

Advanced Calculus

Georges Khater

American University of Beirut, Math 223

January 28, 2026

Contents

1 Metric Spaces	1
1.1 Compact Sets	5
1.2 Complete sets	7
1.3 Total Boundedness	8
1.4 Continuity	12
1.5 Connected sets	16

1 Metric Spaces

Definition 1.1. Let X be a set, a *metric* d is a map

$$d: X \times X \rightarrow \mathbb{R}$$

such that:

- (i) $d(x, y) \geq 0$ with equality iff $x = y$.
- (ii) $d(x, y) = d(y, x)$
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$

Example 1.2. Given a vector space V and norm $\|\cdot\|$ we can define the metric

$$d(x, y) = \|x - y\|$$

Question. Given a vector space V and a metric d on V , does d "come from" a norm? i.e does there exist some norm $\|\cdot\|$ s.t

$$d(x, y) = \|x - y\|$$

The answer in general is no (consider the discrete metric, or even any bounded metric: this contradicts homogeneity).

Notation. Take (X, d) metric space $\delta > 0$ and $x_0 \in X$, the δ -neighborhood of x_0

$$N_\delta(x_0) := \{x \in X : d(x, x_0) < \delta\}$$

Definition 1.3. (X, d) metric space and $E \subset X$

- (i) $x_0 \in X$ is an *interior point* of E iff $\exists \delta > 0$ s.t

$$N_\delta(x_0) \subset E$$

- (ii) $x_0 \in X$ is a *limit point* iff $\forall \delta > 0$

$$N_\delta^*(x_0) \cap E \neq \emptyset$$

(We denote by E' the set of limit points of E)

- (iii) E is *open* iff every $x_0 \in E$ is an interior point of E .

- (iv) E is *closed* iff every $E' \subset E$.

- (v) The *closure* of E denoted \bar{E} is

$$\bar{E} := E \cup E'$$

Review. Fix (X, d) a metric space.

1. $N_\delta(x_0)$ is open $\forall \delta > 0, \forall x_0 \in X$.
2. E is open iff E^c is closed.
3. \bar{E} is closed, in fact \bar{E} is the smallest closed set containing E .
4. The union of any collection of open sets is open.
5. A finite intersection of a collection of open sets is open.
6. Intersection of any collection of closed sets is closed.
7. Finite union of closed is closed.
8. If $x \in E'$ then $N_\delta(x) \cap E'$ is an infinite set.

Definition 1.4. Let x_n be a sequence in X , we say that x_n converges to $x_0 \in X$ iff $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t

$$d(x_n, x_0) < \varepsilon \quad \forall n \geq N$$

1. The limit of a sequence (if exists) is unique.
2. x_n converges to x_0 in (X, d) iff $d(x_n, x_0)$ converges 0 in $(\mathbb{R}, |||)$.

Example 1.5. $x_n = (1/n + 1, e^{-n})$ in \mathbb{R}^2 . Find the limit of x_n in $(\mathbb{R}^2, \|\cdot\|_\infty)$, $(\mathbb{R}^2, \|\cdot\|_1)$, $(\mathbb{R}^2, \|\cdot\|_2)$.

•

$$\|(1/n + 1, e^{-n}, (1, 0))\|_\infty = \max(1/n, e^{-n}) \leq 1/n + e^{-n} \rightarrow 0$$

- $x_0 = (1, 0)$

$$\|(1/n + 1, e^{-n}, -(1, 0))\|_1 = \|(1/n, e^{-n})\|_1 = |1/n| + |e^{-n}| \rightarrow 0$$

•

$$\|(1/n + 1, e^{-n}) - (1, 0)\|_2 = \sqrt{1/n^2 + e^{-2n}} \rightarrow 0$$

Remark 1.6. $x_n \rightarrow x_0$ in $(\mathbb{R}^k, \|\cdot\|_\infty)$

$$(x_{1,n}, x_{2,n}, \dots, x_{k,n}) \rightarrow (x_1, 0, \dots, x_{k,0})$$

Iff

$$x_{1,n} \rightarrow x_{1,0}$$

$$x_{2,n} \rightarrow x_{2,0}$$

\vdots

$$x_{k,n} \rightarrow x_{k,0}$$

in $(\mathbb{R}, \|\cdot\|)$.

Proof. $|x_{i,n} - x_{i,0}| \leq \|x_n - x_0\|_\infty \leq |x_{1,n} - x_{1,0}| + \dots + |x_{k,n} - x_{k,0}|$ Therefore $x_n \rightarrow x_0$ in $(\mathbb{R}^k, \|\cdot\|_\infty)$
 $\iff \|x_n - x_0\|_\infty \rightarrow 0$ and $\iff |x_{i,n} - x_{i,0}| \rightarrow 0 \forall i$ and hence $\iff x_{i,n} \rightarrow x_{i,0} \forall i$. \square

Remark 1.7. Take $x \in \mathbb{R}^n$, then

$$\|x\|_\infty = \max_i (x_i) \leq |x_1| + \dots + |x_n| = \|x\|_1 \leq n \|x\|_\infty$$

Therefore

$$\|x\|_\infty \leq \|x\|_1 \leq n \cdot \|x\|_\infty$$

Similarly, show something for $\|x\|_2$ and $\|x\|_\infty$.

