PHYC 3590 - Advanced Classical Mechanics
Assignment 6

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1 6.16

From problem 6.1 we have $L = R\int_{\theta_1}^{\theta_2}\sqrt{1+\sin^2\theta\phi'(\theta)^2}d\theta$.

$$\begin{aligned}
\frac{\partial f}{\partial \phi'} =& c \\
\frac{\partial f}{\partial \phi'} =& \sqrt{1+\sin^2\theta\phi'(\theta)^2} \\
\frac{\partial f}{\partial \phi'} =& \frac{\sin^2\theta\phi'}{\sqrt{1+\sin^2\theta\phi'(\theta)^2}} \\
\frac{\sin^2\theta\phi'}{\sqrt{1+\sin^2\theta\phi'(\theta)^2}} =& c
\end{aligned}$$

I don't think I fully understand this but choosing the z axis to pass through point one where $\theta = 0$.

2 6.19

The area of the surface of revolution is defined by the equation.

$$\begin{aligned}
A =& 2\pi\int_{x_1}^{x_2} y\sqrt{1+x'(y)^2}dy \\
f =& y\sqrt{1+x'(y)^2} \\
0 =& \frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} \\
0 =& \frac{d}{dy}\frac{\partial f}{\partial x'} \\
y_0 =& \frac{\partial f}{\partial x'} \\
y_0 =& \frac{\partial}{\partial x'}y\sqrt{1+x'^2}
\end{aligned}$$

$$
\begin{aligned}
y_0 =& y\frac{x'}{\sqrt{1+x'^2}} \\
y_0^2(1+x'^2) =& y^2x'^2 \\
y_0^2 =& y^2x'^2 - y_0^2x'^2 \\
y_0^2 =& (y^2 - y_0^2)x'^2 \\
\frac{y_0^2}{y^2 - y_0^2} =& x'^2
\end{aligned}
$$

$$
x' = \frac{y_0}{\sqrt{y^2 - y_0^2}}
$$

I am not sure how to simplify this too the desired equation.

3 6.27

$$
\begin{aligned}
L =& \int ds \\
ds =& \sqrt{dx^2 + dy^2 + dz^2} \\
ds =& \sqrt{x'^2 + y'^2 + z'^2}du \\
L =& \int \sqrt{x'^2 + y'^2 + z'^2}du \\
f =& \sqrt{x'^2 + y'^2 + z'^2} \\
\frac{\partial f}{\partial x'} =& \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} \\
\frac{\partial f}{\partial y'} =& \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}} \\
\frac{\partial f}{\partial z'} =& \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}}
\end{aligned}
$$

The first order derivative of f with respect to x, y, and z are all equal to zero therefore

$$
\begin{aligned}
\frac{d}{dx}\frac{\partial f}{\partial x'} =& 0 \\
\frac{\partial f}{\partial x'} =& c_1 \\
\frac{d}{dx}\frac{\partial f}{\partial y'} =& 0 \\
\frac{\partial f}{\partial y'} =& c_2 \\
\frac{d}{dx}\frac{\partial f}{\partial z'} =& 0 \\
\frac{\partial f}{\partial z'} =& c_3
\end{aligned}
$$

These derivatives are all equal to a constant, therefore x', y' and z' are all equal to constants. This can only be the case if the path is a straight line.

4 7.1

$$
\begin{aligned}
T =& \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\
U =& mgz \\
L =& T - U \\
L =& \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\
\frac{\partial L}{\partial x} =& \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \\
\frac{\partial L}{\partial y} =& \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \\
\frac{\partial L}{\partial z} =& \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} \\
-mg =& \frac{d}{dt}(m\dot{x}) \\
-mg =& m\ddot{x} \\
-g =& \ddot{x}
\end{aligned}
$$

From Newtonian mechanics, you would say that the gravity is the only force acting on the object so the equation would become $F = -mg$, from Newton's second law we have $F = m\ddot{x}$. Combining these two equations, we find the same equation derived above $-mg = m\ddot{x}$.

