

Commutative Algebra

Georges Khater

American University of Beirut, Math 344

February 1, 2026

Contents

1	Introduction.	1
1.1	Multiplicative sets and Prime ideals	1
1.2	The Jacobson radical of a commutative ring	3
1.3	Extension and contraction of ideals	3
1.4	Modules.	4
1.5	Local Rings	5
1.6	Algebras	5
2	Modules	7
2.1	Hom functors and exactness	8
2.2	Presentations of Modules	9
2.3	Tensor Product of Modules (review)	10
2.4	Exactness properties of $- \otimes_A N$	12
3	Localization of rings and modules	14
3.1	Ideals in the Localization	16

1 Introduction.

1.1 Multiplicative sets and Prime ideals

Definition 1.1. A subset $S \subset A$ is called multiplicatively closed iff

1. $1 \in S$
2. if $s, t \in S$, then $st \in S$.

Proposition 1.2. Let S be multiplicatively closed, and let a be an ideal with $a \cap S = \emptyset$. Consider all ideals b s.t

$$\begin{cases} a \subseteq b \\ b \cap S = \emptyset \end{cases}$$

Equivalently $a \subset b \subset A - S$. Then the poset of such ideals satisfies the conditions of Zorn's lemma, and any maximal such element is a prime ideal of A .

Proof. The poset is nonempty because it contains a , every totally ordered subset $\{b_i : i \in I\}$, has an upper bound $\bigcup_{i \in I} b_i$. Therefore by Zorn's lemma there exists some maximal element p of this set.

Suppose that $x, y \in A$ with $xy \in p$ but $x, y \notin p$: Since $x, y \notin b$ then the ideal sums

$$p + \langle x \rangle, p + \langle y \rangle$$

Strictly contain p , so they must contain each an element of S , therefore $\exists s \in S$ of the form

$$s = p_1 + xa_1 \quad p_1 \in p, a_1 \in R$$

and similarly $\exists t \in s$ with

$$t = p_2 + ya_2$$

but then $st \in S$, therefore

$$p_1p_2 + p_1ya_2 + p_2xa_1 + xy a_1 a_2 \in P \implies st \in p$$

contradicting that $p \cap S = \emptyset$.

Therefore p is prime. □

Example 1.3. • A a domain and $S = A - \{0\}$.

- In \mathbb{Z} : $S = \{1, 6, 6^2, \dots\}$, or more generally in any A if $x \in A$ is not nilpotent, then

$$S = \{1, x, x^2, \dots\}, \quad S \cap \{0\} = \emptyset$$

There exists a prime ideal with $p \cap S = \emptyset \implies x \notin p$ (obviously $0 \in p$).

Therefore

$$\{\text{all nilpotent } x\} = \text{nilrad}(0) = \bigcap_{p \text{ prime ideal}} p$$

- $p \subset A$ prime then $A - p$ is multiplicatively closed.

Theorem 1.4. If $a \subset A$ is an ideal, then

$$\text{rad } a := \left\{ x \in A \mid \exists N \geq 1 \text{ with } x^N \in a \right\}$$

satisfies

$$\text{rad } a = \bigcap_{\text{all } p, p \supseteq a} p$$

Proof. • Slick proof: apply the conclusion above to the quotient ring A/a .

- Pedestrian proof: Show both inclusions directly : If $x^N \in a$ and $a \subset p$, then $x \in p$ (for all primes $p \supseteq a$), so $x \in \bigcap_{p \supseteq a} p$.
If $\forall N, x^N \notin a$ then take $S = \{1, x, x^2, \dots\}$ we have $a \cap S = \emptyset$; so $\exists p$ ($p \supseteq a$ and $p \cap S$)

which is prime with $p \supseteq a$ and $p \cap S = \emptyset$ so $\exists p \supseteq a$ with $x \notin p$ so

$$x \notin \bigcap_{p \supseteq a} p$$

□

1.2 The Jacobson radical of a commutative ring

Definition 1.5.

$$\text{Jrad } a = \bigcap_{\substack{\text{all maximal ideals } m \subsetneq A}} \supseteq \text{nilrad } A$$

Proposition 1.6. $x \in \text{Jrad } A \iff \forall y \in A, 1 - xy \text{ is a unit in } A$.

Proof. \implies Let $x \in \text{Jrad } A$, then $\forall m$ maximal ideal, $x \in m$. If $y \in A$ satisfies $1 - xy$ not a unit, then

$$\langle 1 - xy \rangle \subsetneq A$$

so $\exists m$ maximal with $1 - xy \in m$ but $x \in m$. This would force $1 \in m$ since

$$1 - xy + xy$$

\iff Suppose that $\forall y \in A, 1 - xy$ is a unit in A . We want to show that if M is maximal, is $x \in m$? Work in A/m , field: \bar{x} satisfies that $\forall \bar{y} \in A/m$,

$$\bar{1} - \bar{xy} \text{ is a unit in } A/m$$

but in a field, this can only happen if $\bar{x} = \bar{0}$. Therefore $x \in m$, for all m .

□

1.3 Extension and contraction of ideals

Setup. $A \xrightarrow{f} B$ a homomorphism of rings (so B is an A -algebra) and

$$\mathfrak{a} \subseteq A \text{ ideal} \rightarrow \mathfrak{a}^e = \mathfrak{a}B = \langle f(a) : a \in A \rangle$$

$$\mathfrak{b} \subseteq B \rightarrow \mathfrak{b}^c = f^{-1}(\mathfrak{b}) = \ker \left(A \xrightarrow{f} B \xrightarrow{\nu} B/\mathfrak{b} \right)$$

Basic observations

$$\mathfrak{a}^{ec} \supseteq \mathfrak{a}, \quad \mathfrak{b}^{ce} \subseteq \mathfrak{b}$$

but notice that

$$\mathfrak{a} = \mathfrak{a}^e, \quad \mathfrak{b}^{cec} = \mathfrak{b}^c$$

Important fact. if $q \subseteq B$ is prime then so is q^c therefore

$$A/q^c \xrightarrow{\text{subdomain}} B/q$$

Example 1.7. $\mathbb{Z} \rightarrow \mathbb{Q}, (6\mathbb{Z})^e = \mathbb{Q}, (6\mathbb{Z})^{ec} = \mathbb{Z}$.

Example 1.8. $\mathbb{Z} \rightarrow \mathbb{Z}/10\mathbb{Z}$ ($4\mathbb{Z})^e = \frac{4\mathbb{Z}+10\mathbb{Z}}{10\mathbb{Z}} = 2\mathbb{Z}/10\mathbb{Z}$ so

$$(4\mathbb{Z})^{ec} = 2\mathbb{Z}$$

Example 1.9. $A \rightarrow A[x]$ take $\mathfrak{a} \subseteq A$ then $\mathfrak{a}^e = \mathfrak{a}[x]$ which is the set of all polynomials with coefficients in \mathfrak{a} . And $\{0_A\} = \langle x+1 \rangle^c$; $\langle x+1 \rangle^{ce} = \{0_{A[x]}\}$.

1.4 Modules.

View Math 341 Notes, I won't be taking notes here.

