Lecture Notes

Quantum Mechanics II

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1 Vector Spaces

Definition

Linear Vector Space: Collection of objects which follow the rules below...

$$
\begin{aligned}
|v\rangle + |w\rangle \, \varepsilon \, V \\
a(|v\rangle + |w\rangle) =& a|v\rangle + a|w\rangle \\
a(b|v\rangle) =& b(a|v\rangle) \\
|v\rangle + |w\rangle =& |w\rangle + |v\rangle \\
|v\rangle + (|w\rangle + |x\rangle) =& (v\rangle + |w\rangle) + |x\rangle \\
\text{There needs to be a null vector}(|0\rangle) \rightarrow & |v\rangle + |0\rangle = 0 \\
\text{For every vector, v, there is an inverse: } & |v\rangle + |-v\rangle = |0\rangle
\end{aligned}
$$

Definition

The number a,b are elements of a field **F**

$$
\begin{aligned}
F =& R \rightarrow \text{real} \\
F =& C \rightarrow \text{complex}
\end{aligned}
$$

Definition

$$
\begin{aligned}
(a, b, c) + (d, e, f) =& (a + d, b + e, c + f) \\
\alpha(a, b, c) =& (\alpha a, \alpha b, \alpha c) \\
\text{null vector} \rightarrow |0\rangle =& (0, 0, 0) \\
\text{inverse} \rightarrow (-a, -b, -c) =& |-v\rangle \\
(a, b, c) \neq & \text{A Vector Space}
\end{aligned}
$$

Above is not a vector space because it isn't closed under addition.

Definition

Linear Independence: a set of vectors such as $\{|1\rangle, |2\rangle, \ldots, |n\rangle$ is linearly independent if the only solution to $\sum a_i|i\rangle = |0\rangle$ is $a_1 = a_2 = \cdots = a_i = 0$

Example:

$$|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} |2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} |3\rangle = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

This is not an example of linear independence because $|3\rangle + 2|3\rangle - |1\rangle = |0\rangle$

Definition

Dimension: is a vector space of dimension n IFF it can accommodate a max of n LI vectors.

An n-dimensional real vector space $= V^n(R)$

An n-dimensional complex vector space $= V^n(C)$

Theorem

A Basis: is a set of n LI vectors in a n-dimensional vector space.

$$|v\rangle = \sum_{i=n}^{n} v_i |e_i\rangle$$

The expression of a vector, $|v\rangle$ in terms of a particular basis is unique.

Example:

Find the space of all 2×2 matrices.

$$\begin{aligned} v &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ dim(V) &= ??? \\ |1\rangle, |2\rangle, |3\rangle, |4\rangle &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\ \therefore n &= 4 \end{aligned}$$

2 Inner Product Space

To define an analogue to length and angle, we need to define an inner product: $\langle v|w\rangle$.

Inner product is a rule for taking two vectors/ functions and 'getting out' a number.

1. Skew-symmetric: $\langle v|w\rangle = \langle w|v\rangle *$
2. Positive semi definite: $\langle v|v\rangle \geq 0$
3. Linearly in Ket: $\langle v|(a|w\rangle + b|x\rangle) = a\langle v|w\rangle + b\langle v|x\rangle = \langle v|aw + bx\rangle$

Example:

What if a bra vector was a linear super position?

$$\langle aw + bx|v\rangle = \langle v|aw + bx\rangle^* = a^*\langle v|w\rangle^* + b^*\langle v|x\rangle^* = \boxed{a^*\langle w|v\rangle + b^*\langle x|v\rangle}$$

Inner products are antilinear in the bra vector.

$$\langle\Psi|\Psi\rangle = \int \Psi^*(x)\Psi(x)dx$$

Definition:

1. Two vectors are orthogonal if $\langle v|w\rangle = 0$
2. The Norm of vectors is defined as $\sqrt{\langle v|v\rangle} = |v|$
3. A set of unit vectors that are mutually orthogonal are said to constitute an **orthonormal basis**.

