1

Continuum Mechanics as a Foundation of Rheology

Rheology is a science dealing with deformation and flow of matter. Relationships between *stresses* and *deformations* are the fundamental concepts of continuum mechanics, which are discussed in this chapter. The modern history of rheology was marked by the publication of several books[1] in the 1940s, which impacted the education of future generations of rheologists.

1.1 STRESSES

Internal *stresses* are directly related to *forces* applied to a body regardless of their origin. Only in special cases do internal stresses exist in the absence of external forces. These are, for example, thermal stresses caused by temperature inhomogeneity throughout a body or frozen stresses stored as a result of thermal and/or mechanical history of a body treatment caused by its heterogeneity.

1.1.1 GENERAL THEORY

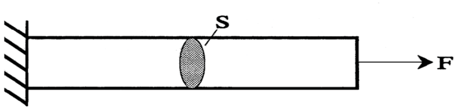

Figure 1.1.1. A bar loaded with a normal force.

Any external force applied to a body leads either to a movement of the body as a whole or to a change of its initial shape. Both may occur simultaneously. The movement of a body in space and/or its rotation around its center of gravity, with no change to its shape is a subject of study by mechanics, and as such is not discussed in this book. The principal focus of our discussion here are changes which occur inside a body on application of an external force. The applied forces create *dynamic reactions* at any point of a body, which are characterized by a physical factor called stress.

Stress can be explained using a simple example. Let us consider a body (a bar). The area of its normal cross-section is S (Fig. 1.1.1). The force, F, is normal to the surface, S. The specific force at any point of the cross-section equals F/S. The ratio is a *normal stress* or a *tensile stress*, σ_E:

$$\sigma_E = F/S \qquad [1.1.1]$$

i.e., *stress is the force per unit of the surface area*. The force at any surface may not be constant, i.e., be a function of coordinates. For example, a train moving on rails presses rails at local zones (where wheels touch the rail). The force is then distributed within the rail according to a complex pattern of stress distribution.

In our case, we do not consider force distribution because we have selected a small surface area, ΔS. A relative (specific) force, ΔF, acting on the area of ΔS is used to calculate the ratio ΔF/ΔS. By decreasing the surface area, we eventually come to its limiting form, as follows:

$$\sigma = \lim \frac{\Delta F}{\Delta S}, \text{ at } \Delta S \to 0, \text{ i.e., } \sigma = \frac{dF}{dS} \qquad [1.1.2]$$

This is a more general and exact definition of stress than given by Eq. 1.1.1 because it is related to a reference point, such as the surface area, ΔS. However, the definition is still not complete. A force at the area ΔS can have any direction, therefore a force is, in fact, a *vector* **F**. This vector can be decomposed into three components along three coordinate axes, in particular it can be decomposed to one perpendicular and two tangential components. The perpendicular component is *normal stress*, and tangential components are *shear stresses*.

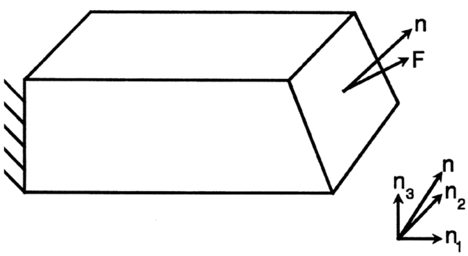

Figure 1.1.2. Definition of the stress tensor – two vectors: force, **F**, and orientation of a surface, **n**.

The selection of a small area ΔS is arbitrary, therefore it is better expressed by vector **n**, which determines *orientation* of ΔS to the normal orientation. The stress is described by a combination of two vectors, **F** and **n**, defined at any reference point. This is shown in Fig. 1.1.2. The stress is a derivative d**F**/d**n**, and it is independent of coordinate axis. Any vector is a physical object existing regardless of the choice of a coordinate system.

In practical applications, it is more convenient to operate with its projections on the coordinate axes rather than with a vector itself. Any vector can be decomposed into its three projections on the orthogonal coordinate axes; let it be Cartesian coordinates, x_1, x_2, and x_3.[2]

It follows from the above explanations that a complete characterization of stress as a physical object requires identification of two vectors: a force and a normal orientation to the surface to which this vector is applied. The physical objects determined in such a manner are called *tensors*, and that is why stress is a value of tensor nature.

Let both vectors, **F** and **n**, defined at any reference point, be represented by their three projections along the orthogonal coordinate axes:

$$\mathbf{n} = n(n_1, n_2, n_3)$$

$$\mathbf{F} = F(F_1, F_2, F_3)$$

Nine values can be obtained from three projections of force on the surfaces determined by the three coordinate vectors. All values of force, F_i (i = 1, 2, 3), must be divided by the surface area to give the components of a stress tensor, σ_{ij}. The first index gives the orientation of a force, and the second index designates the orientation of a surface.

The result is written in the table form (*matrix*), as follows:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad [1.1.3]$$

where the components of the stress tensor, σ_{ij}, mean the following: the first column represents components of a force (a force vector) acts on the plane normal to the x_1 axis, the second column gives the same for x_2 axis, and the third for x_3 axis. The directions to the normal are indicated by the second indices.

The matrix contains all components (projections) of a force vector applied to different planes at an arbitrary point inside a body. In order to emphasize that this set of parameters presents a single physical object, i.e., stress tensor,[3] it is usual to put the table between the brackets.

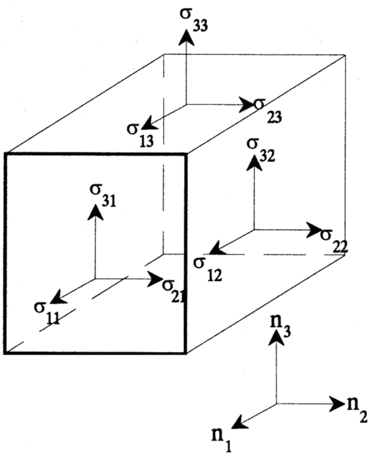

Figure 1.1.3. Three-dimensional stress state, a definition of the stress tensor components.

Fig. 1.1.3 shows all components of the stress tensor acting on a selected point. The components having the same numbers in their index are normal stresses, which are equivalent to the initial definition of the normal stress in Eq. 1.1.1, and all values with different numbers in the index are shear stresses.

All components of the stress tensor are determined at a point, and they can be constant or variable in space (inside a medium). It all depends on the distribution of external forces applied to a body. For example, the force field is homogeneous in Fig. 1.1.1, and thus a stress tensor is constant (inside a body). But the stress field (or stress distribution) is very complex in many other cases, for example, in a liquid flowing inside a channel or in the case of a roof covered with snow.

The values of the stress tensor components depend on the orientation of coordinate axes. They change with rotation of the coordinate axes in space, though the stress state at this point is the same. It is important to remember that, regardless of the choice of the coordinate axes, this is the same physical object, invariant to the choice of the coordinate axis.

Some fundamental facts concerning the stress tensor (and any other tensor) are discussed below.

Comments – operations with tensors

There are several general rules concerning operations with objects of tensor nature.[4] Two of them will be used in this chapter.

The first is the rule of summation of tensors. Tensor **A** is the sum of tensors **B** and **C** if components of **A**, a_{ij}, are the sum of components b_{ij} and c_{ij} with the same indices, i.e., the equality **A** = **B** + **C** means that $a_{ij} = b_{ij} + c_{ij}$.

The second is the rule of multiplication of a tensor by a constant. Tensor **A** equals to the product of a scalar constant k and a tensor **B** if components of **A**, a_{ij}, are equal to kb_{ij}, i.e., the equality **A** = k**B** means that $a_{ij} = kb_{ij}$. The unit tensor (also called the Kronecker delta, δ_{ij}) will be used below. This object is defined as a tensor, for which all diagonal (normal) components are equal to 1 and all shear components (when $i \neq j$) are equal zero.

1.1.2 LAW OF EQUALITY OF CONJUGATED STRESSES

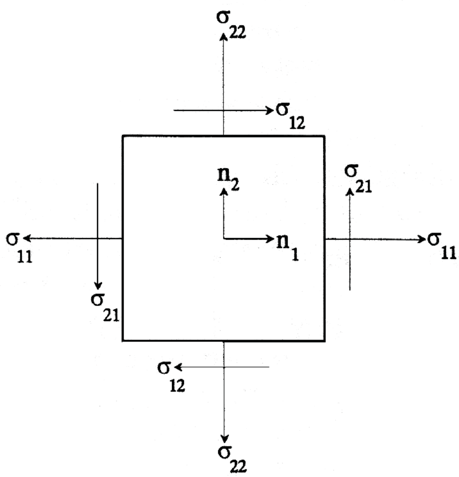

Figure 1.1.4. Two-dimensional (plane) stress state.

Let us consider a plane section of a unit cube in Fig. 1.1.3. The section is shown in Fig. 1.1.4. The rotational equilibrium condition about the central point of the square immediately gives equality

$$\sigma_{12} = \sigma_{21}$$

The same is true for any other pair of shear stresses. The general rule can be formulated as

$$\sigma_{ij} = \sigma_{ji} \qquad [1.1.4]$$

These equalities are known as the *Cauchy rule*.[5]

The result means that only three independent shear components of the stress tensor exist, and the stress state at a point is completely defined by six independent values: three normal, σ_{11}, σ_{22}, and σ_{33}, and three shear stresses, σ_{12}, σ_{13}, and σ_{23}.

However, it is necessary to mention that some special materials may exist, for which the Cauchy rule is invalid. It can happen if there is an inherent *moment of forces* acting inside any element of a medium.

1.1.3 PRINCIPAL STRESSES

The concept of *principal stresses* is a consequence of the dependence of stresses on the orientation of a surface. If stress components change on rotation of the coordinate axes, there must be such orientation of axes, at which the numerical values of these components are extreme (maximum or minimum).

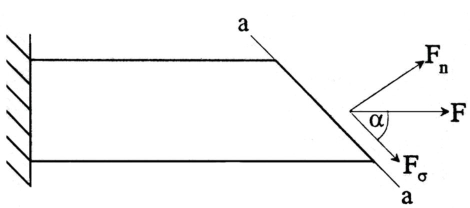

Figure 1.1.5. Stresses on the inclined section of a bar – decomposition of a normal force at the arbitrary oriented surface.

This idea is illustrated by a simple two-dimensional example generated from Fig. 1.1.1. Let the bar be cut at some angle α, as shown in Fig. 1.1.5, and let the force **F** act at this angle α to the plane *aa*. Then, it is easy to calculate two components of the vector **F** – normal and tangential forces, F_n and F_σ, respectively:

$$F_n = F\sin\alpha; \quad F_\sigma = F\cos\alpha$$

Then, the stress tensor components can be found, taking into account that the surface area of the inclined cross-section is $S/\sin\alpha$. Stress components, related to the force per unit of surface area, are found as follows:

normal stress, σ_E

$$\sigma_E = \frac{F_n}{S}\sin\alpha = \frac{F}{S}\sin^2\alpha = \sigma_0\sin^2\alpha \qquad [1.1.5]$$

shear stress, σ

$$\sigma = \frac{F_\sigma}{S}\sin\alpha = \frac{F}{S}\sin\alpha\cos\alpha = \frac{\sigma_0}{2}\sin 2\alpha \qquad [1.1.6]$$

where σ_0 =F/S.