Let A be given and a_1, \dots, a_r ideals in A , we say that these ideals are relatively prime iff $\forall i, j$ with $i \neq j$,

$$a_i + a_j = A \iff \exists x \in a_i, y \in a_j \text{ s.t } x + y = 1$$

Proposition 1.10. *In the above situation we have that*

$$a_1 \cdots a_r = a_1 \cap a_2 \cap \cdots \cap a_r$$

Proof. Induction on r :

$$a_1 a_2 \subseteq a_1 \cap a_2 \quad \text{easy}$$

Conversely if $z \in a_1 \cap a_2$ we have that $x \in a_1, y \in a_2$ s.t $x + y = 1$ then

$$z = z \cdot 1 = zx + zy \in a_1 a_2$$

for the inductive step, we have already $b = a_1 \cap \cdots \cap a_{r-1} = a_1 \cdots a_{r-1}$ need to observe that a_r is still relatively prime to b .

Indeed, we know that

$$a_1 + a_r = A, \dots, a_{r-1} + a_r = A$$

therefore we chose elements in $x_1 \in a_1, \dots, x_{r-1} \in a_{r-1}$ and $y_1, \dots, y_{r-1} \in a_r$ with $x_i + y_i = 1$. We can take

$$\begin{aligned} 1 &= 1^{r-1} \\ &= (x_1 + y_1)(x_2 + y_2) \cdots (x_{r-1} + y_{r-1}) \\ &= x_1 \cdots x_{r-1} + (\text{terms involving some } y_i) \\ &\in a_1 \cdots a_{r-1} + \in a_r \end{aligned}$$

□

Theorem 1.11 (Chinese Remainder Theorem). *Let $a_1 \cdots a_r$ be pairwise relatively prime, consider the homomorphism of A modules*

$$A \xrightarrow{f} A/a_1 \oplus \cdots \oplus A/a_r$$

Hence

$$A/a_1 \oplus \cdots \oplus A/a_r \cong A/\ker = A/a_1 \cdots a_r$$

1.5 Local Rings

Definition 1.12. A ring A is said to be *local* iff it has exactly one maximal ideal (called m).

Example 1.13. • $A = K$ a field, then $m = 0$

- Fix p prime $A \{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z}, b \notin p\mathbb{Z}\} \supset m = \{pc/b \mid c, b \in \mathbb{Z}, p \nmid b\} = pA$. Check that the only ideals are

$$A \supsetneq pA \supsetneq p^2A \supsetneq \dots$$

•

$$\begin{aligned} A &= k[[x_1, x_2]] \\ &= \{\text{formal power series}\} \\ &= \{a_0 + b_1x_1 + b_2x_2 + c_{11}x_1 + \dots\} \end{aligned}$$

Where $a_i, b_j, c_{k,l}, \dots \in K$ with

$$m = \langle x_1, x_2 \rangle = \{\text{formal power series with } a_0 = 0\}$$

This is the only maximal ideal because m is the set of non-units, think about it. Same for the example above.

Proposition 1.14. A is local \iff the set of non-units of A is an ideal (which turns out to be the unique maximal ideal).

Proof. We denote by V the set of non-units of A

- ⇒ Let A be local, let m be its unique maximal ideal we clearly have that $m \subseteq V$. Now take any $v \in V$, then by Zorn's lemma we can find some maximal ideal $I \ni v$. But since m is the unique maximal ideal then we have that $m \ni v$.
- ⇐ Trivial.

□

1.6 Algebras

Definition 1.15. An A -algebra is a ring R (not necc. commutative) with a ring homomorphism $f: A \rightarrow R$ s.t

$$\text{Im } f \subseteq \text{center of } R$$

We will generally stick to the case where R is commutative.

Example 1.16. All rings are \mathbb{Z} -algebras, $M_n(A)$, $A[x_1, \dots, x_r]$. Moreover \mathbb{C} , \mathbb{H} and matrices in \mathbb{C} are all \mathbb{R} algebras

Definition 1.17. A *finitely generated* A -algebra means that $\exists a \subseteq A$ an ideal s.t

$$R \cong A[x_1, \dots, x_n]/a$$

Example 1.18. A \mathbb{Z} -algebra $\mathbb{Z}/10\mathbb{Z}[y_1, y_2]$ where $y_1^2 - 1 = 0$ and $y_2^3 - 2y_1y_2 - 3 = 0$ which is isomorphic to

$$\mathbb{Z}[x_1, x_2]/\langle 10, x_1^2 - 1, x_2^3 - 2x_1x_2 - 3 \rangle$$

Definition 1.19. a, b ideals of A then the *colon ideal* $(a : b)$ is defined to be

$$(a : b) = \{x \in A : xb \subseteq a\} = \bigcap_{y \in b} \text{Ann}(y + a \in A/a)$$

Example 1.20. In \mathbb{Z} ,

$$(4\mathbb{Z} : 10\mathbb{Z}) = 2\mathbb{Z}, \quad (10\mathbb{Z} : 4\mathbb{Z}) = 5\mathbb{Z}$$

Theorem 1.21. Let p_1, \dots, p_r be prime ideals of A and suppose that a satisfies

$$a \subseteq p_1 \cup \dots \cup p_n$$

Then $\exists j$ s.t $a \subseteq p_j$.

(Note that the proof below also works if all the p_i 's are prime except for 1 or 2 p_i 's).

Proof. By induction on n :

- $n = 1$: trivial.
- $n = 2$: Suppose $a \subseteq p_1 \cup p_2$ but $a \not\subseteq p_1$ and $a \not\subseteq p_2$ so $\exists x_1, x_2 \in a$ s.t $x_1 \notin p_1, x_2 \notin p_2$. Therefore we must have $x_1 \in p_2$ and $x_2 \in p_1$, then

$$a \ni z := x_1 + x_2 \notin p_1 \cup p_2$$

(If $z \in p_1$ then $x_1 = z - x_2 \in p_1$).

- Now suppose $n \geq 3$ and we know the result for $n - 1$ primes. Suppose that

$$a \subseteq p_1 \cup \dots \cup p_n$$

if $a \subseteq p_1 \cup \dots \cup p_{n-1}$ (after potentially reordering), then we are okay. So suppose that $a \not\subseteq p_1 \cup \dots \cup p_{n-1}$ for any reordering. so $\exists x_j \in a$ with $x_j \in p_j$ for every j .

Consider $z = x_1 + x_2x_3 \dots x_n$, of course $z \in a$ but $z \notin p_j$ for any j (contradiction)!

Now let $x_1 \in p_1$ and $x_2, \dots, x_n \notin p_1$ and p_1 is prime therefore

$$x_2 \dots x_n \notin p_1$$

hence $z = x_1 + (x_2 \dots x_n) \notin p_1$. Now for p_2 , we have that

$$x_1 \notin p_2, x_2 \in p_2 \implies x_1 + x_2x_3 \dots x_n \notin p_2$$

Similarly for the other p 's.

□

2 Modules

Lemma 2.1 (Nakayama's lemma). *Let A be a local ring, with maximal ideal m ; and Let M be a finitely generated A -module. Suppose that*

$$mM = M$$

(where mM is the set of all finite sums $m_1y_1 + \dots + m_r y_r$ with $m_i \in m$ and $y_i \in M$.)

Then $M = 0$.

