Advanced Calculus

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Contents

1 Metric Spaces

Definition 1.1. Let X be a set, a *metric* d is a map

$$d\colon X \times X \to \mathbb{R}$$

such that:

(i) $d(x, y) \geq 0$ with equality iff $x = y$.

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

Example 1.2. Given a vector space V and norm $\|.\|$ we can define the metric

$$d(x, y) = \|x - y\|$$

Question. Given a vector space V and a metric d on V, does d "come from" a norm? i.e does there exist some norm $\|\|$ s.t

$$d(x, y) = \|x - y\|$$

The answer in general is no (consider the discrete metric, or even any bounded metric: this contradicts homogeneity).

Notation. Take (X, d) metric space $\delta > 0$ and $x_0 \in X$, the *δ-neighborhood* of x_0

$$N_\delta(x_0) := \{x \in X : d(x, x_0) < \delta\}$$

Definition 1.3. (X, d) metric space and $E \subset X$

(i) $x_0 \in X$ is an *interior point* of E iff $\exists \delta > 0$ s.t

$$N_\delta(x_0) \subset E$$

(ii) $x_0 \in X$ is a *limit point* iff $\forall \delta > 0$

$$N_\delta^*(x_0) \cap E \neq \varnothing$$

(We denote by E' the set of limit points of E)

(iii) E is *open* iff every $x_0 \in E$ is an interior point of E.

(iv) E is *closed* iff every $E' \subset E$.

(v) The *closure* of E denoted $\overline{E}$ is

$$\overline{E} := E \cup E'$$

Review. Fix (X, d) a metric space.

1. $N_\delta(x_0)$ is open $\forall \delta > 0,\ \forall x_0 \in X$.
2. E is open iff E^c is closed.
3. $\overline{E}$ is closed, in fact $\overline{E}$ is the smallest closed set containing E.
4. The union of any collection of open sets is open.
5. A finite intersection of a collection of open sets is open.
6. Intersection of any collection of closed sets is closed.
7. Finite union of closed is closed.
8. If $x \in E'$ then $N_\delta(x) \cap E'$ is an infinite set.

Definition 1.4. Let x_n be a sequence in X, we say that x_n converges to $x_0 \in X$ iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t

$$d(x_n, x_0) < \varepsilon \quad \forall n \geq N$$

1. The limit of a sequence (if exists) is unique.
2. x_n converges to x_0 in (X, d) iff $d(x_n, x_0)$ converges 0 in $(\mathbb{R}, \|\|)$.

Example 1.5. $x_n = (1/n + 1, e^{-n})$ in $\mathbb{R}^2$. Find the limit of x_n in $(\mathbb{R}^2, \|.\|_\infty)$, $(\mathbb{R}^2, \|.\|_1)$, $(\mathbb{R}^2, \|.\|_2)$.

- $$\|(1/n + 1, e^{-n}, (1, 0))\|_\infty = \max(1/n, e^{-n}) \leq 1/n + e^{-n} \to 0$$

- $x_0 = (1, 0)$

$$\|(1/n + 1, e^{-n}, -(1, 0))\|_1 = \|(1/n, e^{-n})\|_1 = |1/n| + |e^{-n}| \to 0$$

- $$\|(1/n + 1, e^{-n}) - (1, 0)\|_2 = \sqrt{1/n^2 + e^{-2n}} \to 0$$

Remark 1.6. $x_n \to x_0$ in $(\mathbb{R}^k, \|.\|_\infty)$

$$(x_{1,n}, x_{2,n}, \cdots, x_{k,n}) \to (x1, 0, \cdots, x_{k,0})$$

Iff

$$\begin{aligned} x_{1,n} &\to x_{1,0} \\ x_{2,n} &\to x_{2,0} \\ &\vdots \\ x_{k,n} &\to x_{k,0} \end{aligned}$$

in $(\mathbb{R}, \|.\|)$.

Proof. $|x_{i,n} - x_{i,0} \leq \|x_n - x_0\|_\infty \leq |x_{1,n} - x_{1,0}| + \cdots + |x_{k,n} - x_{k,0}|$ Therefore $x_n \to x_0$ in $(\mathbb{R}^k, \|.\|_\infty)$ $\iff$ $\|x_n - x_0\|_\infty \to 0$ and $\iff$ $|x_{i,n} - x_{i,0}| \to 0$ $\forall i$ and hence $\iff$ $x_{i,n} \to x_{i,0}$ $\forall i$. $\square$

Remark 1.7. Take $x \in \mathbb{R}^n$, then

$$\|x\|_\infty = \max_i(x_i) \leq |x_1| + \cdots + |x_n| = \|x\|_1 \leq n\|x\|_\infty$$

Therefore

$$\|x\|_\infty \leq \|x\|_1 \leq n \cdot \|x\|_\infty$$

Similarly, show something for $\|x\|_2$ and $\|x\|_\infty$.

Definition 1.8. Given a vector space V and norms $\|.\|_a$ and $\|.\|_b$ we say that $\|.\|_a$ and $\|.\|_b$ are equivalent iff $\exists c_1, c_2 > 0 \in \mathbb{R}$ s.t $\forall x \in V$

$$c_1\|x\|_b \leq \|.\|_a \leq c_2\|x\|_b$$

Lemma 1.9. *(X, d) metric space and $E \subseteq X$, then E is closed iff for sequence $x_n \in E$ that converges $\lim_{n\to\infty} x_n \in E$.*

Proof. $\implies$ Assume that there exists a sequence x_n in E that converges to x_0 s.t $x_0 \notin E$, take $\varepsilon > 0$, then since $x_n \to x_0$ then $\exists N \in \mathbb{N}$ s.t

$$x_n \in N_\varepsilon(x_0) \quad \forall n \geq N$$

But $x_n \in E$,

$$N^*_\varepsilon(x_0) \cap E \neq \varnothing$$

Therefore

$$x_0 \in E'$$

Contradiction to the fact that E is closed.

- Take $x_0 \in E'$, let $\varepsilon_n = 1/n$ then

$$\exists x_n \in N_n^*(x_0) \cap E$$

Consider $\{x_n\}_{n\in\mathbb{N}}$ x_n is a sequence in E s.t

$$d(x_n, x_0) < 1/n \implies x_n \to x_0$$

and hence $x_0 \in E$.

□

Theorem 1.10. *Given a vector space V and norms $\|.\|_a$ and $\|.\|_b$ then the following are all equivalent:*

1. *$\|.\|_a$ and $\|.\|_b$ are equivalent ($c_1\|x\|_b \leq \|x\|_a \leq c_2\|x\|_b$).*
2. *$x_n \to x_0$ in $\|.\|_a \iff x_n \to x_0$ in $\|.\|_b$.*
3. *E is closed in $(V, \|.\|_a) \iff E$ is closed in $(V, \|.\|_b)$.*
4. *U is open in $(V, \|.\|_a) \iff U$ is open in $(V, \|.\|_b)$.*

Pro1of.$\implies$ 2

$$c_1\|x_n - x_0\|_b \leq \|x_n - x_0\|_a \leq c_2\|x_n - x_0\| \leq \frac{c_2}{c_1}\|x_n - x_0\|_a$$

Apply Squeeze.

