

Advanced Calculus

Georges Khater

American University of Beirut, Math 223

February 13, 2026

Contents

1	Metric Spaces	1
1.1	Compact Sets	5
1.2	Complete sets	7
1.3	Total Boundedness	8
1.4	Continuity	12
1.5	Connected sets	16
2	Differentiability	19
2.1	Partial derivatives	23
2.2	Chain Rule.	27

1 Metric Spaces

Definition 1.1. Let X be a set, a *metric* d is a map

$$d: X \times X \rightarrow \mathbb{R}$$

such that:

- (i) $d(x, y) \geq 0$ with equality iff $x = y$.
- (ii) $d(x, y) = d(y, x)$
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$

Example 1.2. Given a vector space V and norm $\|\cdot\|$ we can define the metric

$$d(x, y) = \|x - y\|$$

Question. Given a vector space V and a metric d on V , does d "come from" a norm? i.e does there exist some norm $\|\cdot\|$ s.t

$$d(x, y) = \|x - y\|$$

The answer in general is no (consider the discrete metric, or even any bounded metric: this contradicts homogeneity).

Notation. Take (X, d) metric space $\delta > 0$ and $x_0 \in X$, the δ -neighborhood of x_0

$$N_\delta(x_0) := \{x \in X : d(x, x_0) < \delta\}$$

Definition 1.3. (X, d) metric space and $E \subset X$

(i) $x_0 \in X$ is an *interior point* of E iff $\exists \delta > 0$ s.t

$$N_\delta(x_0) \subset E$$

(ii) $x_0 \in X$ is a *limit point* iff $\forall \delta > 0$

$$N_\delta^*(x_0) \cap E \neq \emptyset$$

(We denote by E' the set of limit points of E)

(iii) E is *open* iff every $x_0 \in E$ is an interior point of E .

(iv) E is *closed* iff every $E' \subset E$.

(v) The *closure* of E denoted \overline{E} is

$$\overline{E} := E \cup E'$$

Review. Fix (X, d) a metric space.

1. $N_\delta(x_0)$ is open $\forall \delta > 0, \forall x_0 \in X$.
2. E is open iff E^c is closed.
3. \overline{E} is closed, in fact \overline{E} is the smallest closed set containing E .
4. The union of any collection of open sets is open.
5. A finite intersection of a collection of open sets is open.
6. Intersection of any collection of closed sets is closed.
7. Finite union of closed is closed.
8. If $x \in E'$ then $N_\delta(x) \cap E'$ is an infinite set.

Definition 1.4. Let x_n be a sequence in X , we say that x_n converges to $x_0 \in X$ iff $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t

$$d(x_n, x_0) < \varepsilon \quad \forall n \geq N$$

1. The limit of a sequence (if exists) is unique.
2. x_n converges to x_0 in (X, d) iff $d(x_n, x_0)$ converges 0 in $(\mathbb{R}, |||)$.

Example 1.5. $x_n = (1/n + 1, e^{-n})$ in \mathbb{R}^2 . Find the limit of x_n in $(\mathbb{R}^2, \|\cdot\|_\infty)$, $(\mathbb{R}^2, \|\cdot\|_1)$, $(\mathbb{R}^2, \|\cdot\|_2)$.

•

$$\|(1/n + 1, e^{-n}), (1, 0)\|_\infty = \max(1/n, e^{-n}) \leq 1/n + e^{-n} \rightarrow 0$$

• $x_0 = (1, 0)$

$$\|(1/n + 1, e^{-n}), -(1, 0)\|_1 = \|(1/n, e^{-n})\|_1 = |1/n| + |e^{-n}| \rightarrow 0$$

•

$$\|(1/n + 1, e^{-n}) - (1, 0)\|_2 = \sqrt{1/n^2 + e^{-2n}} \rightarrow 0$$

Remark 1.6. $x_n \rightarrow x_0$ in $(\mathbb{R}^k, \|\cdot\|_\infty)$

$$(x_{1,n}, x_{2,n}, \dots, x_{k,n}) \rightarrow (x_1, 0, \dots, x_{k,0})$$

Iff

$$x_{1,n} \rightarrow x_{1,0}$$

$$x_{2,n} \rightarrow x_{2,0}$$

$$\vdots$$

$$x_{k,n} \rightarrow x_{k,0}$$

in $(\mathbb{R}, \|\cdot\|)$.

Proof. $|x_{i,n} - x_{i,0}| \leq \|x_n - x_0\|_\infty \leq |x_{1,n} - x_{1,0}| + \dots + |x_{k,n} - x_{k,0}|$ Therefore $x_n \rightarrow x_0$ in $(\mathbb{R}^k, \|\cdot\|_\infty)$
 $\iff \|x_n - x_0\|_\infty \rightarrow 0$ and $\iff |x_{i,n} - x_{i,0}| \rightarrow 0 \forall i$ and hence $\iff x_{i,n} \rightarrow x_{i,0} \forall i$. \square

Remark 1.7. Take $x \in \mathbb{R}^n$, then

$$\|x\|_\infty = \max_i(x_i) \leq |x_1| + \dots + |x_n| = \|x\|_1 \leq n\|x\|_\infty$$

Therefore

$$\|x\|_\infty \leq \|x\|_1 \leq n \cdot \|x\|_\infty$$

Similarly, show something for $\|x\|_2$ and $\|x\|_\infty$.

Definition 1.8. Given a vector space V and norms $\|\cdot\|_a$ and $\|\cdot\|_b$ we say that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent iff $\exists c_1, c_2 > 0 \in \mathbb{R}$ s.t $\forall x \in V$

$$c_1\|x\|_b \leq \|\cdot\|_a \leq c_2\|x\|_b$$

Lemma 1.9. (X, d) metric space and $E \subseteq X$, then E is closed iff for sequence $x_n \in E$ that converges $\lim_{n \rightarrow \infty} x_n \in E$.

Proof. \implies Assume that there exists a sequence x_n in E that converges to x_0 s.t $x_0 \notin E$, take $\varepsilon > 0$, then since $x_n \rightarrow x_0$ then $\exists N \in \mathbb{N}$ s.t

$$x_n \in N_\varepsilon(x_0) \quad \forall n \geq N$$

But $x_n \in E$,

$$N_\varepsilon^*(x_0) \cap E \neq \emptyset$$

Therefore

$$x_0 \in E'$$

Contradiction to the fact that E is closed.

- Take $x_0 \in E'$, let $\varepsilon_n = 1/n$ then

$$\exists x_n \in N_{\varepsilon_n}^*(x_0) \cap E$$

Consider $\{x_n\}_{n \in \mathbb{N}}$ x_n is a sequence in E s.t

$$d(x_n, x_0) < 1/n \implies x_n \rightarrow x_0$$

and hence $x_0 \in E$.

□

Theorem 1.10. Given a vector space V and norms $\|\cdot\|_a$ and $\|\cdot\|_b$ then the following are all equivalent:

1. $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent ($c_1\|x\|_b \leq \|x\|_a \leq c_2\|x\|_b$).
2. $x_n \rightarrow x_0$ in $\|\cdot\|_a \iff x_n \rightarrow x_0$ in $\|\cdot\|_b$.
3. E is closed in $(V, \|\cdot\|_a) \iff E$ is closed in $(V, \|\cdot\|_b)$.
4. U is open in $(V, \|\cdot\|_a) \iff U$ is open in $(V, \|\cdot\|_b)$.

Proof. $\implies 2$

$$c_1\|x_n - x_0\|_b \leq \|x_n - x_0\|_a \leq c_2\|x_n - x_0\|_b \leq \frac{c_2}{c_1}\|x_n - x_0\|_a$$

Apply Squeeze.

2 \implies 3 By the Lemma.

3 \implies 4 Trivial.

