ASSIGNMENT - I

ASU ID: 1218506822

(1) $n! \in O(n^n)$

→ Basis step

Assume $c=1, n=1$

$\therefore 1! \leq 1$ (true).

Assume true for $n = t$

Then by Induction hypothesis

$t! \leq t^t$

Under this assumption:

$(t+1)! \leq (t+1)^{t+1}$ must be true.

L.H.S

$(t+1)!$

$= t!\,(t+1)$

$\therefore$ We know that $t! <= t^t$

$\therefore (t+1)! \leq t^t (t+1)$.

$\therefore$ We know that $t^t (t+1) \leq (t+1)^t (t+1)$

$\therefore (t+1)! \leq (t+1)^t (t+1)$

$\therefore (t+1)! \leq (t+1)^{t+1}$

$\therefore$ Therefore $n! \in O(n^n)$ by Mathematical Induction

(2) $2n^2 2^n + n\log n \in \Theta(n^2 2^n)$.

Case (i) $2n^2 2^n + n\log n \leq c_1 n^2 2^n$ for $n \geq n_0$

Assume $c_1 = 3, n_0 = 1$

$\therefore 2n^2 2^n + n\log n \leq 3 \cdot n^2 2^n$, $c = 3, n \geq 1$

let $n=1$ $\quad 2 \cdot 1 \cdot 2 + 0 \leq 3 \cdot 1 \cdot 2$

$4 \leq 6$ (True)

Case (ii) $2n^2 2^n + n\log n \geq c_2 \cdot n^2 2^n$ for $n \geq n_0$

Assume $c_2 = 2, n_0 = 1$

$\therefore 2n^2 2^n + n\log n \geq 2n^2 2^n$, $c = 2, n \geq 1$

let $n=1$ $\quad 2 \cdot 2 + 0 \geq 2 \cdot 2$ (True)

Hence $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$

Therefore $2n^2 2^n + n\log n \in \Theta(n^2 2^n)$.

(3) $n^{1.001} + n\log n \in \Theta(n^{1.001})$

$$c_1 n^{1.001} \le n^{1.001} + n\log n \le c_2 n^{1.001}$$

(normalize the equation)
by dividing $n^{1.001}$

$$c_1 \le \frac{n^{1.001}}{n^{1.001}} + \frac{n\log n}{n^{1.001}} \le c_2$$

$$c_1 \le 1 + \frac{\log n}{n^{0.001}} \le c_2$$

Since $n^{0.001}$ grows so slowly, it seems like $\log n$ grows faster, but for large values of n, n^p $p > 0$, dominates $\log n$.

let $c_1 = 1, c_2 = 3, n_0 = 2^{10^5}$
satisfies the given equation.
for $\forall\ n \ge 2^{10^5}$.

Hence $n^{1.001} + n\log n \in \Theta(n^{1.001})$

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(4) $10n^2+9 = O(n)$

$10n^2+9 \le cn$ for $n > n_0$

divide both sides by cn

$$\frac{10n}{c} + \frac{9}{cn} \le 1$$

$n=1$, $c=100$ $\frac{10\times 1}{100} + \frac{9}{100\times 1} \le 1$ ✓

$n=1000$, $c=100$ $\frac{10\times 10^3}{100} + \frac{9}{10^2 \cdot 10^3} \le 1$ X

By proving through extremes values of n. There exist no +ve constant n_0 and c to define Big 'O'.

