

Algebraic Topology

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1 Bases of the Fundamental Group

1.1 Homotopy equivalence.

Definition 1.1. Let X be a topological space $f, g: X \rightarrow Y$ then we say that f is *homotopic* to g if there is a function $H: X \times I \rightarrow Y$ (called a homotopy) such that

$$H(x, 0) = f(x), \quad H(x, 1) = g(x)$$

Proposition 1.2. *Homotopy is an equivalence relation.*

Lemma 1.3 (Gluing Lemma). *Suppose that $X = A \cup B$ with A, B closed and take $f_1: A \rightarrow Y$, $f_2: B \rightarrow Y$. s.t $f_1(x) = f_2(x)$ for all $x \in A \cap B$. Then*

$$f_3: X \rightarrow Y, \quad f_3(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. Let C be a closed set in Y , then

$$\begin{aligned} f_3^{-1}(C) &= f_3^{-1}(C) \cap (A \cup B) \\ &= (f_3^{-1}(C) \cap A) \cup (f_3^{-1}(C) \cap B) \\ &= f_1^{-1}(C) \cup f_2^{-1}(C) \end{aligned}$$

which is closed in X . \square

Example 1.4. Any two function into \mathbb{R}^n (convex set, start shaped set) are homotopic.

Indeed, take two such $f(x), g(x)$ then let

$$H: X \times Y \rightarrow \mathbb{R}^n, \quad H(x, t) = (1 - t)f(x) + tg(x)$$

Note: we call this *the straight line homotopy*.

Relative Homotopy

Definition 1.5. Let $f, g: X \rightarrow Y$ with $A \subseteq X$ and $f|_A = g|_A$ then we say that f is homotopic to g relative to A if there exists a homotopy $H: X \times I \rightarrow Y$ between f and g

$$H(x, t) = f(x) = g(x) \quad \forall t \in I, \forall x \in A$$

Remark 1.6. Homotopy is a special kind of relative homotopy (for $A = \emptyset$).

Proposition 1.7. *Relative homotopy is an equivalence relation.*

Definition 1.8. Let X a topological space and let $x_1, x_2 \in X$ then a path in X going from x_1 to x_2 is a function $\gamma: I \rightarrow X$ s.t

$$\gamma(0) = x_1, \quad \gamma(1) = x_2$$

Definition 1.9. Suppose that γ_1 and γ_2 are two paths in X whose start point and end point coincide, then we say that γ_1 is path homotopic to γ_2 if γ_1 is homotopic to γ_2 relative to $\{0, 1\}$.

Intuitively this means that we can deform γ_1 into γ_2 without moving the endpoints. i.e $H: I \times I \rightarrow X$ s.t

$$H(s, t) = \begin{cases} \gamma_1(s) & \text{if } t = 0 \\ \gamma_2(s) & \text{if } t = 1 \\ \gamma_1(0) = \gamma_2(0) & \text{if } s = 0 \\ \gamma_1(1) = \gamma_2(1) & \text{if } s = 1 \end{cases}$$

Corollary 1.10. *Path homotopy is an equivalence relation.*

Definition 1.11. $\gamma_1, \gamma_2: I \rightarrow X$ with

$$\gamma_1(1) = \gamma_2(0)$$

we define the *product* of these paths to be the path $\gamma_1 \cdot \gamma_2$ given by

$$\gamma_1 \cdot \gamma_2(s) = \begin{cases} \gamma_1(2s) & \text{if } 0 \leq s \leq 1/2 \\ \gamma_2(2s - 1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Note that this is continuous by the Gluing Lemma.

Proposition 1.12. *Path multiplication is compatible with path homotopy; i.e if*

$$\gamma_1 \sim \gamma'_1, \quad \gamma_2 \sim \gamma'_2$$

then

$$\gamma_1 \cdot \gamma_2 \sim \gamma'_1 \sim \gamma'_2$$

Proof. Let $H_1: I \times I \rightarrow X$ and $H_2: I \times I \rightarrow X$ be the corresponding homotopies then define $H: I \times I \rightarrow X$ by

$$H(s, t) = \begin{cases} H_1(2s, t) & \text{if } 0 \leq t \leq 1/2 \\ H_2(2s - 1, t) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

We have to check that this a valid path homotopy, TODO. □

1.2 The Fundamental group

Definition 1.13. A loop in X based at x_0 is a path whose end points are both x_0 .

Theorem 1.14. Take X a top space and $x_0 \in X$ then the set of path homotopy equivalence classes of the loops based at x_0 is a group where multiplication, identity and inverses are defined as such:

- $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$
- Identity is $[x_0]$
- Inverse of $[\gamma]$ is $[\bar{\gamma}]$ where

$$\bar{\gamma}(s) = \gamma(1 - s)$$

This group is called the Fundamental group of X based at x_0 denoted by $\pi_1(X, x_0)$.

Remark 1.15. Note that multiplication is well defined, since homotopy behaves nicely with path product.

Lemma 1.16. Let $\varphi: I \rightarrow I$ s.t $\varphi(0) = 0$ and $\varphi(1) = 1$. Then for any path γ we have that

$$\gamma \varphi \sim \gamma$$

we call $\gamma \circ \varphi$ a reparametrization of γ .

Proof. Let $H(s, t) = \gamma((1 - t)s + t\varphi(s))$, then clearly

$$H(s, 0) = \gamma(s), \quad H(s, 1) = \gamma(\varphi(s))$$

$$H(0, t) = \gamma(0), \quad \gamma(1, t) = \gamma(1)$$

□

Proof of the theorem.

We have to show that this satisfies the axioms of groups:

- **Associativity.** Take $\gamma_1, \gamma_2, \gamma_3: I \rightarrow X$ s.t

$$\gamma_1(1) = \gamma_2(0), \quad \gamma_2(1) = \gamma_3(0)$$

Then we have that

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(s) = \begin{cases} \gamma_1(4s) & \text{if } 0 \leq s \leq 1/4 \\ \gamma_2(4s - 1) & \text{if } 1/4 \leq s \leq 1/2 \\ \gamma_3(2s - 1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

and

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(s) = \begin{cases} \gamma_1(2s) & \text{if } 0 \leq s \leq 1/2 \\ \gamma_2(4s - 2) & \text{if } 1/2 \leq s \leq 3/4 \\ \gamma_3(4s - 3) & \text{if } 3/4 \leq s \leq 1 \end{cases}$$

We want to find $\varphi: I \rightarrow I$ s.t

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(s) = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(\varphi(s))$$

Indeed, define

$$\varphi(s) = \begin{cases} 2s & \text{if } 0 \leq s \leq 1/4 \\ s + 1/4 & \text{if } 1/4 \leq s \leq 1/2 \\ 1/2s + 1/2 & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Check that this is a valid reparametrization; i.e that the equality above holds.

- **Identity.** We will only show left-identity: right is completely analogous. Take $\gamma: I \rightarrow X$, let $x_0 = \gamma(0)$ and $x_0: I \rightarrow X$ s.t $x_0(s) = x_0$. Then

$$x_0 \cdot \gamma(s) = \begin{cases} x_0 = \gamma(0) & \text{if } 0 \leq s \leq 1/2 \\ \gamma(2s - 1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Then clearly $\gamma(\varphi(s)) = x_0 \cdot \gamma(s)$ via

$$\varphi(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq 1/2 \\ 2s - 1 & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Check that the reparametrization is valid.

