Computer Vision (600.461/600.661)
Homework 4: Feature Matching and Optical Flow

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October 17, 2014

1. **(15 Points) Corner localization via quadratic fit.**

$$E(Q,b,c) = \sum_{u=1} w(x+u)\left[\tfrac{1}{2}(x+u)^T Q(x+u) + b^T(x+u) + c - r(x+u)\right]^2$$

From here, let $x_i = x + u$.

$$\begin{aligned}
\min_{Q,b,c} E(Q,b,c) &= \min_{Q,b,c} \sum_{i=1} w_i \left[\tfrac{1}{2}x_i^T Q x_i + b^T x_i + c - r_i\right]^2 \\
\frac{\partial}{\partial c}E &= \frac{\partial}{\partial c}\sum_{i=1} w_i \left[\tfrac{1}{2}x_i^T Q X_i + b^T x_i + c - r_i\right]^2 \\
0 &= 2\sum_{i=1} w_i \left[\tfrac{1}{2}x_i^T Q X_i + b^T x_i - r_i\right] + 2\sum_{i=1} w_i c \\
c\sum_{i=1} w_i &= -\sum_{i=1} w_i \left[\tfrac{1}{2}x_i^T Q X_i + b^T x_i - r_i\right] \\
c^* &= -\frac{\sum_{i=1} w_i \left[\tfrac{1}{2}x_i^T Q X_i + b^T x_i - r_i\right]}{\sum_{i=1} w_i} \\
c^* &= -w\left(\tfrac{1}{2}\bar{x}^T Q \bar{x} - b^T \bar{x} + \bar{r}\right)
\end{aligned}$$

Now, subbing c^* into the error equation we get,

$$\begin{aligned}
\min_{Q,b,c^*} E(Q,b,c^*) &= \min_{Q,b,c^*} \sum_{i=1} w_i \left[\tfrac{1}{2}x_i^T Q x_i + b^T x_i + c^* - r_i\right]^2 \\
&= \min_{Q,b,c^*} \sum_{i=1} w_i \left[\tfrac{1}{2}x_i^T Q x_i + b^T x_i - \tfrac{1}{2}\bar{x}^T Q \bar{x} - b^T \bar{x} + \bar{r} - r_i\right]^2 \\
&= \min_{Q,b,c^*} \sum_{i=1} w_i \left[\left(\tfrac{1}{2}x_i^T Q x_i - \tfrac{1}{2}\bar{x}^T Q \bar{x}\right) + \left(b^T x_i - b^T \bar{x}\right) - \left(r_i - \bar{r}\right)\right]^2 \\
&= \min_{Q,b,c^*} w\left[\tfrac{1}{2}\tilde{x}^T Q \tilde{x} + b^T \tilde{x} - \tilde{r}\right]^2 \\
\frac{\partial}{\partial b}E &= \frac{\partial}{\partial b} w\left[\tfrac{1}{2}\tilde{x}^T Q \tilde{x} + b^T \tilde{x} - \tilde{r}\right]^2 \\
0 &= 2w\left[\tfrac{1}{2}\tilde{x}^T Q \tilde{x} + b^T \tilde{x} - \tilde{r}\right]\tilde{x}^T \\
0 &= 2w\left[\tfrac{1}{2}\tilde{x}^T Q \tilde{x}\tilde{x}^T + b^T \tilde{x}\tilde{x}^T - \tilde{r}\tilde{x}^T\right] \\
b^* &= \left(\tfrac{1}{2}\tilde{x}^T Q \tilde{x}\tilde{x}^T - \tilde{r}\tilde{x}^T\right)\left(\tilde{x}\tilde{x}^T\right)^{-1}
\end{aligned}$$

Now, subbing b^* into the same equation, this time isolating for our last variable, Q,

$$\min_{Q,b^*,c^*} E(Q,b^*,c^*) = \min_{Q,b^*,c^*} w\left[\tfrac{1}{2}\tilde{x}^T Q \tilde{x} + \left(\left(\tfrac{1}{2}\tilde{x}^T Q \tilde{x}\tilde{x}^T - \tilde{r}\tilde{x}^T\right)\left(\tilde{x}\tilde{x}^T\right)^{-1}\right)^T \tilde{x} - \tilde{r}\right]^2$$

From this form we can see that the solution will be in the form of least squares for Q. Since it is a maximum, Q must be negative semi-definite. Therefore we can constrain the solution to be only negative eigen values, λ.

2. **(20 Points) Feature point matching under a 2D rigid body motion.**

$$E(R,t) = \sum_{j=1} \|y_j - Rx_j - t\|_2^2$$

We know that,from class,

$$\begin{aligned}
\min_{R,t} & E(R,t) : \\
t^* &= \bar{Y} - R\bar{X} \\
R^* &= \min_R \|Y - RX\|_F^2 \\
&= \min_R \|Y\|_F^2 - 2\langle Y, RX\rangle + \|RX\|_F^2 \\
&= \min_R - \langle Y, RX\rangle \\
&= \max_R \langle Y, RX\rangle \\
R^* &= \max_R \text{trace}(Y^T RX)
\end{aligned}$$

We also know from class that R is of the form:

$$\begin{aligned}
R &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
R &= I\cos\theta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin\theta \\
\therefore R^* &= \begin{pmatrix} \cos\theta^* & -\sin\theta^* \\ \sin\theta^* & \cos\theta^* \end{pmatrix}
\end{aligned}$$

