

# Advanced Calculus

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## 1 Metric Spaces

**Definition 1.1.** Let  $X$  be a set, a *metric*  $d$  is a map

$$d: X \times X \rightarrow \mathbb{R}$$

such that:

- (i)  $d(x, y) \geq 0$  with equality iff  $x = y$ .
- (ii)  $d(x, y) = d(y, x)$
- (iii)  $d(x, z) \leq d(x, y) + d(y, z)$

**Example 1.2.** Given a vector space  $V$  and norm  $\|\cdot\|$  we can define the metric

$$d(x, y) = \|x - y\|$$

**Question.** Given a vector space  $V$  and a metric  $d$  on  $V$ , does  $d$  "come from" a norm? i.e does there exist some norm  $\|\cdot\|$  s.t

$$d(x, y) = \|x - y\|$$

The answer in general is no (consider the discrete metric, or even any bounded metric: this contradicts homogeneity).

**Notation.** Take  $(X, d)$  metric space  $\delta > 0$  and  $x_0 \in X$ , the  $\delta$ -neighborhood of  $x_0$

$$N_\delta(x_0) := \{x \in X : d(x, x_0) < \delta\}$$

**Definition 1.3.**  $(X, d)$  metric space and  $E \subset X$

(i)  $x_0 \in X$  is an *interior point* of  $E$  iff  $\exists \delta > 0$  s.t

$$N_\delta(x_0) \subset E$$

(ii)  $x_0 \in X$  is a *limit point* iff  $\forall \delta > 0$

$$N_\delta^*(x_0) \cap E \neq \emptyset$$

(We denote by  $E'$  the set of limit points of  $E$ )

(iii)  $E$  is *open* iff every  $x_0 \in E$  is an interior point of  $E$ .

(iv)  $E$  is *closed* iff every  $E' \subset E$ .

(v) The *closure* of  $E$  denoted  $\overline{E}$  is

$$\overline{E} := E \cup E'$$

**Review.** Fix  $(X, d)$  a metric space.

1.  $N_\delta(x_0)$  is open  $\forall \delta > 0, \forall x_0 \in X$ .
2.  $E$  is open iff  $E^c$  is closed.
3.  $\overline{E}$  is closed, in fact  $\overline{E}$  is the smallest closed set containing  $E$ .
4. The union of any collection of open sets is open.
5. A finite intersection of a collection of open sets is open.
6. Intersection of any collection of closed sets is closed.
7. Finite union of closed is closed.
8. If  $x \in E'$  then  $N_\delta(x) \cap E'$  is an infinite set.

**Definition 1.4.** Let  $x_n$  be a sequence in  $X$ , we say that  $x_n$  converges to  $x_0 \in X$  iff  $\forall \varepsilon > 0 \exists N \in \mathbb{N}$  s.t

$$d(x_n, x_0) < \varepsilon \quad \forall n \geq N$$

1. The limit of a sequence (if exists) is unique.
2.  $x_n$  converges to  $x_0$  in  $(X, d)$  iff  $d(x_n, x_0)$  converges 0 in  $(\mathbb{R}, |||)$ .

**Example 1.5.**  $x_n = (1/n + 1, e^{-n})$  in  $\mathbb{R}^2$ . Find the limit of  $x_n$  in  $(\mathbb{R}^2, \|\cdot\|_\infty)$ ,  $(\mathbb{R}^2, \|\cdot\|_1)$ ,  $(\mathbb{R}^2, \|\cdot\|_2)$ .

•

$$\|(1/n + 1, e^{-n}, (1, 0))\|_\infty = \max(1/n, e^{-n}) \leq 1/n + e^{-n} \rightarrow 0$$

- $x_0 = (1, 0)$

$$\|(1/n + 1, e^{-n}, -(1, 0))\|_1 = \|(1/n, e^{-n})\|_1 = |1/n| + |e^{-n}| \rightarrow 0$$

- 

$$\|(1/n + 1, e^{-n}) - (1, 0)\|_2 = \sqrt{1/n^2 + e^{-2n}} \rightarrow 0$$

**Remark 1.6.**  $x_n \rightarrow x_0$  in  $(\mathbb{R}^k, \|\cdot\|_\infty)$

$$(x_{1,n}, x_{2,n}, \dots, x_{k,n}) \rightarrow (x_1, 0, \dots, x_{k,0})$$

Iff

$$x_{1,n} \rightarrow x_{1,0}$$

$$x_{2,n} \rightarrow x_{2,0}$$

$$\vdots$$

$$x_{k,n} \rightarrow x_{k,0}$$

in  $(\mathbb{R}, \|\cdot\|)$ .

