# Selected Solutions to Hoffman and Kunze's Linear Algebra Second Edition

Greg Kikola

January 12, 2020

# Contents

# Chapter 1

# Linear Equations

## 1.2 Systems of Linear Equations

### 1.2.1 Exercise 1

Verify that the set of complex numbers described in Example 4 is a subfield of $C$.

*Solution.* The set in Example 4 consisted of all complex numbers of the form $x + y\sqrt{2}$, where $x$ and $y$ are rational. Call this set $F$.

Note that $0 = 0 + 0\sqrt{2} \in F$ and $1 = 1 + 0\sqrt{2} \in F$. If $\alpha = a + b\sqrt{2}$ and $\beta = c + d\sqrt{2}$ are any elements of $F$, then

$$\alpha + \beta = (a + c) + (b + d)\sqrt{2} \in F,$$

and

$$-\alpha = -a - b\sqrt{2} \in F.$$

We also have

$$\begin{aligned} \alpha\beta &= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd \\ &= (ac + 2bd) + (ad + bc)\sqrt{2} \in F \end{aligned}$$

and, provided $\alpha$ is nonzero,

$$\alpha^{-1} = \frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2} \in F.$$

Since $F$ contains 0 and 1 and is closed under addition, multiplication, additive inverses, and multiplicative inverses, $F$ is a subfield of $C$. $\square$

### 1.2.2 Exercise 2

Let $F$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} x_1 - x_2 &= 0 \qquad & 3x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 0 \qquad & x_1 + x_2 &= 0 \end{aligned}$$

*Solution.* The systems are equivalent. For the first system, we can write

$$\begin{aligned} x_1 - x_2 &= (3x_1 + x_2) - 2(x_1 + x_2) = 0, \\ 2x_1 + x_2 &= \tfrac{1}{2}(3x_1 + x_2) + \tfrac{1}{2}(x_1 + x_2) = 0. \end{aligned}$$

And for the second system,

$$\begin{aligned} 3x_1 + x_2 &= \tfrac{1}{3}(x_1 - x_2) + \tfrac{4}{3}(2x_1 + x_2) = 0, \\ x_1 + x_2 &= -\tfrac{1}{3}(x_1 - x_2) + \tfrac{2}{3}(2x_1 + x_2) = 0. \end{aligned}$$

□

### 1.2.3 Exercise 3

Test the following systems of equations as in Exercise 1.2.2.

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

*Solution.* For the first system, we have

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= -(x_1 - x_3) + (x_2 + 3x_3) = 0, \\ x_1 + 3x_2 + 8x_3 &= (x_1 - x_3) + 3(x_2 + 3x_3) = 0, \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= \tfrac{1}{2}(x_1 - x_3) + (x_2 + 3x_3) = 0. \end{aligned}$$

For the second system, we have

$$\begin{aligned} x_1 - x_3 &= -\tfrac{3}{4}(-x_1 + x_2 + 4x_3) + \tfrac{1}{4}(x_1 + 3x_2 + 8x_3) + 0(\tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3), \\ x_2 + 3x_3 &= \tfrac{1}{4}(-x_1 + x_2 + 4x_3) + \tfrac{1}{4}(x_1 + 3x_2 + 8x_3) + 0(\tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3). \end{aligned}$$

So, the two systems are equivalent. □

### 1.2.4 Exercise 4

Test the following systems as in Exercise 1.2.2.

$$\begin{aligned} 2x_1 + (-1 + i)x_2 \qquad + x_4 &= 0 \\ 3x_2 - 2ix_3 + 5x_4 &= 0 \end{aligned} \qquad \begin{aligned} \left(1 + \frac{i}{2}\right)x_1 + 8x_2 - ix_3 - x_4 &= 0 \\ \tfrac{2}{3}x_1 - \tfrac{1}{2}x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

*Solution.* Call the equations in the system on the left $L_1$ and $L_2$, and the equations on the right $R_1$ and $R_2$. If $R_1 = aL_1 + bL_2$ then, by equating the coefficients of $x_3$, we get

$$-i = -2ib,$$

which implies that $b = 1/2$. By equating the coefficients of $x_1$, we get

$$1 + \frac{i}{2} = 2a,$$

so that

$$a = \frac{1}{2} + \frac{1}{4}i.$$

Now, comparing the coefficients of $x_4$, we find that

$$-1 = a + 5b = \frac{1}{2} + \frac{1}{4}i + \frac{5}{2} = 3 + \frac{1}{4}i,$$

which is clearly a contradiction. Therefore the two systems are *not* equivalent. □

### 1.2.5 Exercise 5

Let $F$ be a set which contains exactly two elements, 0 and 1. Define an addition and multiplication by the tables:

| $+$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

Verify that the set $F$, together with these two operations, is a field.

*Solution.* From the symmetry in the tables, we see that both operations are commutative.

By considering all eight possibilities, one can see that $(a+b)+c = a+(b+c)$. And one may in a similar way verify that $(ab)c = a(bc)$, so that associativity holds for the two operations.

