Selected Solutions to Hoffman and Kunze's Linear Algebra Second Edition

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July 2, 2020

This document lives at:

https://www.gregkikola.com/projects/guides/

You can find the LaTeX source code on GitHub at:

https://github.com/gkikola/sol-hoffman-kunze

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Preface

This is an unofficial solution guide to the book *Linear Algebra*, Second Edition, by Kenneth Hoffman and Ray Kunze. It is intended for students who are studying linear algebra using Hoffman and Kunze's text. I encourage students who use this guide to first attempt each exercise on their own before looking up the solution, as doing exercises is an essential part of learning mathematics.

In writing this guide, I have avoided using techniques or results before the point at which they are introduced in the text. My solutions should therefore be accessible to someone who is reading through Hoffman and Kunze for the first time.

Given the large number of exercises, errors are unavoidable in a work such as this. I have done my best to proofread each solution, but mistakes will get through nonetheless. If you find one, please feel free to tell me about it via email: gkikola@gmail.com. I appreciate any corrections or feedback.

Please know that this guide is currently unfinished. I am slowly working on adding the remaining chapters, but this will be done at my own pace. If you need a solution to an exercise that I have not included, try typing the problem statement into a web search engine such as Google; it is likely that someone else has already posted a solution.

I am deeply grateful to the authors, Kenneth Hoffman and Ray Kunze, for producing a well-organized and comprehensive book that is a pleasure to read.

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Chapter 1

Linear Equations

1.2 Systems of Linear Equations

1.2.1 Exercise 1

Verify that the set of complex numbers described in Example 4 is a subfield of C.

Solution. The set in Example 4 consisted of all complex numbers of the form $x + y\sqrt{2}$, where x and y are rational. Call this set F.

Note that $0 = 0 + 0\sqrt{2} \in F$ and $1 = 1 + 0\sqrt{2} \in F$. If $\alpha = a + b\sqrt{2}$ and $\beta = c + d\sqrt{2}$ are any elements of F, then

$$\alpha + \beta = (a + c) + (b + d)\sqrt{2} \in F,$$

and

$$-\alpha = -a - b\sqrt{2} \in F.$$

We also have

$$\begin{aligned} \alpha\beta &= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd \\ &= (ac + 2bd) + (ad + bc)\sqrt{2} \in F \end{aligned}$$

and, provided α is nonzero,

$$\alpha^{-1} = \frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2} \in F.$$

Since F contains 0 and 1 and is closed under addition, multiplication, additive inverses, and multiplicative inverses, F is a subfield of C. $\square$

1.2.2 Exercise 2

Let F be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} x_1 - x_2 &= 0 \qquad & 3x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 0 \qquad & x_1 + x_2 &= 0 \end{aligned}$$

Solution. The systems are equivalent. For the first system, we can write

$$\begin{aligned} x_1 - x_2 &= (3x_1 + x_2) - 2(x_1 + x_2) = 0, \\ 2x_1 + x_2 &= \tfrac{1}{2}(3x_1 + x_2) + \tfrac{1}{2}(x_1 + x_2) = 0. \end{aligned}$$

And for the second system,

$$\begin{aligned} 3x_1 + x_2 &= \tfrac{1}{3}(x_1 - x_2) + \tfrac{4}{3}(2x_1 + x_2) = 0, \\ x_1 + x_2 &= -\tfrac{1}{3}(x_1 - x_2) + \tfrac{2}{3}(2x_1 + x_2) = 0. \end{aligned}$$ □

1.2.3 Exercise 3

Test the following systems of equations as in Exercise 1.2.2.

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

Solution. For the first system, we have

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= -(x_1 - x_3) + (x_2 + 3x_3) = 0, \\ x_1 + 3x_2 + 8x_3 &= (x_1 - x_3) + 3(x_2 + 3x_3) = 0, \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= \tfrac{1}{2}(x_1 - x_3) + (x_2 + 3x_3) = 0. \end{aligned}$$

For the second system, we have

$$\begin{aligned} x_1 - x_3 &= -\tfrac{3}{4}(-x_1 + x_2 + 4x_3) + \tfrac{1}{4}(x_1 + 3x_2 + 8x_3) + 0(\tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3), \\ x_2 + 3x_3 &= \tfrac{1}{4}(-x_1 + x_2 + 4x_3) + \tfrac{1}{4}(x_1 + 3x_2 + 8x_3) + 0(\tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3). \end{aligned}$$

So, the two systems are equivalent. □

1.2.4 Exercise 4

Test the following systems as in Exercise 1.2.2.

$$\begin{aligned} 2x_1 + (-1+i)x_2 + x_4 &= 0 \\ 3x_2 - 2ix_3 + 5x_4 &= 0 \end{aligned} \qquad \begin{aligned} \left(1 + \frac{i}{2}\right)x_1 + 8x_2 - ix_3 - x_4 &= 0 \\ \tfrac{2}{3}x_1 - \tfrac{1}{2}x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

Solution. Call the equations in the system on the left L_1 and L_2, and the equations on the right R_1 and R_2. If $R_1 = aL_1 + bL_2$ then, by equating the coefficients of x_3, we get

$$-i = -2ib,$$

which implies that $b = 1/2$. By equating the coefficients of x_1, we get

$$1 + \frac{i}{2} = 2a,$$

so that

$$a = \frac{1}{2} + \frac{1}{4}i.$$

Now, comparing the coefficients of x_4, we find that

$$-1 = a + 5b = \frac{1}{2} + \frac{1}{4}i + \frac{5}{2} = 3 + \frac{1}{4}i,$$

which is clearly a contradiction. Therefore the two systems are *not* equivalent. □

1.2.5 Exercise 5

Let F be a set which contains exactly two elements, 0 and 1. Define an addition and multiplication by the tables:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Verify that the set F, together with these two operations, is a field.

Solution. From the symmetry in the tables, we see that both operations are commutative.

By considering all eight possibilities, one can see that $(a+b)+c = a+(b+c)$. And one may in a similar way verify that $(ab)c = a(bc)$, so that associativity holds for the two operations.

$0 + 0 = 0$ and $0 + 1 = 1$ so F has an additive identity. Similarly, $1 \cdot 0 = 0$ and $1 \cdot 1 = 1$ so F has a multiplicative identity.