$2 \implies 3$ By the Lemma.

$3 \implies 4$ Trivial.

$4 \implies 1$ $B_{\|.\|_a}(0,1)$ open in $(V, \|.\|_a)$ therefore it is open in $(V, \|.\|_b)$. Hence 0 is an interior point in $(V, \|.\|_b)$. Hence $\exists\delta > 0$

$$B_{\|.\|_b}(0,\delta) \subseteq B_{\|.\|_a}(0,1)$$

Let $x \in V$, $x \neq 0$.

$$\frac{\delta}{2} \cdot \frac{x}{\|x\|_b} \in B_{\|.\|_b}(0,\delta) \subseteq B_{\|.\|_a}(0,1)$$

Therefore

$$\|\frac{\delta}{a}\frac{x}{\|x\|_b}\|_a < 1$$

We get

$$\frac{\delta}{2\|x\|_b}\|x_a\| < 1 \implies \|x\|_a \leq \frac{2}{\delta}\|x\|_b \; \forall x \in V$$

□

1.1 Compact Sets

Definition 1.11. (X, d) metric space, $K \subseteq X$. We say that K is *compact* iff for every open cover

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

of K has a finite subcover.

Example 1.12. $(0, 1]$ is not compact in $(R, |.|)$, take

$$\mathcal{G} = \{(1/n, 10)\}_{n \in \mathbb{N}}$$

which has no finite subcover.

Theorem 1.13 (Heine-Borel)**.** *In* $\mathbb{R}$ $[a, b]$ *is compact.*

Example 1.14. If X is finite, then X is compact.

Proof. Let $X = \{x_1, \cdots, x_n\}$ and let

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

be an open cover X. Then for every x_i, fix some G_{α_i} s.t

$$x = \{x_1, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

□

Proposition 1.15. (X, d) *metric space and* $K \subseteq X$ *then every closed subset* E *of* K *is compact.*

Proof. Let $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ be an open cover of E, then

$$E \subseteq \bigcup_{\alpha \in I} G_\alpha$$

Therefore

$$K \subseteq \bigcup G_\alpha \cup E^c, \quad E^c \text{ open}$$

Therefore

$$\mathcal{G}' = \mathcal{G} \cup \{E^c\} \quad \text{is an open cover of } K$$

Therefore it has a finite subcover and

$$K \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_k} \cup E^c$$

Hence

$$E \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_k}$$

Which is a finite subcover of $\mathcal{G}$. □

Definition 1.16. (X, d) metric space and x_n sequence in X and

$$\varphi \colon \mathbb{N} \to \mathbb{N} \quad \text{strictly increasing}$$

We say that $b_n = a_{\varphi(n)}$ is a *subsequence* of a_n.

Proposition 1.17. *x_n converges to $x_0 \iff$ every subsequence x_{n_k} converges to x_0.*

Definition 1.18. (X, d) metric space and $K \subseteq X$. We say that K is sequentially compact iff every sequence $x_n \in K$ has a convergent subsequence in K.

Proposition 1.19. *(X, d) metric space, K sequentially compact iff every infinite $E \subseteq K$ has a limit point in K.*

Proof. $\implies$ Let K be sequentially compact, take $E \subseteq K$ an infinite set. We can extract from E a sequence of distinct elements $x_n \in E \subseteq K$. Therefore x_n has a convergence subsequence x_{n_k} s.t the limit of $x_{n_k} = x_0 \in K$. Moreover, x_{n_k} are distinct in E therefore $x_0 \in E' \cap K$.

$\impliedby$ Let x_n be a sequence in K, take

$$E = \{x_n\}_{n \in \mathbb{N}}$$

We have two cases:

- E **is finite.** Then $\exists x_0$ and $\exists n_1 < n_2 < \cdots$ (infinitely many) s.t

$$x_{n_k} = x_0, \quad \forall k \in \mathbb{N}$$

  Therefore

$$\lim_{k \to \infty} x_{n_k} = x_0 \in E$$

  We found a converging subsequence.
- E **is infinite.** Then by the assumption, we have that E has a limit point x_0 in K.

  $\varepsilon = 1$ $\exists n_1$ s.t $x_{n_1} \in N_1^*(x_0) \cap E$.

  $\varepsilon = 1/2$ $\exists n_2 > n_1$ s.t $x_{n_2} \in N_{1/2}^*(x_0) \cap E$ (since $N_{1/2}^*(x_0) \cap E$ is infinite).

  we can construct $n_k > n_{k-1}$ s.t $x_{n_k} \in N_{1/k}^*(x_0) \cap E$, therefore

$$\lim_{n \to \infty} x_{n_k} = x_0 \in K$$

□

Example 1.20. In $(\mathbb{Q}, |.|)$, $K = \mathbb{Q} \cap [0, 1]$ is closed and bounded. P_n increasing sequence in $\mathbb{Q} \cap [0, 1]$ s.t

$$P_n \to 1/\sqrt{2} \in \mathbb{R}$$

Consider

$$\mathcal{G} = \Big\{(-1, P_1) \cap \mathbb{Q}, (-1, P_2) \cap \mathbb{Q}, \cdots, (-1, P_n) \cap \mathbb{Q}, \cdots, (1/\sqrt{2}, 10) \cap \mathbb{Q}\Big\}$$

Proposition 1.21. *$K \subseteq X$ is compact K is sequentially compact.*

Proof.
- Let $E \subseteq K$ infinite, suppose that $E' \cap K = \varnothing$. Take $x \in K \implies x \notin E'$, therefore $\exists \delta_x > 0$ s.t

$$N_{\delta_x}^*(x) \cap E = \varnothing$$

Let $\mathcal{G} = \{N_{\delta_x}(x)\}_{x \in K}$, is an open cover of K compact, hence it has a finite subcover

$$K \subseteq N_{\delta_{x_1}}(x_1) \cup \cdots \cup N_{\delta_{x_1}}(x_k)$$

But $E \subset K \implies E = (E \cap K)$ hence

$$\begin{aligned} E &\subseteq (N_{\delta_{x_1}}(x_1) \cap E) \cup \cdots \cup (N_{\delta_{x_1}}(x_k) \cap E) \\ &\subseteq \{x_1, \cdots, x_k\} \end{aligned}$$

therefore E is finite, contradiction.