Cheap proof. Suppose $M = \langle x_1, \dots, x_r \rangle$, we can remove redundant x_i 's, therefore we can assume that

$$\forall j, x_j \in \langle x_1, \dots, \hat{j}, \dots, x_r \rangle$$

If $M \neq 0$ we are left with this set of generators with $r \geq 1$. But then $x_1 \in M = mM$ therefore

$$x_1 = \sum_{i=1}^r m_i x_i \quad m_i \in m$$

Therefore $(1 - m_1)x_1 = m_2 z_2 + \dots + m_r x_r$ but $1 - m_1 \notin m$ must be a unit in A ! Therefore

$$x_1 \in \langle x_2, \dots, x_r \rangle \quad \text{impossible}$$

□

Corollary 2.2. Fix (A, m) a local ring. Suppose that $M \supseteq N$, M finitely generated and $M = N + mM$ (Equivalently in M/mM the images of N generate everything). Then $M = N$.

Proof. Apply Nakayama to M/N ; note that

$$m \cdot (M/N) = M/N, \quad M/N \text{ is finitely generated}$$

□

Theorem 2.3 (General Statement of Nakayama's Lemma). *Let A be any (commutative!) ring and let M be a finitely generated A -module, let $a \subseteq A$ be an ideal (usually $a = \text{Jrad } A$) and suppose*

$$aM = M$$

then $\exists b \in 1 + a$ s.t $bM = 0$.

In the case where a is the $\text{Jrad } A$ in which case $b \in 1 + a$ is a unit, so $M = 0$.

Proof. Let $M = \langle x_1, \dots, x_r \rangle$, since $M = aM$, we see that $\exists a_{ij} \in a$ s.t

$$x_1 = a_{11}x_1 + \dots + a_{1r}x_r \in aM$$

$$x_2 = a_{21}x_1 + \dots + a_{2r}x_r \in aM$$

⋮

$$x_r = a_{r1}x_1 + \dots + a_{rr}x_r \in aM$$

Let $P = (a_{ij})_{ij}$ be the matrix of a_{ij} 's, rewrite the system above as

$$\begin{aligned}(1 - a_{11})x_1 - a_{12}x_2 + \cdots - a_{1r}x_r &= 0 \\ a_{21}x_1 + (1 - a_{22})x_2 + \cdots - a_{2r}x_r &= 0 \\ &\vdots\end{aligned}$$

Symbolically we get that

$$(I - P) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Multiplying on the left by the matrix $Q = (I - P)^{\text{adj}} \in M_r(A)$; it has the property that

$$Q(I - P) = \text{diag}(b)$$

where $b = \det(I - P) \in 1 + a$; we deduce

$$bx_1 = 0, bx_2 = 0, \dots, bx_r = 0$$

so b annihilates every generator of M , and therefore annihilates all of M . \square

Example 2.4. We can use this so that if (A, m) then

$$m \supseteq m^2 \supseteq m^3 \supseteq \dots$$

if $m = m^{n+1}$ then $m^n = 0$. Indeed m_i/m_{i+1} is an A/m -module.

2.1 Hom functors and exactness

Quick reminder. M, N are A Modules, therefore $\text{Hom}_A(M, N)$ is an A -module (since A is commutative). Some facts:

- We have a natural isomorphism $\text{Hom}(A, N) \cong N$ (as A -modules).
- Have an natural isomorphism

$$\begin{aligned}\text{Hom}\left(\bigoplus_{\alpha \in I} M_\alpha, N\right) &\cong \prod_{\alpha \in I} \text{Hom}(M_\alpha, N) \\ (\cdots x_\alpha \cdots)_{\text{almost all zeroes}} &\xrightarrow{\varphi} \sum_{\alpha} \varphi_{\alpha}(x_{\alpha})\end{aligned}$$

But also

$$\text{Hom}(M, \prod_{\beta} N_{\beta}) \cong \prod_{\beta} \text{Hom}(M, N_{\beta})$$

$$x \in M \rightarrow (\cdots \psi_{\beta}(x) \cdots) \in \prod_{\beta} N_{\beta}$$

- $0 \rightarrow N' \xrightarrow{f} N \xrightarrow{g} N''$ exact implies that

$$0 \rightarrow \text{Hom}(M, N') \xrightarrow{f_*} \text{Hom}(M, N) \xrightarrow{g_*} \text{Hom}(M, N'')$$

(even if g is surjective, then g_* need not be surjective).

Similarly if

$$M' \xrightarrow{u} M \xrightarrow{w} M'' \rightarrow 0 \quad \text{exact}$$

then

$$0 \rightarrow \text{Hom}(M'', N) \xrightarrow{w^*} \text{Hom}(M, N) \xrightarrow{u^*} \text{Hom}(M', N)$$

(even if U is injective then U^* need not be surjective).

2.2 Presentations of Modules

Let A be a (commutative) ring and M an A -module.

Finite presentation. Say M is generated by x_1, \dots, x_n with relations r_1, \dots, r_k , where

$$r_j : \quad a_{j1}x_1 + \cdots + a_{jn}x_n = 0 \quad (1 \leq j \leq k).$$

Let A^n have basis e_1, \dots, e_n and A^k have basis f_1, \dots, f_k . Define $\eta : A^n \rightarrow M$ by $\eta(e_i) = x_i$ and define $\Phi : A^k \rightarrow A^n$ by

$$\Phi(f_j) = a_{j1}e_1 + \cdots + a_{jn}e_n.$$

Then $\eta \circ \Phi = 0$, and $\text{Im}(\Phi) = \ker(\eta)$ (the relations generate all relations), hence we have an exact sequence

$$A^k \xrightarrow{\Phi} A^n \xrightarrow{\eta} M \rightarrow 0, \quad \text{so } M \cong \text{coker}(\Phi) = A^n / \text{Im}(\Phi).$$

If we write Φ as a matrix, it is the $n \times k$ matrix (a_{ji}) with the convention that the j -th column is the coordinate vector of $\Phi(f_j)$ in the basis (e_i) .

Infinite presentations. Nothing changes: if M is generated by a family $(x_i)_{i \in I}$ with relations indexed by J , one has

$$A^{(J)} \xrightarrow{\Phi} A^{(I)} \xrightarrow{\eta} M \rightarrow 0,$$

where $A^{(I)} = \bigoplus_{i \in I} A$ and $A^{(J)} = \bigoplus_{j \in J} A$ are free modules on those bases.

Computing $\text{Hom}_A(M, N)$ from a presentation

Given an exact sequence

$$A^k \xrightarrow{\Phi} A^n \xrightarrow{\eta} M \rightarrow 0,$$

apply the contravariant left-exact functor $\text{Hom}_A(-, N)$ to obtain the exact sequence

$$0 \rightarrow \text{Hom}_A(M, N) \xrightarrow{\eta^*} \text{Hom}_A(A^n, N) \xrightarrow{\Phi^*} \text{Hom}_A(A^k, N),$$

where $\eta^*(\psi) = \psi \circ \eta$ and $\Phi^*(\varphi) = \varphi \circ \Phi$.

Identifications.

- If $n, k < \infty$ then $\text{Hom}_A(A^n, N) \cong N^n$ and $\text{Hom}_A(A^k, N) \cong N^k$ via

$$\text{Hom}_A(A^n, N) \ni \psi \longleftrightarrow (\psi(e_1), \dots, \psi(e_n)) \in N^n.$$

- More generally, for a free module $A^{(I)} = \bigoplus_{i \in I} A$, one has

$$\text{Hom}_A\left(\bigoplus_{i \in I} A, N\right) \cong \prod_{i \in I} \text{Hom}_A(A, N) \cong \prod_{i \in I} N.$$

(Important: *direct sum* on the left becomes a *direct product* on the right.)

