PHYC 3590 - Advanced Classical Mechanics
Assignment 3

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Problem 3.5

$$\vec{p_1} + \vec{p_2} = \vec{p_1}' + \vec{p_2}' \tag{1}$$
$$\vec{v_2} = \vec{p_2} = 0 \tag{2}$$
$$m_1\vec{v_1} = m_1\vec{v_1}' + m_2\vec{v_2}' \tag{3}$$

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \tag{4}$$
$$v_2 = 0 \tag{5}$$
$$m_1v_1^2 = m_1v_1'^2 + m_2v_2'^2 \tag{6}$$
$$m_1 = m_2 \tag{7}$$
$$v_1^2 = v_1'^2 + v_2'^2 \tag{8}$$
$$\tag{9}$$

From equation (3) we have

$$m_1\vec{v_1} = m_1\vec{v_1}' + m_2\vec{v_2}' \tag{10}$$
$$m_1^2\vec{v_1}^2 = (m_1\vec{v_1}' + m_2\vec{v_2}')^2 \tag{11}$$
$$m_1^2\vec{v_1}^2 = m_1^2\vec{v_1}'^2 + m_2^2\vec{v_2}'^2 + 2m_1m_2\vec{v_1}' \cdot \vec{v_2}' \tag{12}$$
$$m_1 = m_2 = m \tag{13}$$
$$\vec{v_1}^2 = \vec{v_1}'^2 + \vec{v_2}'^2 + 2\vec{v_1}' \cdot \vec{v_2}' \tag{14}$$

Equating equation (8) and (14) we find

$$v_1'^2 + v_2'^2 = \vec{v_1}'^2 + \vec{v_2}'^2 + 2\vec{v_1}' \cdot \vec{v_2}'$$
$$\vec{v_1}' \cdot \vec{v_2}' = 0$$
$$|v_1'| \cdot |v_2'| \cos\theta = 0$$
$$\theta = \boxed{90^o}$$

Problem 3.11

(a)

Show that the equation of motion is $m\dot{v} = -\dot{m}v_{ex} + F^{ext}$.

$$\begin{aligned}
dP =& m\,dv + dm\,v_{ex} \\
\dot{P} =& \frac{dP}{dt} = F^{ext} \\
dP =& F^{ext}dt \\
F^{ext}dt =& m\,dv + dm\,v_{ex} \\
F^{ext} =& m\frac{dv}{dt} + \frac{dm}{dt}v_{ex} \\
m\dot{v} =& -\dot{m}v_{ex} + F^{ext} \\
& \text{QED}
\end{aligned}$$

(b)

$$\begin{aligned}
m\dot{v} =& -\dot{m}v_{ex} - mg \\
(m_0 - kt)\dot{v} =& kv_{ex} - (m_0 - kt)g \\
(m_0 - kt)\frac{dv}{dt} =& kv_{ex} - (m_0 - kt)g \\
dv =& \left(\frac{kv_{ex}}{m_0 - kt} - g\right) dt \\
\int_0^v dv =& \int \left(\frac{kv_{ex}}{m_0 - kt} - g\right) dt \\
v =& \int_0^t \left(\frac{kv_{ex}}{m_0 - kt'} - g\right) dt' \\
v =& \int_0^t \frac{kv_{ex}}{m_0 - kt'} dt' - gt \\
v =& -v_{ex}(\ln|m_0 - kt| - \ln|m_0|) - gt \\
v =& v_{ex}\ln\left(\frac{m_0}{m_0 - kt}\right) - gt \\
v =& \boxed{v_{ex}\ln\left(\frac{m_0}{m}\right) - gt}
\end{aligned}$$

(c)

$$\begin{aligned}
m_0 =& 2 \times 10^6 kg \\
m_f =& 1 \times 10^6 kg \\
v_{ex} =& 3000 m/s \\
t =& 120 s \\
v =& (3000 m/s) \ln\left(\frac{2 \times 10^6 kg}{1 \times 10^6 kg}\right) = \boxed{2079 m/s} \\
v =& (3000 m/s) \ln\left(\frac{2 \times 10^6 kg}{1 \times 10^6 kg}\right) - (9.8 m/s^2)(120 s) = \boxed{903 m/s}
\end{aligned}$$

Without gravity the rocket would be traveling roughly twice as fast.

