int_{c+q-i\infty}^{c+q+i\infty} \frac{\Gamma(q-r)}{-r} x^r dr$. On the left-hand side is the incomplete gamma function and on the right-hand side with using $-r = \frac{\Gamma(1-r)}{\Gamma(-r)}$ is the Meijer G-function, i.e. $\Gamma^+(q, x) = G_{1,2}^{2,0}\left(\begin{matrix} 1 \\ q, 0 \end{matrix} \middle| x\right)$. The claim follows by plugging in $q = -a - 1$ and $x = \frac{\xi}{s}$. □

B Incomplete Beta function

Here we derive the asymptotic behavior for the function $B^-(1 - z; 2\alpha, 3 - 4\alpha)$ as $z \to 0$, which is used to analyze the asymptotic behavior of $\Delta_{\mathcal{P}}(y)$.

Lemma B.1. *We have the following asymptotic results for $B^-(1 - z; 2\alpha, 3 - 4\alpha)$*

1. For $1/2 < \alpha < 3/4$

$$\lim_{z\to 0} B^-(1-z, 2\alpha, 3-4\alpha) = B(2\alpha, 3-4\alpha).$$

2. When $\alpha = 3/4$*,*

$$\lim_{z\to 0} \frac{B^-(1-z, 2\alpha, 3-4\alpha)}{\log(z)} = -1.$$

3. For $\alpha > 3/4$*,*

$$\lim_{z\to 0} z^{4\alpha-3} B^-(1-z, 2\alpha, 3-4\alpha) = \frac{1}{4\alpha-3}.$$

Proof. We use the hypergeometric representation of the incomplete Beta function,

$$B^-(x,a,b) = \frac{x^a}{2a} F(a, 1-b, a+1, x),$$

where F denote the hypergeometric function [REF]. In particular we have that

$$B^-(1-z; 2\alpha, 3-4\alpha) = \frac{(1-z)^{2\alpha}}{2\alpha} F(2\alpha, 4\alpha-2, 2\alpha+1, 1-z).$$

The behavior of $F(a,b,c,1-z)$ as $z \to 0$ depend on the real part of the sum of $c-a-b$ and whether $c = a+b$ [REF]. Since in our case a, b, c will be real it only depends on the sum of $c-a-b$. For $c-a-b > 0$ we have

$$\lim_{z\to 0} F(a,b,c,1-z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{135}$$

if $c = a+b$ then

$$\lim_{z\to 0} \frac{F(a,b,c,1-z)}{\log(z)} = -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}, \tag{136}$$

and finally, when $c-a-b < 0$

$$\lim_{z\to 0} \frac{F(a,b,c,1-z)}{z^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{137}$$

In our case we have,

$$B^-(1-z; 2\alpha, 3-4\alpha) = \frac{(1-z)^{2\alpha}}{2\alpha} F(a,b,c,1-z),$$

with $a := 2\alpha$, $b := 4\alpha - 2$ and $c := 2\alpha + 1$. Therefore,

$$c-a-b = 2\alpha + 1 - 2\alpha - (4\alpha - 2) = 3 - 4\alpha.$$

Now if $\alpha < 3/4$ then $c-a-b > 0$ and hence

$$\lim_{z\to 0} B^-(1-z; 2\alpha, 3-4\alpha) = \frac{1}{2\alpha} \frac{\Gamma(2\alpha+1)\Gamma(3-4\alpha)}{\Gamma(1)\Gamma(3-2\alpha)} = \frac{\Gamma(2\alpha)\Gamma(3-4\alpha)}{\Gamma(3-2\alpha)} = B(2\alpha, 3-4\alpha),$$

where we used that $\Gamma(2\alpha+1) = 2\alpha\Gamma(2\alpha)$.

When $\alpha = 3/4$ then $c-a-b = 0$ and therefore (136) implies that

$$\lim_{z\to 0} \frac{B^-(1-z; 2\alpha, 3-4\alpha)}{\log(z)} = -\frac{1}{2\alpha} \frac{\Gamma(6\alpha-2)}{\Gamma(2\alpha)\Gamma(4\alpha-2)} = -\frac{\Gamma(5/2)}{\frac{3}{2}\Gamma(3/2)} = -1.$$

Finally, when $\alpha > 3/4$, $c-a-b = 3-4\alpha < 0$ and using (137) we get

$$\lim_{z\to 0} z^{4\alpha-3} B^-(1-z, 2\alpha, 3-4\alpha) = \frac{1}{2\alpha} \frac{\Gamma(2\alpha+1)\Gamma(4\alpha-3)}{\Gamma(2\alpha)\Gamma(4\alpha-2)} = \frac{\Gamma(4\alpha-3)}{\Gamma(4\alpha-2)} = \frac{1}{4\alpha-3}.$$

□

C Some results for random variables

We start with the following concentration result which follows from [13, Theorem 4], together with the note directly after it.

Lemma C.1. *Let X_n be a sum of n, possibly dependent, indicators and $c > 0$. Then*

$$\mathbb{P}\left(|X_n - \mathbb{E}\left[X_n\right]| > c\mathbb{E}\left[X_n\right]\right) \leq 2e^{-\frac{c^2\mathbb{E}[X_n]^2}{2}}.$$

Let $H(x) = x\log(x) - x + 1$. Then by a Chernoff bound, see for instance [21, Lemma 1.2],

$$\begin{aligned} \mathbb{P}\left(\mathrm{Po}(\lambda) \geq k\right) &\leq e^{-\lambda H(k/\lambda)} \quad \text{for all } k \geq \lambda \\ \mathbb{P}\left(\mathrm{Po}(\lambda) \leq k\right) &\leq e^{-\lambda H(k/\lambda)} \quad \text{for all } k \leq \lambda. \end{aligned}$$

Note that $H(x) \leq (x-1)^2/2$ for all $0 \leq x \leq 1$. Therefore,

$$\mathbb{P}\left(|\mathrm{Po}(\lambda) - \lambda| \geq x\right) \geq 1 - e^{-\lambda H(1-x/\lambda)} - e^{-\lambda H(1+x/\lambda)}$$

$\mathbb{P}(X > k) \leq e^{-\mu H(\frac{k}{\mu})}$ and for $k < \mu$, $\mathbb{P}(X < k) \leq e^{-\mu H(\frac{k}{\mu})}$, where $H(x) = x\ln x - x + 1$, [21]), it follows that $\mathbb{P}(N \in [n - c\sqrt{n\log n}, n + c\sqrt{n\log n}]) \geq 1 - e^{-nH(\frac{n-c\sqrt{n\log n}}{n})} - e^{-nH(\frac{n+c\sqrt{n\log n}}{n})} \geq 1 - 2e^{-n\frac{c^2 n\log n}{n^2}} = 1 - 2e^{-c^2\log n} = 1 - 2n^{-c^2}$ (where we have used that $H(x) = (x-1)^2$ for x close to 1)

By a Chernoff bound we have

$$\mathbb{P}\left(|\mathrm{Po}(\lambda) - \lambda| \geq x\right) \leq 2e^{-\frac{x^2}{2(\lambda+x)}}. \tag{138}$$

In particular, if $\lambda_n \to \infty$, then, for any $0 < \varepsilon < 1$,

$$\mathbb{P}\left(|\mathrm{Po}(\lambda_n) - \lambda_n| \geq \lambda_n^{\frac{1+\varepsilon}{2}}\right) \leq 2e^{-\frac{\lambda_n^{\varepsilon}}{2\left(1+\lambda_n^{-(1-\varepsilon)/2}\right)}} = O\left(e^{-\lambda_n^{\varepsilon}}\right).$$