$0 + 0 = 0$ and $0 + 1 = 1$ so $F$ has an additive identity. Similarly, $1 \cdot 0 = 0$ and $1 \cdot 1 = 1$ so $F$ has a multiplicative identity.

The additive inverse of 0 is 0 and the additive inverse of 1 is 1. The multiplicative inverse of 1 is 1. So $F$ has inverses.

Lastly, by considering the eight cases, one may verify that $a(b+c) = ab+ac$. Therefore distributivity of multiplication over addition holds and $F$ is a field. □

### 1.2.7 Exercise 7

Prove that each subfield of the field of complex numbers contains every rational number.

*Proof.* Let $F$ be a subfield of $C$ and let $r = m/n$ be any rational number, written in lowest terms. $F$ must contain 0 and 1, so if $r = 0$ then we are done. Now assume $r$ is nonzero.

Since $1 \in F$, and $F$ is closed under addition, we know that $1 + 1 = 2 \in F$. And, if the integer $k$ is in $F$, then $k + 1$ is also in $F$. By induction, we see that all positive integers belong to $F$. We also know that all negative integers are in $F$ because $F$ is closed under additive inverses. So, in particular, $m \in F$ and $n \in F$.

Now $F$ is closed under multiplicative inverses, so $n \in F$ implies $1/n \in F$. Finally, closure under multiplication shows that $m \cdot (1/n) = m/n = r \in F$. Since $r$ was arbitrary, we can conclude that all rational numbers are in $F$. □

### 1.2.8 Exercise 8

Prove that each field of characteristic zero contains a copy of the rational number field.

*Proof.* Let $F$ be a field of characteristic zero. Define the map $f: Q \to F$ (where $Q$ denotes the rational numbers) as follows. Let $f(0) = 0_F$ and $f(1) = 1_F$, where $0_F$ and $1_F$ are the additive and multiplicative identities, respectively, of $F$. Given a positive integer $n$, define $f(n) = f(n-1) + 1_F$ and $f(-n) = -f(n)$. If a rational number $r = m/n$ is not an integer, define $f(r) = f(m) \cdot (f(n))^{-1}$.

First we show that the function $f$ preserves addition and multiplication. A simple induction argument will show that, in the case of integers $m$ and $n$, we have

$$f(m+n) = f(m) + f(n) \quad \text{and} \quad f(mn) = f(m)f(n).$$

Now let $r_1 = m_1/n_1$ and $r_2 = m_2/n_2$ be rational numbers in lowest terms. Then, by the definition of $f$,

$$\begin{aligned}
f(r_1 + r_2) &= f((m_1n_2 + m_2n_1)/(n_1n_2)) \\
&= f(m_1n_2 + m_2n_1)f(n_1n_2)^{-1} \\
&= (f(m_1)f(n_2) + f(m_2)f(n_1))f(n_1)^{-1}f(n_2)^{-1} \\
&= f(m_1)f(n_1)^{-1} + f(m_2)f(n_2)^{-1} \\
&= f(r_1) + f(r_2).
\end{aligned}$$

Likewise,

$$f(r_1r_2) = f(m_1)f(m_2)f(n_1)^{-1}f(n_2)^{-1} = f(r_1)f(r_2).$$

(Formally, this shows that $f$ is a ring homomorphism.)

We will next show that the function $f$ is one-to-one. If $r_1 = m_1/n_1$ and $r_2 = m_2/n_2$ are rational numbers in lowest terms, then $f(r_1) = f(r_2)$ implies

$$f(m_1)f(n_1)^{-1} = f(m_2)f(n_2)^{-1}$$

or

$$f(m_1)f(n_2) = f(m_2)f(n_1).$$

This implies

$$f(m_1n_2) = f(m_2n_1).$$

Now if $m_1n_2 \neq m_2n_1$, then this would imply that $F$ does not have characteristic zero. So $m_1n_2 = m_2n_1$ and so $r_1 = r_2$.

What we have shown is that every rational number corresponds to a distinct element of $F$, and that the operations of addition and multiplication of rational numbers is preserved by this correspondence. So $F$ contains a copy of $Q$. $\square$

# 1.3 Matrices and Elementary Row Operations

## 1.3.1 Exercise 1

Find all solutions to the system of equations

$$\begin{aligned} (1-i)x_1 - ix_2 &= 0 \\ 2x_1 + (1-i)x_2 &= 0. \end{aligned} \tag{1.1}$$

*Solution.* Using elementary row operations, we get

$$\begin{bmatrix} 1-i & -i \\ 2 & 1-i \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1-i & -i \\ 1 & \frac{1}{2}-\frac{1}{2}i \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2}-\frac{1}{2}i \end{bmatrix}.$$

So the system in (1.1) is equivalent to

$$x_1 + \left(\frac{1}{2} - \frac{1}{2}i\right) x_2 = 0.$$

Therefore, if $c$ is any complex scalar, then $x_1 = (-1+i)c$ and $x_2 = 2c$ is a solution to (1.1). □

## 1.3.2 Exercise 2

If

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}$$

find all solutions of $AX = 0$ by row-reducing $A$.

