Commutative Algebra

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Contents

1 Introduction. **1**
- 1.1 Multiplicative sets and Prime ideals . . . 1
- 1.2 The Jacobson radical of a commutative ring . . . 3
- 1.3 Extension and contraction of ideals . . . 3
- 1.4 Modules. . . . 4
- 1.5 Local Rings . . . 5
- 1.6 Algebras . . . 5

2 Modules **7**
- 2.1 Hom functors and exactness . . . 8
- 2.2 Presentations of Modules . . . 9
- 2.3 Tensor Product of Modules (review) . . . 10
- 2.4 Exactness properties of $-\otimes_A N$ . . . 12

3 Localization of rings and modules **14**
- 3.1 Ideals in the Localization . . . 16
- 3.2 The spectrum of a ring. . . . 24

1 Introduction.

1.1 Multiplicative sets and Prime ideals

Definition 1.1. A subset $S \subset A$ is a called multiplicatively closed iff

1. $1 \in S$
2. if $s, t \in S$, then $st \in S$.

Proposition 1.2. *Let S be multiplicatively closed, and let $\mathfrak{a}$ be an ideal with $\mathfrak{a} \cap S = \varnothing$. Consider all ideals $\mathfrak{b}$ s.t*

$$\begin{cases} \mathfrak{a} \subseteq \mathfrak{b} \\ \mathfrak{b} \cap S = \varnothing \end{cases}$$

Equivalently $a \subset b \subset A - S$. Then the poset of such ideals satisfies the conditions of Zorn's lemma, and any maximal such element is a prime ideal of A.

Proof. The poset is nonempty because it contains a, every totally ordered subset $\{b_i : i \in I\}$, has an upper bound $\bigcup_{i \in I} b_i$. Therefore by Zorn's lemma there exists some maximal element p of this set.

Suppose that $x, y \in A$ with $xy \in p$ but $x, y \notin p$: Since $x, y \notin b$ then the ideal sums

$$p + \langle x \rangle,\ p + \langle y \rangle$$

Strictly contain p, so they must contain each an element of S, therefore $\exists s \in S$ of the form

$$s = p_1 + xa_1 \quad p_1 \in p,\ a_1 \in R$$

and similarly $\exists t \in s$ with

$$t = p_2 + ya_2$$

but then $st \in S$, therefore

$$p_1p_2 + p_1ya_2 + p_2xa_1 + xya_1a_2 \in P \implies st \in p$$

contradicting that $p \cap S = \varnothing$.
Therefore p is prime. □

Example 1.3.
- A a domain and $S = A - \{0\}$.
- In $\mathbb{Z}$: $S = \{1, 6, 6^2, \cdots\}$, or more generally in any A if $x \in A$ is not nilpotent, then

  $$S = \left\{1, x, x^2, \cdots\right\}, \quad S \cap \{0\} = \varnothing$$

  There exists a prime ideal with $p \cap S = \varnothing \implies x \notin p$ (obviously $0 \in p$).
  Therefore

  $$\{\text{all nilpotent } x\} = \text{nilrad}(0) = \bigcap_{p \text{ prime ideal}} p$$

- $p \subset A$ prime then $A - p$ is multiplicatively closed.

Theorem 1.4. *If $a \subset A$ is an ideal, then*

$$\text{rad}\, a := \left\{x \in A \mid\ \exists N \geq 1 \text{ with } x^N \in a\right\}$$

satisfies

$$\text{rad}\, a = \bigcap_{\text{all } p,\ p \supseteq a} p$$

Proof.
- Slick proof: apply the conclusion above to the quotient ring A/a.
- Pedestrian proof: Show both inclusions directly : If $x^N \in a$ and $a \subset p$, then $x \in p$ (for all primes $p \supseteq a$), so $x \in \bigcap_{p \supseteq a} p$.
  If $\forall N$, $x^N \notin a$ then take $S = \{1, x, x^2, \cdots\}$ we have $a \cap S = \varnothing$; so $\exists p$ ($p \supseteq a$ and $p \cap S$)

which is prime with $p \supseteq a$ and $p \cap S = \varnothing$ so $\exists p \supseteq a$ with $x \notin p$ so

$$x \notin \bigcap_{p \supseteq a} p$$

$\square$

1.2 The Jacobson radical of a commutative ring

Definition 1.5.

$$\operatorname{Jrad} a = \bigcap_{\text{all maximal ideals } m \subsetneq A} \supseteq \operatorname{nilrad} A$$

Proposition 1.6. $x \in \operatorname{Jrad} A \iff \forall y \in A,\ 1 - xy$ *is a unit in* A.

Proof. $\implies$ Let $x \in \operatorname{Jrad} A$, then $\forall m$ maximal ideal, $x \in m$. If $y \in A$ satisfies $1 - xy$ not a unit, then

$$\langle 1 - xy \rangle \subsetneq A$$

so $\exists m$ maximal with $1 - xy \in m$ but $x \in m$. This would force $1 \in m$ since

$$1 - xy + xy$$

$\impliedby$ Suppose that $\forall y \in A$, $1 - xy$ is a unit in A. We want to show that if M is maximal, is $x \in m$? Work in A/m, field: $\overline{x}$ satisfies that $\forall \overline{y} \in A/m$,

$$\overline{1} - \overline{xy} \text{ is a unit in } A/m$$

but in a field, this can only happen if $\overline{x} = \overline{0}$. Therefore $x \in m$, for all m.

$\square$

1.3 Extension and contraction of ideals

Setup. $A \xrightarrow{f} B$ a homomorphism of rings (so B is an A-algebra) and

$$\mathfrak{a} \subseteq A \text{ ideal} \to \mathfrak{a}^e = \mathfrak{a}B = \langle f(a) \colon a \in A \rangle$$

$$\mathfrak{b} \subseteq B \to \mathfrak{b}^c = f^{-1}(\mathfrak{b}) = \ker\left(A \xrightarrow{f} B \xrightarrow{\nu} B/\mathfrak{b}\right)$$

Basic observations

$$\mathfrak{a}^{ec} \supseteq \mathfrak{a}, \quad \mathfrak{b}^{ce} \subseteq \mathfrak{b}$$

but notice that

$$\mathfrak{a} = \mathfrak{a}^e, \quad \mathfrak{b}^{cec} = \mathfrak{b}^c$$

important fact. if $q \subseteq B$ is prime then so is q^c therefore

$$A/q^c \xrightarrow{\text{subdomain}} B/q$$

Example 1.7. $\mathbb{Z} \to \mathbb{Q}$, $(6\mathbb{Z})^e = \mathbb{Q}$, $(6\mathbb{Z})^{ec} = \mathbb{Z}$.

Example 1.8. $\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z}$ $(4\mathbb{Z})^e = \frac{4\mathbb{Z}+10\mathbb{Z}}{10\mathbb{Z}} = 2\mathbb{Z}/10\mathbb{Z}$ so

$$(4\mathbb{Z})^{ec} = 2\mathbb{Z}$$

Example 1.9. $A \to A[x]$ take $\mathfrak{a} \subseteq A$ then $\mathfrak{a}^e = \mathfrak{a}[x]$ which is the set of all polynomials with coefficients in $\mathfrak{a}$. And $\{0_A\} = \langle x+1 \rangle^c$; $\langle x+1 \rangle^{ce} = \left\{0_{A[x]}\right\}$.

1.4 Modules.

View Math 341 Notes, I won't be taking notes here.

Let A be given and $a_1, \cdots, a_r$ ideals in A, we say that these ideals are relatively prime iff $\forall i, j$ with $i \neq j$,

$$a_i + a_j = A \iff \exists x \in a_i,\ y \in a_j \text{ s.t } x + y = 1$$

Proposition 1.10. *In the above situation we have that*

$$a_1 \cdots a_r = a_1 \cap a_2 \cap \cdots \cap a_r$$

Proof. Induction on r:

$$a_1 a_2 \subseteq a_1 \cap a_2 \quad \text{easy}$$

Conversely if $z \in a_1 \cap a_2$ we have that $x \in a_1,\ y \in a_2$ s.t $x + y = 1$ then

$$z = z \cdot 1 = zx + zy \in a_1 a_2$$

for the inductive step, we have already $b = a_1 \cap \cdots \cap a_{r-1} = a_1 \cdots a_{r-1}$ need to observe that a_r is still relatively prime to b.