*Proof.*  $|x_{i,n} - x_{i,0}| \leq \|x_n - x_0\|_\infty \leq |x_{1,n} - x_{1,0}| + \dots + |x_{k,n} - x_{k,0}|$  Therefore  $x_n \rightarrow x_0$  in  $(\mathbb{R}^k, \|\cdot\|_\infty)$   
 $\iff \|x_n - x_0\|_\infty \rightarrow 0$  and  $\iff |x_{i,n} - x_{i,0}| \rightarrow 0 \forall i$  and hence  $\iff x_{i,n} \rightarrow x_{i,0} \forall i$ .  $\square$

**Remark 1.7.** Take  $x \in \mathbb{R}^n$ , then

$$\|x\|_\infty = \max_i(x_i) \leq |x_1| + \dots + |x_n| = \|x\|_1 \leq n\|x\|_\infty$$

Therefore

$$\|x\|_\infty \leq \|x\|_1 \leq n \cdot \|x\|_\infty$$

Similarly, show something for  $\|x\|_2$  and  $\|x\|_\infty$ .

**Definition 1.8.** Given a vector space  $V$  and norms  $\|\cdot\|_a$  and  $\|\cdot\|_b$  we say that  $\|\cdot\|_a$  and  $\|\cdot\|_b$  are equivalent iff  $\exists c_1, c_2 > 0 \in \mathbb{R}$  s.t  $\forall x \in V$

$$c_1\|x\|_b \leq \|\cdot\|_a \leq c_2\|x\|_b$$

**Lemma 1.9.**  $(X, d)$  metric space and  $E \subseteq X$ , then  $E$  is closed iff for sequence  $x_n \in E$  that converges  $\lim_{n \rightarrow \infty} x_n \in E$ .

*Proof.*  $\implies$  Assume that there exists a sequence  $x_n$  in  $E$  that converges to  $x_0$  s.t  $x_0 \notin E$ , take  $\varepsilon > 0$ , then since  $x_n \rightarrow x_0$  then  $\exists N \in \mathbb{N}$  s.t

$$x_n \in N_\varepsilon(x_0) \quad \forall n \geq N$$

But  $x_n \in E$ ,

$$N_\varepsilon^*(x_0) \cap E \neq \emptyset$$

Therefore

$$x_0 \in E'$$

Contradiction to the fact that  $E$  is closed.

- Take  $x_0 \in E'$ , let  $\varepsilon_n = 1/n$  then

$$\exists x_n \in N_n^*(x_0) \cap E$$

Consider  $\{x_n\}_{n \in \mathbb{N}}$   $x_n$  is a sequence in  $E$  s.t

$$d(x_n, x_0) < 1/n \implies x_n \rightarrow x_0$$

and hence  $x_0 \in E$ .

□

**Theorem 1.10.** Given a vector space  $V$  and norms  $\|\cdot\|_a$  and  $\|\cdot\|_b$  then the following are all equivalent:

1.  $\|\cdot\|_a$  and  $\|\cdot\|_b$  are equivalent ( $c_1\|x\|_b \leq \|x\|_a \leq c_2\|x\|_b$ ).
2.  $x_n \rightarrow x_0$  in  $\|\cdot\|_a \iff x_n \rightarrow x_0$  in  $\|\cdot\|_b$ .
3.  $E$  is closed in  $(V, \|\cdot\|_a) \iff E$  is closed in  $(V, \|\cdot\|_b)$ .
4.  $U$  is open in  $(V, \|\cdot\|_a) \iff U$  is open in  $(V, \|\cdot\|_b)$ .

*Proof.*  $\implies 2$

$$c_1\|x_n - x_0\|_b \leq \|x_n - x_0\|_a \leq c_2\|x_n - x_0\|_b \leq \frac{c_2}{c_1}\|x_n - x_0\|_a$$

Apply Squeeze.

$2 \implies 3$  By the Lemma.

$3 \implies 4$  Trivial.

$4 \implies 1$   $B_{\|\cdot\|_a}(0, 1)$  open in  $(V, \|\cdot\|_a)$  therefore it is open in  $(V, \|\cdot\|_b)$ . Hence 0 is an interior point in  $(V, \|\cdot\|_b)$ . Hence  $\exists \delta > 0$

$$B_{\|\cdot\|_b}(0, \delta) \subseteq B_{\|\cdot\|_a}(0, 1)$$

Let  $x \in V, x \neq 0$ .

$$\frac{\delta}{2} \cdot \frac{x}{\|x\|_b} \in B_{\|\cdot\|_b}(0, \delta) \subseteq B_{\|\cdot\|_a}(0, 1)$$

Therefore

$$\left\| \frac{\delta}{2} \frac{x}{\|x\|_b} \right\|_a < 1$$

We get

$$\frac{\delta}{2\|x\|_b} \|x\|_a < 1 \implies \|x\|_a \leq \frac{2}{\delta} \|x\|_b \quad \forall x \in V$$

□

## 1.1 Compact Sets

**Definition 1.11.**  $(X, d)$  metric space,  $K \subseteq X$ . We say that  $K$  is *compact* iff for every open cover

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

of  $K$  has a finite subcover.

**Example 1.12.**  $(0, 1]$  is not compact in  $(\mathbb{R}, |\cdot|)$ , take

$$\mathcal{G} = \{(1/n, 10)\}_{n \in \mathbb{N}}$$

which has no finite subcover.

