

SOLIDS

4.1 INTRODUCTION AND DEFINITIONS

The concept of *solid* is an idealization of real behavior of numerous materials. Some of them are close to this model, for example, steel and stone. Other materials are far from this ideal model, but in some applications they can also be treated as elastic solids, for example, wood, rubber, and concrete.

In the previous chapters, it was emphasized that any material can be treated as solid or liquid depending on the Deborah number, which is the ratio of inherent time-scale and characteristic time of loading. This is also true for materials listed above. Therefore, the elastic solid is in fact a concept describing *behavior* rather than a particular *material*. The same material may behave as *solid-like* or *liquid-like*, depending on the time-scale of observation. Steel is a typical example in this respect. Nobody doubts that steel is solid in all its applications. However, in many technological operations (such as forging, rolling, drawing) steel *flows*, i.e., it is able to undergo irreversible deformation. This ability of solid material is characterized as *plasticity*.

In spite of departures from the model, it is important to discuss the concept of a solid, or *elastic solid*, because it is a limiting case in rheological properties of real materials.

There is also another aspect of rheological interest in elastic behavior of solid materials. There is a great number of *elastic liquids*, i.e., real liquids which also demonstrate elastic behavior. These are polymeric substances, such as, melts, solutions, and emulsions. The qualitative description of mechanical properties of such liquids must also include characterization of their elasticity. For these reasons, we consider the “pure” case of elastic behavior of a solid, which is a suitable model useful in the rheological equation of state of elastic liquids.

Main concepts and experimental facts on mechanical behavior of solids are included in the framework of rheology. At the same time, considering that rheology is mainly a science devoted to liquids, or flowing media, then this chapter is a secondary subject, though necessary in the general structure of rheology.

The basic idea of a solid is its ability to experience elastic (reversible) deformation. These materials store work done by external forces. They store work in the form of elastic energy and return this energy when forces are removed.

Elastic solids are also treated as materials with a clearly defined relationship between stresses and deformations, i.e., if the stress field is known, then the spatial distribution of deformations is also known, and *vice versa*.

The main point in both concepts is the *absence of time or time-dependent effects*. However, if the deformation is time-dependent (in a permanent stress field), then a charac-

teristic feature of the viscous (dissipative) behavior of matter should be combined with elasticity. Therefore, the material is viscoelastic, as was discussed in Chapter 2.

Formulating the rheological model of an elastic solid, i.e., writing its rheological equation of state (or a constitutive equation), transformation from components of stress and deformation tensors observed in an experiment to an invariant formulation is required. The formulation must not depend on the choice of coordinate system. It can be an expression in the form of *elastic potential* (stored energy), W , as a function of the invariants of stress or deformation tensor or both. Also, this invariant formulation can be a relationship between the invariants of stress and deformation tensors. Both approaches are suitable in formulating a rheological equation of state. The result of the formulation based on the energy concept can be reformulated into the components of stress vs. deformation tensors.

This transition is based on a fundamental expression for stored energy:

$$dW = \sum_{i,j=1}^3 \sigma_{ij} d\epsilon_{ij} \quad [4.1.1]$$

Then, the following formula for stress components, in terms of an elastic potential, can be written:

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} \quad [4.1.2]$$

If W (a function of deformations) is known, then components of the stress tensor can be calculated from Eq. 4.1.2. A method of conversion from stress-deformation function to the elastic potential function is needed to be written using invariants.

In a general case, W can be written as a function of three invariants:

$$W = W(E_1, E_2, E_3) \quad [4.1.3]$$

where E_1 , E_2 , and E_3 are invariants of the tensor of large deformations, as was discussed in section 1.2.

Using an ordinary rule of function differentiation, the following formula can be obtained:

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = \sum_{k=1}^3 \frac{\partial W}{\partial E_k} \frac{\partial E_k}{\partial \epsilon_{ij}} \quad [4.1.4]$$

The last equation gives an answer to the problem formulated above: if a function $W(E_1, E_2, E_3)$ is known, the components of the stress tensor can be found from Eq. 4.1.4.

The above written definitions and equations are applicable to any elastic material regardless of the form of the elastic potential function, Eq. 4.1.3.

4.2 LINEAR ELASTIC (HOOKEAN) MATERIALS

The basic concept and equation of Hookean material was frequently mentioned in the previous chapters (see Eq. 2.1.2) because it correctly represents numerous experimental data of real materials, but with two important limitations:

- the equation is written for one-dimensional deformations (extension)
- the equation is valid for small deformations.

It is thus necessary to formulate the general (invariant) form of Hooke's law. The formulation of Hooke's law for extension is:

$$\sigma_E = E\epsilon = \frac{1}{J}\epsilon \quad [4.2.1]$$

where E is *Young's modulus*, $J = E^{-1}$ is the elastic compliance, σ_E is the normal stress and ϵ is the deformation in extension.

It would be incorrect to write Hooke's law by adding indices in the following form:

$$\sigma_{ij} = E\epsilon_{ij}$$

Such an equation is correct for uniaxial extension ($ij = 11$). But experiments show that in uniaxial extension, the cross-section of the sample is reduced, and this effect is described by *Poisson's ratio*, μ . This experimental fact means that the components of the deformation tensor ϵ_{22} and ϵ_{33} do not equal zero, but the external forces in the directions normal to 11 (22 and 33 directions) are absent, i.e., σ_{22} and σ_{33} equal zero. This is the direct proof that this generalization of Hooke's law is invalid.

An experiment in simple shear shows that a linear relationship between deformations and stresses exists:

$$\sigma = G\gamma \quad [4.2.2]$$

and all-directional (hydrostatic) compression:

$$p = -B\epsilon_v \quad [4.2.3]$$

where σ is the shear stress, γ is the shear deformation, ϵ_v is the volume deformation (relative change in volume of material) and G and B are coefficients:

G – shear modulus

B – bulk modulus of compressibility

which are different than Young's modulus.

It is now necessary to find out whether these constants (E , G , μ , and B) are independent and whether the number of constants is sufficient to characterize the material properties. For other forms of deformation, e.g., biaxial extension, the question arises whether it will be necessary to introduce a "modulus" characterizing linear behavior of material in these types of deformation.

Then the main questions are:

- how a general rheological equation of state should be written to reflect its linear elastic properties in different modes of deformation, which would include all these relationships.
- what is the minimal number of independent characteristic constants ("moduli") describing all types of deformation of material?

In uniaxial extension, not only the *shape* (form), but also the *bulk* (volume) of a body changes, and these two are *independent effects*. It is thus necessary to introduce *at least two independent constants* for characterization of material resistance to volume and shape

changes. This may suggest use of previously proven methods to decompose stress and deformation tensors into spherical (isotropic) and deviatoric parts.

Let us assume that the linear relationships between spherical and deviatoric parts of both tensors exist separately. This assumption gives two independent invariant relationships:

$$I_1 = kE_1 \quad [4.2.4]$$

and

$$\sigma'_{ij} = 2Ge'_{ij} \quad [4.2.5]$$

where I_1 is the first invariant of the stress tensor (sum of normal stresses, which is the measure of *hydrostatic pressure*), E_1 is the first invariant of the deformation tensor (measure of relative volume changes), σ'_{ij} are deviatoric components of the stress tensor, e'_{ij} are deviatoric components of the deformation tensor, and k and G are material constants, the first of them characterizing resistance to volume changes and the second one to the material shape changes.

Hydrostatic pressure is expressed *via* the first invariant of the stress tensor (Eq. 1.1.15) as

$$p = -\frac{I_1}{3} \quad [4.2.6]$$

Then, Eq. 4.2.4 can be rewritten:

$$p = -BE_1 \quad [4.2.7]$$

where $B = -k/3$ is the bulk modulus of elasticity. The two fundamental coefficients, B and G , are sometimes called the *Lamé constants*. The value of G in Eq. 4.2.5 is the shear modulus, and the coefficient 2 in this equation appears due to the formal definition of components of the deformation tensor.

The basic assumption of the generalized (three-dimensional) Hooke law is that Eqs. 4.2.4 and 4.2.5 are valid for *any* type of deformation and that they are the invariant *definition* of Hookean elastic material (body).

Then, it is important to relate the constants in Eqs. 4.2.4 and 4.2.5 with those, which are directly measured in a standard experiment of uniaxial extension, namely, Young's modulus and Poisson's ratio.

Let σ_E be the extensional (normal) stress. Then Eq. 4.2.1 written for all components of the stress and deformation tensors is

$$\varepsilon_{11} = \frac{\sigma_E}{E}; \quad \varepsilon_{22} = \varepsilon_{33} = -\mu\varepsilon_{11}$$

and

$$E_1 = (1 - 2\mu)\frac{\sigma_E}{E}$$

where E is Young's modulus and μ is Poisson's ratio (coefficient).

Hydrostatic pressure, p , in uniaxial extension is

$$p = -\sum_{i=1}^3 \sigma_{ij} = -\frac{\sigma_E}{3}$$

Then, based on Eq. 4.2.4, the following equality can be written

$$\frac{\sigma_E}{3} = B(1-2\mu)\frac{\sigma_E}{E}$$

This gives the desirable relationship between the constants:

$$E = 3(1-2\mu)B$$

Based on the same arguments, the following relationship can be easily obtained:

$$\sigma_E = 2G(1+2\mu)\frac{\sigma_E}{E}$$

and this gives the following relationship between the other constants:

$$E = 2G(1+2\mu)$$

The results obtained in these relationships permit calculation of any pair of constants for any pair of variables, as summarized below.

- for known constants E and μ

$$B = \frac{E}{3(1-2\mu)}, G = \frac{E}{2(1+\mu)} \quad [4.2.8]$$

- for known constants B and G

$$E = \frac{9BG}{3B+G}, \mu = \frac{3B-2G}{6B+2G} \quad [4.2.9]$$

- for known G and μ

$$E = 2G(1+\mu), B = \frac{2G(1+\mu)}{3(1-2\mu)} \quad [4.2.10]$$

- for known E and G

$$B = \frac{EG}{3(3G-E)}, \mu = \frac{E-2G}{2G} \quad [4.2.11]$$

The calculations for any other pairs of constants can also be done.