Indeed, we know that

$$a_1 + a_r = A,\ \cdots, a_{r-1} + a_r = A$$

therefore we chose elements in $x_1 \in a_1, \cdots, x_{r-1} \in a_{r-1}$ and $y_1, \cdots y_{r-1} \in a_r$ with $x_i + y_i = 1$. We can take

$$\begin{aligned} 1 &= 1^{r-1} \\ &= (x_1 + y_1)(x_2 + y_2) \cdots (x_{r-1} + y_{r-1}) \\ &= x_1 \cdots x_{r-1} + (\text{terms involving some } y_i) \\ &= \in a_1 \cdots a_{r-1} + \in a_r \end{aligned}$$

□

Theorem 1.11 (Chinese Remainder Theorem). *Let $a_1 \cdots a_r$ be pairwise relatively prime, consider the homomorphism of A modules*

$$A \xrightarrow{f} A/a_1 \oplus \cdots \oplus A/a_r$$

Hence

$$A/a_1 \oplus \cdots \oplus A/a_r \cong A/\ker = A/a_1 \cdots a_r$$

1.5 Local Rings

Definition 1.12. A ring A is said to be *local* iff it has exactly one maximal ideal (called m).

Example 1.13.
- $A = K$ a field, then $m = 0$
- Fix p prime $A\,\{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z},\ b \notin p\mathbb{Z}\} \supset m = \{pc/b \mid c, b \in \mathbb{Z},\ p \not| b\} = pA$. Check that the only ideals are
$$A \supsetneq pA \supsetneq p^2A \supsetneq \cdots$$
-
$$\begin{aligned} A &= k[[x_1, x_2]] \\ &= \{\text{formal power series}\} \\ &= \{a_0 + b_1x_1 + b_2x_2 + c_{11}x_1 + \cdots\} \end{aligned}$$
Where $a_i, b_j, c_{k.l}, \cdots \in K$ with
$$m = \langle x_1, x_2\rangle = \{\text{formal power series with } a_0 = 0\}$$
This is the only maximal ideal because m is the set of non-units, think about it. Same for the example above.

Proposition 1.14. *A is local $\iff$ the set of non-units of A is an ideal (which turns out to be the unique maximal ideal).*

Proof. We denote by V the set of non-units of A

$\Longrightarrow$ Let A be local, let m be its unique maximal ideal we clearly have that $m \subseteq V$ Now take any $v \in V$, then by Zorn's lemma we can find some maximal ideal $I \ni a$. But since m is the unique maximal ideal then we have that $m \ni a$.

$\Longleftarrow$ Trivial.

□

1.6 Algebras

Definition 1.15. An *A-algebra* is a ring R (not necc. commutative) with a ring homomorphism $f\colon A \to R$ s.t
$$\operatorname{im} f \subseteq \text{ center of } R$$
We will generally stick to the case where R is commutative.

Example 1.16. All rings are $\mathbb{Z}$-algebras, $M_n(A)$, $A[x_1, \cdots, x_r]$. Moreover $\mathbb{C}$, $\mathbb{H}$ and matrices in $\mathbb{C}$ are all $\mathbb{R}$ algebras

Definition 1.17. A *finitely generated A-algebra* means that $\exists a \subseteq A$ an ideal s.t
$$R \cong A[x_1, \cdots, x_n]/a$$

Example 1.18. A $\mathbb{Z}$-algebra $\mathbb{Z}/10\mathbb{Z}[y_1, y_2]$ where $y_1^2-1=0$ and $y_2^3-2y_1y_2-3$ which is isomorphic to

$$\mathbb{Z}[x_1, x_2]/\langle 10, x_1^2-1, x_2^3-2x, x_2-3\rangle$$

Definition 1.19. a, b ideals of A then the *colon ideal* $(a\colon b)$ is defined to be

$$(a\colon b) = \{x \in A\colon xb \subseteq a\} = \bigcap_{y\in b} \mathrm{Ann}(y + a \in A/a)$$

Example 1.20. In $\mathbb{Z}$,

$$(4\mathbb{Z}\colon 10\mathbb{Z}) = 2\mathbb{Z}, \quad (10\mathbb{Z}\colon 4\mathbb{Z}) = 5\mathbb{Z}$$

Theorem 1.21. *Let $p_1, \cdots, p_r$ be prime ideals of A and suppose that a satisfies*

$$a \subseteq p_1 \cup \cdots \cup p_n$$

Then $\exists j$ s.t $a \subseteq p_j$.

(Note that the proof below also works if all the p_i's are prime except except for 1 or 2 p_i's).

Proof. By induction on n:

- $n = 1$: trivial.
- $n = 2$: Suppose $a \subseteq p_1 \cup p_2$ but $a \not\subseteq p_1$ and $a \not\subseteq p_2$ so $\exists x_1, x_2 \in a$ s.t $x_1 \notin p_1$, $x_2 \notin p_2$. Therefore we must have $x_1 \in p_2$ and $x_2 \in p_1$, then

$$a \ni z := x_1 + x_2 \notin p_1 \cup p_2$$

(If $z \in p_1$ then $x_1 = z - x_2 \in p_1$).
- Now suppose $n \geq 3$ and we know the result for $n-1$ primes. Suppose that

$$a \subseteq p_1 \cup \cdots \cup p_n$$

if $a \subseteq p_1 \cup \cdots p_{n-1}$ (after potentially reordering), then we are okay. So suppose that $a \not\subseteq p_1 \cup \cdots \cup p_{n-1}$ for any reordering. so $\exists x_j \in a$ with with $x_j \in p_j$ for every j.
Consider $z = x_1 + x_2x_3 \cdots x_n$, of course $z \in a$ but $z \notin p_j$ for any j (contradiction)!
Now let $x_1 \in p_1$ and $x_2, \cdots, x_n \notin p_1$ and p_1 is prime therefore

$$x_2 \cdots x_n \notin p_1$$

hence $z = x_1 + (x_2 \cdots x_n) \notin p_1$. Now for p_2, we have that

$$x_1 \notin p_2,\ x_2 \in p_2 \implies x_1 + x_2x_3 \cdots x_n \notin p_2$$

Similarly for the other p's.

□

2 Modules

Lemma 2.1 (Nakayama's lemma). *Let A be a local ring, with maximal ideal m; and Let M be a finitely generated A-module. Suppose that*

$$mM = M$$

(where mM is the set of all finite sums $m_1y_1 + \cdots + m_ry_r$ with $m_i \in m$ and $y_i \in M$.)
Then $M = 0$.

Cheap proof. Suppose $M = \langle x_1, \cdots, x_r \rangle$, we can remove redundant x_i's, therefore we can assume that

$$\forall j,\ x_j \in \langle x_1, \cdots, \hat{j}, \cdots, x_r \rangle$$

If $M \neq 0$ we are left with this set of generators with $r \geq 1$. But then $x_1 \in M = mM$ therefore

$$x_1 = \sum_{i=1}^{r} m_i x_i \quad m_i \in m$$

Therefore $(1 - m_1)x_1 = m_2z_2 + \cdots + m_rx_r$ but $1 - m_1 \notin m$ must be a unit in A! Therefore

$$x_1 \in \langle x_2, \cdots, x_r \rangle \quad \text{impossible}$$

$\square$

Corollary 2.2. *Fix (A, m) a local ring. Suppose that $M \supseteq N$, M finitely generated and $M = N + mM$ (Equivalently in M/mM the images of N generate everything). Then $M = N$.*

Proof. Apply Nakayama to M/N; note that

$$m \cdot (M/N) = M/N, \quad M/N \text{ is finitely generated}$$

$\square$

Theorem 2.3 (General Statement of Nakayama's Lemma). *Let A be any (commutative!) ring and let M be a finitely generated A-module, let $a \subseteq A$ be an ideal (usually $a = \text{Jrad}\, A$) and suppose*

$$aM = M$$

then $\exists b \in 1 + a$ s.t $bM = 0$.

In the case where a is the $\text{Jrad}\, A$ in which case $b \in 1 + a$ is a unit, so $M = 0$.