Under the finite identifications, the map $\Phi^* : N^n \rightarrow N^k$ is given explicitly by

$$(y_1, \dots, y_n) \mapsto (z_1, \dots, z_k), \quad z_j = a_{j1}y_1 + \dots + a_{jn}y_n.$$

Therefore

$$\text{Hom}_A(M, N) \cong \ker(N^n \xrightarrow{\Phi^*} N^k).$$

Equivalently: an A -linear map $M \rightarrow N$ is determined by the images of the generators x_i , and the only constraints are that the relations r_j hold after applying the map.

2.3 Tensor Product of Modules (review)

Assume A is commutative (so that left/right issues disappear).

Bilinear maps. For A -modules M, N, P , let $\text{Bil}_A(M \times N, P)$ be the set of maps $\beta : M \times N \rightarrow P$ which are A -linear in each variable:

$$\beta(x_1 + x_2, y) = \beta(x_1, y) + \beta(x_2, y), \quad \beta(ax, y) = a\beta(x, y),$$

and similarly in the second variable.

There is a natural bijection

$$\text{Bil}_A(M \times N, P) \cong_{\text{nat}} \text{Hom}_A(M, \text{Hom}_A(N, P)),$$

sending β to the map $x \mapsto (y \mapsto \beta(x, y))$.

Universal property of the tensor product. The tensor product $M \otimes_A N$ is an A -module equipped with a bilinear map

$$\tau : M \times N \rightarrow M \otimes_A N, \quad (x, y) \mapsto x \otimes y,$$

such that for every A -module P , composition with τ induces a natural isomorphism

$$\text{Hom}_A(M \otimes_A N, P) \cong_{\text{nat}} \text{Bil}_A(M \times N, P).$$

Concrete construction. One can construct $M \otimes_A N$ as the quotient of the free A -module F on symbols (x, y) , by the submodule generated by the bilinearity relations:

$$(x_1 + x_2, y) - (x_1, y) - (x_2, y), \quad (ax, y) - a(x, y), \\ (x, y_1 + y_2) - (x, y_1) - (x, y_2), \quad (x, ay) - a(x, y).$$

The class of (x, y) is denoted $x \otimes y$.

Standard isomorphisms. There are natural isomorphisms

$$A \otimes_A N \cong N, \quad \left(\bigoplus_{\alpha} M_{\alpha} \right) \otimes_A N \cong \bigoplus_{\alpha} (M_{\alpha} \otimes_A N), \quad M \otimes_A N \cong N \otimes_A M,$$

and associativity up to canonical isomorphism:

$$(M \otimes_A N) \otimes_A Z \cong M \otimes_A (N \otimes_A Z).$$

Functionality of \otimes

Given A -linear maps $f : M_1 \rightarrow M_2$ and $g : N_1 \rightarrow N_2$, there is a unique A -linear map

$$f \otimes g : M_1 \otimes_A N_1 \rightarrow M_2 \otimes_A N_2$$

satisfying on simple tensors

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y).$$

Hence for a general tensor $t = \sum_i a_i (x_i \otimes y_i)$,

$$(f \otimes g)(t) = \sum_i a_i (f(x_i) \otimes g(y_i)).$$

Extension of scalars

Let $f : A \rightarrow B$ be a ring homomorphism and let M be an A -module. Then $B \otimes_A M$ is naturally a B -module via multiplication on the left factor:

$$b \cdot (b' \otimes m) := (bb') \otimes m.$$

(Equivalently: the action comes from $\mu_b \otimes 1_M$, where $\mu_b : B \rightarrow B$ is multiplication by b .)

Example 2.5 (Polynomials). If $B = A[x]$, then as an A -module,

$$A[x] \cong \bigoplus_{d \geq 0} A \cdot x^d,$$

so

$$A[x] \otimes_A M \cong \bigoplus_{d \geq 0} (A \cdot x^d) \otimes_A M \cong \bigoplus_{d \geq 0} M.$$

Under this identification, $x^d \otimes m$ corresponds to $m x^d$, and one gets

$$A[x] \otimes_A M \cong M[x]$$

(the module of polynomials with coefficients in M).

If $\mathfrak{a} \subseteq A$ is an ideal (viewed as an A -module), then

$$A[x] \otimes_A \mathfrak{a} \cong \mathfrak{a}[x].$$

Inside $A[x]$, this is the extended ideal $\mathfrak{a}^e := \mathfrak{a} \cdot A[x]$.

Tensoring algebras. If A is commutative and B, C are (commutative) A -algebras, then $B \otimes_A C$ is an A -algebra with multiplication determined by

$$(b \otimes c) \cdot (b' \otimes c') := (bb') \otimes (cc').$$

Example 2.6. There is a natural isomorphism of A -algebras

$$A[x] \otimes_A B \cong B[x].$$

2.4 Exactness properties of $- \otimes_A N$

Right exactness. The functor $- \otimes_A N$ is *right exact*: if

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$$

is exact, then

$$M' \otimes_A N \xrightarrow{f \otimes 1} M \otimes_A N \xrightarrow{g \otimes 1} M'' \otimes_A N \rightarrow 0$$

is exact.

Warning: it need not be left exact (injectivity can fail).

Example: quotients / extended ideals. From $0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A/\mathfrak{a} \rightarrow 0$ we get

$$\mathfrak{a} \otimes_A B \rightarrow A \otimes_A B \rightarrow (A/\mathfrak{a}) \otimes_A B \rightarrow 0.$$

Using $A \otimes_A B \cong B$, the last map identifies $(A/\mathfrak{a}) \otimes_A B$ with

$$B/\mathfrak{a}^e, \quad \mathfrak{a}^e = \mathfrak{a} \cdot B.$$

Computing $M \otimes_A N$ from a presentation

If M has a presentation

$$A^k \xrightarrow{\Phi} A^n \rightarrow M \rightarrow 0,$$

then tensoring with N gives an exact sequence

$$N^k \cong A^k \otimes_A N \xrightarrow{\Phi \otimes 1} A^n \otimes_A N \cong N^n \rightarrow M \otimes_A N \rightarrow 0.$$

Hence

$$M \otimes_A N \cong \text{coker}(\Phi \otimes 1),$$

and under the identifications $A^k \otimes N \cong N^k$, $A^n \otimes N \cong N^n$, the map $\Phi \otimes 1 : N^k \rightarrow N^n$ is given by the *same matrix* (a_{ji}) :

$$(u_1, \dots, u_k) \mapsto \left(\sum_{j=1}^k a_{j1} u_j, \dots, \sum_{j=1}^k a_{jn} u_j \right).$$

Equivalently: $M \otimes_A N$ is generated by $x_i \otimes y$ ($i = 1, \dots, n$, $y \in N$), with relations coming from:

- bilinearity in y (so $x_i \otimes (y + y') = x_i \otimes y + x_i \otimes y'$ and $x_i \otimes (ay) = a(x_i \otimes y)$),
- and the original relations of M : if $\sum_i c_i x_i = 0$ in M , then for every $y \in N$,

$$\sum_i c_i (x_i \otimes y) = 0 \quad \text{in } M \otimes_A N.$$

Conceptual proof of right exactness (via adjunction)

Conceptual proof of right exactness (via adjunction)

Recall the adjunction: for any A -modules X, N, P there is a natural isomorphism

$$\text{Hom}_A(X \otimes_A N, P) \cong \text{Hom}_A(X, \text{Hom}_A(N, P)),$$

equivalently

$$\text{Hom}_A(X \otimes_A N, P) \cong \text{Bil}_A(X \times N, P).$$

Lemma 2.7. Let $Q \xrightarrow{\lambda} R \xrightarrow{\mu} S \rightarrow 0$ be a sequence of A -modules. Assume that for every A -module P the sequence

$$0 \rightarrow \text{Hom}_A(S, P) \xrightarrow{\mu^*} \text{Hom}_A(R, P) \xrightarrow{\lambda^*} \text{Hom}_A(Q, P)$$

is exact. Then $Q \xrightarrow{\lambda} R \xrightarrow{\mu} S \rightarrow 0$ is exact.