□

1.2 Complete sets

Definition 1.22. Given a sequence x_n, we say that x_n is Cauchy iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t

$$d(x_n, x_m) < \varepsilon \quad \text{for } n, m \geq N$$

Proposition 1.23. *If x_n converges, then x_n is Cauchy.*

Proof. Triangle inequality. □

Remark 1.24. The converse is false. In $(\mathbb{Q}, |.|)$, in $(\mathbb{R}, |.|)$ there exists some sequence $x_n \in \mathbb{Q}$ s.t

$$x_n \to \sqrt{2} \quad \text{as } n \to \infty$$

But $x_n \to \sqrt{2}$ implies that x_n is Cauchy in $(\mathbb{R}, |.|)$, therefore x_n is Cauchy in $\mathbb{Q}$. Suppose x_n converges to P in $(\mathbb{Q}, |.|)$, therefore $x_n \to p \in (\mathbb{R}, |.|)$ by uniqueness of limit, $p = \sqrt{2}$, contradiction.

Definition 1.25. (X, d) metric space and $K \subseteq X$, we say that K is complete iff every Cauchy Sequence in K converges in K.

Example 1.26. $(\mathbb{R}, |.|)$ is complete (Math 210)

Proposition 1.27. *If K is complete then K is closed.*

Proof. Let x_n be a sequence in K s.t $x_n \to x_0 \in X$. Since x_n converges, then x_n is Cauchy and K is complete then x_n converges in K, therefore $x_0 \in K$. □

Proposition 1.28. *K complete and $E \subseteq K$ is closed, then E is complete.*

Proof. Let x_n be a Cauchy sequence in E, then x_n is Cauchy in K which is complete, therefore $x_n \to x_0 \in K$. But since E is closed, we get that $x_0 \in E$. □

Proposition 1.29. *(X, d) metric space and x_n Cauchy sequence in $K \subseteq X$. If x_n has a convergent subsequence x_{n_k} then x_n converges to the subsequential limit.*

Proof. Triangle inequality. □

Proposition 1.30. *(X, d) metric space and $K \subseteq X$ is sequentially compact then K is complete (and therefore closed).*

Proof. Let $\{x_n\}$ be a Cauchy sequence in K, K is sequentially compact therefore x_n has a convergent subsequence x_{n_k} that converges in K. Therefore by the previous proposition we get that x_n converges in K. $\square$

Theorem 1.31. *Let V be a finite dimensional inner product space over $\mathbb{R}$, then V is complete w.r.t norm induced by the inner product. Therefore $K \subseteq V$ is complete $\iff$ K is closed.*

(For the converse $K \subseteq V$ which is closed, then K is complete).

Proof. Let $S = \{v_1, \cdots, v_l\}$ be an orthonormal basis of V, take x_n be a cauchy sequence in V. Therefore

$$x_n = r_{1,l}v_1 + \cdots + r_{l,n}v_l$$

Take $r_{1,n} = \langle x_n, v_1 \rangle$ therefore

$$\begin{aligned} |r_{1,m} - r_{1,n}| &= |\langle x_m, v_1 \rangle - \langle x_n, v_1 \rangle| \\ &= |\langle x_m - x_n, v_1 \rangle| \\ &\leq \|x_m - x_n\| \|v_1\| \quad \text{By C.S} \end{aligned}$$

Let $\varepsilon > 0$ then $\exists N \in \mathbb{N}$ s.t

$$\|x_m - x_n\| < \varepsilon \implies r_{1,n} \text{ is Cauchy in } \mathbb{R}$$

THerefore $r_{1,n}$ converges to $r_1 \in \mathbb{R}$. Do the same fore very coordinate, and define

$$x = r_1 v_1 + \cdots + r_n v_n$$

Therefore

$$\begin{aligned} \|x_n - x\| &= \|(r_{1,n} - r_1)v_1 + \cdots + (r_{1,l} - r_l)v_l\| \\ &\leq \|r_{1,n} - r_1\| \|v_1\| + \cdots + \|r_{l,n} - r_l\| \|v_l\| \end{aligned}$$

Therefore

$$x_n \to x$$

$\square$

1.3 Total Boundedness

Definition 1.32. (X, d) metric space and $K \subseteq X$ we say that K is totally bounded iff for every $\varepsilon > 0$ $\exists x_1, \cdots, x_t \in K$ s.t

$$K \subseteq N_\varepsilon(x_1) \cup \cdots \cup N_\varepsilon(x_t)$$

Definition 1.33. $K \subseteq X$ is bounded iff $\exists R > 0$ s.t

$$K \subseteq N_R(x_0) \quad \text{for some } x_0 \in K$$

We say that a sequence is bounded iff

$$E = \{x_n\}_{n \in \mathbb{N}} \quad \text{is bounded}$$

Proposition 1.34. *Cauchy sequences are bounded.*

Proposition 1.35. *$K \subseteq X$ is totally bounded it is bounded.*

Note that the converse is false.

Example 1.36. $(\mathbb{R}, d_{disc})$, $\mathbb{R}$ is bounded since

$$\mathbb{R} \subseteq N_2(0)$$

but it isn't totally bounded, take $\varepsilon = 1/2$ then if it is totally bounded, then $\exists x_1 \cdots x_t \in \mathbb{R}$ s.t

$$\mathbb{R} \subseteq N_{1/2}(x_1) \cup \cdots \cup N_{1/2}(x_t) = \{x_1, \cdots, x_t\}$$

Proof. Take $\varepsilon = 1$ then $K \subseteq N_1(x_1) \cup N_1(x_t)$ for $x_1, \cdots, x_t \in K$. Take x_1 and let $d = \max_{k \in \{1, \cdots, t\}}(x_1, x_t)$, therefore for any $x \in K$ we have that

$$d(x, x_1) \leq d(x, x_i) + d(x_i, x_1)$$

for some x_i s.t $x \in N_1(x_i)$ (Exists by the above), therefore

$$d(x, x_1) \leq 1 + d \implies x \in N_{1+d}(x_1)$$

□

Proposition 1.37. *K is sequentially compact then K is totally bounded.*

Proof. Suppose not, then $\exists \varepsilon_0$ s.t K cannot be covered by finitely many ε_0-neighborhoods. Fix some x_1 then $\exists x_2 \in K \setminus N_{\varepsilon_0}(x_1)$, Similarly there exists some $x_3 \in K \setminus N_{\varepsilon_0}(x_1) \cup N_{\varepsilon_0}(x_2)$ and in general

$$\exists x_n \in K \setminus (N_{\varepsilon_0}(x_1) \cup \cdots \cup N_{\varepsilon_0}(x_{n-1}))$$

Therefore we have that for $n \neq m$

$$d(x_n, x_m) \geq \varepsilon_0$$

Therefore x_n doesn't have a convergent subsequence; contradiction. □

Lemma 1.38. *Every subset of a totally bounded set is totally bounded.*

Proof. $K \subseteq X$ totally bounded, and $E \subseteq K$; take $\varepsilon > 0$, therefore $\exists x_1, x_2, \cdots, x_t \in K$ s.t

$$K \subseteq N_{\varepsilon/2}(x_1) \cup \cdots \cup N_{\varepsilon/2}(x_t)$$

Ignore the neighborhood that don't intersect E, renumbering we have $x_1, \cdots, x_l$ s.t

$$N_{\varepsilon/2}(x_i) \cap E \neq \varnothing$$

We get

$$E \subseteq N_{\varepsilon/2}(x_1) \cup \cdots \cup N_{\varepsilon/2}(x_l)$$

Take $y_i \in N_{\varepsilon/2}(x_i) \cap E$, then

$$N_{\varepsilon}(y_i) \supseteq N_{\varepsilon/2}(x_i)$$

Therefore $E \subseteq N_{\varepsilon}(y_1) \cup \cdots \cup N_{\varepsilon}(y_l)$ and hence E is totally bounded. □

Example 1.39. $[0, 1]$ is totally bounded (since it is compact), therefore $(0, 1)$ is totally bounded.