*Solution.* We get

$$\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{5}{3} & -\frac{1}{3} \\ 0 & -\frac{8}{3} & -\frac{2}{3} \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{1}{5} \\ 0 & -\frac{8}{3} & -\frac{2}{3} \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & \frac{3}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & -\frac{6}{5} \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & \frac{3}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $AX = 0$ has only the trivial solution. □

## 1.3.3 Exercise 3

If

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

find all solutions of $AX = 2X$ and all solutions of $AX = 3X$. (The symbol $cX$ denotes the matrix each entry of which is $c$ times the corresponding entry of $X$.)

*Solution.* The matrix equation $AX = 2X$ corresponds to the system of linear equations

$$\begin{aligned} 6x_1 - 4x_2 \quad\quad &= 2x_1 \\ 4x_1 - 2x_2 \quad\quad &= 2x_2 \\ -1x_1 \quad\quad + 3x_3 &= 2x_3, \end{aligned}$$

or, equivalently,

$$\begin{aligned} 4x_1 - 4x_2 \quad\quad &= 0 \\ 4x_1 - 4x_2 \quad\quad &= 0 \\ -1x_1 \quad\quad + x_3 &= 0. \end{aligned}$$

This system is homogeneous, and can be represented by the equation $BX = 0$, where $B$ is given by

$$B = \begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

$B$ can be row-reduced:

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore any solution of $AX = 2X$ will have the form

$$(x_1, x_2, x_3) = (a, a, a) = a(1, 1, 1),$$

where $a$ is a scalar.

Similarly, the equation $AX = 3X$ can be solved by row-reducing

$$\begin{bmatrix} 3 & -4 & 0 \\ 4 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, solutions of $AX = 3X$ have the form

$$(x_1, x_2, x_3) = (0, 0, b) = b(0, 0, 1),$$

where $b$ is a scalar. □

### 1.3.4 Exercise 4

Find a row-reduced matrix which is row-equivalent to

$$A = \begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix}.$$

*Solution.* Using the elementary row operations, we get

$$\begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & -1+i & 0 \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 1+i & -1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is row-equivalent to $A$. □

### 1.3.5 Exercise 5

Prove that the following two matrices are *not* row-equivalent:

$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix}.$$

*Proof.* By performing row operations on the first matrix, we get

$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & c & 3 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We see that this matrix is row-equivalent to the identity matrix. The corresponding system of equations has only the trivial solution.

For the second matrix, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

The system of equations corresponding to this matrix has nontrivial solutions. Therefore the two matrices are not row-equivalent. □

### 1.3.6 Exercise 6

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a $2 \times 2$ matrix with complex entries. Suppose that $A$ is row-reduced and also that $a + b + c + d = 0$. Prove that there are exactly three such matrices.

*Proof.* One possibility is the zero matrix,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $A$ is not the zero matrix, then it has at least one nonzero row. If it has exactly one nonzero row, then in order to satisfy the given constraints, the nonzero row will have a 1 in the first column and a $-1$ in the second. This gives two possibilities,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Finally, if $A$ has two nonzero rows, then it must be the identity matrix or the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, but neither of these are valid since the sum of the entries is nonzero in each case. Thus there are only the three possibilities given above. □

### 1.3.7 Exercise 7

Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

*Proof.* We can, without loss of generality, assume that the matrix has only two rows, since any additional rows could just be ignored in the procedure that follows. Let this matrix be given by

$$A_0 = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}.$$

First, add $-1$ times row 2 to row 1 to get the matrix

$$A_1 = \begin{bmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 & \cdots & a_n - b_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}.$$

Next, add row 1 to row 2 to get

$$A_2 = \begin{bmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 & \cdots & a_n - b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix},$$

and then add $-1$ times row 2 to row 1, which gives

$$A_3 = \begin{bmatrix} -b_1 & -b_2 & -b_3 & \cdots & -b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}.$$

For the final step, multiply row 1 by $-1$ to get

$$A_4 = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}.$$

We can see that $A_4$ has the same entries as $A_0$ but with the rows interchanged. And only a finite number of elementary row operations of the first two kinds were performed. □

### 1.3.8 Exercise 8

Consider the system of equations $AX = 0$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a $2 \times 2$ matrix over the field $F$. Prove the following.

(a) If every entry of $A$ is 0, then every pair $(x_1, x_2)$ is a solution of $AX = 0$.

*Proof.* This is clear, since the equation $0x_1 + 0x_2 = 0$ is satisfied for any $(x_1, x_2) \in F^2$ (note that in any field, $0x = (1-1)x = x - x = 0$). $\square$

(b) If $ad-bc \neq 0$, the system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$.

*Proof.* First suppose $bd \neq 0$. Then we can perform the following row-reduction.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} ad & bd \\ bc & bd \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} ad-bc & 0 \\ bc & bd \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 \\ bc & bd \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & bd \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, $A$ is row-equivalent to the $2 \times 2$ identity matrix.