4 \implies 1 $B_{\|\cdot\|_a}(0, 1)$ open in $(V, \|\cdot\|_a)$ therefore it is open in $(V, \|\cdot\|_b)$. Hence 0 is an interior point in $(V, \|\cdot\|_b)$. Hence $\exists \delta > 0$

$$B_{\|\cdot\|_b}(0, \delta) \subseteq B_{\|\cdot\|_a}(0, 1)$$

Let $x \in V, x \neq 0$.

$$\frac{\delta}{2} \cdot \frac{x}{\|x\|_b} \in B_{\|\cdot\|_b}(0, \delta) \subseteq B_{\|\cdot\|_a}(0, 1)$$

Therefore

$$\left\| \frac{\delta}{2} \frac{x}{\|x\|_b} \right\|_a < 1$$

We get

$$\frac{\delta}{2\|x\|_b} \|x\|_a < 1 \implies \|x\|_a \leq \frac{2}{\delta} \|x\|_b \forall x \in V$$

□

1.1 Compact Sets

Definition 1.11. (X, d) metric space, $K \subseteq X$. We say that K is *compact* iff for every open cover

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

of K has a finite subcover.

Example 1.12. $(0, 1]$ is not compact in $(\mathbb{R}, |\cdot|)$, take

$$\mathcal{G} = \{(1/n, 10)\}_{n \in \mathbb{N}}$$

which has no finite subcover.

Theorem 1.13 (Heine-Borel). *In \mathbb{R} $[a, b]$ is compact.*

Example 1.14. If X is finite, then X is compact.

Proof. Let $X = \{x_1, \dots, x_n\}$ and let

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

be an open cover X . Then for every x_i , fix some G_{α_i} s.t

$$x = \{x_1, \dots, x_n\} \subseteq \bigcup_{i=1}^n G_{\alpha_i}$$

□

Proposition 1.15. (X, d) metric space and $K \subseteq X$ then every closed subset E of K is compact.

Proof. Let $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ be an open cover of E , then

$$E \subseteq \bigcup_{\alpha \in I} G_\alpha$$

Therefore

$$K \subseteq \bigcup G_\alpha \cup E^c, \quad E^c \text{ open}$$

Therefore

$$\mathcal{G}' = \mathcal{G} \cup \{E^c\} \quad \text{is an open cover of } K$$

Therefore it has a finite subcover and

$$K \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_k} \cup E^c$$

Hence

$$E \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_k}$$

Which is a finite subcover of \mathcal{G} .

□

Definition 1.16. (X, d) metric space and x_n sequence in X and

$$\varphi: \mathbb{N} \rightarrow \mathbb{N} \quad \text{strictly increasing}$$

We say that $b_n = a_{\varphi(n)}$ is a *subsequence* of a_n .

Proposition 1.17. x_n converges to $x_0 \iff$ every subsequence x_{n_k} converges to x_0 .

Definition 1.18. (X, d) metric space and $K \subseteq X$. We say that K is sequentially compact iff every sequence $x_n \in K$ has a convergent subsequence in K .

Proposition 1.19. (X, d) metric space, K sequentially compact iff every infinite $E \subseteq K$ has a limit point in K .

Proof. \implies Let K be sequentially compact, take $E \subseteq K$ an infinite set. We can extract from E a sequence of distinct elements $x_n \in E \subseteq K$. Therefore x_n has a convergence subsequence x_{n_k} s.t the limit of $x_{n_k} = x_0 \in K$. Moreover, x_{n_k} are distinct in E therefore $x_0 \in E' \cap K$.
 \impliedby Let x_n be a sequence in K , take

$$E = \{x_n\}_{n \in \mathbb{N}}$$

We have two cases:

- **E is finite.** Then $\exists x_0$ and $\exists n_1 < n_2 < \dots$ (infinitely many) s.t

$$x_{n_k} = x_0, \quad \forall k \in \mathbb{N}$$

Therefore

$$\lim_{k \rightarrow \infty} x_{n_k} = x_0 \in E$$

We found a converging subsequence.

- **E is infinite.** Then by the assumption, we have that E has a limit point x_0 in K .

$$\varepsilon = 1 \quad \exists n_1 \text{ s.t } x_{n_1} \in N_1^*(x_0) \cap E.$$

$$\varepsilon = 1/2 \quad \exists n_2 > n_1 \text{ s.t } x_{n_2} \in N_{1/2}^*(x_0) \cap E \text{ (since } N_{1/2}^*(x_0) \cap E \text{ is infinite).}$$

we can construct $n_k > n_{k-1}$ s.t $x_{n_k} \in N_{1/k}^*(x_0) \cap E$, therefore

$$\lim_{n \rightarrow \infty} x_{n_k} = x_0 \in K$$

□

Example 1.20. In $(\mathbb{Q}, |\cdot|)$, $K = \mathbb{Q} \cap [0, 1]$ is closed and bounded. P_n increasing sequence in $\mathbb{Q} \cap [0, 1]$ s.t

$$P_n \rightarrow 1/\sqrt{2} \in \mathbb{R}$$

Consider

$$\mathcal{G} = \left\{ (-1, P_1) \cap \mathbb{Q}, (-1, P_2) \cap \mathbb{Q}, \dots, (-1, P_n) \cap \mathbb{Q}, \dots, (1/\sqrt{2}, 10) \cap \mathbb{Q} \right\}$$

Proposition 1.21. $K \subseteq X$ is compact \iff K is sequentially compact.

Proof. • Let $E \subseteq K$ infinite, suppose that $E' \cap K = \emptyset$. Take $x \in K \implies x \notin E'$, therefore $\exists \delta_x > 0$ s.t

$$N_{\delta_x}^*(x) \cap E = \emptyset$$

Let $\mathcal{G} = \{N_{\delta_x}(x)\}_{x \in K}$, is an open cover of K compact, hence it has a finite subcover

$$K \subseteq N_{\delta_{x_1}}(x_1) \cup \cdots \cup N_{\delta_{x_k}}(x_k)$$

But $E \subset K \implies E = (E \cap K)$ hence

$$\begin{aligned} E &\subseteq (N_{\delta_{x_1}}(x_1) \cap E) \cup \cdots \cup (N_{\delta_{x_k}}(x_k) \cap E) \\ &\subseteq \{x_1, \dots, x_k\} \end{aligned}$$

therefore E is finite, contradiction. □

1.2 Complete sets

Definition 1.22. Given a sequence x_n , we say that x_n is Cauchy iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t

$$d(x_n, x_m) < \varepsilon \quad \text{for } n, m \geq N$$

Proposition 1.23. If x_n converges, then x_n is Cauchy.

Proof. Triangle inequality. □

Remark 1.24. The converse is false. In $(\mathbb{Q}, |\cdot|)$, in $(\mathbb{R}, |\cdot|)$ there exists some sequence $x_n \in \mathbb{Q}$ s.t

$$x_n \rightarrow \sqrt{2} \quad \text{as } n \rightarrow \infty$$

But $x_n \rightarrow \sqrt{2}$ implies that x_n is Cauchy in $(\mathbb{R}, |\cdot|)$, therefore x_n is Cauchy in \mathbb{Q} . Suppose x_n converges to P in $(\mathbb{Q}, |\cdot|)$, therefore $x_n \rightarrow p \in (\mathbb{R}, |\cdot|)$ by uniqueness of limit, $p = \sqrt{2}$, contradiction.

Definition 1.25. (X, d) metric space and $K \subseteq X$, we say that K is complete iff every Cauchy Sequence in K converges in K .

Example 1.26. $(\mathbb{R}, |\cdot|)$ is complete (Math 210)

Proposition 1.27. If K is complete then K is closed.

Proof. Let x_n be a sequence in K s.t $x_n \rightarrow x_0 \in X$. Since x_n converges, then x_n is Cauchy and K is complete then x_n converges in K , therefore $x_0 \in K$. □

Proposition 1.28. K complete and $E \subseteq K$ is closed, then E is complete.

