Algebraic Topology

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Contents

1 Bases of the Fundamental Group

1.1 Homotopy equivalence.

Definition 1.1. Let X be a topological space $f, g \colon X \to Y$ then we say that f is *homotopic* to g if there is a function $H \colon X \times I \to Y$ (called a homotopy) such that

$$H(x, 0) = f(x), \quad H(x, 1) = g(x)$$

Proposition 1.2. *Homotopy is an equivalence relation.*

Lemma 1.3 (Gluing Lemma). *Suppose that $X = A \cup B$ with A, B closed and take $f_1 \colon A \to Y$, $f_2 \colon B \to Y$. s.t $f_1(x) = f_2(x)$ for all $x \in A \cap B$. Then*

$$f_3 \colon X \to Y, \quad f_3(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. Let C be a closed set in Y, then

$$\begin{aligned} f_3^{-1}(C) &= f_3^{-1}(C) \cap (A \cup B) \\ &= \left(f_3^{-1}(C) \cap A\right) \cup \left(f_3^{-1} \cap B\right) \\ &= f_1^{-1}(C) \cup f_2^{-1}(C) \end{aligned}$$

which is closed in X. $\square$

Example 1.4. Any two function into $\mathbb{R}^n$ (convex set, start shaped set) are homotopic.

Indeed, take two such $f(x), g(x)$ then let

$$H\colon X \times Y \to \mathbb{R}^n, \quad H(x,t) = (1-t)f(x) + tg(x)$$

Note: we call this *the straight line homotopy*.

Relative Homotopy

Definition 1.5. Let $f, g\colon X \to Y$ with $A \subseteq X$ and $f_{|A} = g_{|A}$ then we say that f is homotopic to g relative to A if there exists a homotopy $H\colon X \times I \to Y$ between f and g

$$H(x,t) = f(x) = g(x) \quad \forall t \in I,\ \forall x \in A$$

Remark 1.6. Homotopy is a special kind of relative homotopy (for $A = \varnothing$).

Proposition 1.7. *Relative homotopy is an equivalence relation.*

Definition 1.8. Let X a topological space and let $x_1, x_2 \in X$ then a path in X going from x_1 to x_2 is a function $\gamma\colon I \to X$ s.t

$$\gamma(0) = x_1, \quad \gamma(1) = x_2$$

Definition 1.9. Suppose that γ_1 and γ_2 are two paths in X whose start point and end point coincide, then we say that γ_1 is path homotopic to γ_2 if γ_1 is homotopic to γ_2 relative to $\{0,1\}$.

Intuitively this means that we can deform γ_1 into γ_2 without moving the endpoints. i.e $H\colon I \times I \to X$ s.t

$$H(s,t) = \begin{cases} \gamma_1(s) & \text{if } t = 0 \\ \gamma_2(s) & \text{if } t = 1 \\ \gamma_1(0) = \gamma_2(0) & \text{if } s = 0 \\ \gamma_1(1) = \gamma_2(1) & \text{if } s = 1 \end{cases}$$

Corollary 1.10. *Path homotopy is an equivalence relation.*

Definition 1.11. $\gamma_1, \gamma_2\colon I \to X$ with

$$\gamma_1(1) = \gamma_2(0)$$

we define the *product* of these paths to be the path $\gamma_1 \cdot \gamma_2$ given by

$$\gamma_1 \cdot \gamma_2(s) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s \leq 1/2 \\ \gamma(2s-1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Note that this is continuous by the Gluing Lemma.

Proposition 1.12. *Path multiplication is compatible with path homotopy; i.e if*

$$\gamma_1 \sim \gamma_1', \quad \gamma_2 \sim \gamma_2'$$

then

$$\gamma_1 \cdot \gamma_2 \sim \gamma_1' \sim \gamma_2'$$

Proof. Let $H_1 \colon I \times I \to X$ and $H_2 \colon I \times I \to X$ be the corresponding homotopies then define $H \colon I \times I \to X$ by

$$H(s,t) = \begin{cases} H_1(2s,t) & \text{if } 0 \leq t \leq 1/2 \\ H_2(2s-1,t) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

We have to check that this a valid path homotopy, TODO. □

1.2 The Fundamental group

Definition 1.13. A loop in X based at x_0 is a path whose end points are both x_0.

Theorem 1.14. *Take X a top space and $x_0 \in X$ then the set of path homotopy equivalence classes of the loops based at x_0 is a group where multiplication, identity and inverses are defined as such:*

- $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$
- *Identity is* $[x_0]$
- *Inverse of* $[\gamma]$ *is* $[\overline{\gamma}]$ *where*

$$\overline{\gamma}(s) = \gamma(1-s)$$

This group is called the Fundamental group of X based at x_0 denoted by $\pi_1(X, x_0)$.

Remark 1.15. Note that multiplication is well defined, since homotopy behaves nicely with path product.

Lemma 1.16. *Let $\varphi \colon I \to I$ s.t $\varphi(0) = 0$ and $\varphi(1) = 1$. Then for any path γ we have that*

$$\gamma\varphi \sim \gamma$$

we call $\gamma \circ \varphi$ a reparametrization of γ.

Proof. Let $H(s,t) = \gamma((1-t)s + t\varphi(s))$, then clearly

$$H(s,0) = \gamma(s), \quad H(s,1) = \gamma(\varphi(s))$$

$$H(0,t) = \gamma(0), \quad \gamma(1,t) = \gamma(1)$$

□

Proof of the theorem.
We have to show that this satisfies the axioms of groups:

- **Associativity.** Take $\gamma_1,\ \gamma_2,\ \gamma_3 \colon I \to X$ s.t

$$\gamma_1(1) = \gamma_2(0), \quad \gamma_2(1) = \gamma_3(0)$$

Then we have that

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(s) = \begin{cases} \gamma_1(4s) & \text{if } 0 \leq s \leq 1/4 \\ \gamma_2(4s-1) & \text{if } 1/4 \leq s \leq 1/2 \\ \gamma_3(2s-1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

and

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(s) = \begin{cases} \gamma_1(2s) & \text{if } 0 \leq s \leq 1/2 \\ \gamma_2(4s-2) & \text{if } 1/2 \leq s \leq 3/4 \\ \gamma_3(4s-3) & \text{if } 3/4 \leq s \leq 1 \end{cases}$$

We want to find $\varphi\colon I \to I$ s.t

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(s) = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(\varphi(s))$$

Indeed, define

$$\varphi(s) = \begin{cases} 2s & \text{if } 0 \leq s \leq 1/4 \\ s + 1/4 & \text{if } 1/4 \leq s \leq 1/2 \\ 1/2s + 1/2 & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Check that this is a valid reparametrization; i.e that the equality above holds.

