Systems Bioengineering 3
Homework 13

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1. (a) Solving for P_n yeilds

$$\begin{aligned}
\beta P_n &= \alpha(n+1)P_{n+1} \\
P_1 &= \frac{\beta}{\alpha}P_0 \\
P_2 &= \frac{\beta}{\alpha}P_1 = \frac{1}{2}\left(\tfrac{\alpha}{\beta}\right)^2 P_0 \\
&\vdots \\
P_n &= \frac{1}{n!}\left(\tfrac{\beta}{\alpha}\right)^n P_0 \\
1 &= \sum_{i=1}^{\infty}\frac{1}{n!}\left(\tfrac{\beta}{\alpha}\right)^n P_0 \\
P_0 &= \frac{1}{\sum_{i=1}^{\infty}\frac{1}{n!}\left(\frac{\beta}{\alpha}\right)^n} \\
P_0 &= e^{-\frac{\beta}{\alpha}} \\
\therefore P_n &= \frac{1}{n!}\left(\tfrac{\beta}{\alpha}\right)^n e^{-\frac{\beta}{\alpha}}
\end{aligned}$$

(b) Expressing $\tilde{P}(\phi)$ as a closed form solutions is done as follows

$$\begin{aligned}
\tilde{P}(\phi) &= \sum_{n=0}^{\infty} e^{-\phi n}P_n \\
&= \sum_{n=0}^{\infty} e^{-\phi n}\frac{1}{n!}\left(\tfrac{\beta}{\alpha}\right)^n e^{-\frac{\beta}{\alpha}} \\
&= e^{-\frac{\beta}{\alpha}}\sum_{n=0}^{\infty}\frac{\left(\frac{\beta e^{-\phi}}{\alpha}\right)^n}{n!} \\
\tilde{P}(\phi) &= e^{\frac{\beta}{\alpha}(e^{-\phi}-1)}
\end{aligned}$$

From this expression, and the definition of the mean and variance (stated below), we compute the mean and variance of this system

$$\begin{aligned}
\langle n \rangle &= \frac{-d}{d\phi} \ln \tilde{P}(\phi)|_{\phi=0} \\
&= \frac{-d}{d\phi} \left(\tfrac{\beta}{\alpha}(e^{-\phi}-1)\right) \\
\langle n \rangle &= \frac{\beta}{\alpha} \\
var(n) = \langle n^2 \rangle - \langle n \rangle^2 &= \frac{d^2}{d\phi^2} \ln \tilde{P}(\phi)|_{\phi=0} \\
&= \frac{d^2}{d\phi^2} \left(\tfrac{\beta}{\alpha}(e^{-\phi}-1)\right) \\
\langle n^2 \rangle - \langle n \rangle^2 &= \frac{\beta}{\alpha}
\end{aligned}$$

(c) From state n, in one transition only states $n+1$ and $n-1$ can be reached; however, over an infinite amount of time any state can be reached by this system.The probability distribution of this system is given by $P(t)$, which can be solved to find the mean time for the first transition, τ.

$$\begin{aligned}
P(\tau) &= (\beta + \alpha n)e^{-(\beta+\alpha n)t} \\
\tau &= \int_0^\infty tP(t)dt \\
\tau &= \frac{1}{\beta + \alpha n}
\end{aligned}$$

(d) Mapping this stochastic dynamic onto a continuous function yeilds the following

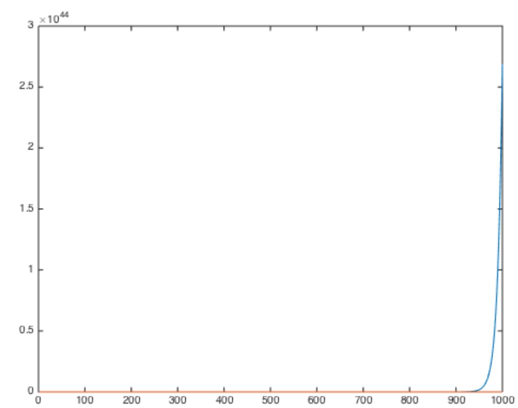
$$\begin{aligned}
\dot{X}(t) &= \beta - \alpha X(t) \\
sX(s) - X(0) &= \frac{\beta}{s} - \alpha X(s) \\
X(s) &= \frac{\beta}{s(s+\alpha)} + \frac{\frac{\beta}{\alpha}}{s+\alpha} \\
X(t) &= \frac{\beta}{\alpha}
\end{aligned}$$

Since $X(t) = \frac{\beta}{\alpha}$ for all t, the mean of this system $\langle t \rangle = \frac{\beta}{\alpha}$, and the variance $\langle t^2 \rangle - \langle t \rangle^2 = 0$.

