

# Algebraic Topology

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## Contents

<b>1</b>	<b>Bases of the Fundamental Group</b>	<b>1</b>
1.1	Homotopy equivalence.	1
1.2	The Fundamental group	3
1.3	Homotopy of spaces	6
1.4	Fundamental group of all spheres	12
1.5	First Algebraic Detour	18
<b>2</b>	<b>The Van Kampen Theorem</b>	<b>26</b>

## 1 Bases of the Fundamental Group

### 1.1 Homotopy equivalence.

**Definition 1.1.** Let  $X$  be a topological space  $f, g: X \rightarrow Y$  then we say that  $f$  is *homotopic* to  $g$  if there is a function  $H: X \times I \rightarrow Y$  (called a homotopy) such that

$$H(x, 0) = f(x), \quad H(x, 1) = g(x)$$

**Proposition 1.2.** *Homotopy is an equivalence relation.*

**Lemma 1.3** (Gluing Lemma). *Suppose that  $X = A \cup B$  with  $A, B$  closed and take  $f_1: A \rightarrow Y$ ,  $f_2: B \rightarrow Y$ . s.t  $f_1(x) = f_2(x)$  for all  $x \in A \cap B$ . Then*

$$f_3: X \rightarrow Y, \quad f_3(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in B \end{cases}$$

*is continuous.*

*Proof.* Let  $C$  be a closed set in  $Y$ , then

$$\begin{aligned} f_3^{-1}(C) &= f_3^{-1}(C) \cap (A \cup B) \\ &= (f_3^{-1}(C) \cap A) \cup (f_3^{-1}(C) \cap B) \\ &= f_1^{-1}(C) \cup f_2^{-1}(C) \end{aligned}$$

which is closed in  $X$ . □

**Example 1.4.** Any two function into  $\mathbb{R}^n$  (convex set, star shaped set) are homotopic.

Indeed, take two such  $f(x), g(x)$  then let

$$H: X \times Y \rightarrow \mathbb{R}^n, \quad H(x, t) = (1 - t)f(x) + tg(x)$$

Note: we call this *the straight line homotopy*.

### Relative Homotopy

**Definition 1.5.** Let  $f, g: X \rightarrow Y$  with  $A \subseteq X$  and  $f|_A = g|_A$  then we say that  $f$  is homotopic to  $g$  relative to  $A$  if there exists a homotopy  $H: X \times I \rightarrow Y$  between  $f$  and  $g$

$$H(x, t) = f(x) = g(x) \quad \forall t \in I, \forall x \in A$$

**Remark 1.6.** Homotopy is a special kind of relative homotopy (for  $A = \emptyset$ ).

**Proposition 1.7.** *Relative homotopy is an equivalence relation.*

**Definition 1.8.** Let  $X$  a topological space and let  $x_1, x_2 \in X$  then a path in  $X$  going from  $x_1$  to  $x_2$  is a function  $\gamma: I \rightarrow X$  s.t

$$\gamma(0) = x_1, \quad \gamma(1) = x_2$$

**Definition 1.9.** Suppose that  $\gamma_1$  and  $\gamma_2$  are two paths in  $X$  whose start point and end point coincide, then we say that  $\gamma_1$  is path homotopic to  $\gamma_2$  if  $\gamma_1$  is homotopic to  $\gamma_2$  relative to  $\{0, 1\}$ .

Intuitively this means that we can deform  $\gamma_1$  into  $\gamma_2$  without moving the endpoints. i.e  $H: I \times I \rightarrow X$  s.t

$$H(s, t) = \begin{cases} \gamma_1(s) & \text{if } t = 0 \\ \gamma_2(s) & \text{if } t = 1 \\ \gamma_1(0) = \gamma_2(0) & \text{if } s = 0 \\ \gamma_1(1) = \gamma_2(1) & \text{if } s = 1 \end{cases}$$

**Corollary 1.10.** *Path homotopy is an equivalence relation.*

**Definition 1.11.**  $\gamma_1, \gamma_2: I \rightarrow X$  with

$$\gamma_1(1) = \gamma_2(0)$$

we define the *product* of these paths to be the path  $\gamma_1 \cdot \gamma_2$  given by

$$\gamma_1 \cdot \gamma_2(s) = \begin{cases} \gamma_1(2s) & \text{if } 0 \leq s \leq 1/2 \\ \gamma_2(2s - 1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Note that this is continuous by the Gluing Lemma.

**Proposition 1.12.** *Path multiplication is compatible with path homotopy; i.e if*

$$\gamma_1 \sim \gamma'_1, \quad \gamma_2 \sim \gamma'_2$$

then

$$\gamma_1 \cdot \gamma_2 \sim \gamma'_1 \cdot \gamma'_2$$

*Proof.* Let  $H_1: I \times I \rightarrow X$  and  $H_2: I \times I \rightarrow X$  be the corresponding homotopies then define  $H: I \times I \rightarrow X$  by

$$H(s, t) = \begin{cases} H_1(2s, t) & \text{if } 0 \leq t \leq 1/2 \\ H_2(2s - 1, t) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

We have to check that this is a valid path homotopy, TODO. □

## 1.2 The Fundamental group

**Definition 1.13.** A loop in  $X$  based at  $x_0$  is a path whose end points are both  $x_0$ .

**Theorem 1.14.** *Take  $X$  a top space and  $x_0 \in X$  then the set of path homotopy equivalence classes of the loops based at  $x_0$  is a group where multiplication, identity and inverses are defined as such:*

- $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$
- Identity is  $[x_0]$
- Inverse of  $[\gamma]$  is  $[\bar{\gamma}]$  where

$$\bar{\gamma}(s) = \gamma(1 - s)$$

*This group is called the Fundamental group of  $X$  based at  $x_0$  denoted by  $\pi_1(X, x_0)$ .*

**Remark 1.15.** Note that multiplication is well defined, since homotopy behaves nicely with path product.

**Lemma 1.16.** *Let  $\varphi: I \rightarrow I$  s.t  $\varphi(0) = 0$  and  $\varphi(1) = 1$ . Then for any path  $\gamma$  we have that*

$$\gamma \varphi \sim \gamma$$

*we call  $\gamma \circ \varphi$  a reparametrization of  $\gamma$ .*

*Proof.* Let  $H(s, t) = \gamma((1 - t)s + t\varphi(s))$ , then clearly

$$H(s, 0) = \gamma(s), \quad H(s, 1) = \gamma(\varphi(s))$$

$$H(0, t) = \gamma(0), \quad \gamma(1, t) = \gamma(1)$$

□

*Proof of the theorem.*

We have to show that this satisfies the axioms of groups:

- **Associativity.** Take  $\gamma_1, \gamma_2, \gamma_3: I \rightarrow X$  s.t

$$\gamma_1(1) = \gamma_2(0), \quad \gamma_2(1) = \gamma_3(0)$$

Then we have that

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(s) = \begin{cases} \gamma_1(4s) & \text{if } 0 \leq s \leq 1/4 \\ \gamma_2(4s - 1) & \text{if } 1/4 \leq s \leq 1/2 \\ \gamma_3(2s - 1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

and

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(s) = \begin{cases} \gamma_1(2s) & \text{if } 0 \leq s \leq 1/2 \\ \gamma_2(4s - 2) & \text{if } 1/2 \leq s \leq 3/4 \\ \gamma_3(4s - 3) & \text{if } 3/4 \leq s \leq 1 \end{cases}$$

We want to find  $\varphi: I \rightarrow I$  s.t

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(s) = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(\varphi(s))$$

Indeed, define

$$\varphi(s) = \begin{cases} 2s & \text{if } 0 \leq s \leq 1/4 \\ s + 1/4 & \text{if } 1/4 \leq s \leq 1/2 \\ 1/2s + 1/2 & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Check that this is a valid reparametrization; i.e that the equality above holds.

- **Identity.** We will only show left-identity: right is completely analogous. Take  $\gamma: I \rightarrow X$ , let  $x_0 = \gamma(0)$  and  $x_0: I \rightarrow X$  s.t  $x_0(s) = x_0$ . Then

$$x_0 \cdot \gamma(s) = \begin{cases} x_0 = \gamma(0) & \text{if } 0 \leq s \leq 1/2 \\ \gamma(2s - 1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Then clearly  $\gamma(\varphi(s)) = x_0 \cdot \gamma(s)$  via

$$\varphi(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq 1/2 \\ 2s - 1 & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Check that the reparametrization is valid.