This set of relationships permits finding any value of material constant from two other values measured experimentally. It is important to note that there are only *two independent constants* which must be measured.

There are several principal physical limitations. If a body is compressed, its volume cannot increase; it can only decrease or may not change at all, if a body is incompressible. It means that there is a principal limitation:

$$B \geq 0$$

Then, it is evident that $E > 0$, and this inequality can be fulfilled only if

$$\mu \leq 0.5$$

Incompressible material is of special interest. For many solids

$$B \gg E$$

i.e., it is much easier to change the shape of a body than its relative volume (density).

In some cases, it is reasonable to accept that $B \rightarrow \infty$, i.e., to suppose that some materials are completely incompressible. It is a good model (or it is almost true) for all liquids and rubbers. This leads to two simple relations:

$$\mu = 0.5$$

and

$$E = 3G$$

This means that *for incompressible media there is only one independent or “free” material constant*.

Reformulating the rheological equation of state for a linear Hookean elastic material in terms of the elastic potential function, let us consider (for a sake of simplicity) an incompressible body. Then, $E_1 = 0$, and W can be a function of E_2 and E_3 only. The simplest is an assumption of a linear relationship between W and E_2 , i.e.,

$$W = -BE_2 \quad [4.2.12]$$

where B is the single independent constant of material.

It is possible to calculate all items entering this equation and to find derivatives:

- with the same indices

$$\frac{\partial E_2}{\partial \varepsilon_{ii}} = -\varepsilon_{ii}$$

- with different indices

$$\frac{\partial E_2}{\partial \varepsilon_{ij}} = -\frac{1}{2}\varepsilon_{ij}$$

Then, combination of these results with Eq. 4.1.4 gives Hooke’s law, i.e., the linear relationship between deviatoric components of the stress and deformation tensors is obtained, as indicated by Eq. 4.2.5. This means that Eq. 4.2.12 is equivalent to the above-formulated concept of Hookean elastic material and this equation can be treated as an invariant definition of an incompressible linear elastic body in the limits of infinitesimal deformations.

Both definitions are equivalent and it seems that Eq. 4.2.12 does not offer any additional advantages in comparison with the standard definition, describing the relation between components of stress and deformation tensors. However, it is not completely true,

and when finite (large) deformations of an elastic body will be discussed, it will be seen that the formulation of the rheological equation of state through an elastic potential function is preferable.

The real range of changes in modulus for some typical materials is as follows:

| Material | Young's modulus |
|-------------------------------|------------------------|
| High modulus, oriented fibers | > 300 GPa |
| Steel | 200 GPa |
| Copper, aluminum, and alloys | 100 GPa |
| Stones | 40 to 60 GPa |
| Engineering plastics | 5 to 20 GPa |
| Ice | 10 GPa |
| Wood | 1 to 10 GPa |
| Leathers | 1 to 100 MPa |
| Rubbers | 0.1 to 5 MPa |
| Polymer and colloid solutions | 1 to 100 Pa |

Young's modulus may vary in the range of more than 11 decimal orders. Poisson's ratio for many solids ranges from 0.3 to 0.4, and for rubbers it is close to 0.5.

The concept of a linear elastic (Hookean) material is the basis of many engineering disciplines, first of all, the theory of elasticity and strength of materials. The basic rheological equations combined with the equilibrium equations (see section 1.1.6) are widely used for solving numerous applied problems. The discussion of all these problems goes beyond the scope of the present book.¹

4.3 LINEAR ANISOTROPIC SOLIDS

In formulating and discussing Hooke's law, it was tacitly assumed that material has the same properties along any arbitrary direction, i.e., that material is *isotropic*. The consequence of this assumption is that only two independent moduli, e.g., extension and shear are modelled. In fact, many real materials are *anisotropic*, i.e., their properties depend on a direction of measurement. The closest examples are oriented fibers, reinforced plastics and wood – their rigidity and strength are very different along the fibers and normal to the fibers. The properties of monocrystals are different in different crystallographic directions due to different intermolecular interactions and different distances between atoms in the crystalline cell.

The behavior of all these materials can be approximated (at least at small deformations) by linear relationships between stresses and deformations. However, it is rather evident that Hooke's law in its standard formulation does not describe properties of anisotropic materials. Then, it is reasonable to make more general suggestions concerning a possible relationship between the components of stress and deformation tensors within the framework of linear approximation.

There are six independent components of the stress tensor, σ_{ij} , and six independent components of the deformation tensor, ε_{ij} . The general *linear* relationship among them is:

$$\sigma_{ij} = \sum_{m,n=1}^3 E_{ijmn} \epsilon_{mn} \quad [4.3.1]$$

where E_{ijmn} are the components of elastic modulus for an anisotropic material and summation takes place by the indices corresponding to the components of the deformation tensor, m and n.

Comment

In order to elucidate Eq. 4.3.1 one of the components of the stress tensor is written below in an expanded form:

$$\sigma_{12} = E_{1211}\epsilon_{11} + E_{1222}\epsilon_{22} + E_{1233}\epsilon_{33} + E_{1212}\epsilon_{12} + E_{1213}\epsilon_{13} + E_{1223}\epsilon_{23}$$

Eq. 4.3.1 can also be written in an inverse form to consider deformation as a function of the components of the stress tensor:

$$\epsilon_{ij} = \sum_{m,n=1}^3 J_{ijmn} \sigma_{mn} \quad [4.3.2]$$

It is evident that this complete formulation includes 36 values of “moduli”, E_{ijmn} , which, in fact, are the components of the *modulus tensor*, or the same number of components of the *compliance tensor*, J_{ijmn} . The correspondence between both is established by the rules of matrix algebra.

Based on some theoretical arguments concerning the mathematical properties of tensors, it is possible to prove that there are not 36 but only 21 independent values of modulus (or compliance).

This is the maximum possible number of independent characteristics of mechanical properties for a linear elastic material (body). The decrease of the number of moduli depends on the type of symmetry of material.

It is necessary to use all 21 constants to describe properties of a crystal of triclinic structure. The increase of the number of axes of symmetry results in a decrease of the number of independent constants. For a monoclinic crystal with the axis of symmetry of the second order, several moduli equal zero, and the number of independent moduli decreases to 13. For a rhombic crystal, only 9 independent moduli exist. The number of independent moduli for a cubic crystal with the axes of symmetry of the fourth order decreases to 3, and coming back to an isotropic material with an infinite number of axes of symmetry, only 2 independent moduli can be determined. The latter relates to amorphous materials or to polycrystals. In polycrystals, the properties of individual crystals are averaged because of the coexistence of a large number of individual anisotropic crystals oriented statistically in space. Elastic properties of material are measured using relatively large samples (at least, much larger than the sizes of individual crystals in a polycrystal body), and therefore the properties of such materials can be described by a model of an ideal Hookean body.

Elastic potential of an anisotropic elastic material is determined by the following sum of products of the components of stress and deformation tensors:

$$W = \sum_{i,j=1}^3 \sum_{m,n=1}^3 E_{ijmn} \sigma_{ij} \varepsilon_{mn} \quad [4.3.3]$$

The tensor E_{ijlm} is symmetrical, and this reduces the number of independent constants characterizing elastic properties of material.

The determination of the components of the modulus tensor is very complex from the experimental point of view. It can be done for real monocrystals, and the results of such kind are not used in rheology, but are of interest in the physics of solids. The situation is somewhat different for anisotropic structures, such as glass-fiber reinforced plastics. These materials are of great technological interest in various applications and therefore quantitative description of their properties is necessary for designing articles made of these materials, because many of them are used for construction of vital importance. These anisotropic materials possess several axes of symmetry and the number of moduli is limited: as a rule, they are oriented along one or two axes and used for producing thin articles (covers, hulls, cases, roofs and so on). Nevertheless measuring constants for such anisotropic materials requires special experimental techniques and is not a trivial task.

4.4 LARGE DEFORMATIONS IN SOLIDS AND NON-LINEARITY

4.4.1 A SINGLE-CONSTANT MODEL

Large elastic deformations result in non-linear dependence between stresses and deformations, and that is why both phenomena (large deformations and non-linearity) appear in the title of this section. Large deformations are not a single cause of non-linearity, as was discussed in section 2.8 for viscoelastic materials and will be illustrated here for elastic solids.

Discussion of large elastic deformations in solids is the most important for rubbers and rubber compounds (*elastomers*) because their main characteristics determine the quality and applicability of these materials and their ability to undergo large recoverable deformations.

The articles made out of rubbery materials work in various applications in which they are subjected to a three-dimensional stress state. The mechanical testing of these (and other) materials is carried out primarily in a unidimensional extension. Therefore, in formulating a constitutive equation for these materials it is necessary to solve the problem, frequently mentioned in this book, of generalization of unidimensional experiments for three-dimensional (invariant) form. And again, it is worth noticing that this problem does not have a unique solution but continues to be the subject of many different attempts, involving personal experience and luck, though some general principles must be fulfilled.²

Discussion of properties of materials in the domain of large deformations is based on some fundamental definitions advanced in section 1.2, where some results related to simple modes of deformations were also considered.

The invariants, E_i , of the large deformation tensor are written, as usual, *via* the principal values of this tensor, ε_i , and the following notation will be used below:

The first (linear) invariant:

$$E_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad [4.4.1]$$

the second (quadratic) invariant:

$$E_2 = \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3 \quad [4.4.2]$$

the third (cubic) invariant:

$$E_3 = \varepsilon_1\varepsilon_2\varepsilon_3 \quad [4.4.3]$$

Rubbery materials are practically incompressible (at least, their bulk modulus is by several orders of magnitude higher than shear modulus) and Poisson's ratio, $\mu = 0.5$. Therefore, the third invariant of the deformation tensor (determining volume changes in deformations) equals 1 and can be excluded from further discussion.

