PHYC 3590 - Advanced Classical Mechanics Assignment 1

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Question 2.10

(a)

These variables are given in the question.

$$\eta = 12\,N \cdot s/m^2$$
$$\rho = 7.8\,g/cm^3$$
$$D = 2\,mm$$

The forces acting on the ball are

$$\vec{F}_{lin} = -\,3\pi\eta Dv\hat{y}$$
$$\vec{F}_{buoyant} = -\,\rho gV\hat{y}$$
$$\vec{F}_g = mg\hat{y}$$

where the ball is falling in the $+y$ direction. For the net force we have

$$ma = mg - 3\pi\eta Dv - \rho gV$$

For v_{ter} we set the a to equal 0.

$$0 = mg - 3\pi\eta Dv_{ter} - \rho gV \tag{1}$$
$$3\pi\eta Dv_{ter} = mg - \rho gV \tag{2}$$
$$v_{ter} = \frac{mg - \rho gV}{3\pi\eta D} \tag{3}$$
$$v_{ter} = \frac{mg - \rho gV}{b} \tag{4}$$

$$ma = mg - bv - \rho gV \tag{5}$$
$$\frac{m}{b}a = \frac{mg - \rho gV}{b} - v \tag{6}$$

Subbing equation (4) into equation (6) gives

$$\frac{m}{b}a = v_{ter} - v$$

Let $u = v - v_{ter}$

$$m\dot{u} = -bu$$

$$u = u_0 \exp\left(-\frac{t}{\tau}\right), \quad \tau = \frac{m}{b}$$

$$v - v_{ter} = (v_0 - v_{ter})\exp\left(-\frac{t}{\tau}\right)$$

$$\tau = \frac{m}{b}$$

$$\tau = \frac{m}{3\pi\eta D}$$

$$m = \rho V = \rho\frac{4}{3}\pi(D/2)^3 = \frac{1}{6}\rho\pi D^3$$

$$\tau = \frac{\rho\pi D^3}{6\cdot 3\pi\eta D} = \boxed{\frac{\rho D^2}{18\eta}}$$

$$\tau = \frac{(7.8g/cm^3)(2mm)^2}{18(12N\cdot s/m^2)}$$

$$\tau = \frac{(7800kg/m^3)(0.002m)^2}{18(12N\cdot s/m^2)}$$

$$\tau = \boxed{1.4\times 10^{-4}\ seconds}$$

95% of v_{ter} is reached after 3τ, which is equal to $\boxed{4.3\times 10^{-4}\ seconds}$.

Solving for v_{ter} numerically, we have

$$v_{ter} = \frac{mg - \rho_{gly}gV}{b}$$

$$v_{ter} = \frac{mg}{3\pi\eta D} - \frac{\rho gV}{3\pi\eta D}$$

$$v_{ter} = \frac{\frac{1}{6}\rho_{ball}\pi D^3 g}{3\pi\eta D} - \frac{\rho_{gly}g(\frac{1}{6}\pi D^3)}{3\pi\eta D}$$

$$v_{ter} = \frac{\frac{1}{6}\rho_{ball}D^2 g}{3\eta} - \frac{\frac{1}{6}\rho_{gly}D^2 g}{3\eta}$$

$$v_{ter} = \frac{D^2 g}{18\eta}(\rho_{ball} - \rho_{gly})$$

$$v_{ter} = \frac{(0.002m)^2(9.81m/s^2)}{18(12N\cdot s/m^2)}(7800kg/m^3 - 1300kg/m^3) = \boxed{1.2\times 10^{-3}m/s}$$

(b)

$$F_{lin} = 3\pi\eta Dv$$

$$R = \frac{Dv\rho}{\eta}$$

$$\frac{F_{quad}}{F_{lin}} = \frac{R}{48}$$

$$F_{quad} = F_{lin}\frac{R}{48}$$

$$F_{quad} = 3\pi\eta Dv\frac{Dv\rho}{48\eta}$$

$$F_{quad} = \boxed{\frac{3\pi\rho D^2 v^2}{48}}$$

Question 2.19

(a)

The only force acting on the projectile is $\vec{w} = -mg\hat{y}$ therefore $a = -g\hat{y}$. The particle also has an initial velocity, $\vec{v}_0 = v_{x_0}\hat{x} + v_{y_0}\hat{y}$. Next to solve for the equation for motion we have

$$x = \int v_{x_0}\, dx = \boxed{v_{x_0}t} \tag{7}$$

$$y = \int v_{y_0}\, dx - \int\int g\, dx\, dx = \boxed{v_{y_0}t - \tfrac{1}{2}gt^2} \tag{8}$$

Next we use equation (7) to solve for t, then plug it into question (8).

$$t = \frac{x}{v_{x_0}}$$

$$y = v_{y_0}\left(\frac{x}{v_{x_0}}\right) - \tfrac{1}{2}g\left(\frac{x}{v_{x_0}}\right)^2$$

$$y = \boxed{\frac{v_{y_0}}{v_{x_0}}x - \frac{g}{2v_{x_0}^2}x^2}$$

(b)