- **Inverse.** Suppose that $\gamma: I \rightarrow X$ is a path, fix $\alpha \in I$. Define the path

$$\gamma_\alpha(s) = \gamma(\alpha s)$$

We want to show that

$$\gamma \cdot \bar{\gamma} \sim \gamma(0)$$

Let $H: I \times I \rightarrow X$ be

$$H(s, t) = \gamma_{1-t} \cdot \overline{\gamma_{1-t}}(s)$$

Then notice that

$$H(s, 0) = \gamma_1 \overline{\gamma_1}(s), \quad H(s, 1) = \gamma_0 \cdot \overline{\gamma_0}(s) = \gamma(0)$$

and

$$H(0, t) = \gamma_{1-t}(0) = \gamma(0), \quad H(1, t) = \gamma(0)$$

Where the last equality comes from the fact that

$$H(s, t) = \begin{cases} \gamma_{1-t}(2s) & \text{if } 0 \leq s \leq 1/2 \\ \overline{\gamma_{1-t}}(2s - 1) = \gamma_{1-t}(2 - 2s) = \gamma((1-t)(2-2s)) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Note that H is continuous since it is clearly continuous on $[0, 1/2] \times I$ and $[1/2, 1] \times I$ + Gluing lemma.

□

Theorem 1.17. Let X be a topological space and let $x_0, x_1 \in X$; take η a path from $x_0 \rightarrow x_1$. Define $\beta_\eta: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ by

$$\beta_\eta[\gamma] = [\eta \cdot \gamma \cdot \bar{\eta}]$$

Then β_η is an isomorphism.

Proof. • **Well defined.** Let $[\gamma_1] = [\gamma_2]$ then

$$\begin{aligned} [\gamma_1] = [\gamma_2] &\implies \gamma_1 \sim \gamma_2 \\ &\implies \eta \cdot \gamma_1 \sim \eta \cdot \gamma_2 \\ &\implies \eta \cdot \gamma_1 \cdot \bar{\eta} \sim \eta \cdot \gamma_2 \cdot \bar{\eta} \\ &\implies [\eta \cdot \gamma_1 \cdot \bar{\eta}] = [\eta \cdot \gamma_2 \cdot \bar{\eta}] \end{aligned}$$

• **Homomorphism.**

$$\begin{aligned} \beta_\eta([\gamma_1] \cdot [\gamma_2]) &= \beta_\eta([\gamma_1 \cdot \gamma_2]) \\ &= [\eta \cdot \gamma_1 \cdot \gamma_2 \cdot \bar{\eta}] \\ &= [\eta \cdot \gamma_1 \cdot x_1 \cdot \gamma_2 \cdot \bar{\eta}] \\ &= [\eta \cdot \gamma_1 \cdot \bar{\eta}] \cdot [\eta \cdot \gamma_2 \cdot \bar{\eta}] \\ &= \beta_\eta[\gamma_1] \cdot \beta_\eta[\gamma_2] \end{aligned}$$

- **Isomorphism.** Notice that

$$\begin{aligned}
 \beta_\eta \circ \beta_{\bar{\eta}}[\gamma] &= [\beta_\eta(\bar{\eta}) \cdot \gamma \cdot \eta] \\
 &= [\eta \cdot \bar{\eta} \cdot \gamma \cdot \eta \cdot \bar{\eta}] \\
 &= [x_0 \cdot \eta \cdot x_0] \\
 &= [\gamma]
 \end{aligned}$$

Similarly $\beta_{\bar{\eta}} \circ \beta_\eta$ is also identity.

□

Corollary 1.18. If X is path connected then $\pi_1(X, x_0)$ is independent of x_0 .

From now on we all space will be path connected (unless stated otherwise explicitly): in this case we will simply write $\pi_1(X)$.

Definition 1.19. If $\pi_1(X) = e$ we say that X is *simply connected*

Proposition 1.20. X is simply connected $\iff \forall x_0, x_1 \in X$ any two paths between x_0 and x_1 are homotopic.

Proof. \Leftarrow Take $x_0 = x_1$, then any two loops are homotopic, in particular every loop is homotopic to the constant loop.

\Rightarrow Let $x_0, x_1 \in X$ and take $\gamma, \gamma': x_0 \rightarrow x_1$. Note that $\gamma \cdot \bar{\gamma}'$ is a loop, therefore $\gamma \cdot \bar{\gamma}' \sim x_0$. Therefore multiplying both sides by γ' we get

$$\begin{aligned}
 (\gamma \cdot \bar{\gamma}') \cdot \gamma' &\sim x_0 \cdot \gamma' \\
 (\gamma \cdot \bar{\gamma}') &\sim \gamma' \\
 \gamma &\sim \gamma'
 \end{aligned}$$

□

Example 1.21. The following spaces are simply connected

- \mathbb{R}^n : Let γ be a loop based at 0, then let

$$H(s, t) = (1 - t)\gamma(s) + t \cdot x_0 = (1 - t)\gamma(s)$$

- Similarly, every convex set is simply connected.
- Similarly, every star-shaped set is simply connected.
- S^n is simply connected for $n \geq 2$ (proof later).

Notice that S^1 is not simply connected (proof later).

1.3 Homotopy of spaces

Let X, Y be topological spaces and $\varphi: X \rightarrow Y$ be a map. Fix $x_0 \in X$, suppose that γ is a loop based at x_0 , then $\varphi(\gamma)$ is a loop based at $\varphi(x_0)$. Moreover, if $\gamma \sim \gamma'$ via homotopy H , then $\varphi(\gamma) \sim \varphi(\gamma')$ via the homotopy $\varphi \circ H$.

Proof. $H: I \times I \rightarrow X$ s.t

$$H(0, t) = x_0 = H(1, t), \quad H(s, 0) = \gamma(s), \quad H(s, 1) = \gamma'(s)$$

Then

$$\varphi \circ H: I \times I \rightarrow Y$$

and

$$\varphi(H(0, t)) = \varphi(x_0) = \varphi(H(1, t)), \quad H(s, 0) = \varphi(\gamma(s)), \quad \varphi(H(s, 1)) = \varphi(\gamma'(s))$$

□

Therefore we get a well defined map from $\pi(X, x_0)$ to $\pi(Y, \varphi(x_0))$, denote by φ_* and defined by

$$\varphi_*([\gamma]) = [\varphi \circ \gamma]$$

Proposition 1.22. • φ_* is a Homomorphism.

- $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.
- $(\mathbb{1})_* = \mathbb{1}_*$
- $\varphi \sim \psi$ relative to x_0 then $\varphi_* = \psi_*$.

Proof. • Take γ_1, γ_2 loops at x_0 , then

$$\varphi(\gamma_1 \cdot \gamma_2) = \varphi(\gamma_1) \cdot \varphi(\gamma_2)$$

because

$$\varphi(\gamma_1 \cdot \gamma_2(s)) = \begin{cases} \varphi(\gamma_1(2s)) & 0 \leq s \leq 1/2 \\ \varphi(\gamma_2(2s - 1)) & 1/2 \leq s \leq 1 \end{cases}$$

but

$$\varphi(\gamma_1) \cdot \varphi(\gamma_2)(s) = \begin{cases} \varphi(\gamma_1(2s)) & 0 \leq s \leq 1/2 \\ \varphi(\gamma_2(2s - 1)) & 1/2 \leq s \leq 1 \end{cases}$$

Therefore

$$\begin{aligned} \varphi_*([\gamma_1] \cdot [\gamma_2]) &= \varphi_*([\gamma_1 \cdot \gamma_2]) \\ &= [\varphi(\gamma_1 \cdot \gamma_2)] = [\varphi(\gamma_1) \cdot \varphi(\gamma_2)] \\ &= [\varphi(\gamma_1)] \cdot [\varphi(\gamma_2)] \\ &= \varphi_*([\gamma_1]) \cdot \varphi_*([\gamma_2]) \end{aligned}$$

- TODO
- TODO
- Let γ be a loop at x_0 and suppose that H is the homotopy between φ and ψ relative to x_0 , therefore $H: X \times I \rightarrow Y$ s.t

$$H(x, 0) = \varphi(x), \quad H(x, 1) = \psi(x), \quad H(x_0, t) = \varphi(x_0) = \psi(x_0)$$

Then consider the map

$$H': I \times I \rightarrow Y, \quad H'(s, t) = H(\gamma(s), t)$$

then

$$H'(s, 0) = H(\gamma(s), 0) = \varphi(\gamma(s)), \quad H'(s, 1) = H(\gamma(s), 1) = \psi(\gamma(s))$$

and

$$H'(0, t) = H(x_0, t) = \varphi(x_0), \quad H'(1, t) = \psi(x_0)$$

Therefore for any γ

$$\varphi_*([\gamma]) = [\varphi(\gamma)] = [\psi(\gamma)] = \psi_*([\gamma])$$

draw the pictures here, it clears things up. \square

Corollary 1.23. $X \cong Y \implies \pi_1(X) \cong \pi_1(Y)$.