- **Inverse.** Suppose that  $\gamma: I \rightarrow X$  is a path, fix  $\alpha \in I$ . Define the path

$$\gamma_\alpha(s) = \gamma(\alpha s)$$

We want to show that

$$\gamma \cdot \bar{\gamma} \sim \gamma(0)$$

Let  $H: I \times I \rightarrow X$  be

$$H(s, t) = \gamma_{1-t} \cdot \overline{\gamma_{1-t}}(s)$$

Then notice that

$$H(s, 0) = \gamma_1 \overline{\gamma_1}(s), \quad H(s, 1) = \gamma_0 \cdot \overline{\gamma_0}(s) = \gamma(0)$$

and

$$H(0, t) = \gamma_{1-t}(0) = \gamma(0), \quad H(1, t) = \gamma(0)$$

Where the last equality comes from the fact that

$$H(s, t) = \begin{cases} \gamma_{1-t}(2s) & \text{if } 0 \leq s \leq 1/2 \\ \overline{\gamma_{1-t}}(2s - 1) = \gamma_{1-t}(2 - 2s) = \gamma((1 - t)(2 - 2s)) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Note that  $H$  is continuous since it is clearly continuous on  $[0, 1/2] \times I$  and  $[1/2, 1] \times I$  + Gluing lemma.

□

**Theorem 1.17.** Let  $X$  be a topological space and let  $x_0, x_1 \in X$ ; take  $\eta$  a path from  $x_0 \rightarrow x_1$ . Define  $\beta_\eta: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$  by

$$\beta_\eta[\gamma] = [\eta \cdot \gamma \cdot \overline{\eta}]$$

Then  $\beta_\eta$  is an isomorphism.

*Proof.* • **Well defined.** Let  $[\gamma_1] = [\gamma_2]$  then

$$\begin{aligned} [\gamma_1] = [\gamma_2] &\implies \gamma_1 \sim \gamma_2 \\ &\implies \eta \cdot \gamma_1 \sim \eta \cdot \gamma_2 \\ &\implies \eta \cdot \gamma_1 \cdot \overline{\eta} \sim \eta \cdot \gamma_2 \cdot \overline{\eta} \\ &\implies [\eta \cdot \gamma_1 \cdot \overline{\eta}] = [\eta \cdot \gamma_2 \cdot \overline{\eta}] \end{aligned}$$

• **Homomorphism.**

$$\begin{aligned} \beta_\eta([\gamma_1] \cdot [\gamma_2]) &= \beta_\eta([\gamma_1 \cdot \gamma_2]) \\ &= [\eta \cdot \gamma_1 \cdot \gamma_2 \cdot \overline{\eta}] \\ &= [\eta \cdot \gamma_1 \cdot x_1 \cdot \gamma_2 \cdot \overline{\eta}] \\ &= [\eta \cdot \gamma_1 \cdot \overline{\eta}] \cdot [\eta \cdot \gamma_2 \cdot \overline{\eta}] \\ &= \beta_\eta[\gamma_1] \cdot \beta_\eta[\gamma_2] \end{aligned}$$

- **Isomorphism.** Notice that

$$\begin{aligned}
 \beta_\eta \circ \beta_{\bar{\eta}}[\gamma] &= [\beta_\eta(\bar{\eta}) \cdot \gamma \cdot \eta] \\
 &= [\eta \cdot \bar{\eta} \cdot \gamma \cdot \eta \cdot \bar{\eta}] \\
 &= [x_0 \cdot \eta \cdot x_0] \\
 &= [\gamma]
 \end{aligned}$$

Similarly  $\beta_{\bar{\eta}} \circ \beta_\eta$  is also identity.

□

**Corollary 1.18.** *If  $X$  is path connected then  $\pi_1(X, x_0)$  is independent of  $X$ .*

*From now on we all space will be path connected (unless stated otherwise explicitly): in this case we will simply write  $\pi_1(X)$ .*

**Definition 1.19.** If  $\pi_1(X) = e$  we say that  $X$  is *simply connected*

**Proposition 1.20.**  $X$  is simply connected  $\iff \forall x_0, x_1 \in X$  any two paths between  $x_0$  and  $x_1$  are homotopic.

*Proof.*  $\Leftarrow$  Take  $x_0 = x_1$ , then any two loops are homotopic, in particular every loop is homotopic to the constant loop.

$\Rightarrow$  Let  $x_0, x_1 \in X$  and take  $\gamma, \gamma': x_0 \rightarrow x_1$ . Note that  $\gamma \cdot \bar{\gamma}'$  is a loop, therefore  $\gamma \cdot \bar{\gamma}' \sim x_0$ . Therefore multiplying both sides by  $\gamma'$  we get

$$\begin{aligned}
 (\gamma \cdot \bar{\gamma}') \cdot \gamma' &\sim x_0 \cdot \gamma' \\
 (\gamma \cdot \bar{\gamma}') &\sim \gamma' \\
 \gamma &\sim \gamma'
 \end{aligned}$$

□

**Example 1.21.** The following spaces are simply connected

- $\mathbb{R}^n$ : Let  $\gamma$  be a loop based at 0, then let

$$H(s, t) = (1 - t)\gamma(s) + t \cdot x_0 = (1 - t)\gamma(s)$$

- Similarly, every convex set is simply connected.
- Similarly, every star-shaped set is simply connected.
- $S^n$  is simply connected for  $n \geq 2$  (proof later).

Notice that  $S^1$  is not simply connected (proof later).

### 1.3 Homotopy of spaces

Let  $X, Y$  be topological spaces and  $\varphi: X \rightarrow Y$  be a map. Fix  $x_0 \in X$ , suppose that  $\gamma$  is a loop based at  $x_0$ , then  $\varphi(\gamma)$  is a loop based at  $\varphi(x_0)$ . Moreover, if  $\gamma \sim \gamma'$  via homotopy  $H$ , then  $\varphi(\gamma) \sim \varphi(\gamma')$  via the homotopy  $\varphi \circ H$ .

*Proof.*  $H: I \times I \rightarrow X$  s.t

$$H(0, t) = x_0 = H(1, t), \quad H(s, 0) = \gamma(s), \quad H(s, 1) = \gamma'(s)$$

Then

$$\varphi \circ H: I \times I \rightarrow Y$$

and

$$\varphi(H(0, t)) = \varphi(x_0) = \varphi(H(1, t)), \quad H(s, 0) = \varphi(\gamma(s)), \quad \varphi(H(s, 1)) = \varphi(\gamma'(s))$$

□

Therefore we get a well defined map from  $\pi(X, x_0)$  to  $\pi(Y, \varphi(x_0))$ , denote by  $\varphi_*$  and defined by

$$\varphi_*([\gamma]) = [\varphi \circ \gamma]$$

**Proposition 1.22.** •  $\varphi_*$  is a Homomorphism.

- $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ .
- $(\mathbb{1})_* = \mathbb{1}_*$
- $\varphi \sim \psi$  relative to  $x_0$  then  $\varphi_* = \psi_*$ .

*Proof.* • Take  $\gamma_1, \gamma_2$  loops at  $x_0$ , then

$$\varphi(\gamma_1 \cdot \gamma_2) = \varphi(\gamma_1) \cdot \varphi(\gamma_2)$$

because

$$\varphi(\gamma_1 \cdot \gamma_2(s)) = \begin{cases} \varphi(\gamma_1(2s)) & 0 \leq s \leq 1/2 \\ \varphi(\gamma_2(2s - 1)) & 1/2 \leq s \leq 1 \end{cases}$$

but

$$\varphi(\gamma_1) \cdot \varphi(\gamma_2)(s) = \begin{cases} \varphi(\gamma_1(2s)) & 0 \leq s \leq 1/2 \\ \varphi(\gamma_2(2s - 1)) & 1/2 \leq s \leq 1 \end{cases}$$

Therefore

$$\begin{aligned} \varphi_*([\gamma_1] \cdot [\gamma_2]) &= \varphi_*([\gamma_1 \cdot \gamma_2]) \\ &= [\varphi(\gamma_1 \cdot \gamma_2)] = [\varphi(\gamma_1) \cdot \varphi(\gamma_2)] \\ &= [\varphi(\gamma_1)] \cdot [\varphi(\gamma_2)] \\ &= \varphi_*([\gamma_1]) \cdot \varphi_*([\gamma_2]) \end{aligned}$$

- TODO
- TODO
- Let  $\gamma$  be a loop at  $x_0$  and suppose that  $H$  is the homotopy between  $\varphi$  and  $\psi$  relative to  $x_0$ , therefore  $H: X \times I \rightarrow Y$  s.t

$$H(x, 0) = \varphi(x), \quad H(x, 1) = \psi(x), \quad H(x_0, t) = \varphi(x_0) = \psi(x_0)$$