Proposition 1.40. (X, d) *Metric space,* $K \subseteq X$ *is complete and totally bounded then* K *is compact.*

Proof. Assume K is not compact, fix $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ an open cover of K with no finite subcover. Let $\varepsilon = 1$, K is totally bounded then K can be covered by finitely many 1-balls with center in K. Hence $\exists x_1 \in K$ s.t $K_1 = N_1(x_1) \cap K$ cannot be covered by finitely many open sets in $\mathcal{G}$. But K_1 is totally bounded (subset of K), now let $\varepsilon = 1/2$, $\exists x_2 \in K_1$ s.t $K_2 = N_{1/2}(x_2) \cap K_1$ cannot be covered by finitely many open sets in $\mathcal{G}$. Similarly, we construct a sequence x_n and

$$K_n = N_{1/n}(x_n) \cap K_{n-1} \quad x_n \in K_{n-1}$$

such that K_n cannot be covered by finitely many G_α's. Notice that

$$K_0 = K \supseteq K_1 \supseteq K_2 \supseteq \cdots$$

Claim. x_n is Cauchy. Indeed, let $\varepsilon > 0$ then $\exists N$ s.t

$$1/N < \varepsilon/2$$

Now take $m, n > N$, then

$$x_n \in K_{n-1} \subseteq K_N, \quad x_m \in K_{m-1} \subseteq K_N$$

therefore $x_n, x_m \in K_N = N_{1/N}(x_N) \cap K_{n-1}$, and hence

$$d(x_n, x_m) \leq 2/N < \varepsilon$$

By completeness of K, we get that x_n converges to some $x_0 \in K$; since $\mathcal{G} = \{G_\alpha\}$ is an open cover of K, there exists $\alpha_0 \in I$ s.t $x_0 \in G_{\alpha_0}$ (open set). So $\exists \varepsilon_0$ s.t

$$N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

But $x_n \to x_0$ then $\exists N_1$ s.t

$$x_N \in N_{\varepsilon_0/2}(x_0) \text{ and } 1/N < \varepsilon_0/2$$

Therefore

$$K_N \subseteq N_{1/N}(x_n) \subseteq N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

contradiction. $\square$

Theorem 1.41. *V finite dimensional product normed space (over $\mathbb{R}$). Let $S = \{v_1, \cdots, v_n\}$ a basis of V. Then the cube*

$$Q_S = \{r_1 v_1 + r_2 v_2 + \cdots + r_n v_n \colon 0 \leq r_i \leq 1\}$$

is compact.

Proof. Let $x^k = r_{1,k} v_1 + r_{2,k} v_2 + \cdots + r_{n,k} v_n$ a sequence in Q_S, now clearly

$$r_{1,k} \text{ is a sequence in } [0,1] \text{, compact set}$$

Therefore $r_{1,k}$ has a convergent subsequence r_{1,k_n}. Similarly $r_{2,k_l} \in [0,1]$ therefore r_{2,k_l} has a convergent subsequence $r_{2,k_{l_t}}$. Doing this n times we get that $\exists$ a subsequence $\{k_j\}$ s.t

$$r_{1,k_j},\ r_{2,k_j}, \cdots,\ r_{n,k_j} \to r_1, r_2, \cdots, r_n$$

take

$$x_0 = r_1 v_1 + \cdots + r_n v_n$$

We can show that

$$\|x_0 - x^{k_j}\| \leq |r_{1,k_j} - r_1| \|v_1\| + \cdots + |r_{n,k_j} - r_n| \|v_n\| \to 0$$

then

$$x^{k_j} \to 0$$

Hence Q_S is (sequentially) compact. $\square$

Theorem 1.42. *Let V be a finite inner product space then*

$$K \subseteq V \textit{ is compact } \iff K \textit{ closed and bounded}$$

Proof.

- K compact $\implies$ complete $\implies$ closed and $\implies$ totally bounded $\implies$ bounded.
- K is bounded, therefore $\exists R > 0$ s.t $K \subseteq B_{\|.\|}(0, R)$. Let $S = \{v_1, \cdots, v_n\}$ be a basis of V and Take $x \in K$ therefore

  $$x = r_1 v_1 + \cdots + r_n v_n$$

  then $r_i = \|\langle x, v_i \rangle\| \leq \|x\| \cdot \|v_i\|$

  $$B_{\|.\|}(0, R) \subseteq \{r_1 v_1 + \cdots + r_n v_n \colon r_i \leq R\}$$

  Therefore $K \subseteq$ some cube and hence K is compact.

$\square$

Proposition 1.43. *In a finite dimensional inner product space we get that bounded $\iff$ totally bounded.*

Proof. $\implies$ Bounded $\implies$ can be put in a cube (compact) $\implies$ Totally bounded.
$\impliedby$ True in general.

□

1.4 Continuity

Definition 1.44. X, Y metric spaces and $E \subseteq X$ and $f\colon E \to Y$ map; $x_0 \in E'$ we say that

$$\lim_{x \to x_0} f(x) = y \in Y$$

iff for every $\varepsilon > 0$ $\exists \delta > 0$ s.t

$$f(x) \in N_\varepsilon(y_0) \text{ for } x \in N^*_\delta(x_0) \cap E$$

Definition 1.45. $f\colon E \to Y$ continuous at x_0 iff for every $\varepsilon > 0$, $\exists \delta > 0$ s.t $f(x) \in N_\varepsilon(x_0)$ for $x \in N_\delta(x_0) \cap E$; i.e

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{for } 0 < d_X(x, x_0) < \delta$$

Remark 1.46. f is continuous at $x_0 \iff x_0$ is an isolated point or $x_0 \in E'$ and $\lim x \to x_0 f(x) = f(x_0)$.