Example 1:

Inner product of $|v\rangle$ and $|w\rangle$.

$$
\begin{aligned}
|v\rangle &= \sum_{i=1}^{n} v_i |e_i\rangle \\
|w\rangle &= \sum_{j=1}^{n} w_j |e_j\rangle \\
\langle v|w\rangle &= \sum_{i=1}^{n} \sum_{j=1}^{n} v_i^* \langle e_i|e_j\rangle w_j \\
\langle e_i|e_j\rangle &= \delta_{ij} \\
\delta_{ij} &= 1 \text{ if i=j} \\
\langle v|w\rangle &= \sum_i v_i^* w_i
\end{aligned}
$$

Example 2:

The expression of $|v\rangle$ in the basis $\{|e_i\rangle\}$ can be written using the projection operator.

$$
\begin{aligned}
\hat{p}|v\rangle &= \sum_i |e_i\rangle\langle e_i|v\rangle \\
v_i &= \langle e_i|v\rangle \\
\hat{p}|v\rangle &= \sum_i v_i|e_i\rangle \\
\hat{p} &= \sum |e_i\rangle\langle e_i| = 1 \\
\hat{p}|v\rangle &= |v\rangle
\end{aligned}
$$

Since $\hat{p}|v\rangle = |v\rangle$, we see that $\hat{p} = 1$ is the identity. The expression $\sum_i |e_i\rangle\langle e_i\rangle = 1$ is a statement of the completeness condition.

Example 3:

$|v\rangle$ in Example 1 can be written as a column vector:

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}$$

Therefore...

$$\langle v|w\rangle = \begin{bmatrix} v_1^* & v_2^* & ... & v_n^* \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ ... \\ w_n \end{bmatrix}$$

Theorem

The triangle inequality: $|v + w| \leq |v| + |w|$

Theorem

The Swartz inequality: $\langle v|w\rangle \leq |v||w|$

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3 Outer Products

The outer product between a ket $|w\rangle$ and a bar $\langle v|$ is:

$$|v_i\rangle\langle w_i| = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} w_1^* & w_2^* & ... & w_n^* \end{bmatrix} = \begin{bmatrix} v_1^* w_1 & v_1^* w_2 & \cdots & v_1^* w_n \\ v_2^* w_1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ v_n^* w_1 & \cdots & \cdots & v_n^* w_n \end{bmatrix}$$

The projector operator in Example 3 is an example of an outer product.

Example:

Consider $\mathbb{V}^3$ with basis $|e_1\rangle$, $|e_2\rangle$, $|e_3\rangle$.

$$|e_2\rangle\langle e_2| = \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\sum_{i=1}^{3}|e_i\rangle\langle e_i| = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \backslash$$

i.e, the outer product between two orthonormal vectors, $|e_i\rangle\langle e_j|$
$|e_i\rangle$, $|e_j\rangle\varepsilon\mathbb{V}^n$ is a $n \times n$ dimensional matrix with zero everywhere except the elements in row i, col j.

4 Dual Vector Spaces

For every vector $|v\rangle$ in $\mathbb{V}^n$ we associate a bra vector $\langle v| = |v\rangle^\dagger$
The basis $\langle v|$ also form a vector space, that is separate and distinct from $|v\rangle$, but related to it.

Example

For ket $|v\rangle = |av_1 + bv_2\rangle$ we associate a bra $\langle v| = \langle av_1 + bv_2| = \langle v_1|a^* + \langle v_2|b_2^*$.

5 Subspaces

Definition:

A subset of vectors $\mathbb{V}^n$ that itself forms a vector space $\precapprox_i^{n_i}$ is called a subspace of $\mathbb{V}^n$.

Definition:

The sum of two vector spaces $\mathbb{V}_i^{n_i}$ and $\mathbb{V}_j^{m_j}$ is written :

$$\mathbb{V}_i^{n_i} \oplus \mathbb{V}_j^{m_j} = \mathbb{V}_k^{P_k}$$

is defined as having (i) all elements of $\mathbb{V}_i^{n_i}$, (ii) all elements $\mathbb{V}_j^{m_j}$ and (iii) all combination of the two.