The following special orientations can be found in a body:

at $\alpha = 90^o$ the normal stress $\sigma_E = \sigma_0$ is at maximum, and the shear stress $\sigma = 0$;
at $\alpha = 45^o$ the normal stress $\sigma_E = \sigma_0/2$ and the shear stress $\sigma = \sigma_0$ is at maximum;
at $\alpha = 0^o$ both σ_E and σ are equal zero, i.e., this plane is free from stresses.

Discussion shows that, at any arbitrary orientation (or direction) of a body, both normal and shear stresses may exist. Moreover, there is always a direction at which the normal or the shear stresses are at maximum. The last observation is very important because various media resist application of *extension* (normal) force or *shear* (tangential) force in different manners. For example, it is difficult to *compress* liquid (compression is achieved by application of negative normal stresses) but it is very easy to shear liquid (to move one layer sliding over another). Another case: when a thin film is stretched, it breaks as a result of action of normal stresses, but shear stresses are practically negligible in this case.

The above examples are only illustrations of a general idea that all components of a stress tensor depend on orientation of a surface because the size of a vector projection depends on orientation of axes in space.

The theory of operations with tensor objects gives the general rules and equations for calculating components of the stress tensor in three-dimensional stress state. The simpler equations for a two-dimensional stress state ("plane stress state") are given by Eqs. 1.1.5 and 1.1.6. Theoretical analysis shows that for any arbitrary stress tensor it is possible to find three orthogonal (i.e., perpendicular to each other) directions, at which normal stresses are extreme and shear stresses are absent (see Fig. 1.1.5). The normal stresses σ_{ii} are maximum along the directions at which shear stresses are absent, $\sigma_{ij} = 0$ ($i \neq j$). These normal stresses are called the principal stresses.

Existence of the principal stresses constitutes a general law for any stress tensor. In fact, it is a particular case of a more general statement: the existence of three principal values is the general law for any tensor.

The concept of the principal stresses permits finding a minimal number of parameters which characterize the stress state at any point. It is more difficult to compare stress states at different points of a body or in different bodies operating with six independent components of the stress tensor acting along different directions. It is much easier to do so dealing with only three normal principal stresses. All numerical values of the components of a stress tensor depend on the choice of the coordinate axes, while the principal stresses do not.

The "threshold" effects on the material behavior can be treated in an unambiguous manner using the principal stresses as a criterion of an event, but not separate components of a stress tensor. It means that physical phenomena caused by application of mechanical forces can be considered in terms of principal stresses. The examples include: phase transition induced by applied forces; heat dissipation in flow; storage of elastic energy; non-sag properties of some semi-liquid materials; rupture of solid bodies; slow movement of snow with sudden transition to avalanche; sand or mud on slopes, etc. The observed physical effects are usually caused by the principal stress which attain maximum value.

1.1.4 INVARIANTS OF A STRESS TENSOR

Knowledge of the principal stresses allows us to distinguish between different stress states of matter (e.g., three different values of the principal stresses or all principal stresses having the same value, etc.).

The principal stresses are characteristic of the stress state of a body (at a given point). They are not influenced by orientation. In other words, they are *invariant* to the choice of orientation.

How to calculate principal stresses if all components of the stress tensor are known for some arbitrary coordinate system is thus an essential practical question. The theory of tensors gives an answer to this question in the form of a cubic algebraic equation:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \qquad [1.1.7]$$

and principal stresses, denoted as σ_1, σ_2, and σ_3 appear to be the three roots of this equation. These roots are evidently expressed through coefficients of Eq. 1.1.7 – I_1, I_2, and I_3. These coefficients are constructed by means of all components of a stress tensor for arbitrary orthogonal orientations in space as:

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \qquad [1.1.7a]$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - (\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) \qquad [1.1.8a]$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{13}\sigma_{23} - (\sigma_{11}\sigma_{23}^2 + \sigma_{22}\sigma_{13}^2 + \sigma_{33}\sigma_{12}^2) \qquad [1.1.9a]$$

The principal stresses σ_1, σ_2, and σ_3 do not depend on orientation of axes of a unit cube (at a point) in space but they are expressed by values of I_1, I_2, and I_3. This leads to the conclusion that I_1, I_2, and I_3 are also invariant with respect to the choice of directions of orientation and that is why they are usually called *invariants of a stress tensor* at a point. According to its structure (the power of the components), I_1 is the first (linear), I_2 is the second (quadratic), and I_3 is the third (cubic) invariant. The invariants can also be expressed *via* the principal stresses only. These formulas are easily written based on Eqs. 1.1.7a -1.1.9a.

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \qquad [1.1.7b]$$

$$I_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 \qquad [1.1.8b]$$

$$I_3 = \sigma_1\sigma_2\sigma_3 \qquad [1.1.9b]$$

Any combination of invariants I_1, I_2, and I_3 is also invariant with respect to the orientation of axes in space. Various mathematical structures of invariants can be derived but it is a fundamental result that three and only three independent values of such kind exist.

Invariants are characteristics of the physical state of matter under the action of forces, either internal or external. This means that neither any stress by itself nor its arbitrary combination but only invariants determine a possibility of occurrence of various physical effects and threshold phenomena, some of which were mentioned above.

The fundamental principle says that the *physical effects must be independent of the choice of a coordinate system* which is quite arbitrary, and that is why invariants (as the combination of principal stresses), which are values independent of a coordinate system, govern physical phenomena, which occur because of the application of mechanical forces.

In many practical applications, a two-dimensional (also known as plane stress) state exists with a stress in the third direction being absent. Thin walled articles having stress-free outer surfaces (e.g., balloons, membranes and covers) are typical examples. The analysis of the two-dimensional stress state is an adequate solution in these cases. "Thin" means that the dimension in the direction normal to the surface is much smaller than in the other two directions.

Examples. Stresses in a thin-wall cylinder

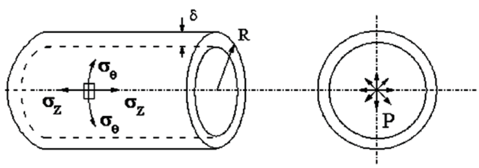

Figure 1.1.6. A thin-wall cylinder loaded by the inner pressure – stresses in the wall.

Internal pressure, p, in a thin-wall cylinder, closed by lids from both sides is typical of vessels working under pressure (chemical reactors, boilers, tubes in tires, and so on). Only normal stresses, σ_θ and σ_z, act inside a wall, where σ_θ is the stress acting in circumferential direction and σ_z is the longitudinal stress (see Fig. 1.1.6). The values of these stresses are calculated from

$$\sigma_\theta = \frac{pR}{\delta}\text{, and } \sigma_z = \frac{pR}{2\delta}$$

where R is the radius of a cylinder, and δ is the thickness of its wall.

This is a typical two-dimensional (plane) stress state, where both components of normal stresses are principal stresses and shear stresses are absent.

A long thin cylinder is twisted by applying a torque, T.

A torque is produced by relative turning, rotating, or twisting of a cylinder (inner or outer with no effect on result). Shear stress, σ, can be calculated from

$$\sigma = \frac{2T}{\pi(2R+\delta)^2\delta}$$

Generalization of plane stress state causes all components containing the index "3" to vanish, which gives the full stress tensor as in Eq. 1.1.10 instead of Eq. 1.1.3:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [1.1.10]$$

In this case, one principal stress, σ_3, is zero and two other, σ_1 and σ_2, are the roots of a quadratic (but not cubic) algebraic equation as follows:

$$\sigma_{1,2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} + \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \qquad [1.1.11]$$

There are two simple cases of the plane stress state: simple (or unidimensional) tension and simple shear. In the first case: $\sigma_1 = \sigma_{11} = \sigma_E$ and $\sigma_2 = 0$. In the second case: $\sigma_{11} = \sigma_{22} = 0$, $\sigma \equiv \sigma_{12}$ and therefore $\sigma_1 = \sigma$ and $\sigma_2 = -\sigma$. The last example demonstrates that, even when only shear stresses are applied, there are two orthogonal planes in a matter where only normal stresses act.

1.1.5 HYDROSTATIC PRESSURE – SPHERICAL TENSOR AND DEVIATOR

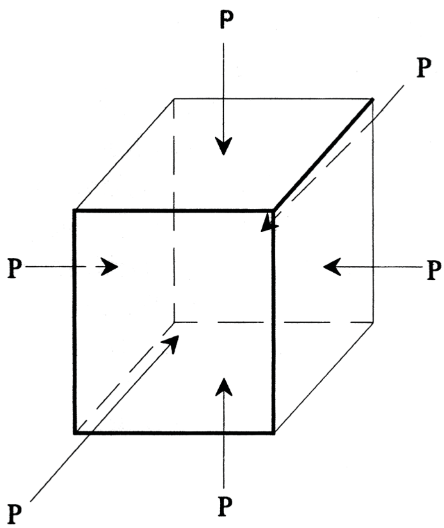

Figure 1.1.7. Hydrostatic pressure, p – all-directional (tri-axial) compression of a unit cube.

It seems pertinent that only normal stresses can change the volume of a body, while shear stresses may distort its form (shape). For this reason, it appears to be reasonable to divide a stress tensor into two components.

Fig. 1.1.7 shows compression of a body under hydrostatic pressure. The main feature of *hydrostatic pressure* is the absence of shear stresses; hence, all stress components with exception of the normal stresses are equal zero, and the stress tensor can be written as follows:

$$\sigma = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \qquad [1.1.12]$$

All principal stresses are the same and equal to -p:

$$\sigma_1 = \sigma_2 = \sigma_3 = -p \qquad [1.1.13]$$

and the "minus" sign shows that the force is directed inside an element of a matter.

Eq. 1.1.12 can be written in a short form using the above discussed rules of operation with tensors:

$$\sigma_{ij} = -p\delta_{ij} \qquad [1.1.14]$$

It means that $\sigma_{ii} = -p$ (the same two indexes of σ) and $\sigma_{ij} = 0$ (if $i \neq j$).

This stress tensor shows that shear stresses are absent at any direction in space. The tensor explains hydrostatic pressure shown in Fig. 1.1.7. The hydrostatic pressure is expressed as

$$p = -\frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = -\frac{I_1}{3} \qquad [1.1.15]$$

The last definition of hydrostatic pressure is true for any stress state, even when σ_{11}, σ_{22}, and σ_{33} are not equal to each other. Eq. 1.1.15 is considered as a general definition of *pressure*, and the stress tensor, Eq. 1.1.12, is called the *spherical stress tensor*.

However, one intriguing question arises: whether the value $-I_1/3$, calculated according to Eq. 1.1.15, and called pressure, has the same physical meaning as pressure used in thermodynamic relationships. Certainly, it is true for hydrostatic pressure when all normal stress components are the same, but this equivalence is assumed to be valid for an arbitrary stress state, though, possibly, it needs separate experimental evidence.