Proof. Let $\varphi: X \rightarrow Y$, $\psi: Y \rightarrow X$ such that

$$\psi \circ \varphi = \text{Id}_X, \quad \varphi \circ \psi = \text{Id}_Y$$

Therefore

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_* = \text{Id}_*$$

similarly

$$\varphi_* \circ \psi_* = \text{Id}_*$$

Therefore

$$\pi_1(X) \cong \pi_1(Y)$$

\square

Remark 1.24. Note that the proof above would still work if we only had that $\psi \circ \varphi \sim \text{Id}$ relative to x_0 and $\varphi \circ \psi \sim \text{Id}$ relative to $\varphi(x_0)$.

Definition 1.25. Let X, Y be two spaces, we say that X is *homotopic* to Y iff there exists some $\varphi: X \rightarrow Y$ and $\psi: Y \rightarrow X$ such that

$$\psi \circ \varphi \sim \text{Id}_X, \quad \varphi \circ \psi \sim \text{Id}_Y$$

Proposition 1.26. Homotopy of spaces is an equivalence relation

Proof. • **Reflexive.** Take $\varphi = \psi = \text{Id}$.

• **Symmetric.** Trivial.

• **Transitive.**

Lemma 1.27. Let $f: X \rightarrow Y$, $f': X \rightarrow Y$ $g, g': Y \rightarrow Z$ with

$$f \sim f', \quad g \sim g'$$

then

$$g \circ f \sim g' \circ f'$$

Proof. Let H be the homotopy between f and f' and H' is the homotopy between g and g' . Then

$$H'': X \times I \rightarrow Z$$

defined by

$$H''(x, t) = H'(H(x, t), t)$$

notice that

$$H''(0, t) = H'(H(x, 0), 0) = H'(f(x), 0) = g \circ f(x)$$

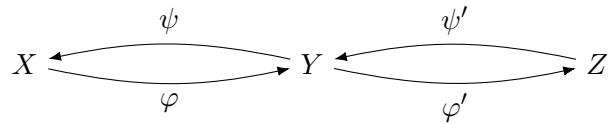
Similarly check the rest. \square

Now suppose we have X, Y, Z spaces with

$$\varphi: X \rightarrow Y, \quad \varphi: Y \rightarrow \mathbb{Z}$$

and

$$\psi: Y \rightarrow X, \quad \psi: Y \rightarrow Z$$



Then

$$\varphi' \circ \varphi: X \rightarrow Z, \quad \psi \circ \psi': Z \rightarrow X$$

s.t

$$\begin{aligned} \psi \circ \psi' \circ \varphi' \circ \varphi &\sim \psi \circ \mathbb{1} \circ \varphi \\ &\sim \mathbb{1} \end{aligned}$$

\square

Terminology. If $X \sim \{\cdot\}$, then we say that X is contractible.

Example 1.28. \mathbb{R}^n is contractible; there is only one map $\varphi: \mathbb{R}^n \rightarrow \{\cdot\}$. Take the map $\psi: \cdot \rightarrow \mathbb{R}^n$ to be

$$\cdot \rightarrow 0$$

Indeed

$$\varphi \circ \psi: \cdot \rightarrow \cdot = \mathbb{1}$$

Moreover, $\psi \circ \varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\psi \circ \varphi(\alpha) = 0$$

let $H: \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$ defined by

$$H(x, t) = t \cdot x$$

Similarly, we get that any convex subset of \mathbb{R}^n is contractible; any star-shaped set is also contractible.

Later on we will see that S^n is not contractible.

Definition 1.29. Let X be a space and $A \subseteq X$, then $r: X \rightarrow A$ is called a *retraction* iff

$$r|_A = \mathbb{1}|_A$$

In this case we say that A is a *retract* of X .

Example 1.30. • Clearly the function $r: \mathbb{R}^n \rightarrow \{x_0\}$ is a retraction.

- $r: \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$, $x \rightarrow \frac{x}{|x|}$

Definition 1.31. X a space, $A \subseteq X$: we say that X deformation retracts to A (or that there is a deformation retraction from x to A) (or that A is a deformation retract of X) iff

- There is a retraction $r: X \rightarrow A$.
- Letting $\iota: A \rightarrow X$ be the inclusion map, we have

$$r \circ \iota = \mathbb{1}|_A$$

and

$$\iota \circ r \sim \mathbb{1}|_A \text{ relative to } A$$

Example 1.32. • \mathbb{R}^n , convex / start shaped subsets deformation retract into a point.

- \mathbb{R}^n deformation retract into D^n : let $r: \mathbb{R}^n \rightarrow D^n$ be defined by

$$r(x) = \begin{cases} x & \text{if } x \in D^n \\ \frac{x}{|x|} & \text{if } x \notin D^n \end{cases}$$

Let $H(x, t) = tr(x) + (1 - t)x$.

- $\mathbb{R}^n \setminus \{0\}$ deformation retracts to S^{n-1} : $r: \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ by

$$r(x) = \frac{x}{|x|}$$

$H(x, t) = (1 - t)x + t \frac{x}{|x|}$ is the homotopy.

There is no deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to any point.

- $I \times I$ (The closed rectangle) deformation retracts to a cup Exercise.

Proposition 1.33. If X deformation retracts to A and A to B , then X deformation retracts to B .

Proof. Exercise. □

Theorem 1.34. $X \sim Y \implies \pi_1(X) \cong \pi_1(Y)$

Lemma 1.35. Let X, Y topological spaces and $\varphi, \psi: X \rightarrow Y$ be two homotopic functions by a homotopy $H: X \times I \rightarrow Y$. Note that $H(x_0, \cdot)$ is a path in Y (denote it by η). Then the following diagram

commutes:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\psi_*} & \pi_1(Y, \psi(x_0)) \\ & \searrow \varphi_* & \downarrow \beta_\eta \\ & & \pi_1(Y, \varphi(x_0)) \end{array}$$

i.e

$$\beta_\eta \circ \psi_* = \varphi_*$$

Proof of lemma. Let $\gamma: I \rightarrow X$ be a loop based at x_0 , let

$$H'(s, t) = \eta_t(s) \cdot H(\gamma(s), t) \cdot \bar{\eta}_t(s)$$

Convince yourself that the endpoints match here; therefore by the Gluing Lemma H' is continuous.