Equation 2.37 from Taylor is

$$y = \frac{v_{y0} + v_{ter}}{v_{x0}} x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

When the air resistance is switched off, v_{ter} and τ go to ∞.

$$\lim_{v_{ter},\tau\to\infty}(y) = ???$$

Taylor expanding the ln term from the equation for y gives

$$y = \frac{v_{y0} + v_{ter}}{v_{x0}} x - v_{ter}\tau\left[\frac{x}{v_{x0}\tau} + \frac{1}{2}\left(\frac{x}{v_{x0}\tau}\right)^2 + \frac{1}{2}\left(\frac{x}{v_{x0}\tau}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0} + v_{ter}}{v_{x0}} x - v_{ter}\left[\frac{x}{v_{x0}} + \frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau^2}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0}}{v_{x0}} x + \frac{g\tau}{v_{x0}} x - g\tau\left[\frac{x}{v_{x0}} + \frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau^2}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0}}{v_{x0}} x + \frac{g\tau}{v_{x0}} x - g\tau\frac{x}{v_{x0}} - g\tau\left[\frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau^2}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0}}{v_{x0}} x - g\tau\left[\frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau^2}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0}}{v_{x0}} x - g\left[\frac{1}{2}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$\lim_{\tau\to\infty}(y) = \boxed{\frac{v_{y0}}{v_{x0}} x - \frac{g}{2v_{x0}^2} x^2}$$

QED

Question 2.37

The first part of this problem is to solve this integral

$$\int \frac{1}{1-u^2}\,du \tag{9}$$

$$= \frac{1}{2}\left[\int \frac{1}{1+u}\,du + \int \frac{1}{1-u}\,du\right] \tag{10}$$

For the left most term we let $v = 1+u$, $du = dv$

$$\int \frac{1}{1+u}\,du = \int \frac{1}{v}\,dv$$

$$= \ln v$$

$$= \boxed{\ln(1+u)}$$

For the right most term we let $v = 1-u$, $du = -dv$

$$\int \frac{1}{1-u}\,du = -\int \frac{1}{v}\,dv$$

$$= -\ln v$$

$$= \boxed{-\ln(1-u)}$$

Subbing the boxed equations int equation (10) gives

$$\frac{1}{2}\left[\ln(1+u) - \ln(1-u)\right] \tag{11}$$

$$= \frac{1}{2}\ln\left[\frac{1+u}{1-u}\right] \tag{12}$$

$$\int \frac{1}{1-u}\,du = \operatorname{arctanh} u \tag{13}$$

From the book we have

$$\frac{dv}{1-v^2/v_{ter}^2} = g\,dt$$

$$\int_0^v \frac{dv}{1-v^2/v_{ter}^2} = \int_0^t g\,dt$$

$$v_{ter}\int_0^u \frac{du'}{1-u'^2} = \int_0^t g\,dt$$

$$u = v/v_{ter}, \quad dv = v_{ter}dt$$

Using the solution given by equation (13), we find

$$v_{ter}\operatorname{arctanh}\frac{v}{v_{ter}} = gt$$

$$\operatorname{arctanh}\frac{v}{v_{ter}} = \frac{gt}{v_{ter}}$$

$$\frac{v}{v_{ter}} = \tanh\left(\frac{gt}{v_{ter}}\right)$$

$$v = \boxed{v_{ter}\tanh\left(\frac{gt}{v_{ter}}\right)}$$

QED

Question 2.52

$$\eta = Ae^{-i\omega t} = ae^{i\delta}e^{-i\omega t} = ae^{i(\delta-\omega t)}$$
$$\eta = a\cos(\delta - \omega t) + ia\sin(\delta - \omega t)$$
$$v_x = a\cos(\delta - \omega t), \quad v_y = a\sin(\delta - \omega t)$$

η moves clockwise with a constant speed of $|a|$ and an angular velocity of $-\omega$.

Question 2.54

Equations (2.68) are

$$\dot{v}_x = \omega v_y$$
$$\dot{v}_y = -\omega v_x$$

Differentiating the first equation in (2.68), we have

$$\ddot{v}_x = \omega \dot{v}_y$$
$$\frac{\ddot{v}_x}{\omega} = \dot{v}_y$$

Subbing this result into the second equation in (2.68) gives

$$\frac{\ddot{v}_x}{\omega} = -\omega v_x$$
$$\ddot{v}_x = -\omega^2 v_x$$
$$v_x = \boxed{A\sin(\omega t) + B\cos(\omega t)}$$

Now solving for v_y we have

$$\omega v_y = \dot{v}_x$$
$$\omega v_y = \frac{d}{dt}[A\sin(\omega t) + B\cos(\omega t)]$$
$$\omega v_y = A\omega\cos(\omega t) - B\omega\sin(\omega t)$$
$$v_y = \boxed{A\cos(\omega t) - B\sin(\omega t)}$$

$$\vec{v} = A\sin(\omega t) + B\cos(\omega t) + i(A\cos(\omega t) - B\sin(\omega t))$$