Definition 1.8. Given a vector space V and norms $\|\cdot\|_a$ and $\|\cdot\|_b$ we say that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent iff $\exists c_1, c_2 > 0 \in \mathbb{R}$ s.t $\forall x \in V$

$$c_1 \|x\|_b \leq \|\cdot\|_a \leq c_2 \|x\|_b$$

Lemma 1.9. (X, d) metric space and $E \subseteq X$, then E is closed iff for sequence $x_n \in E$ that converges $\lim_{n \rightarrow \infty} x_n \in E$.

Proof. \implies Assume that there exists a sequence x_n in E that converges to x_0 s.t $x_0 \notin E$, take $\varepsilon > 0$, then since $x_n \rightarrow x_0$ then $\exists N \in \mathbb{N}$ s.t

$$x_n \in N_\varepsilon(x_0) \quad \forall n \geq N$$

But $x_n \in E$,

$$N_\varepsilon^*(x_0) \cap E \neq \emptyset$$

Therefore

$$x_0 \in E'$$

Contradiction to the fact that E is closed.

- Take $x_0 \in E'$, let $\varepsilon_n = 1/n$ then

$$\exists x_n \in N_n^*(x_0) \cap E$$

Consider $\{x_n\}_{n \in \mathbb{N}}$ x_n is a sequence in E s.t

$$d(x_n, x_0) < 1/n \implies x_n \rightarrow x_0$$

and hence $x_0 \in E$.

□

Theorem 1.10. Given a vector space V and norms $\|\cdot\|_a$ and $\|\cdot\|_b$ then the following are all equivalent:

1. $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent ($c_1\|x\|_b \leq \|x\|_a \leq c_2\|x\|_b$).
2. $x_n \rightarrow x_0$ in $\|\cdot\|_a \iff x_n \rightarrow x_0$ in $\|\cdot\|_b$.
3. E is closed in $(V, \|\cdot\|_a)$ $\iff E$ is closed in $(V, \|\cdot\|_b)$.
4. U is open in $(V, \|\cdot\|_a)$ $\iff U$ is open in $(V, \|\cdot\|_b)$.

Prob. \implies 2

$$c_1\|x_n - x_0\|_b \leq \|x_n - x_0\|_a \leq c_2\|x_n - x_0\|_b \leq \frac{c_2}{c_1}\|x_n - x_0\|_b$$

Apply Squeeze.

2 \implies 3 By the Lemma.

3 \implies 4 Trivial.

4 \implies 1 $B_{\|\cdot\|_a}(0, 1)$ open in $(V, \|\cdot\|_a)$ therefore it is open in $(V, \|\cdot\|_b)$. Hence 0 is an interior point in $(V, \|\cdot\|_b)$. Hence $\exists \delta > 0$

$$B_{\|\cdot\|_b}(0, \delta) \subseteq B_{\|\cdot\|_a}(0, 1)$$

Let $x \in V, x \neq 0$.

$$\frac{\delta}{2} \cdot \frac{x}{\|x\|_b} \in B_{\|\cdot\|_b}(0, \delta) \subseteq B_{\|\cdot\|_a}(0, 1)$$

Therefore

$$\left\| \frac{\delta}{2} \frac{x}{\|x\|_b} \right\|_a < 1$$

We get

$$\frac{\delta}{2\|x\|_b} \|x\|_a < 1 \implies \|x\|_a \leq \frac{2}{\delta} \|x\|_b \quad \forall x \in V$$

□

1.1 Compact Sets

Definition 1.11. (X, d) metric space, $K \subseteq X$. We say that K is *compact* iff for every open cover

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

of K has a finite subcover.

Example 1.12. $(0, 1]$ is not compact in $(\mathbb{R}, |.|)$, take

$$\mathcal{G} = \{(1/n, 10)\}_{n \in \mathbb{N}}$$

which has no finite subcover.

Theorem 1.13 (Heine-Borel). *In \mathbb{R} $[a, b]$ is compact.*

Example 1.14. If X is finite, then X is compact.

Proof. Let $X = \{x_1, \dots, x_n\}$ and let

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

be an open cover X . Then for every x_i , fix some G_{α_i} s.t

$$x = \{x_1, \dots, x_n\} \subseteq \bigcup_{i=1}^n G_{\alpha_i}$$

□

Proposition 1.15. (X, d) metric space and $K \subseteq X$ then every closed subset E of K is compact.

Proof. Let $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ be an open cover of E , then

$$E \subseteq \bigcup_{\alpha \in I} G_\alpha$$

Therefore

$$K \subseteq \bigcup G_\alpha \cup E^c, \quad E^c \text{ open}$$

Therefore

$$\mathcal{G}' = \mathcal{G} \cup \{E^c\} \quad \text{is an open cover of } K$$

Therefore it has a finite subcover and

$$K \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_k} \cup E^c$$

Hence

$$E \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_k}$$

Which is a finite subcover of \mathcal{G} .

□

Definition 1.16. (X, d) metric space and x_n sequence in X and

$$\varphi: \mathbb{N} \rightarrow \mathbb{N} \quad \text{strictly increasing}$$

We say that $b_n = a_{\varphi(n)}$ is a *subsequence* of a_n .

Proposition 1.17. x_n converges to $x_0 \iff$ every subsequence x_{n_k} converges to x_0 .