Proof. Let x_n be a Cauchy sequence in E , then x_n is Cauchy in K which is complete, therefore $x_n \rightarrow x_0 \in K$. But since E is closed, we get that $x_0 \in E$. □

Proposition 1.29. (X, d) metric space and x_n Cauchy sequence in $K \subseteq X$. If x_n has a convergent subsequence x_{n_k} then x_n converges to the subsequential limit.

Proof. Triangle inequality. □

Proposition 1.30. (X, d) metric space and $K \subseteq X$ is sequentially compact then K is complete (and therefore closed).

Proof. Let $\{x_n\}$ be a Cauchy sequence in K , K is sequentially compact therefore x_n has a convergent subsequence x_{n_k} that converges in K . Therefore by the previous proposition we get that x_n converges in K . \square

Theorem 1.31. Let V be a finite dimensional inner product space over \mathbb{R} , then V is complete w.r.t norm induced by the inner product. Therefore $K \subseteq V$ is complete $\iff K$ is closed.

(For the converse $K \subseteq V$ which is closed, then K is complete).

Proof. Let $S = \{v_1, \dots, v_l\}$ be an orthonormal basis of V , take x_n be a cauchy sequence in V . Therefore

$$x_n = r_{1,n}v_1 + \dots + r_{l,n}v_l$$

Take $r_{1,n} = \langle x_n, v_1 \rangle$ therefore

$$\begin{aligned} |r_{1,m} - r_{1,n}| &= |\langle x_m, v_1 \rangle - \langle x_n, v_1 \rangle| \\ &= |\langle x_m - x_n, v_1 \rangle| \\ &\leq \|x_m - x_n\| \|v_1\| \quad \text{By C.S} \end{aligned}$$

Let $\varepsilon > 0$ then $\exists N \in \mathbb{N}$ s.t

$$\|x_m - x_n\| < \varepsilon \implies r_{1,n} \text{ is Cauchy in } \mathbb{R}$$

Therefore $r_{1,n}$ converges to $r_1 \in \mathbb{R}$. Do the same for every coordinate, and define

$$x = r_1v_1 + \dots + r_lv_l$$

Therefore

$$\begin{aligned} \|x_n - x\| &= \|(r_{1,n} - r_1)v_1 + \dots + (r_{l,n} - r_l)v_l\| \\ &\leq \|r_{1,n} - r_1\| \|v_1\| + \dots + \|r_{l,n} - r_l\| \|v_l\| \end{aligned}$$

Therefore

$$x_n \rightarrow x$$

\square

1.3 Total Boundedness

Definition 1.32. (X, d) metric space and $K \subseteq X$ we say that K is totally bounded iff for every $\varepsilon > 0 \exists x_1, \dots, x_t \in K$ s.t

$$K \subseteq N_\varepsilon(x_1) \cup \dots \cup N_\varepsilon(x_t)$$

Definition 1.33. $K \subseteq X$ is bounded iff $\exists R > 0$ s.t

$$K \subseteq N_R(x_0) \quad \text{for some } x_0 \in K$$

We say that a sequence is bounded iff

$$E = \{x_n\}_{n \in \mathbb{N}} \quad \text{is bounded}$$

Proposition 1.34. *Cauchy sequences are bounded.*

Proposition 1.35. $K \subseteq X$ is totally bounded it is bounded.

Note that the converse is false.

Example 1.36. $(\mathbb{R}, d_{disc}), \mathbb{R}$ is bounded since

$$\mathbb{R} \subseteq N_2(0)$$

but it isn't totally bounded, take $\varepsilon = 1/2$ then if it is totally bounded, then $\exists x_1 \cdots x_t \in \mathbb{R}$ s.t

$$\mathbb{R} \subseteq N_{1/2}(x_1) \cup \cdots \cup N_{1/2}(x_t) = \{x_1, \cdots, x_t\}$$

Proof. Take $\varepsilon = 1$ then $K \subseteq N_1(x_1) \cup N_1(x_t)$ for $x_1, \cdots, x_t \in K$. Take x_1 and let $d = \max_{k \in \{1, \dots, t\}}(x_1, x_t)$, therefore for any $x \in K$ we have that

$$d(x, x_1) \leq d(x, x_i) + d(x_i, x_1)$$

for some x_i s.t $x \in N_1(x_i)$ (Exists by the above), therefore

$$d(x, x_1) \leq 1 + d \implies x \in N_{1+d}(x_1)$$

□

Proposition 1.37. K is sequentially compact then K is totally bounded.

Proof. Suppose not, then $\exists \varepsilon_0$ s.t K cannot be covered by finitely many ε_0 -neighborhoods. Fix some x_1 then $\exists x_2 \in K \setminus N_{\varepsilon_0}(x_1)$, Similarly there exists some $x_3 \in K \setminus N_{\varepsilon_0}(x_1) \cup N_{\varepsilon_0}(x_2)$ and in general

$$\exists x_n \in K \setminus (N_{\varepsilon_0}(x_1) \cup \cdots \cup N_{\varepsilon_0}(x_{n-1}))$$

Therefore we have that for $n \neq m$

$$d(x_n, x_m) \geq \varepsilon_0$$

Therefore x_n doesn't have a convergent subsequence; contradiction. □

Lemma 1.38. *Every subset of a totally bounded set is totally bounded.*

Proof. $K \subseteq X$ totally bounded, and $E \subseteq K$; take $\varepsilon > 0$, therefore $\exists x_1, x_2, \cdots, x_t \in K$ s.t

$$K \subseteq N_{\varepsilon/2}(x_1) \cup \cdots \cup N_{\varepsilon/2}(x_t)$$

Ignore the neighborhood that don't intersect E , renumbering we have x_1, \dots, x_l s.t

$$N_{\varepsilon/2}(x_i) \cap E \neq \emptyset$$

We get

$$E \subseteq N_{\varepsilon/2}(x_1) \cup \dots \cup N_{\varepsilon/2}(x_l)$$

Take $y_i \in N_{\varepsilon/2}(x_i) \cap E$, then

$$N_{\varepsilon}(y_i) \supseteq N_{\varepsilon/2}(x_i)$$

Therefore $E \subseteq N_{\varepsilon}(y_1) \cup \dots \cup N_{\varepsilon}(y_l)$ and hence E is totally bounded. \square

Example 1.39. $[0, 1]$ is totally bounded (since it is compact), therefore $(0, 1)$ is totally bounded.

Proposition 1.40. (X, d) Metric space, $K \subseteq X$ is complete and totally bounded then K is compact.

Proof. Assume K is not compact, fix $\mathcal{G} = \{G_{\alpha}\}_{\alpha \in I}$ an open cover of K with no finite subcover. Let $\varepsilon = 1$, K is totally bounded then K can be covered by finitely many 1-balls with center in K . Hence $\exists x_1 \in K$ s.t $K_1 = N_1(x_1) \cap K$ cannot be covered by finitely many open sets in \mathcal{G} . But K_1 is totally bounded (subset of K), now let $\varepsilon = 1/2$, $\exists x_2 \in K_1$ s.t $K_2 = N_{1/2}(x_2) \cap K_1$ cannot be covered by finitely many open sets in \mathcal{G} . Similarly, we construct a sequence x_n and

$$K_n = N_{1/n}(x_n) \cap K_{n-1} \quad x_n \in K_{n-1}$$

such that K_n cannot be covered by finitely many G_{α} 's. Notice that

$$K_0 = K \supseteq K_1 \supseteq K_2 \supseteq \dots$$

Claim. x_n is Cauchy. Indeed, let $\varepsilon > 0$ then $\exists N$ s.t

$$1/N < \varepsilon/2$$

Now take $m, n > N$, then

$$x_n \in K_{n-1} \subseteq K_N, \quad x_m \in K_{m-1} \subseteq K_N$$

therefore $x_n, x_m \in K_N = N_{1/N}(x_N) \cap K_{N-1}$, and hence

$$d(x_n, x_m) \leq 2/N < \varepsilon$$

By completeness of K , we get that x_n converges to some $x_0 \in K$; since $\mathcal{G} = \{G_{\alpha}\}$ is an open cover of K , there exists $\alpha_0 \in I$ s.t $x_0 \in G_{\alpha_0}$ (open set). So $\exists \varepsilon_0$ s.t

$$N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

But $x_n \rightarrow x_0$ then $\exists N_1$ s.t

$$x_N \in N_{\varepsilon_0/2}(x_0) \text{ and } 1/N < \varepsilon_0/2$$

Therefore

$$K_N \subseteq N_{1/N}(x_n) \subseteq N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

contradiction. □

Theorem 1.41. *V finite dimensional product normed space (over \mathbb{R}). Let $S = \{v_1, \dots, v_n\}$ a basis of V . Then the cube*

$$Q_S = \{r_1v_1 + r_2v_2 + \dots + r_nv_n : 0 \leq r_i \leq 1\}$$

is compact.

