Advanced Calculus

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Contents

1 Metric Spaces

Definition 1.1. Let X be a set, a *metric* d is a map

$$d\colon X \times X \to \mathbb{R}$$

such that:

(i) $d(x, y) \geq 0$ with equality iff $x = y$.

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

Example 1.2. Given a vector space V and norm $\|.\|$ we can define the metric

$$d(x, y) = \|x - y\|$$

Question. Given a vector space V and a metric d on V, does d "come from" a norm? i.e does there exist some norm $\|\|$ s.t

$$d(x, y) = \|x - y\|$$

The answer in general is no (consider the discrete metric, or even any bounded metric: this contradicts homogeneity).

Notation. Take (X,d) metric space $\delta > 0$ and $x_0 \in X$, the *δ-neighborhood* of x_0

$$N_\delta(x_0) := \{x \in X : d(x, x_0) < \delta\}$$

Definition 1.3. (X,d) metric space and $E \subset X$

(i) $x_0 \in X$ is an *interior point* of E iff $\exists \delta > 0$ s.t

$$N_\delta(x_0) \subset E$$

(ii) $x_0 \in X$ is a *limit point* iff $\forall \delta > 0$

$$N^*_\delta(x_0) \cap E \neq \varnothing$$

(We denote by E' the set of limit points of E)

(iii) E is *open* iff every $x_0 \in E$ is an interior point of E.

(iv) E is *closed* iff every $E' \subset E$.

(v) The *closure* of E denoted $\overline{E}$ is

$$\overline{E} := E \cup E'$$

Review. Fix (X,d) a metric space.

1. $N_\delta(x_0)$ is open $\forall \delta > 0,\ \forall x_0 \in X$.
2. E is open iff E^c is closed.
3. $\overline{E}$ is closed, in fact $\overline{E}$ is the smallest closed set containing E.
4. The union of any collection of open sets is open.
5. A finite intersection of a collection of open sets is open.
6. Intersection of any collection of closed sets is closed.
7. Finite union of closed is closed.
8. If $x \in E'$ then $N_\delta(x) \cap E'$ is an infinite set.

Definition 1.4. Let x_n be a sequence in X, we say that x_n converges to $x_0 \in X$ iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t

$$d(x_n, x_0) < \varepsilon \quad \forall n \geq N$$

1. The limit of a sequence (if exists) is unique.
2. x_n converges to x_0 in (X,d) iff $d(x_n, x_0)$ converges 0 in $(\mathbb{R}, \|\|)$.

Example 1.5. $x_n = (1/n + 1, e^{-n})$ in $\mathbb{R}^2$. Find the limit of x_n in $(\mathbb{R}^2, \|.\|_\infty)$, $(\mathbb{R}^2, \|.\|_1)$, $(\mathbb{R}^2, \|.\|_2)$.

- $$\|(1/n + 1, e^{-n}, (1,0))\|_\infty = \max(1/n, e^{-n}) \leq 1/n + e^{-n} \to 0$$

- $x_0 = (1, 0)$

$$\|(1/n + 1, e^{-n}, -(1, 0))\|_1 = \|(1/n, e^{-n})\|_1 = |1/n| + |e^{-n}| \to 0$$

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$$\|(1/n + 1, e^{-n}) - (1, 0)\|_2 = \sqrt{1/n^2 + e^{-2n}} \to 0$$

Remark 1.6. $x_n \to x_0$ in $(\mathbb{R}^k, \|.\|_\infty)$

$$(x_{1,n}, x_{2,n}, \cdots, x_{k,n}) \to (x1, 0, \cdots, x_{k,0})$$

Iff

$$\begin{aligned} x_{1,n} &\to x_{1,0} \\ x_{2,n} &\to x_{2,0} \\ &\vdots \\ x_{k,n} &\to x_{k,0} \end{aligned}$$

in $(\mathbb{R}, \|.\|)$.