On the other hand, if $bd = 0$ then one of $b$ or $d$ is zero (but not both). If $b = 0$, then $ad \neq 0$ and we get

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $d = 0$, then $bc \neq 0$ and we have

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} c & 0 \\ a & b \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We see that, in every case, $A$ is row-equivalent to the identity matrix. Therefore $AX = 0$ has only the trivial solution. $\square$

(c) If $ad - bc = 0$ and some entry of $A$ is different from 0, then there is a solution $(x_1^0, x_2^0)$ such that $(x_1, x_2)$ is a solution if and only if there is a scalar $y$ such that $x_1 = yx_1^0$, $x_2 = yx_2^0$.

*Proof.* Since one of the entries $a, b, c, d$ is nonzero, we can assume without loss of generality that $a$ is nonzero (because, if the first row is zero then we could simply interchange the rows and relabel the entries; and if the only nonzero entry occurs in the second column, then we could interchange the columns which would correspond to relabeling $x_1$ and $x_2$).

Keeping in mind that $a$ is nonzero, we perform the following row-reduction.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}.$$

Since $ad - bc = 0$, the second row of this final matrix is zero, and we see that there are nontrivial solutions. If we let

$$x_1^0 = b \quad \text{and} \quad x_2^0 = -a,$$

then $(x_1, x_2)$ is a solution if and only if $x_1 = yx_1^0$ and $x_2 = yx_2^0$ for some $y \in F$. $\square$

## 1.4 Row-Reduced Echelon Matrices

### 1.4.1 Exercise 1

Find all solutions to the following system of equations by row-reducing the coefficient matrix:

$$\begin{aligned} \tfrac{1}{3}x_1 + 2x_2 - \ \ 6x_3 &= 0 \\ -4x_1 \qquad\quad + \ \ 5x_3 &= 0 \\ -3x_1 + 6x_2 - 13x_3 &= 0 \\ -\tfrac{7}{3}x_1 + 2x_2 - \ \ \tfrac{8}{3}x_3 &= 0 \end{aligned}$$

*Solution.* The coefficient matrix reduces as follows:

$$\begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -67 \\ 0 & 16 & -\frac{134}{3} \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 6 & -18 \\ 0 & 1 & -\frac{67}{24} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 16 & -\frac{134}{3} \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting $x_3 = 24t$, we see that all solutions have the form

$$x_1 = 30t, \quad x_2 = 67t, \quad \text{and} \quad x_3 = 24t,$$

where $t$ is an arbitrary scalar. □

### 1.4.2 Exercise 2

Find a row-reduced echelon matrix which is row-equivalent to

$$A = \begin{bmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{bmatrix}.$$

What are the solutions of $AX = 0$?

*Solution.* Performing row-reduction on $A$ gives

$$\begin{bmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -i \\ 0 & 2+2i \\ 0 & i \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -i \\ 0 & 1 \\ 0 & i \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and this last matrix is in row-reduced echelon form. Therefore the homogeneous system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$. □

### 1.4.3 Exercise 3

Describe explicitly all $2 \times 2$ row-reduced echelon matrices.

*Solution.* If a $2 \times 2$ matrix has no nonzero rows, then it is the zero matrix,

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which is in row-reduced echelon form.

Next, if a $2 \times 2$ matrix has exactly one nonzero row, then in order to be in row-reduced echelon form, the nonzero row must be in row 1 and it must start with an entry of 1. There are two possibilities,

$$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where $a$ is an arbitrary scalar.

Lastly, if a $2 \times 2$ matrix in row-reduced echelon form has two nonzero rows, then the diagonal entries must be 1 and the other entries 0, so we get the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

These are the only possibilities. □

### 1.4.4 Exercise 4

Consider the system of equations

$$\begin{aligned} x_1 - \phantom{3}x_2 + 2x_3 &= 1 \\ 2x_1 \phantom{- 3x_2} + 2x_3 &= 1 \\ x_1 - 3x_2 + 4x_3 &= 2. \end{aligned}$$

Does this system have a solution? If so, describe explicitly all solutions.

*Solution.* We perform row-reduction on the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & -2 & 2 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From this we see that the original system of equations has solutions. All solutions are of the form

$$x_1 = -t + \frac{1}{2}, \quad x_2 = t - \frac{1}{2}, \quad \text{and} \quad x_3 = t,$$

for some scalar $t$. □

### 1.4.5 Exercise 5

Give an example of a system of two linear equations in two unknowns which has no solution.

*Solution.* We can find such a system by ensuring that the coefficients in one equation are a multiple of the other, while the constant term is not the same multiple. For example, one such system is

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -3x_1 - 6x_2 &= 5. \end{aligned}$$

This system has no solutions since the augmented matrix is row-equivalent to a matrix in which one row consists of zero entries everywhere but the rightmost column. □

### 1.4.6 Exercise 6

Show that the system

$$\begin{aligned} x_1 - 2x_2 + \phantom{5}x_3 + 2x_4 &= 1 \\ x_1 + \phantom{7}x_2 - \phantom{5}x_3 + \phantom{2}x_4 &= 2 \\ x_1 + 7x_2 - 5x_3 - \phantom{2}x_4 &= 3 \end{aligned}$$

has no solution.