Proposition 1.47. $\lim_{x \to x_0} f(x) = y_0$ *iff for every sequence* $x_n \to x_0$ *we have that* $f(x_n) \to y_0$. *Therefore f is continuous at x_0 iff for every $x_n \to x_0$ we have that $f(x_n) \to f(x_0)$.*

Corollary 1.48. *if f, g are continuous then $f + g,\ cf,\ f \cdot g,\ f/g$ are all continuous at x_0.*

Example 1.49. $f(x, y) = \frac{xy}{x^2+y^2}$ we want to find

$$\lim_{(x,y) \to (0,0)} f(x, y)$$

Take the sequence $(x_n, y_n) = (1/n, 0)$ then

$$\lim_{n \to \infty} (x_n, y_n) = (0, 0)$$

and

$$f(x_n, y_n) = 0 \to 0 \text{ as } n \to \infty$$

Now take $(x_n, y_n) = (1/n, 1/n)$, then

$$\lim_{n \to \infty} (x_n, y_n) = (0, 0)$$

but

$$f(x_n, x_n) = \frac{1/n \cdot 1/n}{1/n^2 + 1/n^2} = 1/2 \to 1/2 \text{ as } n \to \infty$$

Therefore

$$\lim_{n \to \infty} f(x, y) \text{ does not exist}$$

Now we do the same for
$$f(x,y) = \frac{x^2 y}{x^2+y^2}$$
Notice that
$$0 \leq |f(x,y)| = |\frac{x^2}{x^2+y^2}y| \leq |y|$$
but
$$\lim_{(x,y)\to(0,0)} |y| = 0$$
therefore by squeeze we get that
$$\lim_{(x,y)\to 0} |f(x,y)| = 0$$
Hence
$$\lim_{(x,y)\to 0} f(x,y) = 0$$

Some useful inequalities

$$x^2+y^2 \geq x^2$$
$$x^2+y^2 \geq y^2$$
$$x^2+y^2 \geq 2(xy)$$

Example 1.50. Let $(V, \|.\|)$ a normed space and
$$\varphi\colon V \to \mathbb{R}, \quad \varphi(x) = \|x\|$$
Then φ is continuous.

Let x_n a sequence in V that converges to x_0 in $(V, \|.\|)$ therefore
$$\|x_n - x_0\| \to 0$$
We want to show that $\varphi(x_n) \to \varphi(x_0)$ i.e that
$$|\|x_n\| - \|x_0\|| = 0$$
But as shown in PS1, we have that
$$|\|x_n\| - \|x_0\|| \leq \|x_n - x_0\|$$

Proposition 1.51. *$f\colon X \to Y$ is continuos iff $f^{-1}(G)$ open in X for every G open in Y iff $f^{-1}(H)$ is closed in X for every H closed in Y.*

Proof. $\Longrightarrow$ let x_n be a sequence in $f^{-1}(H)$ that converges to x_0 in X. Hence $f(x_n)$ is a sequence in H by continuity of f we have that
$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

Moreover since H is closed, we get that $f(x_0) \in H$. Hence

$$x_0 \in f^{-1}(H)$$

Therefore if f is continuous $f^{-1}(G)$ is open in X for every G open in Y.

$\Longleftarrow$ Now we show that if $f^{-1}(G)$ is open in X for all G open in Y then f is continuous. Fix $x_0 \in X$, let $\varepsilon > 0$ then consider the open set in Y $N_\varepsilon(f(x_0))$ hence $f^{-1}\left(N_\varepsilon(f(x_0))\right)$ is open in X. But $x_0 \in f^{-1}\left(N_\varepsilon(f(x_0))\right)$ hence $\exists \delta > 0$ s.t

$$f\left(N_\delta(x_0)\right) \subseteq N_\varepsilon(f(x_0))$$

□

Example 1.52. $E = \{x, y \in \mathbb{R}^2 \colon y \geq x\}$ is open in $\mathbb{R}^2$, define the continuous function

$$f \colon \mathbb{R}^2 \to \mathbb{R}, f(x, y) = y - x$$

Then

$$E = f^{-1}\left((0, \infty)\right)$$

which is an open set.

Example 1.53. $f \colon X \to Y$ continuous and consider the c-level set with $c \in Y$

$$E = \{x \in X \colon f(x) = c\} = f^{-1}\left(\{c\}\right)$$

is closed in X.

Therefore $E = \left\{(x, y) \colon x^2 + 3y^2 = 1\right\}$ is closed in $\mathbb{R}^2$, S^n is closed.

Remark 1.54. V, W normed spaces and

$$\|.\|_a,\ \|.\|_b \text{ equivalent norms on } V$$

so $f \colon (V, \|.\|_a) \to W$ is continuous iff $f \colon (V, \|.\|_b) \to W$ is continuous.

Equivalently: V, W normed spaces and

$$\|.\|_a,\ \|.\|_b \text{ equivalent norms on } W$$

so $f \colon V \to (W, \|.\|_a)$ is continuous iff $f \colon V \to (W, \|.\|_b)$ is continuous.

Example 1.55. Take $f \colon \mathbb{R}^m \to \mathbb{R}^n$ defined as follows

$$f(x) = (f_1(x), \cdots, f_n(x)) \quad x = (x_1, \cdots, x_m)$$

Then f is continuous iff $f_i \colon \mathbb{R}^m \to R^n$ is continuous for $1 \leq i \leq n$.

Theorem 1.56 (Extreme Value Theorem)**.** *X, Y metric spaces and $K \subseteq X$ compact and $f \colon K \to Y$ continuous. Then $f(K)$ is compact.*

Corollary 1.57. *If $Y = \mathbb{R}$, then the above implies that f attains its maximum and minimum in K.*

Proof. Let y_n be a sequence in $f(K)$, therefore $\exists x_n \in K$ s.t $f(x_n) = y_n$. Since K is compact, x_n has a convergent subsequence x_{n_k} that converges to $x_0 \in K$, using continuity we get that

$$y_{n_k} \to f(x_0) \in f(K)$$

□

Theorem 1.58. *All norms in a finite dimensional vector space are equivalent.*

Proof. V is a finite dimensional vector space, then by PS1 we can define an inner product on V. Let $\|.\|$ be the norm induced by our choice of inner product. Let $S = \{v_1, \cdots, v_n\}$ an orthonormal basis of V. Let $\|.\|_a$ be an arbitrary norm on V, we will show that $\|.\|$ and $\|.\|_a$ are equivalent. Let $x \in V$, then

$$x = r_1 v_1 + \cdots + r_n v_n, \quad |r_i| = \langle x, v_i \rangle \leq \|x\|$$

Therefore $\forall x \in V$

$$\begin{aligned} \|x\|_a &= \|r_1 v_1 + \cdots + r_n v_n\| \\ &\leq |r_1| \|v_1\|_a + \cdots + |r_n| \|v_n\|_a \\ &\leq (\|v_1\|_a + \cdots + \|v_n\|_a) \cdot \|x\| \\ &\leq c_1 \cdot \|x\| \end{aligned}$$

Now define the function $\varphi \colon (V, \|.\|) \to \mathbb{R}$ s.t

$$\varphi(x) = \|x\|_a$$

We claim that φ is continuous, indeed let $x_n \in V$ s.t $x_n \to x_0$ in $\|.\|$. Therefore

$$\begin{aligned} |\varphi(x_n) - \varphi(x_0)| &= |\|x_n\|_a - \|x_0\|_a| \\ &\leq \|x_n - x_0\| \\ &\leq c_1 \|x_n - x_0\| \to 0 \end{aligned}$$

