Midterm - MRM Spring 2016

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1 MRI Physics

a) MRI is considered an insensitive technique because its signal is based on the difference in up- and down-spin protons, which is a very small number (measured in protons per million).

b) From rearranging the Boltzman distribution we have the following expression:

$$\frac{N_{\uparrow} - N_{\downarrow}}{N_{total}} \approx \frac{\gamma h B_0}{2KT}$$

When applying a B_0 of 7T, we find that $\frac{N_{\uparrow}-N_{\downarrow}}{N_{total}} \approx 21 \times 10^{-6}$.

c) From looking at the equation above we can see that increasing our B_0 increases the strength of our MR signal. For that reason we tend to use higher field magnets, since more signal is (almost) always beneficial.

2 Components of MRI Pulse Sequence

a) A pulse sequence is a programmed set of changing magnetic gradients dependent on a number of parameters which determine the quality of the image observed. A pulse sequence will often include an RF pulse and gradient pulses, such as those used for frequency or phase encoding.

b) Gradient fields allow us to discern signal from different regions throughout the volume being imaged. When just bulk B_0 magnetization exists, there is no selectivity based on location within the volume and spatial features cannot be recovered from the signal. Applying gradient fields which encode into either frequency or phase of the signal enables us to recover spatial information from the signal.

c) RF pulses, B_1, are applied in the transverse plane and are used to excite spins aligned with the bulk magnetization field B_0 out of equilibrium so that they may be observed in the MR signal.

d) As three types of magnetic fields are required for MR imaging, we need hardware to generate these fields. Generation of magnetic fields requires, well, magnets. Magnets in the context of MR systems are created by coils. Thus, coils exist to create the required fields. Generally speaking, there exists a B_0 coil to induce bulk magnetization in the entire sample, an RF coil to excite the sample out of equilibrium, and G_z and G_{xy} coils to induce magnetic gradients in the sample and enable spatial localization.

3 Contrast with Spin Echo

a) In a spin echo sequence, we aim to flip the magnetization into the transverse plain, observe the decay of our signal with T_2^*, and then apply a refocusing pulse to observe echos of the transverse magnetization vectors which decay in amplitude according to T_2 at $t = \text{TE}$. The figure below illustrates this process. Initially an RF pulse which tips the magnetization into the transverse plane as well as a slice selection gradient along the z direction. Then, phase and frequency encoding gradients in the y and x directions, respectively, are applied to the sample. The signal is acquired while the frequency gradient is on, effectively sampling a row in k-space. The middle of the applied frequency gradient is when the echo occurs, TE. The time until this entire process is repeated is TR.-