Sketch. Surjectivity of μ follows by taking $P = S / \text{Im}(\mu)$ and considering the natural quotient map. To show $\ker(\mu) \subseteq \text{Im}(\lambda)$, take $P = \text{coker}(\lambda) = R / \text{Im}(\lambda)$ and let $\varphi : R \rightarrow \text{coker}(\lambda)$ be the quotient map. Then $\varphi \circ \lambda = 0$, so $\varphi \in \ker(\lambda^*) = \text{Im}(\mu^*)$, hence $\varphi = \psi \circ \mu$ for some $\psi : S \rightarrow \text{coker}(\lambda)$. If $r \in \ker(\mu)$ then $\varphi(r) = \psi(\mu(r)) = 0$, so $r \in \ker(\varphi) = \text{Im}(\lambda)$. \square

Now suppose $M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is exact. Fix an A -module P and apply the left exact functor $\text{Hom}_A(-, \text{Hom}_A(N, P))$ to obtain an exact sequence

$$0 \rightarrow \text{Hom}_A(M'', \text{Hom}_A(N, P)) \rightarrow \text{Hom}_A(M, \text{Hom}_A(N, P)) \rightarrow \text{Hom}_A(M', \text{Hom}_A(N, P)).$$

Using the adjunction isomorphisms, this becomes

$$0 \rightarrow \text{Hom}_A(M'' \otimes_A N, P) \rightarrow \text{Hom}_A(M \otimes_A N, P) \rightarrow \text{Hom}_A(M' \otimes_A N, P).$$

Since this holds for all P , the lemma implies that

$$M' \otimes_A N \xrightarrow{f \otimes 1} M \otimes_A N \xrightarrow{g \otimes 1} M'' \otimes_A N \rightarrow 0$$

is exact, i.e. $- \otimes_A N$ is right exact.

3 Localization of rings and modules

Intuition for example. Let A be a ring (pretend $A = k[x_1, \dots, x_n]/\mathfrak{a}$ = polynomial functions on $V(\mathfrak{a}) \subseteq k^n$). So $f \in A$ is basically $f: V(\mathfrak{a}) \rightarrow k$, let $V(f) = V(\langle f \rangle + \mathfrak{a})$ be the set where $f = 0$. We can look at

$$V(\mathfrak{a}) - V(f).$$

What are polynomial functions on this? Intuitively they look like fractions

$$a/f^n, \quad a \in A, n \geq 0,$$

i.e. we allow denominators which are powers of f (so we “ignore” what happens on $V(f)$).

We try to define

$$A_f = \left\{ \frac{a}{f^n} : a \in A, n \geq 0 \right\}.$$

Question: when is $\frac{a}{f^n} = \frac{b}{f^m}$? Note we can rewrite as

$$\frac{af^{m+k}}{f^{n+m+k}} = \frac{bf^{n+k}}{f^{n+m+k}}.$$

Therefore $\frac{a}{f^n} = \frac{b}{f^m}$ iff $\exists k$ s.t

$$f^k(af^m) = f^k(bf^n),$$

i.e.

$$f^k(af^m - bf^n) = 0.$$

Remark 3.1. In our hw look at 2.1, this describes exactly $A[x]/\langle xf - 1 \rangle$.

Definition 3.2. Let A be a ring, let $S \subseteq A$ be a multiplicatively closed set, define a relation \sim on $A \times S$ as follows:

$$(a, s) \sim (b, t) \iff \exists u \in S \text{ s.t } u(ta - sb) = 0.$$

Example 3.3. if $f \in A$, the set

$$\{1, f, f^2, \dots\}$$

is multiplicatively closed.

If $\mathfrak{p} \subset A$ is prime, then $A - \mathfrak{p}$ is multiplicatively closed.

Proposition 3.4. \sim is an equivalence relation.

Proof. Exercise. □

Definition 3.5. We define $S^{-1}A$ (also written as A_S) to be the set of equivalence classes of \sim . So an element of $S^{-1}A$ is a symbol

$$\frac{a}{s}.$$

Note that

$$\frac{a}{s} = \frac{b}{t} \iff \exists u \in S \text{ s.t } u(ta - sb) = 0.$$

Proposition 3.6. The operations

$$\begin{aligned}\frac{a}{s} + \frac{b}{t} &= \frac{ta + sb}{st} \\ \frac{a}{s} \cdot \frac{b}{t} &= \frac{ab}{st}\end{aligned}$$

are well defined, and they turn $S^{-1}A$ into a ring with

$$0 = \frac{0}{1}, 1 = \frac{1}{1}.$$

Notation. If $S = \{1, f, f^2, \dots\}$ then we write A_f instead of $S^{-1}A$. If $S = A - \mathfrak{p}$ we write $A_{\mathfrak{p}}$.

Example 3.7. In \mathbb{Z} we write

$$\mathbb{Z}_f = \left\{ \frac{a}{f^n} \in \mathbb{Q} \mid a \in \mathbb{Z}, n \geq 0 \right\}.$$

And

$$\mathbb{Z}_{\langle p \rangle} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}, p \nmid b \right\}.$$

Remark 3.8. There is a ring homomorphism $\varphi: A \rightarrow S^{-1}A$ by $\varphi(a) = a/1$. But note that φ is not necessarily injective.

For example take $(\mathbb{Z}/6\mathbb{Z})$ localized at $S = \{1, 3\}$. Then

$$S^{-1}(\mathbb{Z}/6\mathbb{Z}) = \{0/1, 1/1\} \cong \mathbb{Z}/2\mathbb{Z}.$$

This is because 3 becomes a unit, so the $\mathbb{Z}/3\mathbb{Z}$ -part disappears. Concretely:

$$2/1 = \frac{3 \cdot 2}{3} = 0,$$

hence $\{0, 2, 4\} \subseteq \ker \varphi$. Also

$$\frac{1}{3} = \frac{3 \cdot 1}{3 \cdot 3} = \frac{3}{3} = 1.$$

Using CRT: $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Inverting 3 makes $(1, 0)$ invertible, which forces the second factor to be 0, leaving $\mathbb{Z}/2\mathbb{Z}$.

Example 3.9. Let $A = \mathbb{C}[x, y]/\langle xy \rangle$, these are functions on the union of the two coordinate axes in \mathbb{C}^2 . Localize at x :

$$A_x \cong \mathbb{C}[x, x^{-1}].$$

Therefore

$$\frac{y}{1} = \frac{xy}{x} = 0,$$

so $y \in \ker \varphi$ (intuitively: on $D(x)$ we are away from the y -axis, so y vanishes there).