*Solution.* Row-reduction on the augmented matrix gives

$$\begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 9 & -6 & -3 & 2 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since the first nonzero entry in the bottom row of the last matrix is in the rightmost column, the corresponding system of equations has no solution. Therefore the original system of equations also has no solution. □

### 1.4.7 Exercise 7

Find all solutions of

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + \phantom{2}x_5 &= -2 \\ 2x_1 \phantom{{}- 3x_2} - 4x_3 + 2x_4 + \phantom{2}x_5 &= \phantom{-}3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7. \end{aligned}$$

*Solution.* The augmented matrix can be row-reduced as follows:

$$\begin{bmatrix} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & -3 & -7 & 5 & 2 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 4 & 4 & -4 & -1 & 7 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The columns with a leading 1 correspond to the variables $x_1$, $x_2$, and $x_5$, so these variables will depend on the remaining two variables, which can take any value. Therefore all solutions have the form

$$x_1 = 2s - t + 1, \quad x_2 = t - s + 2, \quad x_3 = s, \quad x_4 = t, \quad \text{and} \quad x_5 = 1,$$

where $s$ and $t$ are arbitrary scalars. □

### 1.4.8 Exercise 8

Let
$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

For which triples $(y_1, y_2, y_3)$ does the system $AX = Y$ have a solution?

*Solution.* We will perform row-reduction on the augmented matrix:

$$\begin{bmatrix} 3 & -1 & 2 & y_1 \\ 2 & 1 & 1 & y_2 \\ 1 & -3 & 0 & y_3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}y_1 \\ 2 & 1 & 1 & y_2 \\ 1 & -3 & 0 & y_3 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}y_1 \\ 0 & \frac{5}{3} & -\frac{1}{3} & -\frac{2}{3}y_1 + y_2 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{1}{3}y_1 + y_3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}y_1 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5}y_1 + \frac{3}{5}y_2 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{1}{3}y_1 + y_3 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}y_1 + \frac{1}{5}y_2 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5}y_1 + \frac{3}{5}y_2 \\ 0 & 0 & -\frac{6}{5} & -\frac{7}{5}y_1 + \frac{8}{5}y_2 + y_3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}y_1 + \frac{1}{5}y_2 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5}y_1 + \frac{3}{5}y_2 \\ 0 & 0 & 1 & \frac{7}{6}y_1 - \frac{4}{3}y_2 - \frac{5}{6}y_3 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2}y_1 + y_2 + \frac{1}{2}y_3 \\ 0 & 1 & 0 & -\frac{1}{6}y_1 + \frac{1}{3}y_2 - \frac{1}{6}y_3 \\ 0 & 0 & 1 & \frac{7}{6}y_1 - \frac{4}{3}y_2 - \frac{5}{6}y_3 \end{bmatrix}.$$

Since every row contains a nonzero entry in the first three columns, the system of equations $AX = Y$ is consistent regardless of the values of $y_1$, $y_2$, and $y_3$. Therefore $AX = Y$ has a unique solution for any triple $(y_1, y_2, y_3)$. □

### 1.4.9 Exercise 9

Let

$$A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

For which $(y_1, y_2, y_3, y_4)$ does the system of equations $AX = Y$ have a solution?

*Solution.* Row-reduction on the augmented matrix gives

$$\begin{bmatrix} 3 & -6 & 2 & -1 & y_1 \\ -2 & 4 & 1 & 3 & y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 1 & -2 & 1 & 0 & y_4 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3}y_1 \\ -2 & 4 & 1 & 3 & y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 1 & -2 & 1 & 0 & y_4 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3}y_1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} & \frac{2}{3}y_1 + y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}y_1 + y_4 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3}y_1 \\ 0 & 0 & 1 & 1 & \frac{2}{7}y_1 + \frac{3}{7}y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}y_1 + y_4 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & \frac{1}{7}y_1 - \frac{2}{7}y_2 \\ 0 & 0 & 1 & 1 & \frac{2}{7}y_1 + \frac{3}{7}y_2 \\ 0 & 0 & 0 & 0 & -\frac{2}{7}y_1 - \frac{3}{7}y_2 + y_3 \\ 0 & 0 & 0 & 0 & -\frac{3}{7}y_1 - \frac{1}{7}y_2 + y_4 \end{bmatrix}.$$

So, in order for the system $AX = Y$ to have a solution, we need $(y_1, y_2, y_3, y_4)$ to satisfy

$$\begin{aligned} -\tfrac{2}{7}y_1 - \tfrac{3}{7}y_2 + y_3 \qquad &= 0 \\ -\tfrac{3}{7}y_1 - \tfrac{1}{7}y_2 \qquad + y_4 &= 0. \end{aligned}$$

To determine the conditions on $Y$, we row-reduce the coefficient matrix for this system.

$$\begin{bmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ -\frac{3}{7} & -\frac{1}{7} & 0 & 1 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{3}{2} & -\frac{7}{2} & 0 \\ -\frac{3}{7} & -\frac{1}{7} & 0 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{3}{2} & -\frac{7}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{bmatrix} \xrightarrow{(1)}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{7}{2} & 0 \\ 0 & 1 & -3 & 2 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -3 & 2 \end{bmatrix}.$$

From this we see that in order for $AX = Y$ to have a solution, $(y_1, y_2, y_3, y_4)$ must take the form

$$(y_1, y_2, y_3, y_4) = (3t - s, 3s - 2t, s, t),$$

where $s$ and $t$ are arbitrary. □

## 1.5 Matrix Multiplication

### 1.5.1 Exercise 1

Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

Compute $ABC$ and $CAB$.