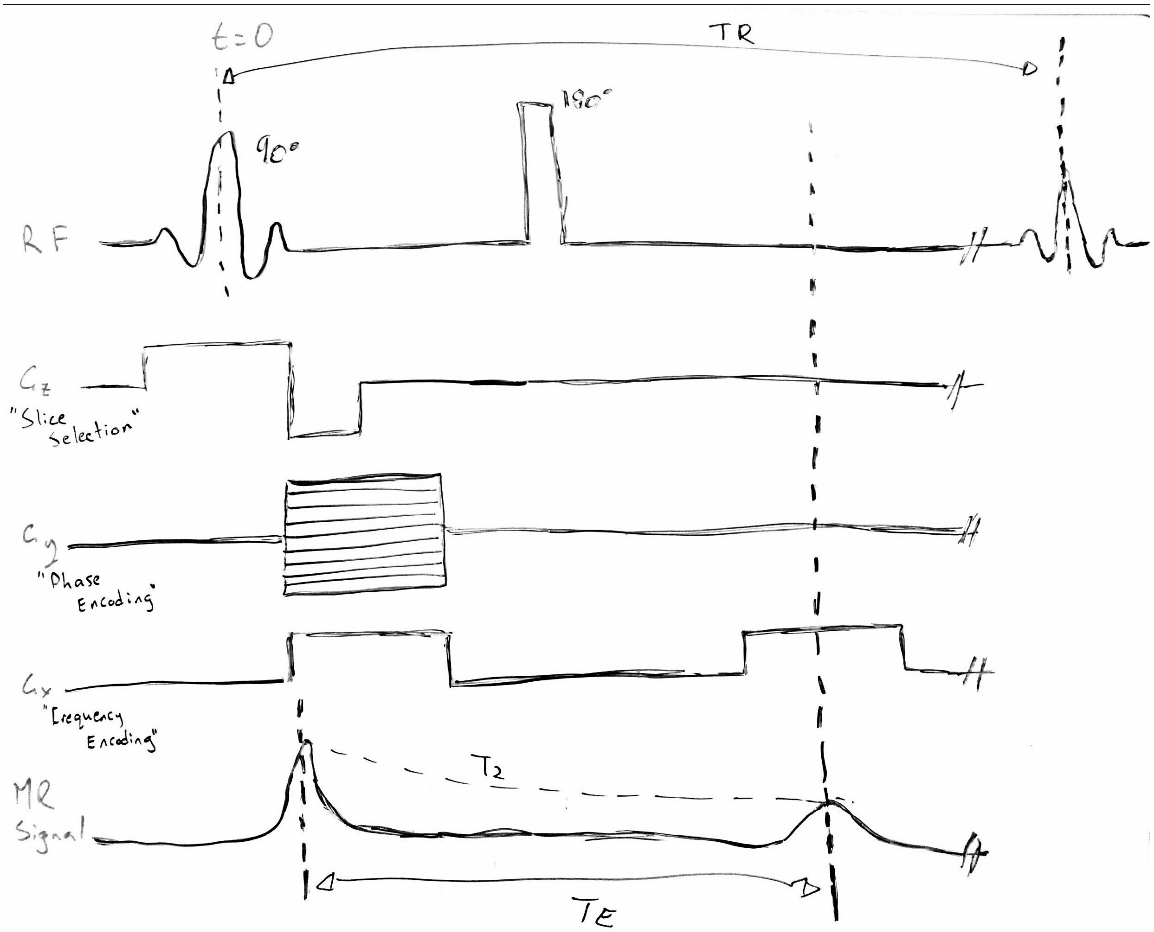

Figure 1: Spin echo pulse sequence

b) In order to achieve T1 contrast, we should use a short TE and a short TR.

c) In order to achieve T2 contrast, we should use a long TE and a long TR.

d) In order to achieve ρ contrast, we should use a short TE and a long TR.

4 RF Pulses

a) In the laboratory frame of reference, as seen in section a) of the figure below, the magnetization vector spirals downward towards the xy plane. The phase is also rotating at a constant velocity about the z axis.

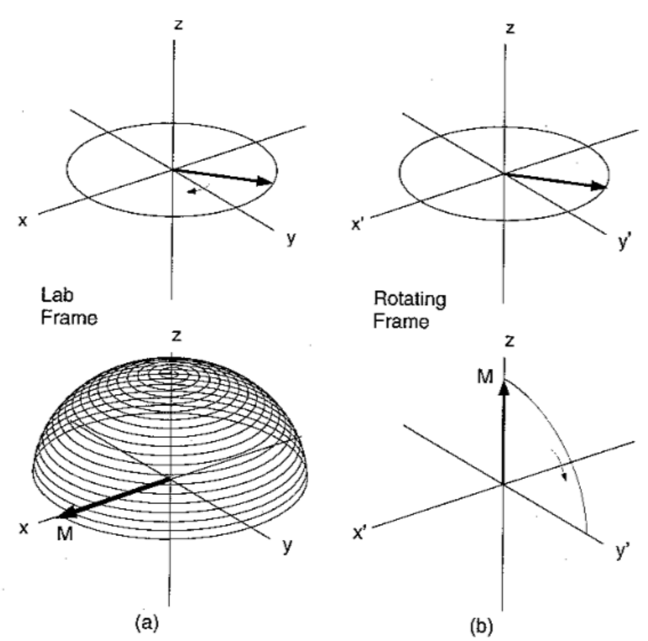

Figure 2: a) Laboratory and b) rotating frames of reference for observing magnetization behaviour. This is Nishimura textbook's Figure 6.4.

b) In the rotating frame of reference, as can be seen in section b) of the figure above, the magnetization vector simply tips onto the transverse plane, and the phase of this vector is constant.

c) An equation that represents a circularly polarized RF pulse as used in MRI is:

$$B_1(t) = B_1 a_1(t) \begin{bmatrix} \cos(\omega_0 t + \phi_1) \\ -\sin(\omega_0 t + \phi_1) \\ 0 \end{bmatrix} = B_1 a_1(t) e^{-i(\omega_0 t + \phi_1)}$$

Where $a_1(t)$ is a unitless pulse envelope with $max|a_1(t)| = 1$. Citation: `http://web.eecs.umich.edu/~fessler/course/516/l/cm-mri-1-33.pdf`

d) The frequency ω_0 determines which tissue will be selected. The middle of the slice will be tissue at the same resonant frequency, and all tissue within the bandwidth of $B_1(t)$ will be excited. The bandwidth is a function of the desired slice thickness and gradient amplitude, given by: $BW_{RF} = \bar{\gamma}\Delta z G_z$. The phase is used to refocus the signal.