We also have the UMP. Let $\iota : A \rightarrow S^{-1}A$ be $\iota(a) = a/1$. Then for any ring B and any ring map $g : A \rightarrow B$ such that $g(S) \subseteq B^\times$, there exists a unique ring map $\tilde{g} : S^{-1}A \rightarrow B$ with $\tilde{g} \circ \iota = g$. Concretely

$$\tilde{g}(a/s) = g(a) g(s)^{-1}.$$

The diagram is:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & S^{-1}A \\ & \searrow g & \downarrow \tilde{g} \\ & & B \end{array}$$

Remark 3.10. $0 \in S \iff S^{-1}A$ is the zero ring.

Soon we will also localize modules; we will get a module over $S^{-1}A$. Define

$$S^{-1}M = \left\{ \frac{x}{s} \mid x \in M, s \in S \right\}$$

with

$$\frac{x}{s} = \frac{y}{t} \iff \exists u \in S \text{ s.t } u(tx - sy) = 0.$$

3.1 Ideals in the Localization

We will especially look at extensions and contractions of ideals for the homomorphism

$$\varphi : A \rightarrow S^{-1}A.$$

Proposition 3.11. If $\mathfrak{b} \subseteq S^{-1}A$ is an ideal, then

$$\mathfrak{b}^{ce} = \mathfrak{b}.$$

Consequently all ideals of a localization are extensions of something.

Proof. $\mathfrak{b} \supseteq \mathfrak{b}^{ce}$ (always).

Conversely, let $\frac{a}{s} \in \mathfrak{b}$. Then $s(\frac{a}{s}) = \frac{a}{1}$ so $a \in \mathfrak{b}^c$, hence $\frac{a}{1} \in \mathfrak{b}^{ce}$ and so

$$\frac{1}{s} \cdot \frac{a}{1} = \frac{a}{s} \in \mathfrak{b}^{ce}.$$

□

Proposition 3.12. Let $\mathfrak{a} \subseteq A$ be an ideal. Then

$$\mathfrak{a}^e = \left\{ \frac{a}{s} \mid a \in \mathfrak{a}, s \in S \right\}$$

(i.e. $S^{-1}\mathfrak{a}$ inside $S^{-1}A$), and

$$\mathfrak{a}^{ec} = \{x \in A \mid \exists s \in S \text{ s.t } sx \in \mathfrak{a}\}.$$

Example 3.13. Think of $A = \mathbb{Z}$ and $S = \{1, 6, 6^2, \dots\}$ therefore

$$S^{-1}A = \mathbb{Z}[1/6].$$

Then $(99\mathbb{Z})^{ec} = 11\mathbb{Z} = (11\mathbb{Z})^{ec}$. Indeed $x \in (99\mathbb{Z})^{ec}$ iff $\exists n$ such that $6^n x \in 99\mathbb{Z}$. For $n \geq 2$, 6^n contributes a factor 3^n , so the 3^2 part is automatic, and the condition becomes exactly $11 \mid x$.

Notice that elements of \mathfrak{a}^e are of the form

$$\varphi(a_1) \cdot \left(\frac{b_1}{s_1}\right) + \dots + \varphi(a_k) \cdot \left(\frac{b_k}{s_k}\right)$$

with $a_i \in \mathfrak{a}$, $b_i \in A$, $s_i \in S$. Take a common denominator

$$s = s_1 \cdots s_k,$$

so $\frac{b_i}{s_i} = \frac{c_i}{s}$ with $c_i = b_i \prod_{j \neq i} s_j$, hence wlog we have

$$\frac{a_1}{1} \cdot \frac{c_1}{s} + \dots + \frac{a_k}{1} \cdot \frac{c_k}{s}.$$

Corollary 3.14. If $\mathfrak{a} = \langle x_1, \dots, x_r \rangle \subset A$ then

$$\mathfrak{a}^e = \left\langle \frac{x_1}{1}, \dots, \frac{x_r}{1} \right\rangle.$$

Corollary 3.15. If A is Noetherian, then so is $S^{-1}A$ (and every ideal of $S^{-1}A$) by the above corollary.

Prime ideals in $S^{-1}A$

Theorem 3.16. 1. If $\mathfrak{q} \subset S^{-1}A$ is prime, then $\mathfrak{p} := \mathfrak{q}^c \subset A$ is prime and

$$\mathfrak{p} \cap S = \emptyset.$$

2. If $\mathfrak{p} \subset A$ is prime and

$$\mathfrak{p} \cap S = \emptyset,$$

then $\mathfrak{p}^e \subset S^{-1}A$ is also prime.

3. Extension and contraction are mutually inverse bijections between the sets

$$\{\mathfrak{p} \subset A \text{ prime} \mid \mathfrak{p} \cap S = \emptyset\} \leftrightarrows \{\mathfrak{q} \subset S^{-1}A \text{ prime}\}.$$

Remark 3.17. This clarifies: if $\mathfrak{a} \subset A$ and $\mathfrak{a} \cap S = \emptyset$, then $\mathfrak{a}^e \subsetneq S^{-1}A$. Hence \mathfrak{a}^e is contained in some maximal ideal $\mathfrak{m} \subset S^{-1}A$. Contracting gives a prime $\mathfrak{p} = \mathfrak{m}^c \subset A$ with $\mathfrak{a} \subseteq \mathfrak{p}$ and $\mathfrak{p} \cap S = \emptyset$.

Proof. 1. We are given \mathfrak{q} is prime, then \mathfrak{q}^c is prime (contraction preserves primality). Also if $\exists s \in S \cap \mathfrak{q}^c$ then $s/1 \in \mathfrak{q}$, but $s/1$ is a unit in $S^{-1}A$ (inverse $1/s$), contradiction since $\mathfrak{q} \subsetneq S^{-1}A$.

2. Suppose we have two elements of $S^{-1}A$ whose product is in \mathfrak{p}^e . Call these elements a/s and b/t with $a, b \in A$ and $s, t \in S$. Then

$$\frac{ab}{st} \in \mathfrak{p}^e.$$

So $\exists u \in S$ such that $uab \in \mathfrak{p}$ (clearing denominators). Since $u \in S$ and $\mathfrak{p} \cap S = \emptyset$, we have $u \notin \mathfrak{p}$; by primality,

$$a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}.$$

Hence a/s or b/t lies in \mathfrak{p}^e .

3. For any ideal $\mathfrak{q} \subset S^{-1}A$, we already proved $\mathfrak{q}^{ce} = \mathfrak{q}$. Now let $\mathfrak{p} \subset A$ be prime with $\mathfrak{p} \cap S = \emptyset$. Clearly $\mathfrak{p}^{ec} \supseteq \mathfrak{p}$. Take $a \in \mathfrak{p}^{ec}$, so $a/1 \in \mathfrak{p}^e$. Then $\exists s \in S$ such that $sa \in \mathfrak{p}$. But $s \notin \mathfrak{p}$, so $a \in \mathfrak{p}$ by primality. Thus $\mathfrak{p}^{ec} = \mathfrak{p}$.