For the plane shear stress state, as was shown above, $\sigma_1 = -\sigma_2 = \sigma$ and $\sigma_3 = 0$. The same conclusion is correct for all other shear components of the stress tensor. It means that in *simple shear*, $I_1 = 0$, i.e., hydrostatic pressure is absent ($p = 0$). This shows that *shear stresses do not influence the volume of a body but may change its shape*.

It is now possible to write down a general expression for any stress tensor by separating the hydrostatic component. In this approach, all shear stresses remain untouched and each diagonal member of the tensor becomes equal to $(\sigma_{ij} - p)$.

This part of the stress tensor (complete tensor minus hydrostatic component) is called a *deviator* or deviatoric part of the stress tensor. It is thought that this part of the tensor is responsible for changes in the shape of a body but not its volume.

Uniaxial extension

The idea of splitting a stress tensor into spherical and deviatoric parts is well illustrated by the example of uniaxial stretching. It results in a body extension and it most likely leads to a volume change of a body. The question arises if uniaxial extension is equivalent to negative hydrostatic pressure? Stress tensor for uniaxial extension is written as

$$\sigma = \begin{bmatrix} \sigma_E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [1.1.16]$$

Similar to Fig. 1.1.1, all other forces except normal force, F, are absent. Therefore, there is no reason for other stress components, except for σ_{11}, and that is why all components in the matrix 1.1.16 equal zero (in particular $\sigma_{22} = \sigma_{33} = 0$), except for $\sigma_{11} = \sigma_E$.

It is now possible to split this tensor into hydrostatic and deviatoric parts, separating hydrostatic pressure (the remaining part of the stress tensor is the deviator). The deviator is simply a difference between full stress tensor and hydrostatic pressure. Then, the stress tensor for uniaxial extension can be written as:

$$\sigma = -p\delta_{ij} + \begin{bmatrix} \frac{2}{3}\sigma_E & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_E & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_E \end{bmatrix} = -p\delta_{ij} + \frac{\sigma_E}{3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad [1.1.17]$$

Any component of the full stress tensor, σ, equals the sum of the components of both addenda with the same indices; for example:

$$\sigma_{11} = -p + \frac{2}{3}\sigma_E = \frac{1}{3}\sigma_E + \frac{2}{3}\sigma_E = \sigma_E$$

and

$$\sigma_{22} = \sigma_{33} = -p - \frac{1}{3}\sigma_E = \frac{1}{3}\sigma_E - \frac{1}{3}\sigma_E = 0$$

The last rearrangements prove that Eqs. 1.1.16 and 1.1.17 are equivalent.

Comparison of Eqs. 1.1.12 and 1.1.17 shows that the *uniaxial extension is not equivalent to hydrostatic pressure* (the sign is not essential in this discussion) as the former leads to the appearance of a deviatoric component of the stress tensor too. In particular, it means that in the case of uniaxial extension it is possible to find such directions in a body where the shear stress exists – contrary to hydrostatic pressure where the shear stresses are principally absent (see discussion of Fig. 1.1.7).

The interpretation of the uniaxial extension as the sum of hydrostatic pressure and deviator explains that one dimensional tension creates not only negative pressure ("negative" means that stresses are oriented outward of unit areas inside a body) but also different normal stresses acting in all directions. This is the physical reason why all dimensions of a body change in the uniaxial tension (increase along the direction of extension but decrease in perpendicular (lateral) directions).

Some comments and examples

The uniaxial extension is important for many technical applications. In particular, this mode of loading is frequently used in material testing. Stress calculations in uniaxial extension need to be done very accurately. At first glance the problem is well represented by Eq. 1.1.1. However, two limitations are essential. First, this equation is valid only far from the ends, because stress distribution near the ends of a sample is determined by details of force application, which is usually not uniform. Thus, the stresses can be calculated from Eq. 1.1.1 only for long samples. Second, the cross-sectional area of sample changes with extension. Therefore Eq. 1.1.1 only describes the initial state of a sample. In technical applications, a stress is often calculated without considering such effects. It is correct then to consider it as some "conventional" or "engineering" stress.

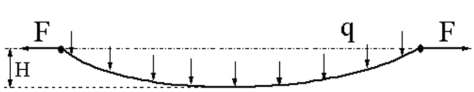

Figure 1.1.8. A sagging fiber loaded by a distributed load.

A loading under own weight of a sample suspended at one end is a special case of uniaxial extension. The normal (extensional) stress is caused by the gravitational force. The maximum stress, σ_{max}, acts at cross-section at which it is suspended. This stress equals $\sigma_{max} = \rho gL$ (where ρ is density, g is gravitational acceleration and L is the length of the sample). The σ_{max} increases as the sample length increases. There exists a length of the sample at which σ_{max} exceeds the material strength. This limiting length, corresponding to σ_{max} can be used as a measure of the material's strength. Such measure is used in engineering practice for characterizing the strength of fibers expressed by a "breaking length".

Analysis of uniaxial loading can be useful in solving many practical problems. For example, let us consider a horizontal fiber (string, rope, etc.) loaded by a distributed load, q (Fig.1.1.8). This distributed load can be its own weight, snow cover, strong wind, etc. Load provokes sagging of a flexible material. The application of the extension force, F, is one possibility of counteracting too extensive, unacceptable sagging. The length between the anchoring points at both sides is L. Then the height of the maximum sagging, H, is calculated from:

$$H = \frac{qL^2}{8F}$$

There is a direct correspondence between extending force and sagging height. The increase of extension force results in decrease of sagging. However, the force cannot be too large because the increase in stress may eventually exceed the strength limit of a fiber.

1.1.6 EQUILIBRIUM (BALANCE) EQUATIONS

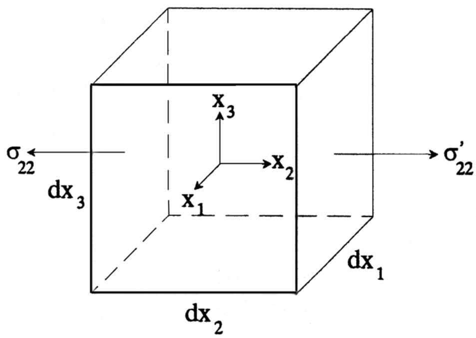

Figure 1.1.9. Components of the stress tensor in Cartesian coordinates – illustration of the stress difference at parallel cube faces along the infinitesimal distance.

The distribution of stresses throughout a body is described by equilibrium (or balance) equations as formulated by Navier,[6] Poisson,[7] and Cauchy[8] in their classical studies. In essence, it is a form of Newton's second law written for a continuum because the sum of all forces at a point equals to the product of mass (of this point) multiplied by acceleration.

A "point" in the theoretical analysis is an elementary (infinitesimal) space with sides oriented along the orthogonal coordinate axes (in Fig. 1.1.9 this space is a cube in the Cartesian coordinates). The idea of the analysis consists of a projection of all the external forces on the faces of the cube along three coordinate axes and their sum equals zero.

Forces are continuously changing at infinitesimal distances along the axis. If there is no special case having discontinuities in force, it is reasonable to think that, for example, a force on the left-hand face of the cube (Fig. 1.1.9) equals to σ_{22}, and on the parallel right-hand face it equals

$$\sigma'_{22} = \sigma_{22} + \frac{\partial \sigma_{22}}{\partial x_2} dx_2$$

The last relationship supposes that stress σ_{22} changes by infinitesimally small value to σ'_{22} at a small distance dx_2.

Other forces also exist and need to be taken into account in formulation of the balance equations. These are forces, presented by the vector $\mathbf{X}(X_1, X_2, X_3)$ per unit volume, and inertia forces equal (per unit volume) to $\rho\mathbf{a}$, where ρ is density of matter and $\mathbf{a}(a_1, a_2, a_3)$ is a vector of acceleration.

Then, writing the sum of projections of all forces (stresses are multiplied by the unit areas of the cube face) and dividing them all by $dx_1dx_2dx_3$ (which is an infinitesimally small value of the higher order), one comes to *equilibrium* (or *balance*) *equations* or *equations of momentum conservation*. For all three coordinate axes, this gives the following system of equations:

$$\begin{aligned}
\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+X_1 &= \rho a_1\\
\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}+X_2 &= \rho a_2\\
\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}+X_3 &= \rho a_3
\end{aligned} \qquad [1.1.18]$$

The system of Eqs. 1.1.18 includes pressure gradients of normal components of the stress tensor. Sometimes, pressure gradient is written separately and in this case σ_{11}, σ_{22}, and σ_{33} must be regarded as deviatoric components of the stress tensor.

For many rheological applications, it is reasonable to treat problems restricted to static equilibrium, and in such cases $\mathbf{a} = 0$. The existence of a body force is important, for example, if movement occurs because of the action of gravity (e.g., sagging paints or sealants from vertical or inclined surfaces, flow of glaciers, etc.). However, in many cases the influence of these forces is negligible and it is possible to assume that $X = 0$.

Then it is possible to omit the last two members of the balance equations and to simplify the system of Eqs. 1.1.18. This simplified (and usually used) system of balance equations is written as follows:

$$\begin{aligned}
\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3} &= 0\\
\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3} &= 0\\
\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3} &= 0
\end{aligned} \qquad [1.1.19]$$

Equilibrium can be considered with respect to different coordinate systems but not restricted to a Cartesian system. The choice of coordinates is only a question of convenience in solving a specific boundary problem. The choice of the coordinate system depends, generally, on the shape and the type of symmetry of a geometrical space which is the most convenient for an application. For example, if the round shells or tubes with one axis of symmetry are discussed, the most convenient coordinate system is cylindrical polar coordinates with r, z, and θ axes.

Components of the stress tensor in the cylindrical (polar) coordinates are shown in Fig. 1.1.10. The static balance equations in the absence of inertia forces, $\mathbf{a} = 0$, and volume – body – forces, $\mathbf{X} = 0$, for the point (or infinitesimal volume element), shown in Fig.

1.1.10, represent equilibrium conditions with respect to r, z, and θ directions. These equations can be written as:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial\sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2\sigma_{\theta r}}{r} = 0 \qquad [1.1.20]$$

$$\frac{\partial\sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} = 0$$

The meaning of the stress tensor components is explained in Fig. 1.1.10.

Some practical cases are symmetrical to the z axis, so that all terms containing $\partial/\partial z$ become zero as does the shear stress $\sigma_{r\theta}$. In some cases, the cylindrical bodies can be very long and variations of stresses along the axis of symmetry are absent (or can be taken as negligibly small). This allows us to continue simplification of the balance equations written in the cylindrical coordinates. In this case the balance equations reduce to

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0 \qquad [1.1.21]$$

All shear components of the stress tensor are absent.

All systems of balance equations, Eqs. 1.1.18 - 1.1.20, contain 6 unknown space functions (formally they contain 9 stress tensor components, but the use of the Cauchy rule decreases this number to 6), i.e., the stress components depend on space coordinates in a inhomogenous stress field. In order to close the system of equations – to make it complete – it is necessary to add the constitutive equations, connecting stress components with deformations. This is the central problem of rheology, because these equations express the rheological properties of matter.