5 Inversion Recovery Spin-Echo: Deriving the Stead-State Equations

a) As steady state has been reached, we are constrained by $M_z\left(\mathrm{TI}^+\right) = M_z\left((\mathrm{TI}+k\tau)^+\right)$ and $M_{xy}\left(\mathrm{TI}^+\right) = M_{xy}\left((\mathrm{TI}+k\tau)^+\right)$ for $k \in 1, 2, 3 \ldots$, or more simply put, that the MR signal is periodic and repeats with time τ.

b) The additional constraint imposed by assuming that $\mathrm{TR} \gg \mathrm{T}_s^*$ is that $M_{xy}\left(\mathrm{TR}^-\right) = 0$.

c) The general equation that represents the solution to the Bloch equations for M_z is:

$$M_z = M_0 + \left(M_z\left(0\right) - M_0\right) e^{-t/T_1}$$

d) The general equation that represents the solution to the Block equations for M_{xy} is:

$$M_{xy} = M_0 e^{-t/T_2}$$

e) With M_z on top and M_{xy} on the bottom, shown below is an approximate waveform for the magnetization amplitudes during this inversion pulse sequence.

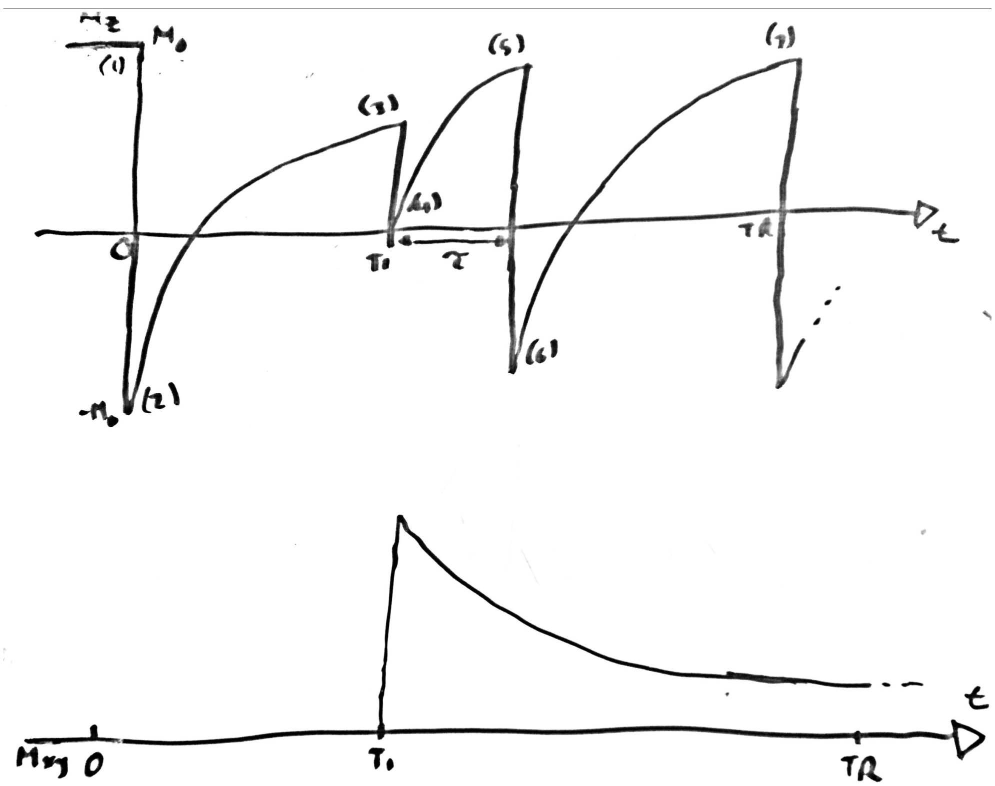

Figure 3: Inversion recovery from equilibrium

f) Overlaid in red is the waveform of the magnetization as started at steady state rather than equilibrium. The steady state waveform overlaps with the original waveform in both cases beginning at time-point (3) as marked on the figure, or, if you prefer, at TI. From this point onward, the waveforms are the same.