□

Proposition 3.18. *There is a natural isomorphism of $S^{-1}A$ -modules*

$$S^{-1}M \cong_{\text{nat}} (S^{-1}A) \otimes_A M, \quad \frac{ax}{s} \leftrightarrows \frac{a}{s} \otimes x.$$

Proof. Define a map

$$\Phi : (S^{-1}A) \otimes_A M \longrightarrow S^{-1}M, \quad \Phi\left(\frac{a}{s} \otimes x\right) = \frac{ax}{s}.$$

First observe that this map exists and is an A -module homomorphism (in fact, it is $S^{-1}A$ -linear once it is well-defined). Indeed, it is induced by the bilinear map

$$\beta : S^{-1}A \times M \rightarrow S^{-1}M, \quad \beta\left(\frac{a}{s}, x\right) = \frac{ax}{s}.$$

We check that β is well-defined. If

$$\frac{a}{s} = \frac{a'}{s'},$$

then by definition there exists $u \in S$ such that

$$u(s'a - sa') = 0.$$

Multiplying by $x \in M$ gives

$$u(s'ax - sa'x) = 0,$$

hence in $S^{-1}M$ we have

$$\frac{ax}{s} = \frac{a'x}{s'}.$$

Therefore β is well-defined, so it induces the A -linear map Φ .

Surjectivity is trivial: every element of $S^{-1}M$ has the form x/s , and

$$\Phi\left(\frac{1}{s} \otimes x\right) = \frac{x}{s}.$$

Reduction to simple tensors. A general element of $(S^{-1}A) \otimes_A M$ is a finite sum

$$t = \frac{a_1}{s_1} \otimes x_1 + \cdots + \frac{a_n}{s_n} \otimes x_n.$$

Passing to a common denominator $s = s_1 \cdots s_n$, we may rewrite

$$t = \frac{b_1}{s} \otimes x_1 + \cdots + \frac{b_n}{s} \otimes x_n = \frac{1}{s} \otimes \left(\sum_i b_i x_i \right).$$

Thus every tensor may be represented in the form $\frac{1}{s} \otimes y$.

Alternate approach (constructing the inverse). Define

$$\Psi : S^{-1}M \rightarrow (S^{-1}A) \otimes_A M, \quad \Psi\left(\frac{x}{s}\right) = \frac{1}{s} \otimes x.$$

We check that Ψ is well-defined. Suppose

$$\frac{x}{s} = \frac{x'}{s'}.$$

Then there exists $u \in S$ such that

$$u(s'x - sx') = 0,$$

i.e.

$$us'x = usx'.$$

Then in the tensor product,

$$\frac{1}{s} \otimes x = \frac{u}{us} \otimes x = \frac{1}{us} \otimes ux = \frac{1}{us'} \otimes ux' = \frac{1}{s'} \otimes x'.$$

Hence Ψ is well-defined.

Finally,

$$(\Phi \circ \Psi)\left(\frac{x}{s}\right) = \Phi\left(\frac{1}{s} \otimes x\right) = \frac{x}{s}$$

and

$$(\Psi \circ \Phi)\left(\frac{a}{s} \otimes x\right) = \Psi\left(\frac{ax}{s}\right) = \frac{1}{s} \otimes ax = \frac{a}{s} \otimes x.$$

Therefore Φ is an isomorphism, natural in M . \square

Remark 3.19. Why is it called *localization*? Recall that there is an order-preserving bijection between primes:

$$\left\{ \mathfrak{p} \subseteq A \mid \mathfrak{p} \cap S = \emptyset \right\} \longleftrightarrow \left\{ \mathfrak{P} \subseteq S^{-1}A \right\}, \quad \mathfrak{p} \mapsto S^{-1}\mathfrak{p}.$$

In particular, if $S = A - \mathfrak{p}$, then

$$A_{\mathfrak{p}} = S^{-1}A$$

is a *local ring*. Indeed, its unique maximal ideal is

$$\mathfrak{m}_{\mathfrak{p}} = \left\{ \frac{a}{s} \in A_{\mathfrak{p}} \mid a \in \mathfrak{p}, s \notin \mathfrak{p} \right\}.$$

Moreover,

$$A_{\mathfrak{p}} - \mathfrak{m}_{\mathfrak{p}} = \left\{ \frac{u}{s} \mid u \notin \mathfrak{p}, s \notin \mathfrak{p} \right\}$$

is exactly the set of units of $A_{\mathfrak{p}}$.

A quick example: let $A = k[x, y, z]$ and $\mathfrak{p} = \langle x, y \rangle$. Then

$$V(\mathfrak{p}) = \text{"the } z\text{-axis".}$$

Also $A/\mathfrak{p} \simeq k[z]$, via

$$f(x, y, z) + \mathfrak{p} \longleftrightarrow f(0, 0, z).$$

Thus

$$A_{\mathfrak{p}} = \left\{ \frac{f(x, y, z)}{s(x, y, z)} \mid s \notin \mathfrak{p} \right\}.$$

The maximal ideal consists of fractions whose numerator vanishes on the z -axis:

$$\mathfrak{m}_{\mathfrak{p}} = \left\{ \frac{f(x, y, z)}{s(x, y, z)} \mid f \in \mathfrak{p} \right\}.$$

Moreover one checks that

$$\frac{A_{\mathfrak{p}}}{\mathfrak{m}_{\mathfrak{p}}} \cong \text{Frac}(A/\mathfrak{p}) \cong k(z),$$

the field of rational functions in z .

Therefore we can define the *localization functor*

$$M \longmapsto S^{-1}M,$$

which is secretly the functor

$$M \longmapsto (S^{-1}A) \otimes_A M.$$

Given a homomorphism $M \xrightarrow{f} N$, we define

$$S^{-1}f : S^{-1}M \rightarrow S^{-1}N, \quad (S^{-1}f)\left(\frac{x}{s}\right) = \frac{f(x)}{s}.$$

Theorem 3.20. *The localization functor is exact. Equivalently, $S^{-1}A$ is a flat A -algebra.*

Proof. Let

$$M \xrightarrow{f} N \xrightarrow{g} P$$

be exact at N , i.e. $\mathfrak{I}(f) = \ker(g)$.

We claim that

$$S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}P$$

is exact at $S^{-1}N$.

Trivially $\Im(S^{-1}f) \subseteq \ker(S^{-1}g)$ since $gf = 0$.

Now let $y/s \in \ker(S^{-1}g)$. Then

$$(S^{-1}g)\left(\frac{y}{s}\right) = \frac{g(y)}{s} = 0$$

in $S^{-1}P$, hence there exists $u \in S$ such that

$$ug(y) = 0, \quad \text{i.e.} \quad g(uy) = 0.$$

Thus $uy \in \ker(g) = \Im(f)$, so $uy = f(x)$ for some $x \in M$. Therefore

$$\frac{y}{s} = \frac{uy}{us} = \frac{f(x)}{us} = (S^{-1}f)\left(\frac{x}{us}\right),$$

proving $\ker(S^{-1}g) \subseteq \Im(S^{-1}f)$. \square

Immediate consequences of exactness.

Let $M \xrightarrow{f} N$ be any homomorphism. Then localization preserves kernels, cokernels, and images:

$$S^{-1}(\ker f) = \ker(S^{-1}f), \quad S^{-1}(\operatorname{coker} f) = \operatorname{coker}(S^{-1}f), \quad S^{-1}(\Im f) = \Im(S^{-1}f).$$

If $N \subseteq M$, then

$$S^{-1}(M/N) \cong (S^{-1}M)/(S^{-1}N),$$

since localizing the short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

remains exact.