The first invariant of the deformation tensor, in the limit of small deformations, has the meaning of volume changes (see Chapter 1.2), and that is why in this case $E_1 = 0$. In the range of large deformations, E_1 does not have such a simple meaning, but the condition of constant volume at deformation of any type permits us to reduce the number of independent invariants to two, because the equality

$$(1 + E)_1(1 + E_2)(1 + E_3) = 1 \quad [4.4.4]$$

is always valid for incompressible materials.

Then, any invariant, expressed as a function of two others, can be excluded. For example, the third invariant, which is expressed by means of Eq. 4.4.4, can be written as

$$E_3 = \frac{1}{(1 + E_1)(1 + E_2)} - 1 \quad [4.4.5]$$

These arguments reduce dependence of W to two independent variables:

$$W = W(E_1, E_2) \quad [4.4.6]$$

An invariant form of Hooke's law providing the linear relationship between stresses and deformations has been formulated above (Eq. 4.2.12) as a linear dependence of an elastic potential on the second invariant of the deformation tensor. In the limits of Hooke's law, E_2 is the second invariant of the tensor of infinitesimal deformations. As discussed in section 1.3, in the case of large deformations, it is necessary to utilize the theory of finite deformations and use some measures of large deformations. It is a natural way of generalization of Hooke's law, though the ambiguity of measures of large deformations may lead to different possibilities of representation of the relationship under discussion.

As the first approximation (or as the first reasonable simple idea), let us assume that an elastic potential is a linear function of the first invariant of the tensor of large deformations:

$$W = AE_1 \quad [4.4.7]$$

Below, the main consequences of this approximation, i.e., Eq. 4.4.7, for different geometries of deformation are discussed. Eq. 4.4.7 can be written in an expanded form as

$$W = A(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = A\left(\frac{\lambda_1^2 - 1}{2} + \frac{\lambda_2^2 - 1}{2} + \frac{\lambda_3^2 - 1}{2}\right) \quad [4.4.8]$$

or

$$W = \frac{A}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \quad [4.4.9]$$

where λ_i are the principal extension ratios.

This formula is equivalent to the linear relationship between an elastic potential, W , and the first invariant, C_1 , of the Cauchy-Green tensor of large deformations

$$W = \frac{A}{2}(C_1 - 3) \quad [4.4.10]$$

The problem of three-dimensional elongation of a body “at a point” can be analyzed by calculating the elastic potential of deformations in the principal axes. From the definition of W , it is easy to show that the elastic potential is expressed *via* principal extension ratios as

$$dW = \sigma_1 \frac{d\lambda_1}{\lambda_1} + \sigma_2 \frac{d\lambda_2}{\lambda_2} + \sigma_3 \frac{d\lambda_3}{\lambda_3} \quad [4.4.11]$$

The following equality is valid for an incompressible material

$$\lambda_1 \lambda_2 \lambda_3 = 1$$

and therefore

$$d(\lambda_1 \lambda_2 \lambda_3) = 0$$

Then, after some simple rearrangements, the following formula for dW can be written

$$dW = (\sigma_1 - \sigma_3) \frac{d\lambda_1}{\lambda_1} + (\sigma_2 - \sigma_3) \frac{d\lambda_2}{\lambda_2} \quad [4.4.12]$$

Moreover, the following expression for dW can be obtained beginning from Eq. 4.4.11 and the condition of material incompressibility:

$$dW = A \left[(\lambda_1^2 - \lambda_3^2) \frac{d\lambda_1}{\lambda_1} + (\lambda_2^2 - \lambda_3^2) \frac{d\lambda_2}{\lambda_2} \right] \quad [4.4.13]$$

A direct comparison of the last formulas for dW (Eqs. 4.4.12 and 4.4.13) gives the following system of equations widely used in the theory of rubber elasticity:

$$\begin{cases} \sigma_1 - \sigma_3 = A(\lambda_1^2 - \lambda_3^2) \\ \sigma_2 - \sigma_3 = A(\lambda_2^2 - \lambda_3^2) \end{cases} \quad [4.4.14]$$

The last system of equations is a solution to the problem of calculation of normal (principal) stresses at known principal elongations.

This solution is not complete because the system of Eqs. 4.4.14 contains only two separate equations for three independent variables, σ_1 , σ_2 , and σ_3 . This result is not unexpected because deformation in an incompressible medium is considered. The last limita-

tion means that, in principle, stresses can be determined up to an uncertain constant: $\sigma_i + C$, where the constant C cannot be determined unambiguously. Superposition of arbitrary hydrostatic pressure changes the stress state of the medium but does not influence its deformation state. As a result, the system of Eq. 4.4.14 determines components of the stress tensor in relation to the constant C that is dependent on hydrostatic pressure.

Therefore, the general solution in determining principal stresses in a three-dimensional deformation state (i.e., when deformations are known or preset) can be written, in accordance to Eqs. 4.4.14, in the following form:

$$\begin{cases} \sigma_1 = A\lambda_1^2 + C \\ \sigma_2 = A\lambda_2^2 + C \\ \sigma_3 = A\lambda_3^2 + C \end{cases} \quad [4.4.15]$$

where the constant C may be found if the hydrostatic pressure is known beforehand.

With this background, it is possible to analyze the main cases of deformations of an elastic solid body with rheological properties obeying Eq. 4.4.7. In uniaxial extension along the axis x_1 , $\sigma_2 = \sigma_3 = 0$, if the extension ratio along the axis x_1 equals λ , the condition of the constant volume of body under deformation results in the following relationship:

$$\lambda_2 = \lambda_3 = \lambda^{-1/2}$$

Then, any of the last two equations of the system Eq. 4.4.15 gives the value of the “free” constant C :

$$C = -A\lambda^{-1}$$

The formula for normal (principal) stress in uniaxial extension for material with rheological properties described by Eq. 4.4.7 is as follows:

$$\sigma_1 = A\left(\lambda^2 - \frac{1}{\lambda}\right) \quad [4.4.16]$$

Comment

Eq. 4.4.16, within the limits of small deformations, degenerates to Hooke's law. The value of the principal elongation is

$$\lambda = \frac{l_0 + \Delta}{l_0}$$

where l_0 is the initial length of a sample and Δ is the increase of the length due to deformations; it is assumed that $\Delta \ll l_0$. Then, by direct substitution of the expression for λ in Eq. 4.4.16, after necessary calculations and neglecting higher order terms of Δ , it is possible to demonstrate that Eq. 4.4.16 degenerates to the following linear relationship

$$\sigma_1 = 3A\frac{\Delta}{l_0} = 3A\varepsilon$$

which is, evidently, Hooke's law with Young's modulus equal to $E = 3A$.

Materials obeying Eq. 4.4.16 are sometimes called *neo-Hookean*, because this equation is the most evident and simplest generalization of Hooke's law for a region of large deformations.

In engineering applications, it is frequently more convenient to use, not the true stress as in Eq. 4.4.16, but the *engineering stress*, f_E , that is the force divided by the initial cross-section of the sample. Simple geometrical arguments show that

$$f_E = \frac{\sigma}{\lambda}$$

and therefore the constitutive equation for uniaxial extension of rubbers is formulated as

$$f_E = \frac{E}{3} \left(\lambda - \frac{1}{\lambda^2} \right) \quad [4.4.17]$$

The reason for changing A to E/3 were discussed in the **Comment** above.

Eq. 4.4.16 is one of the possible methods of representation of experimental data for large deformations of elastic materials. This equation is a consequence of the invariant Eq. 4.4.7. This equation is applied to the analysis of simple shear, which is relatively easy to study by experimental methods. For example, this mode of deformation can be realized by twisting a thin-walled cylinder.

The principal elongations for simple shear were calculated in section 1.2 (see Eq. 1.2.35, where ϵ_{ii} are the principal elongations). Elastic potential for this mode of deformation is

$$W = \frac{1}{2} A \gamma^2 \quad [4.4.18]$$

Shear stress is found from:

$$\sigma = \frac{dW}{dy} = A\gamma \quad [4.4.19]$$

Rheological properties of material, which are described by the invariant Eq. 4.4.7, give linear dependence of shear stress on deformation. Thus, the value of the constant A in this equation has the meaning of shear modulus. In the linear limit of elasticity, modulus in extension, $E = 3A$, but for non-linear domain of large deformations, this simple relationship is not valid, as is clearly seen from Eq. 4.4.16. Therefore, the following conclusions can be made:

- Shear behavior of two different kinds of material (Hookean and those described by the rheological equation of state, Eq. 4.4.7) can be the same, even though they are quite different rheological materials. This is confirmed by the difference in their behavior in extension. It is the direct proof of the thesis that investigation of stress deformation behavior in one mode of loading does not give enough information for estimating the type of rheological model of material.
- Elastic potential (Eq. 4.4.7) predicts non-linear behavior of material in extension and this non-linearity is a direct consequence of large deformations by itself. This rheological equation of state contains only one material constant, which has the

meaning of shear modulus, and can be used for prediction of deformation behavior of material at any mode of loading.

Elastic potential in the form of Eq. 4.4.7 was formulated as a consequence of the molecular (kinetic) statistical theory of rubbery elasticity and is called the *Kuhn-Guth-James-Mark potential*.³ The potential function, expressed by Eq. 4.4.7, was proposed for rubbers. It can be considered as the first approximation describing deformation of rubbers at equilibrium conditions. The last limitation implies that possible time effects are not included in consideration, though effects of such kind are quite typical for rubbers and observed in measuring stress-deformation relationship (see Chapter 2).

4.4.2 MULTI-CONSTANT MODELS

4.4.2.1 Two-constant potential function

Comparison of experimental data obtained for a typical rubbery material with a curve calculated in accordance with Eq. 4.4.17 is presented in Fig. 4.4.1. The experimental curve consists of three sections:

- At small deformations, the stress-deformation relationship is close to linear
- In the intermediate deformation region the relationship is predicted by Eq. 4.4.17
- At very large deformations, stresses increase rapidly and this effect is not described by Eq. 4.4.17.

