p3 GK

d) $\|A\|_F^2 = \sum_{i,j}^{n,m} a_{ij}^2$

$= \sum_{i,j}^{n,m} u_{ij}^2 \sigma_{ij}^2 v_{ij}^2$

$\because \; u_{ij}^2 = u_{ij} u_{ji} = I$ (same for $v_{ij}^2$)

$= \sum_{i,j}^{n,m} \sigma_{ij}^2$

$\because \; \Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_r \end{bmatrix}$ if $i \neq j$, $\sigma_{ij} = 0$
$i, j > r$, $\sigma_{ij} = 0$

$= \sum_k^r \sigma_k^2$

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e) $\min_{\|x\|_2^2=1} \|Ax\|_2^2 = \min_{\|x\|_2^2=1} x^T A^T A x$

$= \min_{\|x\|_2^2=1} x^T V \Sigma U^T U \Sigma V^T x$

$= \min_{\|x\|_2^2=1} x^T (V \Lambda V) x$

↳ This is of the same form as 1·e).

$\therefore$ we can see that $\min_{\|x\|_2^2=1} \|Ax\|_2^2 = \min(\sigma) = \sigma_m$

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3· a) $A = U_r \Sigma_r V_r^T$

$\because \; A A^\dagger A = A, \quad A A^\dagger = I$

$\because \; A A^\dagger = I$

$U_r \Sigma_r V_r^T A^\dagger = I$

$U_r \Sigma_r V_r^T \underbrace{V_r \Sigma_r^{-1} U_r^T}_{A^\dagger} = I$ $\therefore A^\dagger = V_r \Sigma_r^{-1} U_r^T$

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b) $\min \|Ax - b\|_2^2 = \min (Ax-b)^T (Ax-b) = f$

$\frac{df}{dx} = 0 = 2A^T(Ax-b)$

$= 2A^T A x - 2A^T b$

$2A^T A x = 2A^T b$

$x = (A^T A)^{-1} A^T b$ → by definition, $A^\dagger$

$x = A^\dagger b$

$\therefore$ the unique soln is $x^*$ when $b \neq 0$.