Therefore $\lim_{n\to\infty} \varphi(x_n) = \varphi(x_0)$, so φ is continuous. Let $K = \{x \in V \colon \|x\| = 1\}$ closed and bounded in the inner product space $(V, \|.\|)$. Then it is compact in $(V, \|.\|)$ therefore $\varphi(K)$ is compact in $\mathbb{R}$ so it attains its minimum in K. Notice that $\varphi(x) = 0 \iff x = 0$, hence $\forall x \in K,\ \varphi(x) > 0$. Then

$$\min_{x \in K} \varphi(x) > 0$$

Let $x \in V$ and $x \neq 0$ then $\frac{x}{\|x\|} \in K$, therefore

$$\varphi(\frac{x}{|x|}) = \|\frac{x}{\|x\|}\|_a \geq \min \varphi(K)$$

Therefore

$$\|x\|_a \geq \min(\varphi(K))\|x\|$$

□

Corollary 1.59. *If V is finite dimensional then compact iff closed and bounded (independently of the choice of norm).*

1.5 Connected sets

Definition 1.60. (X, d) metric space and $\Omega \subseteq X$, we say that Ω is disconnected iff $\exists U, V \subseteq X$ open sets s.t

$$\Omega = (U \cup V) \cap \Omega$$

With $U \cap \Omega,\ V \cap \Omega \neq \varnothing$ and $U \cap \Omega \cap V = \varnothing$.

Otherwise Ω is connected.

Example 1.61. Let $\Omega = (1, 2] \cup (3, 4)$ disconnected, since

$$\Omega \subseteq (0, 2.5) \cup (2.5, 17)$$

$\mathbb{Q}$ is disconnected in $\mathbb{R}$ since

$$\mathbb{Q} \subseteq (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$$

Proposition 1.62. *X is a connected metric space iff the only sets that are both open and closed are $\varnothing, X$.*

Proof. $\Longrightarrow$ Assume that U is open and closed an $U \notin \{\varnothing, X\}$ then U^c is also not open and closed and not in $\{\varnothing, X\}$ but

$$X = U \cup U^c$$

Therefore X is disconnected.

$\Longleftarrow$ Assume that X is disconnected, then $X = U \cup V$ with U, V open disjoint and nonempty. Therefore $V = U^c$ and U is open then V is closed. Hence V is a clopen set of X but $V \notin \{\varnothing, X\}$ contradiction.

□

Definition 1.63. $I \subseteq \mathbb{R}$ is an interval for $x, y \in I$ s.t $x < y$ then $z \in I,\ \forall x < z < y$.

Theorem 1.64. *$\Omega \subseteq \mathbb{R}$ is connected iff Ω is an interval.*

Proof. $\Longrightarrow$ Assume that Ω is not an interval, therefore $\exists x, y \in \Omega$ s.t $x < z < y$ and $z \notin \Omega$, therefore

$$\Omega \subseteq (-\infty, z) \cup (z, \infty)$$

Therefore Ω is disconnected.

$\Longleftarrow$ Let Ω be an interval. Suppose Ω is not connected, then $\exists U, V$ open and disjoint in Ω in $\mathbb{R}$ s.t

$$\Omega \subseteq U \cup V$$

Then $\exists x \in U \cap \Omega$ and $y \in V \cap \Omega$ and assume wlog that $x < y$; therefore $[x, y] \subseteq \Omega$ consider

$$z = \sup \{[x, y] \cup U\}$$

Note that $z \in [x, y]$ therefore $z \in \Omega$ therefore either $z \in U$ or $z \in V$. Notice that $z > x$ since $x \in U$ an open set. Also notice that $z < y$ since if $z = y \in V$ we have that $\exists \varepsilon_1$ s.t

$$(y - \varepsilon, y) = (z - \varepsilon, z) \subseteq [x, y] \cap V$$

since z is the sup we get that

$$(z - \varepsilon, z) \cap [x, y] \cap U \neq \varnothing$$

contradiction.
Therefore $z \in (x, y)$, suppose that $z \in U$ then $(z, z + \varepsilon_0) \subseteq (x, y) \cap U$ contradiction because $z + \varepsilon_0/2 > z = \sup[x, y] \cap y$. Suppose that $z \in V$ then $\exists \varepsilon_1$ s.t $(z - \varepsilon_1, z) \subseteq (x, y) \cap V$. Since z is the sup then $\exists z^*$ an element s.t

$$z - \varepsilon_1 < z^* < z \text{ and } z^* \in U$$

$z^* \in U \cap V$ contradiction.

□

Theorem 1.65 (Intermediate Value Theorem). *$f \colon X \to Y$ continuous, if X is connected then $f(X)$ is connected. In particular if $Y = \mathbb{R}$ we have the following:*

- *If $y_0 < y_1 \in f(X)$ then $\forall \alpha$ s.t $y_0 < \alpha < y_1$ $f^{-1}(\{c\}) \neq \varnothing$.*

Proof. Suppose that $f(X)$ is disconnected, then $\exists U, V \subseteq Y$ open sets that separate $f(X)$. Therefore

$$\begin{aligned} X &= f^{-1}(f(X)) \\ &= f^{-1}(U \cap V) \cap f(X) \\ &= \left[f^{-1}(U) \cup f^{-1}(V) \right] \cap X \end{aligned}$$

But f is continuous therefore $f^{-1}(U),\ f^{-1}(V)$ are open and disjoint. Now $U \cap f(X) \neq \varnothing$ therefore $f^{-1}(U) \neq \varnothing$, same for V; then X is disconnected. □

Definition 1.66. X is path connected iff for every $a, b \in X$ there exists a continuous function

$$\alpha \colon [0, 1] \to X$$

s.t $\alpha(0) = a,\ \alpha(1) = b$.

Remark 1.67. Let X be a vector space, the case where for every $a, b \in X$ the line segment $(1 - t)a + tb \in X\ \forall t \in [0, 1]$ we say that X is convex.

Example 1.68. $B_R(x_0)$ is always convex and hence path connected in a vector space. Indeed, take $a, b \in B_R(x_0)$ consider

$$\begin{aligned}\|(1-t)a + tb - x_0\| &< (1-t)\|a - x_0\| + t\|b - x_0\| \\ &< (1-t)R + R = R\end{aligned}$$

Proposition 1.69. *(X, d) metric space then X is path connected $\implies$ X is connected.*

Proof. Let X be disconnected then we have U, V nonempty disjoint and open

$$X = U \cup V$$

Then $\exists a \in U$ and $b \in V$, since X is path connected $\exists \alpha\colon [0,1] \to X$ s.t

$$\alpha(0) = a, \quad \alpha(1) = b$$

$[0,1]$ is connected, therefore $\alpha([0,1])$ is also connected but

$$[0,1] = \alpha^{-1}(U) \cup \alpha^{-1}(V)$$

But α is continuous, therefore $\alpha^{-1}(U) \ni 0$ and $\alpha^{-1}(V) \ni V$ which is a contradiction since $[0,1]$ is connected. □

Theorem 1.70. *$(V, \|.\|)$ vector space and $\Omega \subseteq V$ open then Ω is connected iff Ω is path connected.*

Proof. $\impliedby$ True in general.