*Solution.* We get

$$\begin{aligned} ABC &= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \left( \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} CAB &= \begin{bmatrix} 1 & -1 \end{bmatrix} \left( \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}. \end{aligned}$$

□

### 1.5.2 Exercise 2

Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}.$$

Verify directly that $A(AB) = A^2 B$.

*Solution.* We have

$$\begin{aligned} A(AB) &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} A^2B &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}. \end{aligned}$$

So $A(AB) = A^2B$ as expected. □

### 1.5.3 Exercise 3

Find two different $2 \times 2$ matrices $A$ such that $A^2 = 0$ but $A \neq 0$.

*Solution.* Two possibilities are

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Both of these are nonzero matrices that satisfy $A^2 = 0$. □

### 1.5.4 Exercise 4

For the matrix $A$ of Exercise 1.5.2, find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I.$$

*Solution.* We want to reduce

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

to the identity matrix. To start, we can use two elementary row operations of the second kind to get 0 in the bottom two entries of column 1. Performing the same operations on the identity matrix gives

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Then

$$E_2E_1A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & -2 \end{bmatrix}.$$

Next, we can use a row operation of the first kind to make the central entry into a 1:

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so that} \quad E_3E_2E_1A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 3 & -2 \end{bmatrix}.$$

Continuing in this way, we get

$$E_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix},$$

so that

$$E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Then

$$E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{so that} \quad E_6E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally,

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_8 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives

$$E_8E_7E_6E_5E_4E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Thus each of $E_1, E_2, \ldots, E_8$ are elementary matrices, and they are such that $E_8 \cdots E_2E_1A = I$. □

### 1.5.5 Exercise 5

Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}.$$

Is there a matrix $C$ such that $CA = B$?

*Solution.* Suppose there is, and let

$$C = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix}.$$

Then

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}.$$

This leads to the following system of equations:

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 3, & c_4 + 2c_5 + c_6 &= -4, \\ -c_1 + 2c_2 \phantom{+ c_3} &= 1, & -c_4 + 2c_5 \phantom{+ c_6} &= 4. \end{aligned}$$

This system has solutions

$$(c_1, c_2, c_3, c_4, c_5, c_6) = (1 - 2s, 1 - s, 4s, 2t - 4, t, -4t).$$

For example, taking $s = t = -1$, we get the matrix

$$C = \begin{bmatrix} 3 & 2 & -4 \\ -6 & -1 & 4 \end{bmatrix},$$

and one can easily verify that $CA = B$. $\square$

### 1.5.6 Exercise 6

Let $A$ be an $m \times n$ matrix and $B$ an $n \times k$ matrix. Show that the columns of $C = AB$ are linear combinations of the columns of $A$. If $\alpha_1, \ldots, \alpha_n$ are the columns of $A$ and $\gamma_1, \ldots, \gamma_k$ are the columns of $C$, then

$$\gamma_j = \sum_{r=1}^{n} B_{rj} \alpha_r.$$

*Proof.* Let $A, B, C$ be as stated. By the definition of matrix multiplication, we have

$$\begin{aligned} \gamma_j = \begin{bmatrix} A_{11}B_{1j} + A_{12}B_{2j} + \cdots + A_{1n}B_{nj} \\ A_{21}B_{1j} + A_{22}B_{2j} + \cdots + A_{2n}B_{nj} \\ \vdots \\ A_{m1}B_{1j} + A_{m2}B_{2j} + \cdots + A_{mn}B_{nj} \end{bmatrix} \\ = B_{1j} \begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{bmatrix} + B_{2j} \begin{bmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{bmatrix} + \cdots + B_{nj} \begin{bmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{bmatrix} \\ = B_{1j}\alpha_1 + B_{2j}\alpha_2 + \cdots + B_{nj}\alpha_n = \sum_{r=1}^{n} B_{rj}\alpha_r. \end{aligned}$$

Therefore the columns of $C = AB$ are linear combinations of the columns of $A$. $\square$

## 1.6 Invertible Matrices

### 1.6.1 Exercise 1

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}.$$

Find a row-reduced echelon matrix $R$ which is row-equivalent to $A$ and an invertible $3 \times 3$ matrix $P$ such that $R = PA$.