Then consider the map

$$H': I \times I \rightarrow Y, \quad H'(s, t) = H(\gamma(s), t)$$

then

$$H'(s, 0) = H(\gamma(s), 0) = \varphi(\gamma(s)), \quad H'(s, 1) = H(\gamma(s), 1) = \psi(\gamma(s))$$

and

$$H'(0, t) = H(x_0, t) = \varphi(x_0), \quad H'(1, t) = \psi(x_0)$$

Therefore for any  $\gamma$

$$\varphi_*([\gamma]) = [\varphi(\gamma)] = [\psi(\gamma)] = \psi_*([\gamma])$$

draw the pictures here, it clears things up. □

**Corollary 1.23.**  $X \cong Y \implies \pi_1(X) \cong \pi_1(Y)$ .

*Proof.* Let  $\varphi: X \rightarrow Y$ ,  $\psi: Y \rightarrow X$  such that

$$\psi \circ \varphi = \mathbb{1}_X, \quad \varphi \circ \psi = \mathbb{1}_Y$$

Therefore

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_* = \mathbb{1}_*$$

similarly

$$\varphi_* \circ \psi_* = \mathbb{1}_*$$

Therefore

$$\pi_1(X) \cong \pi_1(Y)$$

□

**Remark 1.24.** Note that the proof above would still work if we only had that  $\psi \circ \varphi \sim \mathbb{1}$  relative to  $x_0$  and  $\varphi \circ \psi \sim \mathbb{1}$  relative to  $\varphi(x_0)$ .

**Definition 1.25.** Let  $X, Y$  be two spaces, we say that  $X$  is *homotopic* to  $Y$  iff there exists some  $\varphi: X \rightarrow Y$  and  $\psi: Y \rightarrow X$  such that

$$\psi \circ \varphi \sim \mathbb{1}_X, \quad \varphi \circ \psi \sim \mathbb{1}_Y$$

**Proposition 1.26.** *Homotopy of spaces is an equivalence relation*

*Proof.* • **Reflexive.** Take  $\varphi = \psi = \mathbb{1}$ .

- **Symmetric.** Trivial.
- **Transitive.**

**Lemma 1.27.** Let  $f: X \rightarrow Y$ ,  $f': X \rightarrow Y$ ,  $g, g': Y \rightarrow Z$  with

$$f \sim f', \quad g \sim g'$$



then

$$g \circ f \sim g' \circ f'$$

*Proof.* Let  $H$  be the homotopy between  $f$  and  $f'$  and  $H'$  is the homotopy between  $g$  and  $g'$ . Then

$$H'': X \times I \rightarrow Z$$

defined by

$$H''(x, t) = H'(H(x, t), t)$$

notice that

$$H''(0, t) = H'(H(x, 0), 0) = H'(f(x), 0) = g \circ f(x)$$

Similarly check the rest. □

Now suppose we have  $X, Y, Z$  spaces with

$$\varphi: X \rightarrow Y, \quad \varphi: Y \rightarrow Z$$

and

$$\psi: Y \rightarrow X, \quad \psi': Y \rightarrow Z$$

$$\begin{array}{ccccc} & \xleftarrow{\psi} & & \xleftarrow{\psi'} & \\ X & \xleftrightarrow{\varphi} & Y & \xleftrightarrow{\varphi'} & Z \end{array}$$

Then

$$\varphi' \circ \varphi: X \rightarrow Z, \quad \psi \circ \psi': Z \rightarrow X$$

s.t

$$\begin{aligned} \psi \circ \psi' \circ \varphi' \circ \varphi &\sim \psi \circ \mathbb{1} \circ \varphi \\ &\sim \mathbb{1} \end{aligned}$$

□

**Terminology.** If  $X \sim \{\cdot\}$ , then we say that  $X$  is contractible.

**Example 1.28.**  $\mathbb{R}^n$  is contractible; there is only one map  $\varphi: \mathbb{R}^n \rightarrow \{\cdot\}$ . Take the map  $\psi: \cdot \rightarrow \mathbb{R}^n$  to be

$$\cdot \rightarrow 0$$

Indeed

$$\varphi \circ \psi: \cdot \rightarrow \cdot = \mathbb{1}$$

Moreover,  $\psi \circ \varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\psi \circ \varphi(\alpha) = 0$$

let  $H: \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$  defined by

$$H(x, t) = t \cdot x$$

Similarly, we get that any convex subset of  $\mathbb{R}^n$  is contractible; any star-shaped set is also contractible.

Later on we will see that  $S^n$  is not contractible.

**Definition 1.29.** Let  $X$  be a space and  $A \subseteq X$ , then  $r: X \rightarrow A$  is called a *retraction* iff

$$r|_A = \mathbb{1}_A$$

In this case we say that  $A$  is a *retract* of  $X$ .

**Example 1.30.** • Clearly the function  $r: \mathbb{R}^n \rightarrow \{x_0\}$  is a retraction.

- $r: \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ ,  $x \rightarrow \frac{x}{|x|}$

**Definition 1.31.**  $X$  a space,  $A \subseteq X$ : we say that  $X$  deformation retracts to  $A$  (or that there is a deformation retraction from  $x$  to  $A$ ) (or that  $A$  is a deformation retract of  $X$ ) iff

- There is a retraction  $r: X \rightarrow A$ .
- Letting  $\iota: A \rightarrow X$  be the inclusion map, we have

$$r \circ \iota = \mathbb{1}_A$$

and

$$\iota \circ r \sim \mathbb{1}_X \text{ relative to } A$$

**Example 1.32.** •  $\mathbb{R}^n$ , convex / star shaped subsets deformation retract into a point.

- $\mathbb{R}^n$  deformation retract into  $D^n$ : let  $r: \mathbb{R}^n \rightarrow D^n$  be defined by

$$r(x) = \begin{cases} x & \text{if } x \in D^n \\ \frac{x}{|x|} & \text{if } x \notin D^n \end{cases}$$

Let  $H(x, t) = tr(x) + (1 - t)x$ .

- $\mathbb{R}^n \setminus \{0\}$  deformation retracts to  $S^{n-1}$ :  $r: \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$  by

$$r(x) = \frac{x}{|x|}$$

$H(x, t) = (1 - t)x + t\frac{x}{|x|}$  is the homotopy.

There is no deformation retraction from  $\mathbb{R}^n \setminus \{0\}$  to any point.

- $I \times I$  (The closed rectangle) deformation retracts to a cup Exercise.

**Proposition 1.33.** If  $X$  deformation retracts to  $A$  and  $A$  to  $B$ , then  $X$  deformation retracts to  $B$ .

*Proof.* Exercise. □

**Theorem 1.34.**  $X \sim Y \implies \pi_1(X) \cong \pi_1(Y)$

**Lemma 1.35.** Let  $X, Y$  topological spaces and  $\varphi, \psi: X \rightarrow Y$  be two homotopic functions by a homotopy  $H: X \times I \rightarrow Y$ . Note that  $H(x_0, \cdot)$  is a path in  $Y$  (denote it by  $\eta$ ). Then the following diagram

commutes:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\psi_*} & \pi_1(Y, \psi(x_0)) \\ & \searrow \varphi_* & \downarrow \beta_\eta \\ & & \pi_1(Y, \varphi(x_0)) \end{array}$$

i.e

$$\beta_\eta \circ \psi_* = \varphi_*$$

*Proof of lemma.* Let  $\gamma: I \rightarrow X$  be a loop based at  $x_0$ , let

$$H'(s, t) = \eta_t(s) \cdot H(\gamma(s), t) \cdot \overline{\eta_t}(s)$$

Convince yourself that the endpoints match here; therefore by the Gluing Lemma  $H'$  is continuous.