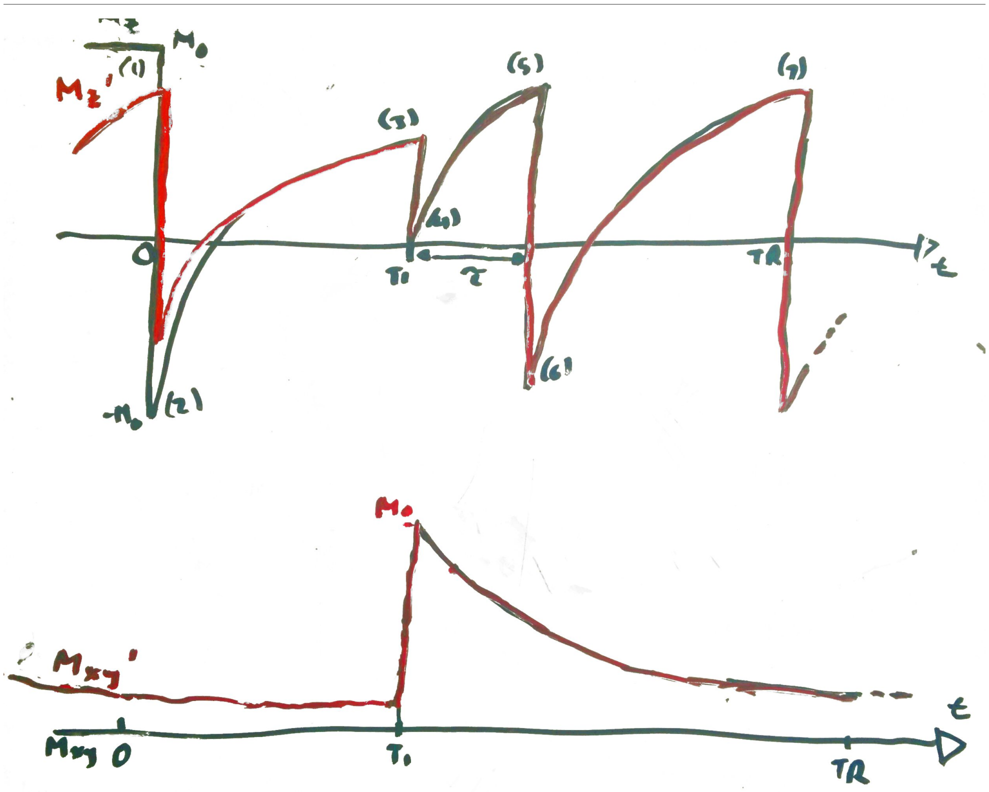

Figure 4: Inversion recovery with overlaid waveforms from steady state

g) To simplify this process, we will analyze the magnetization and derive steady state equations for each numbered step in the waveform depicted above.

Step (1): We are starting at equilibrium, thus the magnetization all lies in the longitudinal direction.

$$M_z\left(0^-\right) = M_0$$
$$M_{xy}\left(0^-\right) = 0$$

Step (2): The initial inversion pulse is applied which flips this magnetization into the negative direction.

$$M_z\left(0^+\right) = -M_0$$
$$M_{xy}\left(0^+\right) = 0$$

Step (3): After the inversion pulse, the signal is slowly recovering towards M_0. However at time TI, the entire signal has not yet recovered. We assume that this recovery does not dip at all into the transverse plane, so that component remains 0.

$$\begin{aligned}
M_z\left(\mathrm{TI}^-\right) &= M_0\left(1-e^{-\mathrm{TI}/T_1}\right) + M_z\left(0^+\right)e^{-\mathrm{TI}/T_1} \\
&= M_0\left(1-e^{-\mathrm{TI}/T_1}\right) - M_0 e^{-\mathrm{TI}/T_1} \\
&= M_0\left(1-2e^{-\mathrm{TI}/T_1}\right) \\
M_{xy}\left(\mathrm{TI}^-\right) &= 0
\end{aligned}$$

Step (4): The 90 degree pulse translates all of the magnetization into the transverse plane, leaving no magnetization in z.

$$\begin{aligned}
M_z\left(\mathrm{TI}^+\right) &= 0 \\
M_{xy}\left(\mathrm{TI}^+\right) &= M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)
\end{aligned}$$

Step (5): For the longitudinal component, this is much like step (3), the magnetization is recovering along the longitudinal direction. The main differences here are the duration of this recovery as well as the initial magnetization when it starts. For the transverse component, we are decaying following the equation given in part d).

$$\begin{aligned}
M_z\left((\mathrm{TI}+\tau)^-\right) &= M_0\left(1-e^{-\tau/T_1}\right) + (0)\, e^{-\tau/T_1} \\
&= M_0\left(1-e^{-\tau/T_1}\right) \\
M_{xy}\left((\mathrm{TI}+\tau)^-\right) &= M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)e^{-\tau/T_2}
\end{aligned}$$

Step (6): Much like step (2) in which the magnetization was inverted, this is what happens here as well. Again, the difference being the amplitude of the waveform which is inverted. The transverse component is unaffected by this pulse.

$$\begin{aligned}
M_z\left((\mathrm{TI}+\tau)^+\right) &= -M_0\left(1-e^{-\tau/T_1}\right) \\
M_{xy}\left((\mathrm{TI}+\tau)^+\right) &= M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)e^{-\tau/T_2}
\end{aligned}$$

