Tutorial 1 Solution

1 a) $\langle \hat{S}_x \rangle = \langle \psi | \hat{S}_x | \psi \rangle = \begin{bmatrix} a^* & b^* \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}$

$$= \frac{\hbar}{2}(a^* b + b^* a)$$

$$= \hbar \, Re(a^* b)$$

$\langle \hat{S}_y \rangle = \langle \psi | \hat{S}_y | \psi \rangle = \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} & -i \\ i & \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = i \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix}$

$$= -i \frac{\hbar}{2}(a^* b - b a^*)$$

$$= -i\hbar \, Im(a^* b)$$

$\langle \hat{S}_z \rangle = \langle \psi | \hat{S}_z | \psi \rangle = \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2}(|a|^2 - |b|^2)$

b) If $|\psi\rangle$ is unnormalized, then the results need to be divided by the norm, $\langle \psi | \psi \rangle = (|a|^2 + |b|^2)$

c) $\hat{S}_x^2 = \left(\frac{\hbar}{2}\right)^2 \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \frac{\hbar^2}{4} \mathbb{1}$

$\hat{S}_y^2 = \left(\frac{\hbar}{2}\right)^2 \begin{bmatrix} & -i \\ i & \end{bmatrix} \begin{bmatrix} & -i \\ i & \end{bmatrix} = \frac{\hbar^2}{2} \mathbb{1}$

$\hat{S}_z^2 = \left(\frac{\hbar}{2}\right)^2 \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = \left(\frac{\hbar}{2}\right)^2 \mathbb{1}$

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3}{4}\hbar^2 \mathbb{1}$$

$$\Rightarrow \langle \psi | \hat{S}^2 | \psi \rangle = \frac{3}{4}\hbar^2 (|a|^2 + |b|^2)$$

$$= \frac{3}{4}\hbar^2$$

This is the expected result since $\hat{S}^2 |\psi\rangle = s(s+1)\hbar^2 |\psi\rangle$

$$= \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 |\psi\rangle$$

$$= \frac{3}{4}\hbar^2 |\psi\rangle$$

2 a) $(+,\hat{z})$-counts $= 25$, $(-,\hat{z})$-counts $= 75$

Therefore there is a 25% prob of finding a Ag-atom in a $|+,z\rangle$ state:

$$|\psi\rangle = \overset{\equiv c_+}{\sqrt{\tfrac{1}{4}}} \, |+,z\rangle + \overset{\equiv c_-}{\sqrt{\tfrac{3}{4}}} \, |+,z\rangle \, , \quad i.e, \ |\langle +,\hat{z} | \psi \rangle|^2 = 25\%$$

b) From part 1a), $\langle \hat{S}_x \rangle = \hbar \,\mathrm{Re}(c_+^* c_-) = \frac{\sqrt{3}}{4}\hbar$

c) $\langle \hat{S}_y \rangle = -\hbar \,\mathrm{Im}(c_+^* c_-) = 0$

Could choose $|\psi\rangle = \frac{1}{2}|+,z\rangle + i\frac{\sqrt{3}}{2}|-,z\rangle$

$\langle \hat{S}_y \rangle = -i\hbar \,\mathrm{Im}\left[\left(\frac{1}{2}\right)\left(\frac{i\sqrt{3}}{2}\right)\right] = \frac{\sqrt{3}}{4}\hbar \rightarrow$ note: $\langle \hat{S}_x \rangle$ is now zero

d) quantum uncertainty in this example is given by the standard deviation of $\hat{S}_z$: $\Delta \hat{S}_z = \sqrt{\langle \hat{S}_z^2 \rangle - \langle S_z \rangle^2}$

$$= \sqrt{\frac{3}{4}\hbar^2 - \left(\frac{\hbar}{2}\left(\frac{1}{2} - \frac{3}{4}\right)\right)^2}$$

$$= \sqrt{\frac{3}{4}\hbar^2 - \frac{\hbar^2}{64}}$$

$$= \frac{\sqrt{47}}{8}\hbar$$

d ii) measurement uncertainty is the error on the values of S_z, i.e. the FWHM of peaks $\simeq 0.07\hbar$. So if we measured S_z of one single atom, out uncertainty would be $\simeq \pm 0.07\hbar$

If we measured <u>many</u> Ag atoms and then fit the data, our uncertainty would be much lower

iii) statistical uncertainty is the error in the measurement of $|c_+|^2$ & $|c_-|^2$, which is $\sim \pm 2$ counts

3a) Answer (i)

b) Nora & Hugo are correct

c) $\hat{S}_n = \hat{S} \cdot \hat{n} = \frac{\hbar}{2}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \sin\theta\cos\phi + \frac{\hbar}{2}\begin{bmatrix} & -i \\ i & \end{bmatrix} \sin\theta\sin\phi + \frac{\hbar}{2}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\cos\theta$

$$= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{bmatrix}$$

$$= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}$$

We know eigenvalues must be $\pm\frac{\hbar}{2}$

$$\hat{S}_n|\xi\rangle = \lambda_1|\xi\rangle$$

$$(\hat{S}_n - \lambda_1 \mathbb{1})|\xi_1\rangle = 0 \qquad |\xi_1\rangle = \begin{bmatrix}\xi_{1+}\\ \xi_{1-}\end{bmatrix}$$

$$\Rightarrow \left(\cancel{\tfrac{\hbar}{2}}\cos\theta - \cancel{\tfrac{\hbar}{2}}\right)\xi_{1+} + \cancel{\tfrac{\hbar}{2}}\sin\theta e^{-i\phi}\xi_{1-} = 0$$

$$\Rightarrow -2\sin^2\left(\tfrac{\theta}{2}\right)\xi_{1+} + 2\sin\theta/2\cos\theta/2\, e^{-i\phi}\xi_{1-} = 0$$

$$\Rightarrow |\xi_1\rangle = \begin{bmatrix}\cos\theta/2\\ \sin\theta/2\, e^{i\phi}\end{bmatrix} = |+, n\rangle$$

For the second eigenvector $|\xi_2\rangle = \begin{bmatrix}\xi_{2+}\\ \xi_{2-}\end{bmatrix}$

$$(\hat{S}_n - \lambda_2 \mathbb{1})|\xi_2\rangle = 0$$

$$\Rightarrow \tfrac{\hbar}{2}\cos\theta - \left(-\tfrac{\hbar}{2}\right)\xi_{2+} + \tfrac{\hbar}{2}\sin\theta e^{-i\phi}\xi_{2-} = 0$$

$$\Rightarrow 2\cos^2\left(\tfrac{\theta}{2}\right)\xi_{2+} + 2\sin\theta/2\cos\theta/2\, e^{-i\phi}\xi_{2+} = 0$$

$$\Rightarrow |\xi_2\rangle = \begin{bmatrix}\sin(\theta/2)\\ -\cos(\theta/2)e^{i\phi}\end{bmatrix} = \begin{bmatrix}\sin\theta/2\, e^{-i\phi}\\ -\cos(\theta/2)\end{bmatrix}\cancel{e^{+i\phi}} = |-, n\rangle$$

$$U = \begin{bmatrix}\xi_1 & \xi_2\\ | & |\end{bmatrix} = \begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi}\\ \sin\frac{\theta}{2}e^{i\phi} & -\cos\left(\frac{\theta}{2}\right)\end{bmatrix}$$