$\implies$ We will show that A is closed by showing that A is open. Take $z \in A^c$ so there is no path joining z to x_0. But Ω is open, so $\exists \delta > 0$ s.t

$$B_\delta(z) \subseteq \Omega$$

Now every point in $B_\delta(z)$ cannot be joined to x_0; Hence

$$B_\delta(z) \subseteq A^c$$

so A^c is open.
Therefore A open, closed and nonempty in Ω a connected space, hence $A = \Omega$.

□

Example 1.71 (Topological sine)**.** $\Omega = \{(x, \sin(1/x)) \mid\ x > 0\} \cup \{(0,0)\}$; let

$$A = \{(x, \sin 1/x)\colon x > 0\}$$

then A is connected (image of a continuous function), with

$$\overline{A} = A \cup (\{0\} \times [-1,1])$$

Therefore $A \subseteq \Omega \subseteq \overline{A}$.

Suppose that Ω was connected, take $x_0 > 0$, suppose that $\exists\alpha\colon [0,1] \to \Omega$ from $(0,0)$ to $(x_0, \sin 1/x_0)$. let $t^* = \sup\{t \in [0,1] \mid x(t)\} = 0$; therefore

$$x(t^*) = 0$$

with $x(t^*) < 1$, hence

$$\alpha(t^*) = (0,0)$$

Take $\varepsilon = 1$, then by the IVT $\exists t^* < t_1 < t^* + 1$ and $t_1 \in [0,1]$ s.t

$$x(t_1) = \frac{1}{(4n_1+1)\pi/2}$$

Take $\varepsilon = 1/2$, then $\exists t_2 \in [0,1]$ s.t $t^* < t_2 < t^* + 1/2$ s.t

$$x(t_2) = \frac{1}{(4n_2+1)\pi/2}$$

We take a sequence $t_k \in [0,1]$ s.t $t^* < t_k < t^* + 1/k$

$$x(t_k) = \frac{1}{(4n_k+1)\pi/2}$$

hence

$$\alpha(t_k) = (\frac{1}{(4n_k+1)\pi/2}, 1)$$

With $t_k \to t^*$, therefore

$$\alpha(t^*) = (0,1)$$

contradicting the continuity of α.

2 Differentiability

Review linear transformations.

Proposition 2.1. *Let V and W normed spaces $T\colon V \to W$ linear transformation, then the following are all equivalent:*

- *T is continuous on V.*
- *T is continuous on 0_V.*
- *$\exists M > 0$ s.t*

$$\|Tx\|_W \le M\|x\|_V$$

Proof. 1 $\implies$ 2 Trivial.

2 $\implies$ 3 T is continuous at 0, hence for $\varepsilon = 1$ $\exists\delta > 0$ s.t

$$\|Tx - T0\|_W < 1 \quad \forall \|x - 0\|_V < \delta$$

Therefore

$$\|Tx\|_W < 1 \quad \forall \|x\|_V < \delta$$

Take any $x \neq 0$ then

$$\|\frac{x}{\|x\|_V} \cdot \delta/2\|_V = \delta/2$$

Therefore

$$\|T\left(\frac{x}{\|x\|_V} \cdot \delta/2\right)\|_W < 1$$

therefore

$$\|Tx\|_W < \frac{2}{\delta}\|x\|$$

$3 \implies 1$

$$\|Tx - Ty\|_W = \|T(x-y)\|_W \leq M\|x-y\|_V$$

□

Example 2.2 (The derivative as a linear transformation)**.** Take $C^1([0,1])$ the space of continuously differentiable functions, with the sup norm

$$\|f\|_\infty = \sup |f(x)| \colon x \in [0,1]$$

and take $C([0,1])$ to be the space of continuous functions with the sup norm. Take

$$\frac{d}{dx} \colon C^1([0,1]) \to C([0,1])$$

Note that this is a linear transformation.

Is it continuous? If it was then $\exists M$ s.t

$$\|f\|_\infty \leq M\|f\|_\infty \quad \forall f$$

take $f_n(x) = x^n$, then

$$\|f_n\|_\infty = 1$$

but $f_n'(x) = nx^{n-1}$ with

$$\|f_n'\|_\infty = n$$

Contradiction to the Archimedean property.

Theorem 2.3. *Let V be finite dimensional, and $T \colon V \to W$ linear transformation then T is continuous.*

Proof. Let $\alpha = \{v_1, \cdots, v_n\}$ be a basis of V. Let $x \in V$, then $\exists r_1, \cdots, r_n$ s.t

$$x = r_1 v_1 + \cdots + r_n v_n$$

Since all norms are equivalent in a finite dimensional vector space, wlog work with the inner product norm and make α an orthonormal basis.

$$\begin{aligned}\|Tx\|_W &= \|r_1 T v_1 + \cdots + r_n T v_n\|_W \\ &\leq |r_1|\|Tv_1\|_W + \cdots + |r_n|\|Tv_n\|\end{aligned}$$

With

$$|r_i| = |\langle x, v_i \rangle| \leq \|x\|$$

Therefore

$$\|Tx\|_W \leq \|x\|_V(\|Tv_1\| + \cdots_{\|} TV_n\|) = M\|x\|$$

□

Motivation. In one dimension, $f\colon \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

if it exists we say that f is differentiable at x_0 and we define

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

In other words, we can say that

$$\begin{aligned} L = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &\iff \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - L \right) = 0 \\ &\iff \lim_{x \to x_0} \left(\frac{f(x) - f(x_0) - L(x - x_0)}{x - x_0} \right) = 0 \\ &\iff \exists L \in \mathbb{R} \text{ s.t } f(x) = f(x_0) + L(x - x_0) + o(|x - x_0|) \text{ with } \frac{o(t)}{|t|} \to 0 \\ &\iff \exists L \in \mathbb{R} \text{ s.t } f(x) = f(x_0) + L(x - x_0) + o(|x - x_0|) \\ &\quad \text{with } \frac{o(|x - x_0|)}{|x - x_0|} \to 0 \text{ as } x \to x_0 \end{aligned}$$

Definition 2.4. V, W normed spaces and $U \subseteq V$ open set with $x_0 \in U$ we say that f is differentiable at x_0 iff there exists a continuous linear transformation $T \in \mathcal{L}(V, W)$ s.t

$$f(x) = f(x_0) + T(x - x_0) + o(\|x - x_0\|_V)$$

or

$$f(x_0 + h) = f(x_0) + T(h) + o(h)$$

with $\frac{o(\|h\|)}{\|h\|_V} \to 0$ as $h \to 0$.