Definition 1.18. (X, d) metric space and $K \subseteq X$. We say that K is sequentially compact iff every sequence $x_n \in K$ has a convergent subsequence in K .

Proposition 1.19. (X, d) metric space, K sequentially compact iff every infinite $E \subseteq K$ has a limit point in K .

Proof. \implies Let K be sequentially compact, take $E \subseteq K$ an infinite set. We can extract from E a sequence of distinct elements $x_n \in E \subseteq K$. Therefore x_n has a convergence subsequence x_{n_k} s.t the limit of $x_{n_k} = x_0 \in K$. Moreover, x_{n_k} are distinct in E therefore $x_0 \in E' \cap K$.

\impliedby Let x_n be a sequence in K , take

$$E = \{x_n\}_{n \in \mathbb{N}}$$

We have two cases:

- **E is finite.** Then $\exists x_0$ and $\exists n_1 < n_2 < \dots$ (infinitely many) s.t

$$x_{n_k} = x_0, \quad \forall k \in \mathbb{N}$$

Therefore

$$\lim_{k \rightarrow \infty} x_{n_k} = x_0 \in E$$

We found a converging subsequence.

- **E is infinite.** Then by the assumption, we have that E has a limit point x_0 in K .

$$\varepsilon = 1 \quad \exists n_1 \text{ s.t } x_{n_1} \in N_1^*(x_0) \cap E.$$

$$\varepsilon = 1/2 \quad \exists n_2 > n_1 \text{ s.t } x_{n_2} \in N_{1/2}^*(x_0) \cap E \text{ (since } N_{1/2}^*(x_0) \cap E \text{ is infinite).}$$

we can construct $n_k > n_{k-1}$ s.t $x_{n_k} \in N_{1/k}^*(x_0) \cap E$, therefore

$$\lim_{n \rightarrow \infty} x_{n_k} = x_0 \in K$$

□

Example 1.20. In $(\mathbb{Q}, |\cdot|)$, $K = \mathbb{Q} \cap [0, 1]$ is closed and bounded. P_n increasing sequence in $\mathbb{Q} \cap [0, 1]$ s.t

$$P_n \rightarrow 1/\sqrt{2} \in \mathbb{R}$$

Consider

$$\mathcal{G} = \left\{ (-1, P_1) \cap \mathbb{Q}, (-1, P_2) \cap \mathbb{Q}, \dots, (-1, P_n) \cap \mathbb{Q}, \dots, (1/\sqrt{2}, 10) \cap \mathbb{Q} \right\}$$

Proposition 1.21. $K \subseteq X$ is compact K is sequentially compact.

Proof. • Let $E \subseteq K$ infinite, suppose that $E' \cap K = \emptyset$. Take $x \in K \implies x \notin E'$, therefore $\exists \delta_x > 0$ s.t

$$N_{\delta_x}^*(x) \cap E = \emptyset$$

Let $\mathcal{G} = \{N_{\delta_x}(x)\}_{x \in K}$, is an open cover of K compact, hence it has a finite subcover

$$K \subseteq N_{\delta_{x_1}}(x_1) \cup \dots \cup N_{\delta_{x_k}}(x_k)$$

But $E \subset K \implies E = (E \cap K)$ hence

$$\begin{aligned} E &\subseteq (N_{\delta_{x_1}}(x_1) \cap E) \cup \dots \cup (N_{\delta_{x_k}}(x_k) \cap E) \\ &\subseteq \{x_1, \dots, x_k\} \end{aligned}$$

therefore E is finite, contradiction.

□

1.2 Complete sets

Definition 1.22. Given a sequence x_n , we say that x_n is Cauchy iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t

$$d(x_n, x_m) < \varepsilon \quad \text{for } n, m \geq N$$

Proposition 1.23. If x_n converges, then x_n is Cauchy.

Proof. Triangle inequality. □

Remark 1.24. The converse is false. In $(\mathbb{Q}, |\cdot|)$, in $(\mathbb{R}, |\cdot|)$ there exists some sequence $x_n \in \mathbb{Q}$ s.t

$$x_n \rightarrow \sqrt{2} \quad \text{as } n \rightarrow \infty$$

But $x_n \rightarrow \sqrt{2}$ implies that x_n is Cauchy in $(\mathbb{R}, |\cdot|)$, therefore x_n is Cauchy in \mathbb{Q} . Suppose x_n converges to P in $(\mathbb{Q}, |\cdot|)$, therefore $x_n \rightarrow p \in (\mathbb{R}, |\cdot|)$ by uniqueness of limit, $p = \sqrt{2}$, contradiction.

Definition 1.25. (X, d) metric space and $K \subseteq X$, we say that K is complete iff every Cauchy Sequence in K converges in K .

Example 1.26. $(\mathbb{R}, |\cdot|)$ is complete (Math 210)

Proposition 1.27. If K is complete then K is closed.

Proof. Let x_n be a sequence in K s.t $x_n \rightarrow x_0 \in X$. Since x_n converges, then x_n is Cauchy and K is complete then x_n converges in K , therefore $x_0 \in K$. □

Proposition 1.28. K complete and $E \subseteq K$ is closed, then E is complete.