- **Identity.** We will only show left-identity: right is completely analogous. Take $\gamma\colon I \to X$, let $x_0 = \gamma(0)$ and $x_0\colon I \to X$ s.t $x_0(s) = x_0$. Then

$$x_0 \cdot \gamma(s) = \begin{cases} x_0 = \gamma(0) & \text{if } 0 \leq s \leq 1/2 \\ \gamma(2s-1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Then clearly $\gamma(\varphi(s)) = x_0 \cdot \gamma(s)$ via

$$\varphi(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq 1/2 \\ 2s-1 & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Check that the reparametrization is valid.

- **Inverse.** Suppose that $\gamma\colon I \to X$ is a path, fix $\alpha \in I$. Define the path

$$\gamma_\alpha(s) = \gamma(\alpha s)$$

We want to show that

$$\gamma \cdot \overline{\gamma} \sim \gamma(0)$$

Let $H\colon I \times I \to X$ be

$$H(s,t) = \gamma_{1-t} \cdot \overline{\gamma_{1-t}}(s)$$

Then notice that

$$H(s,0) = \gamma_1\overline{\gamma_1}(s), \quad H(s,1) = \gamma_0 \cdot \overline{\gamma_0}(s) = \gamma(0)$$

and

$$H(0,t) = \gamma_{1-t}(0) = \gamma(0), \quad H(1,t) = \gamma(0)$$

Where the last equality comes from the fact that

$$H(s,t) = \begin{cases} \gamma_{1-t}(2s) & \text{if } 0 \le s \le 1/2 \\ \overline{\gamma_{1-t}}(2s-1) = \gamma_{1-t}(2-2s) = \gamma((1-t)(2-2s)) & \text{if } 1/2 \le s \le 1 \end{cases}$$

Note that H is continuous since it is clearly continuous on $[0,1/2] \times I$ and $[1/2,1] \times I$ + Gluing lemma.

□

Theorem 1.17. *Let X be a topological space and let $x_0, x_1 \in X$; take η a path from $x_0 \to x_1$. Define $\beta_\eta \colon \pi_1(X,x_1) \to \pi_1(X,x_0)$ by*

$$\beta_\eta[\gamma] = [\eta \cdot \gamma \cdot \overline{\eta}]$$

Then β_η is an isomorphism.

Proof.

- **Well defined.** Let $[\gamma_1] = [\gamma_2]$ then

$$\begin{aligned} [\gamma_1] = [\gamma_2] &\implies \gamma_1 \sim \gamma_2 \\ &\implies \eta \cdot \gamma_1 \sim \eta \cdot \gamma_2 \\ &\implies \eta \cdot \gamma_1 \cdot \overline{\eta} \sim \eta \cdot \gamma_2 \cdot \overline{\eta} \\ &\implies [\eta \cdot \gamma_1 \cdot \overline{\eta}] = [\eta \cdot \gamma_2 \cdot \overline{\eta}] \end{aligned}$$

- **Homomorphism.**

$$\begin{aligned} \beta_\eta\left([\gamma_1] \cdot [\gamma_2]\right) &= \beta_\eta\left([\gamma_1 \cdot \gamma_2]\right) \\ &= [\eta \cdot \gamma_1 \cdot \gamma_2 \cdot \overline{\eta}] \\ &= [\eta \cdot \gamma_1 \cdot x_1 \cdot \gamma_2 \cdot \overline{\eta}] \\ &= [\eta \cdot \gamma_1 \cdot \overline{\eta}] \cdot [\eta \cdot \gamma_2 \cdot \overline{\eta}] \\ &= \beta_\eta[\gamma_1] \cdot \beta_\eta[\gamma_2] \end{aligned}$$

- **Isomorphism.** Notice that

$$\begin{aligned} \beta_\eta \circ \beta_{\overline{\eta}}[\gamma] &= [\beta_\eta(\overline{\eta}) \cdot \gamma \cdot \eta] \\ &= [\eta \cdot \overline{\eta} \cdot \gamma \cdot \eta \cdot \overline{\eta}] \\ &= [x_0 \cdot \eta \cdot x_0] \\ &= [\gamma] \end{aligned}$$

Similarly $\beta_{\overline{\eta}} \circ \beta_\eta$ is also identity.

□

Corollary 1.18. *If X is path connected then $\pi_1(X, x_0)$ is independent of X.*

From now on we all space will be path connected (unless stated otherwise explicitly): in this case we will simply write $\pi_1(X)$.

Definition 1.19. If $\pi_1(X) = e$ we say that X is *simply connected*

Proposition 1.20. *X is simply connected $\iff$ $\forall x_0, x_1 \in X$ any two paths between x_0 and x are homotopic.*

Proof. $\Longleftarrow$ Take $x_0 = x_1$, then any two loops are homotopic, in particular every loop is homotopic to the constant loop.

$\Longrightarrow$ Let $x_0, x_1 \in X$ and take $\gamma, \gamma' \colon x_0 \to x_1$. Note that $\gamma \cdot \overline{\gamma'}$ is a loop, therefore $\gamma \cdot \overline{\gamma'} \sim x_0$. Therefore multiplying both sides by γ' we get

$$\begin{aligned}(\gamma \cdot \overline{\gamma'}) \cdot \gamma' &\sim x_0 \cdot \gamma' \\ (\gamma \cdot \overline{\gamma'}) &\sim \gamma' \\ \gamma &\sim \gamma'\end{aligned}$$

□

Example 1.21. The following spaces are simply connected

- $\mathbb{R}^n$: Let γ be a loop based at 0, then let

$$H(s,t) = (1-t)\gamma(s) + t \cdot x_0 = (1-t)\gamma(s)$$

- Similarly, every convex set is simply connected.
- Similarly, every star-shaped set is simply connected.
- S^n is simply connected for $n \geq 2$ (proof later).

Notice that S^1 is not simply connected (proof later).

1.3 Homotopy of spaces

Let X, Y be topological spaces and $\varphi \colon X \to Y$ be a map. Fix $x_0 \in X$, suppose that γ is a loop based at x_0, then $\varphi(\gamma)$ is a loop based at $\varphi(x_0)$. Moreover, if $\gamma \sim \gamma'$ via homotopy H, then $\varphi(\gamma) \sim \varphi(\gamma')$ via the homotopy $\varphi \circ H$.

