d) $SV = V\Lambda$, where V is the orthonormal basis of S

$SV = \alpha_1 v_1 \lambda_1 + \alpha_2 v_2 \lambda_2 + \dots + \alpha_n v_n \lambda_n$

$\because v_i^T v_j = 0$ for $i \neq j$, $v_i^T v_j = 1$ for $i = j$

[citation]: Wilson, Richard. Math, Caltech. 2010-2011. "Real symmetric matrices"

$V^T S V = \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 + \dots + \alpha_n^2 \lambda_n$

$\therefore$ iff $\lambda_i > 0$ for all $i = 1 \to n$, is $S \succ 0$. Also, iff $\lambda_i \geq 0$ for all $i = 1 \to n$, is $S \succeq 0$. This assumes that not all $\alpha_i = 0$.

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e) $\max\limits_{\|x\|_2^2=1} x^T S x = \lambda_1$, $\min\limits_{\|x\|_2^2=1} x^T S x = \lambda_n$ $\qquad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

$$\mathcal{L}(x, \lambda) = x^T S x + \lambda(1 - x^T x)$$

$$\frac{\delta \mathcal{L}}{\delta x} = 2Sx - 2\lambda x$$

$$0 = 2Sx - 2\lambda x$$

$$Sx = \lambda x$$

$$x^T S x = \lambda \longrightarrow \therefore \min x^T S x = \min(\lambda) = \lambda_n$$

$$\max x^T S x = \max(\lambda) = \lambda_1$$

2.a) $A = U \Sigma V^T$

$AV = U \Sigma V^T V$

$AV = \Sigma U$

$A v_i = \sigma_i u_i$

$A = U \Sigma V^T$

$U^T A = U^T U \Sigma V^T$

$(U^T A = \Sigma V^T)^T$

$A^T U = \Sigma V$

$A^T u_i = \sigma_i v_i$

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b) $A = \begin{bmatrix} u_r & u_{r+1}^{\perp} \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_r^T \\ v_{r+1}^{\perp T} \end{bmatrix}$

$A = U_r \Sigma_r V_r^T$

For u_r, spanning from $i = 1 \to r$, all values of A are included, i.e. range$(A) = \text{Span}(u_r)$ not $\text{Span}(u_n)$ $\because \Sigma_\alpha = 0$, $\alpha = r+1 \to n$.

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c) From the eqn above, we see that $v_{r+1}^{\perp T}$ elements are always eliminated, by being multiplied by zero, $\therefore$ they make up the kernel space of A. They make up the whole kernel as they span from the range of A to n.