Proof. Let x_n be a Cauchy sequence in E , then x_n is Cauchy in K which is complete, therefore $x_n \rightarrow x_0 \in K$. But since E is closed, we get that $x_0 \in E$. □

Proposition 1.29. (X, d) metric space and x_n Cauchy sequence in $K \subseteq X$. If x_n has a convergent subsequence x_{n_k} then x_n converges to the subsequential limit.

Proof. Triangle inequality. □

Proposition 1.30. *(X, d) metric space and $K \subseteq X$ is sequentially compact then K is complete (and therefore closed).*

Proof. Let $\{x_n\}$ be a Cauchy sequence in K , K is sequentially compact therefore x_n has a convergent subsequence x_{n_k} that converges in K . Therefore by the previous proposition we get that x_n converges in K . \square

Theorem 1.31. *Let V be a finite dimensional inner product space over \mathbb{R} , then V is complete w.r.t norm induced by the inner product. Therefore $K \subseteq V$ is complete $\iff K$ is closed.*

(For the converse $K \subseteq V$ which is closed, then K is complete).

Proof. Let $S = \{v_1, \dots, v_l\}$ be an orthonormal basis of V , take x_n be a cauchy sequence in V . Therefore

$$x_n = r_{1,n}v_1 + \dots + r_{l,n}v_l$$

Take $r_{1,n} = \langle x_n, v_1 \rangle$ therefore

$$\begin{aligned} |r_{1,m} - r_{1,n}| &= |\langle x_m, v_1 \rangle - \langle x_n, v_1 \rangle| \\ &= |\langle x_m - x_n, v_1 \rangle| \\ &\leq \|x_m - x_n\| \|v_1\| \quad \text{By C.S} \end{aligned}$$

Let $\varepsilon > 0$ then $\exists N \in \mathbb{N}$ s.t

$$\|x_m - x_n\| < \varepsilon \implies r_{1,n} \text{ is Cauchy in } \mathbb{R}$$

Therefore $r_{1,n}$ converges to $r_1 \in \mathbb{R}$. Do the same for every coordinate, and define

$$x = r_1v_1 + \dots + r_lv_l$$

Therefore

$$\begin{aligned} \|x_n - x\| &= \|(r_{1,n} - r_1)v_1 + \dots + (r_{l,n} - r_l)v_l\| \\ &\leq \|r_{1,n} - r_1\| \|v_1\| + \dots + \|r_{l,n} - r_l\| \|v_l\| \end{aligned}$$

Therefore

$$x_n \rightarrow x$$

\square

1.3 Total Boundedness

Definition 1.32. *(X, d) metric space and $K \subseteq X$ we say that K is totally bounded iff for every $\varepsilon > 0 \exists x_1, \dots, x_t \in K$ s.t*

$$K \subseteq N_\varepsilon(x_1) \cup \dots \cup N_\varepsilon(x_t)$$

Definition 1.33. $K \subseteq X$ is bounded iff $\exists R > 0$ s.t

$$K \subseteq N_R(x_0) \quad \text{for some } x_0 \in K$$

We say that a sequence is bounded iff

$$E = \{x_n\}_{n \in \mathbb{N}} \quad \text{is bounded}$$

Proposition 1.34. Cauchy sequences are bounded.

Proposition 1.35. $K \subseteq X$ is totally bounded if it is bounded.

Note that the converse is false.

Example 1.36. (\mathbb{R}, d_{disc}) , \mathbb{R} is bounded since

$$\mathbb{R} \subseteq N_2(0)$$

but it isn't totally bounded, take $\varepsilon = 1/2$ then if it is totally bounded, then $\exists x_1 \dots x_t \in \mathbb{R}$ s.t

$$\mathbb{R} \subseteq N_{1/2}(x_1) \cup \dots \cup N_{1/2}(x_t) = \{x_1, \dots, x_t\}$$

Proof. Take $\varepsilon = 1$ then $K \subseteq N_1(x_1) \cup N_1(x_t)$ for $x_1, \dots, x_t \in K$. Take x_1 and let $d = \max_{k \in \{1, \dots, t\}}(x_1, x_k)$, therefore for any $x \in K$ we have that

$$d(x, x_1) \leq d(x, x_i) + d(x_i, x_1)$$

for some x_i s.t $x \in N_1(x_i)$ (Exists by the above), therefore

$$d(x, x_1) \leq 1 + d \implies x \in N_{1+d}(x_1)$$

□

Proposition 1.37. K is sequentially compact then K is totally bounded.

Proof. Suppose not, then $\exists \varepsilon_0$ s.t K cannot be covered by finitely many ε_0 -neighborhoods. Fix some x_1 then $\exists x_2 \in K \setminus N_{\varepsilon_0}(x_1)$, Similarly there exists some $x_3 \in K \setminus N_{\varepsilon_0}(x_1) \cup N_{\varepsilon_0}(x_2)$ and in general

$$\exists x_n \in K \setminus (N_{\varepsilon_0}(x_1) \cup \dots \cup N_{\varepsilon_0}(x_{n-1}))$$

Therefore we have that for $n \neq m$

$$d(x_n, x_m) \geq \varepsilon_0$$

Therefore x_n doesn't have a convergent subsequence; contradiction. □

Lemma 1.38. Every subset of a totally bounded set is totally bounded.