The additive inverse of 0 is 0 and the additive inverse of 1 is 1. The multiplicative inverse of 1 is 1. So F has inverses.

Lastly, by considering the eight cases, one may verify that $a(b+c) = ab+ac$. Therefore distributivity of multiplication over addition holds and F is a field. □

1.2.7 Exercise 7

Prove that each subfield of the field of complex numbers contains every rational number.

Proof. Let F be a subfield of C and let $r = m/n$ be any rational number, written in lowest terms. F must contain 0 and 1, so if $r = 0$ then we are done. Now assume r is nonzero.

Since $1 \in F$, and F is closed under addition, we know that $1 + 1 = 2 \in F$. And, if the integer k is in F, then $k + 1$ is also in F. By induction, we see that all positive integers belong to F. We also know that all negative integers are in F because F is closed under additive inverses. So, in particular, $m \in F$ and $n \in F$.

Now F is closed under multiplicative inverses, so $n \in F$ implies $1/n \in F$. Finally, closure under multiplication shows that $m \cdot (1/n) = m/n = r \in F$. Since r was arbitrary, we can conclude that all rational numbers are in F. □

1.2.8 Exercise 8

Prove that each field of characteristic zero contains a copy of the rational number field.

Proof. Let F be a field of characteristic zero. Define the map $f\colon Q \to F$ (where Q denotes the rational numbers) as follows. Let $f(0) = 0_F$ and $f(1) = 1_F$, where 0_F and 1_F are the additive and multiplicative identities, respectively, of F. Given a positive integer n, define $f(n) = f(n-1) + 1_F$ and $f(-n) = -f(n)$. If a rational number $r = m/n$ is not an integer, define $f(r) = f(m) \cdot (f(n))^{-1}$.

First we show that the function f preserves addition and multiplication. A simple induction argument will show that, in the case of integers m and n, we have

$$f(m+n) = f(m) + f(n) \quad \text{and} \quad f(mn) = f(m)f(n).$$

Now let $r_1 = m_1/n_1$ and $r_2 = m_2/n_2$ be rational numbers in lowest terms. Then, by the definition of f,

$$\begin{aligned}
f(r_1 + r_2) &= f((m_1n_2 + m_2n_1)/(n_1n_2)) \\
&= f(m_1n_2 + m_2n_1)f(n_1n_2)^{-1} \\
&= (f(m_1)f(n_2) + f(m_2)f(n_1))f(n_1)^{-1}f(n_2)^{-1} \\
&= f(m_1)f(n_1)^{-1} + f(m_2)f(n_2)^{-1} \\
&= f(r_1) + f(r_2).
\end{aligned}$$

Likewise,

$$f(r_1r_2) = f(m_1)f(m_2)f(n_1)^{-1}f(n_2)^{-1} = f(r_1)f(r_2).$$

(Formally, this shows that f is a ring homomorphism.)

We will next show that the function f is one-to-one. If $r_1 = m_1/n_1$ and $r_2 = m_2/n_2$ are rational numbers in lowest terms, then $f(r_1) = f(r_2)$ implies

$$f(m_1)f(n_1)^{-1} = f(m_2)f(n_2)^{-1}$$

or

$$f(m_1)f(n_2) = f(m_2)f(n_1).$$

This implies

$$f(m_1n_2) = f(m_2n_1).$$

Now if $m_1n_2 \neq m_2n_1$, then this would imply that F does not have characteristic zero. So $m_1n_2 = m_2n_1$ and so $r_1 = r_2$.

What we have shown is that every rational number corresponds to a distinct element of F, and that the operations of addition and multiplication of rational numbers is preserved by this correspondence. So F contains a copy of Q. $\square$

1.3 Matrices and Elementary Row Operations

1.3.1 Exercise 1

Find all solutions to the system of equations

$$\begin{aligned}(1-i)x_1 - ix_2 &= 0\\ 2x_1 + (1-i)x_2 &= 0.\end{aligned} \tag{1.1}$$

Solution. Using elementary row operations, we get

$$\begin{bmatrix}1-i & -i\\ 2 & 1-i\end{bmatrix} \xrightarrow{(1)} \begin{bmatrix}1-i & -i\\ 1 & \frac{1}{2}-\frac{1}{2}i\end{bmatrix} \xrightarrow{(2)} \begin{bmatrix}0 & 0\\ 1 & \frac{1}{2}-\frac{1}{2}i\end{bmatrix}.$$

So the system in (1.1) is equivalent to

$$x_1 + \left(\frac{1}{2} - \frac{1}{2}i\right) x_2 = 0.$$

Therefore, if c is any complex scalar, then $x_1 = (-1+i)c$ and $x_2 = 2c$ is a solution to (1.1). □

1.3.2 Exercise 2

If

$$A = \begin{bmatrix}3 & -1 & 2\\ 2 & 1 & 1\\ 1 & -3 & 0\end{bmatrix}$$

find all solutions of $AX = 0$ by row-reducing A.

Solution. We get

$$\begin{bmatrix}3 & -1 & 2\\ 2 & 1 & 1\\ 1 & -3 & 0\end{bmatrix} \xrightarrow{(1)} \begin{bmatrix}1 & -\frac{1}{3} & \frac{2}{3}\\ 2 & 1 & 1\\ 1 & -3 & 0\end{bmatrix} \xrightarrow{(2)} \begin{bmatrix}1 & -\frac{1}{3} & \frac{2}{3}\\ 0 & \frac{5}{3} & -\frac{1}{3}\\ 0 & -\frac{8}{3} & -\frac{2}{3}\end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix}1 & -\frac{1}{3} & \frac{2}{3}\\ 0 & 1 & -\frac{1}{5}\\ 0 & -\frac{8}{3} & -\frac{2}{3}\end{bmatrix} \xrightarrow{(2)} \begin{bmatrix}1 & 0 & \frac{3}{5}\\ 0 & 1 & -\frac{1}{5}\\ 0 & 0 & -\frac{6}{5}\end{bmatrix} \xrightarrow{(1)} \begin{bmatrix}1 & 0 & \frac{3}{5}\\ 0 & 1 & -\frac{1}{5}\\ 0 & 0 & 1\end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Thus $AX = 0$ has only the trivial solution. □

1.3.3 Exercise 3

If

$$A = \begin{bmatrix}6 & -4 & 0\\ 4 & -2 & 0\\ -1 & 0 & 3\end{bmatrix}$$

find all solutions of $AX = 2X$ and all solutions of $AX = 3X$. (The symbol cX denotes the matrix each entry of which is c times the corresponding entry of X.)