- At  $t = 1$ , we get

$$\eta \cdot \psi(\gamma) \cdot \overline{\eta}$$

- At  $t = 0$ , we get

$$\varphi(x_0) \cdot \varphi(\gamma) \cdot \overline{\varphi(x_0)}$$

So we have that

$$\begin{aligned} \eta \cdot \psi(\gamma) \cdot \overline{\eta} &\sim \varphi(x_0) \cdot \varphi(\gamma) \overline{\varphi(x_0)} \\ &\sim \varphi(\gamma) \end{aligned}$$

and therefore

$$\begin{aligned} [\eta \cdot \psi(\gamma) \cdot \overline{\eta}] &= [\varphi(\gamma)] \\ \beta_\eta[\psi(\gamma)] &= \varphi_*([\gamma]) \\ \beta_\eta(\psi_*([\gamma])) &= \varphi_*([\gamma]) \end{aligned}$$

therefore

$$\varphi_* = \beta_\eta \circ \psi_*$$

□

*Proof of the theorem.*

$$(Y, y_0) \xrightarrow{\psi} (X, x_0) \xrightarrow{\varphi} (Y, y_1) \xrightarrow{\psi} (X, x_1)$$

we know that

$$\psi \circ \varphi \sim \mathbb{1}, \quad \varphi \circ \psi \sim \mathbb{1}$$

Therefore by the lemma

$$(\psi \circ \varphi)_* = \beta, \quad (\varphi \circ \psi)_* = \beta'$$

But  $\beta, \beta'$  are isomorphisms, therefore we get that

$\varphi_*$  is injective and surjective

□

## 1.4 Fundamental group of all spheres

**Theorem 1.36.**

$$\pi_1(S^n) = \begin{cases} e & \text{if } n \geq 2 \\ \mathbb{Z} & \text{if } n = 1 \end{cases}$$

*Proof.* •  $\pi_1(S^2)$ ; there is a homeomorphism between  $S^2 \setminus \{N\}$  and  $\mathbb{R}^2$  ( $N$  is the North pole) called the stereographic projection. i.e  $t \rightarrow (0, 0, 1) + t(x, y, z) = (tx, ty, tx + 1)$  since  $1 + tz = 0 \implies t = \frac{t}{1-z}$  we get the map  $\Sigma$  given by

$$\Sigma(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right)$$

For the inverse map  $t \rightarrow (0, 0, 1) + t(\alpha, \beta, -1) = (t\alpha, t\beta, 1 - t)$ ; on the sphere

$$t^2\alpha^2 + t^2\beta^2 + (1 - t)^2 = 1 \implies t = \frac{2}{\alpha^2 + \beta^2 + 1}$$

so its inverse is given by

$$\Sigma^{-1}\left(\frac{2}{\alpha^2 + \beta^2 + 1}, \frac{2\beta}{\alpha^2 + \beta^2 + 1}, 1 - \frac{2}{\alpha^2 + \beta^2 + 1}\right)$$

TODO, do the same for  $\mathbb{R}^n$ .

We shall take our loops to be based at the south pole  $S$ . Let  $\gamma: I \rightarrow S^2$  be such a loop. If it happens that  $\gamma$  never passes through  $N$ ;  $\Sigma \circ \gamma$  is a loop in  $\mathbb{R}^2$  based at 0. Thus  $H: I \times I \rightarrow S^2$  given by

$$H(s, t) = \Sigma^{-1}((1 - t)\Sigma \circ \gamma(s))$$

gives a homotopy between  $\gamma$  and the constant loop at the south pole.

Now suppose that we have any loop  $\gamma: I \rightarrow S^2$ . Consider the preimage by  $\gamma$  of the open Northern hemisphere, this is open in  $I$  since  $\gamma$  is continuous; in fact it is open in  $(0, 1)$  (and hence an open subset of  $\mathbb{R}$ ). Now any open subset of  $\mathbb{R}$  is the union of countably many disjoint open intervals. Consider now the preimage of  $N$ , it must be closed in  $[0, 1]$  and hence must be compact. Therefore since the preimage of the open Northern hemisphere contains that of  $N$ ; the preimage of  $N$  is an open cover of the preimage of  $N$ . So it has a finite subcover. Thus there are finitely many disjoint open intervals which contain the preimage of  $N$ . Consider one of these intervals  $(a, b)$ . Note that  $\gamma((a, b)) \subseteq N$ ; since  $\gamma$  is continuous we can conclude that  $\gamma(a), \gamma(b) \in \overline{N}$  (the closed Northern hemisphere). Note that  $\gamma(a), \gamma(b)$  must be on the equator; since if they were in the Northern hemisphere we would have that  $a$  belongs to one of the other *disjoint* intervals to  $(a, b)$ . Note that the closed Northern hemisphere is homeomorphic to  $D^2$  by  $(x, y, z) \rightarrow (x, y)$ . but note that

$\partial D^2 = S^1$  which is path connected; since we can find some path  $\eta: I \rightarrow S^1$  going from  $\gamma(a) \rightarrow \gamma(b)$  with  $\eta$  homotopic  $\gamma|_{[a,b]}$ .

Now we can write a homotopy between  $\gamma$  and some loop that skips  $N$ ; see the picture we are using the homotopy above on all  $(a, b)$  like above. This is continuous by the gluing lemma according to our homotopy. So  $\gamma$  is homotopic to a loop that skips  $N$  and so homotopic to the constant loop.

- We will think of  $S^1$  as

$$\{z \in \mathbb{C}: |z| = 1\}$$

We will base our loops at the point  $z = 1$ . We define

$$\Phi: \mathbb{Z} \rightarrow \pi_1(S^1), \quad \Phi(n) = [e^{2\pi i n s}]$$

We also define

$$p: \mathbb{R} \rightarrow S^1, \quad p(x) = e^{2\pi i x}$$

Note that  $[p(ns)] = \Phi(n)$ . We will show that  $\Phi$  is a homomorphism

$$\begin{aligned} \Phi(n) \cdot \Phi(m) &= [p(ns)] \cdot [p(ms)] \\ &= [p(ns) \cdot p(ms)] \\ &= [p(ns) \cdot p(n + ms)] \\ &= [p(ns \cdot (n + ms))] \end{aligned}$$

Note that  $ns \cdot (n + ms)$  is a path  $0 \rightarrow m + n$ , but  $\mathbb{R}$  is simply connected, hence

$$ns \cdot (n + ms) \sim (n + m)s \implies p(ns \cdot (n + ms)) \sim p((n + m) \cdot s)$$

Therefore

$$\Phi(n) \cdot \Phi(m) = [p(n + m)s] = \Phi(n + m)$$

**Claim 1.** For any path  $\gamma$  in  $S^1$  starting at 1, there is a unique path  $\tilde{\gamma} \in \mathbb{R}$  starting at 0, s.t

$$p(\tilde{\gamma}) = \gamma$$

we say that  $\tilde{\gamma}$  is the *lift* of  $\gamma$  starting at 0.

**Claim 2.** Suppose that  $\gamma_1, \gamma_2$  are two paths in  $S^1$  starting at 1, which are homotopic. Then their lifts starting at 0 are also homotopic.

**$\Phi$  is surjective.** Let  $[\gamma] \in \pi_1(S^1)$ , then we know that there is a unique  $\tilde{\gamma}$  in  $\mathbb{R}$  s.t

$$\tilde{\gamma}(0) = 0, \quad p(\tilde{\gamma}) = \gamma$$

Especially  $p(\tilde{\gamma}(1)) = \gamma(1) = 1$ , hence

$$\tilde{\gamma}(1) = n \in \mathbb{Z} \implies \tilde{\gamma} \sim ns$$

Therefore

$$p(\tilde{\gamma}) \sim p(ns) \implies \gamma \sim p(ns) \implies [\gamma] = [p(ns)] = \Phi(n)$$

$\Phi$  is **injective**. Suppose that  $\Phi(n) = \Phi(m)$  therefore

$$[p(ns)] = [p(ms)]$$

We have that the lift of  $p(ns)$  is  $ns$  and the lift of  $p(ms)$  is  $ms$  but

$$p(ns) \sim p(ms) \implies ns \sim ms \implies n = m$$

We will prove the two claims in the following setting:  $\tilde{X}, X$  two spaces,  $p: \tilde{X} \rightarrow X$  s.t  $\forall p \in X, \exists U \ni p$  open then  $p^{-1}(U)$  is a disjoint union of open sets s.t each one of them (say  $V$ ) satisfies

$$p|_V \text{ is a homeomorphism onto } U$$

Such a  $U$  is called an elementary neighborhood or an evenly covered neighborhood.  $p$  is called the projection covering map and

$$(\tilde{X}, p) \text{ is called a covering space of } X$$

The example to take in mind here is  $X = S^1, \tilde{X} = \mathbb{R}$  with  $p$  "wrapping"  $\mathbb{R}$  around  $S^1$ . Indeed consider  $U_1 = S^1 \setminus \{1\}$ , then  $p^{-1}(U_1) = \mathbb{R} \setminus \mathbb{Z}$ ; and  $U_2 = S^1 \setminus \{-1\}$  then  $p^{-1}(U_2) = \mathbb{R} \setminus (1/2\mathbb{Z})$ , so  $(\mathbb{R}, p)$  is indeed a covering space of  $S^1$ .