Proof. $K \subseteq X$ totally bounded, and $E \subseteq K$; take $\varepsilon > 0$, therefore $\exists x_1, x_2, \dots, x_t \in K$ s.t

$$K \subseteq N_{\varepsilon/2}(x_1) \cup \dots \cup N_{\varepsilon/2}(x_t)$$

Ignore the neighborhood that don't intersect E , renumbering we have x_1, \dots, x_l s.t

$$N_{\varepsilon/2}(x_i) \cap E \neq \emptyset$$

We get

$$E \subseteq N_{\varepsilon/2}(x_1) \cup \dots \cup N_{\varepsilon/2}(x_l)$$

Take $y_i \in N_{\varepsilon/2}(x_i) \cap E$, then

$$N_\varepsilon(y_i) \supseteq N_{\varepsilon/2}(x_i)$$

Therefore $E \subseteq N_\varepsilon(y_1) \cup \dots \cup N_\varepsilon(y_l)$ and hence E is totally bounded. \square

Example 1.39. $[0, 1]$ is totally bounded (since it is compact), therefore $(0, 1)$ is totally bounded.

Proposition 1.40. (X, d) Metric space, $K \subseteq X$ is complete and totally bounded then K is compact.

Proof. Assume K is not compact, fix $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ an open cover of K with no finite subcover. Let $\varepsilon = 1$, K is totally bounded then K can be covered by finitely many 1-balls with center in K . Hence $\exists x_1 \in K$ s.t $K_1 = N_1(x_1) \cap K$ cannot be covered by finitely many open sets in \mathcal{G} . But K_1 is totally bounded (subset of K), now let $\varepsilon = 1/2$, $\exists x_2 \in K_1$ s.t $K_2 = N_{1/2}(x_2) \cap K_1$ cannot be covered by finitely many open sets in \mathcal{G} . Similarly, we construct a sequence x_n and

$$K_n = N_{1/n}(x_n) \cap K_{n-1} \quad x_n \in K_{n-1}$$

such that K_n cannot be covered by finitely many G_α 's. Notice that

$$K_0 = K \supseteq K_1 \supseteq K_2 \supseteq \dots$$

Claim. x_n is Cauchy. Indeed, let $\varepsilon > 0$ then $\exists N$ s.t

$$1/N < \varepsilon/2$$

Now take $m, n > N$, then

$$x_n \in K_{n-1} \subseteq K_N, \quad x_m \in K_{m-1} \subseteq K_N$$

therefore $x_n, x_m \in K_N = N_{1/N}(x_N) \cap K_{n-1}$, and hence

$$d(x_n, x_m) \leq 2/N < \varepsilon$$

By completeness of K , we get that x_n converges to some $x_0 \in K$; since $\mathcal{G} = \{G_\alpha\}$ is an open cover of K , there exists $\alpha_0 \in I$ s.t $x_0 \in G_{\alpha_0}$ (open set). So $\exists \varepsilon_0$ s.t

$$N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

But $x_n \rightarrow x_0$ then $\exists N_1$ s.t

$$x_{N_1} \in N_{\varepsilon_0/2}(x_0) \text{ and } 1/N_1 < \varepsilon_0/2$$

Therefore

$$K_N \subseteq N_{1/N}(x_n) \subseteq N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

contradiction. \square

Theorem 1.41. *V finite dimensional product normed space (over \mathbb{R}). Let $S = \{v_1, \dots, v_n\}$ a basis of V . Then the cube*

$$Q_S = \{r_1 v_1 + r_2 v_2 + \dots + r_n v_n : 0 \leq r_i \leq 1\}$$

is compact.

Proof. Let $x^k = r_{1,k} v_1 + r_{2,k} v_2 + \dots + r_{n,k} v_n$ a sequence in Q_S , now clearly

$r_{1,k}$ is a sequence in $[0, 1]$, compact set

Therefore $r_{1,k}$ has a convergent subsequence r_{1,k_n} . Similarly $r_{2,k_l} \in [0, 1]$ therefore r_{2,k_l} has a convergent subsequence $r_{2,k_{l_t}}$. Doing this n times we get that \exists a subsequence $\{k_j\}$ s.t

$$r_{1,k_j}, r_{2,k_j}, \dots, r_{n,k_j} \rightarrow r_1, r_2, \dots, r_n$$

take

$$x_0 = r_1 v_1 + \dots + r_n v_n$$

We can show that

$$\|x_0 - x^{k_j}\| \leq |r_{1,k_j} - r_1| \|v_1\| + \dots + |r_{n,k_j} - r_n| \|v_n\| \rightarrow 0$$

then

$$x^{k_j} \rightarrow 0$$

Hence Q_S is (sequentially) compact. \square

Theorem 1.42. *Let V be a finite inner product space then*

$$K \subseteq V \text{ is compact} \iff K \text{ closed and bounded}$$

Proof. • K compact \implies complete \implies closed and \implies totally bounded \implies bounded.
• K is bounded, therefore $\exists R > 0$ s.t $K \subseteq B_{\|\cdot\|}(0, R)$. Let $S = \{v_1, \dots, v_n\}$ be a basis of V and Take $x \in K$ therefore

$$x = r_1 v_1 + \dots + r_n v_n$$

$$\text{then } r_i = \|\langle x, v_i \rangle\| \leq \|x\| \cdot \|v_i\|$$

$$B_{\|\cdot\|}(0, R) \subseteq \{r_1 v_1 + \dots + r_n v_n : r_i \leq R\}$$

Therefore $K \subseteq$ some cube and hence K is compact. \square

Proposition 1.43. In a finite dimensional inner product space we get that bounded \iff totally bounded.