Iff $\exists T \in \mathcal{L}(V, W)$ s.t

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - T(h)_W}{\|h\|_V} = 0$$

Remark 2.5. If such a T exists, then it is unique.

Proof. Suppose that $\exists T_1, T_2$ continuous linear transformations in $\mathcal{L}(V, W)$ s.t

$$f(x_0 + h) = f(x_0) + T_1(h) + o_1(h), \quad f(x_0 + h) = f(x_0) + T_2(h) + o_2(h)$$

Therefore

$$(T_1 - T_2)(h) = (o_1 - o_2)(h) = o_3(h)$$

take v s.t $\|v\| = 1$, therefore

$$\begin{aligned}(T_2 - T_1)(\lambda v) &= \lambda o_3(\|\lambda v\|)\\ \lambda(T_2 - T_1)(v) = &= \lambda o_3(|\lambda|)\\ (T_2 - T_1)(v) &= \frac{o_3(|\lambda|)}{\lambda}\end{aligned}$$

Therefore as $\lambda \to 0$ we get that

$$(T_2 - T_1)(v) = 0 \quad \forall v \text{ s.t } \|v\| = 1$$

Therefore $T_2 = T_1$ $\square$

Definition 2.6. If f is differentiable at x_0, then we let $Df(x_0)$ to be the linear transformation in $\mathcal{L}(V, W)$ s.t

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o(\|x - x_0\|) \iff \lim_{x \to x_0} \frac{\|f(x) - f(x_0) - Df(x - x_0)\|_W}{\|x - x_0\|_V}$$

Notation. $\mathbb{R}^n = \{x = (x_1, \cdots, x_n) \colon x_i \in \mathbb{R}\}$ as rows vectors, observe $f \colon \mathbb{R}^{1\times n} \to \mathbb{R}^{m\times 1}$ differentiable at x_0 then we have that

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o(|x - x_0|)$$

Then

$$Df(x_0) \in \mathcal{L}(\mathbb{R}^m, R^n)$$

Then $Df(x_0)$ can be represented by a matrix called the *the Jacobian matrix* $[Df(x_0)]$ (an $m \times n$ matrix). Therefore

$$Df(x_0)(x - x_0) = [Df(x_0)](x - x_0)^t$$

Note that we take a row vector and output a column vector.

Example 2.7. In 1 variable we have that

$$[Df(x_0)] = [f'(x_0)]$$

Then

$$Df(x_0) \colon t \to [f'(x_0)] \cdot t = f'(x_0) \cdot t$$

Find the derivative

$$f(x, y) = x^2 + y^2$$

we get that

$$\begin{aligned} f((x,y)+(h_1,h_2)) &= f(x_0+h_1,y_0+h_2) \\ &= (x_0+h_1)^2+(y_0+h_2)^2 \\ &= x_0^2+y_0^2+2y_0h_1+2y_0h2+h_1^2+h_2^2 \\ &= f(x_0,y_0)+(2x_0,2y_0)(h_1,h_2)+o(|h|) \end{aligned}$$

since $\frac{h_1^2+h_2^2}{\sqrt{h_1^2+h_2^2}} \to 0$ so we get

$$Df(x_0,y_0)(v_1,v_2) \to (2x_0,2y_0)\cdot(v_1,v_2)$$

therefore we get

$$[Df(x_0)] = \begin{bmatrix} 2x_0 & 2y_0 \end{bmatrix}$$

Example 2.8. Take $T\colon V \to W$ be a continuous linear transformation we want to find $DT(x)$; but

$$T(x_0+h) = T(x_0)+T(h)+0$$

therefore

$$DT(x_0) = T$$

Proposition 2.9. *$f\colon V \to W$ if f is differentiable at x_0 therefore f is continuous at x_0.*

Proof. Let $f(x) = f(x_0)+Df(x_0)(h)+o(\|x-x_0\|)$ but notice that

$$o(\|x-x_0\|) \to 0$$

but $Df(x_0)$ is continuous hence $\exists M$ s.t

$$\|Df(x_0)(x-x_0)\| \leq M\|x-x_0\| \to 0 \implies \lim_{x\to x_0} f(x) = f(x_0)$$

therefore f is continuous. □

2.1 Partial derivatives

Definition 2.10. $U \subseteq \mathbb{R}^n$ open and $f\colon U \to \mathbb{R}$, then

$$\frac{\partial f}{\partial x_i}(x) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(x+he_i)-f(x)}{h}$$

We call this the partial derivative w.r.t x_i.

Example 2.11.

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

We have shown that this is not continuous at $(0,0)$ hence f is not differentiable at $(0,0)$. But notice that

$$\frac{f(h,0)-f(0,0)}{h} = 0 \to 0 \text{ as } h \to 0$$

therefore $\frac{\partial f}{\partial x}(0,0)$ exists and equal to 0. Similarly we get that $\frac{\partial f}{\partial y}(0,0)$ exists and equal to 0.

What if $f\colon U \to \mathbb{R}$ is differentiable at x_0 then $Df(x_0) \in \mathcal{L}(R^n, \mathbb{R})$ hence $[Df(x_0)]$ is a $1 \times n$ matrix.

$$f(x_0 + he_1) = f(x_0) + Df(x_0)(he_1) + o(|h|)$$

therefore

$$\frac{f(x_0+he_1)-f(x_0)}{h} = Df(x_0)e_1 + \frac{o(h)}{h} \to Df(x_0)e_1$$

therefore $\frac{\partial f}{\partial x_1}$ exists and is the first entry in $[Df(x_0)]$. Similarly we get $\frac{\partial f}{\partial x_i}$ exists and is the i'th entry in $[Df(x_0)]$. We get that

$$[Df(x_0)] = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$$

this is called the gradient of f denoted ∇f.

If f is differentiable then

$$\begin{aligned} f(x) &= f(x_0) + \nabla f(x_0)(x_0 - x)^t + o(|x - x_0|) \\ &= f(x_0) + \nabla f(x_0) \cdot (x - x_0) + o(|x - x_0|) \end{aligned}$$

Theorem 2.12. *$U \subseteq \mathbb{R}^n$ open, and $f\colon U \to \mathbb{R}$ is differentiable iff all partial derivatives exist and*

$$\lim_{h \to 0} \frac{\|f(x_0+h) - f(x_0) - \nabla(x_0) \cdot h\|}{|h|} = 0$$

Example 2.13. $f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{o.w} \end{cases}$ then

$$f_x(0,0) = 0, \quad f_y(0,0) = 0$$

and

$$\frac{f(h_1,h_2) - f(0,0) - \nabla f(0,0) \cdot (h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = \frac{h_1^2 h_2}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}}$$

take $h_1 = 1/n, h_2 = 1/n$ we get that

$$\frac{1}{1\sqrt{2}} \not\to 0$$

hence this is not differentiable.