Where,

$$\begin{aligned}
\theta^* &= \max_\theta \text{trace}(Y^Y RX)^T \\
&= \max_\theta \text{trace}(X^T R^T Y) \\
&= \max_\theta \text{trace}(X^T(I\cos\theta + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sin\theta)Y) \\
&= \max_\theta \text{trace}(X^T Y\cos\theta) + \text{ trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y\sin\theta) \\
0 &= \frac{\partial}{\partial\theta}\left[\text{trace}(X^T Y\cos\theta) + \text{ trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y\sin\theta)\right] \\
\text{trace}(X^T Y)\sin\theta &= \text{ trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y)\cos\theta
\end{aligned}$$

We can now see that

$$\theta^* = \arctan\left(\frac{\text{trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y)}{\text{trace}(X^T Y)}\right)$$

$$\sin(\theta^*) = \frac{\text{trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y)}{\sqrt{\text{trace}(X^T Y)^2 + \text{trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y)^2}}$$

$$\cos(\theta^*) = \frac{\text{trace}(X^T Y)}{\sqrt{\text{trace}(X^T Y)^2 + \text{trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y)^2}}$$

$\therefore$ we can sub in $\sin(\theta^*)$ and $\cos(\theta^*)$ into R and find that

$$R = \frac{\begin{pmatrix} \text{trace}(X^T Y) & -\text{trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y) \\ \text{trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y) & \text{trace}(X^T Y) \end{pmatrix}}{\sqrt{\text{trace}(X^T Y)^2 + \text{trace}(X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y)^2}}$$

3. **(15 Points) Optical flow with changes in illumination.**

$$I(x_u, y+v, t+1) = aI(x,y,t) + b$$

Notice that the above matches the form

$$J(x,y,t) = aI(x,y,t) + b$$

Recognizing the above as a standard affine transform we can conclude that

$$\begin{aligned} b &= \bar{J} - a\bar{I} \\ a &= \tilde{J}\tilde{I}^T(\tilde{I}\tilde{I}^T)^{-1} \end{aligned}$$

Where,

$$\begin{aligned} J &= I(x+u, y+v, t+1) \\ \tilde{J} &= J - \bar{J} \\ \tilde{I} &= I - \bar{I} \end{aligned}$$

Now, applying the brightness constancy contraint (BCC) and making the assumption of small motion,

$$\begin{aligned} I(x+u, y+v, t+1) = aI(x,y,t) + b + uI_x + vI_y \\ 0 &= aI(x,y,t) + b - I(x+u, y+v, t+1) + uI_x + vI_y \\ 0 &= (a-1)I(x,y,t) + b(I(x,y,t) - I(x+u, y+v, t+1)) + uI_x + vI_y \\ 0 &= (a-1)I(x,y,t) + b + I_t + uI_x + vI_y \\ 0 &= (a-1)I(x,y,t) + b + I_t + \nabla I \begin{bmatrix} u & v \end{bmatrix} \end{aligned}$$

From here, we have 1 equation and 2 unknowns, u and v. We now make an assumption that nearby points have a constant optical flow, u and v, and solve for multiple points at the same time. We will choose a 5x5 neighborhood around our point of interest.

$$0 = (a-1)I(p_i) + b + I_t(p_i) + \nabla I(p_i) \begin{bmatrix} u & v \end{bmatrix}$$

$$\begin{bmatrix} I_x(p_1) & I_y(p_1) \\ I_x(p_2) & I_y(p_2) \\ \vdots & \vdots \\ I_x(p_{25}) & I_y(p_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(p_1) \\ I_t(p_2) \\ \vdots \\ I_t(p_{25}) \end{bmatrix} - (a-1) \begin{bmatrix} I(p_1) \\ I(p_2) \\ \vdots \\ I(p_{25}) \end{bmatrix} - \begin{bmatrix} b(p_1) \\ b(p_2) \\ \vdots \\ b(p_{25}) \end{bmatrix}$$

Where we notice the above equation takes the form

$$Ax = b$$

This form is commonly seen and can be solved using the least squares method to give the following general and unque solutions, respectively

$$\min \|Ax - b\|^2$$
$$A^T A x = A^T b$$
$$x = (A^T A)^{-1} A^T b$$

$$\min \left\| \begin{bmatrix} I_x(p_1) & I_y(p_1) \\ I_x(p_2) & I_y(p_2) \\ \vdots & \vdots \\ I_x(p_{25}) & I_y(p_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \left(\begin{bmatrix} I_t(p_1) \\ I_t(p_2) \\ \vdots \\ I_t(p_{25}) \end{bmatrix} - (a-1) \begin{bmatrix} I(p_1) \\ I(p_2) \\ \vdots \\ I(p_{25}) \end{bmatrix} - \begin{bmatrix} b(p_1) \\ b(p_2) \\ \vdots \\ b(p_{25}) \end{bmatrix} \right) \right\|^2$$

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum I_x Q \\ \sum I_y Q \end{bmatrix}$$

Where,

$$Q = \begin{bmatrix} I_t(p_1) \\ I_t(p_2) \\ \vdots \\ I_t(p_{25}) \end{bmatrix} - (a-1) \begin{bmatrix} I(p_1) \\ I(p_2) \\ \vdots \\ I(p_{25}) \end{bmatrix} - \begin{bmatrix} b(p_1) \\ b(p_2) \\ \vdots \\ b(p_{25}) \end{bmatrix}$$