Solution. The matrix equation $AX = 2X$ corresponds to the system of linear equations

$$\begin{aligned} 6x_1 - 4x_2 \qquad &= 2x_1 \\ 4x_1 - 2x_2 \qquad &= 2x_2 \\ -1x_1 \qquad + 3x_3 &= 2x_3, \end{aligned}$$

or, equivalently,

$$\begin{aligned} 4x_1 - 4x_2 \qquad &= 0 \\ 4x_1 - 4x_2 \qquad &= 0 \\ -1x_1 \qquad + x_3 &= 0. \end{aligned}$$

This system is homogeneous, and can be represented by the equation $BX = 0$, where B is given by

$$B = \begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

B can be row-reduced:

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore any solution of $AX = 2X$ will have the form

$$(x_1, x_2, x_3) = (a, a, a) = a(1, 1, 1),$$

where a is a scalar.

Similarly, the equation $AX = 3X$ can be solved by row-reducing

$$\begin{bmatrix} 3 & -4 & 0 \\ 4 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, solutions of $AX = 3X$ have the form

$$(x_1, x_2, x_3) = (0, 0, b) = b(0, 0, 1),$$

where b is a scalar. □

1.3.4 Exercise 4

Find a row-reduced matrix which is row-equivalent to

$$A = \begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix}.$$

Solution. Using the elementary row operations, we get

$$\begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & -1+i & 0 \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 1+i & -1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is row-equivalent to A. □

1.3.5 Exercise 5

Prove that the following two matrices are *not* row-equivalent:

$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix}.$$

Proof. By performing row operations on the first matrix, we get

$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & c & 3 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We see that this matrix is row-equivalent to the identity matrix. The corresponding system of equations has only the trivial solution.

For the second matrix, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

The system of equations corresponding to this matrix has nontrivial solutions. Therefore the two matrices are not row-equivalent. □

1.3.6 Exercise 6

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 matrix with complex entries. Suppose that A is row-reduced and also that $a+b+c+d=0$. Prove that there are exactly three such matrices.

Proof. One possibility is the zero matrix,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If A is not the zero matrix, then it has at least one nonzero row. If it has exactly one nonzero row, then in order to satisfy the given constraints, the nonzero row will have a 1 in the first column and a -1 in the second. This gives two possibilities,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Finally, if A has two nonzero rows, then it must be the identity matrix or the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, but neither of these are valid since the sum of the entries is nonzero in each case. Thus there are only the three possibilities given above. □

1.3.7 Exercise 7

Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Proof. We can, without loss of generality, assume that the matrix has only two rows, since any additional rows could just be ignored in the procedure that follows. Let this matrix be given by

$$A_0 = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}.$$

First, add -1 times row 2 to row 1 to get the matrix

$$A_1 = \begin{bmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 & \cdots & a_n - b_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}.$$

Next, add row 1 to row 2 to get

$$A_2 = \begin{bmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 & \cdots & a_n - b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix},$$

and then add -1 times row 2 to row 1, which gives

$$A_3 = \begin{bmatrix} -b_1 & -b_2 & -b_3 & \cdots & -b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}.$$

For the final step, multiply row 1 by -1 to get

$$A_4 = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}.$$

We can see that A_4 has the same entries as A_0 but with the rows interchanged. And only a finite number of elementary row operations of the first two kinds were performed. □

1.3.8 Exercise 8

Consider the system of equations $AX = 0$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 matrix over the field F. Prove the following.

(a) If every entry of A is 0, then every pair (x_1, x_2) is a solution of $AX = 0$.

Proof. This is clear, since the equation $0x_1 + 0x_2 = 0$ is satisfied for any $(x_1, x_2) \in F^2$ (note that in any field, $0x = (1-1)x = x - x = 0$). ☐

(b) If $ad - bc \neq 0$, the system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$.

Proof. First suppose $bd \neq 0$. Then we can perform the following row-reduction.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} ad & bd \\ bc & bd \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} ad-bc & 0 \\ bc & bd \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 \\ bc & bd \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & bd \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, A is row-equivalent to the 2×2 identity matrix.

On the other hand, if $bd = 0$ then one of b or d is zero (but not both). If $b = 0$, then $ad \neq 0$ and we get

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $d = 0$, then $bc \neq 0$ and we have

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} c & 0 \\ a & b \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We see that, in every case, A is row-equivalent to the identity matrix. Therefore $AX = 0$ has only the trivial solution. ☐

(c) If $ad - bc = 0$ and some entry of A is different from 0, then there is a solution (x_1^0, x_2^0) such that (x_1, x_2) is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$, $x_2 = yx_2^0$.

Proof. Since one of the entries a, b, c, d is nonzero, we can assume without loss of generality that a is nonzero (because, if the first row is zero then we could simply interchange the rows and relabel the entries; and if the only nonzero entry occurs in the second column, then we could interchange the columns which would correspond to relabeling x_1 and x_2).

Keeping in mind that a is nonzero, we perform the following row-reduction.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}.$$

Since $ad - bc = 0$, the second row of this final matrix is zero, and we see that there are nontrivial solutions. If we let

$$x_1^0 = b \quad \text{and} \quad x_2^0 = -a,$$

then (x_1, x_2) is a solution if and only if $x_1 = yx_1^0$ and $x_2 = yx_2^0$ for some $y \in F$. $\square$

1.4 Row-Reduced Echelon Matrices

1.4.1 Exercise 1

Find all solutions to the following system of equations by row-reducing the coefficient matrix:

$$\begin{aligned} \tfrac{1}{3}x_1 + 2x_2 - \ \ 6x_3 &= 0 \\ -4x_1 \qquad\quad + \ \ 5x_3 &= 0 \\ -3x_1 + 6x_2 - 13x_3 &= 0 \\ -\tfrac{7}{3}x_1 + 2x_2 - \ \ \tfrac{8}{3}x_3 &= 0 \end{aligned}$$

Solution. The coefficient matrix reduces as follows:

$$\begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -67 \\ 0 & 16 & -\frac{134}{3} \end{bmatrix} \xrightarrow{(1)}$$

$$\begin{bmatrix} 1 & 6 & -18 \\ 0 & 1 & -\frac{67}{24} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 16 & -\frac{134}{3} \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting $x_3 = 24t$, we see that all solutions have the form

$$x_1 = 30t, \quad x_2 = 67t, \quad \text{and} \quad x_3 = 24t,$$

where t is an arbitrary scalar. □

1.4.2 Exercise 2

Find a row-reduced echelon matrix which is row-equivalent to

$$A = \begin{bmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{bmatrix}.$$

What are the solutions of $AX = 0$?