Example:

$\mathbb{V}_x^1$ is a vector space containing the vectors along $\hat{x}$.
$\mathbb{V}_y^1$ is a vector space containing the vectors along $\hat{y}$.
$\mathbb{V}_x^1 \oplus \mathbb{V}_y^1 = \mathbb{V}_{xy}^2$

Example:

$\mathbb{V}_1^{n_i}$, $\mathbb{V}_2^{n_j}$ are two subspaces where every element of $\mathbb{V}_1^{n_i} \perp \mathbb{V}_2^{n_j}$.
If $\mathbb{V}_1^{n_i} \oplus \mathbb{V}_2^{n_j} = \mathbb{V}^n$, then prove $n_i + n_j = n$.

Consider $|v_\alpha\rangle \in \mathbb{V}_1^{n_i}$ and $|v_\beta\rangle \in \mathbb{V}_2^{n_j}$

$$\begin{aligned}|v_\alpha\rangle + |v_\beta\rangle &= \sum_{i=1}^{n_i} v_{\alpha i}|e_i\rangle + \sum_{j=1}^{n_j} v_{\beta j}|e_j\rangle \\ &= \sum_{i=1}^{n_i+n_j} v_{\alpha+\beta,i}\,|e_i'\rangle, \;\; |e_i'\rangle = \{|e_{i=1}, \ldots, e_{n_i}, e_{j=1}, \ldots, , e_{n_j}\}\end{aligned}$$

Since the $|e_i'\rangle$ are orthogonal, they are linearly independent. The maximum number of linearly independent vectors in $\mathbb{V}^n$ is therefore $n_i + n_j$. Therefore the dimension of $\mathbb{V}^n$ in $n = n_i + n_j$.

6 Operators

An operator converts a ket into another ket. e.g. $|1\rangle, |2\rangle, |3\rangle$ corresponds to $|\hat{x}\rangle, |\hat{y}\rangle, |\hat{z}\rangle$. An operator describing a rotation about $\hat{z}$-axis by 90°, $R_{\pi/2}(\hat{z})$:

$$R_{\pi/2}(\hat{z})|1\rangle = |2\rangle, \;\; R_{\pi/2}(\hat{z})|2\rangle = |-1\rangle, \;\; R_{\pi/2}(\hat{z})|3\rangle = |3\rangle$$

The matrix notation for an operator $\hat{C}$ can be found by operating $\hat{C}$ on the basis vectors $\{|r_j\rangle\}$ and then projecting the result back onto the basis with

$\hat{\mathbb{P}} = \sum_i |e_i\rangle\langle e_i|$.

$$
\begin{aligned}
|a_j\rangle =& \hat{C}|e_j\rangle \\
|a_j\rangle =& \sum_i |e_i\rangle\langle e_i|\hat{C}|e_j\rangle \\
\langle e_i|\hat{C}|e_j\rangle =& C_{ij} \\
C_{ij} =& \begin{bmatrix} \langle e_1|\hat{C}|e_1\rangle & \cdots & \langle e_1|\hat{C}|e_n\rangle \\ \vdots & \ddots & \vdots \\ \langle e_n|\hat{C}|e_1\rangle & \cdots & \langle e_n|\hat{C}|e_n\rangle \end{bmatrix} \\
|a_j\rangle =& \sum_i |e_i\rangle C_{ij}
\end{aligned}
$$

To express the vector $|w\rangle = \hat{C}|v\rangle$, $|v\rangle \in \mathbb{V}^n$, in matrix form, use $\hat{\mathbb{P}}$ twice:

$$
\begin{aligned}
\sum_i |e_i\rangle\langle e_i|w\rangle =& \sum_{ij} |e_i\rangle\langle e_i|\hat{C}|e_j\rangle\langle e_j|v\rangle \\
\sum_i |e_i\rangle w_i =& \sum_{ij} |e_i\rangle C_{ij} v_j
\end{aligned}
$$

In matrix form:

$$
\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}
$$

It is useful to remember that the j^{th} column is the j^{th} transformed basis vector: $\hat{C}|e_j\rangle = \sum_i |e_i\rangle C_{ij}$.

