PHYC 3590xx - Advanced Classical Mechanics Assignment 3

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Problem 3.5

$$\vec{p_1} + \vec{p_2} = \vec{p_1}' + \vec{p_2}' \tag{1}$$
$$\vec{v_2} = \vec{p_2} = 0 \tag{2}$$
$$m_1\vec{v_1} = m_1\vec{v_1}' + m_2\vec{v_2}' \tag{3}$$

$$\tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 = \tfrac{1}{2}m_1 v_1^{'2} + \tfrac{1}{2}m_2 v_2^{'2} \tag{4}$$
$$v_2 = 0 \tag{5}$$
$$m_1 v_1^2 = m_1 v_1^{'2} + m_2 v_2^{'2} \tag{6}$$
$$m_1 = m_2 \tag{7}$$
$$v_1^2 = v_1^{'2} + v_2^{'2} \tag{8}$$
$$\tag{9}$$

From equation (3) we have

$$m_1\vec{v_1} = m_1\vec{v_1}' + m_2\vec{v_2}' \tag{10}$$
$$m_1^2\vec{v_1}^2 = (m_1\vec{v_1}' + m_2\vec{v_2}')^2 \tag{11}$$
$$m_1^2\vec{v_1}^2 = m_1^2\vec{v_1}^{'2} + m_2^2\vec{v_2}^{'2} + 2m_1 m_2 \vec{v_1}' \cdot \vec{v_2}' \tag{12}$$
$$m_1 = m_2 = m \tag{13}$$
$$\vec{v_1}^2 = \vec{v_1}^{'2} + \vec{v_2}^{'2} + 2\vec{v_1}' \cdot \vec{v_2}' \tag{14}$$

Equating equation (8) and (14) we find

$$v_1^{'2} + v_2^{'2} = \vec{v_1}^{'2} + \vec{v_2}^{'2} + 2\vec{v_1}' \cdot \vec{v_2}'$$
$$\vec{v_1}' \cdot \vec{v_2}' = 0$$
$$|v_1'| \cdot |v_2'| \cos\theta = 0$$
$$\theta = \boxed{90^o}$$

Problem 3.11

(a)

Show that the equation of motion is $m\dot{v} = -\dot{m}v_{ex} + F^{ext}$.

$$dP = m\,dv + dm\,v_{ex}$$
$$\dot{P} = \frac{dP}{dt} = F^{ext}$$
$$dP = F^{ext}dt$$
$$F^{ext}dt = m\,dv + dm\,v_{ex}$$
$$F^{ext} = m\frac{dv}{dt} + \frac{dm}{dt}v_{ex}$$
$$m\dot{v} = -\,\dot{m}v_{ex} + F^{ext}$$
QED

(b)

$$m\dot{v} = -\,\dot{m}v_{ex} - mg$$
$$(m_0 - kt)\dot{v} = kv_{ex} - (m_0 - kt)g$$
$$(m_0 - kt)\frac{dv}{dt} = kv_{ex} - (m_0 - kt)g$$
$$dv = \left(\frac{kv_{ex}}{m_0 - kt} - g\right)dt$$
$$\int_0^v dv = \int \left(\frac{kv_{ex}}{m_0 - kt} - g\right)dt$$
$$v = \int_0^t \left(\frac{kv_{ex}}{m_0 - kt'} - g\right)dt'$$
$$v = \int_0^t \frac{kv_{ex}}{m_0 - kt'}dt' - gt$$
$$v = -v_{ex}(\ln|m_0 - kt| - \ln|m_0|) - gt$$
$$v = v_{ex}\ln\left(\frac{m_0}{m_0 - kt}\right) - gt$$
$$\boxed{v = v_{ex}\ln\left(\frac{m_0}{m}\right) - gt}$$

(c)

$$m_0 = 2 \times 10^6 kg$$
$$m_f = 1 \times 10^6 kg$$
$$v_{ex} = 3000 m/s$$
$$t = 120s$$
$$v = (3000 m/s) \ln \left(\frac{2 \times 10^6 kg}{1 \times 10^6 kg} \right) = \boxed{2079 m/s}$$
$$v = (3000 m/s) \ln \left(\frac{2 \times 10^6 kg}{1 \times 10^6 kg} \right) - (9.8 m/s^2)(120s) = \boxed{903 m/s}$$

Without gravity the rocket would be traveling roughly twice as fast.