Solution. Performing row-reduction on A gives

$$\begin{bmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -i \\ 0 & 2+2i \\ 0 & i \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -i \\ 0 & 1 \\ 0 & i \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and this last matrix is in row-reduced echelon form. Therefore the homogeneous system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$. □

1.4.3 Exercise 3

Describe explicitly all 2×2 row-reduced echelon matrices.

Solution. If a 2×2 matrix has no nonzero rows, then it is the zero matrix,

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which is in row-reduced echelon form.

Next, if a 2×2 matrix has exactly one nonzero row, then in order to be in row-reduced echelon form, the nonzero row must be in row 1 and it must start with an entry of 1. There are two possibilities,

$$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where a is an arbitrary scalar.

Lastly, if a 2×2 matrix in row-reduced echelon form has two nonzero rows, then the diagonal entries must be 1 and the other entries 0, so we get the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

These are the only possibilities. □

1.4.4 Exercise 4

Consider the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 2x_3 &= 1 \\ x_1 - 3x_2 + 4x_3 &= 2. \end{aligned}$$

Does this system have a solution? If so, describe explicitly all solutions.

Solution. We perform row-reduction on the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{bmatrix} \xrightarrow{(1)}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & -2 & 2 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From this we see that the original system of equations has solutions. All solutions are of the form

$$x_1 = -t + \frac{1}{2}, \quad x_2 = t - \frac{1}{2}, \quad \text{and} \quad x_3 = t,$$

for some scalar t. □

1.4.5 Exercise 5

Give an example of a system of two linear equations in two unknowns which has no solution.

Solution. We can find such a system by ensuring that the coefficients in one equation are a multiple of the other, while the constant term is not the same multiple. For example, one such system is

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -3x_1 - 6x_2 &= 5. \end{aligned}$$

This system has no solutions since the augmented matrix is row-equivalent to a matrix in which one row consists of zero entries everywhere but the rightmost column. □

1.4.6 Exercise 6

Show that the system

$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 + 7x_2 - 5x_3 - x_4 &= 3 \end{aligned}$$

has no solution.

Solution. Row-reduction on the augmented matrix gives

$$\begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 9 & -6 & -3 & 2 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since the first nonzero entry in the bottom row of the last matrix is in the rightmost column, the corresponding system of equations has no solution. Therefore the original system of equations also has no solution. □

1.4.7 Exercise 7

Find all solutions of

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 &= -2 \\ 2x_1 \phantom{{}- 3x_2} - 4x_3 + 2x_4 + x_5 &= 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7. \end{aligned}$$

Solution. The augmented matrix can be row-reduced as follows:

$$\begin{bmatrix} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & -3 & -7 & 5 & 2 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 4 & 4 & -4 & -1 & 7 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{(1)}$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The columns with a leading 1 correspond to the variables x_1, x_2, and x_5, so these variables will depend on the remaining two variables, which can take any value. Therefore all solutions have the form

$$x_1 = 2s - t + 1, \quad x_2 = t - s + 2, \quad x_3 = s, \quad x_4 = t, \quad \text{and} \quad x_5 = 1,$$

where s and t are arbitrary scalars. □

1.4.8 Exercise 8

Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

For which triples (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

Solution. We will perform row-reduction on the augmented matrix:

$$\begin{bmatrix} 3 & -1 & 2 & y_1 \\ 2 & 1 & 1 & y_2 \\ 1 & -3 & 0 & y_3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}y_1 \\ 2 & 1 & 1 & y_2 \\ 1 & -3 & 0 & y_3 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}y_1 \\ 0 & \frac{5}{3} & -\frac{1}{3} & -\frac{2}{3}y_1 + y_2 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{1}{3}y_1 + y_3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}y_1 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5}y_1 + \frac{3}{5}y_2 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{1}{3}y_1 + y_3 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}y_1 + \frac{1}{5}y_2 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5}y_1 + \frac{3}{5}y_2 \\ 0 & 0 & -\frac{6}{5} & -\frac{7}{5}y_1 + \frac{8}{5}y_2 + y_3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}y_1 + \frac{1}{5}y_2 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5}y_1 + \frac{3}{5}y_2 \\ 0 & 0 & 1 & \frac{7}{6}y_1 - \frac{4}{3}y_2 - \frac{5}{6}y_3 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2}y_1 + y_2 + \frac{1}{2}y_3 \\ 0 & 1 & 0 & -\frac{1}{6}y_1 + \frac{1}{3}y_2 - \frac{1}{6}y_3 \\ 0 & 0 & 1 & \frac{7}{6}y_1 - \frac{4}{3}y_2 - \frac{5}{6}y_3 \end{bmatrix}.$$

Since every row contains a nonzero entry in the first three columns, the system of equations $AX = Y$ is consistent regardless of the values of y_1, y_2, and y_3. Therefore $AX = Y$ has a unique solution for any triple (y_1, y_2, y_3). □

1.4.9 Exercise 9

Let

$$A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

For which (y_1, y_2, y_3, y_4) does the system of equations $AX = Y$ have a solution?

