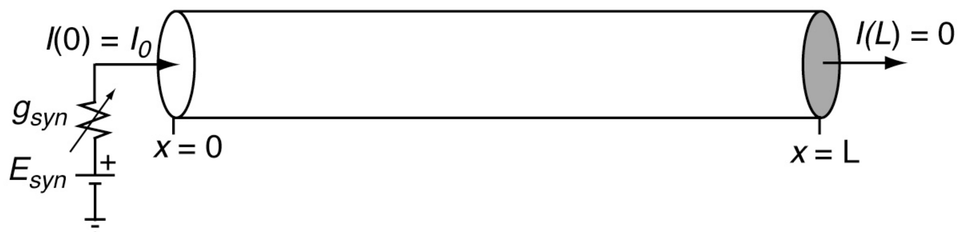
Systems Biology II 580.422 2015 Homework 3

Problem 1

In this problem, you will work out the solutions to the linear cable equation for a finite cable. Consider the problem sketched below. The cable has length L cm. At its left end, a current $I_0(t)$ is injected into the cable by a synapse (as drawn) or by an electrode. Ignore the details of the source of the current for the first part of the problem. At its right end, the cable is terminated by a closed circuit, $I(L) = 0$. This closed end boundary condition is often used for the end of a dendritic tree, assuming that no current flows out of the ends of the terminal branches of the tree.



The problem that needs to the solved is the following, using the same notation as in the course notes and lecture slides:

$$\lambda^2 \frac{\partial^2 V}{\partial x^2} = \tau \frac{\partial V}{\partial t} + V$$
$$V(x, t=0) = 0$$
$$I_i(x=0,t) = I_0(t) \quad \text{and} \quad I_i(L) = 0$$

Consistent with the usual requirements for boundary conditions, there is an initial condition in time (V is at the resting potential everywhere) and two conditions in space.

Part a) Show that the spatial boundary conditions are the same as the following (Hint: go back to the derivation of the cable equation handout, p. 7). Recall that $I_i(x,t)$ is the axial current at point x.

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = -r_i I_0(t) \quad \text{and} \quad \left.\frac{\partial V}{\partial x}\right|_{x=L} = 0$$

Part b) Write the Laplace transform w.r.t. time t for the problem defined above. To do this, Laplace transform the differential equation and the boundary conditions. Explain what happens to the initial condition in time. Assume in doing this that the order of Laplace transformation and any other linear operation (like differentiation w.r.t. x) can be interchanged. That is the Laplace transform of $\partial V(x,t)/\partial x$ is equal to $\partial \bar{V}(s,x)/\partial x$. Note that no functional form for $I_0(t)$ has been given yet.

The result of part b) should be a homogeneous ordinary differential equation in x with two boundary conditions.

Part c) Write the solution of the O.D.E. derived in part b) and apply the boundary conditions. The result should be something like

$$\bar{V}(x,s) = A(s)\cosh\left[g(s)x\right]$$

Write functions for *A(s)* and *g(s)*. You can write the solution in terms of exponentials, but it is more compact to use cosh(x) = $(e^x+e^{-x})/2$ and sinh(x) = $(e^x-e^{-x})/2$.

Part d) This is a difficult Laplace transform to invert analytically, but some understanding of the solution can be gotten by considering a special case. Consider the D.C. steady state, i.e. the solution in response to a steady current long after it was applied, long enough that all transients have died away and $\partial V/\partial t = 0$. This condition can be derived from your solution to part c) by using the final value theorem in which

$$V(x,t \to \infty) = \lim_{s \to 0} s\bar{V}(x,s)$$

Derive an equation for the steady potential $V(x,t \to \infty)$ in the cylinder.

Part e) Plot the steady potential as $V(x,\infty)/\lambda r_i I_0$ for x = [0, L] in cylinders of length L = 0.5λ, λ, 2λ, and 4λ. Compare the result with the potential, derived in class and in the handouts, in an infinite cylinder in the same D.C. steady state.