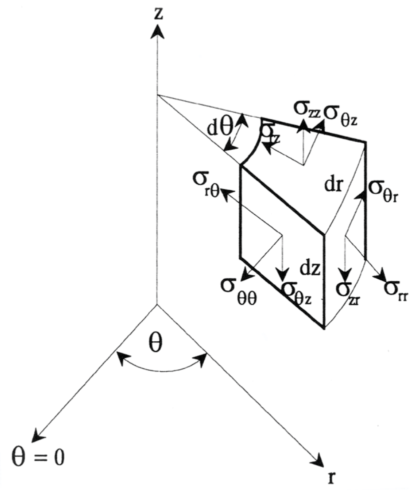

Figure 1.1.10. Components of the stress tensor in cylindrical (polar) coordinate system.

1.2 DEFORMATIONS

1.2.1 DEFORMATIONS AND DISPLACEMENTS

1.2.1.1 Deformations

The result of action of external forces can either be movement of a body in space or a change to its shape. Continuum mechanics is interested in changes occurring inside a body. The change of body shape is essentially the change of *distances* between different sites inside material, and this phenomenon is called *deformation*. Deformation is just a geometrical concept and all interpretations of this concept have clear geometrical images.

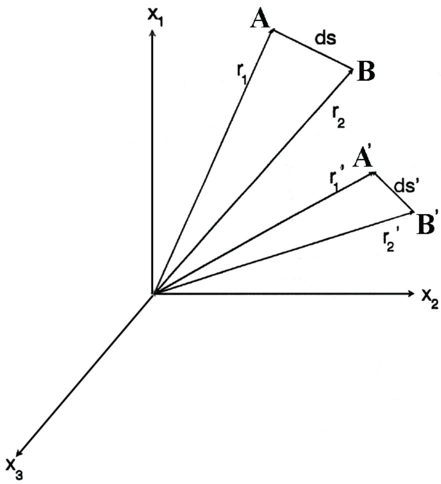

Figure 1.2.1. Displacements of two points in a body – the origin of deformations.

The change of distances between points inside a body can be monitored by following changes of very small (infinitesimally small) distances between two neighboring points.

Let the initial distance between two points A and B in material be ds (Fig. 1.2.1). For some reason, they both move and their new positions become A' and B'. Their movement in space is not of interest by itself for continuum mechanics. Only the new distance between them, which becomes, ds', is of interest. Moreover, the absolute value of the difference $(ds' - ds)$ is also not important for continuum mechanics, because the initial length ds might be arbitrary. Only the relative change of the distance between two points is relevant, and it is determined as

$$\varepsilon = \frac{ds' - ds}{ds} \qquad [1.2.1]$$

The distance between the points A and B is infinitesimal. Assuming that a body after deformation remains continuous (between sites A and B), the distance between points A' and B' is still infinitesimal.

The definition, Eq. 1.2.1, is not tied to any coordinate system, and it means that ε is a scalar object. However, this value can be expressed through components of tensor of deformation (or strain).[9]

Square of ds is calculated as:

$$(ds)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 \qquad [1.2.2]$$

Here, three Cartesian coordinates are defined as x_1, x_2, and x_3.

The square of the length $(ds')^2$ is calculated from

$$(ds')^2 = (dx'_1)^2 + (dx'_2)^2 + (dx'_3)^2 \qquad [1.2.3]$$

The coordinates of the new position of the A'B' length are calculated from

$$dx'_1 = \left(1 + \frac{\partial u_1}{\partial x_1}\right)dx_1 + \frac{\partial u_1}{\partial x_2}dx_2 + \frac{\partial u_1}{\partial x_3}dx_3$$

$$dx'_2 = \frac{\partial u_2}{\partial x_1}dx_1 + \left(1 + \frac{\partial u_2}{\partial x_2}\right)dx_2 + \frac{\partial u_2}{\partial x_3}dx_3 \qquad [1.2.4]$$

$$dx'_3 = \frac{\partial u_3}{\partial x_1}dx_1 + \frac{\partial u_3}{\partial x_2}dx_2 + \left(1 + \frac{\partial u_3}{\partial x_3}\right)dx_3$$

With neglecting the terms of higher orders than dx, it is easy to calculate the difference $(ds')^2 - (ds)^2$, which equals to

$$(ds')^2 - (ds)^2 = 2[\varepsilon_{11}(dx_1)^2 + \varepsilon_{22}(dx_2)^2 + \varepsilon_{33}(dx_3)^2] + \\ + 4[\varepsilon_{12}dx_1dx_2 + \varepsilon_{13}dx_1dx_3 + \varepsilon_{23}dx_2dx_3] \quad [1.2.5]$$

The change of length is expressed by six values of ε_{ij}, which can be expressed in a symmetric form as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial x_j} + \frac{\partial u_2}{\partial x_i}\frac{\partial u_2}{\partial x_j} + \frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_j}\right) \quad [1.2.6]$$

The indices i and j are used instead of 1, 2 and 3 for a brevity to not repeat Eq. 1.2.6 for six components of ε_{ij}.

The values ε_{ij} are not equal to the change of length of the distance AB but are only the *measures* of this change.

It is possible to prove that the values are components of a tensor, and this tensor is called the *tensor of large deformations*.[10] The complete expression for ε_{ij} consists of linear (first term in Eq. 1.2.6) and quadratic (second term in Eq. 1.2.6) terms.

If derivatives in Eq. 1.2.6 are small (<< 1) and their pairs of products, which enter into the second right hand side term in Eq. 1.2.6, are negligibly smaller than derivatives, relationships can be further simplified. The derivatives can be omitted, and only the first term of equation remains. This only holds true for small deformations, and that is why the tensor consisting only the first derivatives is called a *tensor of small* or *infinitesimal deformation* (or *strain*). This tensor, d_{ij}, can be written as follows:

$$d = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \quad [1.2.7]$$

The first row of the tensor d_{ij} represents the projections of deformations along the x_1 axis, and so on.

It is worth repeating that the components of ε_{ij} tensor, as defined by Eq. 1.2.1, were calculated based on pure geometrical arguments. The final result of these calculations is given by Eq. 1.2.6, or in the case of small deformations, by Eq. 1.2.7.

In the separate sections, the small deformation tensor, d_{ij}, and the complete (large deformation) tensor, ε_{ij}, will be further discussed.

1.2.1.2 DISPLACEMENTS

As shown in section 1.1.1, any tensor can be defined by two vectors. It is similar with deformations. The position of any site (point) in a body is characterized by its radius vector, **r**. Two sites are involved in the definition of deformation, A and B. Therefore it is necessary to introduce two vectors: $\mathbf{r}_1$ for the point A and $\mathbf{r}_2$ for the point B.

The value u_i, entering Eq. 1.2.6 via expressions for ε_{ij}, characterizes projections of the displacement vector, **u**, which represent the movement of the site A into its new position A'.

The quantitative determination of deformation can be accomplished by following *displacement*, $\mathbf{u} = (d\mathbf{r}_1 - d\mathbf{r}_2)$. The result of subtraction of two vectors is also a vector, and it can be expressed by its three projections: $\mathbf{u}(u_1, u_2, u_3)$. Relative displacement is expressed as $(d\mathbf{r}_1 - d\mathbf{r}_2)/d\mathbf{r}_1$. This object – contrary to the vector **u** – is characterized not only by its length but also by its orientation in space. Since two vectors, **u** and $\mathbf{x}(x_1, x_2, x_3)$, describe the relative displacement, the latter is of tensorial nature. Indeed, deformation and relative displacement are tensors and components of both tensors can be calculated through the derivative $d\mathbf{u}/d\mathbf{x}$. It is also pertinent that there are nine such values (three projections of vector **u** and three of vector **x**), as it could be expected for a tensor.

The values of all derivatives are dimensionless and they are expressed in absolute numbers or percents.

The tensor of relative displacement, **g**, is, by definition,

$$\mathbf{g} = \mathrm{grad}\,\mathbf{u} \qquad [1.2.8]$$

and it can be written via the components, g_{ij}, of this tensor

$$\mathbf{g} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \qquad [1.2.9]$$

The first row includes derivatives of the u_1-component of displacement along the three coordinate axis, the second row is the same for the u_2-component, and the third, for the u_3-component of the vector **u**.

The displacement tensor, defined by Eq. 1.2.9, is *not* deformation.

It is quite evident that the tensors d_{ij} and g_{ij} are not equivalent. The difference between them becomes clear if one decomposes the components of the tensor g_{ij} into two parts in the following manner:

$$g_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \qquad [1.2.10]$$

The first, the so-called symmetrical, part of the tensor g_{ij}, coincides with the deformation tensor d_{ij}, but it is evident that the deformations are something different than the displacements.

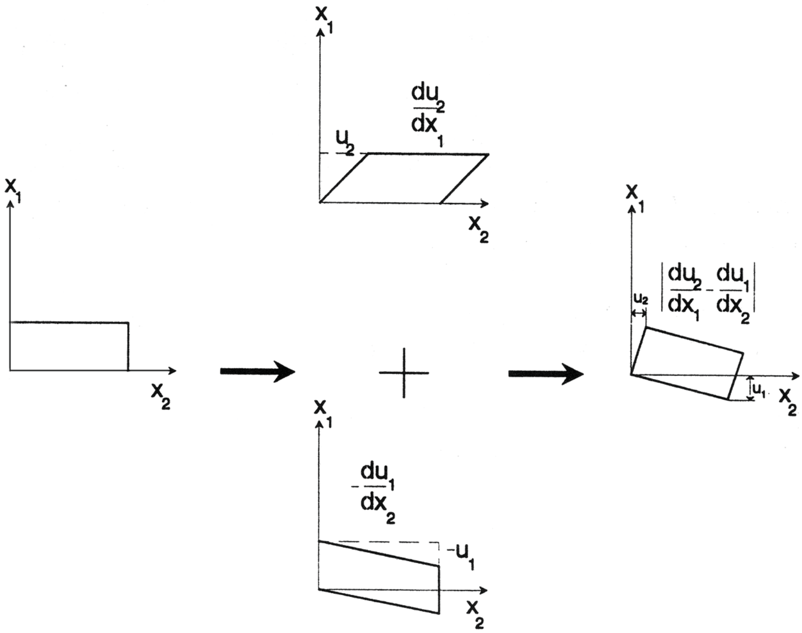

Figure 1.2.2. Superposition of two shear deformations leading to rotation of an element in a body.