Solution. Row-reduction on the augmented matrix gives

$$\begin{bmatrix} 3 & -6 & 2 & -1 & y_1 \\ -2 & 4 & 1 & 3 & y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 1 & -2 & 1 & 0 & y_4 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3}y_1 \\ -2 & 4 & 1 & 3 & y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 1 & -2 & 1 & 0 & y_4 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3}y_1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} & \frac{2}{3}y_1 + y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}y_1 + y_4 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3}y_1 \\ 0 & 0 & 1 & 1 & \frac{2}{7}y_1 + \frac{3}{7}y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}y_1 + y_4 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & \frac{1}{7}y_1 - \frac{2}{7}y_2 \\ 0 & 0 & 1 & 1 & \frac{2}{7}y_1 + \frac{3}{7}y_2 \\ 0 & 0 & 0 & 0 & -\frac{2}{7}y_1 - \frac{3}{7}y_2 + y_3 \\ 0 & 0 & 0 & 0 & -\frac{3}{7}y_1 - \frac{1}{7}y_2 + y_4 \end{bmatrix}.$$

So, in order for the system $AX = Y$ to have a solution, we need (y_1, y_2, y_3, y_4) to satisfy

$$\begin{aligned} -\tfrac{2}{7}y_1 - \tfrac{3}{7}y_2 + y_3 \qquad &= 0 \\ -\tfrac{3}{7}y_1 - \tfrac{1}{7}y_2 \qquad + y_4 &= 0. \end{aligned}$$

To determine the conditions on Y, we row-reduce the coefficient matrix for this system.

$$\begin{bmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ -\frac{3}{7} & -\frac{1}{7} & 0 & 1 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{3}{2} & -\frac{7}{2} & 0 \\ -\frac{3}{7} & -\frac{1}{7} & 0 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{3}{2} & -\frac{7}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{bmatrix} \xrightarrow{(1)}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{7}{2} & 0 \\ 0 & 1 & -3 & 2 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -3 & 2 \end{bmatrix}.$$

From this we see that in order for $AX = Y$ to have a solution, (y_1, y_2, y_3, y_4) must take the form

$$(y_1, y_2, y_3, y_4) = (3t - s, 3s - 2t, s, t),$$

where s and t are arbitrary. □

1.5 Matrix Multiplication

1.5.1 Exercise 1

Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

Compute ABC and CAB.

Solution. We get

$$\begin{aligned} ABC &= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} CAB &= \begin{bmatrix} 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}. \end{aligned}$$

□

1.5.2 Exercise 2

Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}.$$

Verify directly that $A(AB) = A^2B$.

Solution. We have

$$\begin{aligned} A(AB) &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} A^2B &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}. \end{aligned}$$

So $A(AB) = A^2B$ as expected. □

1.5.3 Exercise 3

Find two different 2×2 matrices A such that $A^2 = 0$ but $A \neq 0$.

Solution. Two possibilities are

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Both of these are nonzero matrices that satisfy $A^2 = 0$. □

1.5.4 Exercise 4

For the matrix A of Exercise 1.5.2, find elementary matrices $E_1, E_2, \dots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I.$$

Solution. We want to reduce

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

to the identity matrix. To start, we can use two elementary row operations of the second kind to get 0 in the bottom two entries of column 1. Performing the same operations on the identity matrix gives

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Then

$$E_2E_1A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & -2 \end{bmatrix}.$$

Next, we can use a row operation of the first kind to make the central entry into a 1:

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so that} \quad E_3E_2E_1A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 3 & -2 \end{bmatrix}.$$

Continuing in this way, we get

$$E_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix},$$

so that

$$E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Then

$$E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{so that} \quad E_6E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally,

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_8 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives

$$E_8E_7E_6E_5E_4E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Thus each of $E_1, E_2, \ldots, E_8$ are elementary matrices, and they are such that $E_8 \cdots E_2E_1A = I$. □

1.5.5 Exercise 5

Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}.$$

Is there a matrix C such that $CA = B$?

Solution. Suppose there is, and let

$$C = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix}.$$

Then

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}.$$

This leads to the following system of equations:

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 3, & c_4 + 2c_5 + c_6 &= -4, \\ -c_1 + 2c_2 \phantom{{}+c_3} &= 1, & -c_4 + 2c_5 \phantom{{}+c_6} &= 4. \end{aligned}$$

This system has solutions

$$(c_1, c_2, c_3, c_4, c_5, c_6) = (1 - 2s, 1 - s, 4s, 2t - 4, t, -4t).$$

For example, taking $s = t = -1$, we get the matrix

$$C = \begin{bmatrix} 3 & 2 & -4 \\ -6 & -1 & 4 \end{bmatrix},$$

and one can easily verify that $CA = B$. □

1.5.6 Exercise 6

Let A be an $m \times n$ matrix and B an $n \times k$ matrix. Show that the columns of $C = AB$ are linear combinations of the columns of A. If $\alpha_1, \ldots, \alpha_n$ are the columns of A and $\gamma_1, \ldots, \gamma_k$ are the columns of C, then

$$\gamma_j = \sum_{r=1}^{n} B_{rj} \alpha_r.$$

Proof. Let A, B, C be as stated. By the definition of matrix multiplication, we have

$$\begin{aligned} \gamma_j &= \begin{bmatrix} A_{11}B_{1j} + A_{12}B_{2j} + \cdots + A_{1n}B_{nj} \\ A_{21}B_{1j} + A_{22}B_{2j} + \cdots + A_{2n}B_{nj} \\ \vdots \\ A_{m1}B_{1j} + A_{m2}B_{2j} + \cdots + A_{mn}B_{nj} \end{bmatrix} \\ &= B_{1j} \begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{bmatrix} + B_{2j} \begin{bmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{bmatrix} + \cdots + B_{nj} \begin{bmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{bmatrix} \\ &= B_{1j}\alpha_1 + B_{2j}\alpha_2 + \cdots + B_{nj}\alpha_n = \sum_{r=1}^{n} B_{rj}\alpha_r. \end{aligned}$$

Therefore the columns of $C = AB$ are linear combinations of the columns of A. □

1.6 Invertible Matrices

1.6.1 Exercise 1

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}.$$

Find a row-reduced echelon matrix R which is row-equivalent to A and an invertible 3×3 matrix P such that $R = PA$.