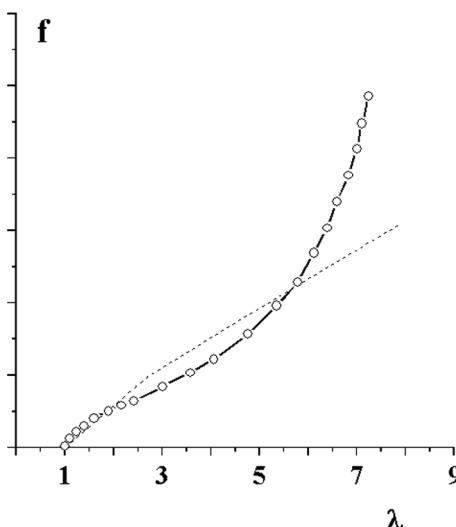


Figure 4.4.1. Typical dependence of engineering stress, f , on relative elongation, λ (experimental data points for a soft rubber and its theoretical representation by Eq. 4.4.17 (dashed line)).

It is necessary to develop more comprehensive methods for description of experimental data. For this purpose, rheological equations of state can be developed based on Eq. 4.4.7. Representation of rheological properties of solid materials via an elastic potential function $W(C_1, C_2)$ is equivalent (after some formal mathematical transformations) to representing it as a function $W(C_1, C_1^{-1})$, i.e., it is possible to use first invariants of the Cauchy-Green and the Finger tensors of large deformations. There is a great freedom in varying any conceivable form of this function in an attempt to fit various experimental data. Formally, no limitation in approximating function $W(C_1, C_1^{-1})$ exists because any approximation obeys the general principle of invariance. Certainly, in real practice it is desirable to search for approximations having the simplest form, possibly a linear form.

The linear form is the simplest, and one of the examples of linear elastic potential is given by Eq. 4.4.7, but it contains only one argument, the first invariant E_1 , (or C_1 as in Eq. 4.4.10). Then the next possible approximation is a combination of linear functions in the form

$$W = AE_1 + BE_2 \quad [4.4.20]$$

where A and B are material constants.

This elastic potential can be rewritten *via* the principal values of extension ratios. After mathematical transformation, the following formula is obtained:

$$W = G_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + G_2\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}\right) \quad [4.4.21]$$

where “new” constants, G_1 and G_2 , are expressed by “old” ones, A and B, as

$$G_1 = \frac{A - B}{2}, \quad G_2 = \frac{B}{4}$$

and the final result of the rearrangements is the expression of W, as a function of the first invariants of both the Cauchy-Green and the Finger tensors given by

$$W = G_1(C_1 - 3) + G_2(C_1^{-1} - 3) \quad [4.4.22]$$

This result confirms the equivalence of functions $W(C_1, C_2)$ and $W(C_1, C_1^{-1})$. An elastic potential in the form of Eq 4.4.22 was proposed by Mooney⁴ and independently by Rivlin⁵ on the basis of his general approach to the construction of elastic potential functions for large deformations. The function $W(C_1, C_1^{-1})$ in Eq. 4.4.22 is known as the *Mooney-Rivlin potential*.

The role of the second term in Eq. 4.4.22 is illustrated by stress-deformation dependencies for uniaxial extension and shear. In uniaxial extension

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda_3 = \lambda^{-1/2}$$

where λ is an extension ratio.

Using formulae for components of stresses discussed above, it is possible to show that normal stress in the direction of stretching, σ_E , is

$$\sigma_E = 2\left[G_1\left(\lambda^2 - \frac{1}{\lambda}\right) + G_2\left(\lambda - \frac{1}{\lambda^2}\right)\right] \quad [4.4.23]$$

with two other stress components equal zero.

The acting force can be related to the initial cross-section of the stretched sample. This engineering stress, f_E , is written in the following manner

$$f_E = 2\left(\lambda - \frac{1}{\lambda^2}\right)\left(G_1 + G_2\frac{1}{\lambda}\right) \quad [4.4.24]$$

The structure of Eq. 4.4.24 demonstrates that the addition of the second term in Eq. 4.4.22 leads to the appearance of the “correction” term in Eq. 4.4.24. If $G_2 = 0$, Eq. 4.4.24 becomes the well known formula for the stress related to one member of elastic potential function (Eq. 4.4.7). The difference is because of $2G_2/\lambda$ in Eq. 4.4.24. The influence of this correction can be estimated, bearing in mind that according to the experimental data,

$G_2 \approx 0.1 G_1$. However, addition of the second term is of principal value, especially considering that other relationships between both constants are not excluded.

Treating experimental data in terms of Eq. 4.4.24 assumes that they ought to be presented in the coordinates f_M vs. λ^{-1} , where

$$f_M = \frac{f}{\lambda - \lambda^{-2}} \quad [4.4.25]$$

It is expected that the experimental data points, presented in these coordinates, will lie on a straight line. It is also worth mentioning that the function f_M is, in fact,

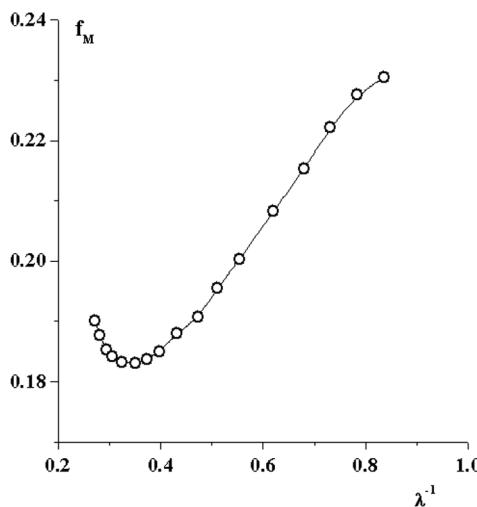
$$f_M = \frac{\partial W}{\partial C_1} + \frac{1}{\lambda} \frac{\partial W}{\partial C_2} \quad [4.4.26]$$

Fig. 4.4.2 is an illustration of this approach. The dependence under discussion gives a straight line at $\lambda^{-1} > 0.4$ (i.e. at $\lambda < 2.5$). At higher degrees of extension, strong divergence from the two-constant potential is evident. However, $\lambda = 2.5$ is already a large deformation, and it is important that at least in this deformation range Eq. 4.4.24 is valid.

Is it possible to be certain that this potential function is a correct image of real changes (at least at $\lambda < 2.5$)? In order to answer this question another type of deformation other than uniaxial extension must be examined. Let it be shear deformation. It is easy to prove that Eq. 4.4.21 leads to the following dependence of shear stress, σ , on shear deformation, γ ,

$$\sigma = 2(G_1 + G_2)\gamma \quad [4.4.27]$$

where the sum $(G_1 + G_2)$ is the shear modulus.



The constitutive equation, expressed by Eq. 4.4.20, predicts linear behavior in shear, though it is non-linear in extension.

Repeatedly, this is a proof that the same rheological behavior in deformations of one type (in this case, linear stress-deformation in shear) does not mean that the type of deformation under other conditions (in this case, stress-deformation in uniaxial extension) must be the same. The result demonstrates again that the data obtained in experiments of one type cannot be a criterion for selection of rheological equation of state (constitutive equation) and cannot be used for unambiguous prediction of stress-deformation dependencies in different modes of deformations.

Figure 4.4.2. Linearization of experimental data in coordinates of Eq. 4.4.24. [Adapted, by permission, from R.S. Rivlin, D.W. Saunders, *Phil. Trans. Roy. Soc., A243*, 251 (1951)].

4.4.2.2 Multi-member series

Eqs. 4.4.20 and 4.4.21 can be formally treated as the first approximation (the first

order terms of a series) of some non-linear functions $W(C_1, C_2)$ or $W(C_1, C_1^{-1})$. If one adds higher order terms with their own material constants, one may expect to improve the correspondence between the theoretical predictions and the experimental data. In particular, it is important for shear studies because the linear relationship in Eq. 4.4.27 is not a realistic result.

If linear functions (Eq. 4.4.7 or Eq. 4.4.20) are not sufficient (and they are not for three-dimensional deformations), the quadratic term can be added and the expression for elastic potential is

$$W = AE_1 + BE_2 + ME_1^2 = G_1(C_1 - 3) + G_2(C_1^{-1} - 3) + G_3(C_1 - 3)^2 \quad [4.4.28]$$

where A , B , M , G_1 , G_2 , and G_3 are material constants and the latter three are expressed via A , B , and M .

Based on Eq. 4.4.28, the following formula can be derived for the normal engineering stress in uniaxial extension:

$$f_E = 2\left(G_1 + G_2 \frac{1}{\lambda} + G_3 \lambda^2\right)\left(\lambda - \frac{1}{\lambda^2}\right) \quad [4.4.29]$$

It is evident that Eq. 4.4.29 contains a new quadratic term with its own material constant in a “correction” term, compared to Eq. 4.4.24. Certainly, three empirical constants allow us to fit experimental data points much better than one or even two “free” constants in Eq. 4.4.24, and that is why Eq. 4.4.29 gives satisfactory approximation of different experimental data. Moreover, Eq. 4.4.29 predicts nonlinear dependence of stress-deformation function in a simple shear.

The most general *phenomenological* form of an elastic potential function is

$$W = \sum_{\alpha, \beta, \gamma=0} G_{\alpha, \beta, \gamma} (C_1 - 3)^\alpha (C_2 - 3)^\beta (C_3 - 3)^\gamma \quad [4.4.30]$$

where $G_{\alpha, \beta, \gamma}$ are empirical constants and $G_{000} = 0$ because elastic energy of undeformed body is assumed to be zero.