Proof. $|x_{i,n} - x_{i,0} \leq \|x_n - x_0\|_\infty \leq |x_{1,n} - x_{1,0}| + \cdots + |x_{k,n} - x_{k,0}|$ Therefore $x_n \to x_0$ in $(\mathbb{R}^k, \|.\|_\infty)$ $\iff$ $\|x_n - x_0\|_\infty \to 0$ and $\iff$ $|x_{i,n} - x_{i,0}| \to 0\ \forall i$ and hence $\iff$ $x_{i,n} \to x_{i,0}\ \forall i$. $\square$

Remark 1.7. Take $x \in \mathbb{R}^n$, then

$$\|x\|_\infty = \max_i(x_i) \leq |x_1| + \cdots + |x_n| = \|x\|_1 \leq n\|x\|_\infty$$

Therefore

$$\|x\|_\infty \leq \|x\|_1 \leq n \cdot \|x\|_\infty$$

Similarly, show something for $\|x\|_2$ and $\|x\|_\infty$.

Definition 1.8. Given a vector space V and norms $\|.\|_a$ and $\|.\|_b$ we say that $\|.\|_a$ and $\|.\|_b$ are equivalent iff $\exists c_1, c_2 > 0 \in \mathbb{R}$ s.t $\forall x \in V$

$$c_1\|x\|_b \leq \|.\|_a \leq c_2\|x\|_b$$

Lemma 1.9. *(X, d) metric space and $E \subseteq X$, then E is closed iff for sequence $x_n \in E$ that converges $\lim_{n\to\infty} x_n \in E$.*

Proof. $\implies$ Assume that there exists a sequence x_n in E that converges to x_0 s.t $x_0 \notin E$, take $\varepsilon > 0$, then since $x_n \to x_0$ then $\exists N \in \mathbb{N}$ s.t

$$x_n \in N_\varepsilon(x_0) \quad \forall n \geq N$$

But $x_n \in E$,

$$N^*_\varepsilon(x_0) \cap E \neq \varnothing$$

Therefore

$$x_0 \in E'$$

Contradiction to the fact that E is closed.

- Take $x_0 \in E'$, let $\varepsilon_n = 1/n$ then

$$\exists x_n \in N_n^*(x_0) \cap E$$

Consider $\{x_n\}_{n\in\mathbb{N}}$ x_n is a sequence in E s.t

$$d(x_n, x_0) < 1/n \implies x_n \to x_0$$

and hence $x_0 \in E$.

□

Theorem 1.10. *Given a vector space V and norms $\|.\|_a$ and $\|.\|_b$ then the following are all equivalent:*

1. *$\|.\|_a$ and $\|.\|_b$ are equivalent ($c_1\|x\|_b \leq \|x\|_a \leq c_2\|x\|_b$).*
2. *$x_n \to x_0$ in $\|.\|_a \iff x_n \to x_0$ in $\|.\|_b$.*
3. *E is closed in $(V, \|.\|_a) \iff E$ is closed in $(V, \|.\|_b)$.*
4. *U is open in $(V, \|.\|_a) \iff U$ is open in $(V, \|.\|_b)$.*

Pro1of.$\implies 2$

$$c_1\|x_n - x_0\|_b \leq \|x_n - x_0\|_a \leq c_2\|x_n - x_0\| \leq \frac{c_2}{c_1}\|x_n - x_0\|_a$$

Apply Squeeze.

$2 \implies 3$ By the Lemma.

$3 \implies 4$ Trivial.

$4 \implies 1$ $B_{\|.\|_a}(0,1)$ open in $(V, \|.\|_a)$ therefore it is open in $(V, \|.\|_b)$. Hence 0 is an interior point in $(V, \|.\|_b)$. Hence $\exists\delta > 0$

$$B_{\|.\|_b}(0,\delta) \subseteq B_{\|.\|_a}(0,1)$$

Let $x \in V$, $x \neq 0$.

$$\frac{\delta}{2} \cdot \frac{x}{\|x\|_b} \in B_{\|.\|_b}(0,\delta) \subseteq B_{\|.\|_a}(0,1)$$

Therefore

$$\|\frac{\delta}{a}\frac{x}{\|x\|_b}\|_a < 1$$

We get

$$\frac{\delta}{2\|x\|_b}\|x_a\| < 1 \implies \|x\|_a \leq \frac{2}{\delta}\|x\|_b \; \forall x \in V$$

□

1.1 Compact Sets

Definition 1.11. (X, d) metric space, $K \subseteq X$. We say that K is *compact* iff for every open cover

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

of K has a finite subcover.