**Theorem 1.13** (Heine-Borel). *In  $\mathbb{R}$   $[a, b]$  is compact.*

**Example 1.14.** If  $X$  is finite, then  $X$  is compact.

*Proof.* Let  $X = \{x_1, \dots, x_n\}$  and let

$$\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$$

be an open cover  $X$ . Then for every  $x_i$ , fix some  $G_{\alpha_i}$  s.t

$$x = \{x_1, \dots, x_n\} \subseteq \bigcup_{i=1}^n G_{\alpha_i}$$

□

**Proposition 1.15.**  $(X, d)$  metric space and  $K \subseteq X$  then every closed subset  $E$  of  $K$  is compact.

*Proof.* Let  $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$  be an open cover of  $E$ , then

$$E \subseteq \bigcup_{\alpha \in I} G_\alpha$$

Therefore

$$K \subseteq \bigcup G_\alpha \cup E^c, \quad E^c \text{ open}$$

Therefore

$$\mathcal{G}' = \mathcal{G} \cup \{E^c\} \quad \text{is an open cover of } K$$

Therefore it has a finite subcover and

$$K \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_k} \cup E^c$$

Hence

$$E \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_k}$$

Which is a finite subcover of  $\mathcal{G}$ .

□

**Definition 1.16.**  $(X, d)$  metric space and  $x_n$  sequence in  $X$  and

$$\varphi: \mathbb{N} \rightarrow \mathbb{N} \quad \text{strictly increasing}$$

We say that  $b_n = a_{\varphi(n)}$  is a *subsequence* of  $a_n$ .

**Proposition 1.17.**  $x_n$  converges to  $x_0 \iff$  every subsequence  $x_{n_k}$  converges to  $x_0$ .

**Definition 1.18.**  $(X, d)$  metric space and  $K \subseteq X$ . We say that  $K$  is sequentially compact iff every sequence  $x_n \in K$  has a convergent subsequence in  $K$ .

**Proposition 1.19.**  $(X, d)$  metric space,  $K$  sequentially compact iff every infinite  $E \subseteq K$  has a limit point in  $K$ .

*Proof.*  $\implies$  Let  $K$  be sequentially compact, take  $E \subseteq K$  an infinite set. We can extract from  $E$  a sequence of distinct elements  $x_n \in E \subseteq K$ . Therefore  $x_n$  has a convergence subsequence  $x_{n_k}$  s.t the limit of  $x_{n_k} = x_0 \in K$ . Moreover,  $x_{n_k}$  are distinct in  $E$  therefore  $x_0 \in E' \cap K$ .  
 $\impliedby$  Let  $x_n$  be a sequence in  $K$ , take

$$E = \{x_n\}_{n \in \mathbb{N}}$$

We have two cases:

- **$E$  is finite.** Then  $\exists x_0$  and  $\exists n_1 < n_2 < \dots$  (infinitely many) s.t

$$x_{n_k} = x_0, \quad \forall k \in \mathbb{N}$$

Therefore

$$\lim_{k \rightarrow \infty} x_{n_k} = x_0 \in E$$

We found a converging subsequence.

- **$E$  is infinite.** Then by the assumption, we have that  $E$  has a limit point  $x_0$  in  $K$ .

$$\varepsilon = 1 \quad \exists n_1 \text{ s.t } x_{n_1} \in N_1^*(x_0) \cap E.$$

$$\varepsilon = 1/2 \quad \exists n_2 > n_1 \text{ s.t } x_{n_2} \in N_{1/2}^*(x_0) \cap E \text{ (since } N_{1/2}^*(x_0) \cap E \text{ is infinite).}$$

we can construct  $n_k > n_{k-1}$  s.t  $x_{n_k} \in N_{1/k}^*(x_0) \cap E$ , therefore

$$\lim_{n \rightarrow \infty} x_{n_k} = x_0 \in K$$

□

**Example 1.20.** In  $(\mathbb{Q}, |\cdot|)$ ,  $K = \mathbb{Q} \cap [0, 1]$  is closed and bounded.  $P_n$  increasing sequence in  $\mathbb{Q} \cap [0, 1]$  s.t

$$P_n \rightarrow 1/\sqrt{2} \in \mathbb{R}$$

Consider

$$\mathcal{G} = \left\{ (-1, P_1) \cap \mathbb{Q}, (-1, P_2) \cap \mathbb{Q}, \dots, (-1, P_n) \cap \mathbb{Q}, \dots, (1/\sqrt{2}, 10) \cap \mathbb{Q} \right\}$$