For an incompressible material, $C_3 = 0$, and a simpler general form of the elastic potential can be written:

$$W = \sum_{\alpha, \beta=0} G_{\alpha, \beta} (C_1 - 3)^\alpha (C_2 - 3)^\beta \quad [4.4.31]$$

A very interesting and principal problem arises concerning the practical possibility of finding W as a function of invariants. The general answer to this problem is that it is necessary to compare results of experiments carried out at *different geometrical schemes of loading*. For example, a normal stress in uniaxial extension for an arbitrary function $W(C_1, C_2)$ can be expressed as

$$f_E = 2\left(\lambda - \frac{1}{\lambda^2}\right)\left(\frac{\partial W}{\partial C_1} + \frac{1}{\lambda} \frac{\partial W}{\partial C_2}\right) \quad [4.4.32]$$

as is also seen from Eqs. 4.4.25 and 4.4.26.

The general form of the dependence of shear stress on deformation for simple shear (its measure is γ) can be formulated as

$$\sigma = 2\gamma \left(\frac{\partial W}{\partial C_1} + \frac{\partial W}{\partial C_2} \right) \quad [4.4.33]$$

The dependencies of f_M and $\sigma/2\gamma$ on the elastic potential function are expressed in different manners. From comparison of dependencies of f_M (see Eq. 4.4.25) and $\sigma/2\gamma$ on the right-hand sides of Eqs. 4.4.32 and 4.4.33, it is possible to find the dependence $W(C_2)$ as

$$\frac{\partial W}{\partial C_2} = \frac{\frac{\sigma}{2\gamma} - f_M}{1 - \lambda^{-1}}$$

Then the dependence $W(C_1)$ is found from Eq. 4.4.32 or Eq. 4.4.33.

Other types of fundamental experiments can also be used to find an elastic potential function $W(C_1, C_2)$ but it is essential that at least two different geometrical schemes of loading are used and compared.

The idea that a function $W(C_1, C_2)$ can be expanded into a power series, and that it is possible to use any desirable number of terms of series, permits, by adding new arbitrary (“free”) empirical constants, a reasonable degree of correspondence between a theoretical (phenomenological) curve and experimental data.

The same concept can be realized in a somewhat different way. From the very beginning, it was assumed that coefficients used, even in a very general formulation of an elastic potential, Eq. 4.4.30, are constant but in reality the material properties are not constant and may depend on deformation. The coefficients must be expressed as dependencies of “modulus” A in Eq. 4.4.7 on invariants of the deformation tensor. It means that nonlinearity appears not only as a consequence of large deformations by itself but also as a function of some physical phenomenon (for example, structure transformation happening in the course, or as a consequence, of large deformations), i.e., the simplest quasi-linear potential (Eq. 4.4.7) is not sufficient for fitting experimental data, when physical nonlinear effects are encountered.

4.4.2.3 General presentation

It was proven that the most general constitutive equation for elastic materials can be written as⁶

$$\sigma = \alpha_0 \delta + \alpha_1 C + \alpha_2 C^2 \quad [4.4.34]$$

In this equation, tensor values of stress, σ , unit tensor, δ (see comments in section 1.1.1) and the Cauchy-Green tensor, C , are used. The tensor C can be equivalently substituted by the tensor C^{-1} .

It is important that the coefficients, α_0 , α_1 , and α_2 in Eq. 4.4.34 are arbitrary scalar functions of the deformation invariants. For the particular case of incompressible material (such as rubbers) $\alpha_0 = -p$ (i.e., it is hydrostatic pressure). Also, it is easy to treat Eqs. 4.4.7 and 4.4.20 as particular cases of the more general Eq. 4.4.34.

Both approaches expand the function $W(C_1, C_2)$ into a power series or treat the material parameters α_1 and α_2 in Eq. 4.4.34 as functions of invariants of the deformation ten-

sor. These are formal presentations of elastic properties of solids. Eq. 4.4.7 can be based on some reasonable physical arguments (“*statistical theory of rubber elasticity*”).⁷ Its generalization, in spite of numerous theoretical attempts, has no such universally accepted physical ground and must be treated as an empirical relationship invented for fitting the experimental data.

The last remark in this section regards time effects. *Time must not be mentioned* in this section at all, because this concept is not consistent with the idea of elastic (instantaneous) reaction of material to the applied force. This fundamental idea already has been emphasized by stating that stress-deformation relationships discussed in this chapter are valid for *equilibrium* conditions. However, there is a great difference between instantaneous and equilibrium reactions and the gap between both is the field of time-dependent effects. Moreover, large deformations and rubbery elasticity are relevant to polymeric materials and various time-dependent effects (relaxation, etc.) are typical of these materials. That is why it is important (though in some cases difficult) to separate time-dependent effects and distinguish “pure” (equilibrium) stress-deformation dependence.

The formulation of elastic potential function should be based on fundamental molecular arguments determining the structure of the elastic potential function. In the simplest case, such an approach was used in the single-constant Kuhn-Guth-James-Mark potential, which was previously discussed. This potential function is based on affine transformation of a network created by macromolecular chains. Later a two-constant potential was proposed that incorporated a concept of limited extensibility of macromolecular chains:⁸

$$W = K_1 E_1 + K_2 \ln[(E_2 + 3)/3] \quad [4.4.35]$$

where K_1 and K_2 are constants and the small deformation limit of elastic modulus (Hooke's modulus) E is expressed as

$$E = 6K_1 + 2K_2$$

This concept is useful in explanation of the effect of hardening in extension of polymer melts (see section 3.7.3). The construction of general non-linear models of viscoelastic liquids also requires such molecular-based models of non-linear elasticity (see section 2.8.2).

4.4.2.4 Elastic potential of the power-law type

A method of introducing multi-component elastic potential function is based on the generalized (nonlinear) measures of deformations. The following assumption for the $W(\lambda_i)$ function was proposed:⁹

$$W = \frac{2G}{n} I_E + BI_E^m \quad [4.4.36]$$

where G , m , n and B are material constants, and I_E is the first invariant (the sum) of the generalized measure of deformations, E_α , which is expressed as

$$E_\alpha = \frac{1}{n} (\lambda_\alpha^n - 1)$$

This approach leads to non-linear dependencies in deformation of any arbitrary type. The potential function, including extension and simple shear, is based on comparison of invariant dependencies of stresses and deformations, including geometry of deformations of different types. The calculations or two-dimensional model gives the following equation:

$$\sigma_\alpha - \sigma_\beta = (\lambda_\alpha^n - \lambda_\beta^n) \left(\frac{2G}{n} + mBI_E^{m-1} \right) \quad [4.4.37]$$

where indices α and β define principal stresses and extension ratios defined in principal axes.

The left-hand side of this equality will be noted as X and the right-hand side as Y . Then the dependence of X on Y is expected to be a straight line inclined by 45° to both axes regardless of the geometry of deformation. Experimental results for two different materials and four geometries of deformation are presented in Fig. 4.4.3 in the coordinates $X-Y$. It is seen that the power-law type elastic potential fits the experimental data in the invariant form, and that is why this equation can be considered as acceptable.

This approach is useful for fitting experimental data obtained in study of different rubber-like materials.

4.4.3 THE POYNTING EFFECT

Large deformations, even uniaxial, lead to three-dimensional effects. In liquids, it is the Weissenberg effect (see section 3.4.2). In solids, a similar effect is called the *Poynting effect*.¹⁰

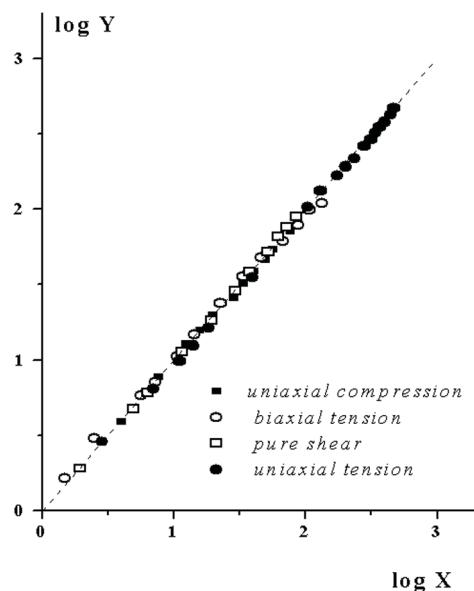
The experiments demonstrated that twisting thin wires affects their length. This effect becomes noticeable at relatively large angles of twisting, for example, after several turns of wire.

The characteristic (dimensionless) geometrical size of wire is the ratio of its radius to length, R/L . This ratio is smaller than 1. The values of relative deformations are also small, but they are comparable with values of $(R/L)^2$. The Poynting effect is a typical phenomenon of the second order. The changes in length are proportional to

Figure 4.4.3. Experimental data for two materials and four types of deformation presented in the coordinates of Eq. 4.4.36. [Adapted, by permission, from B.J. Blatz, S.C. Sharda, N.W. Tschoegl, *Trans. Soc. Rheol.*, **18**, 145 (1974)].

the squared value of $R\Omega/L$, where Ω is an angle of twisting.

The Poynting effect applies to large deformations in shear. Shearing does not cause volume changes. However, this is not true in large deformations. That is why the other outcome of the Poynting effect is volume change. It is analogous to *dilatancy* (volume change in shear) in liquid-like materials.



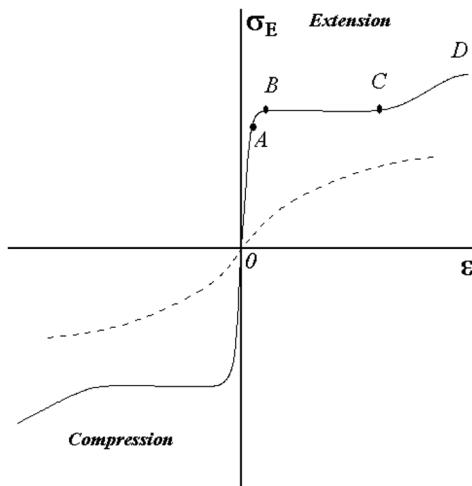


Figure 4.5.1. Typical deformation curves for mild steel (solid line) and non-ferrous metals (dashed line). Characteristic points are marked.

Figure 4.5.2. The deformation curve with a peak and plateau of plasticity.