5 7.4

Using Lagrangian mechanics we have

$$
\begin{aligned}
U =& mgh = mgy\sin\alpha \\
T =& \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\
L =& \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy\sin\alpha \\
\frac{\partial L}{\partial x} =& \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \\
0 =& \frac{d}{dt}(m\dot{x}) \\
& \boxed{m\ddot{x} = 0} \\
\frac{\partial L}{\partial y} =& \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \\
-mg\sin\alpha =& \frac{d}{dt}(m\dot{y}) \\
& \boxed{m\ddot{y} = -mg\sin\alpha}
\end{aligned}
$$

Using Newtonian mechanics we have

$$
\begin{aligned}
F =& m\ddot{y} && (1)\\
F =& -mg\sin\alpha && (2)\\
m\ddot{y} =& -mg\sin\alpha && (3)
\end{aligned}
$$

The solution using Lagrangian mechanics agrees with the Newtonian solution.

6 7.8

6.1 (a)

$$
\begin{aligned}
U =& \tfrac{1}{2}kx^2 = \tfrac{1}{2}k(x_1 - x_2 - l)^2\\
T =& \tfrac{1}{2}m\dot{x_1}^2 + \tfrac{1}{2}m\dot{x_2}^2\\
L =& \tfrac{1}{2}m\dot{x_1}^2 + \tfrac{1}{2}m\dot{x_2}^2 - \tfrac{1}{2}k(x_1 - x_2 - l)^2
\end{aligned}
$$

6.2 (b)

$$
\begin{aligned}
X =& \tfrac{1}{2}(x_1 + x_2)\\
x =& x_1 - x_2 - l\\
x_2 =& x_1 - x - l\\
X =& \tfrac{1}{2}(x_1 + x_1 - x - l)\\
x_1 =& \boxed{X + \tfrac{1}{2}(x + l)}\\
x_1 =& x_2 + x + l\\
X =& \tfrac{1}{2}(x_2 + x + l + x_2)\\
x_2 =& \boxed{X - \tfrac{1}{2}(x + l)}\\
L =& \tfrac{1}{2}m(\dot{X} + \tfrac{1}{2}\dot{x})^2 + \tfrac{1}{2}m(\dot{X} - \tfrac{1}{2}\dot{x})^2 - \tfrac{1}{2}k[X - \tfrac{1}{2}(x + l) - X + \tfrac{1}{2}(x + l) - l]^2\\
L =& \tfrac{1}{2}m(\dot{X} + \tfrac{1}{2}\dot{x})^2 + \tfrac{1}{2}m(\dot{X} - \tfrac{1}{2}\dot{x})^2 - \tfrac{1}{2}kx^2\\
L =& \tfrac{1}{2}m(\dot{X}^2 + \tfrac{1}{2}\dot{x}\dot{X} + \tfrac{1}{2}\dot{x}^2) + \tfrac{1}{2}m(\dot{X}^2 - \tfrac{1}{2}\dot{x}\dot{X} + \tfrac{1}{2}\dot{x}^2) - \tfrac{1}{2}kx^2\\
L =& \tfrac{1}{2}m(\dot{X}^2 + \tfrac{1}{2}\dot{x}^2) + \tfrac{1}{2}m(\dot{X}^2 + \tfrac{1}{2}\dot{x}^2) - \tfrac{1}{2}kx^2\\
L =& \boxed{m\dot{X}^2 + \tfrac{1}{4}m\dot{x}^2 - \tfrac{1}{2}kx^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial X} =& \frac{d}{dt}\frac{\partial L}{\partial \dot{X}}\\
0 =& 2m\ddot{X}\\
\frac{\partial L}{\partial x} =& \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\\
-kx =& \frac{1}{2}m\ddot{x}
\end{aligned}
$$

6.3 (c)

$$
\begin{aligned}
0 =& 2m\ddot{X} && (4)\\
\dot{X} =& V && (5)\\
X =& Vt + V_0 && (6)
\end{aligned}
$$

This solution says that X is changing with a constant velocity, V.

$$
\begin{aligned}
0 =& \ddot{x} + \frac{2k}{m}x\\
&\text{has a general solution of}\\
x =& A\cos\left(\sqrt{\frac{2k}{m}}t\right) + B\sin\left(\sqrt{\frac{2k}{m}}t\right)
\end{aligned}
$$

The solution says that x is oscillating.