Proof. Let $M = \langle x_1, \cdots, x_r \rangle$, since $M = aM$, we see that $\exists a_{ij} \in a$ s.t

$$\begin{aligned}
x_1 &= a_{11}x_1 + \cdots + a_{1r}x_r \in aM \\
x_2 &= a_{21}x_1 + \cdots + a_{2r}x_r \in a_M \\
&\vdots \\
x_r &= a_{r1}x_1 + \cdots + a_{rr}x_r \in a_M
\end{aligned}$$

Let $P = (a_{ij})_{ij}$ be the matrix of a_{ij}'s, rewrite the system above as

$$\begin{aligned}(1-a_{11})x_1 - a_{12}x_2 + \cdots - a_{1r}x_r &= 0\\ a_{21}x_1 + (1-a_{22})x_2 + \cdots - a_{2r}x_r &= 0\\ &\vdots\end{aligned}$$

Symbolically we get that

$$(I-P)\begin{pmatrix}x_1\\x_2\\\vdots\\x_r\end{pmatrix} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$$

Multiplying on the left by the matrix $Q = (I-P)^{adj} \in M_r(A)$; it has the property that

$$Q(I-P) = \operatorname{diag}(b)$$

where $b = \det(I-P) \in 1 + a$; we deduce

$$bx_1 = 0,\ bx_2 = 0, \cdots,\ bx_r = 0$$

so b annihilates every generator of M, and therefore annihilates all of M. $\square$

Example 2.4. We can use this so that if (A, m) then

$$m \supseteq m^2 \supseteq m^3 \supseteq \cdots$$

if $m = m^{n+1}$ then $m^n = 0$. Indeed m_i/m_{i+1} is an A/m-module.

2.1 Hom functors and exactness

Quick reminder. M, N are A Modules, therefore $\operatorname{Hom}_A(M, N)$ is an A-module (since A is commutative). Some facts:

- We have a natural isomorphism $\operatorname{Hom}(A, N) \cong N$ (as A-modules).
- Have an natural isomorphism

$$\operatorname{Hom}\left(\bigoplus_{\alpha\in I} M_\alpha, N\right) \cong \prod_{\alpha\in I}\operatorname{Hom}(M_\alpha, N)$$

$$(\cdots x_\alpha \cdots)_{\text{almost all zeroes}} \xrightarrow{\varphi} \sum_\alpha \varphi_\alpha(x_\alpha)$$

But also

$$\operatorname{Hom}(M, \prod_\beta N_\beta) \cong \prod_\beta \operatorname{Hom}(M, N_\beta)$$

$$x \in M \to (\cdots \psi_\beta(x) \cdots) \in \prod_\beta N_\beta$$

- $0 \to N' \xrightarrow{f} N \xrightarrow{g} N''$ exact implies that

$$0 \to \mathrm{Hom}(M, N') \xrightarrow{f_*} \mathrm{Hom}(M, N) \xrightarrow{g_*} \mathrm{Hom}(M, N'')$$

  (even if g is surjective, then g_* need not be surjective).
  Similarly if
$$M' \xrightarrow{u} M \xrightarrow{w} M'' \to 0 \quad \text{exact}$$
  then
$$0 \to \mathrm{Hom}(M'', N) \xrightarrow{w^*} \mathrm{Hom}(M, N) \xrightarrow{u^*} \mathrm{Hom}(M', N)$$

  (even if U is injective then U^* need not be surjective).

2.2 Presentations of Modules

Let A be a (commutative) ring and M an A-module.

Finite presentation. Say M is generated by $x_1, \dots, x_n$ with relations $r_1, \dots, r_k$, where

$$r_j : \quad a_{j1}x_1 + \cdots + a_{jn}x_n = 0 \qquad (1 \leq j \leq k).$$

Let A^n have basis $e_1, \dots, e_n$ and A^k have basis $f_1, \dots, f_k$. Define $\eta : A^n \to M$ by $\eta(e_i) = x_i$ and define $\Phi : A^k \to A^n$ by
$$\Phi(f_j) = a_{j1}e_1 + \cdots + a_{jn}e_n.$$

Then $\eta \circ \Phi = 0$, and $\mathrm{im}(\Phi) = \ker(\eta)$ (the relations generate all relations), hence we have an exact sequence
$$A^k \xrightarrow{\Phi} A^n \xrightarrow{\eta} M \to 0, \qquad \text{so } M \cong \mathrm{coker}(\Phi) = A^n / \mathrm{im}(\Phi).$$

If we write Φ as a matrix, it is the $n \times k$ matrix (a_{ji}) with the convention that the j-th column is the coordinate vector of $\Phi(f_j)$ in the basis (e_i).

Infinite presentations. Nothing changes: if M is generated by a family $(x_i)_{i \in I}$ with relations indexed by J, one has
$$A^{(J)} \xrightarrow{\Phi} A^{(I)} \xrightarrow{\eta} M \to 0,$$

where $A^{(I)} = \bigoplus_{i \in I} A$ and $A^{(J)} = \bigoplus_{j \in J} A$ are free modules on those bases.

Computing $\mathrm{Hom}_A(M, N)$ from a presentation

Given an exact sequence
$$A^k \xrightarrow{\Phi} A^n \xrightarrow{\eta} M \to 0,$$

apply the contravariant left-exact functor $\mathrm{Hom}_A(-, N)$ to obtain the exact sequence

$$0 \to \mathrm{Hom}_A(M, N) \xrightarrow{\eta^*} \mathrm{Hom}_A(A^n, N) \xrightarrow{\Phi^*} \mathrm{Hom}_A(A^k, N),$$

where $\eta^*(\psi) = \psi \circ \eta$ and $\Phi^*(\varphi) = \varphi \circ \Phi$.

Identifications.

- If $n, k < \infty$ then $\mathrm{Hom}_A(A^n, N) \cong N^n$ and $\mathrm{Hom}_A(A^k, N) \cong N^k$ via

$$\mathrm{Hom}_A(A^n, N) \ni \psi \longleftrightarrow (\psi(e_1), \dots, \psi(e_n)) \in N^n.$$

- More generally, for a free module $A^{(I)} = \bigoplus_{i \in I} A$, one has

$$\mathrm{Hom}_A\Big(\bigoplus_{i\in I} A,\ N\Big) \cong \prod_{i\in I} \mathrm{Hom}_A(A, N) \cong \prod_{i\in I} N.$$

(important: *direct sum* on the left becomes a *direct product* on the right.)

Under the finite identifications, the map $\Phi^* : N^n \to N^k$ is given explicitly by

$$(y_1, \dots, y_n) \longmapsto (z_1, \dots, z_k), \qquad z_j = a_{j1}y_1 + \cdots + a_{jn}y_n.$$

Therefore

$$\mathrm{Hom}_A(M, N) \cong \ker(N^n \xrightarrow{\Phi^*} N^k).$$

Equivalently: an A-linear map $M \to N$ is determined by the images of the generators x_i, and the only constraints are that the relations r_j hold after applying the map.

2.3 Tensor Product of Modules (review)

Assume A is commutative (so that left/right issues disappear).

Bilinear maps. For A-modules M, N, P, let $\mathrm{Bil}_A(M \times N, P)$ be the set of maps $\beta : M \times N \to P$ which are A-linear in each variable:

$$\beta(x_1 + x_2, y) = \beta(x_1, y) + \beta(x_2, y), \quad \beta(ax, y) = a\beta(x, y),$$

and similarly in the second variable.

There is a natural bijection

$$\mathrm{Bil}_A(M \times N, P) \ \cong_{\mathrm{nat}}\ \mathrm{Hom}_A\left(M, \mathrm{Hom}_A(N, P)\right),$$

sending β to the map $x \mapsto (y \mapsto \beta(x, y))$.

Universal property of the tensor product. The tensor product $M \otimes_A N$ is an A-module equipped with a bilinear map

$$\tau : M \times N \to M \otimes_A N, \qquad (x, y) \mapsto x \otimes y,$$

such that for every A-module P, composition with τ induces a natural isomorphism

$$\mathrm{Hom}_A(M \otimes_A N,\ P) \ \cong_{\mathrm{nat}}\ \mathrm{Bil}_A(M \times N,\ P).$$

Concrete construction. One can construct $M \otimes_A N$ as the quotient of the free A-module F on symbols (x, y), by the submodule generated by the bilinearity relations:

$$(x_1 + x_2, y) - (x_1, y) - (x_2, y), \qquad (ax, y) - a(x, y),$$
$$(x, y_1 + y_2) - (x, y_1) - (x, y_2), \qquad (x, ay) - a(x, y).$$

The class of (x, y) is denoted $x \otimes y$.