Hence $10n^2+9 \notin O(n)$

(5) $n^2 \log n = \Theta(n^2)$

(i) let $n = 2^K$, K is any positive Integer

L.H.S

$n^2 \log n = (2^K)^2 \log_2 2^K$

$= K(2^K)^2$

$n^2 = 2^{K^2}$ let $c=1$, $n=2^K$

$\therefore K(2^{K^2}) \geq 2^{K^2}$

$\therefore (n^2)\log n \geq n^2$

$n^2 \log n = \Omega(n^2)$ holds true

(ii) $n^2 \log n \le c_1 \cdot n^2$

Apply log on both sides

let $c_1=1$

$\log n^2 + \log(\log n) \le 2\log n$

$2\log n + \log(\log n) \le 2\log n$

for $n=2$

$2+0 \le 2$ ✓

for $n=3$

$2\log_2 3 + \log(\log_2 3) \le 2\log_2 3$ X

for $n=1000$

$2\log_2 10^3 + \log(\log_2 10^3) \le 2\log_2 10^3$

$\Rightarrow 6\log_2 10 + \log(3\log_2 10) \le 6\log_2 10$ X

Hence $n^2 \log n$ doesnot belong to $\Theta(n^2)$ as there exists no n_0 and c_1 value to define Big Θ.

Therefore $n^2 \log n \notin \Theta(n^2)$.

(vi) $n^3 2^n + 6n^2 3^n = O(n^3 2^n)$

$n^3 2^n + 6n^2 3^n \le c \cdot n^3 2^n$

divide by $n^3 2^n$ on both sides

$$1 + \frac{6}{n}\left(\frac{3}{2}\right)^n \le c.$$

let $c=100$, $n=1$

$1 + \frac{6}{1}\left(\frac{3}{2}\right)^1 \le 100$

$10 \le 100$ ✓

let $n = 10^3$

$1 + \frac{6}{10^3}\left(\frac{3}{2}\right)^{1000} \le 100$ X

Hence, there exists no c and n_0 value to define big 'O'.

Therefore $n^3 2^n + 6n^2 3^n$ doesnot belong to $O(n^3 2^n)$

(2)

(i) $16n^2$

10^{10} operations in 1 sec.

No. of operations in one hr

$= 60 \times 60 \times 10^{10}$

$= 36 \times \cancel{10^{4}}\ 10^{12}$

largest input size n for

$n^2 = 36 \times 10^{12}$

$n = \underline{6 \times 10^6}$ Ans

(ii). $n^3 = 36 \times 10^{12}$

$n = \sqrt[3]{36 \times 10^{12}}$

$n = 3.3019 \times 10^4$

$= \underline{33019}$ Ans

(iii) $100n^2 = 36 \times 10^{12}$

$n^2 = 36 \times 10^{10}$

$n = \sqrt{36 \times 10^{10}}$

$n = 6 \times 10^5$ // Ans.

(iv) $2^n = 36 \times 10^{12}$

$n = \log_2 (36 \times 10^{12})$

$n = \log_2 36 + \log_2 10^{12}$

$n = 2 \log_2 6 + 12 \log_2 10$

$= 2 \times 2.584 + 12 \times 3.321$

$= 5.168 + 39.852 = \underline{45 \text{ approx}}$

Therefore the largest input size 'n' for

n^2 is 6×10^6 operations

n^3 is 33019 operations

$100n^2$ is 6×10^5 operations

2^n is 45 operations.

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Hamiltonian Cycle Problem:

Hamiltonian Cycle: It is a path through a graph that starts and ends at the same vertex and includes every other vertex exactly once.

Hamiltonian Cycle Problem:

- Instance: A graph $G = (V, E)$;

Problem: Does Graph G contains a Hamiltonian cycle?

Travelling Sales man Problem:

For a given weighted graph $G' = (V', E')$, with non-negative weights, and integer K', The Travelling Salesman Problem is to find whether G' contains a simple cycle of length $<= K$ that passes through all the vertices. The graph must be complete graph.