The sense of this difference or the meaning of the second, the so-called antisymmetrical, part of the displacement tensor is explained in Fig. 1.2.2. Let us follow the deformation of an infinitesimal two-dimensional (plane) body element drawn here as a rectangle in the left diagram. Two displacements, u_2 and u_1, having gradients du_2/dx_1 and $-du_1/dx_2$, may occur as shown in the central part of Fig. 1.2.2. Now, let us superimpose these two displacements, as shown in the right diagram. It is evident from Fig. 1.2.2 that the summation of du_2/dx_1 and $-du_1/dx_2$ does not lead to deformation but to rotation of the body element. It means that the second term in Eq. 1.2.10 represents rotation, but not deformation. It can be written in the following manner:

$$g_{ij} = d_{ij} + \theta_{ij} \qquad [1.2.11]$$

where θ_{ij} is given by the following equation:

$$\theta_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \qquad [1.2.12]$$

These values are the components of the *tensor of rotations* (*vorticity*) of infinitesimal volumes inside a body. Thus displacement at any point of a body is a sum of deformation and rotation.

1.2.2 INFINITESIMAL DEFORMATIONS: PRINCIPAL VALUES AND INVARIANTS

Pure geometrical analysis demonstrates that the diagonal components of tensor d_{ij} expressed by Eq. 1.2.7 are equivalent to relative elongations (extension ratios) and non-diagonal components are shear or changes of angles between two orthogonal lines at a point.

The tensor of small (infinitesimal) deformations has all the general features of any other tensor, for example, the stress tensor discussed above. In particular, it is possible to calculate the principal values and invariants of this tensor using the same equations as for the stress tensor, only with changes in symbols. However, the invariants of the deformation tensor have a definite geometrical interpretation.

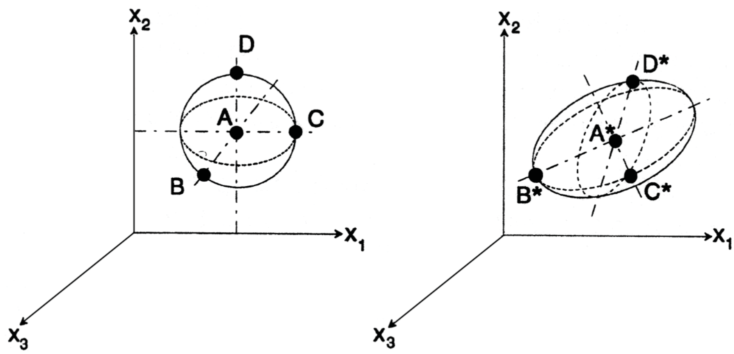

Figure 1.2.3. Transformation of a sphere into an ellipsoid as a consequence of three principal deformations along their axes.

For the infinitesimal deformation tensor, the principal deformations, d_1, d_2, and d_3, are extensions in three orthogonal directions. It can be illustrated by deformations in the vicinity of some arbitrary point. Let us represent an infinitely small volume in a body as a sphere (Fig. 1.2.3) with a center positioned at a point A and radius of the sphere dr (infinitesimal small length). The coordinates of the central point A are x_1, x_2, and x_3. As a result of movements and displacements, the following changes have taken place in a body: the point A has moved to a new position A*, the directions of the radii AB, AC, and AD have changed to the directions A*B*, A*C*, and A*D*, respectively. As a result, the sphere itself has transformed into an ellipsoid with semi-axes of length $(1 + d_1)dr$, $(1 + d_2)dr$, and $(1 + d_3)dr$, respectively.

The deformations characterize the change in shape of a volume element of the body on transition from a sphere to an ellipsoid. Besides they determine the relative change in volume, ε_V, which can be written as follows:

$$\varepsilon_V = \frac{V_{ell} - V_{sph}}{V_{sph}}$$

Simple calculation shows that

$$\varepsilon_V = (1 + d_1)(1 + d_2)(1 + d_3) - 1 \quad [1.2.13]$$

It is very easy to show that ε_V is expressed by invariants of the deformation tensor. The change in volume must not be associated with a choice of the coordinate system, and the invariants do not depend on the coordinate axes. Therefore, only invariants can determine the change of volume. The relationship between invariants is simple if deformations are small and it is possible to neglect quadratic terms in Eq. 1.2.13. Eq. 1.2.13 gives the following result:

$$\varepsilon_V = d_1 + d_2 + d_3 \quad [1.2.14]$$

i.e., *volumetric changes are equal to the first (linear) invariant* of the tensor of infinitesimal deformations and that is its physical meaning.

The volumetric changes in deformation can also be represented by extension ratios, λ_i. For this purpose, let us (conditionally) cut out a small rectangular parallel slab, at a some site in a body, oriented along the principal axes. Let the length of its edges be a, b and c before deformation, and let them become a*, b*, and c* as a result of deformation. Then, the extension ratios are:

$$\lambda_1 = a^*/a;\ \lambda_2 = b^*/b;\ \lambda_3 = c^*/c$$

The volume change is calculated as

$$\frac{V^* - V}{V} = \frac{\Delta V}{V} = \frac{a^*b^*c^*}{abc} - 1 = \lambda_1\lambda_2\lambda_3 - 1 \quad [1.2.15]$$

The last equation shows a very simple rule of constancy of volume in deformations of any type:

$$\lambda_1\lambda_2\lambda_3 = 1 \quad [1.2.16]$$

Like any other tensor, the deformation tensor, d_{ij}, can be decomposed into spherical and deviatoric parts. The first invariant is the volume change. It is possible to write:

$$d_{ij} = \frac{\varepsilon_V}{3}\delta_{ij} + d_{ij}^{(dev)} \quad [1.2.17]$$

The second term, on the right-hand side of the equation, is a deviatoric part, $d_{ij}^{(dev)}$ of the d_{ij} tensor which describes shape transformations occurring without changes in volume. Splitting the deformation tensor, d_{ij}, into spherical and deviatoric parts corresponds to separating the complete deformation into changes of volume and shape.

1.2.3 LARGE (FINITE) DEFORMATIONS

The difference between small (*infinitesimal*) and large (*finite*) deformations depends on values of derivatives in Eq. 1.2.6. If all derivatives are much smaller than 1, the quadratic terms, i.e., products of derivatives (in parentheses), can be neglected and the tensor d_{ij} is used instead of ε_{ij}.

In the discussion of the concept of large deformations, it is always assumed that a *reference state* of deformation can be established. In this sense, flow of liquid may not be considered as a deformation because all states are equivalent. Liquid does not have an initial (or reference) state. That is why only materials having *memory* of their initial state are important in determining deformations. Having such an approach, it is very easy to illustrate the essential difference between small and large deformations, using the simplest model of uniaxial extension from Fig. 1.2.4. Let a fiber (or a bar) of the length l_0 be stretched by Δl. The simple question is: what is the deformation in this case?

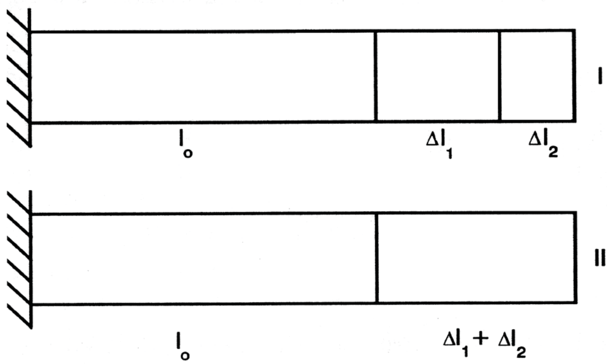

Figure 1.2.4. Two ways of realization of large deformation – two-step extension (I) or one-step extension (II).

In the first case, let $l_0 = 1$ and $\Delta l = 0.1$. The so-called *engineering measure* of deformation is

$$\varepsilon^* = \frac{\Delta l}{l_0} \qquad [1.2.18]$$

and in the example under discussion $\varepsilon^* = 0.1$ or 10%.

The reasoning becomes more complex if Δl is comparable with l, for example, let Δl to be equal 1. An engineering measure of deformation is the characteristic of the change of specimen length, and $\varepsilon^* = 1$ (or 100%). But this approach to the definition of deformation contains an inherent contradiction. Let us compare two situations, drawn in Fig. 1.2.4. In the first case (case I), the increase in the length occurs in two consequent steps: initially by Δl_1 and then, separately, Δl_2. Then, the deformation in the first step is $\varepsilon_1^* = \Delta l_1/l_0$ and, in the second step, it is $\varepsilon_2^* = \Delta l_2/l_1$, because the initial length of the sample in the second step is l_1. The total deformation, ε_{total}, is the sum of both deformations, as follows

$$\varepsilon^{I}_{total} = \varepsilon_1^* + \varepsilon_2^* = \frac{\Delta l_1}{l_0} + \frac{\Delta l_2}{l_1} = \frac{l_0(\Delta l_1 + \Delta l_2) + \Delta l_1^2}{l_0 l_1}$$

where l_0 is the initial length of the sample and $l_1 = l_0 + \Delta l_1$.

In the second case (case II), the increase of the length is achieved just in one step. This increase equals $(\Delta l_1 + \Delta l_2)$ and the total deformation is calculated from

$$\varepsilon^{II}_{total} = \frac{\Delta l_1 + \Delta l_2}{l_0} = \frac{l_0(\Delta l_1 + \Delta l_2) + \Delta l_1^2 + \Delta l_1 \Delta l_2}{l_0 l_1}$$

If elongations are small ($\Delta l_1 << 1$, and $\Delta l_2 << 1$), the difference between ε^{I}_{total} and ε^{II}_{total} is negligible. However, if it is not so, then the above written formulas clearly demonstrate that $\varepsilon^{I}_{total} \neq \varepsilon^{II}_{total}$, i.e., the final results of extension are different. This contradicts the physical meaning of the experiment interpretation: in reality the final result is the same in both cases and the sample does not "know" which way it was brought to the final state, whereas calculations show a difference. This contradiction appears only as a result of large deformations, because if deformations are small, the quadratic terms in formulas for ε_{total} are negligible in comparison to the linear terms.

It becomes apparent that there is a need to introduce such measure of deformation that does not depend on the sequence of operations. Such measure is called a logarithmic or the Hencky strain measure,[11] ε^H, which is defined by:

$$\varepsilon^H = \ln\left(\frac{l_0 + \Delta l}{l_0}\right) \qquad [1.2.19]$$

It is easy to prove that large deformations analyzed by this measure obey the law of additivity. Therefore, in the example discussed above, the resulting deformation, determined by the Hencky strain measure, does not depend on the history of deformation, as required:

$$\varepsilon^{I}_{total} = \ln\left(\frac{l_0 + \Delta l_1}{l_0}\right) + \ln\left[\frac{l_0 + (\Delta l_1 + \Delta l_2)}{l_0 + \Delta l_1}\right] = \ln\left[\frac{l_0 + (\Delta l_1 + \Delta l_2)}{l_0}\right]$$

and

$$\varepsilon^{II}_{total} = \ln\left[\frac{l_0 + (\Delta l_1 + \Delta l_2)}{l_0}\right]$$

Some other measures of large deformations are also used in the rheological literature. For example, let a fiber (or a bar) of the initial length l_0 be stretched and increase its length by Δl. The extension ratio, λ, equals $(\Delta l + l_0)/l_0$. Then

$$\frac{du_1}{dx_1} = \lambda - 1$$

and according to the definition given by Eq. 1.2.6

$$\varepsilon_{11} = \frac{du_1}{dx_1} + \frac{1}{2}\left(\frac{du_1}{dx_1}\right)^2 = (\lambda - 1) + \frac{1}{2}(\lambda - 1)^2 = \frac{1}{2}(\lambda^2 - 1) \qquad [1.2.20]$$

This measure of large deformations was introduced by George Green.[12] The large deformation tensor, used in continuum mechanics and based on the definition expressed by Eq. 1.2.6, is called a *Cauchy-Green tensor*, C_{ij}, and it is defined as

$$C_{ij} = \delta_{ij} + 2\varepsilon_{ij} \qquad [1.2.21]$$

where δ_{ij} is the Kronecker delta.