Standard isomorphisms. There are natural isomorphisms

$$A \otimes_A N \cong N, \qquad \Big(\bigoplus_\alpha M_\alpha\Big) \otimes_A N \cong \bigoplus_\alpha (M_\alpha \otimes_A N), \qquad M \otimes_A N \cong N \otimes_A M,$$

and associativity up to canonical isomorphism:

$$(M \otimes_A N) \otimes_A Z \cong M \otimes_A (N \otimes_A Z).$$

Functoriality of $\otimes$

Given A-linear maps $f : M_1 \to M_2$ and $g : N_1 \to N_2$, there is a unique A-linear map

$$f \otimes g : \; M_1 \otimes_A N_1 \to M_2 \otimes_A N_2$$

satisfying on simple tensors

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y).$$

Hence for a general tensor $t = \sum_i a_i(x_i \otimes y_i)$,

$$(f \otimes g)(t) = \sum_i a_i\big(f(x_i) \otimes g(y_i)\big).$$

Extension of scalars

Let $f : A \to B$ be a ring homomorphism and let M be an A-module. Then $B \otimes_A M$ is naturally a *B-module* via multiplication on the left factor:

$$b \cdot (b' \otimes m) := (bb') \otimes m.$$

(Equivalently: the action comes from $\mu_b \otimes 1_M$, where $\mu_b : B \to B$ is multiplication by b.)

Example 2.5 (Polynomials)**.** If $B = A[x]$, then as an A-module,

$$A[x] \cong \bigoplus_{d \geq 0} A \cdot x^d,$$

so

$$A[x] \otimes_A M \cong \bigoplus_{d \geq 0} (A \cdot x^d) \otimes_A M \cong \bigoplus_{d \geq 0} M.$$

Under this identification, $x^d \otimes m$ corresponds to $m\, x^d$, and one gets

$$A[x] \otimes_A M \cong M[x]$$

(the module of polynomials with coefficients in M).

If $\mathfrak{a} \subseteq A$ is an ideal (viewed as an A-module), then

$$A[x] \otimes_A \mathfrak{a} \cong \mathfrak{a}[x].$$

Inside $A[x]$, this is the extended ideal $\mathfrak{a}^e := \mathfrak{a} \cdot A[x]$.

Tensoring algebras. If A is commutative and B, C are (commutative) A-algebras, then $B \otimes_A C$ is an A-algebra with multiplication determined by

$$(b \otimes c) \cdot (b' \otimes c') := (bb') \otimes (cc').$$

Example 2.6. There is a natural isomorphism of A-algebras

$$A[x] \otimes_A B \cong B[x].$$

2.4 Exactness properties of $- \otimes_A N$

Right exactness. The functor $- \otimes_A N$ is *right exact*: if

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact, then

$$M' \otimes_A N \xrightarrow{f \otimes 1} M \otimes_A N \xrightarrow{g \otimes 1} M'' \otimes_A N \to 0$$

is exact.

Warning: it need not be left exact (injectivity can fail).

Example: quotients / extended ideals. From $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ we get

$$\mathfrak{a} \otimes_A B \to A \otimes_A B \to (A/\mathfrak{a}) \otimes_A B \to 0.$$

Using $A \otimes_A B \cong B$, the last map identifies $(A/\mathfrak{a}) \otimes_A B$ with

$$B/\mathfrak{a}^e, \qquad \mathfrak{a}^e = \mathfrak{a} \cdot B.$$

Computing $M \otimes_A N$ from a presentation

If M has a presentation

$$A^k \xrightarrow{\Phi} A^n \to M \to 0,$$

then tensoring with N gives an exact sequence

$$N^k \cong A^k \otimes_A N \xrightarrow{\Phi\otimes 1} A^n \otimes_A N \cong N^n \to M \otimes_A N \to 0.$$

Hence

$$M \otimes_A N \cong \operatorname{coker}(\Phi \otimes 1),$$

and under the identifications $A^k \otimes N \cong N^k$, $A^n \otimes N \cong N^n$, the map $\Phi \otimes 1 : N^k \to N^n$ is given by the *same matrix* (a_{ji}):

$$(u_1, \dots, u_k) \longmapsto \Big(\sum_{j=1}^{k} a_{j1} u_j,\ \dots,\ \sum_{j=1}^{k} a_{jn} u_j\Big).$$

Equivalently: $M \otimes_A N$ is generated by $x_i \otimes y$ ($i = 1, \dots, n$, $y \in N$), with relations coming from:

- bilinearity in y (so $x_i \otimes (y + y') = x_i \otimes y + x_i \otimes y'$ and $x_i \otimes (ay) = a(x_i \otimes y)$),
- and the original relations of M: if $\sum_i c_i x_i = 0$ in M, then for every $y \in N$,

$$\sum_i c_i (x_i \otimes y) = 0 \quad \text{in } M \otimes_A N.$$

Conceptual proof of right exactness (via adjunction)

Conceptual proof of right exactness (via adjunction)

Recall the adjunction: for any A-modules X, N, P there is a natural isomorphism

$$\operatorname{Hom}_A(X \otimes_A N,\ P) \ \cong\ \operatorname{Hom}_A\left(X, \operatorname{Hom}_A(N, P)\right),$$

equivalently

$$\operatorname{Hom}_A(X \otimes_A N,\ P) \ \cong\ \operatorname{Bil}_A(X \times N,\ P).$$

Lemma 2.7. *Let $Q \xrightarrow{\lambda} R \xrightarrow{\mu} S \to 0$ be a sequence of A-modules. Assume that for every A-module P the sequence*

$$0 \to \operatorname{Hom}_A(S, P) \xrightarrow{\mu^*} \operatorname{Hom}_A(R, P) \xrightarrow{\lambda^*} \operatorname{Hom}_A(Q, P)$$

is exact. Then $Q \xrightarrow{\lambda} R \xrightarrow{\mu} S \to 0$ is exact.

Sketch. Surjectivity of μ follows by taking $P = S/\operatorname{im}(\mu)$ and considering the natural quotient map. To show $\ker(\mu) \subseteq \operatorname{im}(\lambda)$, take $P = \operatorname{coker}(\lambda) = R/\operatorname{im}(\lambda)$ and let $\varphi : R \to \operatorname{coker}(\lambda)$ be the quotient map. Then $\varphi \circ \lambda = 0$, so $\varphi \in \ker(\lambda^*) = \operatorname{im}(\mu^*)$, hence $\varphi = \psi \circ \mu$ for some $\psi : S \to \operatorname{coker}(\lambda)$. If $r \in \ker(\mu)$ then $\varphi(r) = \psi(\mu(r)) = 0$, so $r \in \ker(\varphi) = \operatorname{im}(\lambda)$. □

Now suppose $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact. Fix an A-module P and apply the left exact functor $\operatorname{Hom}_A(-, \operatorname{Hom}_A(N, P))$ to obtain an exact sequence

$$0 \to \operatorname{Hom}_A(M'', \operatorname{Hom}_A(N, P)) \to \operatorname{Hom}_A(M, \operatorname{Hom}_A(N, P)) \to \operatorname{Hom}_A(M', \operatorname{Hom}_A(N, P)).$$

Using the adjunction isomorphisms, this becomes

$$0 \to \mathrm{Hom}_A(M'' \otimes_A N, P) \to \mathrm{Hom}_A(M \otimes_A N, P) \to \mathrm{Hom}_A(M' \otimes_A N, P).$$

Since this holds for all P, the lemma implies that

$$M' \otimes_A N \xrightarrow{f\otimes 1} M \otimes_A N \xrightarrow{g\otimes 1} M'' \otimes_A N \to 0$$

is exact, i.e. $- \otimes_A N$ is right exact.