Proof. \implies Bounded \implies can be put in a cube (compact) \implies Totally bounded.

\iff True in general.

□

1.4 Continuity

Definition 1.44. X, Y metric spaces and $E \subseteq X$ and $f: E \rightarrow Y$ map; $x_0 \in E'$ we say that

$$\lim_{x \rightarrow x_0} f(x) = y \in Y$$

iff for every $\varepsilon > 0 \exists \delta > 0$ s.t

$$f(x) \in N_\varepsilon(y_0) \text{ for } x \in N_\delta^*(x_0) \cap E$$

Definition 1.45. $f: E \rightarrow Y$ continuous at x_0 iff for every $\varepsilon > 0, \exists \delta > 0$ s.t $f(x) \in N_\varepsilon(x_0)$ for $x \in N_\delta(x_0) \cap E$; i.e

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{for } 0 < d_X(x, x_0) < \delta$$

Remark 1.46. f is continuous at $x_0 \iff x_0$ is an isolated point or $x_0 \in E'$ and $\lim x \rightarrow x_0 f(x) = f(x_0)$.

Proposition 1.47. $\lim_{x \rightarrow x_0} f(x) = y_0$ iff for every sequence $x_n \rightarrow x_0$ we have that $f(x_n) \rightarrow y_0$. Therefore f is continuous at x_0 iff for every $x_n \rightarrow x_0$ we have that $f(x_n) \rightarrow f(x_0)$.

Corollary 1.48. if f, g are continuous then $f + g, cf, f \cdot g, f/g$ are all continuous at x_0 .

Example 1.49. $f(x, y) = \frac{xy}{x^2+y^2}$ we want to find

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

Take the sequence $(x_n, y_n) = (1/n, 0)$ then

$$\lim_{n \rightarrow \infty} (x_n, y_n) = (0, 0)$$

and

$$f(x_n, y_n) = 0 \rightarrow 0 \text{ as } n \rightarrow \infty$$

Now take $(x_n, y_n) = (1/n, 1/n)$, then

$$\lim_{n \rightarrow \infty} (x_n, y_n) = (0, 0)$$

but

$$f(x_n, x_n) = \frac{1/n \cdot 1/n}{1/n^2 + 1/n^2} = 1/2 \rightarrow 1/2 \text{ as } n \rightarrow \infty$$

Therefore

$$\lim_{n \rightarrow \infty} f(x, y) \text{ does not exist}$$

Now we do the same for

$$f(x, y) = \frac{x^2 y}{x^2 + y^2}$$

Notice that

$$0 \leq |f(x, y)| = \left| \frac{x^2}{x^2 + y^2} y \right| \leq |y|$$

but

$$\lim_{(x,y) \rightarrow (0,0)} |y| = 0$$

therefore by squeeze we get that

$$\lim_{(x,y) \rightarrow 0} |f(x, y)| = 0$$

Hence

$$\lim_{(x,y) \rightarrow 0} f(x, y) = 0$$

Some useful inequalities

$$\begin{aligned} x^2 + y^2 &\geq x^2 \\ x^2 + y^2 &\geq y^2 \\ x^2 + y^2 &\geq 2(xy) \end{aligned}$$

Example 1.50. Let $(V, \|\cdot\|)$ a normed space and

$$\varphi: V \rightarrow \mathbb{R}, \quad \varphi(x) = \|x\|$$

Then φ is continuous.

Let x_n a sequence in V that converges to x_0 in $(V, \|\cdot\|)$ therefore

$$\|x_n - x_0\| \rightarrow 0$$

We want to show that $\varphi(x_n) \rightarrow \varphi(x_0)$ i.e that

$$\|\|x_n\| - \|x_0\|\| = 0$$

But as shown in PS1, we have that

$$\|\|x_n\| - \|x_0\|\| \leq \|x_n - x_0\|$$

Proposition 1.51. $f: X \rightarrow Y$ is continuos iff $f^{-1}(G)$ open in X for every G open in Y iff $f^{-1}(H)$ is closed in X for every H closed in Y .

Proof. \implies let x_n be a sequence in $f^{-1}(H)$ that converges to x_0 in X . Hence $f(x_n)$ is a sequence in H by continuity of f we have that

$$\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

Moreover since H is closed, we get that $f(x_0) \in H$. Hence

$$x_0 \in f^{-1}(H)$$

Therefore if f is continuous $f^{-1}(G)$ is open in X for every G open in Y .

\Leftarrow Now we show that if $f^{-1}(G)$ is open in X for all G open in Y then f is continuous. Fix $x_0 \in X$, let $\varepsilon > 0$ then consider the open set in Y $N_\varepsilon(f(x_0))$ hence $f^{-1}(N_\varepsilon(f(x_0)))$ is open in X . But $x_0 \in f^{-1}(N_\varepsilon(f(x_0)))$ hence $\exists \delta > 0$ s.t

$$f(N_\delta(x_0)) \subseteq N_\varepsilon(f(x_0))$$

□

Example 1.52. $E = \{x, y \in \mathbb{R}^2 : y \geq x\}$ is open in \mathbb{R}^2 , define the continuous function

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = y - x$$

Then

$$E = f^{-1}((0, \infty))$$

which is an open set.