**Claim 3.** Suppose that  $(\tilde{X}, p)$  is a covering space of  $X$ , and let  $f: Y \times I \rightarrow X$  and suppose we have some  $\tilde{f}: Y \times \{0\} \rightarrow \tilde{X}$  s.t

$$p \circ \tilde{f} = f$$

then  $\tilde{f}$  can be extended *uniquely* to  $Y \times I$  s.t  $p \circ \tilde{f} = f$  (draw the picture).

**Proof that claim 3 implies claim 1 and 2.** For claim 1, let  $Y = \{\cdot\}$ , in this case

$$\{\cdot\} \times I \cong I$$

so we have that  $f: I \rightarrow X$  and  $\tilde{f}: 0 \rightarrow \tilde{X}$  s.t  $\tilde{f}$  can be uniquely extended to  $I$ ; which is precisely claim 1 where we are given a path in  $X$  and the starting point of the lift is  $\tilde{X}$ . For claim 2, take  $Y = I$  and  $f$  to be the homotopy between the two paths in  $X$ ; from claim 1 we can lift the homotopy at the initial time; claim 3 lifts the homotopy to  $\tilde{X}$ . By uniqueness of lifts the sides of the homotopy lift to the constant paths and the top of the homotopy gives a lift of the second path starting where the lift of the first path started.

**Proof of claim 3.** Let  $y \in Y, \forall t \in I$  there is an open neighborhood  $O_t$  of  $t$  in  $I$  and an open neighborhood  $N_t$  of  $y$  in  $Y$  s.t

$$f(N_t \times O_t) \subseteq \text{an elementary neighborhood}$$

$\{y\} \times I$  is compact and  $\{N_t \times O_t\}_{t \in I}$  is an open cover, therefore  $\exists t_1, \dots, t_k \in I$  s.t

$$\{y\} \times I \subseteq (N_{t_1} \times O_{t_1}) \cup \dots \cup (N_{t_k} \times O_{t_k})$$

Let  $N = \bigcap_{i=1}^k N_{t_i}$ , and let  $\delta > 0$  be the Lebesgue number of the cover  $O_{t_1}, \dots, O_{t_k}$ . Therefore any closed interval of length  $\leq \delta$  is contained in one of the  $O_t$ 's. Consider  $N \times [0, \delta]$  we know that  $f(N \times [0, \delta]) \subset U$  an elementary neighborhood. But  $p \circ \tilde{f} = f$ , therefore  $\tilde{f}(N \times 0) \subset p^{-1}(U)$ , let  $\tilde{U}$  be the "part" of  $p^{-1}(U)$  which contains  $\tilde{f}(y, 0)$ . Now by taking the preimage of  $\tilde{U}$  by  $\tilde{f}$  we get some open subset of  $Y$  which is also an open subset of  $N$ ; replace  $N$  with this smaller subset call it  $N$ . Now  $p$  is a homeomorphism between  $\tilde{U}$  and  $U$ , so it has an inverse  $p^{-1}: U \rightarrow \tilde{U}$ , define

$$\tilde{f} = p^{-1} \circ f$$

This way, we have extended  $\tilde{f}$  to  $N \times [0, \delta]$ . Now do the same for  $N \times [\delta, 2\delta]$  by looking at  $\tilde{f}(N \times \{\delta\})$  and making sure that it maps to only one of the preimages of the new elementary neighborhood, we shrink  $N$  again and extend  $\tilde{f}$  by letting it be  $p^{-1} \circ f$ . Note that by construction we have that  $p \circ \tilde{f} = f$ , and it is continuous by the Gluing lemma. Hence we know that  $\forall y \in Y, \exists N_y$  s.t  $\tilde{f}$  is extended continuously to include  $N_y \times I$  with  $p \circ \tilde{f} = f$ . Let us show that lifts of paths are unique; indeed suppose that we have  $\gamma: I \rightarrow X$  a path in  $X$ , and let  $\tilde{\gamma}_1, \tilde{\gamma}_2$  be two lifts of  $\gamma$  (i.e  $p \circ \tilde{\gamma}_1 = p \circ \tilde{\gamma}_2 = \gamma$  and they both have the same start point). Therefore  $\exists \delta > 0$  s.t any closed interval of length  $\delta$  is mapped by  $\gamma$  into an elementary neighborhood. Consider the interval  $[0, \delta]$ , we know that

$$\gamma([0, \delta]) \subset \text{an elementary neighborhood } U$$

This implies that  $\tilde{\gamma}_1([0, \delta]) \subset p^{-1}(U)$  and  $\tilde{\gamma}_2([0, \delta]) \subset p^{-1}(U)$ ; since  $[0, \delta]$  is connected, we know that

$$\gamma_1([0, \delta]), \gamma_2([0, \delta]) \subset \text{a single } \tilde{U}_1, \tilde{U}_2$$

since  $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$  we get  $\tilde{U}_1 = \tilde{U}_2$ . But since  $p \circ \tilde{\gamma}_1 = p \circ \tilde{\gamma}_2$  and since  $p$  is a homeomorphism, then

$$\tilde{U}_1 = \tilde{U}_2 \rightarrow 0$$

Proceeding similarly, we get that  $\tilde{\gamma}_1 = \tilde{\gamma}_2$ .

Note that restricting  $f$  to a line like  $\{y\} \times I$  gives a path in  $X$ , and restricting  $\tilde{f}$  to this line gives us a lift of this path. So if we use  $\tilde{f}$ 's coming from different  $N_y \times I$ 's, then they must coincide on the intersection because the intersection is made of lines as above. So  $\tilde{f}$  and  $\tilde{f}$  is unique. And  $\tilde{f}$  is continuous using the gluing lemma for open sets. □

**Theorem 1.37** (Fundamental theorem of Algebra). *Let  $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$  then*

$$p(z_0) = 0 \text{ for some } z_0 \in \mathbb{C}$$

*Proof.* Assume that  $p(z) \neq 0$  consider

$$H_1(s, t) = \frac{\frac{p(re^{2\pi is})}{|p(re^{2\pi is})|}}{\frac{p(tr)}{|p(tr)|}}$$

Note that this gives a homotopy between the constant loop in  $S^1$  at 1 and  $\frac{\frac{p(re^{2\pi is})}{|p(re^{2\pi is})|}}{\frac{p(r)}{|p(r)|}}$ . Choose  $r > 0$  to be s.t  $r > 1$  and  $r > |a_{n-1}| + \dots + |a_0|$ , then notice that if  $z$  is restricted to the circle of radius  $r$ , we have that

$$\begin{aligned} |z|^n &= r^n = r^{n-1} \cdot r > r^{n-1}(|a_{n-1}| + \dots + |a_0|) \\ &> r^{n-1}|a_{n-1}| + \dots + r^{n-1}|a_0| \\ &> r^{n-1}|a_{n-1}| + r^{n-2}|a_{n-2}| + \dots + |a_0| \\ &= |a_{n-1}z^{n-1}| + \dots + |a_0| \\ &\geq |a_{n-1}z^{n-1} + \dots + a_0| \end{aligned}$$

it follows that the following polynomials depending on  $t$  never vanish on the circle of radius  $r$ :

$$q_t(z) = z^n + t(a_{n-1}z^{n-1} + \dots + a_0)$$

This is because for  $|z| = r$  we have that

$$q_t(z) \geq |z^n| - |t(a_{n-1}z^{n-1} + \dots + a_0)| > 0$$

Now consider  $H_2(s, t)$  to be

$$H_2(s, t) = \frac{\frac{q_t(re^{2\pi is})}{|q_t(re^{2\pi is})|}}{\frac{q_t(r)}{|p(r)|}}$$

Note that this gives a homotopy between the loop which we just created (write it out TODO); and

$$\frac{\frac{r^n e^{2\pi ins}}{r^n}}{r^n / r^n} = e^{2\pi ins}$$

Therefore using both  $H_1$  and  $H_2$  we have a homotopy between the constant loop and  $e^{2\pi ins}$  by contradiction.  $\square$

**Theorem 1.38** (Brouwer's fixed point).  $f: D^n \rightarrow D^n$  then  $f$  has a fixed point, i.e  $\exists p_0 \in D^n$  s.t

$$f(p_0) = p_0$$

*Proof.* • **Case  $n = 1$ .** Let  $f: D^1 \rightarrow D^1$ , if  $f(-1) = -1$  we are done same if  $f(1) = 1$ . Otherwise, we get that  $f(-1) > -1$ ,  $f(1) < 1$  therefore  $f(x) - x$  is  $> 0$  at  $-1$  and  $< 0$  at  $1$ . Therefore by the IVT,  $\exists x_0 \in (-1, 1)$  s.t

$$f(x_0) - x_0 = 0$$



- **Case**  $n = 2$ . Assume that  $f: D^2 \rightarrow D^2$  s.t  $f(x) \neq x$  for all  $x \in D^2$ . Define  $g: D^2 \rightarrow S^1$  as follows: for every  $x \in D^2$ , take the line from  $f(x)$  to  $x$  then  $g(x)$  is the intersection between this line and  $S^1$ . Note that  $g$  is continuous and it is also a retraction from  $D^2 \rightarrow S^1$ . Take the loop  $e^{2\pi is}$  in  $D^2$ , then  $e^{2\pi is} \sim$  constant in  $D^2$ . But

$$g(e^{2\pi is}) = e^{2\pi is} \not\sim \text{constant loop in } S^1$$

Contradiction.