Proof. Let $x^k = r_{1,k}v_1 + r_{2,k}v_2 + \dots + r_{n,k}v_n$ a sequence in Q_S , now clearly

$$r_{1,k} \text{ is a sequence in } [0, 1], \text{ compact set}$$

Therefore $r_{1,k}$ has a convergent subsequence $r_{1,k_{l_1}}$. Similarly $r_{2,k_{l_1}} \in [0, 1]$ therefore $r_{2,k_{l_1}}$ has a convergent subsequence $r_{2,k_{l_2}}$. Doing this n times we get that \exists a subsequence $\{k_j\}$ s.t

$$r_{1,k_j}, r_{2,k_j}, \dots, r_{n,k_j} \rightarrow r_1, r_2, \dots, r_n$$

take

$$x_0 = r_1v_1 + \dots + r_nv_n$$

We can show that

$$\|x_0 - x^{k_j}\| \leq |r_{1,k_j} - r_1|\|v_1\| + \dots + |r_{n,k_j} - r_n|\|v_n\| \rightarrow 0$$

then

$$x^{k_j} \rightarrow 0$$

Hence Q_S is (sequentially) compact. □

Theorem 1.42. *Let V be a finite inner product space then*

$$K \subseteq V \text{ is compact} \iff K \text{ closed and bounded}$$

Proof. • $K \text{ compact} \implies \text{complete} \implies \text{closed and} \implies \text{totally bounded} \implies \text{bounded}.$

- K is bounded, therefore $\exists R > 0$ s.t $K \subseteq B_{\|\cdot\|}(0, R)$. Let $S = \{v_1, \dots, v_n\}$ be a basis of V and Take $x \in K$ therefore

$$x = r_1v_1 + \dots + r_nv_n$$

$$\text{then } r_i = \|\langle x, v_i \rangle\| \leq \|x\| \cdot \|v_i\|$$

$$B_{\|\cdot\|}(0, R) \subseteq \{r_1v_1 + \dots + r_nv_n : r_i \leq R\}$$

Therefore $K \subseteq$ some cube and hence K is compact. □

Proposition 1.43. *In a finite dimensional inner product space we get that bounded \iff totally bounded.*

Proof. \implies Bounded \implies can be put in a cube (compact) \implies Totally bounded.

\impliedby True in general.

□

1.4 Continuity

Definition 1.44. X, Y metric spaces and $E \subseteq X$ and $f: E \rightarrow Y$ map; $x_0 \in E'$ we say that

$$\lim_{x \rightarrow x_0} f(x) = y \in Y$$

iff for every $\varepsilon > 0 \exists \delta > 0$ s.t

$$f(x) \in N_\varepsilon(y_0) \text{ for } x \in N_\delta^*(x_0) \cap E$$

Definition 1.45. $f: E \rightarrow Y$ continuous at x_0 iff for every $\varepsilon > 0, \exists \delta > 0$ s.t $f(x) \in N_\varepsilon(x_0)$ for $x \in N_\delta(x_0) \cap E$; i.e

$$d_Y(f(x), f(x_0)) < \varepsilon \text{ for } 0 < d_X(x, x_0) < \delta$$

Remark 1.46. f is continuous at $x_0 \iff x_0$ is an isolated point or $x_0 \in E'$ and $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

Proposition 1.47. $\lim_{x \rightarrow x_0} f(x) = y_0$ iff for every sequence $x_n \rightarrow x_0$ we have that $f(x_n) \rightarrow y_0$. Therefore f is continuous at x_0 iff for every $x_n \rightarrow x_0$ we have that $f(x_n) \rightarrow f(x_0)$.

Corollary 1.48. if f, g are continuous then $f + g, cf, f \cdot g, f/g$ are all continuous at x_0 .

Example 1.49. $f(x, y) = \frac{xy}{x^2 + y^2}$ we want to find

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

Take the sequence $(x_n, y_n) = (1/n, 0)$ then

$$\lim_{n \rightarrow \infty} (x_n, y_n) = (0, 0)$$

and

$$f(x_n, y_n) = 0 \rightarrow 0 \text{ as } n \rightarrow \infty$$

Now take $(x_n, y_n) = (1/n, 1/n)$, then

$$\lim_{n \rightarrow \infty} (x_n, y_n) = (0, 0)$$

but

$$f(x_n, y_n) = \frac{1/n \cdot 1/n}{1/n^2 + 1/n^2} = 1/2 \rightarrow 1/2 \text{ as } n \rightarrow \infty$$

Therefore

$$\lim_{n \rightarrow \infty} f(x, y) \text{ does not exist}$$

Now we do the same for

$$f(x, y) = \frac{x^2 y}{x^2 + y^2}$$

Notice that

$$0 \leq |f(x, y)| = \left| \frac{x^2}{x^2 + y^2} y \right| \leq |y|$$

but

$$\lim_{(x, y) \rightarrow (0, 0)} |y| = 0$$

therefore by squeeze we get that

$$\lim_{(x, y) \rightarrow 0} |f(x, y)| = 0$$

Hence

$$\lim_{(x, y) \rightarrow 0} f(x, y) = 0$$

Some useful inequalities

$$x^2 + y^2 \geq x^2$$

$$x^2 + y^2 \geq y^2$$

$$x^2 + y^2 \geq 2(xy)$$

Example 1.50. Let $(V, \|\cdot\|)$ a normed space and

$$\varphi: V \rightarrow \mathbb{R}, \quad \varphi(x) = \|x\|$$

Then φ is continuous.

Let x_n a sequence in V that converges to x_0 in $(V, \|\cdot\|)$ therefore

$$\|x_n - x_0\| \rightarrow 0$$

We want to show that $\varphi(x_n) \rightarrow \varphi(x_0)$ i.e that

$$|\|x_n\| - \|x_0\|| = 0$$

But as shown in PS1, we have that

$$|\|x_n\| - \|x_0\|| \leq \|x_n - x_0\|$$

Proposition 1.51. $f: X \rightarrow Y$ is continuous iff $f^{-1}(G)$ open in X for every G open in Y iff $f^{-1}(H)$ is closed in X for every H closed in Y .

Proof. \implies let x_n be a sequence in $f^{-1}(H)$ that converges to x_0 in X . Hence $f(x_n)$ is a sequence in H by continuity of f we have that

$$\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

Moreover since H is closed, we get that $f(x_0) \in H$. Hence

$$x_0 \in f^{-1}(H)$$

Therefore if f is continuous $f^{-1}(G)$ is open in X for every G open in Y .