Example 1.12. $(0, 1]$ is not compact in $(R, |.|)$, take

$$\mathcal{G} = \{(1/n, 10)\}_{n \in \mathbb{N}}$$

which has no finite subcover.

Theorem 1.13 (Heine-Borel). *In* $\mathbb{R}$ $[a, b]$ *is compact.*

Example 1.14. If X is finite, then X is compact.

Proof. Let $X = \{x_1, \cdots, x_n\}$ and let

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

be an open cover X. Then for every x_i, fix some G_{α_i} s.t

$$x = \{x_1, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

□

Proposition 1.15. (X, d) *metric space and* $K \subseteq X$ *then every closed subset* E *of* K *is compact.*

Proof. Let $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ be an open cover of E, then

$$E \subseteq \bigcup_{\alpha \in I} G_\alpha$$

Therefore

$$K \subseteq \bigcup G_\alpha \cup E^c, \quad E^c \text{ open}$$

Therefore

$$\mathcal{G}' = \mathcal{G} \cup \{E^c\} \quad \text{is an open cover of } K$$

Therefore it has a finite subcover and

$$K \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_k} \cup E^c$$

Hence

$$E \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_k}$$

Which is a finite subcover of $\mathcal{G}$. □

Definition 1.16. (X, d) metric space and x_n sequence in X and

$$\varphi \colon \mathbb{N} \to \mathbb{N} \quad \text{strictly increasing}$$

We say that $b_n = a_{\varphi(n)}$ is a *subsequence* of a_n.

Proposition 1.17. *x_n converges to $x_0 \iff$ every subsequence x_{n_k} converges to x_0.*

Definition 1.18. (X, d) metric space and $K \subseteq X$. We say that K is sequentially compact iff every sequence $x_n \in K$ has a convergent subsequence in K.

Proposition 1.19. *(X, d) metric space, K sequentially compact iff every infinite $E \subseteq K$ has a limit point in K.*

Proof. $\implies$ Let K be sequentially compact, take $E \subseteq K$ an infinite set. We can extract from E a sequence of distinct elements $x_n \in E \subseteq K$. Therefore x_n has a convergence subsequence x_{n_k} s.t the limit of $x_{n_k} = x_0 \in K$. Moreover, x_{n_k} are distinct in E therefore $x_0 \in E' \cap K$.

$\impliedby$ Let x_n be a sequence in K, take

$$E = \{x_n\}_{n \in \mathbb{N}}$$

We have two cases:

- E **is finite.** Then $\exists x_0$ and $\exists n_1 < n_2 < \cdots$ (infinitely many) s.t

$$x_{n_k} = x_0, \quad \forall k \in \mathbb{N}$$

Therefore

$$\lim_{k \to \infty} x_{n_k} = x_0 \in E$$

We found a converging subsequence.

- E **is infinite.** Then by the assumption, we have that E has a limit point x_0 in K.

$\varepsilon = 1$ $\exists n_1$ s.t $x_{n_1} \in N_1^*(x_0) \cap E$.