Another way of saying this is that

$$g_*: \pi_1(D^2) \rightarrow \pi_1(S^1)$$

but  $\pi_1(D^2)$  is trivial, but its image is not the identity.

□

**Corollary 1.39.** *There is no retraction  $D^2 \rightarrow S^1$ .*

**Theorem 1.40** (Borsuk-Ulam Theorem). *Let  $f: S^n \rightarrow \mathbb{R}^n$  then there  $\exists \in S^n$  s.t*

$$f(x) = f(-x)$$

*Proof.* •  $n = 1$ . We have  $f: S^1 \rightarrow \mathbb{R}$ , if  $f(1) = f(-1)$  we are done. Otherwise wlog assume  $f(1) > f(-1)$ ; consider

$$g(x) = f(x) - f(-x)$$

then  $g$  is continuous with  $g(1) > 0$  and  $g(-1) < 0$  therefore by the IVT applied to the upper hemisphere,  $\exists x_0 \in S^1 \cap \{(x, y) \mid y \geq 0\}$  s.t  $g(x_0) = 0$ .

- $n = 2$ . Suppose that  $f: S^2 \rightarrow \mathbb{R}^2$  with  $f(x) \neq f(-x)$  for all  $x$ , let  $g: S^2 \rightarrow S^1$  be defined by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

note that  $g(-x) = -g(x)$ . Also wlog we are going to assume that

$$g(1, 0, 0) = 1$$

Consider now the loop

$$\gamma': I \rightarrow S^2, \quad \gamma'(s) = (\cos(2\pi s), \sin(2\pi s), 0)$$

Let  $\gamma: I \rightarrow S^1$  be  $g \circ \gamma'$  we know that  $\pi^1(S^2) = e$  therefore

$$\gamma' \sim \text{constant}$$

and hence

$$\gamma \sim \text{constant in } S^1$$

Note that  $\gamma'(s + 1/2) = -\gamma'(s)$  so it follows that  $\gamma(s + 1/2) = -\gamma(s)$ . Consider the lift  $\tilde{\gamma}$  of

$\gamma$  starting at 0. Then (do math), we get that

$$\tilde{\gamma}(s + 1/2) - \tilde{\gamma}(s) = \frac{n(s)}{2}$$

Where  $n(s)$  is odd; but since LHS is continuous  $n(s)$  must be a constant  $n$ . i.e

$$\tilde{\gamma}(s + 1/2) = \tilde{\gamma}(s) + n/2$$

Hence

$$\tilde{\gamma}(1) = \tilde{\gamma}(1/2) + n/2 = \tilde{\gamma}(0) + n = n$$

Since  $\gamma$  is homotopic to the constant,  $\gamma(1) = 0$  but  $n$  is odd: contradiction.

□

## 1.5 First Algebraic Detour

Let  $\{G_\alpha\}_{\alpha \in \mathcal{A}}$  be a collection of groups, we can make a group out of the product in two ways

1.  $\times_{\alpha \in \mathcal{A}} G_\alpha$  is the set of all elements in the set theoretic product with multiplication defined pointwise.
2. **The weak product.**  $\pi_{\alpha \in \mathcal{A}} G_\alpha$  is the subgroup of the product s.t for each element only finitely many components are nontrivial.

**Theorem 1.41** (Universal Property of the Weak Product of Abelian Groups). *Suppose that we have a collection  $\{G_\alpha\}$  all abelian, then  $\prod_\alpha G_\alpha$  satisfies the following property: Suppose we have homomorphisms  $f_\alpha: G_\alpha \rightarrow A$  where  $A$  is abelian then  $\exists! f: \prod_\alpha G_\alpha \rightarrow A$  a homeomorphism s.t the following diagram commutes for every  $\alpha$ :*

$$\begin{array}{ccc} G_\alpha & \xrightarrow{\iota_\alpha} & \prod_{\beta} G_\beta \\ & \searrow f_\alpha & \downarrow f \\ & & A \end{array}$$

Where

$$\iota_\alpha(g) = \{g_\beta\} = \begin{cases} g & \text{if } \alpha = p \\ e & \text{if } \alpha \neq p \end{cases}$$

*Proof.* Let  $f(\{g_\alpha\}) = \prod_\alpha f_\alpha(g_\alpha)$ . Note that this product is well defined since only finitely many  $f_\alpha(g_\alpha)$ 's are nontrivial. Moreover, the order is irrelevant since  $A$  is Abelian. Clearly we have that  $f$  satisfies the diagram. We also have

$$f(\{g_\alpha\} \{h_\alpha\}) = f(\{g_\alpha h_\alpha\}) = \prod_\alpha f_\alpha(g_\alpha h_\alpha) = \prod_\alpha f_\alpha(g_\alpha) \prod_\alpha f_\alpha(h_\alpha)$$

Where the last step is justified since  $A$  is abelian.

Note that  $\prod_{\alpha} G_{\alpha}$  is generated by the  $i_{\alpha}(G_{\alpha})$ 's, but note that any  $f'$  that satisfies the diagram must agree with our  $f$  on each of these  $G_{\alpha}$ 's (both give us  $f_{\alpha}(G_{\alpha})$ ) therefore  $f' = f$ .  $\square$

**Remark 1.42.** Note that the universal property of the weak product defines it (up to isomorphism). Indeed, suppose that we have a collection  $\{G_{\alpha}\}$  all abelian, and some abelian  $G$  that satisfies the following property: For any  $A$  and any homomorphisms  $f_{\alpha}: G_{\alpha} \rightarrow A$  and homomorphisms  $\iota_{\alpha}: G_{\alpha} \rightarrow G$  where  $A$  is abelian and  $\exists! f: G \rightarrow A$  a homomorphism s.t the following diagram commutes for every  $\alpha$ :

$$\begin{array}{ccc} G_{\alpha} & \xrightarrow{\iota_{\alpha}} & G \\ & \searrow f_{\alpha} & \downarrow f \\ & & A \end{array}$$

Then  $G \cong \prod_{\alpha \in \mathcal{A}} G_{\alpha}$ .

*Proof.* Draw the commutative diagram.  $\square$

**Definition 1.43.** Let  $S$  be a set. The *free abelian group on  $S$*  is an abelian group  $F(S)$  together with a map

$$\iota: S \rightarrow F(S)$$

such that for every abelian group  $A$  and every map  $\chi: S \rightarrow A$  there exists a unique group homomorphism

$$\varphi: F(S) \rightarrow A$$

making the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F(S) \\ & \searrow \chi & \downarrow \varphi \\ & & A \end{array}$$

**Example 1.44.** Let  $S = \{x\}$ , let  $F(S) = \mathbb{Z}$  and  $j: \{x\} \rightarrow \mathbb{Z}$  be defined by  $j(x) = 1$ . Pick any  $A$  abelian, let  $\chi: S \rightarrow A$  let  $a = \chi(x)$  and define

$$\varphi: \mathbb{Z} \rightarrow A, \quad \varphi(n) = a^n$$

indeed,  $\varphi(j(x)) = a = \chi(x)$  and  $\varphi$  is unique since it  $\varphi'(1) = a$  and 1 generates  $\mathbb{Z}$ .

Generally we write elements of  $F(\{x\})$  as  $nx$ .