*Solution.* We can perform elementary row operations on $A$, while performing the same operations on $I$, in order to find $R$ and $P$:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 4 & 5 \\ 0 & -4 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 8 & 11 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & 1 & 2 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}, \quad \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}.$$

Therefore,

$$R = \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}, \quad P = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & -2 & 3 \\ 2 & 0 & -2 \\ 1 & 2 & 1 \end{bmatrix},$$

and $R = PA$. □

### 1.6.2 Exercise 2

Do Exercise 1.6.1, but with

$$A = \begin{bmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{bmatrix}.$$

*Solution.* We proceed as before:

$$\begin{bmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & -i \\ 2 & 0 & i \\ i & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & -i \\ 0 & 6 & 3i \\ 0 & 1+3i & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -i & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & -i \\ 0 & 1 & \frac{1}{2}i \\ 0 & 1+3i & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & -i & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & \frac{1}{2}i \\ 0 & 1 & \frac{1}{2}i \\ 0 & 0 & \frac{3}{2}-\frac{1}{2}i \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{6}-\frac{1}{2}i & \frac{1}{3} & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & \frac{1}{2}i \\ 0 & 1 & \frac{1}{2}i \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3}i & \frac{1}{5}+\frac{1}{15}i & \frac{3}{5}+\frac{1}{5}i \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} & \frac{1}{30}-\frac{1}{10}i & \frac{1}{10}-\frac{3}{10}i \\ 0 & -\frac{3}{10}-\frac{1}{10}i & \frac{1}{10}-\frac{3}{10}i \\ -\frac{1}{3}i & \frac{1}{5}+\frac{1}{15}i & \frac{3}{5}+\frac{1}{5}i \end{bmatrix}.$$

So

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I, \quad P = \frac{1}{30}\begin{bmatrix} 10 & 1-3i & 3-9i \\ 0 & -9-3i & 3-9i \\ -10i & 6+2i & 18+6i \end{bmatrix},$$

and $R = PA$. □

### 1.6.3 Exercise 3

For each of the two matrices

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

use elementary row operations to discover whether it is invertible, and to find the inverse in case it is.

*Solution.* For the first matrix, row-reduction gives

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 0 & -11 & 4 \\ 0 & -11 & 4 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 0 & -11 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

and we see that the original matrix is not invertible since it is row-equivalent to a matrix having a row of zeros.

For the second matrix, we get

$$\begin{bmatrix}1 & -1 & 2\\ 3 & 2 & 4\\ 0 & 1 & -2\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}1 & -1 & 2\\ 0 & 5 & -2\\ 0 & 1 & -2\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & -2\\ 0 & 5 & -2\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ -3 & 1 & 0\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 8\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ -3 & 1 & -5\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ -\frac{3}{8} & \frac{1}{8} & -\frac{5}{8}\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 1\\ -\frac{3}{4} & \frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{8} & \frac{1}{8} & -\frac{5}{8}\end{bmatrix}.$$

From this we see that the original matrix is invertible and its inverse is the matrix

$$\frac{1}{8}\begin{bmatrix}8 & 0 & 8\\ -6 & 2 & -2\\ -3 & 1 & -5\end{bmatrix}.$$ □

### 1.6.4 Exercise 4

Let

$$A = \begin{bmatrix}5 & 0 & 0\\ 1 & 5 & 0\\ 0 & 1 & 5\end{bmatrix}.$$

For which $X$ does there exist a scalar $c$ such that $AX = cX$?

*Solution.* Let

$$X = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}.$$

Then $AX = cX$ implies

$$\begin{aligned} 5x_1 &= cx_1\\ x_1 + 5x_2 &= cx_2\\ x_2 + 5x_3 &= cx_3, \end{aligned}$$

and this is a homogeneous system of equations with coefficient matrix

$$B = \begin{bmatrix} 5-c & 0 & 0 \\ 1 & 5-c & 0 \\ 0 & 1 & 5-c \end{bmatrix}.$$

If $c = 5$ then $(x_1, x_2, x_3) = (0, 0, t)$ for some scalar $t$, so this gives one possibility for $X$. If we assume $c \neq 5$, then the matrix $B$ can be row-reduced to the identity matrix, so that $X = 0$ is then the only possibility. Therefore, there is a scalar $c$ with $AX = cX$ if and only if

$$X = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix},$$

for some arbitrary scalar $t$. □

### 1.6.5 Exercise 5

Discover whether

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

is invertible, and find $A^{-1}$ if it exists.

*Solution.* We proceed in the usual way:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

Thus $A$ is invertible and

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$ □

### 1.6.6 Exercise 6

Suppose $A$ is a $2 \times 1$ matrix and that $B$ is a $1 \times 2$ matrix. Prove that $C = AB$ is not invertible.

*Proof.* Let

$$A = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \end{bmatrix}$$

so that

$$C = AB = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}.$$

Suppose $C$ has an inverse. Then $C$ is row-equivalent to the identity matrix, and so cannot be row-equivalent to a matrix having a row of zeros. Consequently, each of $a$, $b$, $c$, and $d$ must be nonzero, since otherwise $C$ would be row-equivalent to such a matrix.