(d)

If $\dot{m}\, v_{ext} < mg$, the rocket would remain stationary until the enough fuel mass was ejected so that $\dot{m}\, v_{ext} > mg$. Then the rocket would lift off.

Problem 3.21

$$\begin{aligned}
CM =& \frac{1}{m} \int y\, dm \\
dm =& \sigma\, dA, \quad y = r \sin\theta \\
CM =& \frac{1}{m} \int r \sin\theta \sigma dA \\
\sigma =& \frac{m}{A} = \frac{m}{1/2 \pi R^2} = \frac{2m}{\pi R^2} \\
dA =& r\, dr\, d\theta \\
CM =& \int\int \frac{2}{\pi R^2} r^2\, dr \sin\theta\, d\theta \\
CM =& \frac{2}{\pi R^2} \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \\
CM =& \frac{2}{\pi R^2} (R^3/3)(2) \\
CM =& \boxed{\frac{4R}{3\pi}}
\end{aligned}$$

Problem 3.27

(a)

$$
\begin{aligned}
l =& \vec{r} \times \vec{p} \\
\vec{p} =& m\vec{v} \\
l =& \vec{r} \times m\vec{v} \\
\vec{v} =& \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \\
l =& \vec{r} \times m(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) \\
l =& r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) \\
l =& r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) \\
l =& r\hat{r} \times mr\dot{\phi}\hat{\phi} \\
\omega =& \dot{\phi} \\
l =& r\hat{r} \times mr\omega\hat{\phi} \\
l =& \boxed{mr^2\omega(\hat{r} \times \hat{\phi})}
\end{aligned}
$$

(b)

Show that $\frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{l}{2m}$. Ler r be the distance between the planet and the sun, let x be the distance the planet is traveling perpendicular to r.

$$
\begin{aligned}
A =& \frac{1}{2}rx \\
dA =& \frac{1}{2}r\,dx \\
x =& r\sin\phi \\
dx =& r\,d\phi \\
dA =& \frac{1}{2}r\,r\,d\phi \\
\frac{dA}{dt} =& \frac{1}{2}r^2\frac{d\phi}{dt} \\
\frac{dA}{dt} =& \boxed{\frac{1}{2}r^2\omega}
\end{aligned}
$$

From part (a) we have

$$
\begin{aligned}
l =& mr^2\omega && (15) \\
\frac{l}{2m} =& \frac{1}{2}r^2\omega && (16) \\
\frac{dA}{dt} =& \frac{1}{2}r^2\omega && (17) \\
\frac{dA}{dt} =& \boxed{\frac{l}{2m}} && (18)
\end{aligned}
$$

Problem 3.32

$$
\begin{aligned}
I =& \sum m_\alpha \rho_\alpha^2 \\
I =& \int_v Density \cdot \rho^2 \, dV \\
dV =& r^2 dr \sin\theta d\theta d\phi \\
I =& \int_v Density \cdot \rho^2 \, r^2 dr \sin\theta d\theta d\phi \\
Density =& \frac{m}{V} = \frac{3m}{4\pi R^3} \\
I =& \int_v \frac{3m}{4\pi R^3} \rho^2 \, r^2 dr \sin\theta d\theta d\phi \\
\rho^2 =& r^2 \sin^2\theta \\
I =& \int_v \frac{3m}{4\pi R^3} r^2 \sin^2\theta \, r^2 dr \sin\theta d\theta d\phi \\
I =& \frac{3m}{4\pi R^3} \int_0^R r^4 dr \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\phi \\
I =& \frac{3m}{4\pi R^3} (R^5/5)(2\pi) \int_0^\pi \sin^3\theta d\theta \\
\int_0^\pi \sin^3\theta d\theta =& \int_0^\pi (1-\cos^2\theta)\sin\theta d\theta \\
=& \int_1^{-1} u^2 - 1 du, \;\; u = \cos\theta, \;\; du = -\sin\theta d\theta \\
=& [u^3/3]_1^{-1} - [u]_1^{-1} = \frac{4}{3} \\
I =& \frac{3m}{4\pi R^3} (R^5/5)(2\pi)\frac{4}{3} \\
I =& \frac{2}{5} m R^2 \\
& \text{QED}
\end{aligned}
$$