Step (7): We again have a signal recovering towards positive longitudinal magnetization. The expressions for the $M_z(t)$ at times TI+2τ and TR follow the same form, so the general form is solved for with values finally being plugged in. Here, again, the transverse component continues to decay.

$$\begin{aligned}
M_z(t) &= M_0\left(1-e^{-t/T_1}\right) + M_z\left((\mathrm{TI}+\tau)^+\right)e^{-t/T_1} \\
&= M_0\left(1-e^{-t/T_1}\right) - M_0\left(1-e^{-\tau/T_1}\right)e^{-t/T_1} \\
&= M_0\left(1-e^{-t/T_1}\right) - M_0\left(e^{-t/T_1}-e^{-(\tau+t)/T_1}\right) \\
&= M_0\left(1-2e^{-t/T_1}+e^{-(\tau+t)/T_1}\right) \\
M_z\left((\mathrm{TI}+2\tau)\right) &= M_0\left(1-2e^{-\tau/T_1}+e^{-2\tau/T_1}\right) \\
M_z\left(\mathrm{TR}^-\right) &= M_0\left(1-2e^{-(\mathrm{TR}-\mathrm{TI}-\tau)/T_1}+e^{-(\mathrm{TR}-\mathrm{TI})/T_1}\right)
\end{aligned}$$

$$M_{xy}\left(\mathrm{TI}+2\tau\right)=M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)e^{-2\tau/T_2}$$
$$M_{xy}\left(\mathrm{TR}^-\right)=M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)e^{-(\mathrm{TR}-\mathrm{TI})/T_2}$$

As requested, the final forms at each of these stages and times is tabulated below.

t	$M_z(t)$	$M_{xy}(t)$
0^+	$-M_0$	0
TI^-	$M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)$	0
TI^+	0	$M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)$
$(\mathrm{TI}+\tau)^-$	$M_0\left(1-e^{-\tau/T_1}\right)$	$M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)e^{-\tau/T_2}$
$(\mathrm{TI}+\tau)^+$	$-M_0\left(1-e^{-\tau/T_1}\right)$	$M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)e^{-\tau/T_2}$
$\mathrm{TI}+2\tau$	$M_0\left(1-2e^{-\tau/T_1}+e^{-2\tau/T_1}\right)$	$M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)e^{-2\tau/T_2}$
TR^-	$M_0\left(1-2e^{-(\mathrm{TR}-\mathrm{TI}-\tau)/T_1}+e^{-(\mathrm{TR}-\mathrm{TI})/T_1}\right)$	$M_0\left(1-2e^{-\mathrm{TI}/T_1}\right)e^{-(\mathrm{TR}-\mathrm{TI})/T_2}$

Table 1: Longitudinal magnetization equations for inversion recovery

h) In order for the signal sampled to be zero in the steady state, the transverse magnetization must be exactly zero at TE. To solve for this, we can set the equation solved for $M_{xy}\left(\mathrm{TI}+2\tau\right)$ above in step (7) to zero, and solve for TI. As this algebra looked like it could have been potentially ugly (I realized later it wasn't so bad), I used MATLAB's symbolic toolbox to solve it, and have copied the code below as Figure 5. The final answer to this expression is:

$$\mathrm{TI}=T_1\ln\left(2\right)$$

```
syms tau m0 t1 tr ti t2;    % Declare variables
f = m0*(1-2*exp(-ti/t1))*exp(-2*tau/t2);    % Define function
value = f == 0; % Define target condition
my_ti = solve(value, ti);   % Solve target condition for TI
pretty(my_ti)   % Prettyyyy
```

Figure 5: Code snippet used for solving TI

i) Seeing as $\mathrm{TE}=2\tau$, and TR is large, the sequence shown here most closely resembles that of a T2 weighted image.

j) If we changed the orientation of the refocusing RF pulse to be in the x' direction instead of the y' direction, the readout would be 180 degrees out of phase. (i.e. the echo would occur along the $-y$-axis instead of the y-axis). In practice it is more common to apply the refocusing pulse along the perpendicular direction the initial RF pulse