\Leftarrow Now we show that if $f^{-1}(G)$ is open in X for all G open in Y then f is continuous. Fix $x_0 \in X$, let $\varepsilon > 0$ then consider the open set in Y $N_\varepsilon(f(x_0))$ hence $f^{-1}(N_\varepsilon(f(x_0)))$ is open in X . But $x_0 \in f^{-1}(N_\varepsilon(f(x_0)))$ hence $\exists \delta > 0$ s.t

$$f(N_\delta(x_0)) \subseteq N_\varepsilon(f(x_0))$$

□

Example 1.52. $E = \{x, y \in \mathbb{R}^2 : y \geq x\}$ is open in \mathbb{R}^2 , define the continuous function

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = y - x$$

Then

$$E = f^{-1}((0, \infty))$$

which is an open set.

Example 1.53. $f: X \rightarrow Y$ continuous and consider the c -level set with $c \in Y$

$$E = \{x \in X : f(x) = c\} = f^{-1}(\{c\})$$

is closed in X .

Therefore $E = \{(x, y) : x^2 + 3y^2 = 1\}$ is closed in \mathbb{R}^2 , S^n is closed.

Remark 1.54. V, W normed spaces and

$$\|\cdot\|_a, \|\cdot\|_b \text{ equivalent norms on } V$$

so $f: (V, \|\cdot\|_a) \rightarrow W$ is continuous iff $f: (V, \|\cdot\|_b) \rightarrow W$ is continuous.

Equivalently: V, W normed spaces and

$$\|\cdot\|_a, \|\cdot\|_b \text{ equivalent norms on } W$$

so $f: V \rightarrow (W, \|\cdot\|_a)$ is continuous iff $f: V \rightarrow (W, \|\cdot\|_b)$ is continuous.

Example 1.55. Take $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ defined as follows

$$f(x) = (f_1(x), \dots, f_n(x)) \quad x = (x_1, \dots, x_m)$$

Then f is continuous iff $f_i: \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous for $1 \leq i \leq n$.

Theorem 1.56 (Extreme Value Theorem). X, Y metric spaces and $K \subseteq X$ compact and $f: K \rightarrow Y$ continuous. Then $f(K)$ is compact.

Corollary 1.57. *If $Y = \mathbb{R}$, then the above implies that f attains its maximum and minimum in K .*

Proof. Let y_n be a sequence in $f(K)$, therefore $\exists x_n \in K$ s.t $f(x_n) = y_n$. Since K is compact, x_n has a convergent subsequence x_{n_k} that converges to $x_0 \in K$, using continuity we get that

$$y_{n_k} \rightarrow f(x_0) \in f(K)$$

□

Theorem 1.58. *All norms in a finite dimensional vector space are equivalent.*

Proof. V is a finite dimensional vector space, then by PS1 we can define an inner product on V . Let $\|\cdot\|$ be the norm induced by our choice of inner product. Let $S = \{v_1, \dots, v_n\}$ an orthonormal basis of V . Let $\|\cdot\|_a$ be an arbitrary norm on V , we will show that $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent. Let $x \in V$, then

$$x = r_1 v_1 + \dots + r_n v_n, \quad |r_i| = \langle x, v_i \rangle \leq \|x\|$$

Therefore $\forall x \in V$

$$\begin{aligned} \|x\|_a &= \|r_1 v_1 + \dots + r_n v_n\| \\ &\leq |r_1| \|v_1\|_a + \dots + |r_n| \|v_n\|_a \\ &\leq (\|v_1\|_a + \dots + \|v_n\|_a) \cdot \|x\| \\ &\leq c_1 \cdot \|x\| \end{aligned}$$

Now define the function $\varphi: (V, \|\cdot\|) \rightarrow \mathbb{R}$ s.t

$$\varphi(x) = \|x\|_a$$

We claim that φ is continuous, indeed let $x_n \in V$ s.t $x_n \rightarrow x_0$ in $\|\cdot\|$. Therefore

$$\begin{aligned} |\varphi(x_n) - \varphi(x_0)| &= |\|x_n\|_a - \|x_0\|_a| \\ &\leq \|x_n - x_0\| \\ &\leq c_1 \|x_n - x_0\| \rightarrow 0 \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \varphi(x_n) = \varphi(x_0)$, so φ is continuous. Let $K = \{x \in V: \|x\| = 1\}$ closed and bounded in the inner product space $(V, \|\cdot\|)$. Then it is compact in $(V, \|\cdot\|)$ therefore $\varphi(K)$ is compact in \mathbb{R} so it attains its minimum in K . Notice that $\varphi(x) = 0 \iff x = 0$, hence $\forall x \in K, \varphi(x) > 0$. Then

$$\min_{x \in K} \varphi(x) > 0$$

Let $x \in V$ and $x \neq 0$ then $\frac{x}{\|x\|} \in K$, therefore

$$\varphi\left(\frac{x}{\|x\|}\right) = \left\|\frac{x}{\|x\|}\right\|_a \geq \min \varphi(K)$$

Therefore

$$\|x\|_a \geq \min(\varphi(K))\|x\|$$

□

Corollary 1.59. *If V is finite dimensional then compact iff closed and bounded (independently of the choice of norm).*

1.5 Connected sets

Definition 1.60. (X, d) metric space and $\Omega \subseteq X$, we say that Ω is disconnected iff $\exists U, V \subseteq X$ open sets s.t

$$\Omega = (U \cup V) \cap \Omega$$

With $U \cap \Omega, V \cap \Omega \neq \emptyset$ and $U \cap \Omega \cap V = \emptyset$.

Otherwise Ω is connected.

Example 1.61. Let $\Omega = (1, 2] \cup (3, 4)$ disconnected, since

$$\Omega \subseteq (0, 2.5) \cup (2.5, 17)$$

\mathbb{Q} is disconnected in \mathbb{R} since

$$\mathbb{Q} \subseteq (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$$

Proposition 1.62. X is a connected metric space iff the only sets that are both open and closed are \emptyset, X .

Proof. \implies Assume that U is open and closed and $U \notin \{\emptyset, X\}$ then U^c is also not open and closed and not in $\{\emptyset, X\}$ but

$$X = U \cup U^c$$

Therefore X is disconnected.

\Leftarrow Assume that X is disconnected, then $X = U \cup V$ with U, V open disjoint and nonempty. Therefore $V = U^c$ and U is open then V is closed. Hence V is a clopen set of X but $V \notin \{\emptyset, X\}$ contradiction.

□

Definition 1.63. $I \subseteq \mathbb{R}$ is an interval for $x, y \in I$ s.t $x < y$ then $z \in I, \forall x < z < y$.

Theorem 1.64. $\Omega \subseteq \mathbb{R}$ is connected iff Ω is an interval.

Proof. \implies Assume that Ω is not an interval, therefore $\exists x, y \in \Omega$ s.t $x < z < y$ and $z \notin \Omega$, therefore

$$\Omega \subseteq (-\infty, z) \cup (z, \infty)$$

Therefore Ω is disconnected.

\Leftarrow Let Ω be an interval. Suppose Ω is not connected, then $\exists U, V$ open and disjoint in Ω in \mathbb{R} s.t

$$\Omega \subseteq U \cup V$$

Then $\exists x \in U \cap \Omega$ and $y \in V \cap \Omega$ and assume wlog that $x < y$; therefore $[x, y] \subseteq \Omega$ consider

$$z = \sup \{[x, y] \cup U\}$$

Note that $z \in [x, y]$ therefore $z \in \Omega$ therefore either $z \in U$ or $z \in V$. Notice that $z > x$ since $x \in U$ an open set. Also notice that $z < y$ since if $z = y \in V$ we have that $\exists \varepsilon_1$ s.t

$$(y - \varepsilon, y) = (z - \varepsilon, z) \subseteq [x, y] \cap V$$

since z is the sup we get that

$$(z - \varepsilon, z) \cap [x, y] \cap U \neq \emptyset$$

contradiction.

