Lecture Notes

Quantum Mechanics II

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1 Vector Spaces

Definition

Linear Vector Space: Collection of objects which follow the rules below...

$$\begin{aligned}|v\rangle + |w\rangle \; \varepsilon \, V \\ a(|v\rangle + |w\rangle) =& a|v\rangle + a|w\rangle \\ a(b|v\rangle) =& b(a|v\rangle) \\ |v\rangle + |w\rangle =& |w\rangle + |v\rangle \\ |v\rangle + (|w\rangle + |x\rangle) =& (v\rangle + |w\rangle) + |x\rangle \\ \text{There needs to be a null vector}(|0\rangle) \rightarrow |v\rangle + |0\rangle = 0 \\ \text{For every vector, v, there is an inverse: } |v\rangle + |-v\rangle = |0\rangle\end{aligned}$$

Definition

The number a,b are elements of a field **F**

$$\begin{aligned}F =& R \rightarrow \text{real} \\ F =& C \rightarrow \text{complex}\end{aligned}$$

Definition

$$\begin{aligned}(a, b, c) + (d, e, f) =& (a + d, b + e, c + f) \\ \alpha(a, b, c) =& (\alpha a, \alpha b, \alpha c) \\ \text{null vector} \rightarrow |0\rangle =& (0, 0, 0) \\ \text{inverse} \rightarrow (-a, -b, -c) =& |-v\rangle \\ (a, b, c) \neq& \text{A Vector Space}\end{aligned}$$

Above is not a vector space because it isn't closed under addition.

Definition

Linear Independence: a set of vectors such as $\{|1\rangle, |2\rangle, \ldots, |n\rangle$ is linearly independent if the only solution to $\sum a_i|i\rangle = |0\rangle$ is $a_1 = a_2 = \cdots = a_i = 0$

Example:

$$|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} |2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} |3\rangle = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

This is not an example of linear independence because $|3\rangle + 2|3\rangle - |1\rangle = |0\rangle$

Definition

Dimension: is a vector space of dimension n IFF it can accommodate a max of n LI vectors.

An n-dimensional real vector space $= V^n(R)$

An n-dimensional complex vector space $= V^n(C)$

Theorem

A Basis: is a set of n LI vectors in a n-dimensional vector space.

$$|v\rangle = \sum_{i=n}^{n} v_i |e_i\rangle$$

The expression of a vector, $|v\rangle$ in terms of a particular basis is unique.

Example:

Find the space of all 2×2 matrices.

$$\begin{aligned} v =& \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ dim(V) =& ??? \\ |1\rangle, |2\rangle, |3\rangle, |4\rangle =& \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\ \therefore n =& 4 \end{aligned}$$

2 Inner Product Space

To define an analogue to length and angle, we need to define an inner product: $\langle v|w\rangle$.

Inner product is a rule for taking two vectors/ functions and 'getting out' a number.

1. Skew-symmetric: $\langle v|w\rangle = \langle w|v\rangle *$
2. Positive semi definite: $\langle v|v\rangle \geq 0$
3. Linearly in Ket: $\langle v|(a|w\rangle + b|x\rangle) = a\langle v|w\rangle + b\langle v|x\rangle = \langle v|aw + bx\rangle$

Example:

What if a bra vector was a linear super position?

$$\langle aw + bx|v\rangle = \langle v|aw + bx\rangle^* = a^*\langle v|w\rangle^* + b^*\langle v|x\rangle^* = \boxed{a^*\langle w|v\rangle + b^*\langle x|v\rangle}$$

Inner products are antilinear in the bra vector.

$$\langle \Psi|\Psi\rangle = \int \Psi^*(x)\Psi(x)dx$$

Definition:

1. Two vectors are orthogonal if $\langle v|w\rangle = 0$
2. The Norm of vectors is defined as $\sqrt{\langle v|v\rangle} = |v|$
3. A set of unit vectors that are mutually orthogonal are said to constitute an **orthonormal basis**.

Example:

Inner product of $|v\rangle$ and $|w\rangle$.

$$\begin{aligned}
|v\rangle &= \sum_{i=1}^{n} v_i |e_i\rangle \\
|w\rangle &= \sum_{j=1}^{n} w_j |e_j\rangle \\
\langle v|w\rangle &= \sum_{i=1}^{n} \sum_{j=1}^{n} v_i^* \langle e_i|e_j\rangle w_j \\
\langle e_i|e_j\rangle &= \delta_{ij} \\
\delta_{ij} &= 1 \text{ if i=j} \\
\langle v|w\rangle &= \sum_{i} v_i^* w_i
\end{aligned}$$

Example:

The expression of $|v\rangle$ in the basis $\{|e_i\rangle\}$ can be written using the projection operator.

$$\begin{aligned}
\hat{p}|v\rangle &= \sum_{i} |e_i\rangle\langle e_i|v\rangle \\
v_i &= \langle e_i|v\rangle \\
\hat{p}|v\rangle &= \sum_{i} v_i |e_i\rangle \\
\hat{p} &= \sum |e_i\rangle\langle e_i| = 1
\end{aligned}$$

This is the statement of the completeness of the basis.