4.5 LIMITS OF ELASTICITY

4.5.1 STANDARD EXPERIMENT – MAIN DEFINITIONS

The *limit of elasticity* is a result of standard experiments in uniaxial extension of solid samples. Two typical cases are shown in Figs. 4.5.1 and 4.5.2. Sometimes, the normal stress-deformation dependence, $\sigma_E(\epsilon)$, is monotonous (e.g., for mild, low carbon, steel or copper, as in Fig. 4.5.1). In other cases, a sharp maximum is observed on the $\sigma_E(\epsilon)$ dependence (e.g., for many metallic materials and plastics, as in Fig. 4.5.2).

The point A in these figures corresponds to the *limit of proportionality*, and the OA part of the full curve is a domain of validity of Hooke's law. Elastic deformations continue up to the point B, which is called a *limit of elasticity*. After this point, irreversible deformations are detected. Deformations of this kind increase greatly along the plateau in the range BC (or B'C) and this is known as a domain of *plasticity* or *plastic deformation*. The maximum stress (points A and B in Fig. 4.5.2) is called a *limit of plasticity*, or *yield stress*.¹¹ The $\sigma_E(\epsilon)$ curve ends at point D, marking a failure of sample, and this point is called *ultimate* or *tensile strength* of material. Deformation, corresponding to this stress, is called the *ultimate elongation* or *elongation at break*.

There is a large number of *fragile* or *brittle* solid materials which break close to the point B (e.g., reinforced plastics, many ceramics, inorganic glass, pig iron, monocrystals, etc.).

Two phenomena define the limits of elasticity:

- *elastic-to-plastic transition*
- *failure*.

It is noteworthy that both effects are observed not only in extension, but also in deformation of any geometry, for example, in uniaxial compression, shear, combination of twisting and extension, etc. Only all-dimensional (hydrostatic) compression occupies a

special position. Quantitative measures of conditions, corresponding to the limit of elasticity, are different depending on the deformation mode.

The elastic-to-plastic transition, as well as fracture, can be considered as some critical event on the deformation-stress dependence. Therefore, both phenomena can be considered in analogous terms, especially when the three-dimensional effects are discussed and the general criteria are searched for.

4.5.2 PLASTICITY

The transition from elastic to plastic behavior of solids is a very important phenomenon from two points of view:

- The articles made out of solids must store memory of their shape in application. It is possible only if deformation force loading an article does not exceed the limit of elasticity. If it does, an article is damaged. In engineering applications this transition is a crucial part of design. The limit of elasticity (or “limiting stress”) is determined by standard methods for all materials. This is the most important engineering property of material used as a starting-point in design and applied calculations.
- Many technological operations cause plastic deformation of material that is forced to accept a new form (e.g., stamping, rolling, drawing through a die, and so on). Technological operations require elastic-to-plastic transition.

In geology and geophysics, it is widely accepted that soils and rocks can be plastically deformed under action of giant forces. Displacement of glaciers is another important example of plastic deformation of solids. Quantitative description of plasticity in geology is not yet well developed (mainly, due to experimental difficulties in determination of material properties of these media), but its importance is appreciated.

Plasticity of solids is similar to viscoplastic behavior of “liquid” materials (see Chapter 3). The following differences and similarities between viscoplastic “liquids” and plastic solids can be distinguished.

- In the domain of small deformations preceding plasticity, elastic deformation of viscoplastic “liquid” is negligible and it does not affect applications, whereas small deformations in solids are critically important because they determine application properties of material. Elasticity of viscoplastic “liquids” also exists and can be measured, though the elastic modulus in viscoplastic materials is usually very small in comparison with the elastic modulus of solids.
- Plastic deformation in solids is limited by sample failure, whereas plastic deformation of liquids (i.e., their flow) may continue for an unlimited time. However, it is not an absolute difference, considering that the conclusion is based on comparison of different deformation modes – extension for solids and shear for liquids. The extension of viscoplastic materials is also limited by failure.

From a rheological point of view, all materials are elastoplastic and they differ quantitatively but not qualitatively. The difference between elastic-plastic solids and viscoplastic liquids consists of the following:

- values of elastic modulus in the domain of small deformations preceding plasticity are of the order 10^{10} - 10^{11} Pa for typical solids, but not higher than 10^3 Pa for typical viscoplastic materials

- values of stresses in the domain of plastic flow do not exceed 10^6 Pa (depending on viscosity and shear rate) for flowing liquids, but they are of the order of 10^{10} Pa for metals and 10^7 - 10^8 Pa for plastics.

The difference between plastic behavior of “solids” and “liquids” reaches several decimal orders of magnitude of characteristic parameters. But this difference is not absolute, and it is possible to find examples corresponding to intermediate cases.

Though plastic behavior of different materials is principally similar for a rheologist, viscoplastic “liquids” and plastic solids occupy different domains of applications. Therefore, they are treated separately: viscoplastic materials as a special kind of liquid and plasticity of solids as a part of the continuum mechanics of solids.¹²

4.5.3 CRITERIA OF PLASTICITY AND FAILURE

The limit of elasticity for elastic-to-plastic transition or failure is a critical point on the stress-deformation diagram if time-dependent effects are not taken into account (the time effects are typically assumed to be absent in the concept of solids). This critical point can be found in experiments of standard geometry. The principal question is how to construct the three-dimensional (invariant) criterion which can be used for an arbitrary stress state of material. This is important in applications because stress states in real situations are three-dimensional.

The generalized criteria of plasticity and failure ought to be formulated in the analogous form, because both of them are critical phenomena. For an isotropic material the criterion of transition must be a symmetrical function of principal stresses. The role of the average normal stress (i.e., hydrostatic pressure) is negligible. Therefore, it is sufficient to use two invariants of the stress tensor. Then, the critical state criterion must depend on the difference of principal stresses. It leads to formulation of two criteria mostly used for applied purposes.

The search for the criterion of transition (or failure) in an arbitrary three-dimensional stress state is based on the concept of the equivalency of this stress state to some unidimensional state determined by a single critical (or limiting) value of stress.

4.5.3.1 Maximum shear stress

Material transforms to the state of plasticity (flow) or breaks at a stressed state, when the maximum shear stress exceeds some critical value, σ^* .¹³ This condition is called the *Tresca-Saint-Venant criterion*.¹⁴

For uniaxial extension, at normal stress, σ_E , shear stress equals $\sigma_E/2$ (see section 1.1) and therefore the condition of plasticity in extension is

$$\sigma^* = 0.5\sigma_E^* \quad [4.5.1]$$

It means that the plasticity state (or failure) is reached when shear stress equals $0.5\sigma_E^*$ for σ_E^* measured in uniaxial extension.

For a three-dimensional stress state, three maximum (“invariant”) shear stresses $\sigma_{1,max}$, $\sigma_{2,max}$, and $\sigma_{3,max}$, are calculated as

$$\begin{cases} \sigma_{1, \max} = \left| \frac{\sigma_2 - \sigma_3}{2} \right| \\ \sigma_{2, \max} = \left| \frac{\sigma_1 - \sigma_3}{2} \right| \\ \sigma_{3, \max} = \left| \frac{\sigma_1 - \sigma_2}{2} \right| \end{cases} \quad [4.5.2]$$

One of these differences is maximum (by its absolute value) and it determines the critical condition. According to this approach, σ^* equals the difference of two normal stresses, maximum and minimum. The middle value of principal normal stress has no influence on the limiting state.

As an example, let us calculate maximum shear stress in superposition of shear stress, σ , and normal stress, σ_E . Then the condition of limiting state, according to the results of calculations of maximum shear stress, is:

$$\sigma^* = \sqrt{\sigma_E^2 + 4\sigma^2} \quad [4.5.3]$$

The critical (limiting) state is reached when and if the right-hand side of this equation becomes equal the critical number – acceptable value of shear stress.

4.5.3.2 The intensity of shear stresses (“energetic” criterion)

This approach accounts for three differences of principal normal stresses or three principal shear stresses, defined by Eqs 4.5.2.¹⁵ The criterion is formulated as

$$\sigma_E^* = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]} \quad [4.5.4]$$

This condition is called the *Huber-von Mises criterion*. The value T, is defined as

$$T = \sqrt{\frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]} \quad [4.5.5]$$

In the theory of stresses it is sometimes called the *intensity of shear stresses*. Both values, σ_E and T, can be expressed by “average” value of shear stress, σ_{av} , which is

$$\sigma_{av} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$$

Then, the criterion is written as

$$\sigma_E^* = \sqrt{3T} = \sqrt{\frac{9}{2}}\sigma_{av} \quad [4.5.6]$$

If $T = \sigma$, the following relationship between σ^* and σ_E^* takes place:

$$\sigma^* = 0.577\sigma_E^* \quad [4.5.7]$$

It is close, though not equivalent, to the criterion of maximum shear stress in Eq. 4.5.1.

In the deformation of elastic solids, the work done is stored in the form of elastic potential energy. The total energy of deformation can be separated into components related to shape and volume changes. Then, accurate calculations show that the part of energy which is responsible for shape changes, W_{sh} , is expressed as:

$$W_{sh} = \frac{1+\mu}{3E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3) = \frac{1+\mu}{E}T^2 \quad [4.5.8]$$

The last equation shows that the condition of a critical state is directly related to the stored energy, not to the total energy, but only to the part that is responsible for shape changes. This is the reason for calling the criterion an *energetic criterion*. Volume changes do not influence the state of material.

4.5.3.3 Maximum normal stress

The above-discussed cases were based mainly on estimation of shear stresses, which determine conditions of the limiting state. This is directly related to the concept of shape changes of solid materials. However, volume changes in extension of solids also take place and sometimes they lead to transitions. Neck formation in uniaxial extension of polymers is an example of such an effect (see Fig. 2.8.3). Though different explanations of the mechanism of neck formation were proposed and discussed, the following seems adequate:¹⁶ a neck forms when a relative volume of body increases up to a certain level. This formation of an extra free volume provides the conditions for relaxation transition. This is a transition from an isotropic to an oriented state. The role of volume changes is dominating.