Similarly,

$$S^{-1}(M_1 \oplus M_2) \cong S^{-1}M_1 \oplus S^{-1}M_2.$$

The direct sum maps fit into the standard diagram:

$$\begin{array}{ccccc} M_1 & \xrightarrow{i_1} & M_1 \oplus M_2 & \xleftarrow{\pi_2} & M_2 \\ & \swarrow p_1 & & & \end{array}$$

with relations

$$p_1 i_1 = 1, \quad p_2 i_2 = 1, \quad p_1 i_2 = 0, \quad p_2 i_1 = 0, \quad i_1 p_1 + i_2 p_2 = 1_{M_1 \oplus M_2}.$$

More interestingly, if $N_1, N_2 \subseteq M$, then

$$S^{-1}(N_1 \cap N_2) = S^{-1}N_1 \cap S^{-1}N_2, \quad S^{-1}(N_1 + N_2) = S^{-1}N_1 + S^{-1}N_2.$$

Indeed we use the exact sequence

$$0 \rightarrow N_1 \cap N_2 \rightarrow N_1 \oplus N_2 \xrightarrow{(y_1, y_2) \mapsto y_1 - y_2} N_1 + N_2 \rightarrow 0,$$

where $z \mapsto (z, z)$ is the inclusion.

Proposition 3.21. *Let M, N be A -modules. Then $\text{Hom}_A(M, N)$ is an A -module, and there is a natural map*

$$S^{-1}\text{Hom}_A(M, N) \xrightarrow{\varphi} \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N),$$

given by

$$\frac{f}{s} \longmapsto \left(\frac{x}{t} \mapsto \frac{f(x)}{st} \right).$$

Moreover:

- If M is finitely generated, then φ is injective.
- If M is finitely presented, then φ is bijective.

(Sketch: finite generation controls kernels, finite presentation controls surjectivity.)

Local vs global properties

We know: for $x \in M$, when is $x/1 = 0$ in $S^{-1}M$? Precisely when

$$\exists s \in S \text{ such that } sx = 0, \quad \text{i.e.} \quad (x) \cap S \neq \emptyset.$$

Fix $x \in M$. Which primes \mathfrak{p} satisfy $x/1 \neq 0$ in $M_{\mathfrak{p}}$? These are exactly

$$\{\mathfrak{p} \mid (x) \subseteq \mathfrak{p}\}.$$

Example 3.22. Let $A = \mathbb{Z}$ and $M = \mathbb{Z}/20\mathbb{Z}$. Then

$$([1]) = 20\mathbb{Z} \subseteq 2\mathbb{Z}, 5\mathbb{Z}, \quad ([4]) = 5\mathbb{Z}.$$

Hence

$$M_{(2)} \cong \mathbb{Z}/4\mathbb{Z}, \quad M_{(5)} \cong \mathbb{Z}/5\mathbb{Z}.$$

Example 3.23. Let $A = \mathbb{C}[x, y]$ and maximal ideals

$$\mathfrak{m}_{a,b} = \langle x - a, y - b \rangle.$$

If

$$M = A/\langle x^2 + y^2 - 1 \rangle,$$

then $[1]$ lives at $\mathfrak{m}_{a,b}$ iff

$$x^2 + y^2 - 1 \in \mathfrak{m}_{a,b} \iff a^2 + b^2 = 1.$$

Example 3.24. Consider $[x] \in k[x, y]/\langle x, y \rangle$ as a $k[x, y]$ -module element. Then

$$([x]) = \langle y \rangle,$$

so $[x]$ lives exactly along the x -axis.

Local-global principles

Proposition 3.25. Let M be an A -module. Then the following are equivalent:

- $M = 0$.
- For all primes \mathfrak{p} , $M_{\mathfrak{p}} = 0$.
- For all maximal ideals \mathfrak{m} , $M_{\mathfrak{m}} = 0$.

Equivalently, for $x \in M$, the following are equivalent:

- $x = 0$.
- For all primes \mathfrak{p} , $x/1 = 0$ in $M_{\mathfrak{p}}$.
- For all maximal ideals \mathfrak{m} , $x/1 = 0$ in $M_{\mathfrak{m}}$.

Proof. (1) \implies (2) \implies (3) is trivial.

For (3) \implies (1): assume $M_{\mathfrak{m}} = 0$ for all maximal ideals. Let $x \in M$ with $x \neq 0$. Then (x) is a proper ideal, hence contained in some maximal ideal \mathfrak{m} :

$$(x) \subseteq \mathfrak{m}.$$

But $x/1 = 0$ in $M_{\mathfrak{m}}$ means there exists $s \notin \mathfrak{m}$ such that $sx = 0$. Thus $s \in (x) \subseteq \mathfrak{m}$, contradiction. Therefore $x = 0$, so $M = 0$. \square

Consequences.

Proposition 3.26. A homomorphism $f : M \rightarrow N$ is injective (resp. surjective, resp. bijective) iff for all primes \mathfrak{p} (equivalently, all maximals \mathfrak{m}),

$$f_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$$

is injective (resp. surjective, resp. bijective).

Proof. Apply localization to the exact sequences involving $\ker f$ and $\text{coker } f$. \square

Proposition 3.27. If $N \subseteq M$, then

$$N = M \iff N_{\mathfrak{p}} = M_{\mathfrak{p}} \forall \mathfrak{p} \iff N_{\mathfrak{m}} = M_{\mathfrak{m}} \forall \mathfrak{m}.$$

Proof. Apply the previous proposition to M/N . \square

Proposition 3.28. If $x \in M$ satisfies $x/1 = 0$ in $M_{\mathfrak{m}}$, then the image of x in

$$M/\mathfrak{m}M \cong (A/\mathfrak{m}) \otimes_A M$$

is zero.

Proof. If $x/1 = 0$ in $M_{\mathfrak{m}}$, then $\exists s \notin \mathfrak{m}$ with $sx = 0$. In $M/\mathfrak{m}M$, we have $s[x] = 0$, but also $\mathfrak{m}[x] = 0$. Since $s \notin \mathfrak{m}$, the ideal $\mathfrak{m} + \langle s \rangle = A$, hence $[x] = 0$. \square

Theorem 3.29. Suppose M is a finitely generated A -module. Then

$$M = 0 \iff \forall \mathfrak{m}, M/\mathfrak{m}M = 0.$$

Proof. Assume $M/\mathfrak{m}M = 0$ for all maximal ideals \mathfrak{m} .

Localizing at \mathfrak{m} , we get

$$M_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}M_{\mathfrak{m}} \cong (M/\mathfrak{m}M)_{\mathfrak{m}} = 0.$$

Since $M_{\mathfrak{m}}$ is finitely generated over the local ring $A_{\mathfrak{m}}$, Nakayama's lemma implies

$$M_{\mathfrak{m}} = 0 \quad \forall \mathfrak{m}.$$

By the local-global principle, $M = 0$. \square

Remark 3.30. Finite generation is essential. For example, if $A = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$, then multiplication by n is surjective for all n , hence

$$(\mathbb{Q}/\mathbb{Z})/p(\mathbb{Q}/\mathbb{Z}) = 0 \quad \forall p,$$

but $\mathbb{Q}/\mathbb{Z} \neq 0$.