Proof. $H \colon I \times I \to X$ s.t

$$H(0,t) = x_0 = H(1,t), \quad H(s,0) = \gamma(s), \quad H(s,1) = \gamma'(s)$$

Then

$$\varphi \circ H \colon I \times I \to Y$$

and

$$\varphi(H(0,t)) = \varphi(x_0) = \varphi(H(1,t)), \quad H(s,0) = \varphi(\gamma(s)), \quad \varphi(H(s,1)) = \varphi(\gamma'(s))$$

□

Therefore we get a well defined map from $\pi(X, x_0)$ to $\pi(Y, \varphi(x_0))$, denote by φ_* and defined by

$$\varphi_*([\gamma]) = [\varphi \circ \gamma]$$

Proposition 1.22.
- *φ_* is a Homomorphism.*
- $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.
- $(\mathbb{1})_* = \mathbb{1}_*$
- *$\varphi \sim \psi$ relative to x_0 then $\varphi_* = \psi_*$.*

Proof.
- Take γ_1, γ_2 loops at x_0, then

$$\varphi(\gamma_1 \cdot \gamma_2) = \varphi(\gamma_1) \cdot \varphi(\gamma_2)$$

because

$$\varphi(\gamma_1 \cdot \gamma_2(s)) = \begin{cases} \varphi(\gamma_1(2s)) & 0 \leq s \leq 1/2 \\ \varphi(\gamma_2(2s-1)) & 1/2 \leq s \leq 1 \end{cases}$$

but

$$\varphi(\gamma_1) \cdot \varphi(\gamma_2)(s) = \begin{cases} \varphi(\gamma_1(2s)) & 0 \leq s \leq 1/2 \\ \varphi(\gamma_2(2s-1) & 1/2 \leq s \leq 1) \end{cases}$$

Therefore

$$\begin{aligned} \varphi_*([\gamma_1] \cdot [\gamma_2]) &= \varphi_*([\gamma_1 \cdot \gamma_2]) \\ &= [\varphi(\gamma_1 \cdot \gamma_2)] = [\varphi(\gamma_1) \cdot \varphi(\gamma_2)] \\ &= [\varphi(\gamma_1)] \cdot [\varphi(\gamma_2)] \\ &= \varphi_*([\gamma_1]) \cdot \varphi_*([\gamma_2]) \end{aligned}$$

- TODO
- TODO
- Let γ be a loop at x_0 and suppose that H is the homotopy between φ and ψ relative to x_0, therefore $H\colon X \times I \to Y$ s.t

$$H(x, 0) = \varphi(x), \quad H(x, 1) = \psi(x), \quad H(x_0, t) = \varphi(x_0) = \psi(x_0)$$

Then consider the map

$$H'\colon I \times I \to Y, \quad H'(s, t) = H(\gamma(s), t)$$

then

$$H'(s, 0) = H(\gamma(s), 0) = \varphi(\gamma(s)), \quad H'(s, 1) = H(\gamma(s), 1) = \psi(\gamma(s))$$

and

$$H'(0, t) = H(x_0, t) = \varphi(x_0), \quad H'(1, t) = \psi(x_0)$$

Therefore for any γ
$$\varphi_*([\gamma]) = [\varphi(\gamma)] = [\psi(\gamma)] = \psi_*([\gamma])$$

draw the pictures here, it clears things up. $\square$

Corollary 1.23. $X \cong Y \implies \pi_1(X) \cong \pi_1(Y)$.

Proof. Let $\varphi\colon X \to Y, \quad \psi\colon Y \to X$ such that
$$\psi \circ \varphi = \mathbb{1}_X, \quad \varphi \circ \psi = \mathbb{1}_Y$$

Therefore
$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_* = \mathbb{1}_*$$

similarly
$$\varphi_* \circ \psi_* = \mathbb{1}_*$$

Therefore
$$\pi_1(X) \cong \pi_1(Y)$$

$\square$

Remark 1.24. Note that the proof above would still work if we only had that $\psi \circ \varphi \sim \mathbb{1}$ relative to x_0 and $\varphi \circ \psi \sim \mathbb{1}$ relative to $\varphi(x_0)$.

Definition 1.25. Let X, Y be two spaces, we say that X is *homotopic* to Y iff there exists some $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ such that
$$\psi \circ \varphi \sim \mathbb{1}_X, \quad \varphi \circ \psi \mathbb{1}_Y$$

Proposition 1.26. *Homotopy of spaces is an equivalence relation*

Proof.
- **Reflexive.** Take $\varphi = \psi = \mathbb{1}$.
- **Symmetric.** Trivial.
- **Transitive.**

Lemma 1.27. *Let $f\colon X \to Y$, $f'\colon X \to Y$ $g, g'\colon Y \to Z$ with*
$$f \sim f', \quad g \sim g'$$

then
$$g \circ f \sim g' \circ f'$$

Proof. Let H be the homotopy between f and f' and H' is the homotopy between g and g'. Then
$$H''\colon X \times I \to Z$$

defined by
$$H''(x,t) = H'(H(x,t),t)$$

notice that

$$H''(0,t) = H'(H(x,0),0) = H'(f(x),0) = g \circ f(x)$$

Similarly check the rest. □

Now suppose we have X, Y, Z spaces with

$$\varphi \colon X \to Y, \quad \varphi \colon Y \to \mathbb{Z}$$

and

$$\psi \colon Y \to X, \quad \psi' Y \to Z$$

$$\begin{tikzcd} X \arrow[r, bend right, "\varphi"'] & Y \arrow[l, bend right, "\psi"'] \arrow[r, bend right, "\varphi'"'] & Z \arrow[l, bend right, "\psi'"'] \end{tikzcd}$$

Then

$$\varphi' \circ \varphi \colon X \to Z, \quad \psi \circ \psi' \colon Z \to X$$

s.t

$$\begin{aligned} \psi \circ \psi' \circ \varphi' \circ \varphi &\sim \psi \circ \mathbb{1} \circ \varphi \\ &\sim \mathbb{1} \end{aligned}$$

□

Terminology. If $X \sim \{\cdot\}$, then we say that X is contractible.

Example 1.28. $\mathbb{R}^n$ is contractible; there is only one map $\varphi \colon \mathbb{R}^n \to \{\cdot\}$. Take the map $\psi \colon \cdot \to \mathbb{R}^n$ to be

$$\cdot \to 0$$

Indeed

$$\varphi \circ \psi \colon \cdot \to \cdot = \mathbb{1}$$

Moreover, $\psi \circ \varphi \colon \mathbb{R}^n \to \mathbb{R}^n$

$$\psi \circ \varphi(\alpha) = 0$$

let $H \colon \mathbb{R}^n \times i \to \mathbb{R}^n$ defined by

$$H(x,t) = t \cdot x$$

Similarly, we get that any convex subset of $\mathbb{R}^n$ is contractible; any star-shaped set is also contractible.

Later on we will see that S^n is not contractible.

Definition 1.29. Let X be a space and $A \subseteq X$, then $r \colon X \to A$ is called a *retraction* iff

$$r_{|A} = \mathbb{1}_{|A}$$

In this case we say that A is a *retract* of X.