**Proposition 1.21.**  $K \subseteq X$  is compact  $\iff$   $K$  is sequentially compact.

*Proof.* • Let  $E \subseteq K$  infinite, suppose that  $E' \cap K = \emptyset$ . Take  $x \in K \implies x \notin E'$ , therefore  $\exists \delta_x > 0$  s.t

$$N_{\delta_x}^*(x) \cap E = \emptyset$$

Let  $\mathcal{G} = \{N_{\delta_x}(x)\}_{x \in K}$ , is an open cover of  $K$  compact, hence it has a finite subcover

$$K \subseteq N_{\delta_{x_1}}(x_1) \cup \cdots \cup N_{\delta_{x_k}}(x_k)$$

But  $E \subset K \implies E = (E \cap K)$  hence

$$\begin{aligned} E &\subseteq (N_{\delta_{x_1}}(x_1) \cap E) \cup \cdots \cup (N_{\delta_{x_k}}(x_k) \cap E) \\ &\subseteq \{x_1, \dots, x_k\} \end{aligned}$$

therefore  $E$  is finite, contradiction. □

## 1.2 Complete sets

**Definition 1.22.** Given a sequence  $x_n$ , we say that  $x_n$  is Cauchy iff for every  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  s.t

$$d(x_n, x_m) < \varepsilon \quad \text{for } n, m \geq N$$

**Proposition 1.23.** If  $x_n$  converges, then  $x_n$  is Cauchy.

*Proof.* Triangle inequality. □

**Remark 1.24.** The converse is false. In  $(\mathbb{Q}, |\cdot|)$ , in  $(\mathbb{R}, |\cdot|)$  there exists some sequence  $x_n \in \mathbb{Q}$  s.t

$$x_n \rightarrow \sqrt{2} \quad \text{as } n \rightarrow \infty$$

But  $x_n \rightarrow \sqrt{2}$  implies that  $x_n$  is Cauchy in  $(\mathbb{R}, |\cdot|)$ , therefore  $x_n$  is Cauchy in  $\mathbb{Q}$ . Suppose  $x_n$  converges to  $P$  in  $(\mathbb{Q}, |\cdot|)$ , therefore  $x_n \rightarrow p \in (\mathbb{R}, |\cdot|)$  by uniqueness of limit,  $p = \sqrt{2}$ , contradiction.

**Definition 1.25.**  $(X, d)$  metric space and  $K \subseteq X$ , we say that  $K$  is complete iff every Cauchy Sequence in  $K$  converges in  $K$ .

**Example 1.26.**  $(\mathbb{R}, |\cdot|)$  is complete (Math 210)

**Proposition 1.27.** If  $K$  is complete then  $K$  is closed.

*Proof.* Let  $x_n$  be a sequence in  $K$  s.t  $x_n \rightarrow x_0 \in X$ . Since  $x_n$  converges, then  $x_n$  is Cauchy and  $K$  is complete then  $x_n$  converges in  $K$ , therefore  $x_0 \in K$ . □

**Proposition 1.28.**  $K$  complete and  $E \subseteq K$  is closed, then  $E$  is complete.

*Proof.* Let  $x_n$  be a Cauchy sequence in  $E$ , then  $x_n$  is Cauchy in  $K$  which is complete, therefore  $x_n \rightarrow x_0 \in K$ . But since  $E$  is closed, we get that  $x_0 \in E$ . □

**Proposition 1.29.**  $(X, d)$  metric space and  $x_n$  Cauchy sequence in  $K \subseteq X$ . If  $x_n$  has a convergent subsequence  $x_{n_k}$  then  $x_n$  converges to the subsequential limit.

*Proof.* Triangle inequality. □

**Proposition 1.30.**  $(X, d)$  metric space and  $K \subseteq X$  is sequentially compact then  $K$  is complete (and therefore closed).

*Proof.* Let  $\{x_n\}$  be a Cauchy sequence in  $K$ ,  $K$  is sequentially compact therefore  $x_n$  has a convergent subsequence  $x_{n_k}$  that converges in  $K$ . Therefore by the previous proposition we get that  $x_n$  converges in  $K$ .  $\square$

**Theorem 1.31.** Let  $V$  be a finite dimensional inner product space over  $\mathbb{R}$ , then  $V$  is complete w.r.t norm induced by the inner product. Therefore  $K \subseteq V$  is complete  $\iff K$  is closed.

(For the converse  $K \subseteq V$  which is closed, then  $K$  is complete).

*Proof.* Let  $S = \{v_1, \dots, v_l\}$  be an orthonormal basis of  $V$ , take  $x_n$  be a cauchy sequence in  $V$ . Therefore

$$x_n = r_{1,n}v_1 + \dots + r_{l,n}v_l$$

Take  $r_{1,n} = \langle x_n, v_1 \rangle$  therefore

$$\begin{aligned} |r_{1,m} - r_{1,n}| &= |\langle x_m, v_1 \rangle - \langle x_n, v_1 \rangle| \\ &= |\langle x_m - x_n, v_1 \rangle| \\ &\leq \|x_m - x_n\| \|v_1\| \quad \text{By C.S} \end{aligned}$$

Let  $\varepsilon > 0$  then  $\exists N \in \mathbb{N}$  s.t

$$\|x_m - x_n\| < \varepsilon \implies r_{1,n} \text{ is Cauchy in } \mathbb{R}$$

Therefore  $r_{1,n}$  converges to  $r_1 \in \mathbb{R}$ . Do the same for every coordinate, and define

$$x = r_1v_1 + \dots + r_lv_l$$

Therefore

$$\begin{aligned} \|x_n - x\| &= \|(r_{1,n} - r_1)v_1 + \dots + (r_{l,n} - r_l)v_l\| \\ &\leq \|r_{1,n} - r_1\| \|v_1\| + \dots + \|r_{l,n} - r_l\| \|v_l\| \end{aligned}$$

