PHYC 3590 - Advanced Classical Mechanics
Assignment 2

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Question 2.10

(a)

These variables are given in the question.

$$\begin{aligned}\eta =& 12\, N \cdot s/m^2 \\ \rho =& 7.8\, g/cm^3 \\ D =& 2\, mm\end{aligned}$$

The forces acting on the ball are

$$\begin{aligned}\vec{F}_{lin} =& -3\pi\eta Dv\hat{y} \\ \vec{F}_{buoyant} =& -\rho gV\hat{y} \\ \vec{F}_g =& mg\hat{y}\end{aligned}$$

where the ball is falling in the $+y$ direction. For the net force we have

$$ma = mg - 3\pi\eta Dv - \rho gV$$

For v_{ter} we set the a to equal 0.

$$\begin{aligned}0 =& mg - 3\pi\eta Dv_{ter} - \rho gV && (1) \\ 3\pi\eta Dv_{ter} =& mg - \rho gV && (2) \\ v_{ter} =& \frac{mg - \rho gV}{3\pi\eta D} && (3) \\ v_{ter} =& \frac{mg - \rho gV}{b} && (4)\end{aligned}$$

$$\begin{aligned}ma =& mg - bv - \rho gV && (5) \\ \frac{m}{b}a =& \frac{mg - \rho gV}{b} - v && (6)\end{aligned}$$

Subbing equation (4) into equation (6) gives

$$\frac{m}{b}a = v_{ter} - v$$

Let $u = v - v_{ter}$

$$\begin{aligned} m\dot{u} =& -bu \\ u =& u_0 \exp\left(-\frac{t}{\tau}\right), \ \tau = \frac{m}{b} \\ v - v_{ter} =& (v_0 - v_{ter}) \exp\left(-\frac{t}{\tau}\right) \end{aligned}$$

$$\begin{aligned} \tau =& \frac{m}{b} \\ \tau =& \frac{m}{3\pi\eta D} \\ m =& \rho V = \rho\frac{4}{3}\pi(D/2)^3 = \frac{1}{6}\rho\pi D^3 \\ \tau =& \frac{\rho\pi D^3}{6 \cdot 3\pi\eta D} = \boxed{\frac{\rho D^2}{18\eta}} \\ \tau =& \frac{(7.8g/cm^3)(2mm)^2}{18(12N \cdot s/m^2)} \\ \tau =& \frac{(7800kg/m^3)(0.002m)^2}{18(12N \cdot s/m^2)} \\ \tau =& \boxed{1.4 \times 10^{-4}\ seconds} \end{aligned}$$

95% of v_{ter} is reached after 3τ, which is equal to $\boxed{4.3 \times 10^{-4}\ seconds}$.

Solving for v_{ter} numerically, we have

$$\begin{aligned} v_{ter} =& \frac{mg - \rho_{gly} gV}{b} \\ v_{ter} =& \frac{mg}{3\pi\eta D} - \frac{\rho gV}{3\pi\eta D} \\ v_{ter} =& \frac{\frac{1}{6}\rho_{ball}\pi D^3 g}{3\pi\eta D} - \frac{\rho_{gly} g(\frac{1}{6}\pi D^3)}{3\pi\eta D} \\ v_{ter} =& \frac{\frac{1}{6}\rho_{ball} D^2 g}{3\eta} - \frac{\frac{1}{6}\rho_{gly} D^2 g}{3\eta} \\ v_{ter} =& \frac{D^2 g}{18\eta}(\rho_{ball} - \rho_{gly}) \\ v_{ter} =& \frac{(0.002m)^2(9.81m/s^2)}{18(12N \cdot s/m^2)}(7800kg/m^3 - 1300kg/m^3) = \boxed{1.2 \times 10^{-3} m/s} \end{aligned}$$

(b)

$$\begin{aligned}
F_{lin} =& 3\pi\eta D v \\
R =& \frac{Dv\rho}{\eta} \\
\frac{F_{quad}}{F_{lin}} =& \frac{R}{48} \\
\frac{F_{quad}}{F_{lin}} =& \frac{Dv\rho}{48\eta} \\
\frac{F_{quad}}{F_{lin}} =& \frac{(0.002mm)(1.2\times10^{-3}m/s)(1.3g/cm^3)}{48(12N\cdot s/m^2)} \\
\frac{F_{quad}}{F_{lin}} =& \frac{(0.002m)(1.2\times10^{-3}m/s)(1300kg/m^3)}{48(12N\cdot s/m^2)} \\
\frac{F_{quad}}{F_{lin}} =& \boxed{5.4\times10^{-6}}
\end{aligned}$$

$\frac{F_{quad}}{F_{lin}}$ is small, therefore F_{quad} is negligible. This means it was a good approximation to only consider linear drag.

Question 2.19

(a)

The only force acting on the projectile is $\vec{w} = -mg\hat{y}$ therefore $a = -g\hat{y}$. The particle also has an initial velocity, $\vec{v}_0 = v_{x_0}\hat{x} + v_{y_0}\hat{y}$. Next to solve for the equation for motion we have

$$x = \int v_{x_0}\, dx = \boxed{v_{x_0} t} \tag{7}$$

$$y = \int v_{y_0}\, dx - \int\int g\ dx\, dx = \boxed{v_{y_0} t - \tfrac{1}{2} g t^2} \tag{8}$$

Next we use equation (7) to solve for t, then plug it into question (8).

$$t = \frac{x}{v_{x_0}}$$

$$y = v_{y_0}\left(\frac{x}{v_{x_0}}\right) - \tfrac{1}{2} g \left(\frac{x}{v_{x_0}}\right)^2$$

$$y = \boxed{\frac{v_{y_0}}{v_{x_0}} x - \frac{g}{2v_{x_0}^2} x^2}$$

(b)