$\varepsilon = 1/2$ $\exists n_2 > n_1$ s.t $x_{n_2} \in N_{1/2}^*(x_0) \cap E$ (since $N_{1/2}^*(x_0) \cap E$ is infinite).

we can construct $n_k > n_{k-1}$ s.t $x_{n_k} \in N_{1/k}^*(x_0) \cap E$, therefore

$$\lim_{n \to \infty} x_{n_k} = x_0 \in K$$

□

Example 1.20. In $(\mathbb{Q}, |.|)$, $K = \mathbb{Q} \cap [0, 1]$ is closed and bounded. P_n increasing sequence in $\mathbb{Q} \cap [0, 1]$ s.t

$$P_n \to 1/\sqrt{2} \in \mathbb{R}$$

Consider

$$\mathcal{G} = \Big\{(-1, P_1) \cap \mathbb{Q}, (-1, P_2) \cap \mathbb{Q}, \cdots, (-1, P_n) \cap \mathbb{Q}, \cdots, (1/\sqrt{2}, 10) \cap \mathbb{Q}\Big\}$$

Proposition 1.21. *$K \subseteq X$ is compact K is sequentially compact.*

Proof.

- Let $E \subseteq K$ infinite, suppose that $E' \cap K = \varnothing$. Take $x \in K \implies x \notin E'$, therefore $\exists \delta_x > 0$ s.t

$$N_{\delta_x}^*(x) \cap E = \varnothing$$

Let $\mathcal{G} = \{N_{\delta_x}(x)\}_{x\in K}$, is an open cover of K compact, hence it has a finite subcover

$$K \subseteq N_{\delta_{x_1}}(x_1) \cup \cdots \cup N_{\delta_{x_1}}(x_k)$$

But $E \subset K \implies E = (E \cap K)$ hence

$$\begin{aligned} E &\subseteq (N_{\delta_{x_1}}(x_1) \cap E) \cup \cdots \cup (N_{\delta_{x_1}}(x_k) \cap E) \\ &\subseteq \{x_1, \cdots, x_k\} \end{aligned}$$

therefore E is finite, contradiction.

□

1.2 Complete sets

Definition 1.22. Given a sequence x_n, we say that x_n is Cauchy iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t

$$d(x_n, x_m) < \varepsilon \quad \text{for } n, m \geq N$$

Proposition 1.23. *If x_n converges, then x_n is Cauchy.*

Proof. Triangle inequality. □

Remark 1.24. The converse is false. In $(\mathbb{Q}, |.|)$, in $(\mathbb{R}, |.|)$ there exists some sequence $x_n \in \mathbb{Q}$ s.t

$$x_n \to \sqrt{2} \quad \text{as } n \to \infty$$

But $x_n \to \sqrt{2}$ implies that x_n is Cauchy in $(\mathbb{R}, |.|)$, therefore x_n is Cauchy in $\mathbb{Q}$. Suppose x_n converges to P in $(\mathbb{Q}, |.|)$, therefore $x_n \to p \in (\mathbb{R}, |.|)$ by uniqueness of limit, $p = \sqrt{2}$, contradiction.

Definition 1.25. (X, d) metric space and $K \subseteq X$, we say that K is complete iff every Cauchy Sequence in K converges in K.

Example 1.26. $(\mathbb{R}, |.|)$ is complete (Math 210)

Proposition 1.27. *If K is complete then K is closed.*

Proof. Let x_n be a sequence in K s.t $x_n \to x_0 \in X$. Since x_n converges, then x_n is Cauchy and K is complete then x_n converges in K, therefore $x_0 \in K$. □

Proposition 1.28. *K complete and $E \subseteq K$ is closed, then E is complete.*

Proof. Let x_n be a Cauchy sequence in E, then x_n is Cauchy in K which is complete, therefore $x_n \to x_0 \in K$. But since E is closed, we get that $x_0 \in E$. □

Proposition 1.29. *(X, d) metric space and x_n Cauchy sequence in $K \subseteq X$. If x_n has a convergent subsequence x_{n_k} then x_n converges to the subsequential limit.*

Proof. Triangle inequality. □

Proposition 1.30. *(X, d) metric space and $K \subseteq X$ is sequentially compact then K is complete (and therefore closed).*

Proof. Let $\{x_n\}$ be a Cauchy sequence in K, K is sequentially compact therefore x_n has a convergent subsequence x_{n_k} that converges in K. Therefore by the previous proposition we get that x_n converges in K. $\square$

Theorem 1.31. *Let V be a finite dimensional inner product space over $\mathbb{R}$, then V is complete w.r.t norm induced by the inner product. Therefore $K \subseteq V$ is complete $\iff$ K is closed.*

(For the converse $K \subseteq V$ which is closed, then K is complete).