Example 1.30.
- Clearly the function $r\colon \mathbb{R}^n \to \{x_0\}$ is a retraction.
- $r\colon \mathbb{R}^n \setminus \{0\} \to S^{n-1}, \quad x \to \frac{x}{|x|}$

Definition 1.31. X a space, $A \subseteq X$: we say that X deformation retracts to A (or that there is a deformation retraction from x to A) (or that A is a deformation retract of X) iff
- There is a retraction $r\colon X \to A$.
- Letting $\iota\colon A \to X$ be the inclusion map, we have
$$r \circ \iota = \mathbb{1}_{|A}$$
and
$$\iota \circ r \sim \mathbb{1}_{|A} \text{ relative to } A$$

Example 1.32.
- $\mathbb{R}^n$, convex / start shaped subsets deformation retract into a point.
- $\mathbb{R}^n$ deformation retract into D^n: let $r\colon \mathbb{R}^n \to D^n$ be defined by
$$r(x) = \begin{cases} x & \text{if } x \in D^n \\ \frac{x}{|x|} & \text{if } x \notin D^n \end{cases}$$
Let $H(x,t) = tr(x) + (1-t)x$.
- $\mathbb{R}^n \setminus \{0\}$ deformation retracts to S^{n-1}: $r\colon \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ by
$$r(x) = \frac{x}{|x|}$$
$H(x,t) = (1-t)x + t\frac{x}{|x|}$ is the homotopy.
There is no deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to any point.
- $I \times I$ (The closed rectangle) deformation retracts to a cup Exercise.

Proposition 1.33. *If X deformation retracts to A and A to B, then X deformation retracts to B.*

Proof. Exercise. □

Theorem 1.34. $X \sim Y \implies \pi_1(X) \cong \pi_1(Y)$

Lemma 1.35. *Let X, Y topological spaces and $\varphi, \psi\colon X \to Y$ be two homotopic functions by a homotopy $H\colon X \times I \to Y$. Note that $H(x_0, \cdot)$ is a path in Y (denote it by η). Then the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "\psi_*"] \arrow[dr, "\varphi_*"'] & \pi_1(Y,\psi(x_0)) \arrow[d, "\beta_\eta"] \\ & \pi_1(Y,\varphi(x_0)) \end{tikzcd}$$

i.e
$$\beta_\eta \circ \psi_* = \varphi_*$$

Proof of lemma. Let $\gamma\colon I \to X$ be a loop based at x_0, let

$$H'(s,t) = \eta_t(s) \cdot H(\gamma(s),t) \cdot \overline{\eta_t}(s)$$

Convince yourself that the endpoints match here; therefore by the Gluing Lemma H' is continuous.

- At $t = 1$, we get
$$\eta \cdot \psi(\gamma) \cdot \overline{\eta}$$
- At $t = 0$, we get
$$\varphi(x_0) \cdot \varphi(\gamma) \cdot \overline{\varphi(x_0)}$$

So we have that

$$\begin{aligned}\eta \cdot \psi(\gamma) \cdot \overline{\eta} &\sim \varphi(x_0) \cdot \varphi(\gamma)\overline{\varphi}(x_0) \\ &\sim \varphi(\gamma)\end{aligned}$$

and therefore

$$\begin{aligned}[\eta \cdot \psi(\gamma) \cdot \overline{\eta}] &= [\varphi(\gamma)] \\ \beta_\eta[\psi(\gamma)] &= \varphi_*([\gamma]) \\ \beta_\eta(\psi_*([\gamma])) &= \varphi_*([\gamma])\end{aligned}$$

therefore

$$\varphi_* = \beta_\eta \circ \psi_*$$

□

Proof of the theorem.

$$(Y,y_0) \xrightarrow{\quad\psi\quad} (X,x_0) \xrightarrow{\quad\varphi\quad} (Y,y_1) \xrightarrow{\quad\psi\quad} (X,x_1)$$

we know that

$$\psi \circ \varphi \sim \mathbb{1}, \quad \varphi \circ \psi \sim \mathbb{1}$$

Therefore by the lemma

$$(\psi \circ \varphi)_* = \beta, \quad (\varphi \circ \psi)_* = \beta'$$

But β, β' are isomorphisms, therefore we get that

$$\varphi_* \text{ is injective and surjective}$$

□

1.4 Fundamental group of all spheres

Theorem 1.36.

$$\pi_1(S^n) = \begin{cases} e & \text{if } n \geq 2 \\ \mathbb{Z} & \text{if } n = 1 \end{cases}$$

Proof.

- $\pi_1(S^2)$; there is a homeomorphism between $S^2 \setminus \{N\}$ and $\mathbb{R}^2$ (N is the North pole) called the stereographic projection. i.e $t \to (0,0,1) + t(x,y,z) = (tx, ty, tx+1)$ since $1 + tz = 0 \implies t = \frac{t}{1-z}$ we get the map Σ given by

$$\Sigma(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z})$$

For the inverse map $t \to (0,0,1) + t(\alpha, \beta, -1) = (t\alpha, t\beta, 1-t)$; on the sphere

$$t^2\alpha^2 + t^2\beta^2 + (1-t)^2 = 1 \implies t = \frac{2}{\alpha^2 + \beta^2 + 1}$$

so its inverse is given by

$$\Sigma^{-1}(\frac{2}{\alpha^2 + \beta^2 + 1}, \frac{2\beta}{\alpha^2 + \beta^2 + 1}, 1 - \frac{2}{\alpha^2 + \beta^2 + 1})$$

TODO, do the same for $\mathbb{R}^n$.
We shall take our loops to be based at the south pole S. Let $\gamma\colon I \to S^2$ be such a loop. If it happens that γ never passes through N; $\Sigma \circ \gamma$ is a loop in $\mathbb{R}^2$ based at 0. Thus $H\colon I \times I \to S^2$ given by

$$H(s,t) = \Sigma^{-1}\left((1-t)\Sigma \circ \gamma(s)\right)$$

gives a homotopy between γ and the constant loop at the south pole.
Now suppose that we have any loop $\gamma\colon I \to S^2$. Consider the preimage by γ of the open Northern hemisphere, this is open in I since γ is continuous; in fact it is open in $(0,1)$ (and hence an open subset of $\mathbb{R}$). Now any open subset of $\mathbb{R}$ is the union of countably many disjoint open intervals. Consider now the preimage of N, it must be closed in $[0,1]$ and hence must be compact. Therefore since the preimage of the open Northern hemisphere contains that of N; the preimage of N is an open cover of the preimage of N. So it has a finite subcover. Thus there are finitely many disjoint open intervals which contain the preimage of N. Consider one of these intervals (a,b). Note that $\gamma((a,b)) \subseteq N$; since γ is continuous we can conclude that $\gamma(a), \gamma(b) \in \overline{N}$ (the closed Northern hemisphere). Note that $\gamma(a), \gamma(b)$ must be on the equator; since if they were in the Northern hemisphere we would have that a belongs to one of the other *disjoint* intervals to (a,b). Note that the closed Northern hemisphere is homeomorphic to D^2 by $(x,y,z) \to (x,y)$. but note that $\partial D^2 = S^1$ which is path connected; since we can find some path $\eta\colon I \to S^1$ going from $\gamma(a) \to \gamma(b)$ with η homotopic $\gamma_{|[a,b]}$.
Now we can write a homotopy between γ and some loop that skips N; see the picture we are using the homotopy above on all (a,b) like above. This is continuous by the gluing lemma according to our homotopy. So γ is homotopic to a loop that skips N and so homotopic to the constant loop.