Similar to the tensor defined by Eq. 1.2.6, a Cauchy-Green tensor of finite deformations characterizes the change in distance between two arbitrary sites at a "point":

Another tensor of large deformations is also frequently used. This is the inverse (or reciprocal) tensor to the Cauchy-Green tensor, C_{ij}, named the *Finger tensor*, C_{ij}^{-1}.[13] According to the definition, the relationship between both tensors is

$$C_{ij}C_{ij}^{-1} = \delta_{ij} \qquad [1.2.22]$$

The principal components of the large deformation tensor, ε_i, are expressed by equation equivalent to Eq. 1.2.20:

$$\varepsilon_i = \frac{1}{2}(\lambda_i^2 - 1) \qquad [1.2.23]$$

where λ_i are the principal elongation ratios.

The principal values of the tensors C_{ij} and C_{ij}^{-1} are also expressed *via* the principal elongation ratios:

$$C_i = \lambda_i^2\text{; and } C_i^{-1} = \lambda_i^{-2} \qquad [1.2.24]$$

The first invariants of both tensors, $C_{I,inv}$, are as follows:

$$C_{I,\,inv} = \lambda_1^2\lambda_2^2\lambda_3^2\text{; and } C_{I,\,inv}^{-1} = \lambda_1^{-2}\lambda_2^{-2}\lambda_3^{-2} \qquad [1.2.25]$$

Introducing different measures of deformations does not exclude the main question regarding the initial state – point of reference of the deformed state. The importance of this question has already been demonstrated by the example of large deformations in uniaxial extensions. For static states, this problem can be solved by introducing the Hencky measure of deformations. The same problem appears and becomes more pertinent for a continuously moving medium where the position of deformed elements of a body is changing in time and it is necessary to describe the process or the rate of deformation. This problem will be discussed in more detail in Section 1.3.2.

1.2.4 SPECIAL CASES OF DEFORMATIONS – UNIAXIAL ELONGATION AND SIMPLE SHEAR

1.2.4.1 Uniaxial elongation and Poisson's ratio

Experiments show that a sample being stretched in the axial direction changes dimensions in the lateral direction. The relation between relative changes of dimensions in the lateral and the axial directions cannot be established on the basis of pure geometrical arguments because this relation reflects an inherent, independent property of material. The ratio of relative lateral contraction to the relative longitudinal extension is the quantitative characteristic of material property. This property of material is called *Poisson's ratio*.[7]

Let the radius of the cross-section of a bar in the initial state be r_0, and the length, l_0. If its elongation is Δl, and, as a result of stretching, the radius is decreased by Δr, then, by definition, Poisson's ratio, μ, is:

$$\mu = \frac{\Delta r / r_0}{\Delta l / l_0} \qquad [1.2.26]$$

It is now easy to calculate the volume change, resulting from uniaxial stretching. The relative change of volume, $\Delta V/V_0$, is

$$\frac{\Delta V}{V_0} = \frac{(r + \Delta r_0)^2 (l_0 + \Delta l) - r_0^2 l_0}{r_0^2 l_0} \qquad [1.2.27]$$

where $V_0 = \pi r_0^2 l_0$ is the initial volume of a sample (in a non-deformed state).

For small deformations $\Delta l << l_0$ and consequently $\Delta r << r_0$. In this case, Eq. 1.2.27 gives

$$\frac{\Delta V}{V_0} = 1 - 2\mu \qquad [1.2.28]$$

Poisson's ratio is a measure of volume changes during small deformations. From Eq. 1.2.28, one can see that deformations occur without volume changes when $\mu = 0.5$. For solid materials, $\mu < 0.5$, (for many solid materials $\mu \approx 0.3 - 0.35$). This means that their elongation is accompanied by an increase in specific volume. Only some rubbers and polymer melts $\mu \cong 0.5$ deform without volume changes.

The concept of Poisson's ratio allows one to use the general method of decomposing the deformation tensor, d_{ij}, into spherical and deviatoric terms for uniaxial extension. If $\lambda \ll 1$, and deformation in uniaxial extension equals ε^*, the tensor of infinitesimal deformations, Eq. 1.2.7, for such cases can be written as:

$$d = \begin{bmatrix} \varepsilon^* & 0 & 0 \\ 0 & -\mu\varepsilon^* & 0 \\ 0 & 0 & -\mu\varepsilon^* \end{bmatrix} = \frac{1-2\mu}{3}\varepsilon^*\delta_{ij} + \frac{1+\mu}{3}\varepsilon^* \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad [1.2.29]$$

The structure of this sum is very similar to the structure of the stress tensor decomposed into two parts (compare with the analogous procedure in section 1.1).

More precise analysis of Eq. 1.2.27 shows, however, that for large deformations Eq. 1.2.28 is not valid, and the rule of $\mu = 0.5$, as the condition for maintaining the constant volume at stretching, has no general meaning. Indeed, preserving a formal definition of Eq. 1.2.26 for Poisson's ratio, according to Eq. 1.2.27, the condition for $V = 0$ is:

$$1 - 2\mu(1 + \varepsilon) + \mu^2 \varepsilon(1 + \varepsilon) = 0 \qquad [1.2.30]$$

If $\varepsilon << 1$, Eq 1.2.30 is converted to an ordinary condition $\mu = 0.5$, but in the more general case it is not true.

Example – Poisson's ratio in finite deformations

Let the bar be stretched by 9 times (e.g., rubber ribbon or melted fiber). It means that $l/l_0 = 8$ and the volume can remain unchanged if the final radius becomes equal to 1/3 of its initial value. Then $\Delta r/r_0 = 2/3$. In this case, according to Eq 1.2.30, and following the formal definition, Eq. 1.2.25, $\mu = 1/12$.

This example shows that adaptation of infinitesimal deformation mechanics ($\mu = 0.5$ as a necessary condition for the constant volume at extension) to the domain of large deformations must not be done in a straightforward manner.

1.2.4.2 Simple shear and pure shear

Movement of all fluids and liquid-like materials is based on the model of sliding of neighboring layers relative to each other. This is a case of *simple shear*. Simple shear is also realized in several modes of deformations of solids, such as, for example, twisting long tubes or wires.

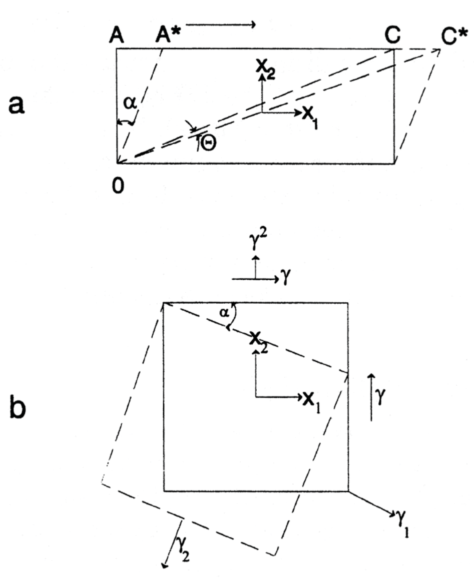

Figure 1.2.5. Small (a) and large (b) deformations in simple shear.

The schemes of two-dimensional (plane) simple shear for an element of a body in small deformation, and for a general case of arbitrary deformation, are shown in Figs. 1.2.5 a and b, respectively. Along the direction of shear marked by an arrow, a displacement, u_1, takes place. Its gradient, du_1/dx_2, is determined by the slope which is denoted as:

$$\gamma = \tan\alpha = \frac{du_1}{dx_2} \qquad [1.2.31]$$

Since the length of linear elements, which were directed before deformation in the x_2 direction, is changed in shear, one more displacement component, u_2, appears. It is related to the change in the length of the segment OA, which, after displacement, becomes equal to OA*:

$$\frac{OA^* - OA}{OA} = (1 + \gamma^2)^{1/2} - 1 \qquad [1.2.32]$$

The value of $\gamma = du_1/dx_2$ in a simple shear determines all components of the tensor at large deformations. According to the definition of the ε_{ij} tensor, its components are:

$$\varepsilon_{12} = \varepsilon_{21} = 0.5\gamma; \ \varepsilon_{22} = 0.5\gamma^2 \qquad [1.2.33]$$

This tensor is graphically illustrated in Fig. 1.2.5 b in which the components of the tensor, ε_{ij}, are marked by arrows (factor 0.5 is omitted in drawing this figure). The appearance of a diagonal component in the deformation tensor in a simple shear is a direct consequence of large deformations. It is a second-order effect because ε_{22} is proportional to γ^2 and its value becomes negligible if $\gamma << 1$. This phenomenon is known as the *Poynting effect*,[14] which is observed in wire twisting (their length slightly changes). Twisting is an example of shear deformation, and the observed change of the length is regarded as relative to the ε_{22} component of the deformation tensor.

Shear produces a shift between the direction of shear, x_1, and the orientation of principal axis. The shift is denoted by an angle, α, as shown in Fig. 1.2.5 b. The angle can be calculated from:

$$\alpha = \frac{1}{2}\arctan(\gamma/2) \qquad [1.2.34]$$

The main components of the deformation tensor may be written as follows:

$$\varepsilon_{11} = \cot\alpha; \ \varepsilon_{22} = \tan\alpha; \ \varepsilon_{33} = 1 \qquad [1.2.35]$$

The results obtained from Eqs. 1.2.35 indicate that in simple shear no volume change occurs because the product $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$

Expressions for the components of the Cauchy-Green and the Finger tensors in shear are important for future discussion concerning rheological models of elastic bodies of different types. Direct calculations give the following expressions for C_{ij} and C_{ij}^{-1}:

$$C = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C^{-1} = \begin{bmatrix} 1+\gamma^2 & -\gamma & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [1.2.36]$$

In a simple shear, not only the lengths of linear elements change (e.g., along the principal directions), but rotation of the elements of a body also takes place. This effect is well seen in Fig. 1.2.5, where the angle of rotation, θ, of the diagonal element from OC to OC* position, is shown.

Shear deformation in this figure is due to displacement, AA*, and α is its gradient. Any gradient of displacement consists of deformation and rotation, which in general form is expressed by Eq. 1.2.11. For a small displacement, the angle of rotation, $\theta = \alpha/2$, is used, unlike for large deformations, where the general Eq. 1.2.10 is applicable.