- At $t = 1$, we get

$$\eta \cdot \psi(\gamma) \cdot \bar{\eta}$$

- At $t = 0$, we get

$$\varphi(x_0) \cdot \varphi(\gamma) \cdot \overline{\varphi(x_0)}$$

So we have that

$$\begin{aligned} \eta \cdot \psi(\gamma) \cdot \bar{\eta} &\sim \varphi(x_0) \cdot \varphi(\gamma) \overline{\varphi(x_0)} \\ &\sim \varphi(\gamma) \end{aligned}$$

and therefore

$$\begin{aligned} [\eta \cdot \psi(\gamma) \cdot \bar{\eta}] &= [\varphi(\gamma)] \\ \beta_\eta[\psi(\gamma)] &= \varphi_*([\gamma]) \\ \beta_\eta(\psi_*([\gamma])) &= \varphi_*([\gamma]) \end{aligned}$$

therefore

$$\varphi_* = \beta_\eta \circ \psi_*$$

□

Proof of the theorem.

$$(Y, y_0) \xrightarrow{\psi} (X, x_0) \xrightarrow{\varphi} (Y, y_1) \xrightarrow{\psi} (X, x_1)$$

we know that

$$\psi \circ \varphi \sim \mathbb{1}, \quad \varphi \circ \psi \sim \mathbb{1}$$

Therefore by the lemma

$$(\psi \circ \varphi)_* = \beta, \quad (\varphi \circ \psi)_* = \beta'$$

But β, β' are isomorphisms, therefore we get that

φ_* is injective and surjective

□

1.4 Fundamental group of all spheres

Theorem 1.36.

$$\pi_1(S^n) = \begin{cases} e & \text{if } n \geq 2 \\ \mathbb{Z} & \text{if } n = 1 \end{cases}$$

Proof. • $\pi_1(S^2)$; there is a homeomorphism between $S^2 \setminus \{N\}$ and \mathbb{R}^2 (N is the North pole) called the stereographic projection. i.e $t \rightarrow (0, 0, 1) + t(x, y, z) = (tx, ty, tx + 1)$ since $1 + tz = 0 \implies t = \frac{t}{1-z}$ we get the map Σ given by

$$\Sigma(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

For the inverse map $t \rightarrow (0, 0, 1) + t(\alpha, \beta, -1) = (t\alpha, t\beta, 1-t)$; on the sphere

$$t^2\alpha^2 + t^2\beta^2 + (1-t)^2 = 1 \implies t = \frac{2}{\alpha^2 + \beta^2 + 1}$$

so its inverse is given by

$$\Sigma^{-1}\left(\frac{2}{\alpha^2 + \beta^2 + 1}, \frac{2\beta}{\alpha^2 + \beta^2 + 1}, 1 - \frac{2}{\alpha^2 + \beta^2 + 1}\right)$$

TODO, do the same for \mathbb{R}^n .

We shall take our loops to be based at the south pole S . Let $\gamma: I \rightarrow S^2$ be such a loop. If it happens that γ never passes through N ; $\Sigma \circ \gamma$ is a loop in \mathbb{R}^2 based at 0. Thus $H: I \times I \rightarrow S^2$ given by

$$H(s, t) = \Sigma^{-1}((1-t)\Sigma \circ \gamma(s))$$

gives a homotopy between γ and the constant loop at the south pole.

Now suppose that we have any loop $\gamma: I \rightarrow S^2$. Consider the preimage by γ of the open Northern hemisphere, this is open in I since γ is continuous; in fact it is open in $(0, 1)$ (and hence an open subset of \mathbb{R}). Now any open subset of \mathbb{R} is the union of countably many disjoint open intervals. Consider now the preimage of N , it must be closed in $[0, 1]$ and hence must be compact. Therefore since the preimage of the open Northern hemisphere contains that of N ; the preimage of N is an open cover of the preimage of N . So it has a finite subcover. Thus there are finitely many disjoint open intervals which contain the preimage of N . Consider one of these intervals (a, b) . Note that $\gamma((a, b)) \subseteq N$; since γ is continuous we can conclude that $\gamma(a), \gamma(b) \in \bar{N}$ (the closed Northern hemisphere). Note that $\gamma(a), \gamma(b)$ must be on the equator; since if they were in the Northern hemisphere we would have that a belongs to one of the other *disjoint* intervals to (a, b) . Note that the closed Northern hemisphere is homeomorphic to D^2 by $(x, y, z) \rightarrow (x, y)$. but note that

$\partial D^2 = S^1$ which is path connected; since we can find some path $\eta: I \rightarrow S^1$ going from $\gamma(a) \rightarrow \gamma(b)$ with η homotopic $\gamma|_{[a,b]}$.

Now we can write a homotopy between γ and some loop that skips N ; see the picture we are using the homotopy above on all (a, b) like above. This is continuous by the gluing lemma according to our homotopy. So γ is homotopic to a loop that skips N and so homotopic to the constant loop.

- We will think of S^1 as

$$\{z \in \mathbb{C}: |z| = 1\}$$

We will base our loops at the point $z = 1$. We define

$$\Phi: \mathbb{Z} \rightarrow \pi_1(S^1), \quad \Phi(n) = [e^{2\pi ins}]$$

We also define

$$p: \mathbb{R} \rightarrow S^1, \quad p(x) = e^{2\pi ix}$$

Note that $[p(ns)] = \Phi(n)$. We will show that Φ is a homomorphism

$$\begin{aligned} \Phi(n) \cdot \Phi(m) &= [p(ns)] \cdot [p(ms)] \\ &= [p(ns) \cdot p(ms)] \\ &= [p(ns) \cdot p(n + ms)] \\ &= [p(ns \cdot (n + ms))] \end{aligned}$$

Note that $ns \cdot (n + ms)$ is a path $0 \rightarrow m + n$, but \mathbb{R} is simply connected, hence

$$ns \cdot (n + ms) \sim (n + m)s \implies p(ns \cdot (n + ms)) \sim p((n + m) \cdot s)$$

Therefore

$$\Phi(n) \cdot \Phi(m) = [p(n + m)s] = \Phi(n + m)$$

Claim 1. For any path γ in S^1 starting at 1, there is a unique path $\tilde{\gamma} \in \mathbb{R}$ starting at 0, s.t

$$p(\tilde{\gamma}) = \gamma$$

we say that $\tilde{\gamma}$ is the *lift* of γ starting at 0.

Claim 2. Suppose that γ_1, γ_2 are two paths in S^1 starting at 1, which are homotopic. Then their lifts starting at 0 are also homotopic.

Φ is surjective. Let $[\gamma] \in \pi_1(S^1)$, then we know that there is a unique $\tilde{\gamma}$ in \mathbb{R} s.t

$$\tilde{\gamma}(0) = 0, \quad p(\tilde{\gamma}) = \gamma$$

Especially $p(\tilde{\gamma}(1)) = \gamma(1) = 1$, hence

$$\tilde{\gamma}(1) = n \in \mathbb{Z} \implies \tilde{\gamma} \sim ns$$

Therefore

$$p(\tilde{\gamma}) \sim p(ns) \implies \gamma \sim p(ns) \implies [\gamma] = [p(ns)] = \Phi(n)$$

Φ is injective. Suppose that $\Phi(n) = \Phi(m)$ therefore

$$[p(ns)] = [p(ms)]$$

We have that the lift of $p(ns)$ is ns and the lift of $p(ms)$ is ms but

$$p(ns) \sim p(ms) \implies ns \sim ms \implies n = m$$

We will prove the two claims in the following setting: \tilde{X}, X two spaces, $p: \tilde{X} \rightarrow X$ s.t $\forall p \in X, \exists U \ni p$ open then $p^{-1}(U)$ is a disjoint union of open sets s.t each one of them (say V) satisfies

$$p|_V \text{ is a homeomorphism onto } U$$

Such a U is called an elementary neighborhood or an evenly covered neighborhood. p is called the projection covering map and

(\tilde{X}, p) is called a covering space of X

The example to take in mind here is $X = S^1, \tilde{X} = \mathbb{R}$ with p "wrapping" \mathbb{R} around S^1 . Indeed consider $U_1 = S^1 \setminus \{1\}$, then $p^{-1}(U_1) \mathbb{R} \setminus \mathbb{Z}$; and $U_2 = S^1 \setminus \{-1\}$ then $p^{-1}(U_2)$ and $U_2 = \mathbb{R} \setminus (1/2\mathbb{Z})$, so (\mathbb{R}, p) is indeed a covering space of S^1 .