- We will think of S^1 as
$$\{z \in \mathbb{C} \colon |z| = 1\}$$
We will base our loops at the point $z = 1$. We define
$$\Phi \colon \mathbb{Z} \to \pi_1(S^1), \quad \Phi(n) = [e^{2\pi i n s}]$$
We also define
$$p \colon \mathbb{R} \to S^1, \quad p(x) = e^{2\pi i x}$$
Note that $[p(ns)] = \Phi(n)$. We will show that Φ is a homomorphism
$$\begin{aligned}\Phi(n) \cdot \Phi(m) &= [p(ns)] \cdot [p(ms)] \\ &= [p(ns) \cdot p(ms)] \\ &= [p(ns) \cdot p(n + ms)] \\ &= [p(ns \cdot (n + ms))]\end{aligned}$$
Note that $ns \cdot (n + ms)$ is a path $0 \to m + n$, but $\mathbb{R}$ is simply connected, hence
$$ns \cdot (n + ms) \sim (n + m)s \implies p(ns \cdot (n + ms)) \sim p((n + m) \cdot s)$$
Therefore
$$\Phi(n) \cdot \Phi(m) = [p(n + m)s] = \Phi(n + m)$$
Claim 1. For any path γ in S^1 starting at 1, there is a unique path $\tilde{\gamma} \in \mathbb{R}$ starting at 0, s.t
$$p(\tilde{\gamma}) = \gamma$$
we say that $\tilde{\gamma}$ is the *lift* of γ starting at 0.
Claim 2. Suppose that γ_1, γ_2 are two paths in S^1 starting at 1, which are homotopic. Then their lifts starting at 0 are also homotopic.
Φ **is surjective.** Let $[\gamma] \in \pi_1(S^1)$, then we know that there is a unique $\tilde{\gamma}$ in $\mathbb{R}$ s.t
$$\tilde{\gamma}(0) = 0, \quad p(\tilde{\gamma}) = \gamma$$
Especially $p(\tilde{\gamma}(1)) = \gamma(1) = 1$, hence
$$\tilde{\gamma}(1) = n \in \mathbb{Z} \implies \tilde{\gamma} \sim ns$$
Therefore
$$p(\tilde{\gamma}) \sim p(ns) \implies \gamma \sim p(ns) \implies [\gamma] = [p(ns)] = \Phi(n)$$
Φ **is injective.** Suppose that $\Phi(n) = \Phi(m)$ therefore
$$[p(ns)] = [p(ms)]$$

We have that the lift of $p(ns)$ is ns and the lift of $p(ms)$ is ms but

$$p(ns) \sim p(ms) \implies ns \sim ms \implies n = m$$

We will prove the two claims in the following setting: $\tilde{X}, X$ two spaces, $p\colon \tilde{X} \to X$ s.t $\forall p \in X,\ \exists U \ni p$ open then $p^{-1}(U)$ is a disjoint union of open sets s.t each one of them (say V) satisfies

$$p_{|V} \quad \text{is a homeomorphism onto } U$$

Such a U is called an elementary neighborhood or an evenly covered neighborhood. p is called the projection covering map and

$$(\tilde{X}, p) \text{ is called a covering space of } X$$

The example to take in mind here is $X = S^1$, $\tilde{X} = \mathbb{R}$ with p "wrapping" $\mathbb{R}$ around S^1. Indeed consider $U_1 = S^1 \setminus \{1\}$, then $p^{-1}(U_1)\ \mathbb{R} \setminus \mathbb{Z}$; and $U_2 = S^1 \setminus \{-1\}$ then $p^{-1}(U_2)$ and $U_2 = \mathbb{R} \setminus (1/2\mathbb{Z})$, so $(\mathbb{R}, p)$ is indeed a covering space of S^1.
Claim 3. Suppose that $(\tilde{X}, p)$ is a covering space of X, and let $f\colon Y \times I \to X$ and suppose we have some $\tilde{f}\colon Y \times \{0\} \to \tilde{X}$ s.t

$$p \circ \tilde{f} = f$$

then $\tilde{f}$ can be extended *uniquely* to $Y \times I$ s.t $p \circ \tilde{f} = f$ (draw the picture).
Proof that claim 3 implies claim 1 and 2. For claim 1, let $Y = \{\cdot\}$, in this case

$$\{\cdot\} \times I \cong I$$

so we have that $f\colon I \to X$ and $\tilde{f}\colon 0 \to \tilde{X}$ s.t $\tilde{f}$ can be uniquely extended to I; which is precisely claim 1 where we are given a path in X and the starting point of the lift is $\tilde{X}$. For claim 2, take $Y = I$ and f to be the homotopy between the two paths in X; from claim 1 we can lift the homotopy at the initial time; claim 3 lifts the homotopy to $\tilde{X}$. By uniqueness of lifts the sides of the homotopy lift to the constant paths and the top of the homotopy gives a lift of the second path starting where the lift of the first path started.
Proof of claim 3. Let $y \in Y$, $\forall t \in I$ there is an open neighborhood O_t of t in I and an open neighborhood N_t of y in Y s.t

$$f(N_t \times O_t) \subseteq \text{ an elementary neighborhood}$$

$\{y\} \times I$ is compact and $\{N_t \times O_t\}_{t \in I}$ is an open cover, therefore $\exists t_1, \cdots, t_k \in I$ s.t

$$\{y\} \times I \subseteq (N_{t_1} \times O_{t_1}) \cup \cdots \cup (N_{t_k} \times O_{t_k})$$