Therefore

$$x_n \rightarrow x$$

$\square$

### 1.3 Total Boundedness

**Definition 1.32.**  $(X, d)$  metric space and  $K \subseteq X$  we say that  $K$  is totally bounded iff for every  $\varepsilon > 0 \exists x_1, \dots, x_t \in K$  s.t

$$K \subseteq N_\varepsilon(x_1) \cup \dots \cup N_\varepsilon(x_t)$$



**Definition 1.33.**  $K \subseteq X$  is bounded iff  $\exists R > 0$  s.t

$$K \subseteq N_R(x_0) \quad \text{for some } x_0 \in K$$

We say that a sequence is bounded iff

$$E = \{x_n\}_{n \in \mathbb{N}} \quad \text{is bounded}$$

**Proposition 1.34.** *Cauchy sequences are bounded.*

**Proposition 1.35.**  $K \subseteq X$  is totally bounded it is bounded.

Note that the converse is false.

**Example 1.36.**  $(\mathbb{R}, d_{disc}), \mathbb{R}$  is bounded since

$$\mathbb{R} \subseteq N_2(0)$$

but it isn't totally bounded, take  $\varepsilon = 1/2$  then if it is totally bounded, then  $\exists x_1 \cdots x_t \in \mathbb{R}$  s.t

$$\mathbb{R} \subseteq N_{1/2}(x_1) \cup \cdots \cup N_{1/2}(x_t) = \{x_1, \cdots, x_t\}$$

*Proof.* Take  $\varepsilon = 1$  then  $K \subseteq N_1(x_1) \cup N_1(x_t)$  for  $x_1, \cdots, x_t \in K$ . Take  $x_1$  and let  $d = \max_{k \in \{1, \dots, t\}}(x_1, x_t)$ , therefore for any  $x \in K$  we have that

$$d(x, x_1) \leq d(x, x_i) + d(x_i, x_1)$$

for some  $x_i$  s.t  $x \in N_1(x_i)$  (Exists by the above), therefore

$$d(x, x_1) \leq 1 + d \implies x \in N_{1+d}(x_1)$$

□

**Proposition 1.37.**  $K$  is sequentially compact then  $K$  is totally bounded.

*Proof.* Suppose not, then  $\exists \varepsilon_0$  s.t  $K$  cannot be covered by finitely many  $\varepsilon_0$ -neighborhoods. Fix some  $x_1$  then  $\exists x_2 \in K \setminus N_{\varepsilon_0}(x_1)$ , Similarly there exists some  $x_3 \in K \setminus N_{\varepsilon_0}(x_1) \cup N_{\varepsilon_0}(x_2)$  and in general

$$\exists x_n \in K \setminus (N_{\varepsilon_0}(x_1) \cup \cdots \cup N_{\varepsilon_0}(x_{n-1}))$$

Therefore we have that for  $n \neq m$

$$d(x_n, x_m) \geq \varepsilon_0$$

Therefore  $x_n$  doesn't have a convergent subsequence; contradiction. □

**Lemma 1.38.** *Every subset of a totally bounded set is totally bounded.*

*Proof.*  $K \subseteq X$  totally bounded, and  $E \subseteq K$ ; take  $\varepsilon > 0$ , therefore  $\exists x_1, x_2, \cdots, x_t \in K$  s.t

$$K \subseteq N_{\varepsilon/2}(x_1) \cup \cdots \cup N_{\varepsilon/2}(x_t)$$

Ignore the neighborhood that don't intersect  $E$ , renumbering we have  $x_1, \dots, x_l$  s.t

$$N_{\varepsilon/2}(x_i) \cap E \neq \emptyset$$

We get

$$E \subseteq N_{\varepsilon/2}(x_1) \cup \dots \cup N_{\varepsilon/2}(x_l)$$

Take  $y_i \in N_{\varepsilon/2}(x_i) \cap E$ , then

$$N_{\varepsilon}(y_i) \supseteq N_{\varepsilon/2}(x_i)$$

Therefore  $E \subseteq N_{\varepsilon}(y_1) \cup \dots \cup N_{\varepsilon}(y_l)$  and hence  $E$  is totally bounded.  $\square$

**Example 1.39.**  $[0, 1]$  is totally bounded (since it is compact), therefore  $(0, 1)$  is totally bounded.

**Proposition 1.40.**  $(X, d)$  Metric space,  $K \subseteq X$  is complete and totally bounded then  $K$  is compact.

*Proof.* Assume  $K$  is not compact, fix  $\mathcal{G} = \{G_{\alpha}\}_{\alpha \in I}$  an open cover of  $K$  with no finite subcover. Let  $\varepsilon = 1$ ,  $K$  is totally bounded then  $K$  can be covered by finitely many 1-balls with center in  $K$ . Hence  $\exists x_1 \in K$  s.t  $K_1 = N_1(x_1) \cap K$  cannot be covered by finitely many open sets in  $\mathcal{G}$ . But  $K_1$  is totally bounded (subset of  $K$ ), now let  $\varepsilon = 1/2$ ,  $\exists x_2 \in K_1$  s.t  $K_2 = N_{1/2}(x_2) \cap K_1$  cannot be covered by finitely many open sets in  $\mathcal{G}$ . Similarly, we construct a sequence  $x_n$  and