**Theorem 1.45.** The free abelian group of  $S$  is unique up to isomorphism.

*Proof.* Draw the picture; initial object is unique.  $\square$

**Theorem 1.46.** Suppose that  $S = \bigcup_{\alpha \in \mathcal{A}} S_{\alpha}$  with  $\alpha \neq \alpha' \implies S_{\alpha} \cap S_{\alpha'} = \emptyset$ , then

$$F(S) \cong \prod_{\alpha} F(S_{\alpha})$$

*Proof.*

$$\begin{array}{ccccc}
 S_\alpha & \xrightarrow{j_{S_\alpha}} & F(S_\alpha) & \xrightarrow{\iota_\alpha} & \prod_{\beta \in \mathcal{A}} F(S_\beta) \\
 \downarrow i_\alpha & & \downarrow \varphi_\alpha & & \downarrow \Phi \\
 S & \xrightarrow{j_S} & F(S) & \xrightarrow{\bar{\chi}} & A \\
 & \searrow & \swarrow & & \\
 & & \chi & & 
 \end{array}$$

□

**Corollary 1.47.** For any  $S$ , note that

$$F(S) \cong \prod_{x \in S} F(\{x\}) \cong \prod_{x \in S} \mathbb{Z}$$

We write elements in  $F(S)$  as

$$n_1 x_1 + \cdots + n_k x_k$$

This notation is useful because multiplication in  $F(S)$  corresponds to addition of coefficients.

**Theorem 1.48.** Any abelian group  $A$  is the homomorphic image of a free abelian group.

*Proof.* Let  $S = A$  and let  $\chi: S \rightarrow A = 1$ ; draw the diagram the proof is done. □

**Example 1.49.** Let  $\mathbb{Z}_2$ , take  $A = \mathbb{Z}_2$  and find the  $\varphi: F(A) \rightarrow A$ . But also let  $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}_2$  defined by

$$\varphi(n) = n \pmod{2}$$

**Definition 1.50.** let  $A$  be an abelian group suppose that  $F(S)$  is free abelian on  $S$  and

$$\varphi: F(S) \rightarrow A \quad \text{surjective}$$

Every nontrivial element of the kernel of  $\varphi$  is said to be a non-trivial relation on the generators of  $A$ . Moreover if  $r_1, \dots, r_k$  are nontrivial relation and  $r_{k+1} \in \langle r_1, \dots, r_k \rangle$ , then we say that  $r_{k+1}$  is a consequence of  $r_1, \dots, r_k$ . If  $r_1, \dots, r_k$  generate the whole kernel, we say that we have a presentation of  $A$  in terms of generators and relations.

**Example 1.51.** In the example above write  $\mathbb{Z} = F(\{x\})$ , think of  $A$  as being generated by  $x$  with the following relations imposed:

$$x + x = 0, \quad x + x + x + x = 0, \quad -x - x = 0, \dots$$

Note that  $x + x + x + x$  is a consequence of  $x + x$ . Indeed, since all relations are a consequence of  $x + x = 0$ , therefore  $\mathbb{Z}_2$  is generated by a single element with a single relation.

## Free Groups

**Definition 1.52** (Free group). Let  $S$  be a set. A *free group* on  $S$ , denoted  $F^*(S)$  (often written  $F(S)$ ), is a group together with a function

$$j : S \rightarrow F^*(S)$$

such that for every group  $G$  and every function

$$\chi : S \rightarrow G$$

there exists a unique group homomorphism

$$\varphi : F^*(S) \rightarrow G$$

making the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{j} & F^*(S) \\ & \searrow \chi & \downarrow \varphi \\ & & G \end{array}$$

i.e.  $\varphi \circ j = \chi$ .

Equivalently,  $(F^*(S), j)$  is the initial object in the category of groups equipped with a map from  $S$ .

**Theorem 1.53** (Uniqueness). *The free group on  $S$  is unique up to isomorphism.*

*Proof.* Let  $(F_1, j_1)$  and  $(F_2, j_2)$  be two free groups on  $S$ . By the universal property, there exist unique homomorphisms

$$\phi : F_1 \rightarrow F_2, \quad \psi : F_2 \rightarrow F_1$$

such that  $\phi \circ j_1 = j_2$  and  $\psi \circ j_2 = j_1$ . Then  $\psi \circ \phi : F_1 \rightarrow F_1$  is a homomorphism satisfying  $(\psi \circ \phi) \circ j_1 = j_1$ . By uniqueness it must be the identity, hence  $\psi \circ \phi = \text{id}$ . Similarly  $\phi \circ \psi = \text{id}$ , so  $F_1 \cong F_2$ .  $\square$

**Example 1.54.** If  $S = \{x\}$  then  $F^*(S) \cong \mathbb{Z}$ .

*Proof.* Define  $j(x) = 1 \in \mathbb{Z}$ . Given any group  $G$  and map  $\chi : S \rightarrow G$ , write  $\chi(x) = g$ . Define

$$\varphi : \mathbb{Z} \rightarrow G, \quad \varphi(n) = g^n.$$

This is the unique homomorphism with  $\varphi \circ j = \chi$ , hence  $\mathbb{Z}$  satisfies the universal mapping property.  $\square$

## Free Products

**Definition 1.55** (Free product). Let  $\{G_\alpha\}_{\alpha \in \mathcal{A}}$  be a family of groups. The *free product*

$$\prod_{\alpha \in \mathcal{A}}^* G_\alpha$$

is a group together with homomorphisms

$$i_\alpha : G_\alpha \rightarrow \prod_{\alpha}^* G_\alpha$$

such that for any group  $G$  and any family of homomorphisms

$$f_\alpha : G_\alpha \rightarrow G$$

there exists a unique homomorphism

$$f : \prod_{\alpha}^* G_\alpha \rightarrow G$$

making the diagrams commute:

$$\begin{array}{ccc} G_\alpha & \xrightarrow{i_\alpha} & \prod_{\alpha}^* G_\alpha \\ & \searrow f_\alpha & \downarrow f \\ & & G \end{array}$$

i.e.  $f \circ i_\alpha = f_\alpha$  for all  $\alpha$ .

## Free Groups as Free Products

**Theorem 1.56.** Suppose  $S = \bigsqcup_{\alpha} S_\alpha$  is a disjoint union of sets. Then

$$F^*(S) \cong \prod_{\alpha}^* F^*(S_\alpha).$$

*Proof.* Both sides satisfy the same universal mapping property for maps from  $S$ . Hence they are isomorphic by uniqueness of universal objects.  $\square$

**Corollary 1.57.** Let  $S$  be any set. Then

$$F^*(S) \cong \prod_{s \in S}^* F^*({s}) \cong \prod_{s \in S}^* \mathbb{Z}.$$

**Corollary 1.58.** Every group is a homomorphic image of a free group.

*Proof.* Let  $G$  be a group and take  $S = G$  as a set. The identity map  $S \rightarrow G$  extends uniquely to a homomorphism

$$F^*(S) \rightarrow G$$

which is surjective.  $\square$

## Generators and Relations

**Definition 1.59.** Let  $G$  be a group and suppose

$$f : F^*(S) \twoheadrightarrow G$$

is a surjective homomorphism.

- The elements of  $S$  (via their images in  $G$ ) are called generators.
- The nontrivial elements of  $\ker(f)$  are called relations.
- If  $r_1, \dots, r_k \in F^*(S)$  then  $r_{k+1}$  is a consequence of them if

$$r_{k+1} \in \langle\langle r_1, \dots, r_k \rangle\rangle,$$

the smallest normal subgroup containing  $r_1, \dots, r_k$ .

## Existence of Free Products

**Theorem 1.60.** *Free products exist.*

*Construction via reduced words.* Let  $\mathcal{A} = \bigsqcup_{\alpha} G_{\alpha}$  be the disjoint union of the underlying sets.

**Words.** A word is a finite sequence

$$w = g_1 g_2 \cdots g_n \quad \text{with } g_i \in G_{\alpha_i}.$$

A word is *reduced* if:

1.  $g_i \neq e$  for all  $i$ ,
2. adjacent letters come from different groups:  $\alpha_i \neq \alpha_{i+1}$ .

The empty word is denoted by  $e$ .

**Multiplication.** Define multiplication by:

- concatenation of words,
- followed by reduction:
  - multiply adjacent elements from the same group,
  - remove identities,
  - cancel inverse pairs.

**Group axioms.**

- Identity: the empty word.
- Inverse:

$$(g_1 \cdots g_n)^{-1} = g_n^{-1} \cdots g_1^{-1}.$$

- **Associativity.** Let  $W$  denote the set of reduced words. For each letter  $g$  (i.e. an element of some  $G_\alpha$ ), define the map

$$L_g : W \rightarrow W, \quad L_g(w) = g \cdot w,$$

where the product on the right means concatenation followed by reduction.