Equation 2.37 from Taylor is

$$y = \frac{v_{y0} + v_{ter}}{v_{x0}} x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

When the air resistance is switched off, v_{ter} and τ go to ∞.

$$\lim_{v_{ter},\tau\to\infty} (y) = ???$$

Taylor expanding the ln term from the equation for y gives

$$y = \frac{v_{y0} + v_{ter}}{v_{x0}} x - v_{ter}\tau \left[\frac{x}{v_{x0}\tau} + \frac{1}{2}\left(\frac{x}{v_{x0}\tau}\right)^2 + \frac{1}{2}\left(\frac{x}{v_{x0}\tau}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0} + v_{ter}}{v_{x0}} x - v_{ter} \left[\frac{x}{v_{x0}} + \frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau^2}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0}}{v_{x0}} x + \frac{g\tau}{v_{x0}} x - g\tau \left[\frac{x}{v_{x0}} + \frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau^2}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0}}{v_{x0}} x + \frac{g\tau}{v_{x0}} x - g\tau \frac{x}{v_{x0}} - g\tau \left[\frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau^2}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0}}{v_{x0}} x - g\tau \left[\frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau^2}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$y = \frac{v_{y0}}{v_{x0}} x - g \left[\frac{1}{2}\left(\frac{x}{v_{x0}}\right)^2 + \frac{1}{2\tau}\left(\frac{x}{v_{x0}}\right)^3 + \ldots\right]$$

$$\lim_{\tau\to\infty} (y) = \boxed{\frac{v_{y0}}{v_{x0}} x - \frac{g}{2v_{x0}^2} x^2}$$

QED

Question 2.37

The first part of this problem is to solve this integral

$$\int \frac{1}{1-u^2}\, du \tag{9}$$

$$= \frac{1}{2}\left[\int \frac{1}{1+u} du + \int \frac{1}{1-u} du\right] \tag{10}$$

For the left most term we let $v = 1+u$, $du = dv$

$$\begin{aligned}\int \frac{1}{1+u} du &= \int \frac{1}{v} dv \\ &= \ln v \\ &= \boxed{\ln(1+u)}\end{aligned}$$

For the right most term we let $v = 1-u$, $du = -dv$

$$\begin{aligned}\int \frac{1}{1-u} du &= -\int \frac{1}{v} dv \\ &= -\ln v \\ &= \boxed{-\ln(1-u)}\end{aligned}$$

Subbing the boxed equations int equation (10) gives

$$\frac{1}{2}\left[\ln(1+u) - \ln(1-u)\right] \tag{11}$$

$$= \frac{1}{2}\ln\left[\frac{1+u}{1-u}\right] \tag{12}$$

$$\int \frac{1}{1-u} du = \operatorname{arctanh} u \tag{13}$$

From the book we have

$$\begin{aligned}\frac{dv}{1 - v^2/v_{ter}^2} &= g\, dt \\ \int_0^v \frac{dv}{1 - v^2/v_{ter}^2} &= \int_0^t g\, dt \\ v_{ter} \int_0^u \frac{du'}{1 - u'^2} &= \int_0^t g\, dt \\ u = v/v_{ter},\ dv &= v_{ter} dt\end{aligned}$$

Using the solution given by equation (13), we find

$$\begin{aligned}v_{ter} \operatorname{arctanh} \frac{v}{v_{ter}} &= gt \\ \operatorname{arctanh} \frac{v}{v_{ter}} &= \frac{gt}{v_{ter}} \\ \frac{v}{v_{ter}} &= \tanh\left(\frac{gt}{v_{ter}}\right) \\ v &= \boxed{v_{ter} \tanh\left(\frac{gt}{v_{ter}}\right)}\end{aligned}$$

QED

Question 2.52

$$
\begin{aligned}
\eta =& Ae^{-i\omega t} = ae^{i\delta}e^{-i\omega t} = ae^{i(\delta-\omega t)} \\
\eta =& a\cos(\delta-\omega t) + ia\sin(\delta-\omega t) \\
v_x =& a\cos(\delta-\omega t),\ v_y = a\sin(\delta-\omega t)
\end{aligned}
$$

η moves clockwise with a constant speed of $|a|$ and an angular velocity of $-\omega$.

Question 2.54

Equations (2.68) are

$$
\begin{aligned}
\dot{v_x} =& \omega v_y \\
\dot{v_y} =& -\omega v_x
\end{aligned}
$$

Differentiating the first equation in (2.68), we have

$$
\begin{aligned}
\ddot{v_x} =& \omega \dot{v_y} \\
\frac{\ddot{v_x}}{\omega} =& \dot{v_y}
\end{aligned}
$$

Subbing this result into the second equation in (2.68) gives

$$
\begin{aligned}
\frac{\ddot{v_x}}{\omega} =& -\omega v_x \\
\ddot{v_x} =& -\omega^2 v_x \\
v_x =& \boxed{A\sin(\omega t) + B\cos(\omega t)}
\end{aligned}
$$

Now solving for v_y we have

$$
\begin{aligned}
\omega v_y =& \dot{v_x} \\
\omega v_y =& \frac{d}{dt}[A\sin(\omega t) + B\cos(\omega t)] \\
\omega v_y =& A\omega\cos(\omega t) - B\omega\sin(\omega t) \\
v_y =& \boxed{A\cos(\omega t) - B\sin(\omega t)}
\end{aligned}
$$

$$
\vec{v} = A\sin(\omega t) + B\cos(\omega t) + i(A\cos(\omega t) - B\sin(\omega t))
$$