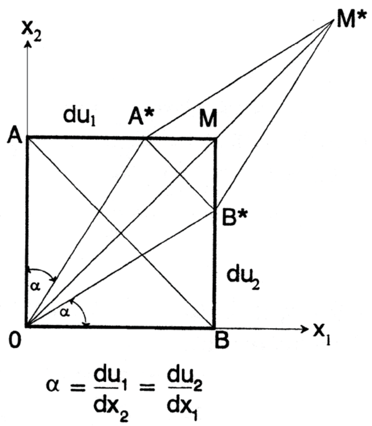

Figure 1.2.6. Pure shear of an element of a body.

It is possible to find such shear conditions where no rotation occurs. This case, called *pure shear*, is based on the definition of θ_{ij} from Eq. 1.2.12. $\theta_{ij} = 0$ if all differences of the displacement gradients equal zero. For the simple shear, this condition has the following form: $du_1/dx_2 = du_2/dx_1$.

A geometrical image of pure shear is drawn in Fig. 1.2.6. In pure shear, the diagonal AB of the small square (at some point) moves, due to deformation, into new position A*B*, parallel to its initial position, and the diagonal OM does not change its position at all, being only extended to OM*. Therefore, no element of the body undergoes rotation.

Fig. 1.2.6 can be obtained in a different way. It is quite evident that transition from the square OAMB to the rhomb OA*M*B* can be achieved by pressing the square along the direction AB, with simultaneous stretching along the direction OM. It means that pure shear can be realized through the superposition of two uniaxial extension deformations (with different signs).

The difference between simple shear and pure shear is the same as the difference between deformation and displacement. This difference may appear important in formulations of constitutive equations describing rheological properties and behavior of real materials.

1.3 KINEMATICS OF DEFORMATIONS

1.3.1 RATES OF DEFORMATION AND VORTICITY

Motion of a body is characterized by *velocity*, which is a vector. If velocity at any given point of a body is the same, it means that the body moves as a whole and no deformation takes place. The deformation appears only as a consequence of a velocity gradient at a "point", which means that two neighboring locations (the distance between them being infinitesimal) move with different velocities. If velocity is **v** (a vector value), the components of its gradient, a = d**v**/d**r**, are calculated as

$$a_{ij} = \frac{dv_i}{dr_j} \qquad [1.3.1]$$

The space coordinates are described by radius-vector, **r**. Thus **a** is a tensor with components a_{ij} determined by two vectors (**v** and **r**). The velocity is the rate of displacement, i.e., **v** = d**u**/dt. The relationship between gradient of velocity, a_{ij} = d**v**/d**r**, and gradient of displacement, g_{ij} = d**u**/d**r**, can be established from the following rearrangements:

$$a_{ij} = \frac{dv}{dr} = \frac{d}{dr}\left(\frac{du}{dt}\right) = \frac{d^2u}{drdt} = \frac{d}{dt}\left(\frac{du}{dr}\right) = \frac{dg_{ij}}{dt} \qquad [1.3.2]$$

In section 1.2.1, it was established that the whole gradient of displacement is not controlling deformation, only its symmetric part. The same is true for the deformation rate. The reasoning is the same as above. Differentiation with respect to a scalar – time, d/dt, adds nothing new to the result. By decomposing tensor a_{ij} into symmetrical and antisymmetrical components,

$$a_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial v_j} - \frac{\partial v_j}{\partial v_i}\right) \qquad [1.3.3]$$

one obtains

$$a_{ij} = D_{ij} + \omega_{ij} \qquad [1.3.4]$$

where D_{ij} is the *rate of deformation* tensor, and ω_{ij} is the so-called *vorticity tensor.*

As in the previous case, the rate of deformation tensor characterizes local changes of shape. The deformation is related to the first term of Eq. 1.3.4, while the vorticity tensor describes the rate of rotation of local elements of a body without their deformation.

The difference between the tensors a_{ij} and D_{ij} (which is quite similar to the difference between the tensors g_{ij} and d_{ij}) can be illustrated by a simple example. Let us analyze the rotation of a solid (non-deformable) body around some axes. The velocity, **v**, at a point located at the distance, **r**, from the axis of rotation equals ωr, where ω is the constant angular velocity. Thus, v = ωr, and the gradient of velocity, grad v = dv/dr, is evidently equal ω. It means that, during rotation of a solid body, the gradient of velocity does exist and there is no deformation because (as initially assumed) the body is non-deformable.

This example is also valid for any rotational movement, for example, for a circular movement of liquid placed between the stationary inner and the rotating outer cylinders (for any liquid point the difference between gradient of velocity and rate of deformation exists). In the latter case:

$$\frac{dv}{dr} = \frac{d(\omega r)}{dr} = \omega + \frac{rd\omega}{dr} \qquad [1.3.5]$$

Rate of deformation equals the second member of the sum, $r(d\omega/dr)$, whereas the first member, ω, represents superimposed rotation which does not influence deformation of matter placed between the cylinders. Indeed, it is possible to add a constant angular velocity of rotation, Ω, to both cylinders (to force both to rotate with the same constant angular velocity added to the rotation of the outer cylinder with the angular velocity, ω). It will increase the velocity gradient by this value but will not change the deformation rate.

1.3.2 DEFORMATION RATES WHEN DEFORMATIONS ARE LARGE

Some difficulties in calculation are encountered in the case of large deformations. At the end of section 1.2.3 it was pointed out that description of large (finite) deformations requires special attention and monitoring of a continuously moving medium, because positions of deformed elements are changing with time.

Substantial derivative

Similar solutions are required in classical hydrodynamics, when, for example, temperature effect due to heat exchange is included or material is transformed by chemical reaction. Such processes happen in media in motion. The problem is solved by using the so-called *material* or *substantial derivative*, D/Dt, which can be written for an arbitrary variable, Y, which depends on time and a site position moving in space, as:

$$\frac{DY(x_i, t)}{Dt} = \frac{\partial Y}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial Y}{\partial x_k} \qquad [1.3.6]$$

The first term represents local changes of the Y value, whereas the second term describes the movement of this local site in three-dimensional space.

In the theory of large deformations, it is important to know the rate of deformation in a fixed and moving coordinate system. Changes occur in a traveling element of material, which deforms along its replacement. This is called the *principle of material indifference*, which states that all physical phenomena must not depend on a coordinate system used for their mathematical formulation.[15]

As a result of large deformations, material elements can travel far away from their initial position, and that is why it is important to apply proper rules of transition from the reference state. Similar to the discussion of large uniaxial extension (see section 1.2.3), it is also important to choose different reference states in such a manner that they will not lead to an ambiguous estimation of deformation. An observer who measures properties of a material is always positioned in a fixed (unmovable) coordinate system. Hence, the general approach consists of formulating ideas concerning possible rheological behavior of a material for a moving (and deforming) element of a medium, recalculating them into a fixed coordinate system, and then comparing the results with an experiment.

This is true for the rate of deformation. There are many mathematical avenues to transform the rate of deformation tensor into a fixed coordinate system and, depending on selection, various forms of time derivatives were proposed, one of them is Eq. 1.3.6. In some theoretical studies, kinematic tensors of a higher order were introduced, which are time derivatives of the Cauchy-Green or the Finger tensors. They are used when the rheological behavior of a material depends on higher derivatives of deformation. In Chapter 2, devoted to properties of viscoelastic materials, it will be demonstrated that their behavior can be modeled by equations containing a sum of n-th order time derivatives of deformation (the so-called rheological equations of a differential type).

The physical meaning of substantial time derivative, D/Dt, requires that the derivative is calculated for a moving medium in which a material point follows time changes and leaves its initial position. The most popular are the *Rivlin-Ericksen*, $A_n(t)$,[16] and the *White-Metzner*, $B_n(t)$,[17] *tensors* of the n-th order. They are determined as

$$A_n(t) = \frac{D^n C_{ij}(t)}{Dt^n} \qquad [1.3.7]$$

and

$$B_n(t) = -\frac{D^n C_{ij}^{-1}(t)}{Dt^n} \qquad [1.3.8]$$

where C_{ij} and C_{ij}^{-1} are the Cauchy-Green and the Finger tensors, respectively.

The use of various measures of large deformations and different types of their time derivatives permits to make qualitative predictions concerning all possible effects in mechanical behavior of different materials. It is the global task of an experiment to evaluate possible models and to find the simplest of them which can adequately describe numerous physical phenomena observed in real materials in an unambiguous manner.

The practical application of the above discussed approaches for formulating constitutive equations for different materials and using them in solving dynamic (boundary) problems are considered in more details in Chapters 2 and 6 of this book.

1.4 SUMMARY – CONTINUUM MECHANICS IN RHEOLOGY

1.4.1 GENERAL PRINCIPLES

Classical continuum mechanics is one of the milestones of rheology. Rheology, dealing with *properties of matter*, regards these properties as relationships between *stresses* and *deformations*, which are the fundamental concepts of continuum mechanics.

The idea of a continuum, as well as mathematical operations used in mechanics, assume that there is a continuous transition and movement from point to point. A "*point*" is understood as a mathematical object of infinitesimally small size. However, it is necessary to accept the following contradiction: a "physical" point is different than a mathematical point.

Almost everybody is convinced that matter consists of molecules and intermolecular empty spaces, which means that in reality, any material body is heterogeneous. At the same time, an observer is sure that he "sees" a body of matter as a homogeneous continuous mass without holes and empty spaces. The obvious way out of these contradictory evidences lies in the idea of the *geometrical scale of observation*.

This scale must be large enough to distinguish individual molecules or their segments. The characteristic order of size of a molecule (its cross-section or length of several bonds) is 1 nm. Then, only when dealing with the sizes of the order of at least 10 nm, one may neglect molecular structure and treat a body as *homogeneous*. It means that a characteristic volume is of an order larger than 10^3 nm^3. This is a real size of a physical "point", which is quite different from a philosophical or geometrical point. The latter is an infinitely small object of zero size. The physical "point" contains 10^4 molecules or segments of macromolecule, and throughout its volume all molecular size fluctuations are averaged. The number of molecules in such a point is large enough for smoothing and averaging procedures.

In many cases, especially when discussing properties of a single-component material, it is possible to neglect the inherent structure of medium and the difference between ideas of the "physical" and "mathematical" concepts of a point as immaterial.

Having in mind the real scale of a physical point, it is supposed that it is permissible to apply methods of mathematical analysis of infinitesimal quantities (which formally relate to a geometrical point) to a physical medium. The formal extrapolation of physics-based analysis to infinitely small sizes tacitly avoids the incorrectness of this operation, and the only justification for this is the fact that in almost all practical applications, nobody is interested in what really happens in a very small volume.

However, there are at least three important principal exceptions.