But, since $a$ and $b$ are nonzero, we can multiply the second row of $C$ by $a/b$ to get the row-equivalent matrix

$$\begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} ac & ad \\ ac & ad \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} ac & bd \\ 0 & 0 \end{bmatrix},$$

which is clearly not invertible. Therefore $C$ cannot have an inverse. □

### 1.6.7 Exercise 7

Let $A$ be an $n \times n$ (square) matrix. Prove the following two statements:

(a) If $A$ is invertible and $AB = 0$ for some $n \times n$ matrix $B$, then $B = 0$.

*Proof.* Since $AB = 0$ and $A$ is invertible, we can multiply on the left by $A^{-1}$ to get

$$B = A^{-1}0.$$

But the product on the right is clearly the $n \times n$ zero matrix, so $B = 0$. □

(b) If $A$ is not invertible, then there exists an $n \times n$ matrix $B$ such that $AB = 0$ but $B \neq 0$.

*Proof.* If $A$ is not invertible, then the homogeneous system of equations $AX = 0$ has a nontrivial solution $X_0$. Let $B$ be the matrix whose first column is $X_0$ and whose other entries are all zero, and consider the product $AB$.

The entries in the first column of $AB$ must be zero since the first column is just $AX_0$, and the remaining entries must be zero since all other columns are the product of $A$ with a zero column. Thus the proof is complete. □

### 1.6.8 Exercise 8

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Prove, using elementary row operations, that $A$ is invertible if and only if

$$ad - bc \neq 0.$$

*Proof.* First, if $A$ is invertible, then one of $a$, $b$, $c$, or $d$ must be nonzero. If $a \neq 0$, then we can reduce

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix},$$

and we must have $ad - bc \neq 0$ since otherwise $A$ could not be row-equivalent to the identity matrix, contradicting Theorem 12.

If, instead, $a = 0$ then we must have $b \neq 0$ since otherwise $A$ would have a row of zeros and could not be row-equivalent to the identity matrix. So we can proceed:

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix},$$

and we see that we must have $c \neq 0$. Thus $ad - bc = -bc \neq 0$. This completes the first half of the proof.

Conversely, assume that $ad - bc \neq 0$. If $d \neq 0$ then $A$ can be reduced to get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} ad & bd \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} ad - bc & 0 \\ c & d \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $A$ is row-equivalent to the identity matrix. On the other hand, if $d = 0$ then $b$ must be nonzero and we get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} a & b \\ -bc & -bd \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} a & b \\ ad - bc & 0 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that $A$ is again row-equivalent to the identity matrix. In either case, $A$ must be invertible by Theorem 12. □

### 1.6.9 Exercise 9

An $n \times n$ matrix $A$ is called **upper-triangular** if $A_{ij} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0.

*Proof.* Let $A$ be an $n \times n$ upper-triangular matrix.

First, suppose every entry on the main diagonal of $A$ is nonzero, and consider the homogeneous linear system $AX = 0$:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= 0 \\ A_{22}x_2 + \cdots + A_{2n}x_n &= 0 \\ &\vdots \\ A_{nn}x_n &= 0. \end{aligned}$$

Since $A_{nn}$ is nonzero, the last equation implies that $x_n = 0$. Then, since $A_{n-1,n-1}$ is nonzero, the second-to-last equation implies that $x_{n-1} = 0$. Continuing in this way, we see that $x_i = 0$ for each $i = 1, 2, \ldots, n$. Therefore the system $AX = 0$ has only the trivial solution, hence $A$ is invertible.

Conversely, suppose $A$ is invertible. Then $A$ cannot contain any zero rows, nor can $A$ be row-equivalent to a matrix with a row of zeros. This implies that $A_{nn} \neq 0$. Consider $A_{n-1,n-1}$. If $A_{n-1,n-1}$ is zero, then by dividing row $n$ by $A_{nn}$, and then by adding $-A_{n-1,n}$ times row $n$ to row $n-1$, we see that $A$ is row-equivalent to a matrix whose $(n-1)$st row is all zeros. This is a contradiction, so $A_{n-1,n-1} \neq 0$. In the same manner, we can show that $A_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$. Thus all entries on the main diagonal of $A$ are nonzero. □

### 1.6.11 Exercise 11

Let $A$ be an $m \times n$ matrix. Show that by means of a finite number of elementary row and/or column operations one can pass from $A$ to a matrix $R$ which is both 'row-reduced echelon' and 'column-reduced echelon,' i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$ if $i > r$. Show that $R = PAQ$, where $P$ is an invertible $m \times m$ matrix and $Q$ is an invertible $n \times n$ matrix.

*Proof.* By Theorem 5, $A$ is row-equivalent to a row-reduced echelon matrix $R_0$. And, by the second corollary to Theorem 12, there is an invertible $m \times m$ matrix $P$ such that $R_0 = PA$.

Results that are analogous to Theorems 5 and 12 (with similar proofs) hold for column-reduced echelon matrices, so there is a matrix $R$ which is column-equivalent to $R_0$ and an invertible $n \times n$ matrix $Q$ such that $R = R_0Q$. Then $R = PAQ$ and we see that, through a finite number of elementary row and/or column operations, $A$ passes to a matrix $R$ that is both row- and column-reduced echelon. □