3 Localization of rings and modules

Intuition for example. Let A be a ring (pretend $A = k[x_1, \cdots, x_n]/\mathfrak{a}$ = polynomial functions on $V(\mathfrak{a}) \subseteq k^n$). So $f \in A$ is basically $f\colon V(\mathfrak{a}) \to k$, let $V(f) = V(\langle f \rangle + \mathfrak{a})$ be the set where $f = 0$. We can look at

$$V(\mathfrak{a}) - V(f).$$

What are polynomial functions on this? Intuitively they look like fractions

$$a/f^n, \qquad a \in A,\ n \geq 0,$$

i.e. we allow denominators which are powers of f (so we "ignore" what happens on $V(f)$).

We try to define

$$A_f = \left\{ \frac{a}{f^n} : a \in A,\ n \geq 0 \right\}.$$

Question: when is $\frac{a}{f^n} = \frac{b}{f^m}$? Note we can rewrite as

$$\frac{af^{m+k}}{f^{n+m+k}} = \frac{bf^{n+k}}{f^{n+m+k}}.$$

Therefore $\frac{a}{f^n} = \frac{b}{f^m}$ iff $\exists k$ s.t

$$f^k(af^m) = f^k(bf^n),$$

i.e.

$$f^k(af^m - bf^n) = 0.$$

Remark 3.1. In our hw look at 2.1, this describes exactly $A[x]/\langle xf - 1\rangle$.

Definition 3.2. Let A be a ring, let $S \subseteq A$ be a multiplicatively closed set, define a relation $\sim$ on $A \times S$ as follows:

$$(a, s) \sim (b, t) \iff \exists u \in S \text{ s.t } u(ta - sb) = 0.$$

Example 3.3. if $f \in A$, the set

$$\left\{1, f, f^2, \cdots\right\}$$

is multiplicatively closed.

If $\mathfrak{p} \subset A$ is prime, then $A - \mathfrak{p}$ is multiplicatively closed.

Proposition 3.4. *$\sim$ is an equivalence relation.*

Proof. Exercise. $\square$

Definition 3.5. We define $S^{-1}A$ (also written as A_S) to be the set of equivalence classes of $\sim$. So an element of $S^{-1}A$ is a symbol

$$\frac{a}{s}.$$

Note that

$$\frac{a}{s} = \frac{b}{t} \iff \exists u \in S \text{ s.t } u(ta - sb) = 0.$$

Proposition 3.6. *The operations*

$$\frac{a}{s} + \frac{b}{t} = \frac{ta + sb}{st}$$
$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

are well defined, and they turn $S^{-1}A$ into a ring with

$$0 = \frac{0}{1},\ 1 = \frac{1}{1}.$$

Notation. If $S = \{1, f, f^2, \cdots\}$ then we write A_f instead of $S^{-1}A$. If $S = A - \mathfrak{p}$ we write $A_{\mathfrak{p}}$.

Example 3.7. In $\mathbb{Z}$ we write

$$\mathbb{Z}_f = \left\{ \frac{a}{f^n} \in \mathbb{Q} \mid\ a \in \mathbb{Z},\ n \geq 0 \right\}.$$

And

$$\mathbb{Z}_{\langle p \rangle} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid\ a, b \in \mathbb{Z},\ p \nmid b \right\}.$$

Remark 3.8. There is a ring homomorphism $\varphi\colon A \to S^{-1}A$ by $\varphi(a) = a/1$. But note that φ is not necessarily injective.

For example take $(\mathbb{Z}/6\mathbb{Z})$ localized at $S = \{1, 3\}$. Then

$$S^{-1}(\mathbb{Z}/6\mathbb{Z}) = \{0/1, 1/1\} \cong \mathbb{Z}/2\mathbb{Z}.$$

This is because 3 becomes a unit, so the $\mathbb{Z}/3\mathbb{Z}$-part disappears. Concretely:

$$2/1 = \frac{3 \cdot 2}{3} = 0,$$

hence $\{0, 2, 4\} \subseteq \ker \varphi$. Also

$$\frac{1}{3} = \frac{3 \cdot 1}{3 \cdot 3} = \frac{3}{3} = \frac{1}{1}.$$

Using CRT: $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Inverting 3 makes $(1, 0)$ invertible, which forces the second factor to be 0, leaving $\mathbb{Z}/2\mathbb{Z}$.

Example 3.9. Let $A = \mathbb{C}[x, y]/\langle xy \rangle$, these are functions on the union of the two coordinate axes in $\mathbb{C}^2$. Localize at x:

$$A_x \cong \mathbb{C}[x, x^{-1}].$$

Therefore

$$\frac{y}{1} = \frac{xy}{x} = 0,$$

so $y \in \ker\varphi$ (intuitively: on $D(x)$ we are away from the y-axis, so y vanishes there).

We also have the UMP. Let $\iota : A \to S^{-1}A$ be $\iota(a) = a/1$. Then for any ring B and any ring map $g : A \to B$ such that $g(S) \subseteq B^\times$, there exists a unique ring map $\tilde{g} : S^{-1}A \to B$ with $\tilde{g} \circ \iota = g$. Concretely

$$\tilde{g}(a/s) = g(a)\, g(s)^{-1}.$$

The diagram is:

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "g"'] & S^{-1}A \arrow[d, dashed, "\tilde{g}"] \\ & B \end{tikzcd}$$

Remark 3.10. $0 \in S \iff S^{-1}A$ is the zero ring.

Soon we will also localize modules; we will get a module over $S^{-1}A$. Define

$$S^{-1}M = \left\{ \frac{x}{s} \mid x \in M,\ s \in S \right\}$$

with

$$\frac{x}{s} = \frac{y}{t} \iff \exists u \in S \text{ s.t } u(tx - sy) = 0.$$

3.1 Ideals in the Localization

We will especially look at extensions and contractions of ideals for the homomorphism

$$\varphi\colon A \to S^{-1}A.$$

Proposition 3.11. *If $\mathfrak{b} \subseteq S^{-1}A$ is an ideal, then*

$$\mathfrak{b}^{ce} = \mathfrak{b}.$$

Consequently all ideals of a localization are extensions of something.

Proof. $\mathfrak{b} \supseteq \mathfrak{b}^{ce}$ (always).

Conversely, let $\frac{a}{s} \in \mathfrak{b}$. Then $s(\frac{a}{s}) = \frac{a}{1}$ so $a \in \mathfrak{b}^c$, hence $\frac{a}{1} \in \mathfrak{b}^{ce}$ and so

$$\frac{1}{s} \cdot \frac{a}{1} = \frac{a}{s} \in \mathfrak{b}^{ce}.$$

□

Proposition 3.12. *Let $\mathfrak{a} \subseteq A$ be an ideal. Then*

$$\mathfrak{a}^e = \left\{ \frac{a}{s} \mid a \in \mathfrak{a},\ s \in S \right\}$$

(i.e. $S^{-1}\mathfrak{a}$ inside $S^{-1}A$), and

$$\mathfrak{a}^{ec} = \{x \in A \mid \exists s \in S \text{ s.t } sx \in \mathfrak{a}\}.$$

Example 3.13. Think of $A = \mathbb{Z}$ and $S = \{1, 6, 6^2, \cdots\}$ therefore

$$S^{-1}A = \mathbb{Z}[1/6].$$

Then $(99\mathbb{Z})^{ec} = 11\mathbb{Z} = (11\mathbb{Z})^{ec}$. Indeed $x \in (99\mathbb{Z})^{ec}$ iff $\exists n$ such that $6^n x \in 99\mathbb{Z}$. For $n \geq 2$, 6^n contributes a factor 3^n, so the 3^2 part is automatic, and the condition becomes exactly $11 \mid x$.