$$K_n = N_{1/n}(x_n) \cap K_{n-1} \quad x_n \in K_{n-1}$$

such that  $K_n$  cannot be covered by finitely many  $G_{\alpha}$ 's. Notice that

$$K_0 = K \supseteq K_1 \supseteq K_2 \supseteq \dots$$

**Claim.**  $x_n$  is Cauchy. Indeed, let  $\varepsilon > 0$  then  $\exists N$  s.t

$$1/N < \varepsilon/2$$

Now take  $m, n > N$ , then

$$x_n \in K_{n-1} \subseteq K_N, \quad x_m \in K_{m-1} \subseteq K_N$$

therefore  $x_n, x_m \in K_N = N_{1/N}(x_N) \cap K_{N-1}$ , and hence

$$d(x_n, x_m) \leq 2/N < \varepsilon$$

By completeness of  $K$ , we get that  $x_n$  converges to some  $x_0 \in K$ ; since  $\mathcal{G} = \{G_{\alpha}\}$  is an open cover of  $K$ , there exists  $\alpha_0 \in I$  s.t  $x_0 \in G_{\alpha_0}$  (open set). So  $\exists \varepsilon_0$  s.t

$$N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

But  $x_n \rightarrow x_0$  then  $\exists N_1$  s.t

$$x_N \in N_{\varepsilon_0/2}(x_0) \text{ and } 1/N < \varepsilon_0/2$$

Therefore

$$K_N \subseteq N_{1/N}(x_n) \subseteq N_{\varepsilon_0}(x_0) \subseteq G_{\alpha_0}$$

contradiction. □

**Theorem 1.41.** *V finite dimensional product normed space (over  $\mathbb{R}$ ). Let  $S = \{v_1, \dots, v_n\}$  a basis of  $V$ . Then the cube*

$$Q_S = \{r_1v_1 + r_2v_2 + \dots + r_nv_n : 0 \leq r_i \leq 1\}$$

*is compact.*

*Proof.* Let  $x^k = r_{1,k}v_1 + r_{2,k}v_2 + \dots + r_{n,k}v_n$  a sequence in  $Q_S$ , now clearly

$$r_{1,k} \text{ is a sequence in } [0, 1], \text{ compact set}$$

Therefore  $r_{1,k}$  has a convergent subsequence  $r_{1,k_{l_1}}$ . Similarly  $r_{2,k_l} \in [0, 1]$  therefore  $r_{2,k_l}$  has a convergent subsequence  $r_{2,k_{l_2}}$ . Doing this  $n$  times we get that  $\exists$  a subsequence  $\{k_j\}$  s.t

$$r_{1,k_j}, r_{2,k_j}, \dots, r_{n,k_j} \rightarrow r_1, r_2, \dots, r_n$$

take

$$x_0 = r_1v_1 + \dots + r_nv_n$$

We can show that

$$\|x_0 - x^{k_j}\| \leq |r_{1,k_j} - r_1|\|v_1\| + \dots + |r_{n,k_j} - r_n|\|v_n\| \rightarrow 0$$

then

$$x^{k_j} \rightarrow 0$$

Hence  $Q_S$  is (sequentially) compact. □

**Theorem 1.42.** *Let  $V$  be a finite inner product space then*

$$K \subseteq V \text{ is compact} \iff K \text{ closed and bounded}$$

*Proof.* •  $K \text{ compact} \implies \text{complete} \implies \text{closed and} \implies \text{totally bounded} \implies \text{bounded}.$

- $K$  is bounded, therefore  $\exists R > 0$  s.t  $K \subseteq B_{\|\cdot\|}(0, R)$ . Let  $S = \{v_1, \dots, v_n\}$  be a basis of  $V$  and Take  $x \in K$  therefore

$$x = r_1v_1 + \dots + r_nv_n$$

$$\text{then } r_i = \|\langle x, v_i \rangle\| \leq \|x\| \cdot \|v_i\|$$

$$B_{\|\cdot\|}(0, R) \subseteq \{r_1v_1 + \dots + r_nv_n : r_i \leq R\}$$

Therefore  $K \subseteq$  some cube and hence  $K$  is compact. □

**Proposition 1.43.** *In a finite dimensional inner product space we get that bounded  $\iff$  totally bounded.*

*Proof.*  $\implies$  Bounded  $\implies$  can be put in a cube (compact)  $\implies$  Totally bounded.

$\impliedby$  True in general.