1. A central physical problem exists in explanation of macro-observations of the molecular structure of matter. One would like to understand what happens to a molecule or how intermolecular interactions occur; then going through micro-volumes containing numerous molecules and averaging molecular phenomena, one would come to the macro properties of a body.
2. In some applications we use "zero" size. If geometrical shapes under consideration have sharp angles and the size at the corner of any angle (formally) equals zero, extrapolation of calculations results in such "zero" volume and sometimes leads to infinite values, and this is out of the realm of physical meaning. The analysis of problems of this kind requires special methods.
3. There are many materials which cannot be considered as homogeneous in principle. It is therefore necessary to consider their *structure*, i.e., such materials are *heterogeneous* by definition. For example, a medium can be a statistical or a regular mixture of some components with step-like transitions between them. Typical examples of such heterogeneity are suspensions and filled polymers, sometimes with well-arranged (in reinforced plastics) structure. In some applications, the structure of heterogeneous materials may be out of interest and it is possible to continue to treat the medium as homogeneous, averaging inner differences in relation to much higher geometrical scale. For example, for many astronomic observations, the Sun and the Earth are regarded as quite homogeneous and moreover can be treated as "points". In other cases, the role of heterogeneity can be important and it may become a determining factor (for example, for reinforced plastics), but, in any case, the scale of such heterogeneity has to be much larger than the characteristic molecular sizes.

1.4.2 OBJECTS OF CONTINUUM AS TENSORS

Continuum mechanics of any material operates with some fundamental concepts characterizing dynamic (*stresses*), geometrical (*deformations*) and kinematic (*deformation rates*) situation at a point (*site*). In this approach, a "point" is always understood in a mathematical sense as infinitesimal small objects. All these concepts are physical objects existing regardless of the choice of the coordinate system.

Stress is a measure of forces acting on a point and it is defined as a relative force or a force related to the unit area. Stress values depend on the direction of the applied force and the orientation of a surface on which acting forces are considered. Stress is an object of *tensor* nature.

Stresses determine deformation of matter, and, in limiting cases, when they overcome some threshold, they lead to *transitions* and eventually to rupture of material.

Stresses can be *normal* (perpendicular) and *shear* (tangential) to the surface where they act.

The stress tensor is written via its *components* – projections of the force on the coordinate axes.

It is always possible to calculate components of the stress tensor for any direction and to find such *principal* directions and principal normal stresses, which are extreme with shear stresses absent in those directions.

There are three particular combinations of any arbitrary stress tensor which do not depend on choice of axes or their orientation in space. These combinations are called *invariants*. Independence of these combinations of the stress components on the choice of the coordinate system is the evidence of the existence of stress as a physical quantity regardless of the coordinate system.

The stress tensor can be divided into two parts, one of which (known as *spherical*), being hydrostatic pressure, is responsible for the volume changes, and the other part (called *deviatoric*) is responsible for shape (or form) changes of a body (at a point). The spherical part of the stress tensor (and the first invariant of this tensor) determines the hydrostatic *pressure* (all-directional, tri-axial, compression acting on a body).

Calculation of stresses throughout a body is realized by solving differential equations with appropriate boundary conditions. These equations represent the law of equilibrium (or balance) of all forces applied at a point.

Due to different reasons and, in particular, due to the action of external forces, the points in a body can move in space and this is known as *displacement*. If displacements are inhomogenous throughout a body (i.e., different at different points) *relative* displacements appear and they lead to *deformations*, which are determined as the changes of infinitesimal distances between points inside a body.

Displacement is a *vector*, but relative displacements, as well as deformations, similar to stress, are the quantities of tensor nature, because their presence is controlled by two vectors. The relative displacement is described by radius vectors of two points for which the displacement is considered, and the deformation is characterized by means of a vector of displacement and a radius vector at a point, where the displacement occurs.

Deformation is only a part of the relative displacement, the latter also includes rotation of elements of a body as a whole.

Deformations can be small (or *infinitesimally small*) or large (or *finite*). The boundary between them is determined by the value of relative displacement (or gradient of displacement), which is a dimensionless value. If this value is small ($\ll 1$), it is reasonable to neglect the square of this value in comparison with the value itself. One can thus neglect all quadratic terms included in the definition of deformation. In this case, deformations can be treated as infinitesimally small.

The deformation tensor can be divided into two parts: the spherical part, which represents volume changes, and the deviatoric part, which is a characteristic of the shape transformations.

If *large deformations* are considered, some new effects appear. First of all, deformations occur at a site which moves, as a result of displacement, and vacates its initial position. Description of all occurrences (including deformation itself) must be done in relation to a moving point. An observer, carrying out experiments, follows its behavior and treats the results of measurements in a fixed coordinate system. Hence, it is necessary to know the rules of transformations and the tensor values used for projecting deformations from a moving to a fixed coordinate system.

Large deformations are characterized by special measures of deformation, such as the *Hencky measure* (a logarithmic measure subjective to additivity rule), and the *Cauchy-Green* and the *Finger tensors* of large deformations.

The tensors of deformations, similar to any other tensors, have principal axes along which the principal values of this tensor are calculated. Besides, three invariants of the deformation tensors are calculated by the standard rules of operations with tensors. The geometrical sense of the first invariant of the deformation tensor is volume change caused by deformation.

The kinematic picture of the relative movement of points of a continuum is characterized by the time derivative of displacement of a point (*its velocity*), time derivative of relative displacement (*gradient of velocity*), and time derivative of deformation (*rate of deformation*). Time derivatives of tensors are also tensors. For calculation of the rate of deformation, special rules exist which take into consideration large deformations and movements of a deforming site in space. Gradient of velocity is the sum of the rate of deformation and *vorticity* tensors of elements of a body, which – due to displacements – can rotate simultaneously with deformation.

Two special cases of deformations are of interest: uniaxial longitudinal extension and simple shear. In the process of extension, a body undergoes lateral compression. The ratio of relative changes of lateral and longitudinal sizes is called *Poisson's ratio*, which is an inherent property of a material. For the range of small deformations the volume of a body remains unchanged if Poisson's ratio equals 0.5. In simple shear, volume changes are not taking place at all. However, at large shear deformations, diagonal components of the deformation tensor appear, and they lead to some second-order effects.

Simple shear is accompanied by rotation of elementary volumes in space. In order to exclude rotation, it is necessary to apply *pure shear* in which rotation does not exist. This type of deformation is equivalent to a two-dimensional superposition of extension and compression in mutually perpendicular directions.

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2 Sometimes the coordinate system used in this book will be marked as x, y, and z, which are equivalent to x_1, x_2, and x_3, respectively.

3 Stress tensor is a particular case of a more general concept of tensors. This particular case is called tensor of the second rank.

4 Those who are interested to understand more about tensor analysis may use textbooks, such as for example, R. Aris, **Vectors, Tensors and the Basic Equations of Fluid Mechanics**, *Prentice Hall*, Englewood

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5 A.L. Cauchy (1789-1857) – French mathematician and physicist. One of the founders of the modern mathematical analysis and mechanics. His main publication on the subject under discussion: A. L. Cauchy, De la pression ou tension dans un corps solide, *Ex. de Math.*, 1827.

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9 Some authors try to distinguish between "strain" and "deformation". The College Dictionary has the following definition: "strain is a deformation of a body or a structure as a result of an applied force". This shows that both terms have the same meaning.

10 A.J.B. de St Venant was the first who realized the necessity to consider the concept of large (finite) deformations. A. J. B. de St. Venant, Mémoire sur équilibre des corps solides, dous les limits de leur élasticité, et sur les conditions de lèur resistance quond es désplacements êpouvés par leurs points ne sont par trés petit, *C. R. Acad. Sci.*, Paris, **24**, 1847. Later the fundamental ideas of the theory of large deformations were formulated by F. D. Murnaghan, *Ann. L. Mech.*, **59**, 235 (1937). Also the following paper contained many ideas, which became commonly accepted and used afterwards: M. Biot, Theory of elasticity with large displacements and rotations, Proc. 5 th Int. Congr. Appl. Mech., 1939.

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12 A.E. Green published (together with R.S. Rivlin) a number of important works in the field of continuum mechanics of solids capable of large deformations. See: A. E. Green, R. S. Rivlin, *Arch. Ration. Mech. Anal.*, **1**, 1 (1957) and **4**, 387 (1960). His general approach to the theory of large deformations was summarized in the book: A.E. Green, J.E. Adkins, **Large Elastic Deformations and Non-linear Continuum Mechanics**, *Clarendon Press*, Oxford, 1960. George Green, *Cambridge Phil. Soc. Trans.*, **7** (1839) (see **Mathematical Papers of the Late George Green**, edited by N.M. Ferrers, London, 1871)

13 J. Finger, *Sitzungsberichte Acad. Wiss. Wien*, (IIa), **103**, 163 (1894).

14 J.H. Poynting, *Proc. Roy. Soc. London*, A **82**, 546 (1909); A **86**, 534 (1912). This is, probably, the first experimental evidence of the second-order effect in finite deformations.

15 The fundamental principles which must be fulfilled in formulating the rheological constitutive equations in the case of finite deformations were advanced in a very clear and closed form in an important paper of J.G. Oldroyd, *Proc. Roy. Soc.*, A**200**, 523 (1950). This publication stimulated a lot of investigations on the mechanics of continuum of non-linear materials. However, it is worth to mention the earlier precursors in this field: pioneering works of S. K. Zaremba, *Bull. Acad. Sci. Cracovie*, **85**, 380 and 594 (1903) and G. Jaumann, **Grundlagen der Bewegungslehre**, *Springer*, Leipzig, 1905.

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QUESTIONS FOR CHAPTER 1

QUESTION 1-1

What is the equilibrium state of liquid and solid in the absence of stresses?

QUESTION 1-2

What are the possible limits of Poisson's ratio, μ? Can its value exceed 0.5? Can it be negative?

QUESTION 1-3

What are the pressure and the shear stresses in the stress state created by the following normal stresses: $\sigma_{11} = \sigma_0$; $\sigma_{22} = -\sigma_0$ and $\sigma_{33} = 0$? What are shear stresses in this case?

QUESTION 1-4

Calculate stresses acting on a thread being suspended by its end and stretched by its own weight.

QUESTION 1-5

Analyze a situation where a horizontal long flexible engineering element (fiber, bar, etc.) is loaded along its length by a distributed force, q (i.e., force, normal to the bar, per the unit of a length).

QUESTION 1-6

In section 1.3.1, the difference between the gradient of velocity and the rate of deformation is explained. What is the situation with these values for uniaxial extension?

QUESTION 1-7

Calculate the stresses in a hemispherical cup loaded by its own weight. Such a case is part of many engineering designs, for example, in a spherical roof covering a large area of a stadium or a warehouse.

QUESTION 1-8

Let liquid be placed between two coaxial cylinders with radii R_o (outer) and R_i (inner). The gap between cylinders $\Delta = R_o - R_i$ is small in comparison with the cylinder radii. Let the outer cylinder rotate with an angular velocity, Ω. Then, the assembly of both cylinders begins to rotate with the same angular velocity, ω. What are the shear rates and gradients of velocity in these two cases?

QUESTION 1-9

A cylindrical thread of length l_0 is fixed at one end and stretched at the other end. What must be the time dependence of velocity, v(t), of stretching that is sufficient to maintain a constant deformation rate, $\dot{\varepsilon}_0$ = const?

QUESTION 1-10

Put-forth your arguments proving the possibility to neglect shear stresses in a thin-wall cylinder as in the Example in section 1.1.4.

Answers can be found in a special section entitled Answers.