6 Pulse Sequence: 1-Sided Radial Imaging vs. Cartesian Imaging

a) Shown in the figure below is the pulse sequence required to image k-space radially with 8 spokes each at 45 degree separation. The figure shows a single RF pulse to TR period. As it was too cluttered to annotate fully, I will describe the sequence and provide values here. Initially, like usual, a depolarizing RF pulse is applies as well as a slice selection gradient, G_z, with duration $2A$. The rephasing component of the slice selection gradient lasts for half of that time, A. Next, for each spoke, a pair of gradient pulses, G_x and G_y are applied. The magnitude of these gradients combined using Pythagorean Theorem and the angle of the spoke is always $G_{x,max} = G_{y,max} = G_r$, to ensure that the readout time is constant for each spoke. The duration of these gradients is B. The ADC is on for the duration of these gradients, collecting data outwards from the origin. Immediately following the sampled portion of the gradients, the gradients are reversed for time B so that we return to the origin of k-space prior to sampling the next spoke. Different colours - paired across G_x and G_y - indicate each set of gradients (1 applied per RF$\rightarrow$TR period) for the desired 8 spokes. The amplitude each of G_x and G_y can take the following values: $G_r, G_r/\sqrt{2}, 0, -G_r/\sqrt{2}$, and $-G_r$, though as mentioned earlier the absolute value of the combined magnitude must always be G_r.

b) When using a two-sided Cartesian sampling pattern and specifying FOV_x, FOV_y, Δx, and Δy, we can compute many remaining image parameters. The number of phase encode (y) and readout (x) samples are, as derived from Nishimura equation 5.81:

$$N_x = \frac{\text{FOV}_x}{\Delta x}$$
$$N_y = \frac{\text{FOV}_y}{\Delta y}$$

The resolution in k-space (i.e. Δk_x and Δk_y) can be found from Nishimura equation 5.75 as:

$$\Delta k_x = \frac{1}{\text{FOV}_x}$$
$$\Delta k_y = \frac{1}{\text{FOV}_y}$$

The maximum and minimum values in k-space in both the k_x and k_y directions, as seen from Nishimura equations 5.85 and 5.87 are:

$$k_{x,max} = \frac{1}{2\Delta x}$$
$$k_{y,max} = \frac{1}{2\Delta y}$$
$$k_{x,min} = -k_{x,max}$$
$$k_{y,min} = -k_{y,max}$$