Let $N = \bigcap_{i=1}^k N_{t_i}$, and let $\delta > 0$ be the Lebesgue number of the cover $O_{t_1}, \cdots, O_{t_k}$. Therefore any closed interval of length $\leq \delta$ is contained in one of the O_t's. Consider $N \times [0, \delta]$ we know that $f(N \times [0, \delta]) \subset U$ an elementary neighborhood. But $p \circ \tilde{f} = f$, therefore $\tilde{f}(N \times 0) \subset p^{-1}(U)$, let $\tilde{U}$ be the "part" of $p^{-1}(U)$ which contains $\tilde{f}(y, 0)$. Now by taking the preimage of $\tilde{U}$ by $\tilde{f}$ we get some open subset of Y which is also an open subset of N; replace N with this smaller subset call it N. Now p is a homeomorphism between $\tilde{U}$ and U, so it has an inverse $p^{-1}\colon U \to \tilde{U}$, define

$$\tilde{f} = p^{-1} \circ f$$

This way, we have extended $\tilde{f}$ to $N \times [0, \delta]$. Now do the same for $N \times [\delta, 2\delta]$ by looking at $\tilde{f}(N \times \{\delta\})$ and making sure that it maps to only one of the preimages of the new elementary neighborhood, we shrink N again and extend $\tilde{f}$ by letting it be $p^{-1} \circ f$.
Note that by construction we have that $p \circ \tilde{f} = f$, and it is continuous by the Gluing lemma. Hence we know that $\forall y \in Y$, $\exists N_y$ s.t $\tilde{f}$ is extended continuously to include $N_y \times I$ with $p \circ \tilde{f} = f$. Let us show that lifts of paths are unique; indeed suppose that we have $\gamma\colon I \to X$ a path in X, and let $\tilde{\gamma}_1,\ \tilde{\gamma}_2$ be two lifts of γ (i.e $p \circ \tilde{\gamma}_1 = p\tilde{\gamma}_2 = \gamma$ and they both have the same start point). Therefore $\exists \delta > 0$ s.t any closed interval of length δ is mapped by γ into an elementary neighborhood. Consider the interval $[0, \delta]$, we know that

$$\gamma\left([0, \delta]\right) \subset \text{ an elementary neighborhood } U$$

This implies that $\tilde{\gamma}_1([0, \delta]) \subset p^{-1}(U)$ and $\tilde{\gamma}_2([0, \delta]) \subset p^{-1}(U)$; since $[0, \delta]$ is connected, we know that

$$\gamma_1([0, \delta]),\ \gamma_2([0, \delta]) \subset \text{ a single } \tilde{U}_1, \tilde{U}_2$$

since $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$ we get $\tilde{U}_1 = \tilde{U}_2$. But since $p \circ \tilde{\gamma}_1 = p \circ \tilde{\gamma}_2$ and since p is a homeomorphism, then

$$\tilde{U}_1 = \tilde{U}_2 \to 0$$

Proceeding similarly, we get that $\tilde{\gamma}_1 = \tilde{\gamma}_2$.
Note that restricting f to a line like $\{y\} \times I$ gives a path in X, and restricting $\tilde{f}$ to this line gives us a lift of this path. So if we use $\tilde{f}$'s coming from different $N_y \times I$'s, then they must coincide on the intersection because the intersection is made of lines as above. So $\tilde{f}$ and $\tilde{f}$ is unique. And $\tilde{f}$ is continuous using the gluing lemma for open sets.

□

Theorem 1.37 (Fundamental theorem of Algebra)**.** *Let* $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ *then*

$$p(z_0) = 0 \textit{ for some } z_0 \in \mathbb{C}$$

Proof. Assume that $p(z) \neq 0$ consider

$$H_1(s, t) = \frac{\frac{p(tre^{2\pi i s})}{|p(tre^{2\pi i s|})}}{\frac{p(tr)}{|p(tr)|}}$$

Note that this gives a homotopy between the constant loop in S^1 at 1 and $\frac{\frac{p(re^{2\pi i s})}{|p(re^{2\pi i s})|}}{\frac{p(r)}{|r|}}$. Choose $r > 0$ to be s.t $r > 1$ and $r > |a_{n-1}| + \cdots + |a_0|$, then notice that if z is restricted to the circle of radius r, we have that

$$\begin{aligned}|z|^n = r^n = r^{n-1}\cdot r > r^{n-1}(|a_{n-1}| + \cdots + |a_0|)\\ > r^{n-1}|a_{n-1}| + \cdots + r^{n-1}|a_0|\\ > r^{n-1}|a_{n-1}| + r^{n-2}|a_{n-2}| + \cdots + |a_0|\\ = |a_{n-1}z^{n-1}| + \cdots + |a_0|\\ \geq |a_{n-1}z^{n-1} + \cdots + a_0|\end{aligned}$$

it follows that the following polynomials depending on t never vanish on the circle of radius r:

$$q_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$$

This is because for $|z| = r$ we have that

$$q_t(z) \geq |z^n| - |t(a_{n-1}z^{n-1} + \cdots + a_0)| > 0$$

Now consider $H_2(s,t)$ to be

$$H_2(s,t) = \frac{\frac{q_t(re^{2\pi is})}{|q_t(re^{2\pi is}|)}}{\frac{q_t(r)}{|p(r)|}}$$

Note that this gives a homotopy between the loop which we just created (write it out TODO); and

$$\frac{\frac{r^n e^{2\pi ins}}{r^n}}{r^n/r^n} = e^{2\pi ns}$$

Therefore using both H_1 and H_2 we have a homotopy between the constant loop and $e^{2\pi ins}$ by contradiction. □

Theorem 1.38 (Brouwer's fixed point). *$f\colon D^n \to D^n$ then f has a fixed point, i.e $\exists p_0 \in D^n$ s.t*

$$f(p_0) = p_0$$

Proof.

- **Case $n = 1$.** Let $f\colon D^1 \to D^1$, if $f(-1) = -1$ we are done same if $f(1) = 1$. Otherwise, we get that $f(-1) > -1,\ f(1) < 1$ therefore $f(x) - x$ is > 0 at -1 and < 0 at 1. Therefore by the IVT, $\exists x_0 \in (-1,1)$ s.t

$$f(x_0) - x_0 = 0$$

- **Case $n = 2$.** Assume that $f\colon D^2 \to D^2$ s.t $f(x) \neq x$ for all $x \in D^2$. Define $g\colon D^2 \to S^1$ as follows: for every $x \in D^2$, take the line from $f(x)$ to x then $g(x)$ is the intersection between this line and S^1. Note that g is continuous and it is also a retraction from $D^2 \to S^1$. Take the loop $e^{2\pi is}$ in D^2, then $e^{2\pi is} \sim$ constant in D^2. But

$$g(e^{2\pi is}) = e^{2\pi is} \not\sim \text{ constant loop in } S^1$$

  Contradiction.

Another way of saying this is that

$$g_*\colon \pi_1(D^2) \to \pi(S^1)$$

but $\pi_1(D^2)$ is trivial, but its image is not the identity.

□

Corollary 1.39. *There is no retraction $D^2 \to S^1$.*

Theorem 1.40 (Borsuk-Ulam Theorem)**.** *Let $f\colon S^n \to \mathbb{R}^n$ then there $\exists \in S^n$ s.t*

$$f(x) = f(-x)$$

Proof.

