Project Problem 3

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For our project this week, we began scaffolding the code for the simulation. The start of our scaffolding began by looking at the MHD equations and deciding the best way to go about calculating. The analysis starts with:

$$
\begin{aligned}
\frac{d\boldsymbol{B}}{dt} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta^2 \nabla^2 \boldsymbol{B} \\
\frac{d\boldsymbol{v}}{dt} &= -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nabla P + (\boldsymbol{J} \times \boldsymbol{B}) + \nu^2 \nabla^2 \boldsymbol{v} \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\nabla \cdot \boldsymbol{v} &= 0
\end{aligned}
$$

Where $\boldsymbol{B}$ is the magnetic field in Gaussian units, $\boldsymbol{v}$ is the velocity of the fluid, $\boldsymbol{J}$ is the current density, P is the pressure, η is the resistivity of the plasma, and ν is the viscosity of the plasma. Leveraging the divergence properties, using $\boldsymbol{J} = \nabla \times \boldsymbol{B}$ in Gaussian, and using vector identity 8 from the front cover of Griffiths, we got the first two equations to be:

$$
\begin{aligned}
\frac{d\boldsymbol{B}}{dt} &= (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} + \eta^2 \nabla^2 \boldsymbol{B} \\
\frac{d\boldsymbol{v}}{dt} &= -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nabla P + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu^2 \nabla^2 \boldsymbol{v}
\end{aligned}
$$

We realized that from these equations, our best course of action would be to use the Euler step method to solve numerically for $\boldsymbol{B}$ and $\boldsymbol{v}$. The scaffold of our code is similar to ones using Euler step in PHY 481:

- An array containing time values in discrete steps will be made
- Arrays for the components of $\boldsymbol{B}$ and $\boldsymbol{v}$ will be made containing zeros eveywhere expect for the intial value
- $\boldsymbol{B}$ and $\boldsymbol{v}$ will be computed for the i+1$^{\text{st}}$ time point by computing the derivative using $\boldsymbol{B}$ and $\boldsymbol{v}$ at the i$^{\text{th}}$ time point and adding that change on to those $\boldsymbol{B}$ and $\boldsymbol{v}$

These two vectors contain the interesting physics of the problem and provide us with a method to check our algorithm. The relationship between the kinetic energy, E_v, and the magnetic energy, E_B, can be found from plugging these variables into their respective energy equations. There is a characteristic plot for dynamos, so if our model can replicate the behavior seen by previous research, then we know our solver is on the right track. Taylor-Green (TG) flow has specific initial conditions for the velocity and the magnetic field and the ideal case sets $\eta = \nu = 0$. The initial conditions for TG flow are:

$$\begin{aligned}
v_x &= v_0 sin(x)cos(y)cos(z) \\
v_y &= -v_0 cos(x)sin(y)cos(z) \\
B_x &= B_0 cos(x)sin(y)sin(z) \\
B_y &= B_0 sin(x)cos(y)sin(z) \\
B_z &= -2B_0 sin(x)sin(y)cos(z)
\end{aligned}$$

What was left to determine is what P should be. From our reading, P should be dependent on the plasma. Some plasmas are such that the fluid dynamics dominates this term and others are dominated by magnetic effects. For testing out right now, we are using the equation:

$$P = -\frac{\rho}{4}(cos(2x) + cos(2y))(F(t))^2$$

Pulled from reading about the Navier-Stokes equation in fluid dynamics. $F(t) = e^{-2\nu t}$, so for ideal TG flow, this goes to 1. With all of the functions figured out, the next step is to work on making the code and getting the right plots for the behavior of the energy. Both of us plan to work on developing the code and making sure it is working properly.

We each have individual action items, as well. Since we know that we are making a lot of assumptions about our plasma to test the model, it seems like it would be a good idea to use physical values for these parameters and functions that one would encounter in an astrophysical situation. Garrett will look into physical values for η and ν, as well as P to find out how we could improve upon the simulations once it is up and running. Katie will begin looking into research where MHD has been applied and what was learned from those calculations. This will helps us in preparing a our poster presentation when we want to provide some actual examples of this being used.

For a background graphic, we will most likely include the four main equations of MHD, the TG initial conditions, and information about how we derived the final equations to be used in our code. Additionally, we will probably include the scaffolding steps that we listed out here for our code. These equations and the method for computing them contain the meat of the theory in this project.

The first week involved the two of us talking and trying to figure out what exactly we wanted to do. Katie contributed the idea for the project and the initial resources to start looking into. Garrett caught up on the relevant aspects of MHD once the project was decided upon. Last week, the two of us talked out our overall plan to approach the project and worked on the listed steps. This week, the two of us again met and planned out the project problem. Both of us wanted to be a part of the coding, so we decided to share that job and split the research portion in two for this week. Both of contributed to scaffolding the code and reworking the MHD equations to get them into a form that could be coded, as wel as to figuring out what the conditions we should test in our first run should be. After this week, I think we both of us are on the same page and have a good idea of how this work should be split up.