Claim 3. Suppose that (\tilde{X}, p) is a covering space of X , and let $f: Y \times I \rightarrow X$ and suppose we have some $\tilde{f}: Y \times \{0\} \rightarrow \tilde{X}$ s.t

$$p \circ \tilde{f} = f$$

then \tilde{f} can be extended *uniquely* to $Y \times I$ s.t $p \circ \tilde{f} = f$ (draw the picture).

Proof that claim 3 implies claim 1 and 2. For claim 1, let $Y = \{\cdot\}$, in this case

$$\{\cdot\} \times I \cong I$$

so we have that $f: I \rightarrow X$ and $\tilde{f}: 0 \rightarrow \tilde{X}$ s.t \tilde{f} can be uniquely extended to I ; which is precisely claim 1 where we are given a path in X and the starting point of the lift is \tilde{X} . For claim 2, take $Y = I$ and f to be the homotopy between the two paths in X ; from claim 1 we can lift the homotopy at the initial time; claim 3 lifts the homotopy to \tilde{X} . By uniqueness of lifts the sides of the homotopy lift to the constant paths and the top of the homotopy gives a lift of the second path starting where the lift of the first path started.

Proof of claim 3. Let $y \in Y, \forall t \in I$ there is an open neighborhood O_t of t in I and an open neighborhood N_t of y in Y s.t

$$f(N_t \times O_t) \subseteq \text{an elementary neighborhood}$$

$\{y\} \times I$ is compact and $\{N_t \times O_t\}_{t \in I}$ is an open cover, therefore $\exists t_1, \dots, t_k \in I$ s.t

$$\{y\} \times I \subseteq (N_{t_1} \times O_{t_1}) \cup \dots \cup (N_{t_k} \times O_{t_k})$$

Let $N = \bigcap_{i=1}^k N_{t_i}$, and let $\delta > 0$ be the Lebesgue number of the cover O_{t_1}, \dots, O_{t_k} . Therefore any closed interval of length $\leq \delta$ is contained in one of the O_t 's. Consider $N \times [0, \delta]$ we know that $f(N \times [0, \delta]) \subset U$ an elementary neighborhood. But $p \circ \tilde{f} = f$, therefore $\tilde{f}(N \times 0) \subset p^{-1}(U)$, let \tilde{U} be the "part" of $p^{-1}(U)$ which contains $\tilde{f}(y, 0)$. Now by taking the preimage of \tilde{U} by \tilde{f} we get some open subset of Y which is also an open subset of N ; replace N with this smaller subset call it N . Now p is a homeomorphism between \tilde{U} and U , so it has an inverse $p^{-1}: U \rightarrow \tilde{U}$, define

$$\tilde{f} = p^{-1} \circ f$$

This way, we have extended \tilde{f} to $N \times [0, \delta]$. Now do the same for $N \times [\delta, 2\delta]$ by looking at $\tilde{f}(N \times \{\delta\})$ and making sure that it maps to only one of the preimages of the new elementary neighborhood, we shrink N again and extend \tilde{f} by letting it be $p^{-1} \circ f$. Note that by construction we have that $p \circ \tilde{f} = f$, and it is continuous by the Gluing lemma. Hence we know that $\forall y \in Y, \exists N_y$ s.t \tilde{f} is extended continuously to include $N_y \times I$ with $p \circ \tilde{f} = f$. Let us show that lifts of paths are unique; indeed suppose that we have $\gamma: I \rightarrow X$ a path in X , and let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be two lifts of γ (i.e $p \circ \tilde{\gamma}_1 = p \circ \tilde{\gamma}_2 = \gamma$ and they both have the same start point). Therefore $\exists \delta > 0$ s.t any closed interval of length δ is mapped by γ into an elementary neighborhood. Consider the interval $[0, \delta]$, we know that

$$\gamma([0, \delta]) \subset \text{an elementary neighborhood } U$$

This implies that $\tilde{\gamma}_1([0, \delta]) \subset p^{-1}(U)$ and $\tilde{\gamma}_2([0, \delta]) \subset p^{-1}(U)$; since $[0, \delta]$ is connected, we know that

$$\tilde{\gamma}_1([0, \delta]), \tilde{\gamma}_2([0, \delta]) \subset \text{a single } \tilde{U}_1, \tilde{U}_2$$

since $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$ we get $\tilde{U}_1 = \tilde{U}_2$. But since $p \circ \tilde{\gamma}_1 = p \circ \tilde{\gamma}_2$ and since p is a homeomorphism, then

$$\tilde{U}_1 = \tilde{U}_2 \rightarrow 0$$

Proceeding similarly, we get that $\tilde{\gamma}_1 = \tilde{\gamma}_2$.

Note that restricting f to a line like $\{y\} \times I$ gives a path in X , and restricting \tilde{f} to this line gives us a lift of this path. So if we use \tilde{f} 's coming from different $N_y \times I$'s, then they must coincide on the intersection because the intersection is made of lines as above. So \tilde{f} and \tilde{f} is unique. And \tilde{f} is continuous using the gluing lemma for open sets.

□

Theorem 1.37 (Fundamental theorem of Algebra). *Let $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ then*

$$p(z_0) = 0 \text{ for some } z_0 \in \mathbb{C}$$

Proof. Assume that $p(z) \neq 0$ consider

$$H_1(s, t) = \frac{\frac{p(tre^{2\pi i s})}{|p(tre^{2\pi i s})|}}{\frac{p(tr)}{|p(tr)|}}$$

Note that this gives a homotopy between the constant loop in S^1 at 1 and $\frac{p(re^{2\pi i s})}{|p(re^{2\pi i s})|}$. Choose $r > 0$ to be s.t $r > 1$ and $r > |a_{n-1}| + \dots + |a_0|$, then notice that if z is restricted to the circle of radius r , we have that

$$\begin{aligned} |z|^n &= r^n = r^{n-1} \cdot r > r^{n-1}(|a_{n-1}| + \dots + |a_0|) \\ &> r^{n-1}|a_{n-1}| + \dots + r^{n-1}|a_0| \\ &> r^{n-1}|a_{n-1}| + r^{n-2}|a_{n-2}| + \dots + |a_0| \\ &= |a_{n-1}z^{n-1}| + \dots + |a_0| \\ &\geq |a_{n-1}z^{n-1} + \dots + a_0| \end{aligned}$$

it follows that the following polynomials depending on t never vanish on the circle of radius r :

$$q_t(z) = z^n + t(a_{n-1}z^{n-1} + \dots + a_0)$$

This is because for $|z| = r$ we have that

$$q_t(z) \geq |z^n| - |t(a_{n-1}z^{n-1} + \dots + a_0)| > 0$$

Now consider $H_2(s, t)$ to be

$$H_2(s, t) = \frac{\frac{q_t(re^{2\pi i s})}{|q_t(re^{2\pi i s})|}}{\frac{q_t(r)}{|p(r)|}}$$

Note that this gives a homotopy between the loop which we just created (write it out TODO); and

$$\frac{\frac{r^n e^{2\pi i ns}}{r^n}}{\frac{r^n}{r^n}} = e^{2\pi i ns}$$

Therefore using both H_1 and H_2 we have a homotopy between the constant loop and $e^{2\pi i ns}$ by contradiction. \square

Theorem 1.38 (Brouwer's fixed point). $f: D^n \rightarrow D^n$ then f has a fixed point, i.e $\exists p_0 \in D^n$ s.t

$$f(p_0) = p_0$$

Proof. • **Case $n = 1$.** Let $f: D^1 \rightarrow D^1$, if $f(-1) = -1$ we are done same if $f(1) = 1$.