- $n = 1$. We have $f\colon S^1 \to \mathbb{R}$, if $f(1) = f(-1)$ we are done. Otherwise wlog assume $f(1) > f(-1)$; consider
$$g(x) = f(x) - f(-x)$$
then g is continuous with $g(1) > 0$ and $g(-1) < 0$ therefore by the IVT applied to the upper hemisphere, $\exists x_0 \in S^1 \cap \{(x, y) \mid y \geq 0\}$ s.t $g(x_0) = 0$.
- $n = 2$. Suppose that $f\colon S^2 \to \mathbb{R}^2$ with $f(x) \neq f(-x)$ for all x, let $g\colon S^2 \to S^1$ be defined by
$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$
note that $g(-x) = -g(x)$. Also wlog we are going to assume that
$$g(1, 0, 0) = 1$$
Consider now the loop
$$\gamma'\colon I \to S^2, \quad \gamma'(s) = (\cos(2\pi s), \sin(2\pi s), 0)$$
Let $\gamma\colon I \to S^1$ be $g \circ \gamma'$ we know that $\pi^1(S^2) = e$ therefore
$$\gamma' \sim \text{ constant}$$
and hence
$$\gamma \sim \text{ constant in } S^1$$
Note that $\gamma'(s + 1/2) = -\gamma'(s)$ so it follows that $\gamma(s + 1/2) = -\gamma(s)$. Consider the lift $\tilde{\gamma}$ of γ starting at 0. Then (do math), we get that
$$\tilde{\gamma}(s + 1/2) - \tilde{\gamma}(s) = \frac{n(s)}{2}$$
Where $n(s)$ is odd; but since LHS is continuous $n(s)$ must be a constant n. i.e
$$\tilde{\gamma}(s + 1/2) = \tilde{\gamma}(s) + n/2$$

Hence

$$\tilde{\gamma}(1) = \tilde{\gamma}(1/2) + n/2 = \tilde{\gamma}(0) + n = n$$

Since γ is homotopic to the constant, $\gamma(1) = 0$ but n is odd: contradiction.

□

1.5 First Algebraic Detour

Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of groups, we can make a group out of the product in two ways

1. $\bigtimes_{\alpha \in \mathcal{A}} G_\alpha$ is the set of all elements in the set theoretic product with multiplication defined pointwise.

2. **The weak product.** $\pi_{\alpha \in \mathcal{A}} G_\alpha$ is the subgroup of the product s.t for each element only finitely many components are nontrivial.

Theorem 1.41 (Universal Property of the Weak Product of Abelian Groups)**.** *Suppose that we have a collection $\{G_\alpha\}$ all abelian, then $\prod_\alpha G_\alpha$ satisfies the following property: Suppose we have homomorphisms $f_\alpha \colon G_\alpha \to A$ where A is abelian then $\exists! f \colon \prod_\alpha G_\alpha \to A$ a homeomorphism s.t the following diagram commutes for every α:*

$$\begin{tikzcd}
G_\alpha \arrow[r, "\iota_\alpha"] \arrow[dr, "f_\alpha"'] & \prod_\beta G_\beta \arrow[d, "f"] \\
& A
\end{tikzcd}$$

Where

$$\iota_\alpha(g) = \{g_\beta\} = \begin{cases} g & \text{if } \alpha = p \\ e & \text{if } \alpha \neq p \end{cases}$$

Proof. Let $f(\{g_\alpha\}) = \prod_\alpha f_\alpha(g_\alpha)$. Note that this product is well defined since only finitely many $f_\alpha(g_\alpha)$'s are nontrivial. Moreover, the order is irrelevant since A is Abelian. Clearly we have that f satisfies the diagram. We also have

$$f(\{g_\alpha\}\{h_\alpha\}) = f(\{g_\alpha h_\alpha\}) = \prod_\alpha f_\alpha(g_\alpha h_\alpha) = \prod f_\alpha(g_\alpha) \prod f(h_\alpha)$$

Where the last step is justified since A is abelian.

Note that $\prod_\alpha G_\alpha$ is generated by the $i_\alpha(G_\alpha)$'s, but note that any f' that satisfies the diagram must agree with our f on each of these G_α's (both give us $f_\alpha(G_\alpha)$) therefore $f' = f$. □

Remark 1.42. Note that the universal property of the weak product defines it (up to isomorphism). Indeed, suppose that we have a collection $\{G_\alpha\}$ all abelian, and some abelian G that satisfies the following property: For any A and any homomorphisms $f_\alpha \colon G_\alpha \to A$ and homomorphisms $\iota_\alpha \colon G_\alpha \to G$ where A is abelian and $\exists! f \colon G \to A$ a homomorphism s.t the following diagram commutes for every α:

$$\begin{tikzcd} G_\alpha \arrow[r, "\iota_\alpha"] \arrow[dr, "f_\alpha"'] & G \arrow[d, "f"] \\ & A \end{tikzcd}$$

Then $G \cong \prod_{\alpha\in\mathcal{A}} G_\alpha$.

Proof. Draw the commutative diagram. □

Definition 1.43. Let S be a set, then the free abelian group over S is an abelian group denoted by $F(S)$ together with a map $\iota\colon S \to F(S)$ s.t for any map $\chi\colon i\colon S \to A$ (A abelian) there is a unique homomorphism $\varphi\colon F(S) \to A$ s.t the following diagram commutes

It is easy to see that j must be injective.

Example 1.44. Let $S = \{x\}$, let $F(S) = \mathbb{Z}$ and $j\colon \{x\} \to \mathbb{Z}$ be defined by $j(x) = 1$. Pick any A abelian, let $\chi\colon S \to A$ let $a = \xi(x)$ an define

$$\varphi\colon \mathbb{Z} \to A, \quad \varphi(n) = a^n$$

indeed, $\varphi(j(x)) = a = \xi(x)$ abd φ is unique since it $\varphi'(1) = a$ and 1 generates $\mathbb{Z}$.

Generally we write elements of $F(\{x\})$ as nx.

Theorem 1.45. *The free abelian group of S is unique up to isomorphism.*

Proof. Draw the picture; final element is unique. □

Theorem 1.46. *Suppose that $S = \bigcup_{\alpha\in\mathcal{A}} = S_\alpha$ with $\alpha \neq \alpha' \implies S_\alpha \cap S'_\alpha = \varnothing$, then*

$$F(S) \cong \prod_\alpha F(S_\alpha)$$

Proof.

$$\begin{tikzcd} S_\alpha \arrow[r, "j_\alpha"] \arrow[d, hook, "i_\alpha"'] \arrow[drr, bend left=15, "\chi_\alpha"] & F(S_\alpha) \arrow[r, "\iota_\alpha"] \arrow[dr, "\varphi_\alpha"] & \prod^*_\beta F(S_\beta) \arrow[d, dashed, "\Phi"] \\ S \arrow[ur, bend right, "j"] \arrow[rr, bend left=10, "\chi"] & & A \end{tikzcd}$$

□

Corollary 1.47. *For any S, note that*

$$F(S) \cong \prod_{x\in S} F(\{x\}) \cong \prod_{x\in S} \mathbb{Z}$$

We write elements in $F(S)$ as

$$n_1x_1 + \cdots + n_kx_k$$

This notation is useful because multiplication in $F(S)$ corresponds to addition of coefficients.