(d)

If $\dot{m} \, v_{ext} < mg$, the rocket would remain stationary until the enough fuel mass was ejected so that $\dot{m} \, v_{ext} > mg$. Then the rocket would lift off.

Problem 3.21

$$CM = \frac{1}{m} \int y \, dm$$
$$dm = \sigma \, dA, \quad y = r \sin \theta$$
$$CM = \frac{1}{m} \int r \sin \theta \sigma dA$$
$$\sigma = \frac{m}{A} = \frac{m}{1/2 \pi R^2} = \frac{2m}{\pi R^2}$$
$$dA = r \, dr \, d\theta$$
$$CM = \int \int \frac{2}{\pi R^2} r^2 \, dr \sin \theta \, d\theta$$
$$CM = \frac{2}{\pi R^2} \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta$$
$$CM = \frac{2}{\pi R^2} (R^3/3)(2)$$
$$CM = \boxed{\frac{4R}{3\pi}}$$

Problem 3.27

(a)

$$l = \vec{r} \times \vec{p}$$
$$\vec{p} = m\vec{v}$$
$$l = \vec{r} \times m\vec{v}$$
$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$
$$l = \vec{r} \times m(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$$
$$l = r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$$
$$l = r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$$
$$l = r\hat{r} \times mr\dot{\phi}\hat{\phi}$$
$$\omega = \dot{\phi}$$
$$l = r\hat{r} \times mr\omega\hat{\phi}$$
$$l = \boxed{mr^2\omega(\hat{r} \times \hat{\phi})}$$

(b)

Show that $\frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{l}{2m}$. Ler r be the distance between the planet and the sun, let x be the distance the planet is traveling perpendicular to r.

$$A = \frac{1}{2}rx$$
$$dA = \frac{1}{2}r\,dx$$
$$x = r\sin\phi$$
$$dx = r\,d\phi$$
$$dA = \frac{1}{2}r\,r\,d\phi$$
$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\phi}{dt}$$
$$\frac{dA}{dt} = \boxed{\frac{1}{2}r^2\omega}$$

From part (a) we have

$$l = mr^2\omega \tag{15}$$

$$\frac{l}{2m} = \frac{1}{2}r^2\omega \tag{16}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\omega \tag{17}$$

$$\frac{dA}{dt} = \boxed{\frac{l}{2m}} \tag{18}$$

Problem 3.32

$$I = \sum m_\alpha \rho_\alpha^2$$

$$I = \int_v Density \cdot \rho^2 \, dV$$

$$dV = r^2 dr \sin \theta d\theta d\phi$$

$$I = \int_v Density \cdot \rho^2 \, r^2 dr \sin \theta d\theta d\phi$$

$$Density = \frac{m}{V} = \frac{3m}{4\pi R^3}$$

$$I = \int_v \frac{3m}{4\pi R^3} \rho^2 \, r^2 dr \sin \theta d\theta d\phi$$

$$\rho^2 = r^2 \sin^2 \theta$$

$$I = \int_v \frac{3m}{4\pi R^3} r^2 \sin^2 \theta \, r^2 dr \sin \theta d\theta d\phi$$

$$I = \frac{3m}{4\pi R^3} \int_0^R r^4 dr \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} d\phi$$

$$I = \frac{3m}{4\pi R^3} (R^5/5)(2\pi) \int_0^\pi \sin^3 \theta d\theta$$

$$\int_0^\pi \sin^3 \theta d\theta = \int_0^\pi (1 - \cos^2 \theta) \sin \theta d\theta$$

$$= \int_1^{-1} u^2 - 1 du, \ u = \cos \theta, \ du = -\sin \theta d\theta$$

$$= [u^3/3]_1^{-1} - [u]_1^{-1} = \frac{4}{3}$$

$$I = \frac{3m}{4\pi R^3} (R^5/5)(2\pi) \frac{4}{3}$$

$$I = \frac{2}{5} mR^2$$

$$\text{QED}$$