Proof. Let $S = \{v_1, \cdots, v_l\}$ be an orthonormal basis of V, take x_n be a cauchy sequence in V. Therefore

$$x_n = r_{1,l}v_1 + \cdots + r_{l,n}v_l$$

Take $r_{1,n} = \langle x_n, v_1 \rangle$ therefore

$$\begin{aligned} |r_{1,m} - r_{1,n}| &= |\langle x_m, v_1 \rangle - \langle x_n, v_1 \rangle| \\ &= |\langle x_m - x_n, v_1 \rangle| \\ &\leq \|x_m - x_n\| \|v_1\| \quad \text{By C.S} \end{aligned}$$

Let $\varepsilon > 0$ then $\exists N \in \mathbb{N}$ s.t

$$\|x_m - x_n\| < \varepsilon \implies r_{1,n} \text{ is Cauchy in } \mathbb{R}$$

THerefore $r_{1,n}$ converges to $r_1 \in \mathbb{R}$. Do the same fore very coordinate, and define

$$x = r_1 v_1 + \cdots + r_n v_n$$

Therefore

$$\begin{aligned} \|x_n - x\| &= \|(r_{1,n} - r_1)v_1 + \cdots + (r_{1,l} - r_l)v_l\| \\ &\leq \|r_{1,n} - r_1\| \|v_1\| + \cdots + \|r_{l,n} - r_l\| \|v_l\| \end{aligned}$$

Therefore

$$x_n \to x$$

$\square$

1.3 Total Boundedness

Definition 1.32. (X, d) metric space and $K \subseteq X$ we say that K is totally bounded iff for every $\varepsilon > 0$ $\exists x_1, \cdots, x_t \in K$ s.t

$$K \subseteq N_\varepsilon(x_1) \cup \cdots \cup N_\varepsilon(x_t)$$

Definition 1.33. $K \subseteq X$ is bounded iff $\exists R > 0$ s.t

$$K \subseteq N_R(x_0) \quad \text{for some } x_0 \in K$$

We say that a sequence is bounded iff

$$E = \{x_n\}_{n \in \mathbb{N}} \quad \text{is bounded}$$

Proposition 1.34. *Cauchy sequences are bounded.*

Proposition 1.35. *$K \subseteq X$ is totally bounded it is bounded.*

Note that the converse is false.

Example 1.36. $(\mathbb{R}, d_{disc})$, $\mathbb{R}$ is bounded since

$$\mathbb{R} \subseteq N_2(0)$$

but it isn't totally bounded, take $\varepsilon = 1/2$ then if it is totally bounded, then $\exists x_1 \cdots x_t \in \mathbb{R}$ s.t

$$\mathbb{R} \subseteq N_{1/2}(x_1) \cup \cdots \cup N_{1/2}(x_t) = \{x_1, \cdots, x_t\}$$

Proof. Take $\varepsilon = 1$ then $K \subseteq N_1(x_1) \cup N_1(x_t)$ for $x_1, \cdots, x_t \in K$. Take x_1 and let $d = \max_{k \in \{1, \cdots, t\}}(x_1, x_t)$, therefore for any $x \in K$ we have that

$$d(x, x_1) \leq d(x, x_i) + d(x_i, x_1)$$

for some x_i s.t $x \in N_1(x_i)$ (Exists by the above), therefore

$$d(x, x_1) \leq 1 + d \implies x \in N_{1+d}(x_1)$$

□

Proposition 1.37. *K is sequentially compact then K is totally bounded.*

Proof. Suppose not, then $\exists \varepsilon_0$ s.t K cannot be covered by finitely many ε_0-neighborhoods. Fix some x_1 then $\exists x_2 \in K \setminus N_{\varepsilon_0}(x_1)$, Similarly there exists some $x_3 \in K \setminus N_{\varepsilon_0}(x_1) \cup N_{\varepsilon_0}(x_2)$ and in general

$$\exists x_n \in K \setminus (N_{\varepsilon_0}(x_1) \cup \cdots \cup N_{\varepsilon_0}(x_{n-1}))$$