Notice that elements of $\mathfrak{a}^e$ are of the form

$$\varphi(a_1) \cdot (\frac{b_1}{s_1}) + \cdots + \varphi(a_k) \cdot (\frac{b_k}{s_k})$$

with $a_i \in \mathfrak{a},\ b_i \in A,\ s_i \in S$. Take a common denominator

$$s = s_1 \cdots s_k,$$

so $\frac{b_i}{s_i} = \frac{c_i}{s}$ with $c_i = b_i \prod_{j \neq i} s_j$, hence wlog we have

$$\frac{a_1}{1} \cdot \frac{c_1}{s} + \cdots + \frac{a_k}{1} \cdot \frac{c_k}{s}.$$

Corollary 3.14. *If $\mathfrak{a} = \langle x_1, \cdots, x_r \rangle \subset A$ then*

$$\mathfrak{a}^e = \langle \frac{x_1}{1}, \cdots, \frac{x_r}{1} \rangle.$$

Corollary 3.15. *If A is Noetherian, then so is $S^{-1}A$ (and every ideal of $S^{-1}A$) by the above corollary.*

Prime ideals in $S^{-1}A$

Theorem 3.16. *1. If $\mathfrak{q} \subset S^{-1}A$ is prime, then $\mathfrak{p} := \mathfrak{q}^c \subset A$ is prime and*

$$\mathfrak{p} \cap S = \varnothing.$$

2. *If $\mathfrak{p} \subset A$ is prime and*

$$\mathfrak{p} \cap S = \varnothing,$$

then $\mathfrak{p}^e \subset S^{-1}A$ is also prime.

3. *Extension and contraction are mutually inverse bijections between the sets*

$$\{\mathfrak{p} \subset A \text{ prime} \mid \mathfrak{p} \cap S = \varnothing\} \leftrightarrows \left\{\mathfrak{q} \subset S^{-1}A \text{ prime}\right\}.$$

Remark 3.17. This clarifies: if $\mathfrak{a} \subset A$ and $\mathfrak{a} \cap S = \varnothing$, then $\mathfrak{a}^e \subsetneq S^{-1}A$. Hence $\mathfrak{a}^e$ is contained in some maximal ideal $\mathfrak{m} \subset S^{-1}A$. Contracting gives a prime $\mathfrak{p} = \mathfrak{m}^c \subset A$ with $\mathfrak{a} \subseteq \mathfrak{p}$ and $\mathfrak{p} \cap S = \varnothing$.

Proof. 1. We are given $\mathfrak{q}$ is prime, then $\mathfrak{q}^c$ is prime (contraction preserves primality). Also if $\exists s \in S \cap \mathfrak{q}^c$ then $s/1 \in \mathfrak{q}$, but $s/1$ is a unit in $S^{-1}A$ (inverse $1/s$), contradiction since $\mathfrak{q} \subsetneq S^{-1}A$.

2. Suppose we have two elements of $S^{-1}A$ whose product is in $\mathfrak{p}^e$. Call these elements a/s and b/t with $a, b \in A$ and $s, t \in S$. Then

$$\frac{ab}{st} \in \mathfrak{p}^e.$$

So $\exists u \in S$ such that $uab \in \mathfrak{p}$ (clearing denominators). Since $u \in S$ and $\mathfrak{p} \cap S = \varnothing$, we have $u \notin \mathfrak{p}$; by primality,

$$a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}.$$

Hence a/s or b/t lies in $\mathfrak{p}^e$.

3. For any ideal $\mathfrak{q} \subset S^{-1}A$, we already proved $\mathfrak{q}^{ce} = \mathfrak{q}$. Now let $\mathfrak{p} \subset A$ be prime with $\mathfrak{p} \cap S = \varnothing$. Clearly $\mathfrak{p}^{ec} \supseteq \mathfrak{p}$. Take $a \in \mathfrak{p}^{ec}$, so $a/1 \in \mathfrak{p}^e$. Then $\exists s \in S$ such that $sa \in \mathfrak{p}$. But $s \notin \mathfrak{p}$, so $a \in \mathfrak{p}$ by primality. Thus $\mathfrak{p}^{ec} = \mathfrak{p}$.

□

Proposition 3.18. *There is a natural isomorphism of $S^{-1}A$-modules*

$$S^{-1}M \cong_{\text{nat}} (S^{-1}A) \otimes_A M, \qquad \frac{ax}{s} \leftrightarrows \frac{a}{s} \otimes x.$$

Proof. Define a map

$$\Phi : (S^{-1}A) \otimes_A M \longrightarrow S^{-1}M, \qquad \Phi\left(\frac{a}{s} \otimes x\right) = \frac{ax}{s}.$$

First observe that this map exists and is an A-module homomorphism (in fact, it is $S^{-1}A$-linear once it is well-defined). Indeed, it is induced by the bilinear map

$$\beta : S^{-1}A \times M \to S^{-1}M, \qquad \beta\left(\frac{a}{s}, x\right) = \frac{ax}{s}.$$

We check that β is well-defined. If

$$\frac{a}{s} = \frac{a'}{s'},$$

then by definition there exists $u \in S$ such that

$$u(s'a - sa') = 0.$$

Multiplying by $x \in M$ gives

$$u(s'ax - sa'x) = 0,$$

hence in $S^{-1}M$ we have

$$\frac{ax}{s} = \frac{a'x}{s'}.$$

Therefore β is well-defined, so it induces the A-linear map Φ.

Surjectivity is trivial: every element of $S^{-1}M$ has the form x/s, and

$$\Phi\left(\frac{1}{s}\otimes x\right)=\frac{x}{s}.$$

Reduction to simple tensors. A general element of $(S^{-1}A)\otimes_A M$ is a finite sum

$$t=\frac{a_1}{s_1}\otimes x_1+\cdots+\frac{a_n}{s_n}\otimes x_n.$$

Passing to a common denominator $s=s_1\cdots s_n$, we may rewrite

$$t=\frac{b_1}{s}\otimes x_1+\cdots+\frac{b_n}{s}\otimes x_n=\frac{1}{s}\otimes\Big(\sum_i b_i x_i\Big).$$

Thus every tensor may be represented in the form $\frac{1}{s}\otimes y$.

Alternate approach (constructing the inverse). Define

$$\Psi : S^{-1}M\to(S^{-1}A)\otimes_A M,\qquad \Psi\left(\frac{x}{s}\right)=\frac{1}{s}\otimes x.$$

We check that Ψ is well-defined. Suppose

$$\frac{x}{s}=\frac{x'}{s'}.$$

Then there exists $u\in S$ such that

$$u(s'x-sx')=0,$$

i.e.

$$us'x=usx'.$$

Then in the tensor product,

$$\frac{1}{s}\otimes x=\frac{u}{us}\otimes x=\frac{1}{us}\otimes ux=\frac{1}{us'}\otimes ux'=\frac{1}{s'}\otimes x'.$$

Hence Ψ is well-defined.

Finally,

$$(\Phi\circ\Psi)\left(\frac{x}{s}\right)=\Phi\left(\frac{1}{s}\otimes x\right)=\frac{x}{s}$$

and

$$(\Psi\circ\Phi)\left(\frac{a}{s}\otimes x\right)=\Psi\left(\frac{ax}{s}\right)=\frac{1}{s}\otimes ax=\frac{a}{s}\otimes x.$$

Therefore Φ is an isomorphism, natural in M. □

Remark 3.19. Why is it called *localization*? Recall that there is an order-preserving bijection between primes:

$$\Big\{\mathfrak{p}\subseteq A\mid \mathfrak{p}\cap S=\varnothing\Big\}\;\longleftrightarrow\;\Big\{\mathfrak{p}\subseteq S^{-1}A\Big\},\qquad \mathfrak{p}\mapsto S^{-1}\mathfrak{p}.$$

In particular, if $S = A - \mathfrak{p}$, then
$$A_\mathfrak{p} = S^{-1}A$$
is a *local ring*. Indeed, its unique maximal ideal is
$$\mathfrak{m}_\mathfrak{p} = \left\{ \frac{a}{s} \in A_\mathfrak{p} \;\middle|\; a \in \mathfrak{p},\ s \notin \mathfrak{p} \right\}.$$
Moreover,
$$A_\mathfrak{p} - \mathfrak{m}_\mathfrak{p} = \left\{ \frac{u}{s} \;\middle|\; u \notin \mathfrak{p},\ s \notin \mathfrak{p} \right\}$$
is exactly the set of units of $A_\mathfrak{p}$.