Let  $w = h_1 \cdots h_n$  be a reduced word. Then

$$L_{g^{-1}} \circ L_g(h_1 \cdots h_n) = L_{g^{-1}}(gh_1 \cdots h_n).$$

We consider cases:

- If  $g$  and  $h_1$  belong to different groups, no reduction occurs, and

$$L_{g^{-1}}(gh_1 \cdots h_n) = h_1 \cdots h_n.$$

- If  $g$  and  $h_1$  belong to the same group, two subcases arise:

- \* If  $h_1 \neq g^{-1}$ , then multiplication inside the group replaces  $gh_1$  by a single element, and applying  $L_{g^{-1}}$  restores  $h_1 \cdots h_n$  after reduction.
- \* If  $h_1 = g^{-1}$ , cancellation occurs and

$$L_{g^{-1}}(gh_1 \cdots h_n) = g^{-1}h_2 \cdots h_n = h_1h_2 \cdots h_n.$$

Hence  $L_{g^{-1}} \circ L_g = \text{id}$ , so  $L_g$  is a bijection with inverse  $L_{g^{-1}}$ .

Now let  $w = g_1 \cdots g_n$  be a reduced word and define

$$L_w := L_{g_1} \circ \cdots \circ L_{g_n}.$$

Then  $L_w$  is a bijection as a composition of bijections.

Moreover, the map

$$w \mapsto L_w$$

is injective. Indeed, if  $w_1 \neq w_2$ , then

$$L_{w_1}(e) = w_1 \neq w_2 = L_{w_2}(e),$$

where  $e$  denotes the empty word.

Suppose

$$w_1 = g_1 \cdots g_n h_1 \cdots h_m, \quad w_2 = h_m^{-1} \cdots h_1^{-1} k_1 \cdots k_\ell,$$

so that reduction gives

$$w_1 \cdot w_2 = g_1 \cdots g_n k_1 \cdots k_\ell.$$

Then

$$L_{w_1 \cdot w_2} = L_{g_1} \circ \cdots \circ L_{g_n} \circ L_{k_1} \circ \cdots \circ L_{k_\ell}.$$



On the other hand,

$$L_{w_1} \circ L_{w_2} = L_{g_1} \circ \cdots \circ L_{g_n} \circ L_{h_1} \circ \cdots \circ L_{h_m} \circ L_{h_m^{-1}} \circ \cdots \circ L_{h_1^{-1}} \circ L_{k_1} \circ \cdots \circ L_{k_\ell},$$

and the middle terms cancel since  $L_{h_i} \circ L_{h_i^{-1}} = \text{id}$ . Thus

$$L_{w_1} \circ L_{w_2} = L_{w_1 \cdot w_2}.$$

Finally,

$$L_{(w_1 \cdot w_2) \cdot w_3} = (L_{w_1} \circ L_{w_2}) \circ L_{w_3} = L_{w_1} \circ (L_{w_2} \circ L_{w_3}) = L_{w_1 \cdot (w_2 \cdot w_3)}.$$

Since the map  $w \mapsto L_w$  is injective, we conclude

$$(w_1 \cdot w_2) \cdot w_3 = w_1 \cdot (w_2 \cdot w_3),$$

establishing associativity.

We still need to show that this group respects the UMP of the Free product. Indeed, we define

$$i_\alpha: G_\alpha \rightarrow W, \quad i_\alpha(g) = g$$

This is a valid homomorphism since

$$i_\alpha(g \cdot h) = (gh) = g \cdot h = i_\alpha(g) \cdot i_\alpha(h)$$

Suppose that we have  $G$  and

$$f_\alpha: G_\alpha \rightarrow G$$

Let  $f: w \rightarrow G$  be defined as follows:

$$f(g_1 \cdots g_n) = f_{\alpha_1}(g_1) f_{\alpha_2}(g_2) \cdots f_{\alpha_n}(g_n)$$

Note that  $f$  does make the diagram commute, take  $g \in G_\alpha$  we get

$$f_\alpha(g) = f(i_\alpha(g))$$

We still need to show that it is a homomorphism, let  $w = g_1 \cdots g_n h_1 \cdots h_m$  and  $w_2 = h_m^{-1} \cdots h_1^{-1} k_1 \cdots k_\ell$  s.t  $w_1 \cdot w_2 = g_1 \cdots g_n k_1 \cdots k_\ell$  then

$$\begin{aligned} f(w_1) f(w_2) &= f_{\alpha_1}(g_1) \cdots f_{\alpha_n}(g_n) f_{\beta_1}(h_1) \cdots f_{\beta_m}(h_m) f_{\beta_m}(h_m^{-1}) \cdots f_{\beta_1}(h_1) f_{\gamma_1}(k_1) \cdots f_{\gamma_\ell}(k_\ell) \\ &= f(w_1 \cdot w_2) \end{aligned}$$

Note that any homomorphism that makes the diagram commute must agree with  $f$  on the one letter words, which generate the product. Therefore  $f$  is indeed unique.  $\square$

## 2 The Van Kampen Theorem

Let  $X$  be a space with

$$X = \bigcup_{\alpha \in \mathcal{A}} A_\alpha, \quad \bigcap_{\alpha \in \mathcal{A}} A_\alpha \neq \emptyset$$

Fix  $x_0 \in \bigcap_{\alpha \in \mathcal{A}} A_\alpha$ , therefore all the fundamental groups below will be based at  $x_0$ . Let  $\iota_\alpha: A_\alpha \rightarrow X$  be the usual inclusion map. So we get

$$\iota_\alpha^*: \pi_1(A_\alpha) \rightarrow \pi_1(X)$$

Thus we get a unique homomorphism

$$\Phi: \prod_{\alpha \in \mathcal{A}}^* \pi_1(A_\alpha) \rightarrow \pi_1(X)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(A_\alpha) & \xrightarrow{i_\alpha} & \prod_{\alpha} \pi_1(A_\alpha) \\ & \searrow \iota_\alpha^* & \downarrow \Phi \\ & & \pi_1(X) \end{array}$$

**Theorem 2.1** (The Van Kampen Theorem). *Suppose we have the setting above, and  $A_\alpha$ 's are open,  $A_\alpha \cap A_\beta$  is path connected,  $A_\alpha \cap A_\beta \cap A_\gamma$  is path connected. Then  $\Phi: \prod_{\alpha} \pi_1(A_\alpha) \rightarrow \pi_1(X)$  is surjective and  $\ker \Phi$  is the smallest normal subgroup generated by elements of the form*

$$\left[ i_\alpha \left( \iota_{\alpha\beta}^*([\gamma]) \right) \right]^{-1} \left[ i_\beta \left( \iota_{\beta\alpha}^*([\gamma]) \right) \right]$$

Where

$$[\gamma] \in \pi_1(A_\alpha \cap A_\beta)$$

and

$$\iota_{\alpha\beta}: A_\alpha \cap A_\beta \rightarrow A_\alpha \quad \text{inclusion}$$

**Remark 2.2.** Note that

$$\begin{aligned} \Phi \left[ i_\alpha \left( \iota_{\alpha\beta}^*([\gamma]) \right) \right] &= \iota_\alpha^* \left( \iota_{\beta\alpha}^*([\gamma]) \right) \\ &= (\iota_\beta \circ \iota_{\beta\alpha})^*([\gamma]) \end{aligned}$$

Similarly

$$\begin{aligned} \Phi \left[ i_\beta \left( \iota_{\beta\alpha}^*([\gamma]) \right) \right] &= \iota_\beta^* \left( \iota_{\alpha\beta}^*([\gamma]) \right) \\ &= (\iota_\alpha \circ \iota_{\alpha\beta})^*([\gamma]) \end{aligned}$$

Since  $\iota_\beta \circ \iota_{\beta\alpha} = \iota_\alpha \circ \iota_{\alpha\beta}$  we have

$$(\iota_\beta \circ \iota_{\beta\alpha})^* = (\iota_\alpha \circ \iota_{\alpha\beta})^*$$

So a priori we have that if  $H$  is the smallest normal subgroup generated by these, we have that (in general)

$$H \subset \ker \Phi$$

The theorem gives us

$$\ker \phi = H$$

We will now demonstrate that all the conditions above are necessary:

**Example 2.3.** Take  $S_1$  as the open union of two arcs (more than half-arcs). Note that their intersection is not path connected. Well

$$\pi_1(S_1) = \mathbb{Z}, \pi_1(P_1) = \pi_1(P_2) = e$$

**Example 2.4.** View picture, lots of drawing.