□

## 1.4 Continuity

**Definition 1.44.**  $X, Y$  metric spaces and  $E \subseteq X$  and  $f: E \rightarrow Y$  map;  $x_0 \in E'$  we say that

$$\lim_{x \rightarrow x_0} f(x) = y \in Y$$

iff for every  $\varepsilon > 0 \exists \delta > 0$  s.t

$$f(x) \in N_\varepsilon(y_0) \text{ for } x \in N_\delta^*(x_0) \cap E$$

**Definition 1.45.**  $f: E \rightarrow Y$  continuous at  $x_0$  iff for every  $\varepsilon > 0, \exists \delta > 0$  s.t  $f(x) \in N_\varepsilon(y_0)$  for  $x \in N_\delta(x_0) \cap E$ ; i.e

$$d_Y(f(x), f(x_0)) < \varepsilon \text{ for } 0 < d_X(x, x_0) < \delta$$

**Remark 1.46.**  $f$  is continuous at  $x_0 \iff x_0$  is an isolated point or  $x_0 \in E'$  and  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .

**Proposition 1.47.**  $\lim_{x \rightarrow x_0} f(x) = y_0$  iff for every sequence  $x_n \rightarrow x_0$  we have that  $f(x_n) \rightarrow y_0$ . Therefore  $f$  is continuous at  $x_0$  iff for every  $x_n \rightarrow x_0$  we have that  $f(x_n) \rightarrow f(x_0)$ .

**Corollary 1.48.** if  $f, g$  are continuous then  $f + g, cf, f \cdot g, f/g$  are all continuous at  $x_0$ .

**Example 1.49.**  $f(x, y) = \frac{xy}{x^2 + y^2}$  we want to find

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

Take the sequence  $(x_n, y_n) = (1/n, 0)$  then

$$\lim_{n \rightarrow \infty} (x_n, y_n) = (0, 0)$$

and

$$f(x_n, y_n) = 0 \rightarrow 0 \text{ as } n \rightarrow \infty$$

Now take  $(x_n, y_n) = (1/n, 1/n)$ , then

$$\lim_{n \rightarrow \infty} (x_n, y_n) = (0, 0)$$

but

$$f(x_n, y_n) = \frac{1/n \cdot 1/n}{1/n^2 + 1/n^2} = 1/2 \rightarrow 1/2 \text{ as } n \rightarrow \infty$$

Therefore

$$\lim_{n \rightarrow \infty} f(x, y) \text{ does not exist}$$

Now we do the same for

$$f(x, y) = \frac{x^2 y}{x^2 + y^2}$$

Notice that

$$0 \leq |f(x, y)| = \left| \frac{x^2}{x^2 + y^2} y \right| \leq |y|$$

but

$$\lim_{(x,y) \rightarrow (0,0)} |y| = 0$$

therefore by squeeze we get that

$$\lim_{(x,y) \rightarrow 0} |f(x, y)| = 0$$

Hence

$$\lim_{(x,y) \rightarrow 0} f(x, y) = 0$$

### Some useful inequalities

$$x^2 + y^2 \geq x^2$$

$$x^2 + y^2 \geq y^2$$

$$x^2 + y^2 \geq 2(xy)$$

**Example 1.50.** Let  $(V, \|\cdot\|)$  a normed space and

$$\varphi: V \rightarrow \mathbb{R}, \quad \varphi(x) = \|x\|$$

Then  $\varphi$  is continuous.

Let  $x_n$  a sequence in  $V$  that converges to  $x_0$  in  $(V, \|\cdot\|)$  therefore

$$\|x_n - x_0\| \rightarrow 0$$

We want to show that  $\varphi(x_n) \rightarrow \varphi(x_0)$  i.e that

$$|\|x_n\| - \|x_0\|| = 0$$

But as shown in PS1, we have that

$$|\|x_n\| - \|x_0\|| \leq \|x_n - x_0\|$$

**Proposition 1.51.**  $f: X \rightarrow Y$  is continuous iff  $f^{-1}(G)$  open in  $X$  for every  $G$  open in  $Y$  iff  $f^{-1}(H)$  is closed in  $X$  for every  $H$  closed in  $Y$ .

*Proof.*  $\implies$  let  $x_n$  be a sequence in  $f^{-1}(H)$  that converges to  $x_0$  in  $X$ . Hence  $f(x_n)$  is a sequence in  $H$  by continuity of  $f$  we have that

$$\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

Moreover since  $H$  is closed, we get that  $f(x_0) \in H$ . Hence

$$x_0 \in f^{-1}(H)$$

Therefore if  $f$  is continuous  $f^{-1}(G)$  is open in  $X$  for every  $G$  open in  $Y$ .

$\Leftarrow$  Now we show that if  $f^{-1}(G)$  is open in  $X$  for all  $G$  open in  $Y$  then  $f$  is continuous. Fix  $x_0 \in X$ , let  $\varepsilon > 0$  then consider the open set in  $Y$   $N_\varepsilon(f(x_0))$  hence  $f^{-1}(N_\varepsilon(f(x_0)))$  is open in  $X$ . But  $x_0 \in f^{-1}(N_\varepsilon(f(x_0)))$  hence  $\exists \delta > 0$  s.t

$$f(N_\delta(x_0)) \subseteq N_\varepsilon(f(x_0))$$

□

**Example 1.52.**  $E = \{x, y \in \mathbb{R}^2 : y \geq x\}$  is open in  $\mathbb{R}^2$ , define the continuous function

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = y - x$$

Then

$$E = f^{-1}((0, \infty))$$

which is an open set.