Otherwise, we get that $f(-1) > -1$, $f(1) < 1$ therefore $f(x) - x$ is > 0 at -1 and < 0 at 1 . Therefore by the IVT, $\exists x_0 \in (-1, 1)$ s.t

$$f(x_0) - x_0 = 0$$

- **Case $n = 2$.** Assume that $f: D^2 \rightarrow D^2$ s.t $f(x) \neq x$ for all $x \in D^2$. Define $g: D^2 \rightarrow S^1$ as follows: for every $x \in D^2$, take the line from $f(x)$ to x then $g(x)$ is the intersection between this line and S^1 . Note that g is continuous and it is also a retraction from $D^2 \rightarrow S^1$. Take the loop $e^{2\pi is}$ in D^2 , then $e^{2\pi is} \sim \text{constant in } D^2$. But

$$g(e^{2\pi is}) = e^{2\pi is} \not\sim \text{constant loop in } S^1$$

Contradiction.

Another way of saying this is that

$$g_*: \pi_1(D^2) \rightarrow \pi(S^1)$$

but $\pi_1(D^2)$ is trivial, but its image is not the identity.

□

Corollary 1.39. *There is no retraction $D^2 \rightarrow S^1$.*

Theorem 1.40 (Borsuk-Ulam Theorem). *Let $f: S^n \rightarrow \mathbb{R}^n$ then there $\exists \in S^n$ s.t*

$$f(x) = f(-x)$$

Proof. • $n = 1$. We have $f: S^1 \rightarrow \mathbb{R}$, if $f(1) = f(-1)$ we are done. Otherwise wlog assume $f(1) > f(-1)$; consider

$$g(x) = f(x) - f(-x)$$

then g is continuous with $g(1) > 0$ and $g(-1) < 0$ therefore by the IVT applied to the upper hemisphere, $\exists x_0 \in S^1 \cap \{(x, y) \mid y \geq 0\}$ s.t $g(x_0) = 0$.

- $n = 2$. Suppose that $f: S^2 \rightarrow \mathbb{R}^2$ with $f(x) \neq f(-x)$ for all x , let $g: S^2 \rightarrow S^1$ be defined by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

note that $g(-x) = -g(x)$. Also wlog we are going to assume that

$$g(1, 0, 0) = 1$$

Consider now the loop

$$\gamma': I \rightarrow S^2, \quad \gamma'(s) = (\cos(2\pi s), \sin(2\pi s), 0)$$

Let $\gamma: I \rightarrow S^1$ be $g \circ \gamma'$ we know that $\pi^1(S^2) = e$ therefore

$$\gamma' \sim \text{constant}$$

and hence

$$\gamma \sim \text{constant in } S^1$$

Note that $\gamma'(s + 1/2) = -\gamma'(s)$ so it follows that $\gamma(s + 1/2) = -\gamma(s)$. Consider the lift $\tilde{\gamma}$ of

γ starting at 0. Then (do math), we get that

$$\tilde{\gamma}(s + 1/2) - \tilde{\gamma}(s) = \frac{n(s)}{2}$$

Where $n(s)$ is odd; but since LHS is continuous $n(s)$ must be a constant n . i.e

$$\tilde{\gamma}(s + 1/2) = \tilde{\gamma}(s) + n/2$$

Hence

$$\tilde{\gamma}(1) = \tilde{\gamma}(1/2) + n/2 = \tilde{\gamma}(0) + n = n$$

Since γ is homotopic to the constant, $\gamma(1) = 0$ but n is odd: contradiction.

□

1.5 First Algebraic Detour

Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of groups, we can make a group out of the product in two ways

1. $\times_{\alpha \in \mathcal{A}} G_\alpha$ is the set of all elements in the set theoretic product with multiplication defined pointwise.
2. **The weak product.** $\pi_{\alpha \in \mathcal{A}} G_\alpha$ is the subgroup of the product s.t for each element only finitely many components are nontrivial.

Theorem 1.41 (Universal Property of the Weak Product of Abelian Groups). *Suppose that we have a collection $\{G_\alpha\}$ all abelian, then $\prod_\alpha G_\alpha$ satisfies the following property: Suppose we have homomorphisms $f_\alpha: G_\alpha \rightarrow A$ where A is abelian then $\exists! f: \prod_\alpha G_\alpha \rightarrow A$ a homeomorphism s.t the following diagram commutes for every α :*

$$\begin{array}{ccc} G_\alpha & \xrightarrow{\iota_\alpha} & \prod_\beta G_\beta \\ & \searrow f_\alpha & \downarrow f \\ & & A \end{array}$$

Where

$$\iota_\alpha(g) = \{g_\beta\} = \begin{cases} g & \text{if } \alpha = p \\ e & \text{if } \alpha \neq p \end{cases}$$

Proof. Let $f(\{g_\alpha\}) = \prod_\alpha f_\alpha(g_\alpha)$. Note that this product is well defined since only finitely many $f_\alpha(g_\alpha)$'s are nontrivial. Moreover, the order is irrelevant since A is Abelian. Clearly we have that f satisfies the diagram. We also have

$$f(\{g_\alpha\} \{h_\alpha\}) = f(\{g_\alpha h_\alpha\}) = \prod_\alpha f_\alpha(g_\alpha h_\alpha) = \prod_\alpha f_\alpha(g_\alpha) \prod_\alpha f(h_\alpha)$$

Where the last step is justified since A is abelian.

Note that $\prod_{\alpha} G_{\alpha}$ is generated by the $i_{\alpha}(G_{\alpha})$'s, but note that any f' that satisfies the diagram must agree with our f on each of these G_{α} 's (both give us $f_{\alpha}(G_{\alpha})$) therefore $f' = f$. \square

Remark 1.42. Note that the universal property of the weak product defines it (up to isomorphism). Indeed, suppose that we have a collection $\{G_{\alpha}\}$ all abelian, and some abelian G that satisfies the following property: For any A and any homomorphisms $f_{\alpha}: G_{\alpha} \rightarrow A$ and homomorphisms $\iota_{\alpha}: G_{\alpha} \rightarrow G$ where A is abelian and $\exists! f: G \rightarrow A$ a homomorphism s.t the following diagram commutes for every α :

$$\begin{array}{ccc} G_{\alpha} & \xrightarrow{\iota_{\alpha}} & G \\ & \searrow f_{\alpha} & \downarrow f \\ & & A \end{array}$$

Then $G \cong \prod_{\alpha \in \mathcal{A}} G_{\alpha}$.

Proof. Draw the commutative diagram. \square

Definition 1.43. Let S be a set. The *free abelian group on S* is an abelian group $F(S)$ together with a map

$$\iota: S \rightarrow F(S)$$

such that for every abelian group A and every map $\chi: S \rightarrow A$ there exists a unique group homomorphism

$$\varphi: F(S) \rightarrow A$$

making the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F(S) \\ & \searrow \chi & \downarrow \varphi \\ & & A \end{array}$$

Example 1.44. Let $S = \{x\}$, let $F(S) = \mathbb{Z}$ and $j: \{x\} \rightarrow \mathbb{Z}$ be defined by $j(x) = 1$. Pick any A abelian, let $\chi: S \rightarrow A$ let $a = \chi(x)$ and define

$$\varphi: \mathbb{Z} \rightarrow A, \quad \varphi(n) = a^n$$

indeed, $\varphi(j(x)) = a = \chi(x)$ and φ is unique since it $\varphi'(1) = a$ and 1 generates \mathbb{Z} .

Generally we write elements of $F(\{x\})$ as nx .

Theorem 1.45. The free abelian group of S is unique up to isomorphism.