Therefore we have that for $n \neq m$

$$d(x_n, x_m) \geq \varepsilon_0$$

Therefore x_n doesn't have a convergent subsequence; contradiction. □

Lemma 1.38. *Every subset of a totally bounded set is totally bounded.*

Proof. $K \subseteq X$ totally bounded, and $E \subseteq K$; take $\varepsilon > 0$, therefore $\exists x_1, x_2, \cdots, x_t \in K$ s.t

$$K \subseteq N_{\varepsilon/2}(x_1) \cup \cdots \cup N_{\varepsilon/2}(x_t)$$

Ignore the neighborhood that don't intersect E, renumbering we have $x_1, \cdots, x_l$ s.t

$$N_{\varepsilon/2}(x_i) \cap E \neq \varnothing$$

We get

$$E \subseteq N_{\varepsilon/2}(x_1) \cup \cdots \cup N_{\varepsilon/2}(x_l)$$

Take $y_i \in N_{\varepsilon/2}(x_i) \cap E$, then

$$N_\varepsilon(y_i) \supseteq N_{\varepsilon/2}(x_i)$$

Therefore $E \subseteq N_\varepsilon(y_1) \cup \cdots \cup N_\varepsilon(y_l)$ and hence E is totally bounded. □

Example 1.39. $[0, 1]$ is totally bounded (since it is compact), therefore $(0, 1)$ is totally bounded.

Proposition 1.40. *(X, d) Metric space, $K \subseteq X$ is complete and totally bounded then K is compact.*

Proof. Assume K is not compact, fix $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ an open cover of K with no finite subcover. Let $\varepsilon = 1$, K is totally bounded then K can be covered by finitely many 1-balls with center in K. Hence $\exists x_1 \in K$ s.t $K_1 = N_1(x_1) \cap K$ cannot be covered by finitely many open sets in $\mathcal{G}$. But K_1 is totally bounded (subset of K), now let $\varepsilon = 1/2$, $\exists x_2 \in K_1$ s.t $K_2 = N_{1/2}(x_2) \cap K_1$ cannot be covered by finitely many open sets in $\mathcal{G}$. Similarly, we construct a sequence x_n and

$$K_n = N_{1/n}(x_n) \cap K_{n-1} \quad x_n \in K_{n-1}$$

such that K_n cannot be covered by finitely many G_α's. Notice that

$$K_0 = K \supseteq K_1 \supseteq K_2 \supseteq \cdots$$

Claim. x_n is Cauchy. Indeed, let $\varepsilon > 0$ then $\exists N$ s.t

$$1/N < \varepsilon/2$$

Now take $m, n > N$, then

$$x_n \in K_{n-1} \subseteq K_N, \quad x_m \in K_{m-1} \subseteq K_N$$

therefore $x_n, x_m \in K_N = N_{1/N}(x_N) \cap K_{n-1}$, and hence

$$d(x_n, x_m) \leq 2/N < \varepsilon$$

By completeness of K, we get that x_n converges to some $x_0 \in K$; since $\mathcal{G} = \{G_\alpha\}$ is an open cover of K, there exists $\alpha_0 \in I$ s.t $x_0 \in G_{\alpha_0}$ (open set). So $\exists \varepsilon_0$ s.t

$$N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

But $x_n \to x_0$ then $\exists N_1$ s.t

$$x_N \in N_{\varepsilon_0/2}(x_0) \text{ and } 1/N < \varepsilon_0/2$$

Therefore

$$K_N \subseteq N_{1/N}(x_n) \subseteq N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

contradiction. □

Theorem 1.41. *V finite dimensional product normed space (over $\mathbb{R}$). Let $S = \{v_1, \cdots, v_n\}$ a basis of V. Then the cube*

$$Q_S = \{r_1 v_1 + r_2 v_2 + \cdots + r_n v_n \colon 0 \leq r_i \leq 1\}$$

is compact.