7 Product of Operators

$$
\begin{aligned}
(AB)_{ij} =& \langle e_i|\hat{A}\hat{B}|e_j\rangle \\
\sum_k |e_k\rangle\langle e_k| =& \mathbb{P} = 1 \\
(AB)_{ij} =& \sum_k \langle e_i|\hat{A}|e_k\rangle\langle e_k|\hat{B}|e_j\rangle \\
(AB)_{ij} =& \sum_k A_{ik} B_{kj}
\end{aligned}
$$

8 Adjoint Operator

For a scalar a, $a|v\rangle = |av\rangle$ and the corresponding $\langle av| = \langle v|a^*$
For an operator $\hat{C}$, $\hat{C}|v\rangle = |\hat{C}v\rangle$, we define the adjoint operator $\hat{C}^\dagger$ such that $\langle\hat{C}v| = \langle v|C^\dagger$. The form of $C^\dagger$ is the best demonstrated acting on a basis:

$$\begin{aligned}
C^\dagger_{ij} &= \langle e_i|\hat{C}^\dagger|e_j\rangle \\
C^\dagger_{ij} &= \langle \hat{C}e_i|e_j\rangle \\
C^\dagger_{ij} &= \langle e_j|\hat{C}e_i\rangle^* \\
C^\dagger_{ij} &= \langle e_j|\hat{C}|e_i\rangle^* \\
C^\dagger_{ij} &= C_{ij}
\end{aligned}$$

Definition:

Hermitian Operator: An operator is hermitian if $\hat{C}^\dagger = \hat{C}$

Anti-hermitian Operator: an operator is anti-hermitian if $\hat{C}^\dagger = -\hat{C}$

Unitary Operator: an operator $\hat{U}$ is unitary if $U^\dagger U = I$, $i.e.\, U^\dagger = U^{-1}$
Example: a rotation matrix is **unitary** if $R^\dagger_\theta(\hat{z}) = R_{-\theta}(\hat{z}) = R^{-1}_\theta(\hat{z})$.

9 Hermitian Operators

Theorem:

The eigenvalues of a hermitian operator are real.

Proof: Consider an operator with eigenvalues w:

$$\begin{aligned}
\Omega|w\rangle &= w|w\rangle \\
\langle w|\Omega|w\rangle &= w\langle w|w\rangle,\ eq.1 \\
\langle w|\Omega w\rangle &= w\langle w|w\rangle \\
\langle w|\Omega w\rangle^* &= w^*\langle w|w\rangle \\
\langle \Omega w|w\rangle &= w^*\langle w|w\rangle \\
\langle w|\Omega^\dagger|w\rangle &= w^*\langle w|w\rangle,\ eq.2
\end{aligned}$$

If $\hat{\Omega}$ is hermitian, $\hat{\Omega}^\dagger = \hat{\Omega} \to w^* = w \to w \in \mathbb{R}$.

Theorem:

For every Hermitian matrix, there exists an orthonormal set of eigenvectors. It is diagonal in this basis an it has its eigenvalues as diagonal elements.

Theorem:

For two commuting hermitian operators $\hat{\Omega}, \hat{\Lambda}$ there exists a basis of eigenvectors that diagonalize them both:

$$\begin{aligned}\hat{\Omega}|w\rangle =& w|w\rangle \\ \hat{\Lambda}|\lambda\rangle =& \lambda|\lambda\rangle\end{aligned}$$

Because $\left[\hat{\Omega}, \hat{\Lambda}\right] = 0...$

$$\hat{\Lambda}\hat{\Omega}|\lambda\rangle = \hat{\Omega}\hat{\Lambda}|\lambda\rangle = \lambda(\Omega|\lambda)$$

Therefore $\hat{\Omega}|\lambda\rangle$ is an eigenvector of $\hat{\Lambda}$. The eigenvectors are, however unique up to a scale factor $\rightarrow \hat{\Omega}|\lambda\rangle = w|\lambda\rangle$. Therefore $|\lambda\rangle$ is an eigenvector of both $\hat{\Omega}$ and $\hat{\Lambda}$, and therefore diagonalize them both.