Solution. We can perform elementary row operations on A, while performing the same operations on I, in order to find R and P:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 4 & 5 \\ 0 & -4 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 8 & 11 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & 1 & 2 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}, \quad \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}.$$

Therefore,

$$R = \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}, \quad P = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & -2 & 3 \\ 2 & 0 & -2 \\ 1 & 2 & 1 \end{bmatrix},$$

and $R = PA$. □

1.6.2 Exercise 2

Do Exercise 1.6.1, but with

$$A = \begin{bmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{bmatrix}.$$

Solution. We proceed as before:

$$\begin{bmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -i \\ 2 & 0 & i \\ i & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -i \\ 0 & 6 & 3i \\ 0 & 1+3i & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -i & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -i \\ 0 & 1 & \frac{1}{2}i \\ 0 & 1+3i & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & -i & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2}i \\ 0 & 1 & \frac{1}{2}i \\ 0 & 0 & \frac{3}{2}-\frac{1}{2}i \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{6}-\frac{1}{2}i & \frac{1}{3} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2}i \\ 0 & 1 & \frac{1}{2}i \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3}i & \frac{1}{5}+\frac{1}{15}i & \frac{3}{5}+\frac{1}{5}i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} & \frac{1}{30}-\frac{1}{10}i & \frac{1}{10}-\frac{3}{10}i \\ 0 & -\frac{3}{10}-\frac{1}{10}i & \frac{1}{10}-\frac{3}{10}i \\ -\frac{1}{3}i & \frac{1}{5}+\frac{1}{15}i & \frac{3}{5}+\frac{1}{5}i \end{bmatrix}.$$

So

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I, \quad P = \frac{1}{30}\begin{bmatrix} 10 & 1-3i & 3-9i \\ 0 & -9-3i & 3-9i \\ -10i & 6+2i & 18+6i \end{bmatrix},$$

and $R = PA$. □

1.6.3 Exercise 3

For each of the two matrices

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

use elementary row operations to discover whether it is invertible, and to find the inverse in case it is.

Solution. For the first matrix, row-reduction gives

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 0 & -11 & 4 \\ 0 & -11 & 4 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 0 & -11 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

and we see that the original matrix is not invertible since it is row-equivalent to a matrix having a row of zeros.

For the second matrix, we get

$$\begin{bmatrix}1 & -1 & 2\\ 3 & 2 & 4\\ 0 & 1 & -2\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}1 & -1 & 2\\ 0 & 5 & -2\\ 0 & 1 & -2\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & -2\\ 0 & 5 & -2\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ -3 & 1 & 0\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 8\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ -3 & 1 & -5\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ -\frac{3}{8} & \frac{1}{8} & -\frac{5}{8}\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 1\\ -\frac{3}{4} & \frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{8} & \frac{1}{8} & -\frac{5}{8}\end{bmatrix}.$$

From this we see that the original matrix is invertible and its inverse is the matrix

$$\frac{1}{8}\begin{bmatrix}8 & 0 & 8\\ -6 & 2 & -2\\ -3 & 1 & -5\end{bmatrix}.$$

□

1.6.4 Exercise 4

Let

$$A = \begin{bmatrix}5 & 0 & 0\\ 1 & 5 & 0\\ 0 & 1 & 5\end{bmatrix}.$$

For which X does there exist a scalar c such that $AX = cX$?

Solution. Let

$$X = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}.$$

Then $AX = cX$ implies

$$\begin{aligned} 5x_1 &= cx_1\\ x_1 + 5x_2 &= cx_2\\ x_2 + 5x_3 &= cx_3, \end{aligned}$$

and this is a homogeneous system of equations with coefficient matrix

$$B = \begin{bmatrix} 5-c & 0 & 0 \\ 1 & 5-c & 0 \\ 0 & 1 & 5-c \end{bmatrix}.$$

If $c = 5$ then $(x_1, x_2, x_3) = (0, 0, t)$ for some scalar t, so this gives one possibility for X. If we assume $c \neq 5$, then the matrix B can be row-reduced to the identity matrix, so that $X = 0$ is then the only possibility. Therefore, there is a scalar c with $AX = cX$ if and only if

$$X = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix},$$

for some arbitrary scalar t. □

1.6.5 Exercise 5

Discover whether

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

is invertible, and find A^{-1} if it exists.

Solution. We proceed in the usual way:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

Thus A is invertible and

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

□

1.6.6 Exercise 6

Suppose A is a 2×1 matrix and that B is a 1×2 matrix. Prove that $C = AB$ is not invertible.

Proof. Let
$$A = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \end{bmatrix}$$
so that
$$C = AB = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}.$$
Suppose C has an inverse. Then C is row-equivalent to the identity matrix, and so cannot be row-equivalent to a matrix having a row of zeros. Consequently, each of a, b, c, and d must be nonzero, since otherwise C would be row-equivalent to such a matrix.

But, since a and b are nonzero, we can multiply the second row of C by a/b to get the row-equivalent matrix
$$\begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} ac & ad \\ ac & ad \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} ac & bd \\ 0 & 0 \end{bmatrix},$$
which is clearly not invertible. Therefore C cannot have an inverse. □

1.6.7 Exercise 7

Let A be an $n \times n$ (square) matrix. Prove the following two statements:

(a) If A is invertible and $AB = 0$ for some $n \times n$ matrix B, then $B = 0$.

Proof. Since $AB = 0$ and A is invertible, we can multiply on the left by A^{-1} to get
$$B = A^{-1}0.$$
But the product on the right is clearly the $n \times n$ zero matrix, so $B = 0$. □

(b) If A is not invertible, then there exists an $n \times n$ matrix B such that $AB = 0$ but $B \neq 0$.

Proof. If A is not invertible, then the homogeneous system of equations $AX = 0$ has a nontrivial solution X_0. Let B be the matrix whose first column is X_0 and whose other entries are all zero, and consider the product AB.

The entries in the first column of AB must be zero since the first column is just AX_0, and the remaining entries must be zero since all other columns are the product of A with a zero column. Thus the proof is complete. □

1.6.8 Exercise 8

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Prove, using elementary row operations, that A is invertible if and only if

$$ad - bc \neq 0.$$

Proof. First, if A is invertible, then one of a, b, c, or d must be nonzero. If $a \neq 0$, then we can reduce

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix},$$

and we must have $ad - bc \neq 0$ since otherwise A could not be row-equivalent to the identity matrix, contradicting Theorem 12.

If, instead, $a = 0$ then we must have $b \neq 0$ since otherwise A would have a row of zeros and could not be row-equivalent to the identity matrix. So we can proceed:

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix},$$

and we see that we must have $c \neq 0$. Thus $ad - bc = -bc \neq 0$. This completes the first half of the proof.