Therefore $z \in (x, y)$, suppose that $z \in U$ then $(z, z + \varepsilon_0) \subseteq (x, y) \cap U$ contradiction because $z + \varepsilon_0/2 > z = \sup[x, y] \cap y$. Suppose that $z \in V$ then $\exists \varepsilon_1$ s.t $(z - \varepsilon_1, z) \subseteq (x, y) \cap V$. Since z is the sup then $\exists z^*$ an element s.t

$$z - \varepsilon_1 < z^* < z \text{ and } z^* \in U$$

$z^* \in U \cap V$ contradiction. □

Theorem 1.65 (Intermediate Value Theorem). $f: X \rightarrow Y$ continuous, if X is connected then $f(X)$ is connected. In particular if $Y = \mathbb{R}$ we have the following:

- If $y_0 < y_1 \in f(X)$ then $\forall \alpha$ s.t $y_0 < \alpha < y_1$ $f^{-1}(\{\alpha\}) \neq \emptyset$.

Proof. Suppose that $f(X)$ is disconnected, then $\exists U, V \subseteq Y$ open sets that separate $f(X)$. Therefore

$$\begin{aligned} X &= f^{-1}(f(X)) \\ &= f^{-1}(U \cup V) \cap f(X) \\ &= [f^{-1}(U) \cup f^{-1}(V)] \cap X \end{aligned}$$

But f is continuous therefore $f^{-1}(U)$, $f^{-1}(V)$ are open and disjoint. Now $U \cap f(X) \neq \emptyset$ therefore $f^{-1}(U) \neq \emptyset$, same for V ; then X is disconnected. □

Definition 1.66. X is path connected iff for every $a, b \in X$ there exists a continuous function

$$\alpha: [0, 1] \rightarrow X$$

s.t $\alpha(0) = a$, $\alpha(1) = b$.

Remark 1.67. Let X be a vector space, the case where for every $a, b \in X$ the line segment $(1 - t)a + tb \in X \forall t \in [0, 1]$ we say that X is convex.

Example 1.68. $B_R(x_0)$ is always convex and hence path connected in a vector space. Indeed, take $a, b \in B_R(x_0)$ consider

$$\begin{aligned}\|(1-t)a + tb - x_0\| &< (1-t)\|a - x_0\| + t\|b - x_0\| \\ &< (1-t)R + R = R\end{aligned}$$

Proposition 1.69. (X, d) metric space then X is path connected $\implies X$ is connected.

Proof. Let X be disconnected then we have U, V nonempty disjoint and open

$$X = U \cup V$$

Then $\exists a \in U$ and $b \in V$, since X is path connected $\exists \alpha: [0, 1] \rightarrow X$ s.t

$$\alpha(0) = a, \quad \alpha(1) = b$$

$[0, 1]$ is connected, therefore $\alpha([0, 1])$ is also connected but

$$[0, 1] = \alpha^{-1}(U) \cup \alpha^{-1}(V)$$

But α is continuous, therefore $\alpha^{-1}(U) \ni 0$ and $\alpha^{-1}(V) \ni 1$ which is a contradiction since $[0, 1]$ is connected. \square

Theorem 1.70. $(V, \|\cdot\|)$ vector space and $\Omega \subseteq V$ open then Ω is connected iff Ω is path connected.

Proof. \Leftarrow True in general.

\implies We will show that A is closed by showing that A is open. Take $z \in A^c$ so there is no path joining z to x_0 . But Ω is open, so $\exists \delta > 0$ s.t

$$B_\delta(z) \subseteq \Omega$$

Now every point in $B_\delta(z)$ cannot be joined to x_0 ; Hence

$$B_\delta(z) \subseteq A^c$$

so A^c is open.

Therefore A open, closed and nonempty in Ω a connected space, hence $A = \Omega$. \square

Example 1.71 (Topological sine). $\Omega = \{(x, \sin(1/x)) \mid x > 0\} \cup \{(0, 0)\}$; let

$$A = \{(x, \sin 1/x) : x > 0\}$$

then A is connected (image of a continuous function), with

$$\overline{A} = A \cup (\{0\} \times [-1, 1])$$

Therefore $A \subseteq \Omega \subseteq \overline{A}$.

Suppose that Ω was connected, take $x_0 > 0$, suppose that $\exists \alpha: [0, 1] \rightarrow \Omega$ from $(0, 0)$ to $(x_0, \sin 1/x_0)$. let $t^* = \sup \{t \in [0, 1] \mid x(t) = 0\}$; therefore

$$x(t^*) = 0$$

with $x(t^*) < 1$, hence

$$\alpha(t^*) = (0, 0)$$

Take $\varepsilon = 1$, then by the IVT $\exists t^* < t_1 < t^* + 1$ and $t_1 \in [0, 1]$ s.t

$$x(t_1) = \frac{1}{(4n_1 + 1)\pi/2}$$

Take $\varepsilon = 1/2$, then $\exists t_2 \in [0, 1]$ s.t $t^* < t_2 < t^* + 1/2$ s.t

$$x(t_2) = \frac{1}{(4n_2 + 1)\pi/2}$$

We take a sequence $t_k \in [0, 1]$ s.t $t^* < t_k < t^* + 1/k$

$$x(t_k) = \frac{1}{(4n_k + 1)\pi/2}$$

hence

$$\alpha(t_k) = \left(\frac{1}{(4n_k + 1)\pi/2}, 1\right)$$

With $t_k \rightarrow t^*$, therefore

$$\alpha(t^*) = (0, 1)$$

contradicting the continuity of α .

2 Differentiability

Review linear transformations.

Proposition 2.1. *Let V and W normed spaces $T: V \rightarrow W$ linear transformation, then the following are all equivalent:*

- T is continuous on V .
- T is continuous on 0_V .
- $\exists M > 0$ s.t

$$\|Tx\|_W \leq M\|x\|_V$$

Proof. \implies 2 Trivial.

2 \implies 3 T is continuous at 0, hence for $\varepsilon = 1 \exists \delta > 0$ s.t

$$\|Tx - T0\|_W < 1 \quad \forall \|x - 0\|_V < \delta$$

Therefore

$$\|Tx\|_W < 1 \quad \forall \|x\|_V < \delta$$

Take any $x \neq 0$ then

$$\left\| \frac{x}{\|x\|_V} \cdot \delta/2 \right\|_V = \delta/2$$

Therefore

$$\left\| T \left(\frac{x}{\|x\|_V} \cdot \delta/2 \right) \right\|_W < 1$$

therefore

$$\|Tx\|_W < \frac{2}{\delta} \|x\|_V$$

3 \implies 1

$$\|Tx - Ty\|_W = \|T(x - y)\|_W \leq M \|x - y\|_V$$

□

Example 2.2 (The derivative as a linear transformation). Take $C^1([0, 1])$ the space of continuously differentiable functions, with the sup norm

$$\|f\|_\infty = \sup |f(x)| : x \in [0, 1]$$

and take $C([0, 1])$ to be the space of continuous functions with the sup norm. Take

$$\frac{d}{dx} : C^1([0, 1]) \rightarrow C([0, 1])$$

Note that this is a linear transformation.

Is it continuous? If it was then $\exists M$ s.t

$$\|f\|_\infty \leq M \|f\|_\infty \quad \forall f$$

take $f_n(x) = x^n$, then

$$\|f_n\|_\infty = 1$$

but $f'_n(x) = nx^{n-1}$ with

$$\|f'_n\|_\infty = n$$

Contradiction to the Archimedean property.

Theorem 2.3. Let V be finite dimensional, and $T : V \rightarrow W$ linear transformation then T is continuous.