Example 1.53. $f: X \rightarrow Y$ continuous and consider the c -level set with $c \in Y$

$$E = \{x \in X : f(x) = c\} = f^{-1}(\{c\})$$

is closed in X .

Therefore $E = \{(x, y) : x^2 + 3y^2 = 1\}$ is closed in \mathbb{R}^2 , S^n is closed.

Remark 1.54. V, W normed spaces and

$$\|\cdot\|_a, \|\cdot\|_b \text{ equivalent norms on } V$$

so $f: (V, \|\cdot\|_a) \rightarrow W$ is continuous iff $f: (V, \|\cdot\|_b) \rightarrow W$ is continuous.

Equivalently: V, W normed spaces and

$$\|\cdot\|_a, \|\cdot\|_b \text{ equivalent norms on } W$$

so $f: V \rightarrow (W, \|\cdot\|_a)$ is continuous iff $f: V \rightarrow (W, \|\cdot\|_b)$ is continuous.

Example 1.55. Take $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ defined as follows

$$f(x) = (f_1(x), \dots, f_n(x)) \quad x = (x_1, \dots, x_m)$$

Then f is continuous iff $f_i: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is continuous for $1 \leq i \leq n$.

Theorem 1.56 (Extreme Value Theorem). X, Y metric spaces and $K \subseteq X$ compact and $f: K \rightarrow Y$ continuous. Then $f(K)$ is compact.

Corollary 1.57. If $Y = \mathbb{R}$, then the above implies that f attains its maximum and minimum in K .

Proof. Let y_n be a sequence in $f(K)$, therefore $\exists x_n \in K$ s.t $f(x_n) = y_n$. Since K is compact, x_n has a convergent subsequence x_{n_k} that converges to $x_0 \in K$, using continuity we get that

$$y_{n_k} \rightarrow f(x_0) \in f(K)$$

□

Theorem 1.58. All norms in a finite dimensional vector space are equivalent.

Proof. V is a finite dimensional vector space, then by PS1 we can define an inner product on V . Let $\|\cdot\|$ be the norm induced by our choice of inner product. Let $S = \{v_1, \dots, v_n\}$ an orthonormal basis of V . Let $\|\cdot\|_a$ be an arbitrary norm on V , we will show that $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent. Let $x \in V$, then

$$x = r_1 v_1 + \dots + r_n v_n, \quad |r_i| = \langle x, v_i \rangle \leq \|x\|$$

Therefore $\forall x \in V$

$$\begin{aligned} \|x\|_a &= \|r_1 v_1 + \dots + r_n v_n\| \\ &\leq |r_1| \|v_1\|_a + \dots + |r_n| \|v_n\|_a \\ &\leq (\|v_1\|_a + \dots + \|v_n\|_a) \cdot \|x\| \\ &\leq c_1 \cdot \|x\| \end{aligned}$$

Now define the function $\varphi: (V, \|\cdot\|) \rightarrow \mathbb{R}$ s.t

$$\varphi(x) = \|x\|_a$$

We claim that φ is continuous, indeed let $x_n \in V$ s.t $x_n \rightarrow x_0$ in $\|\cdot\|$. Therefore

$$\begin{aligned} |\varphi(x_n) - \varphi(x_0)| &= \|\|x_n\|_a - \|x_0\|_a\| \\ &\leq \|x_n - x_0\| \\ &\leq c_1 \|x_n - x_0\| \rightarrow 0 \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \varphi(x_n) = \varphi(x_0)$, so φ is continuous. Let $K = \{x \in V : \|x\| = 1\}$ closed and bounded in the inner product space $(V, \|\cdot\|)$. Then it is compact in $(V, \|\cdot\|)$ therefore $\varphi(K)$ is compact in \mathbb{R} so it attains its minimum in K . Notice that $\varphi(x) = 0 \iff x = 0$, hence $\forall x \in K, \varphi(x) > 0$. Then

$$\min_{x \in K} \varphi(x) > 0$$

Let $x \in V$ and $x \neq 0$ then $\frac{x}{\|x\|} \in K$, therefore

$$\varphi\left(\frac{x}{\|x\|}\right) = \left\| \frac{x}{\|x\|} \right\|_a \geq \min \varphi(K)$$

Therefore

$$\|x\|_a \geq \min(\varphi(K))\|x\|$$

□

Corollary 1.59. *If V is finite dimensional then compact iff closed and bounded (independently of the choice of norm).*

1.5 Connected sets

Definition 1.60. (X, d) metric space and $\Omega \subseteq X$, we say that Ω is disconnected iff $\exists U, V \subseteq X$ open sets s.t

$$\Omega = (U \cup V) \cap \Omega$$

With $U \cap \Omega, V \cap \Omega \neq \emptyset$ and $U \cap \Omega \cap V = \emptyset$.

Otherwise Ω is connected.