Conversely, assume that $ad - bc \neq 0$. If $d \neq 0$ then A can be reduced to get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} ad & bd \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} ad - bc & 0 \\ c & d \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and A is row-equivalent to the identity matrix. On the other hand, if $d = 0$ then b and c must be nonzero and we get

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that A is again row-equivalent to the identity matrix. In either case, A must be invertible by Theorem 12. □

1.6.9 Exercise 9

An $n \times n$ matrix A is called **upper-triangular** if $A_{ij} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0.

Proof. Let A be an $n \times n$ upper-triangular matrix.

First, suppose every entry on the main diagonal of A is nonzero, and consider the homogeneous linear system $AX = 0$:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= 0 \\ A_{22}x_2 + \cdots + A_{2n}x_n &= 0 \\ &\vdots \\ A_{nn}x_n &= 0. \end{aligned}$$

Since A_{nn} is nonzero, the last equation implies that $x_n = 0$. Then, since $A_{n-1,n-1}$ is nonzero, the second-to-last equation implies that $x_{n-1} = 0$. Continuing in this way, we see that $x_i = 0$ for each $i = 1, 2, \ldots, n$. Therefore the system $AX = 0$ has only the trivial solution, hence A is invertible.

Conversely, suppose A is invertible. Then A cannot contain any zero rows, nor can A be row-equivalent to a matrix with a row of zeros. This implies that $A_{nn} \neq 0$. Consider $A_{n-1,n-1}$. If $A_{n-1,n-1}$ is zero, then by dividing row n by A_{nn}, and then by adding $-A_{n-1,n}$ times row n to row $n-1$, we see that A is row-equivalent to a matrix whose $(n-1)$st row is all zeros. This is a contradiction, so $A_{n-1,n-1} \neq 0$. In the same manner, we can show that $A_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$. Thus all entries on the main diagonal of A are nonzero. □

1.6.11 Exercise 11

Let A be an $m \times n$ matrix. Show that by means of a finite number of elementary row and/or column operations one can pass from A to a matrix R which is both 'row-reduced echelon' and 'column-reduced echelon,' i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$ if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

Proof. By Theorem 5, A is row-equivalent to a row-reduced echelon matrix R_0. And, by the second corollary to Theorem 12, there is an invertible $m \times m$ matrix P such that $R_0 = PA$.

Results that are analogous to Theorems 5 and 12 (with similar proofs) hold for column-reduced echelon matrices, so there is a matrix R which is column-equivalent to R_0 and an invertible $n \times n$ matrix Q such that $R = R_0 Q$. Then $R = PAQ$ and we see that, through a finite number of elementary row and/or column operations, A passes to a matrix R that is both row- and column-reduced echelon. □

Chapter 2

Vector Spaces

2.1 Vector Spaces

2.1.1 Exercise 1

If F is a field, verify that F^n (as defined in Example 1) is a vector space over the field F.

Proof. We need to check that addition and scalar multiplication, as defined in Example 1, satisfy conditions (3) and (4) of the definition. Let

$$\begin{aligned}\alpha &= (\alpha_1, \alpha_2, \dots, \alpha_n), \\ \beta &= (\beta_1, \beta_2, \dots, \beta_n),\end{aligned}$$

and

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$$

be arbitrary vectors in F^n. From the commutativity of addition in F, we have

$$\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n) = (\beta_1 + \alpha_1, \dots, \beta_n + \alpha_n) = \beta + \alpha,$$

so addition is commutative in F^n. Similarly, by associativity of addition in F, we have

$$\begin{aligned}\alpha + (\beta + \gamma) &= (\alpha_1 + (\beta_1 + \gamma_1), \dots, \alpha_n + (\beta_n + \gamma_n)) \\ &= ((\alpha_1 + \beta_1) + \gamma_1, \dots, (\alpha_n + \beta_n) + \gamma_n) \\ &= (\alpha + \beta) + \gamma,\end{aligned}$$

and associativity holds in F^n. The unique 0 vector is

$$0 = (0, 0, \dots, 0),$$

and it is clear that $\alpha + 0 = \alpha$. The unique additive inverse of α is given by

$$-\alpha = (-\alpha_1, -\alpha_2, \dots, -\alpha_n),$$

and certainly $\alpha + (-\alpha) = 0$. The conditions in (3) are satisfied.

Now let c and d be scalars in F. Then

$$\begin{aligned}
1\alpha &= (1\alpha_1, \ldots, 1\alpha_n) = (\alpha_1, \ldots, \alpha_n) = \alpha, \\
(cd)\alpha &= ((cd)\alpha_1, \ldots, (cd)\alpha_n) = (c(d\alpha_1), \ldots, c(d\alpha_n)) = c(d\alpha), \\
c(\alpha + \beta) &= (c(\alpha_1 + \beta_1), \ldots, c(\alpha_n + \beta_n)) \\
&= (c\alpha_1 + c\beta_1, \ldots, c\alpha_n + c\beta_n) \\
&= c\alpha + c\beta,
\end{aligned}$$

and

$$\begin{aligned}
(c + d)\alpha &= ((c + d)\alpha_1, \ldots, (c + d)\alpha_n) \\
&= (c\alpha_1 + d\alpha_1, \ldots, c\alpha_n + d\alpha_n) \\
&= c\alpha + d\alpha,
\end{aligned}$$

so the conditions in (4) are satisfied. Therefore F^n is a vector space over F. ☐

2.1.2 Exercise 2

If V is a vector space over the field F, verify that

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4$$

for all vectors α_1, α_2, α_3, and α_4 in V.

Proof. We only need to make use of commutativity and associativity of vector addition:

$$\begin{aligned}
(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) &= (\alpha_2 + \alpha_1) + (\alpha_3 + \alpha_4) \\
&= \alpha_2 + [\alpha_1 + (\alpha_3 + \alpha_4)] \\
&= \alpha_2 + [(\alpha_1 + \alpha_3) + \alpha_4] \\
&= [\alpha_2 + (\alpha_1 + \alpha_3)] + \alpha_4 \\
&= [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4.
\end{aligned}$$

☐

2.1.3 Exercise 3

If C is the field of complex numbers, which vectors in C^3 are linear combinations of $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$?