The following condition of the limiting state can be introduced. The criterion of the condition is a critical level of normal stress, σ^* . For a two-axis stress state, when shear, σ , and normal, σ_E , stresses act simultaneously, the criterion of the limiting state is written as

$$\sigma^* = \frac{1}{2}[\sigma_E + \sqrt{(\sigma_E^2 + 4\sigma^2)}] \quad [4.5.9]$$

Normal and shear stresses influence the condition of the limiting state to a relatively equivalent extent.

4.5.3.4 Maximum deformation

Physical meaning of this approach is close to the previous one. However, the quantitative expression for the critical stress appears somewhat different. The detailed calculations show that for superposition of shear, σ , and normal, σ_E , stresses, the condition of the limiting state is written as

$$\sigma^* = \left[\frac{1}{2}(\sigma_E + \sqrt{\sigma_E^2 + 4\sigma^2}) - \frac{1}{2}\mu(\sigma_E - \sqrt{\sigma_E^2 + 4\sigma^2}) \right] \quad [4.5.10]$$

In this case, too, there is a complicated superimposing influence of normal and shear stresses on the critical state of material.

4.5.3.5 Complex criteria

Summarizing the above-mentioned, one can see that two concepts of a critical state condition exist: the dominating role of either shear or normal stresses is assumed. In the first

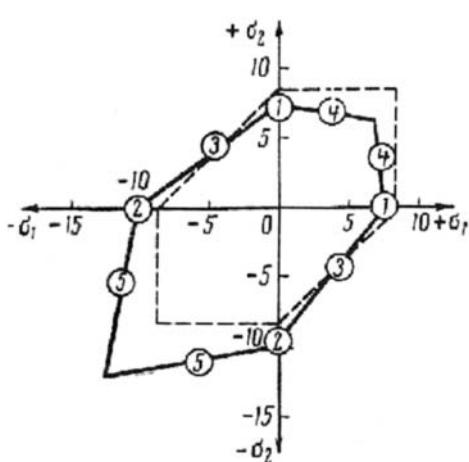


Figure 4.5.3. Limiting states of polystyrene as predicted by the maximum shear stress criterion (dashed line) and measured experimentally. [Adapted, by permission, from W. Whitney, R.D. Andrews, *J. Polymer Sci., C*, **16**, 2981(1967)].

stresses. In some cases, the angle of inclination of these lines is not exactly 45° , and this is the direct proof of the superposition of both mechanisms – shearing and extension.

A comparison of experimental results measured in a planar (two-dimensional) deformation field and predictions of the maximum shear stress criterion are shown in Fig. 4.5.3. The points marked are 1 – related to uniaxial extension, 2 – compression, 3 – shear in twisting, 4 – biaxial extension, and 5 – biaxial compression. Points marked as 3 are the direct measures of the critical shear stress, σ^* . Starting from this value, it is possible to construct the complete contour of limiting states, Eq. 4.5.1. This contour is drawn by a dashed line. The solid line is the contour built using the experimental data points.

It is evident that the experimental data do not exactly correspond to the theoretical predictions. The main reason in the case of data presented in Fig. 4.5.3 is the difference of limiting states in extension and compression (i.e., in two different uniaxial stress states). This phenomenon is well known for many real materials, such as stones, pig iron, and others. This phenomenon is the reflection of the role of hydrostatic pressure on a limiting state. The correct formulation of a criterion requires introduction of a normal stress factor in addition to the maximum shear stress. Then, the critical condition can be written as:

$$\frac{\sigma_1 - \sigma_3}{2} = \sigma^* + \sigma_E \tan \phi \quad [4.5.11]$$

The introduction of the angle, ϕ , reflects the inclinations of the direction of the maximum resistance from the line of the action of maximum shear stresses in uniaxial extension. This direction is oriented by the angle $\theta = (\pi/4 + \phi/2)$ to the axis of extension. According to the experimental data for polystyrene (Fig. 4.5.3), the angle ϕ equals 13° and $\theta = 51^\circ 30'$.

case, shear sliding (shape changes without volume increase) determines the critical state, and in the second, volume effects are a dominating factor and the critical state is reached due to the volume transition or separation of one part of material from another.

It is reasonable to suggest that in reality both effects are superimposed and the dominating mechanism of the critical state depends on the nature of material and/or the geometry of loading. Some situations are rather evident. For example, neck formation is a consequence of volume effects and thus of normal stresses. During extension of many ferrous metals, the lines on the sample surface (so-called *Luders lines*) inclined by 45° to the axis of extension are formed. This is a result of sliding of structure elements caused by action of shear

A general criterion of limiting states must reflect the influence of both shear and normal stresses. Normal stress modifies the conditions of transition (or break) according to the following linear relationship obtained from modification of the Eq. 4.5.4:¹⁷

$$\sigma^* = \sigma_{av} + \frac{v}{3} I_1 \quad [4.5.12]$$

where I_1 is the first invariant of the stress tensor (see Eq. 1.1.7) and v is an empirical factor. The condition $v = 0$ or deformation mode with $I_1 = 0$ correspond to the criterion of the limiting intensity of shear stresses. If $I_1 > 0$, i.e., material is stretched and sample can resist lower shear stresses. Hydrostatic compression increases an ability to withdraw at higher shear stresses.

A criterion accounting for different resistances to compression and extension is presented in the following way¹⁸

$$9\sigma_{av}^2 + 6(C - M)\sigma_E = 2CM \quad [4.5.13]$$

where C and M are limiting values of normal stress in compression and extension, respectively. If $C = M$, material is isotropic and, again, the criterion of maximum intensity of shear stresses appears valid. However, if $C > M$ (as usual), the situation appears more complex.

The applicability of Eq. 4.5.13 to the analysis of experimental data is illustrated in Fig. 4.5.4. The asymmetry of curve (due to the difference of properties in compression and extension) is evident. Eq. 4.5.13 fits the experimental data very well. It is worth mentioning that Eq. 4.5.12 also describes these experimental points in a satisfactory manner. Both equations are empirical and it is not possible to say which one of them is better.

The criteria discussed above are related to isotropic materials. Numerous materials of engineering interest are anisotropic in their structure, for example, reinforced plastics or monocrystals. Criteria of the limiting state also can be constructed for such materials, based on the same approaches that were used for isotropic materials. An analytical form of such criteria is complicated, though the physical principles are the same.

4.5.4 STRUCTURE EFFECTS

Departing from the limits of elasticity inevitably has an effect on structure and, thus, changes in mechanical (rheological) properties of matter. Such effects are, in many cases, irreversible. However, this irreversibility may be, at least, partly reversible on rest, though the rates of physical processes in solids are much slower than in liquids because of much longer relaxation times. The characteristic relaxation times in solids are at least in hours. In some cases, the increase of temperature and long storage at elevated temperature (*annealing*) may

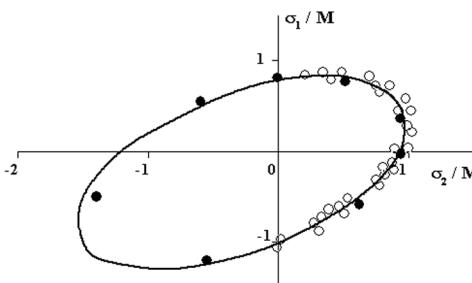


Figure 4.5.4. Comparison of theoretical predictions based on Eq. 4.5.13 (solid line) with experimental data for polystyrene (solid circles) and some other polymers (open circles). The ratio of C/M is chosen to be equal 1.3. Normal stresses σ_1 and σ_2 are normalized by limiting extension stress M . [Adapted by permission from S.S. Sternstein, L. Ongchin, *ACS Polymer Preprints*, **10**, 1117 (1969)].

accelerate the process of structure restoration and make it observable. On the other hand, the damage of structure at large deformations may be so extensive that its complete restoration is not possible even with high temperature treatment. This is why the initial state (shape) of material after plastic deformation cannot be reached by applying the same stresses but with the opposite sign.

Example

The car body deformed by a single impact in accident regrettably cannot be repaired by a single impact in the opposite direction.

The structure destruction and its (partial or complete) restoration in the case of a solid is similar to *thixotropy* of liquid-like materials, though this term is rarely used for solids. The concept of *structure* is more definite and clear for solids than for liquids, because the structure of solids can easily be inspected by direct optical, electronic, or X-ray methods. The mechanisms of structure rearrangement are also easy to investigate in many cases, though these mechanisms may be very different. One may be caused by sliding along crystal defects (dislocations), another by destroying relatively large structure aggregates (e.g., spherulites), or by orientation, deformation-induced phase transitions, and so on.

The reversibility of structure rearrangements suggests that this process proceeds in time, and it is necessary to consider kinetics of this phenomenon.

It is reasonable to distinguish between time-dependent effects of two types:

- a domain of viscoelastic effects consisting of superposition of energy storage and dissipation (see Chapter 2)
- the structure effects caused by changes in macrostructure of materials induced by deformations. Effects of such kind are observed primarily by irreproducibility of the results of material properties measurements in repeated experiments and their slow (complete or partial) restoration at rest.

Some typical structural phenomena leading to the changes of mechanical properties in the domain of nonlinear inelastic deformations are discussed below. It is noteworthy to remind one that these effects are *not viscoelastic* phenomena but a *structure time-dependent* effect. The situations cannot be well defined in various cases and it is not always possible to establish clear boundaries between effects of these two different classes.

4.5.4.1 Strengthening

A crystalline material after additional loading, beyond the limit of elasticity (in a plastic zone), becomes more rigid on repeated loading. The limit of elasticity increases and the plasticity of material becomes suppressed. Also, impact strength and resistance to shock resistance decreases. This effect is called *strengthening* or *stress hardening*. Strengthening of surface appears as a result of mechanical treatment in some technological operations and can be a purposeful technological procedure to improve properties of products.

Strengthening in plastic deformation decreases the resistance of material to loading in the opposite direction (this is known as the *Bauschinger effect*). Strengthening partly or completely disappears after a prolonged rest or annealing at elevated temperature, which is similar to thixotropic effects.