A quick example: let $A = k[x, y, z]$ and $\mathfrak{p} = \langle x, y \rangle$. Then
$$V(\mathfrak{p}) = \text{“the } z\text{-axis”}.$$
Also $A/\mathfrak{p} \simeq k[z]$, via
$$f(x, y, z) + \mathfrak{p} \;\longleftrightarrow\; f(0, 0, z).$$
Thus
$$A_\mathfrak{p} = \left\{ \frac{f(x, y, z)}{s(x, y, z)} \;\middle|\; s \notin \mathfrak{p} \right\}.$$
The maximal ideal consists of fractions whose numerator vanishes on the z-axis:
$$\mathfrak{m}_\mathfrak{p} = \left\{ \frac{f(x, y, z)}{s(x, y, z)} \;\middle|\; f \in \mathfrak{p} \right\}.$$

Moreover one checks that
$$\frac{A_\mathfrak{p}}{\mathfrak{m}_\mathfrak{p}} \;\cong\; \mathrm{Frac}(A/\mathfrak{p}) \;\cong\; k(z),$$
the field of rational functions in z.

Therefore we can define the *localization functor*
$$M \;\longmapsto\; S^{-1}M,$$
which is secretly the functor
$$M \;\longmapsto\; (S^{-1}A) \otimes_A M.$$

Given a homomorphism $M \xrightarrow{f} N$, we define
$$S^{-1}f : S^{-1}M \to S^{-1}N, \qquad (S^{-1}f)\left(\frac{x}{s}\right) = \frac{f(x)}{s}.$$

Theorem 3.20. *The localization functor is exact. Equivalently, $S^{-1}A$ is a flat A-algebra.*

Proof. Let
$$M \xrightarrow{f} N \xrightarrow{g} P$$
be exact at N, i.e. $\mathrm{im}(f) = \ker(g)$.

We claim that

$$S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}P$$

is exact at $S^{-1}N$.

Trivially $\operatorname{im}(S^{-1}f) \subseteq \ker(S^{-1}g)$ since $gf = 0$.

Now let $y/s \in \ker(S^{-1}g)$. Then

$$(S^{-1}g)\left(\frac{y}{s}\right) = \frac{g(y)}{s} = 0$$

in $S^{-1}P$, hence there exists $u \in S$ such that

$$ug(y) = 0, \quad \text{i.e.} \quad g(uy) = 0.$$

Thus $uy \in \ker(g) = \operatorname{im}(f)$, so $uy = f(x)$ for some $x \in M$. Therefore

$$\frac{y}{s} = \frac{uy}{us} = \frac{f(x)}{us} = (S^{-1}f)\left(\frac{x}{us}\right),$$

proving $\ker(S^{-1}g) \subseteq \operatorname{im}(S^{-1}f)$. □

immediate consequences of exactness.

Let $M \xrightarrow{f} N$ be any homomorphism. Then localization preserves kernels, cokernels, and images:

$$S^{-1}(\ker f) = \ker(S^{-1}f), \qquad S^{-1}(\operatorname{coker} f) = \operatorname{coker}(S^{-1}f), \qquad S^{-1}(\operatorname{im} f) = \operatorname{im}(S^{-1}f).$$

If $N \subseteq M$, then

$$S^{-1}(M/N) \;\cong\; (S^{-1}M)/(S^{-1}N),$$

since localizing the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

remains exact.

Similarly,

$$S^{-1}(M_1 \oplus M_2) \;\cong\; S^{-1}M_1 \oplus S^{-1}M_2.$$

The direct sum maps fit into the standard diagram:

$$\begin{tikzcd} M_1 \arrow[r, "i_1"] & M_1 \oplus M_2 \arrow[l, bend left, "p_1"] \arrow[r, bend right, "p_2"'] & M_2 \arrow[l, "i_2"'] \end{tikzcd}$$

with relations

$$p_1 i_1 = 1, \quad p_2 i_2 = 1, \quad p_1 i_2 = 0, \quad p_2 i_1 = 0, \qquad i_1 p_1 + i_2 p_2 = 1_{M_1 \oplus M_2}.$$

More interestingly, if $N_1, N_2 \subseteq M$, then

$$S^{-1}(N_1 \cap N_2) = S^{-1}N_1 \cap S^{-1}N_2, \qquad S^{-1}(N_1 + N_2) = S^{-1}N_1 + S^{-1}N_2.$$

Indeed we use the exact sequence

$$0 \to N_1 \cap N_2 \to N_1 \oplus N_2 \xrightarrow{(y_1,y_2)\mapsto y_1-y_2} N_1 + N_2 \to 0,$$

where $z \mapsto (z, z)$ is the inclusion.

Proposition 3.21. *Let M, N be A-modules. Then $\operatorname{Hom}_A(M, N)$ is an A-module, and there is a natural map*

$$S^{-1}\operatorname{Hom}_A(M, N) \xrightarrow{\varphi} \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N),$$

given by

$$\frac{f}{s} \longmapsto \Big(\frac{x}{t} \mapsto \frac{f(x)}{st}\Big).$$

Moreover:

- *If M is finitely generated, then φ is injective.*
- *If M is finitely presented, then φ is bijective.*

(Sketch: finite generation controls kernels, finite presentation controls surjectivity.)

Local vs global properties

We know: for $x \in M$, when is $x/1 = 0$ in $S^{-1}M$? Precisely when

$$\exists s \in S \text{ such that } sx = 0, \qquad \text{i.e.} \qquad \operatorname{Ann}(x) \cap S \neq \varnothing.$$

Fix $x \in M$. Which primes $\mathfrak{p}$ satisfy $x/1 \neq 0$ in $M_\mathfrak{p}$? These are exactly

$$\{\mathfrak{p} \mid \operatorname{Ann}(x) \subseteq \mathfrak{p}\}.$$

Example 3.22. Let $A = \mathbb{Z}$ and $M = \mathbb{Z}/20\mathbb{Z}$. Then

$$\operatorname{Ann}([1]) = 20\mathbb{Z} \subseteq 2\mathbb{Z}, 5\mathbb{Z}, \qquad \operatorname{Ann}([4]) = 5\mathbb{Z}.$$

Hence

$$M_{(2)} \cong \mathbb{Z}/4\mathbb{Z}, \qquad M_{(5)} \cong \mathbb{Z}/5\mathbb{Z}.$$

Example 3.23. Let $A = \mathbb{C}[x, y]$ and maximal ideals

$$\mathfrak{m}_{a,b} = \langle x - a,\ y - b\rangle.$$

If

$$M = A/\langle x^2 + y^2 - 1\rangle,$$

then $[1]$ lives at $\mathfrak{m}_{a,b}$ iff

$$x^2 + y^2 - 1 \in \mathfrak{m}_{a,b} \iff a^2 + b^2 = 1.$$

Example 3.24. Consider $[x] \in k[x,y]/\langle x,y\rangle$ as a $k[x,y]$-module element. Then

$$\mathrm{Ann}([x]) = \langle y\rangle,$$

so $[x]$ lives exactly along the x-axis.

Local-global principles

Proposition 3.25. *Let M be an A-module. Then the following are equivalent:*

- $M = 0$.
- *For all primes $\mathfrak{p}$, $M_{\mathfrak{p}} = 0$.*
- *For all maximal ideals $\mathfrak{m}$, $M_{\mathfrak{m}} = 0$.*

Equivalently, for $x \in M$, the following are equivalent:

- $x = 0$.
- *For all primes $\mathfrak{p}$, $x/1 = 0$ in $M_{\mathfrak{p}}$.*
- *For all maximal ideals $\mathfrak{m}$, $x/1 = 0$ in $M_{\mathfrak{m}}$.*

Proof. $(1) \implies (2) \implies (3)$ is trivial.

For $(3) \implies (1)$: assume $M_{\mathfrak{m}} = 0$ for all maximal ideals. Let $x \in M$ with $x \neq 0$. Then $\mathrm{Ann}(x)$ is a proper ideal, hence contained in some maximal ideal $\mathfrak{m}$:

$$\mathrm{Ann}(x) \subseteq \mathfrak{m}.$$

But $x/1 = 0$ in $M_{\mathfrak{m}}$ means there exists $s \notin \mathfrak{m}$ such that $sx = 0$. Thus $s \in \mathrm{Ann}(x) \subseteq \mathfrak{m}$, contradiction. Therefore $x = 0$, so $M = 0$. □

Consequences.

Proposition 3.26. *A homomorphism $f : M \to N$ is injective (resp. surjective, resp. bijective) iff for all primes $\mathfrak{p}$ (equivalently, all maximals $\mathfrak{m}$),*

$$f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$$

is injective (resp. surjective, resp. bijective).

