580.439/639 Homework #9

Not to be handed in

Problem 1

Consider learning rules similar to the Perceptron Learning Rule (PLR) using a *linear* perceptron, in which $v = \vec{w} \cdot \vec{x}$. The bias term has been absorbed into the weight vector $\vec{w}$ and there is no squashing function. Consider the problem of supervised learning to make this perceptron give responses $\left\{ v^p \middle| p = 1 \ldots P \right\}$ to input patterns $\left\{ \vec{u}^p \middle| p = 1 \ldots P \right\}$. The dimension of the weight and input vectors is N.

Part a) Define the *overlap matrix* $\mathbf{Q} = [q_{jk}]$ as $q_{jk} = \sum_{i=1}^{N} u_i^j u_i^k$, where u_j^i is the j^{th} component of the i^{th} pattern vector. Let the perceptron's weights be

$$\vec{w} = \sum_{j=1}^{P} \sum_{k=1}^{P} v^j \left(\mathbf{Q}^{-1} \right)_{jk} \vec{u}^k \quad ,$$

where $\left(\mathbf{Q}^{-1} \right)_{jk}$ means the jk^{th} element of the inverse of $\mathbf{Q}$. Write a matrix equation for $\mathbf{Q}$ in terms of the NxP pattern matrix $\mathbf{U}$ whose columns are the input patterns $\vec{u}^p$ and show that this weight vector gives output v^l to input pattern $\vec{u}^l$.

Part b) Under what conditions on the input patterns does the analysis of part a) work?

Part c) Define the error in a linear perceptron as the difference between the output and the desired output summed across the input patterns:

$$E = \frac{1}{2} \sum_{j=1}^{P} \left(\vec{w} \cdot \vec{u}^j - v^j \right)^2$$

The weights could be found by gradient descent in which $\vec{w}_{new} = \vec{w}_{old} - \varepsilon \vec{\nabla} E$ where ε is the learning rate. Show that this leads to a learning rule that looks the same as the PLR (except for the missing sgn[] functions).

Problem 2

Part a) Show that the function below behaves like a Lyapunov function for the Hopfield network discussed in class, in that H decreases monotonically when the network's state values change in time. S_j is the output value of the j^{th} neuron as usual.

$$H = -\frac{1}{2} \sum_{i,j} w_{ij} S_i S_j \qquad (*)$$

where the summation is taken over all pairs of states. To do this, you will need to assume that $w_{ij} = w_{ji}$. Make it clear why this assumption is needed. Hint: consider H'-H, the change in the energy function when a single state flips, that is H changes to H' when S_k changes to S_k' and no other states change. Show that H'-H is always negative when S_k changes according to the discrete update rule for the Hopfield net:

$$S_k' = \text{sgn}\left[\sum_{j=1}^{N} w_{kj}\, S_j\right]$$

Have you proven that Eqn. (*) is a Lyapunov function, as defined in class?

Part b) To be a Lyapunov function, H must be minimized at an equilibrium point. We want the equilibrium points to be at patterns stored in the net. Suppose a Hopfield net has only one pattern $\vec{u} = [u_1, u_2, \cdots, u_N]$ stored. Consider the function

$$H = -\frac{1}{2N}\left(\sum_{i=1}^{N} S_i u_i\right)^2 \qquad (**)$$

Argue that this is minimized when S_i=u_i, i.e. the pattern is an equilibrium point. Is the minimum unique (i.e. can you find another equilibrium point)? Now use the definition of weights for the Hopfield net to show that (*) and (**) are the same. For the case of a net with many patterns stored, (**) could be summed over all the patterns; would the same result hold? Is the function (**) positive definite? If not, what can be done to make it so?