Proof. Let $\alpha = \{v_1, \dots, v_n\}$ be a basis of V . Let $x \in V$, then $\exists r_1, \dots, r_n$ s.t

$$x = r_1 v_1 + \dots + r_n v_n$$

Since all norms are equivalent in a finite dimensional vector space, wlog work with the inner product norm and make α an orthonormal basis.

$$\begin{aligned} \|Tx\|_W &= \|r_1 T v_1 + \dots + r_n T v_n\|_W \\ &\leq |r_1| \|T v_1\|_W + \dots + |r_n| \|T v_n\|_W \end{aligned}$$

With

$$|r_i| = |\langle x, v_i \rangle| \leq \|x\|$$

Therefore

$$\|Tx\|_W \leq \|x\|_V (\|Tv_1\| + \cdots + \|TV_n\|) = M\|x\|$$

□

Motivation. In one dimension, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}$

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

if it exists we say that f is differentiable at x_0 and we define

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

In other words, we can say that

$$\begin{aligned} L = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} &\iff \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - L \right) = 0 \\ &\iff \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0) - L(x - x_0)}{x - x_0} \right) = 0 \\ &\iff \exists L \in \mathbb{R} \text{ s.t } f(x) = f(x_0) + L(x - x_0) + o(|x - x_0|) \text{ with } \frac{o(t)}{|t|} \rightarrow 0 \\ &\iff \exists L \in \mathbb{R} \text{ s.t } f(x) = f(x_0) + L(x - x_0) + o(|x - x_0|) \\ &\quad \text{with } \frac{o(|x - x_0|)}{|x - x_0|} \rightarrow 0 \text{ as } x \rightarrow x_0 \end{aligned}$$

Definition 2.4. V, W normed spaces and $U \subseteq V$ open set with $x_0 \in U$ we say that f is differentiable at x_0 iff there exists a continuous linear transformation $T \in \mathcal{L}(V, W)$ s.t

$$f(x) = f(x_0) + T(x - x_0) + o(\|x - x_0\|_V)$$

or

$$f(x_0 + h) = f(x_0) + T(h) + o(h)$$

with $\frac{o(\|h\|_V)}{\|h\|_V} \rightarrow 0$ as $h \rightarrow 0$.

Iff $\exists T \in \mathcal{L}(V, W)$ s.t

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0) - T(h)_W}{\|h\|_V} = 0$$

Remark 2.5. If such a T exists, then it is unique.

Proof. Suppose that $\exists T_1, T_2$ continuous linear transformations in $\mathcal{L}(V, W)$ s.t

$$f(x_0 + h) = f(x_0) + T_1(h) + o_1(h), \quad f(x_0 + h) = f(x_0) + T_2(h) + o_2(h)$$

Therefore

$$(T_1 - T_2)(h) = (o_1 - o_2)(h) = o_3(h)$$

take v s.t $\|v\| = 1$, therefore

$$\begin{aligned}(T_2 - T_1)(\lambda v) &= \lambda o_3(\|\lambda v\|) \\ \lambda(T_2 - T_1)(v) &= \lambda o_3(|\lambda|) \\ (T_2 - T_1)(v) &= \frac{o_3(|\lambda|)}{\lambda}\end{aligned}$$

Therefore as $\lambda \rightarrow 0$ we get that

$$(T_2 - T_1)(v) = 0 \quad \forall v \text{ s.t } \|v\| = 1$$

Therefore $T_2 = T_1$ □

Definition 2.6. If f is differentiable at x_0 , then we let $Df(x_0)$ to be the linear transformation in $\mathcal{L}(V, W)$ s.t

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o(\|x - x_0\|) \iff \lim_{x \rightarrow x_0} \frac{\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|_W}{\|x - x_0\|_V}$$

Notation. $\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_i \in \mathbb{R}\}$ as rows vectors, observe $f: \mathbb{R}^{1 \times n} \rightarrow \mathbb{R}^{m \times 1}$ differentiable at x_0 then we have that

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o(\|x - x_0\|)$$

Then

$$Df(x_0) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

Then $Df(x_0)$ can be represented by a matrix called the *Jacobian matrix* $[Df(x_0)]$ (an $m \times n$ matrix). Therefore

$$Df(x_0)(x - x_0) = [Df(x_0)](x - x_0)^t$$

Note that we take a row vector and output a column vector.

Example 2.7. In 1 variable we have that

$$[Df(x_0)] = [f'(x_0)]$$

Then

$$Df(x_0): t \rightarrow [f'(x_0)] \cdot t = f'(x_0) \cdot t$$

Find the derivative

$$f(x, y) = x^2 + y^2$$

we get that

$$\begin{aligned}
 f((x, y) + (h_1, h_2)) &= f(x_0 + h_1, y_0 + h_2) \\
 &= (x_0 + h_1)^2 + (y_0 + h_2)^2 \\
 &= x_0^2 + y_0^2 + 2y_0h_1 + 2y_0h_2 + h_1^2 + h_2^2 \\
 &= f(x_0, y_0) + (2x_0, 2y_0)(h_1, h_2) + o(|h|)
 \end{aligned}$$

since $\frac{h_1^2 + h_2^2}{\sqrt{h_1^2 + h_2^2}} \rightarrow 0$ so we get

$$Df(x_0, y_0): (v_1, v_2) \rightarrow (2x_0, 2y_0) \cdot (v_1, v_2)$$

therefore we get

$$[Df(x_0)] = \begin{bmatrix} 2x_0 & 2y_0 \end{bmatrix}$$

Example 2.8. Take $T: V \rightarrow W$ be a continuous linear transformation we want to find $DT(x)$; but

$$T(x_0 + h) = T(x_0) + T(h) + 0$$

therefore

$$DT(x_0) = T$$

Proposition 2.9. $f: V \rightarrow W$ if f is differentiable at x_0 therefore f is continuous at x_0 .

Proof. Let $f(x) = f(x_0) + Df(x_0)(h) + o(\|x - x_0\|)$ but notice that

$$o(\|x - x_0\|) \rightarrow 0$$

but $Df(x_0)$ is continuous hence $\exists M$ s.t

$$\|Df(x_0)(x - x_0)\| \leq M\|x - x_0\| \rightarrow 0 \implies \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

therefore f is continuous. □

2.1 Partial derivatives

Definition 2.10. $U \subseteq \mathbb{R}^n$ open and $f: U \rightarrow \mathbb{R}$, then

$$\frac{\partial f}{\partial x_i}(x) = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(x + he_i) - f(x)}{h}$$

We call this the partial derivative w.r.t x_i .

Example 2.11.

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We have shown that this is not continuous at $(0, 0)$ hence f is not differentiable at $(0, 0)$. But notice that

$$\frac{f(h, 0) - f(0, 0)}{h} = 0 \rightarrow 0 \text{ as } h \rightarrow 0$$

therefore $\frac{\partial f}{\partial x}(0, 0)$ exists and equal to 0. Similarly we get that $\frac{\partial f}{\partial y}(0, 0)$ exists and equal to 0.

What if $f: U \rightarrow \mathbb{R}$ is differentiable at x_0 then $Df(x_0) \in \mathcal{L}(R^n, \mathbb{R})$ hence $[Df(x_0)]$ is a $1 \times n$ matrix.

$$f(x_0 + he_1) = f(x_0) + Df(x_0)(he_1) + o(|h|)$$

therefore

$$\frac{f(x_0 + he_1) - f(x_0)}{h} = Df(x_0)e_1 + \frac{o(h)}{h} \rightarrow Df(x_0)e_1$$

therefore $\frac{\partial f}{\partial x_1}$ exists and is the first entry in $[Df(x_0)]$. Similarly we get $\frac{\partial f}{\partial x_i}$ exists and is the i 'th entry in $[Df(x_0)]$. We get that

$$[Df(x_0)] = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

this is called the gradient of f denoted ∇f .