Proof. Apply localization to the exact sequences involving $\ker f$ and $\operatorname{coker} f$. □

Proposition 3.27. *If $N \subseteq M$, then*

$$N = M \iff N_{\mathfrak{p}} = M_{\mathfrak{p}}\ \forall \mathfrak{p} \iff N_{\mathfrak{m}} = M_{\mathfrak{m}}\ \forall \mathfrak{m}.$$

Proof. Apply the previous proposition to M/N. □

Proposition 3.28. *If $x \in M$ satisfies $x/1 = 0$ in $M_{\mathfrak{m}}$, then the image of x in*

$$M/\mathfrak{m}M \;\cong\; (A/\mathfrak{m}) \otimes_A M$$

is zero.

Proof. If $x/1 = 0$ in $M_\mathfrak{m}$, then $\exists s \notin \mathfrak{m}$ with $sx = 0$. In $M/\mathfrak{m}M$, we have $s[x] = 0$, but also $\mathfrak{m}[x] = 0$. Since $s \notin \mathfrak{m}$, the ideal $\mathfrak{m} + \langle s \rangle = A$, hence $[x] = 0$. □

Theorem 3.29. *Suppose M is a finitely generated A-module. Then*

$$M = 0 \iff \forall \mathfrak{m},\ M/\mathfrak{m}M = 0.$$

Proof. Assume $M/\mathfrak{m}M = 0$ for all maximal ideals $\mathfrak{m}$.

Localizing at $\mathfrak{m}$, we get

$$M_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}M_\mathfrak{m} \;\cong\; (M/\mathfrak{m}M)_\mathfrak{m} = 0.$$

Since $M_\mathfrak{m}$ is finitely generated over the local ring $A_\mathfrak{m}$, Nakayama's lemma implies

$$M_\mathfrak{m} = 0 \quad \forall \mathfrak{m}.$$

By the local-global principle, $M = 0$. □

Remark 3.30. Finite generation is essential. For example, if $A = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$, then multiplication by n is surjective for all n, hence

$$(\mathbb{Q}/\mathbb{Z})/p(\mathbb{Q}/\mathbb{Z}) = 0 \quad \forall p,$$

but $\mathbb{Q}/\mathbb{Z} \neq 0$.

3.2 The spectrum of a ring.

Definition 3.31. Let A be a ring. The *spectrum* of A is the set of all prime ideals:

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

The *maximal spectrum* is the subset of maximal ideals:

$$\mathrm{maxSpec}(A) = \{\mathfrak{m} \subseteq A \mid \mathfrak{m} \text{ is maximal}\}.$$

Example 3.32. Let $A = \mathbb{C}[x, y]$. Then $\mathrm{Spec}(A)$ consists of:

1. Maximal ideals
$$\mathfrak{m}_{a,b} = \langle x - a,\ y - b \rangle,$$
which correspond to points $(a, b) \in \mathbb{C}^2$.

2. Prime ideals of height 1, generated by irreducible polynomials:
$$\langle f(x, y) \rangle, \qquad f \text{ irreducible},$$
which correspond to irreducible affine curves.

3. The zero ideal (0), which corresponds to the generic point of A^2.

Let $X = \mathrm{Spec}(A)$. We now define a topology on X, called the *Zariski topology*.

Closed sets

Definition 3.33. If $\mathfrak{a} \subseteq A$ is an ideal, define its *vanishing set*:

$$V(\mathfrak{a}) = \{\mathfrak{p} \in X \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

More generally, for any subset $\Sigma \subseteq A$, define

$$V(\Sigma) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \Sigma \subseteq \mathfrak{p}\}.$$

Equivalently,

$$V(\Sigma) = V(\langle \Sigma \rangle).$$

Basic open sets

Definition 3.34. For $f \in A$, define the *distinguished open set*

$$U_f = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}.$$

Then

$$U_f = X \setminus V(\langle f \rangle).$$

Intuitively, U_f is the set of primes where f does *not* vanish.

Proposition 3.35. *The sets U_f form a basis of open sets, and*

$$U_f \cap U_g = U_{fg}.$$

Moreover, for any $\Sigma \subseteq A$,

$$X \setminus V(\Sigma) = \bigcup_{f \in \Sigma} U_f.$$

The closed sets form a topology

Proposition 3.36. *The sets $V(\mathfrak{a})$ satisfy the axioms for closed sets:*

1. $$V(0) = X, \qquad V(1) = \varnothing.$$

2. *For ideals $\mathfrak{a}_i$,*
$$\bigcap_{i \in I} V(\mathfrak{a}_i) = V\Big(\sum_{i \in I} \mathfrak{a}_i\Big).$$

3. *For ideals $\mathfrak{a}, \mathfrak{b}$,*
$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}).$$

Useful special cases

- $$V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b}).$$

- $$V(f_1 \cdots f_n) = V(f_1) \cup \cdots \cup V(f_n).$$
- If $\mathfrak{a} = \langle f_1, \ldots, f_k \rangle$, then
$$V(\mathfrak{a}) = X \setminus (U_{f_1} \cup \cdots \cup U_{f_1}).$$

Restriction via localization

Proposition 3.37. *There are natural homeomorphisms:*

$$\mathrm{Spec}(A/\mathfrak{a}) \cong V(\mathfrak{a}), \qquad \mathrm{Spec}(A_f) \cong U_f.$$

Radicals and closed sets

Remark 3.38. For any ideal $\mathfrak{a}$,
$$V(\mathfrak{a}) = V(\mathrm{rad}\,\mathfrak{a}).$$

Thus,
$$\mathfrak{a} \subseteq \mathfrak{b} \implies V(\mathfrak{a}) \supseteq V(\mathfrak{b}).$$

Proposition 3.39.
$$V(\mathfrak{a}) \supseteq V(\mathfrak{b}) \quad \Longleftrightarrow \quad \mathrm{rad}\,\mathfrak{a} \subseteq \mathrm{rad}\,\mathfrak{b}.$$

Corollary 3.40.
$$V(\mathfrak{a}) = V(\mathfrak{b}) \quad \Longleftrightarrow \quad \mathrm{rad}\,\mathfrak{a} = \mathrm{rad}\,\mathfrak{b}.$$

Thus closed sets in $\mathrm{Spec}(A)$ *correspond bijectively to radical ideals.*

The Zariski topology is not Hausdorff

Example 3.41. For $A = \mathbb{C}[x]$,

$$\mathrm{Spec}(A) = \{(0)\} \cup \{\langle x - a \rangle \mid a \in \mathbb{C}\}.$$

Every nonempty open set contains (0), so the space is not Hausdorff.

Example 3.42. For $A = \mathbb{Z}$,

$$\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) \mid p \text{ prime}\}.$$

The basic opens are
$$U_n = \mathrm{Spec}(\mathbb{Z}) \setminus \{(p) \mid p \mid n\}.$$

So U_n removes only finitely many primes. Every nonempty open set contains (0).

Quasi-compactness

Proposition 3.43. $\mathrm{Spec}(A)$ *is quasi-compact.*

Proof. Suppose

$$\operatorname{Spec}(A) = \bigcup_{i \in I} U_{f_i}.$$

Then

$$\varnothing = \bigcap_{i \in I} V(f_i) = V(\langle f_i \mid i \in I \rangle).$$

Thus

$$1 \in \langle f_i \mid i \in I \rangle,$$

so finitely many $f_{i_1}, \ldots, f_{i_k}$ generate the unit ideal, hence

$$\operatorname{Spec}(A) = U_{f i_1} \cup \cdots \cup U_{f_{i_k}}.$$

□

Corollary 3.44. *If A is a Noetherian ring, then all subsets of* $\operatorname{Spec} A$ *are quasi-compact.*

Restriction to open sets

Proposition 3.45. *Let M be an A-module and $x \in M$.*

1. *If $x/1 = 0$ in $M_\mathfrak{p}$, then there exists $f \notin \mathfrak{p}$ such that $x/1 = 0$ in M_f.*
2. *If $\operatorname{Spec}(A) = U_{f_1} \cup \cdots \cup U_{f_n}$ and $x/1 = 0$ in each M_{f_i}, then $x = 0$ in M.*