Example 1.61. Let $\Omega = (1, 2] \cup (3, 4)$ disconnected, since

$$\Omega \subseteq (0, 2.5) \cup (2.5, 17)$$

\mathbb{Q} is disconnected in \mathbb{R} since

$$\mathbb{Q} \subseteq (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$$

Proposition 1.62. *X is a connected metric space iff the only sets that are both open and closed are \emptyset, X .*

Proof. \implies Assume that U is open and closed an $U \notin \{\emptyset, X\}$ then U^c is also not open and closed and not in $\{\emptyset, X\}$ but

$$X = U \cup U^c$$

Therefore X is disconnected.

\impliedby Assume that X is disconnected, then $X = U \cup V$ with U, V open disjoint and nonempty. Therefore $V = U^c$ and U is open then V is closed. Hence V is a clopen set of X but $V \notin \{\emptyset, X\}$ contradiction.

□

Definition 1.63. $I \subseteq \mathbb{R}$ is an interval for $x, y \in I$ s.t $x < y$ then $z \in I, \forall x < z < y$.

Theorem 1.64. $\Omega \subseteq \mathbb{R}$ is connected iff Ω is an interval.

Proof. \implies Assume that Ω is not an interval, therefore $\exists x, y \in \Omega$ s.t $x < z < y$ and $z \notin \Omega$, therefore

$$\Omega \subseteq (-\infty, z) \cup (z, \infty)$$

Therefore Ω is disconnected.

\impliedby Let Ω be an interval. Suppose Ω is not connected, then $\exists U, V$ open and disjoint in Ω in \mathbb{R} s.t

$$\Omega \subseteq U \cup V$$

Then $\exists x \in U \cap \Omega$ and $y \in V \cap \Omega$ and assume wlog that $x < y$; therefore $[x, y] \subseteq \Omega$ consider

$$z = \sup \{[x, y] \cup U\}$$

Note that $z \in [x, y]$ therefore $z \in \Omega$ therefore either $z \in U$ or $z \in V$. Notice that $z > x$ since $x \in U$ an open set. Also notice that $z < y$ since if $z = y \in V$ we have that $\exists \varepsilon_1$ s.t

$$(y - \varepsilon, y) = (z - \varepsilon, z) \subseteq [x, y] \cap V$$

since z is the sup we get that

$$(z - \varepsilon, z) \cap [x, y] \cap U \neq \emptyset$$

contradiction.

Therefore $z \in (x, y)$, suppose that $z \in U$ then $(z, z + \varepsilon_0) \subseteq (x, y) \cap U$ contradiction because $z + \varepsilon_0/2 > z = \sup[x, y] \cap y$. Suppose that $z \in V$ then $\exists \varepsilon_1$ s.t $(z - \varepsilon_1, z) \subseteq (x, y) \cap V$. Since z is the sup then $\exists z^*$ an element s.t

$$z - \varepsilon_1 < z^* < z \text{ and } z^* \in U$$

$z^* \in U \cap V$ contradiction. □

Theorem 1.65 (Intermediate Value Theorem). *f: X → Y continuous, if X is connected then f(X) is connected. In particular if Y = ℝ we have the following:*

- If $y_0 < y_1 \in f(X)$ then $\forall \alpha$ s.t $y_0 < \alpha < y_1$ $f^{-1}(\{\alpha\}) \neq \emptyset$.

Proof. Suppose that $f(X)$ is disconnected, then $\exists U, V \subseteq Y$ open sets that separate $f(X)$. Therefore

$$\begin{aligned} X &= f^{-1}(f(X)) \\ &= f^{-1}(U \cap V) \cap f(X) \\ &= [f^{-1}(U) \cup f^{-1}(V)] \cap X \end{aligned}$$

But f is continuous therefore $f^{-1}(U)$, $f^{-1}(V)$ are open and disjoint. Now $U \cap f(X) \neq \emptyset$ therefore $f^{-1}(U) \neq \emptyset$, same for V ; then X is disconnected. □

Definition 1.66. *X is path connected iff for every $a, b \in X$ there exists a continuous function*

$$\alpha: [0, 1] \rightarrow X$$

s.t $\alpha(0) = a$, $\alpha(1) = b$.

Remark 1.67. Let X be a vector space, the case where for every $a, b \in X$ the line segment $(1 - t)a + tb \in X \forall t \in [0, 1]$ we say that X is convex.

Example 1.68. $B_R(x_0)$ is always convex and hence path connected in a vector space. Indeed, take $a, b \in B_R(x_0)$ consider

$$\begin{aligned}\|(1-t)a + tb - x_0\| &< (1-t)\|a - x_0\| + t\|b - x_0\| \\ &< (1-t)R + R = R\end{aligned}$$

Proposition 1.69. (X, d) metric space then X is path connected $\implies X$ is connected.

Proof. Let X be disconnected then we have U, V nonempty disjoint and open

$$X = U \cup V$$

Then $\exists a \in U$ and $b \in V$, since X is path connected $\exists \alpha: [0, 1] \rightarrow X$ s.t

$$\alpha(0) = a, \quad \alpha(1) = b$$

$[0, 1]$ is connected, therefore $\alpha([0, 1])$ is also connected but

$$[0, 1] = \alpha^{-1}(U) \cup \alpha^{-1}(V)$$

But α is continuous, therefore $\alpha^{-1}(U) \ni 0$ and $\alpha^{-1}(V) \ni 1$ which is a contradiction since $[0, 1]$ is connected. \square

Theorem 1.70. V vector space and $\Omega \subseteq V$ open then Ω is connected iff Ω is path connected.