Theorem 1.48. *Any abelian group A is the homomorphic image of a free abelian group.*

Proof. Let $S = A$ and let $\chi\colon S \to A = \mathbb{1}$; draw the diagram the proof is done. □

Example 1.49. Let $\mathbb{Z}_2$, take $A = \mathbb{Z}_2$ and find the $\varphi\colon F(A) \to A$. But also let $\varphi\colon \mathbb{Z} \to \mathbb{Z}_2$ defined by

$$\varphi(n) = n \mod 2$$

Definition 1.50. let A be an abelian group suppose that $F(S)$ is free abelian on S and

$$\varphi\colon F(S) \to A \quad \text{surjective}$$

Every nontrivial element of the kernel of φ is said to be a non-trivial relation on the generators of A. Moreover if $r_1, \cdots, r_k$ are nontrivial relation and $r_{k+1} \in \langle r_1, \cdots, r_k \rangle$, then we say that r_{k+1} is a consequence of $r_1, \cdots, r_k$. If $r_1, \cdots, r_k$ generate the whole kernel, we say that we have a presentation of A in terms of generators and relations.

Example 1.51. In the example above write $\mathbb{Z} = F(\{x\})$, think of A as being generated by x with the following relations imposed:

$$x + x = 0, \quad x + x + x + x = 0, \quad -x - x = 0, \cdots$$

Note that $x + x + x + x$ is a consequence of $x + x$. Indeed, since all relations are a consequence of $x + x = 0$, therefore $\mathbb{Z}_2$ is generated by a single element with a single relation.

Free Groups

Definition 1.52 (Free group)**.** Let S be a set. A *free group* on S, denoted $F^*(S)$ (often written $F(S)$), is a group together with a function

$$j : S \to F^*(S)$$

such that for every group G and every function

$$\chi : S \to G$$

there exists a unique group homomorphism

$$\varphi : F^*(S) \to G$$

making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "j"] \arrow[dr, "\chi"'] & F^*(S) \arrow[d, "\varphi"] \\ & G \end{tikzcd}$$

i.e. $\varphi \circ j = \chi$.

Equivalently, $(F^*(S), j)$ is the initial object in the category of groups equipped with a map from S.

Theorem 1.53 (Uniqueness)**.** *The free group on S is unique up to isomorphism.*

Proof. Let (F_1, j_1) and (F_2, j_2) be two free groups on S. By the universal property, there exist unique homomorphisms

$$\phi : F_1 \to F_2, \qquad \psi : F_2 \to F_1$$

such that $\phi \circ j_1 = j_2$ and $\psi \circ j_2 = j_1$. Then $\psi \circ \phi : F_1 \to F_1$ is a homomorphism satisfying $(\psi \circ \phi) \circ j_1 = j_1$. By uniqueness it must be the identity, hence $\psi \circ \phi = \mathrm{id}$. Similarly $\phi \circ \psi = \mathrm{id}$, so $F_1 \cong F_2$. ☐

Example 1.54. If $S = \{x\}$ then $F^*(S) \cong \mathbb{Z}$.

Proof. Define $j(x) = 1 \in \mathbb{Z}$. Given any group G and map $\chi : S \to G$, write $\chi(x) = g$. Define

$$\varphi : \mathbb{Z} \to G, \qquad \varphi(n) = g^n.$$

This is the unique homomorphism with $\varphi \circ j = \chi$, hence $\mathbb{Z}$ satisfies the universal mapping property. ☐

Free Products

Definition 1.55 (Free product)**.** Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of groups. The *free product*

$$\prod_{\alpha \in \mathcal{A}}^{*} G_\alpha$$

is a group together with homomorphisms

$$i_\alpha : G_\alpha \to \prod_{\alpha}^{*} G_\alpha$$

such that for any group G and any family of homomorphisms

$$f_\alpha : G_\alpha \to G$$

there exists a unique homomorphism

$$f : \prod_{\alpha}^{*} G_\alpha \to G$$

making the diagrams commute:

$$\begin{tikzcd} G_\alpha \arrow[r, "i_\alpha"] \arrow[dr, "f_\alpha"'] & \prod^*_\alpha G_\alpha \arrow[d, "f"] \\ & G \end{tikzcd}$$

i.e. $f \circ i_\alpha = f_\alpha$ for all α.

Free Groups as Free Products

Theorem 1.56. *Suppose $S = \bigsqcup_\alpha S_\alpha$ is a disjoint union of sets. Then*

$$F^*(S) \cong \prod_\alpha^* F^*(S_\alpha).$$

Proof. Both sides satisfy the same universal mapping property for maps from S. Hence they are isomorphic by uniqueness of universal objects. □

Corollary 1.57. *Let S be any set. Then*

$$F^*(S) \cong \prod_{s\in S}^* F^*(\{s\}) \cong \prod_{s\in S}^* \mathbb{Z}.$$

Corollary 1.58. *Every group is a homomorphic image of a free group.*

Proof. Let G be a group and take $S = G$ as a set. The identity map $S \to G$ extends uniquely to a homomorphism

$$F^*(S) \to G$$

which is surjective. □

Generators and Relations

Definition 1.59. Let G be a group and suppose

$$f : F^*(S) \twoheadrightarrow G$$

is a surjective homomorphism.

- The elements of S (via their images in G) are called generators.
- The nontrivial elements of $\ker(f)$ are called relations.
- If $r_1, \ldots, r_k \in F^*(S)$ then r_{k+1} is a consequence of them if

  $$r_{k+1} \in \langle\langle r_1, \ldots, r_k \rangle\rangle,$$

  the smallest normal subgroup containing $r_1, \ldots, r_k$.

Existence of Free Products

Theorem 1.60. *Free products exist.*

Construction via reduced words. Let $\mathcal{A} = \bigsqcup_\alpha G_\alpha$ be the disjoint union of the underlying sets.

Words. A word is a finite sequence

$$w = g_1 g_2 \cdots g_n \quad \text{with } g_i \in G_{\alpha_i}.$$

A word is *reduced* if:

1. $g_i \neq e$ for all i,
2. adjacent letters come from different groups: $\alpha_i \neq \alpha_{i+1}$.

The empty word is denoted by e.

Multiplication. Define multiplication by:

- concatenation of words,
- followed by reduction:
  - multiply adjacent elements from the same group,
  - remove identities,
  - cancel inverse pairs.

Group axioms.

- Identity: the empty word.
- Inverse:

$$(g_1 \cdots g_n)^{-1} = g_n^{-1} \cdots g_1^{-1}.$$

- Associativity follows from associativity of concatenation and uniqueness of reduction to reduced form.

Thus reduced words form a group, which satisfies the universal mapping property of the free product. ☐