2. (a) As can be seen from part 2b) above, the variance of n takes the form of $\frac{\beta}{\alpha}$, so the variance here is $S(0) = 10$. Since the relaxation of $X(t)$ is defined, we can compute $S(t)$.

$$\begin{aligned}
X(t) &= \frac{\beta}{\alpha}(1 - e^{-\alpha t}) + X(0)e^{-\alpha t} \\
S(t) &= \frac{X(t) - \mu}{X(0) - \mu} S(0) \\
&= S(0)e^{-\alpha t} \\
&= \frac{\beta}{\alpha} e^{-\alpha t} \\
&= 10e^{-\alpha t}
\end{aligned}$$

(b) The value of $\langle n \rangle$ is approximately 10. You can see from the figure below that the two plots are identical to one another - the two being the analytical and experimental solutions to this equation.

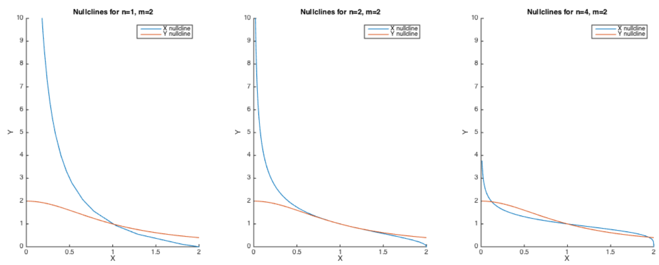


(c)

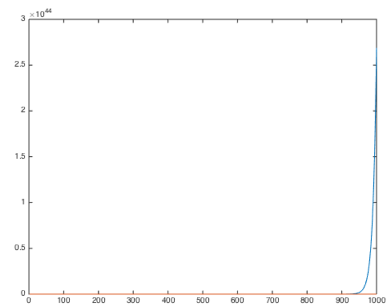
3. (a) The nullclines for the equations given above were found to be:

$$\begin{aligned}
Y(t) &= \left(\frac{2}{X(t)} - 1\right)^{\frac{1}{n}}; &\quad& \text{nullcline for } \dot{X}(t) = 0 \\
Y(t) &= \frac{2}{1 + X(t)^m}; && \text{nullcline for } \dot{Y}(t) = 0
\end{aligned}$$

The nullclines of plotted for values of $n = (1, 2, 4)$ and $m = (2)$ are as follows.



(b) For the case when $n = 4$ and $m = 2$, we notice that the nullclines cross at three locations. If we compute the roots of the system $Z = Y_{null} - X_{null}$ we can find the locations where the two nullclines intersect. Performing this analysis the roots were found to be: $(0.1247, 1.9694)$, $(1, 1)$, $(1.9397, 0.4200)$, where the center/symmetric fixed point occurs at $(1, 1)$. Shown below is a plot of this system with the fixed points found above marked on the plot.



(c) The Jacobian of this system can be computed as follows.

$$
\begin{aligned}
J &= \begin{bmatrix} \frac{\partial \dot{X}}{\partial X} & \frac{\partial \dot{X}}{\partial Y} \\ \\ \frac{\partial \dot{Y}}{\partial X} & \frac{\partial \dot{Y}}{\partial Y} \end{bmatrix} \\
&= \begin{bmatrix} -1 & \frac{-2nY^{n-1}}{(1+Y^n)^2} \\ \\ \frac{-2nX^{m-1}}{(1+X^m)^2} & -1 \end{bmatrix}
\end{aligned}
$$

We can simplify the above, and evaluate it at the fixed point $(1,1)$

$$J = \begin{bmatrix} -1 & \frac{nX^2Y^{n-1}}{2} \\ \\ \frac{mY^2X^{m-1}}{2} & -1 \end{bmatrix}_{(1,1)}$$
$$J = \begin{bmatrix} -1 & \frac{n}{2} \\ \\ \frac{m}{2} & -1 \end{bmatrix}$$

The eigen values can be computed by solving $0 = |J - \lambda I|$

$$0 = \left| \begin{bmatrix} -1 & \frac{n}{2} \\ \\ \frac{m}{2} & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ \\ 0 & \lambda \end{bmatrix} \right|$$
$$0 = \begin{vmatrix} -1-\lambda & \frac{n}{2} \\ \\ \frac{m}{2} & -1-\lambda \end{vmatrix}$$
$$0 = (-1-\lambda)^2 - \frac{mn}{4}$$
$$\lambda = -1 \pm \sqrt{\frac{mn}{4}}$$

In order for the fixed point to be unstable, the following condition must apply

$$0 < \lambda$$
$$0 < -1 + \sqrt{\frac{mn}{4}}$$
$$1 < \sqrt{\frac{mn}{4}}$$
$$4 < mn$$

The condition on the Hill coefficients that permits patterning is a large number of components combine between the two systems. If the product of the system compontents is larger than 4, the system will be unstable and pattern at the symmetric fixed point.