4.5.4.2 Thixotropy

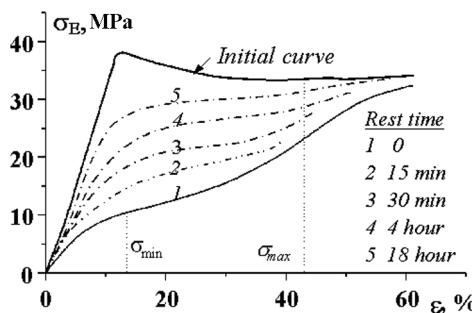


Figure 4.5.5. Thixotropy in deformation and rest of crystalline polypropylene – structure destruction and restoration. [Adopted, by permission, from G.P. Andrianova, Nguen Vin-Chii, *Vysokomol. Soedin.* (Polymers – in Russian), **14A**, 1545 (1972)].

The effect of reversible changes of material properties in the range of large deformations was first described for filled rubbers (known as the *Mullins effect*)¹⁹ and later for deforming crystalline polymers (see Fig. 4.5.5). The plateau part of the deformation curve decreases from σ_{\max} at an initial experiment to σ_{\min} in the repeated experiments carried out just after one another. However, during prolonged rest (several hours) this plateau value of stress returns to its initial value.

Structure destruction and restoration is a typical thixotropic phenomenon. It is very difficult to establish, which state of material is in equilibrium due to very slow structure transformations.

REFERENCES

- 1 There are many fundamental and applied textbooks devoted to solving the problem of linear elastic bodies and engineering designs. See, for example, classical and standard books: A.E.H. Love, *A Treatise on the Mathematic Theory of Elasticity*, 4th Ed., Dover, NY, 1927; S. P. Timoshenko and G. H. MacCullough, *Elements of Strength of Materials*, van Nostrand, NY, 1949; S. P. Timoshenko and J. M. Gere, *Mechanics of Materials*, van Nostrand Reinhold, NY, 1972; B. Sandor, *Strength of Materials*, Prentice Hall, Englewood Cliffs, 1978; E. P. Popov and T. A. Balan, *Engineering Mechanics of Solids*, Prentice Hall, Upper Saddle River, 1999.
- 2 A general approach to solving this problem was advanced by R.S. Rivlin, *Proc. Roy. Soc.*, **A193**, 260 (1948); *Phil. Trans. Roy. Soc. Lond.*, **A240**, 459 (1948); *J. Rat. Mech. Anal.*, **4**, 681 (1955) and **5**, 179 (1956). R. S. Rivlin, *Phil. Trans. Roy. Soc.*, **A241**, 379 (1948) and **A242**, 173 (1949); *Proceed. Cambridge Phil Soc.*, **45**, 485 (1949); *Proceed. Royal Soc.*, **A195**, 563 (1949). The review of “**Forty Years of Non-linear Continuum Mechanics**” was presented in: R.S. Rivlin, *Advances in Rheology*, v. 1, 1, Eds. B. Mena, A. Garcia-Rejon, C. Rangel-Nafaila, IX Intern. Congress on Rheology, Mexico, 1984. The general theory of large deformations is also summarized in the monograph: B.W. Ogden, *Non-linear Elastic Deformations*, Ellis Harwood, Chichester, 1984 and the latest survey is G. Saccomandi, R.W. Ogden, *Mechanics and Thermodynamics of Rubberlike Solids*, Springer, 2004.
- 3 This theory was proposed and developed in the following publications: W. Kuhn, *Koll. Zs.*, **68**, 2 (1934); E. Guth, H. Mark, *Monatsch. Chem.*, **65**, 93 (1934); E. Guth, *J. Appl. Phys.*, **10**, 201 (1939); E. Guth, H.M. James, *J. Ind. Eng. Chem.*, **33**, 634 (1941); H.M. James, E. Guth, *J. Ind. Eng. Chem.*, **34**, 1365 (1942) and *J. Chem. Phys.*, **11**, 455 (1943); E. Guth, H.M. James, H. Mark, *Adv. Colloid Sci.*, **2**, 253 (1946). One can find a very good review of the fundamental concepts in rubber elasticity in a classical monograph: L.R.G. Treloar, *The Physics of Rubber Elasticity*, Oxford Univ. Press, 3rd ed., 1975.
- 4 M. Mooney, *J. Appl. Phys.*, **11**, 582 (1940).
- 5 See the publications by Rivlin of 1948 in Ref.2.
- 6 M. Reiner, *Am. J. Math.*, **70**, 433 (1948). R.S. Rivlin, Proc. 1st Symp. Naval Struct. Mech., 169 (1960), Pergamon. Oxford. The absence of higher terms of C are based on mathematical arguments, in particular on the Hamilton-Cayley theorem, proving that higher terms can be degenerated to linear and quadratic terms of a tensor.
- 7 See Ref. 3.
- 8 M.H. Wagner, *J. Rheol.*, **38**, 655 (1994); E. Fried, *J. Mech. Phys. Solids*, **50**, 571 (2002). See also A.N. Gent, *J. Rheol.*, **49**, 271 (2005).
- 9 B.J. Blatz, S.C. Sharda, N.W. Tschoegl, *Trans. Soc. Rheol.*, **18**, 145 (1974).
- 10 H.H. Poynting, *Proc. Roy. Soc. London*, **A82**, 546 (1909); **86**, 534 (1912). It was the first experimental

demonstration of the second-order effects in the mechanics of continuum.

- 11 Some authors treat the stress corresponding to the plateau B'C on the $\sigma(e)$ curve as the yield stress.
- 12 The theory of plasticity of solids was first developed by A. Saint-Venant (1797-1866) – French engineer and scientist – as a separate branch of mechanics. This theory is set forth in the following classical monographs: A. Nadai, **Plasticity**, 1931 and **A Theory of Flow and Fracture of Solids**, 1950, McGraw Hill Co, New York; R. Hill, **Plasticity**, 1950, Clarendon Press, Oxford; W. Prager, P.G.Hodge, **Theory of Perfectly Plastic Solids**, 1951, Wiley & Sons, New York.
- 13 French engineer H. Tresca (1864) was the first who introduced this idea. The complete mathematical formulation of this concept was developed by A. Saint-Venant.
- 14 Sometimes this approach is also called the Mohr criterion.
- 15 This criterion was first proposed by Huber (1904) and became popular due to publications of R. von Mises (1913).
- 16 G.P. Andrianova, V.A. Kargin, *Vysokomol. Soedin.* (Polymers - in Russian), **12A**, 3 (1970).
- 17 S.S. Sternstein, L. Ongchin, *ACS Polymer Preprints*, **10**, 1117 (1969).
- 18 R. Raghava, R.M. Caddel, G.S. Yeh, *J. Mater. Sci.*, **8**, 225 (1973).
- 19 L. Mullins, *Rubber Chem. Technol.*, **42**, 339 (1969).

QUESTIONS FOR CHAPTER 4

QUESTION 4-1

Values of Young's modulus, E, and the bulk modulus of compressibility, B, are measured. Find shear modulus for a Hookean solid.

Additional question

Show that for an incompressible material the last equation transfers to the relationship between extensional and shear modulus known for rubber-like materials.

QUESTION 4-2

A bar is placed between two rigid walls. Its temperature is 20°C. Then the bar is heated to 200°C. What are the stresses that appear in the bar?

Additional question

Using standard values of parameters for steel, estimate the level of stresses. Ordinary values of the parameters of material are: $\alpha = 1.2 \cdot 10^{-5} \text{ K}^{-1}$, $E = 2.1 \cdot 10^5 \text{ MPa}$. It was assumed that $\Delta T = 180\text{K}$. Then direct calculation gives: $\sigma \approx -450 \text{ MPa}$.

QUESTION 4-3

Analyze the stress field in torsion of a cylindrical shaft caused by torque, T. This occurs in transmitting torque in a gear box.

QUESTION 4-4

Compare the stress state in torsion of a solid cylindrical bar of radius R_o and a tube with the same outer radius and the inner radius equal R_i . What is the increase of the maximum shear stress produced by decrease in cross-section of bar caused by changing solid cross-section to a tube?

QUESTION 4-5

Calculate the principal stresses and maximum shear stress, if torque, T, and the stretching force, F, act simultaneously on the shaft of radius R.

Additional question

Are these results valid for shafts made out of rubber?

QUESTION 4-6

A shaft is twisted with a torque T , as in Question 4-3. However the torque is high enough to produce stresses exceeding the yield stress, σ_Y , of material. Describe the stress situation along the shaft radius.

Additional question

What will be the deformations after unloading the shaft?

QUESTION 4-7

Prove that at small deformation, Hooke's law is the limit of rubber elasticity equation.

QUESTION 4-8

A rubber-like strip is stretched by the applied force $F = 0.2\text{N}$. The area of the cross-section of strip is $S = 1 \text{ mm}^2$. The elastic modulus, E , was measured at small deformations and it equals $3 \cdot 10^5 \text{ Pa}$. What is the elongation of the strip? What would be the estimated elongation if one would use Hooke's law for calculations?

Additional question 1

Why does the coefficient $1/3$ appear in this equation?

Additional question 2

Why was Eq. 4.4.17 used for calculations but not Eq. 4.4.16?

QUESTION 4-9

According to Hooke's law, the use of compression instead of extension leads to the symmetrical change of normal stresses. Is it the same for a rubbery material with rheological properties characterized by Eq.4.4.17?

Additional question

Can the last result be treated as a proof of anisotropy of material, i.e., existence of different values of elastic modulus in extension and compression, as is known for some other engineering materials, for example, concrete?

QUESTION 4-10

How are time effects taken into account in formulation of the constitutive equation for large deformations, e.g., Eqs. 4.4.7 or 4.4.20?

QUESTION 4-11

A cylindrical rod of radius R was studied in uniaxial extension. It was found that it can work below the critical force F^* . Then, this rod was used as a shaft working at torsion deformation mode. What is the limiting value of the torque, T^* , that can be applied to the shaft?

Answers can be found in a special section entitled Answers.