Proof. Draw the picture; initial object is unique. \square

Theorem 1.46. Suppose that $S = \bigcup_{\alpha \in \mathcal{A}} S_{\alpha}$ with $\alpha \neq \alpha' \implies S_{\alpha} \cap S'_{\alpha'} = \emptyset$, then

$$F(S) \cong \prod_{\alpha} F(S_{\alpha})$$

Proof.

$$\begin{array}{ccccc}
 S_\alpha & \xrightarrow{j_{S_\alpha}} & F(S_\alpha) & \xrightarrow{\iota_\alpha} & \prod_{\beta \in \mathcal{A}} F(S_\beta) \\
 i_\alpha \downarrow & & \varphi_\alpha \downarrow & & \downarrow \Phi \\
 S & \xrightarrow{j_S} & F(S) & \xrightarrow{\bar{\chi}} & A
 \end{array}$$

□

Corollary 1.47. For any S , note that

$$F(S) \cong \prod_{x \in S} F(\{x\}) \cong \prod_{x \in S} \mathbb{Z}$$

We write elements in $F(S)$ as

$$n_1 x_1 + \cdots + n_k x_k$$

This notation is useful because multiplication in $F(S)$ corresponds to addition of coefficients.

Theorem 1.48. Any abelian group A is the homomorphic image of a free abelian group.

Proof. Let $S = A$ and let $\chi: S \rightarrow A = \mathbb{1}$; draw the diagram the proof is done. □

Example 1.49. Let \mathbb{Z}_2 , take $A = \mathbb{Z}_2$ and find the $\varphi: F(A) \rightarrow A$. But also let $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}_2$ defined by

$$\varphi(n) = n \pmod{2}$$

Definition 1.50. let A be an abelian group suppose that $F(S)$ is free abelian on S and

$$\varphi: F(S) \rightarrow A \text{ surjective}$$

Every nontrivial element of the kernel of φ is said to be a non-trivial relation on the generators of A . Moreover if r_1, \dots, r_k are nontrivial relation and $r_{k+1} \in \langle r_1, \dots, r_k \rangle$, then we say that r_{k+1} is a consequence of r_1, \dots, r_k . If r_1, \dots, r_k generate the whole kernel, we say that we have a presentation of A in terms of generators and relations.

Example 1.51. In the example above write $\mathbb{Z} = F(\{x\})$, think of A as being generated by x with the following relations imposed:

$$x + x = 0, \quad x + x + x + x = 0, \quad -x - x = 0, \dots$$

Note that $x + x + x + x$ is a consequence of $x + x$. Indeed, since all relations are a consequence of $x + x = 0$, therefore \mathbb{Z}_2 is generated by a single element with a single relation.

Free Groups

Definition 1.52 (Free group). Let S be a set. A *free group* on S , denoted $F^*(S)$ (often written $F(S)$), is a group together with a function

$$j : S \rightarrow F^*(S)$$

such that for every group G and every function

$$\chi : S \rightarrow G$$

there exists a unique group homomorphism

$$\varphi : F^*(S) \rightarrow G$$

making the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{j} & F^*(S) \\ & \searrow \chi & \downarrow \varphi \\ & & G \end{array}$$

i.e. $\varphi \circ j = \chi$.

Equivalently, $(F^*(S), j)$ is the initial object in the category of groups equipped with a map from S .

Theorem 1.53 (Uniqueness). *The free group on S is unique up to isomorphism.*

Proof. Let (F_1, j_1) and (F_2, j_2) be two free groups on S . By the universal property, there exist unique homomorphisms

$$\phi : F_1 \rightarrow F_2, \quad \psi : F_2 \rightarrow F_1$$

such that $\phi \circ j_1 = j_2$ and $\psi \circ j_2 = j_1$. Then $\psi \circ \phi : F_1 \rightarrow F_1$ is a homomorphism satisfying $(\psi \circ \phi) \circ j_1 = j_1$. By uniqueness it must be the identity, hence $\psi \circ \phi = \text{id}$. Similarly $\phi \circ \psi = \text{id}$, so $F_1 \cong F_2$. \square

Example 1.54. If $S = \{x\}$ then $F^*(S) \cong \mathbb{Z}$.

Proof. Define $j(x) = 1 \in \mathbb{Z}$. Given any group G and map $\chi : S \rightarrow G$, write $\chi(x) = g$. Define

$$\varphi : \mathbb{Z} \rightarrow G, \quad \varphi(n) = g^n.$$

This is the unique homomorphism with $\varphi \circ j = \chi$, hence \mathbb{Z} satisfies the universal mapping property. \square

Free Products

Definition 1.55 (Free product). Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of groups. The *free product*

$$\prod_{\alpha \in \mathcal{A}}^* G_\alpha$$

is a group together with homomorphisms

$$i_\alpha : G_\alpha \rightarrow \prod_{\alpha}^* G_\alpha$$

such that for any group G and any family of homomorphisms

$$f_\alpha : G_\alpha \rightarrow G$$

there exists a unique homomorphism

$$f : \prod_{\alpha}^* G_\alpha \rightarrow G$$

making the diagrams commute:

$$\begin{array}{ccc} G_\alpha & \xrightarrow{i_\alpha} & \prod_{\alpha}^* G_\alpha \\ & \searrow f_\alpha & \downarrow f \\ & & G \end{array}$$

i.e. $f \circ i_\alpha = f_\alpha$ for all α .

Free Groups as Free Products

Theorem 1.56. Suppose $S = \bigsqcup_\alpha S_\alpha$ is a disjoint union of sets. Then

$$F^*(S) \cong \prod_{\alpha}^* F^*(S_\alpha).$$

Proof. Both sides satisfy the same universal mapping property for maps from S . Hence they are isomorphic by uniqueness of universal objects. \square

Corollary 1.57. Let S be any set. Then

$$F^*(S) \cong \prod_{s \in S}^* F^*(\{s\}) \cong \prod_{s \in S}^* \mathbb{Z}.$$

Corollary 1.58. Every group is a homomorphic image of a free group.

Proof. Let G be a group and take $S = G$ as a set. The identity map $S \rightarrow G$ extends uniquely to a homomorphism

$$F^*(S) \rightarrow G$$

which is surjective. \square

Generators and Relations

Definition 1.59. Let G be a group and suppose

$$f : F^*(S) \twoheadrightarrow G$$

is a surjective homomorphism.

- The elements of S (via their images in G) are called generators.
- The nontrivial elements of $\ker(f)$ are called relations.
- If $r_1, \dots, r_k \in F^*(S)$ then r_{k+1} is a consequence of them if

$$r_{k+1} \in \langle\langle r_1, \dots, r_k \rangle\rangle,$$

the smallest normal subgroup containing r_1, \dots, r_k .

Existence of Free Products

Theorem 1.60. *Free products exist.*

Construction via reduced words. Let $\mathcal{A} = \bigsqcup_\alpha G_\alpha$ be the disjoint union of the underlying sets.

Words. A word is a finite sequence

$$w = g_1 g_2 \cdots g_n \quad \text{with } g_i \in G_{\alpha_i}.$$

A word is *reduced* if:

1. $g_i \neq e$ for all i ,
2. adjacent letters come from different groups: $\alpha_i \neq \alpha_{i+1}$.

The empty word is denoted by e .

Multiplication. Define multiplication by:

- concatenation of words,
- followed by reduction:
 - multiply adjacent elements from the same group,
 - remove identities,
 - cancel inverse pairs.

Group axioms.

- Identity: the empty word.
- Inverse:

$$(g_1 \cdots g_n)^{-1} = g_n^{-1} \cdots g_1^{-1}.$$

- **Associativity.** Let W denote the set of reduced words. For each letter g (i.e. an element of some G_α), define the map

$$L_g : W \rightarrow W, \quad L_g(w) = g \cdot w,$$

where the product on the right means concatenation followed by reduction.