If f is differentiable then

$$\begin{aligned} f(x) &= f(x_0) + \nabla f(x_0)(x - x_0)^t + o(|x - x_0|) \\ &= f(x_0) + \nabla f(x_0) \cdot (x - x_0) + o(|x - x_0|) \end{aligned}$$

Theorem 2.12. $U \subseteq \mathbb{R}^n$ open, and $f: U \rightarrow \mathbb{R}$ is differentiable iff all partial derivatives exist and

$$\lim_{h \rightarrow 0} \frac{\|f(x_0 + h) - f(x_0) - \nabla f(x_0) \cdot h\|}{|h|} = 0$$

Example 2.13. $f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{o.w} \end{cases}$ then

$$f_x(0, 0) = 0, \quad f_y(0, 0) = 0$$

and

$$\frac{f(h_1, h_2) - f(0, 0) - \nabla f(0, 0) \cdot (h_1, h_2)}{\sqrt{h_1^2 + h_2^2}} = \frac{h_1^2 h_2}{(h_1^2 + h_2^2) \sqrt{h_1^2 + h_2^2}}$$

take $h_1 = 1/n, h_2 = 1/n$ we get that

$$\frac{1}{1\sqrt{2}} \not\rightarrow 0$$

hence this is not differentiable.

Example 2.14. $f(x, y) = x \cdot \sqrt{y}$ for $x \in \mathbb{R}, y \geq 0$ then

$$\frac{\partial f}{\partial x} = \sqrt{y} \quad x \in \mathbb{R}, y \geq 0$$

$$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y}} \quad x \in \mathbb{R}, y > 0$$

But what happens to $\frac{\partial f}{\partial y}$ at $(x, 0)$?

$$\begin{aligned} \frac{\partial f}{\partial y}(x, 0) &= \lim_{h \rightarrow 0} \frac{f(x, h) - f(x, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x\sqrt{h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{x}{\sqrt{h}} \end{aligned}$$

If $x \neq 0$ we have that

$$\lim_{h \rightarrow 0} \frac{x}{\sqrt{h}} \text{ doesn't exist}$$

hence $\frac{\partial f}{\partial y}$ doesn't exist at $(x, 0)$ for $x \neq 0$. But for $x = 0$ we have that

$$\frac{\partial f}{\partial y}(0, 0) = 0$$

Now we ask, is $f(x, y) = x\sqrt{y}$ differentiable at $(0, 0)$. Indeed

$$\nabla f(0, 0) = (f_x(0, 0), f_y(0, 0)) = (0, 0)$$

and

$$\begin{aligned} \frac{|f((0, 0) + (h_1, h_2)) - f(0, 0) - (0, 0) \cdot (h_1, h_2)|}{\|h\|} &= \frac{|h_1\sqrt{h_2}|}{\sqrt{h_1^2 + h_2^2}} \\ &= \sqrt{\frac{h_1^2|h_2|}{h_1^2 + h_2^2}} \\ &\leq \sqrt{|h_2|} \rightarrow 0 \end{aligned}$$

therefore f is differentiable at $(0, 0)$.

Theorem 2.15. $U \subseteq \mathbb{R}^n$ and $f: U \rightarrow \mathbb{R}^m$ with $f = (f_1, \dots, f_m)$ then f is differentiable iff $f_1, \dots, f_m: U \rightarrow \mathbb{R}$ are all differentiable. Then

$$[Df(x_0)]_{m \times n} = \begin{pmatrix} \nabla f_1(x_0) \\ \nabla f_2(x_0) \\ \vdots \\ \nabla f_m(x_0) \end{pmatrix}$$

Therefore

$$Df(x_0) \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \nabla f_1(x_0) \cdot h \\ \nabla f_2(x_0) \cdot h \\ \vdots \\ \nabla f_m(x_0) \cdot h \end{pmatrix}$$

Example 2.16. $f(x, y) = (x^2y^2, x^2 + y^2, x^2 - y^2)$ then

$$[Df(x, y)] = \begin{pmatrix} 2xy^2 & 2x^2y \\ 2x & 2y \\ 2x & -2y \end{pmatrix}$$

Therefore

$$[Df(1, 2)] = \begin{pmatrix} 2 & 4 \\ 2 & 4 \\ 2 & -4 \end{pmatrix}$$

Hence

$$Df(1, 2): \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \rightarrow \begin{pmatrix} 8h_1 + 4h_2 \\ 2h_1 + 4h_2 \\ 2h_1 - 4h_2 \end{pmatrix}$$

Proof. • Suppose that f is differentiable, then

$$f(x_0 + h) = f(x_0) + Df(x_0)h + o(\|h\|)$$

Taking the dot with $e_i \in \mathbb{R}^m$ we get

$$f_i(x_0 + h) = f_i(x_0) + Df(x_0)(h) \cdot e_i + o(\|h\|)$$

Therefore f_i is differentiable with

$$Df_i(x_0): h \rightarrow (Df(x_0)(h)) \cdot e_i$$

• Suppose that each f_i is differentiable then

$$f_i(x_0 + h) = f_i(x_0) + Df_i(x_0)h + o_i(\|h\|)$$

Therefore

$$f(x_0 + h) = f(x_0) + (Df_1(x_0)h, \dots, Df_m(x_0)h) + (o_1(\|h\|), \dots, o_m(\|h\|))$$

Hence f is differentiable with

$$Df(x_0) = \begin{pmatrix} Df_1(x_0) \\ Df_2(x_0) \\ \vdots \\ Df_m(x_0) \end{pmatrix}$$

but $Df_i(x_0) = \nabla f_i(x_0)$. i.e

$$Df(x_0): h \rightarrow (\nabla f_i(x_0) \cdot h)_{1 \leq i \leq m}$$

□

Remark 2.17. f, g differentiable, then $f + \alpha g$ is differentiable, and

$$D(f + \alpha g) = Df + \alpha Dg$$

2.2 Chain Rule.

Theorem 2.18. Let $U \subseteq \mathbb{R}^n$ and $f: U \rightarrow \mathbb{R}^m$ and $V \subset \mathbb{R}^m$ open and $g: V \rightarrow \mathbb{R}$, take $x_0 \in U$ s.t $f(x_0) \in V$.

If f is differentiable at x_0 , and g is differentiable at $f(x_0)$. Then $h := g \circ f$ is differentiable at x_0 with

$$Dh(x_0) = Dg(f(x_0)) \circ Df(x_0)$$

In matrix form

$$[Dh(x_0)] = [Dg(f(x_0))][Df(x_0)]$$

Example 2.19. $h: \mathbb{R}^m \rightarrow \mathbb{R}$ with

$$h(u, v) = g(x(u, v), y(u, v))$$

Let $f(u, v) = (x(u, v), y(u, v))$ Therefore

$$[Dh(u, v)] = [Dg(x(u, v), y(u, v))][Df(u, v)]$$

$$\begin{pmatrix} \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Therefore we get that

$$\begin{pmatrix} \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial u} & \frac{\partial g}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial v} \end{pmatrix}$$

Example 2.20. $\gamma: [a, b] \rightarrow \mathbb{R}^n$ and $f: \mathbb{R}^n \rightarrow \mathbb{R}$ with

$$h(t) = f(\gamma(t))$$

Then

$$[Dh(t)] = [Df(\gamma(t))][D\gamma(t)]$$

$$\begin{aligned} [h'(t)] &= (\nabla f(\gamma(t))) \begin{pmatrix} \gamma'_1 \\ \gamma'_2 \\ \vdots \\ \gamma'_n \end{pmatrix} \\ &= \nabla f(\gamma(t)) \cdot \gamma'(t) \end{aligned}$$

i.e $h: [a, b] \rightarrow \mathbb{R}$.

Consequences.

1. Mean value theorem for $\mathbb{R}^n \rightarrow \mathbb{R}$

2. $Df(x) = 0 \implies f$ is constant (in an open and connected domain).
3. If $\frac{\partial f_i}{\partial x_i}$ exist and are continuous at x_0 then f is differentiable at x_0 .