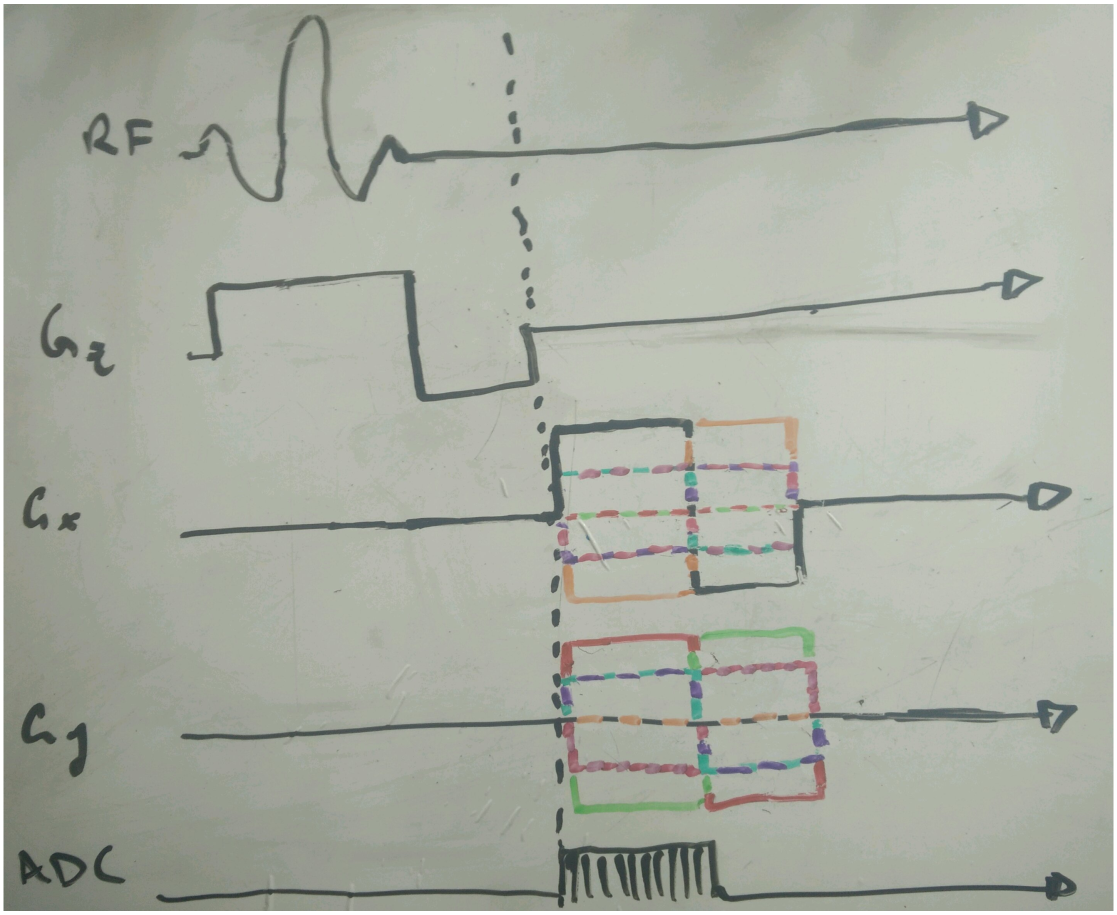

Figure 6: Radial sampling of k-space. Each colour indicates a single readout session in both the G_x and G_y waveforms. For instance, the purple waveform goes from $-G_r/\sqrt{2}$ to $G_r/\sqrt{2}$ in G_x and from $G_r/\sqrt{2}$ to $-G_r/\sqrt{2}$ in G_y, resulting in the diagonal spoke in the upper-left quadrant of k-space.

c) As we are now obtaining data in a radial sequence, the FOV is not constant as we move further from the origin. We wish to obtain at least as high a FOV as the user specified, we must choose parameters for Δk_ϕ and Δk_r such that at the worst case scenario we match this. Radially, the sequencing is identical to sequencing a Cartesian slice, therefore the parameters are as follows, where FOV and Δx were the values defined in the Cartesian system:

$$\Delta k_r = \frac{1}{\text{FOV}}$$
$$N_r = \frac{\text{FOV}}{2\Delta x}$$

In the azimuthal direction, the field of the view will be the lowest at the boundary of the image. Therefore, we must set Δk_ϕ to be at it's largest, Δk_x. For N_ϕ, we need to ensure that we sample at least as many trajectories as we did in the Cartesian system to obtain the same FOV. If we sample N_y columns of data in the top half plane, we must sample just as many over the half-circle we will be ranging over in our radial system. Here, we multiply a

π factor by N_y (equivalently N_x) to ensure that happens. Thus, our equations here are:

$$\Delta k_\phi = \frac{1}{\text{FOV}}$$
$$N_\phi = \pi \frac{\text{FOV}}{\Delta x}$$

d) Looking at the relationships between the definitions of our N_r, N_ϕ, and N_x, we can derive explicit relationships. We notice that our definitions all contain the fraction $\frac{\text{FOV}}{\Delta x}$, but with the N_r and N_ϕ values containing an additional factor of $1/2$ and π, respectively. The expressions mapping N_r and N_ϕ to N_x can then be written as:

$$N_x = 2N_r$$
$$N_x = \frac{N_\phi}{\pi}$$

We recall that N_r is half of N_x because we are only sampling outwards from the origin, not fully across our space in the radial sequence, thus only require half of the samples to obtain the same FOV. We also recall that N_phi is π times larger than N_x because we would like N_x observations over the upper half circle, naturally spanning π radians. We can then see from these relationships that though we are able to take half of the samples radially in the radial sequence, we take considerably more samples in the azimuthal direction. This is because in order to achieve at least an equal FOV everywhere in the image azimuthally, we have a higher than desired FOV near the center. However, the total number of samples in each case is N_x^2. When we compute this, we notice that the Cartesian method requires, obviously, N_x^2 samples, and the radial requires $2N_x^2/\pi$ samples. Thus, it is more efficient to sample in a radial system than a Cartesian one.

e) Solving the equations established for a two-sided Cartesian approach, as done in part b), with FOV = 30 cm and $\Delta x = \Delta y = 1$ mm yields:

$$N_x = N_y = \frac{300 \text{ mm}}{1 \text{ mm}} = 300$$
$$\Delta k_x = \Delta k_y = \frac{1}{300 \text{ mm}} = 0.003\bar{3} \text{ mm}^{-1}$$
$$k_{x,max} = k_{y,max} = \frac{1}{2 \times 1 \text{ mm}} = 0.5 \text{ mm}^{-1}$$
$$k_{x,min} = k_{y,min} = -0.5 \text{ mm}^{-1}$$

For the case in which we used a one-sided radial approach, as done in part c), and the values given above, we find:

$$N_r = \frac{300 \text{ mm}}{2 \cdot 1 \text{ mm}} = 150$$
$$\Delta k_r = \frac{1}{300 \text{ mm}} = 0.003\bar{3} \text{ mm}^{-1}$$
$$N_\phi = \pi \frac{300 \text{ mm}}{1 \text{ mm}} = 150\pi \approx 471$$
$$\Delta k_\phi = \frac{1}{300 \text{ mm}} = 0.003\bar{3} \text{ mm}^{-1}$$

f) The z dimension is not explicitly represented in k-space. The z component of the image is captured through slice selection.