$\therefore$ length of cycle = Sum of weights of all the edges in cycle

Instance: A finite set $C = \{c_1, c_2, ---, c_m\}$ of cities, a distance $d(c_i, c_j) \in Z^+$ for each pair of cities $c_i, c_j \in C$ and a bound $B \in Z^+$ (where Z^+ denotes +ve integers)

Problem: Is there a tour (cycle) of all the cities in C having total length no more than B, that is an ordering $< C_{\pi(1)}, C_{\pi(2)}, ---, C_{\pi(m)}$ of C such that

$$\sum_{i=1}^{m-1} d(C_{\pi(i)}, C_{\pi(i+1)}) + d(C_{\pi(m)}, C_{\pi(1)}) \le B$$

Reduction of Hamiltonian Cycle Problem to Traveling Salesman Problem in Polynomial Time

To reduce the Hamiltonian Cycle problem to the Traveling Salesman problem for a Graph $G = (V, E)$.

(1) Make the Graph as a Complete Graph, by connecting the edges. All pairs of vertices through edge that were not connected in G.

(2) let the new graph be $G' = (V', E')$ where $V' = V$ and $E' = (u, v)$ for any $u, v \in V'$

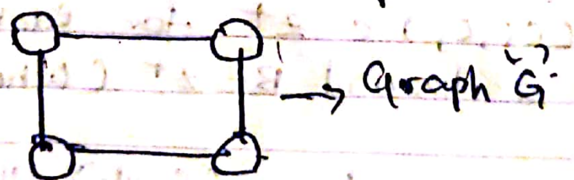
For edges in G' that were also present in G, we assign weight '1'. For other edges we assign '2'.

i.e, $\forall e = (u,v) \in E'$,

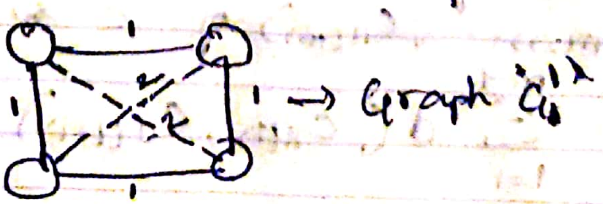
$dist(e) = 1$, If $(u,v) \in E$ — ①

$dist(e) = 2$, If $(u,v) \notin E$ — ②

Consider a graph 'G' as an input to Hamiltonian Cycle



Construct a graph 'G'' which is complete graph

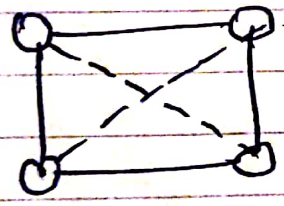


Mark the edges according to ① & ②

The dotted edges are not present in Graph 'G' and has weight '2' claim:- The Graph G contains a Hamilton cycle, if and only if there is a tour of all the cities in G', that has a total length <= 4 (Bound) - B.

(1) If there is a 'tour' or 'cycle' that passes through all vertices exactly once and has length <= 4, in graph G', the cycle contains only edges that were originally present in graph G. (
Hence, there exists a Hamiltonian cycle in G.

(2) If there exist a Hamiltonian Cycle in a Graph G, it form a cycle in G' with length = 4, since the weights of all edges is '1'
Hence there exists a Solution for Traveling Salesman problem in G' with length <= 4. (B)

G' :-  → G' has a cycle passing through all vertices once and has length <= 4.
This cycle is Hamiltonian cycle in G.

∴ claim Hence, given the Hamiltonian cycle is Hard (NP-complete), the decision Version of TSP is also hard in polynomial Time.

4) Consider

$f_2(n) = (2n)^{0.7} = (2)^{0.7}(n)^{0.7} = O(n^{0.7})$

$f_3(n) = n^3 + 100 = O(n^3)$

$f_1(n) = n^{3.4} = O(n^{3.4})$

~~Com~~Arranging the polynomial functions in order

$O(n^{0.7}) < O(n^3) < O(n^{3.4})$

$f_2(n) < f_3(n) < f_1(n)$

And $f_4(n) < f_5(n)$ because the functions are exponential & fast growing

$\therefore O(20^n) < O(250^n)$

$\therefore$ Therefore the Asymptotic order is

$f_2(n) < f_3(n) < f_1(n) < f_4(n) < f_5(n)$