Proof. Let $x^k = r_{1,k} v_1 + r_{2,k} v_2 + \cdots + r_{n,k} v_n$ a sequence in Q_S, now clearly

$$r_{1,k} \text{ is a sequence in } [0,1] \text{, compact set}$$

Therefore $r_{1,k}$ has a convergent subsequence r_{1,k_n}. Similarly $r_{2,k_l} \in [0,1]$ therefore r_{2,k_l} has a convergent subsequence $r_{2,k_{l_t}}$. Doing this n times we get that $\exists$ a subsequence $\{k_j\}$ s.t

$$r_{1,k_j},\ r_{2,k_j}, \cdots,\ r_{n,k_j} \to r_1, r_2, \cdots, r_n$$

take

$$x_0 = r_1 v_1 + \cdots + r_n v_n$$

We can show that

$$\|x_0 - x^{k_j}\| \leq |r_{1,k_j} - r_1|\|v_1\| + \cdots + |r_{n,k_j} - r_n|\|v_n\| \to 0$$

then

$$x^{k_j} \to 0$$

Hence Q_S is (sequentially) compact. □

Theorem 1.42. *Let V be a finite inner product space then*

$$K \subseteq V \textit{ is compact} \iff K \textit{ closed and bounded}$$

Proof.
- K compact $\implies$ complete $\implies$ closed and $\implies$ totally bounded $\implies$ bounded.
- K is bounded, therefore $\exists R > 0$ s.t $K \subseteq B_{\|.\|}(0, R)$. Let $S = \{v_1, \cdots, v_n\}$ be a basis of V and Take $x \in K$ therefore

  $$x = r_1 v_1 + \cdots + r_n v_n$$

  then $r_i = \|\langle x, v_i \rangle\| \leq \|x\| \cdot \|v_i\|$

  $$B_{\|.\|}(0, R) \subseteq \{r_1 v_1 + \cdots + r_n v_n \colon r_i \leq R\}$$

  Therefore $K \subseteq$ some cube and hence K is compact.

□

Proposition 1.43. *In a finite dimensional inner product space we get that bounded $\iff$ totally bounded.*

Proof. $\implies$ Bounded $\implies$ can be put in a cube (compact) $\implies$ Totally bounded.
$\impliedby$ True in general.

□

1.4 Continuity

Definition 1.44. X, Y metric spaces and $E \subseteq X$ and $f\colon E \to Y$ map; $x_0 \in E'$ we say that

$$\lim_{x \to x_0} f(x) = y \in Y$$

iff for every $\varepsilon > 0$ $\exists \delta > 0$ s.t

$$f(x) \in N_\varepsilon(y_0) \text{ for } x \in N^*_\delta(x_0) \cap E$$

Definition 1.45. $f\colon E \to Y$ continuous at x_0 iff for every $\varepsilon > 0$, $\exists \delta > 0$ s.t $f(x) \in N_\varepsilon(x_0)$ for $x \in N_\delta(x_0) \cap E$; i.e

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{for } 0 < d_X(x, x_0) < \delta$$

Remark 1.46. f is continuous at $x_0 \iff x_0$ is an isolated point or $x_0 \in E'$ and $\lim x \to x_0 f(x) = f(x_0)$.

Proposition 1.47. $\lim_{x \to x_0} = y_0$ *iff for every sequence $x_n \to x_0$ we have that $f(x_n) \to y_0$. Therefore f is continuous at x_0 iff for every $x_n \to x_0$ we have that $f(x_n) \to x_0$.*

Corollary 1.48. *if f, g are continuous then $f + g,\ cf,\ f \cdot g,\ f/g$ are all continuous at x_0.*