**Example 1.53.**  $f: X \rightarrow Y$  continuous and consider the  $c$ -level set with  $c \in Y$

$$E = \{x \in X : f(x) = c\} = f^{-1}(\{c\})$$

is closed in  $X$ .

Therefore  $E = \{(x, y) : x^2 + 3y^2 = 1\}$  is closed in  $\mathbb{R}^2$ ,  $S^n$  is closed.

**Remark 1.54.**  $V, W$  normed spaces and

$$\|\cdot\|_a, \|\cdot\|_b \text{ equivalent norms on } V$$

so  $f: (V, \|\cdot\|_a) \rightarrow W$  is continuous iff  $f: (V, \|\cdot\|_b) \rightarrow W$  is continuous.

Equivalently:  $V, W$  normed spaces and

$$\|\cdot\|_a, \|\cdot\|_b \text{ equivalent norms on } W$$

so  $f: V \rightarrow (W, \|\cdot\|_a)$  is continuous iff  $f: V \rightarrow (W, \|\cdot\|_b)$  is continuous.

**Example 1.55.** Take  $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$  defined as follows

$$f(x) = (f_1(x), \dots, f_n(x)) \quad x = (x_1, \dots, x_m)$$

Then  $f$  is continuous iff  $f_i: \mathbb{R}^m \rightarrow \mathbb{R}$  is continuous for  $1 \leq i \leq n$ .

**Theorem 1.56** (Extreme Value Theorem).  $X, Y$  metric spaces and  $K \subseteq X$  compact and  $f: K \rightarrow Y$  continuous. Then  $f(K)$  is compact.

**Corollary 1.57.** *If  $Y = \mathbb{R}$ , then the above implies that  $f$  attains its maximum and minimum in  $K$ .*

*Proof.* Let  $y_n$  be a sequence in  $f(K)$ , therefore  $\exists x_n \in K$  s.t  $f(x_n) = y_n$ . Since  $K$  is compact,  $x_n$  has a convergent subsequence  $x_{n_k}$  that converges to  $x_0 \in K$ , using continuity we get that

$$y_{n_k} \rightarrow f(x_0) \in f(K)$$

□

**Theorem 1.58.** *All norms in a finite dimensional vector space are equivalent.*

*Proof.*  $V$  is a finite dimensional vector space, then by PS1 we can define an inner product on  $V$ . Let  $\|\cdot\|$  be the norm induced by our choice of inner product. Let  $S = \{v_1, \dots, v_n\}$  an orthonormal basis of  $V$ . Let  $\|\cdot\|_a$  be an arbitrary norm on  $V$ , we will show that  $\|\cdot\|$  and  $\|\cdot\|_a$  are equivalent. Let  $x \in V$ , then

$$x = r_1 v_1 + \dots + r_n v_n, \quad |r_i| = \langle x, v_i \rangle \leq \|x\|$$

Therefore  $\forall x \in V$

$$\begin{aligned} \|x\|_a &= \|r_1 v_1 + \dots + r_n v_n\| \\ &\leq |r_1| \|v_1\|_a + \dots + |r_n| \|v_n\|_a \\ &\leq (\|v_1\|_a + \dots + \|v_n\|_a) \cdot \|x\| \\ &\leq c_1 \cdot \|x\| \end{aligned}$$

Now define the function  $\varphi: (V, \|\cdot\|) \rightarrow \mathbb{R}$  s.t

$$\varphi(x) = \|x\|_a$$

We claim that  $\varphi$  is continuous, indeed let  $x_n \in V$  s.t  $x_n \rightarrow x_0$  in  $\|\cdot\|$ . Therefore

$$\begin{aligned} |\varphi(x_n) - \varphi(x_0)| &= |\|x_n\|_a - \|x_0\|_a| \\ &\leq \|x_n - x_0\| \\ &\leq c_1 \|x_n - x_0\| \rightarrow 0 \end{aligned}$$

Therefore  $\lim_{n \rightarrow \infty} \varphi(x_n) = \varphi(x_0)$ , so  $\varphi$  is continuous. Let  $K = \{x \in V: \|x\| = 1\}$  closed and bounded in the inner product space  $(V, \|\cdot\|)$ . Then it is compact in  $(V, \|\cdot\|)$  therefore  $\varphi(K)$  is compact in  $\mathbb{R}$  so it attains its minimum in  $K$ . Notice that  $\varphi(x) = 0 \iff x = 0$ , hence  $\forall x \in K, \varphi(x) > 0$ . Then

$$\min_{x \in K} \varphi(x) > 0$$

Let  $x \in V$  and  $x \neq 0$  then  $\frac{x}{\|x\|} \in K$ , therefore

$$\varphi\left(\frac{x}{\|x\|}\right) = \left\|\frac{x}{\|x\|}\right\|_a \geq \min \varphi(K)$$

Therefore

$$\|x\|_a \geq \min(\varphi(K))\|x\|$$

□

**Corollary 1.59.** *If  $V$  is finite dimensional then compact iff closed and bounded (independently of the choice of norm).*