Let $w = h_1 \cdots h_n$ be a reduced word. Then

$$L_{g^{-1}} \circ L_g(h_1 \cdots h_n) = L_{g^{-1}}(gh_1 \cdots h_n).$$

We consider cases:

- If g and h_1 belong to different groups, no reduction occurs, and

$$L_{g^{-1}}(gh_1 \cdots h_n) = h_1 \cdots h_n.$$

- If g and h_1 belong to the same group, two subcases arise:

- * If $h_1 \neq g^{-1}$, then multiplication inside the group replaces gh_1 by a single element, and applying $L_{g^{-1}}$ restores $h_1 \cdots h_n$ after reduction.
- * If $h_1 = g^{-1}$, cancellation occurs and

$$L_{g^{-1}}(gh_1 \cdots h_n) = g^{-1}h_2 \cdots h_n = h_1h_2 \cdots h_n.$$

Hence $L_{g^{-1}} \circ L_g = \text{id}$, so L_g is a bijection with inverse $L_{g^{-1}}$.

Now let $w = g_1 \cdots g_n$ be a reduced word and define

$$L_w := L_{g_1} \circ \cdots \circ L_{g_n}.$$

Then L_w is a bijection as a composition of bijections.

Moreover, the map

$$w \longmapsto L_w$$

is injective. Indeed, if $w_1 \neq w_2$, then

$$L_{w_1}(e) = w_1 \neq w_2 = L_{w_2}(e),$$

where e denotes the empty word.

Suppose

$$w_1 = g_1 \cdots g_n h_1 \cdots h_m, \quad w_2 = h_m^{-1} \cdots h_1^{-1} k_1 \cdots k_\ell,$$

so that reduction gives

$$w_1 \cdot w_2 = g_1 \cdots g_n k_1 \cdots k_\ell.$$

Then

$$L_{w_1 \cdot w_2} = L_{g_1} \circ \cdots \circ L_{g_n} \circ L_{k_1} \circ \cdots \circ L_{k_\ell}.$$

On the other hand,

$$L_{w_1} \circ L_{w_2} = L_{g_1} \circ \cdots \circ L_{g_n} \circ L_{h_1} \circ \cdots \circ L_{h_m} \circ L_{h_m^{-1}} \circ \cdots \circ L_{h_1^{-1}} \circ L_{k_1} \circ \cdots \circ L_{k_\ell},$$

and the middle terms cancel since $L_{h_i} \circ L_{h_i^{-1}} = \text{id}$. Thus

$$L_{w_1} \circ L_{w_2} = L_{w_1 \cdot w_2}.$$

Finally,

$$L_{(w_1 \cdot w_2) \cdot w_3} = (L_{w_1} \circ L_{w_2}) \circ L_{w_3} = L_{w_1} \circ (L_{w_2} \circ L_{w_3}) = L_{w_1 \cdot (w_2 \cdot w_3)}.$$

Since the map $w \mapsto L_w$ is injective, we conclude

$$(w_1 \cdot w_2) \cdot w_3 = w_1 \cdot (w_2 \cdot w_3),$$

establishing associativity.

We still need to show that this group respects the UMP of the Free product. Indeed, we define

$$i_\alpha: G_\alpha \rightarrow W, \quad i_\alpha(g) = g$$

This is a valid homomorphism since

$$i_\alpha(g \cdot h) = (gh) = g \cdot h = i_\alpha(g) \cdot i_\alpha(h)$$

Suppose that we have G and

$$f_\alpha: G_\alpha \rightarrow G$$

Let $f: w \rightarrow G$ be defined as follows:

$$f(g_1 \cdots g_n) = f_{\alpha_1}(g_1) f_{\alpha_2}(g_2) \cdots f_{\alpha_n}(g_n)$$

Note that f does make the diagram commute, take $g \in G_\alpha$ we get

$$f_\alpha(g) = f(i_\alpha(g))$$

We still need to show that it is a homomorphism, let $w = g_1 \cdots g_n h_1 \cdots h_m$ and $w_2 = h_m^{-1} \cdots h_1^{-1} k_1 \cdots k_l$ s.t $w_1 \cdot w_2 = g_1 \cdots g_n k_1 \cdots k_l$ then

$$\begin{aligned} f(w_1) f(w_2) &= f_{\alpha_1}(g_1) \cdots f_{\alpha_n}(g_n) f_{\beta_1}(h_1) \cdots f_{\beta_m}(h_m) f_{\beta_m}(h_m^{-1}) \cdots f_{\beta_1}(h_1) f_{\gamma_1}(k_1) \cdots f_{\gamma_l}(k_l) \\ &= f(w_1 \cdot w_2) \end{aligned}$$

Note that any homomorphism that makes the diagram commute must agree with f on the one letter words, which generate the product. Therefore f is indeed unique. \square

2 The Van Kampen Theorem

Let X be a space with

$$X = \bigcup_{\alpha \in \mathcal{A}} A_\alpha, \quad \bigcap_{\alpha \in \mathcal{A}} A_\alpha \neq \emptyset$$

Fix $x_0 \in \bigcap_{\alpha \in \mathcal{A}} A_\alpha$, therefore all the fundamental groups below will be based at x_0 . Let $\iota_\alpha: A_\alpha \rightarrow X$ be the usual inclusion map. So we get

$$\iota_\alpha^*: \pi_1(A_\alpha) \rightarrow \pi_1(X)$$

Thus we get a unique homomorphism

$$\Phi: \prod_{\alpha \in \mathcal{A}}^* \pi_1(A_\alpha) \rightarrow \pi_1(X)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(A_\alpha) & \xrightarrow{i_\alpha^*} & \prod_{\alpha} \pi_1(A_\alpha) \\ & \downarrow \iota_\alpha^* & \\ & \pi_1(X) & \downarrow \Phi \end{array}$$

Theorem 2.1 (The Van Kampen Theorem). *Suppose we have the setting above, and A_α 's are open, $A_\alpha \cap A_\beta$ is path connected, $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected. Then $\Phi: \prod_{\alpha} \pi_1(A_\alpha) \rightarrow \pi_1(X)$ is surjective and $\ker \Phi$ is the smallest normal subgroup generated by elements of the form*

$$\left[i_\alpha \left(\iota_{\alpha\beta}^*([\gamma]) \right) \right]^{-1} \left[i_\beta \left(\iota_{\beta\alpha}^*([\gamma]) \right) \right]$$

Where

$$[\gamma] \in \pi_1(A_\alpha \cap A_\beta)$$

and

$$\iota_{\alpha\beta}: A_\alpha \cap A_\beta \rightarrow A_\alpha \quad \text{inclusion}$$

Remark 2.2. Note that

$$\begin{aligned} \Phi \left[i_\alpha \left(\iota_{\alpha\beta}^*([\gamma]) \right) \right] &= \iota_\beta^* \left(\iota_{\beta\alpha}^*([\gamma]) \right) \\ &= (\iota_\beta \circ \iota_{\beta\alpha})^*(\gamma) \end{aligned}$$

Similarly

$$\begin{aligned} \Phi \left[i_\beta \left(\iota_{\beta\alpha}^*([\gamma]) \right) \right] &= \iota_\alpha^* (\iota_{\alpha\beta}^*([\gamma])) \\ &= (\iota_\alpha \circ \iota_{\alpha\beta})^* [\gamma] \end{aligned}$$

Since $\iota_\beta \circ \iota_{\beta\alpha} = \iota_\alpha \circ \iota_{\alpha\beta}$ we have

$$(\iota_\beta \circ \iota_{\beta\alpha})^* = (\iota_\alpha \circ \iota_{\alpha\beta})^*$$

So a priori we have that if H is the smallest normal subgroup generated by these, we have that (in general)

$$H \subset \ker \Phi$$

The theorem gives us

$$\ker \phi = H$$

We will now demonstrate that all the conditions above are necessary:

Example 2.3. Take S_1 as the open union of two arcs (more than half-arcs). Note that their intersection is not path connected. Well

$$\pi_1(S_1) = \mathbb{Z}, \pi_1(P_1) = \pi_1(P_2) = e$$

Example 2.4. View picture, lots of drawing.