7 Encoding Matrices

a) The phase encoding matrix of the 3 pixel image shown is:

$$E = \begin{bmatrix} 1 & 1 & 1 \\ e^{i\frac{2\pi}{3}1} & 1 & e^{-i\frac{2\pi}{3}1} \\ e^{i\frac{2\pi}{3}2} & 1 & e^{-i\frac{2\pi}{3}2} \end{bmatrix}$$

b) The general form of the elements in an encoding matrix are $E_{m,n} = e^{-i\frac{2\pi}{\text{FOV}_y}mn}$, where m and n are the row and column indices in the matrix, and FOV_y is the field of view in the image. The amount of phase accrued at a given pixel is related to the relative position of the pixel to the others in the image. For the first encoding (row) no pixels accrue any phase. The first pixel will never accrue phase, but the remaining pixels will all be affected in subsequent encodings. Each subsequent encoding step will spread an additional 2π radians of phase throughout the image evenly, maximizing the spread of the magnetization vectors for each pixel.

c) As there is an additional pixel in the new image, there must be an extra column in the encoding matrix. This is due to the fact that each row represents a separate phase encoding, each column a pixel being observed, and as we are now observing the effect of one more pixel in each observation. As the FOV_y did not increase, we are now adding more than 2π radians of phase in each observation. Since N_y wasn't increased we are not able to add another row to our matrix, either. The new matrix is:

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-2i\frac{2\pi}{3}} & 1 & e^{-i\frac{2\pi}{3}} & e^{-i\frac{4\pi}{3}} \\ e^{-2i\frac{4\pi}{3}} & 1 & e^{-i\frac{4\pi}{3}} & e^{-i\frac{8\pi}{3}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{i\frac{2\pi}{3}} & 1 & e^{-i\frac{2\pi}{3}} & e^{-i\frac{4\pi}{3}} \\ e^{i\frac{4\pi}{3}} & 1 & e^{-i\frac{4\pi}{3}} & e^{-i\frac{8\pi}{3}} \end{bmatrix}$$

d) Adding a pixel but, importantly, not changing the FOV or increasing the number of observations, N_{pe}, will add a term to each row of the matrix that is in fact phase shifted identically to another pixel in the image. We notice this in the matrix above, where our A0 and A3 pixels have in fact the same terms. This is the source of our fold-over aliasing, as both of those pixels contain information about the other, as well. Additionally, this matrix will now be rank-incomplete, and the condition number will go up. This means that the matrix difficult to invert, and we will have a poor reconstruction of the image.

8 Traversing k-Space

a) The gradient intensity in space is akin to velocity in k-space. As the gradient increases, Δk_x increases, resulting in a higher velocity through k-space. The slew rate of the pulse is akin to the acceleration in k-space. As the gradient is reaching it's amplitude, the value of Δk_x is changing, resulting in unequal k-space pixel sizes. If the slew rate is higher, then the acceleration is higher on transitions, and the Δk_x values converge to those achieved by the maximum amplitude at every point in the image. The position in k-space is the integral under the gradient curve. The longer the gradient is played, the further you are along the k-space image.

b) Having an ideal gradient means that the slew rate and gradient maximum amplitude are infinite. Translating this to k-space, we notice that with the acceleration would be infinite on transient events (i.e. a gradient turning on) and zero the remainder of the time. This means that whenever a gradient is applied, the desired gradient amplitude is immediately reached. Derived from this, velocity is then constant, meaning that at all points in the image Δk_x is equal. Again integrating this value, this means that all k-space pixels (i.e. positions) are equally spaced and sized.

c) As non-idealities exist in practice, the slew rate and maximum gradient amplitude are finite. This means that when a gradient is turned on the acceleration is finite non-zero. As a result, the velocity in k-space is changing, resulting in different Δk_x values throughout k-space. The position in k-space then is not traversed evenly, and that all positions are not equal size - in particular, those near the boundary of k-space experience smaller gradient amplitudes, meaning that they are of a lower resolution.

d) Of course there exist physical limitations which prevent the slew rate of the magnets to be ideal as well as the gradient amplitudes to be infinite - these are due to properties of the materials used in the component design. However, there also exist practical reasons why we do not want these idealities in MR. If gradient amplitudes are too large they will cause the subject to twitch, as their body is reacting to different parts of them undergoing different magnetization. Slew rates also cannot be too high or they will cause uncomfortable twitching sensations in the subjects, as makes sense given that the gradients on their own cause this reaction and the slew rate is the change in gradient with respect to time. MRI was designed as a medical imaging modality, and thus it is important to consider the effect on the comfort of the subjects.