Solution. A vector $\alpha = (y_1, y_2, y_3)$ is a linear combination of $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$ if there are scalars x_1, x_2, x_3 such that

$$x_1(1, 0, -1) + x_2(0, 1, 1) + x_3(1, 1, 1) = \alpha,$$

which leads to the following system of equations:

$$\begin{aligned}
x_1 \qquad + x_3 &= y_1 \\
x_2 + x_3 &= y_2 \\
-x_1 + x_2 + x_3 &= y_3.
\end{aligned}$$

Since the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

is row-equivalent to the identity matrix, this system of equations has a solution for each α. Therefore, all vectors in C^3 are linear combinations of the vectors $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$. $\square$

2.1.4 Exercise 4

Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$\begin{aligned} (x, y) + (x_1, y_1) &= (x + x_1, y + y_1) \\ c(x, y) &= (cx, y). \end{aligned}$$

Is V, with these operations, a vector space over the field of real numbers?

Solution. No, V is not a vector space. Most of the conditions are satisfied, but distributivity over scalar addition fails when y is nonzero:

$$(c + d)(x, y) = ((c + d)x, y) = (cx + dx, y)$$

but

$$c(x, y) + d(x, y) = (cx, y) + (dx, y) = (cx + dx, 2y).$$

$\square$

2.1.5 Exercise 5

On R^n, define two operations

$$\begin{aligned} \alpha \oplus \beta &= \alpha - \beta \\ c \cdot \alpha &= -c\alpha. \end{aligned}$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(R^n, \oplus, \cdot)$?

Solution. Commutativity of $\oplus$ fails, since $\alpha - \beta$ is not, in general, equal to $\beta - \alpha$. Associativity of $\oplus$ also fails since, for nonzero γ,

$$(\alpha - \beta) - \gamma \neq \alpha - \beta + \gamma = \alpha - (\beta - \gamma).$$

The usual zero vector still works, since $\alpha \oplus 0 = \alpha - 0 = \alpha$. Additive inverses also exist, but they are not the usual ones. Instead, each vector is its own inverse, since $\alpha \oplus \alpha = \alpha - \alpha = 0$.

For multiplication $\cdot$, it is not the case that $1 \cdot \alpha = \alpha$ since, for nonzero α, $1 \cdot \alpha = -1\alpha \neq \alpha$. Associativity with scalar multiplication does not hold either, since

$$(c_1 c_2) \cdot \alpha = -c_1 c_2 \alpha$$

while

$$c_1 \cdot (c_2 \cdot \alpha) = c_1 \cdot (-c_2 \alpha) = c_1 c_2 \alpha.$$

For the distributive properties, we have

$$\begin{aligned} c \cdot (\alpha \oplus \beta) &= c \cdot (\alpha - \beta) \\ &= -c(\alpha - \beta) \\ &= -c\alpha + c\beta \end{aligned}$$

and

$$\begin{aligned} c \cdot \alpha \oplus c \cdot \beta &= -c\alpha - (-c\beta) \\ &= -c\alpha + c\beta, \end{aligned}$$

so the first distributive property holds. And

$$\begin{aligned} (c_1 + c_2) \cdot \alpha &= -(c_1 + c_2)\alpha \\ &= -c_1\alpha - c_2\alpha, \end{aligned}$$

while

$$\begin{aligned} c_1 \cdot \alpha \oplus c_2 \cdot \alpha &= -c_1\alpha - (-c_2\alpha) \\ &= -c_1\alpha + c_2\alpha, \end{aligned}$$

so the second distributive property fails.

To summarize, in the vector space definition, only properties (c) and (d) of (3) and (c) of (4) hold. □

2.1.6 Exercise 6

Let V be the set of all complex-valued functions f on the real line such that (for all t in R)

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that V, with the operations

$$\begin{aligned} (f + g)(t) &= f(t) + g(t) \\ (cf)(t) &= cf(t) \end{aligned}$$

is a vector space over the field of *real* numbers. Give an example of a function in V which is not real-valued.

Solution. Commutativity and associativity of addition follow from the properties of addition in C. Note that the zero function is in V. If $f \in V$, then the function $-f$ given by

$$(-f)(t) = -f(t)$$

is in V since

$$-f(-t) = -\overline{f(t)} = \overline{-f(t)} = \overline{(-f)(t)}.$$

And $f + (-f)$ is the zero function.

For scalar multiplication, we have

$$(1f)(t) = f(t),$$

so the first property is satisfied. And

$$((cd)f)(t) = (cd)f(t) = c((df)(t)) = (c(df))(t),$$

so the second property is satisfied. And distributivity holds, since

$$\begin{aligned}(c(f+g))(t) &= c(f(t)+g(t)) \\ &= cf(t)+cg(t) \\ &= (cf)(t)+(cg)(t) \\ &= (cf+cg)(t)\end{aligned}$$

and

$$\begin{aligned}((c+d)f)(t) &= (c+d)f(t) \\ &= cf(t)+df(t) \\ &= (cf)(t)+(df)(t) \\ &= (cf+df)(t).\end{aligned}$$

Therefore V is a vector space over R.

For an example of a function in V, consider the function f from R to C given by

$$f(t) = ti.$$

Then $f(-t) = -ti = \overline{ti} = \overline{f(t)}$ as required. □

2.1.7 Exercise 7

Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers. Define

$$\begin{aligned}(x,y)+(x_1,y_1) &= (x+x_1,0) \\ c(x,y) &= (cx,0).\end{aligned}$$

Is V, with these operations, a vector space?

Solution. No, V is not a vector space since $1\alpha = \alpha$ does not hold for all α in V. For example, $1(1,1) = (1,0) \neq (1,1)$. V also fails to have an additive identity. □

2.2 Subspaces

2.2.1 Exercise 1

Which of the following sets of vectors $\alpha = (a_1, \ldots, a_n)$ in R^n are subspaces of R^n $(n \geq 3)$?

(a) all α such that $a_1 \geq 0$

Solution. This is not a subspace since it is not closed under scalar multiplication (take any negative scalar). □

(b) all α such that $a_1 + 3a_2 = a_3$

Solution. This is a subspace: Let $\beta = (b_1, b_2, \ldots, b_n)$. Then consider the vector $c\alpha + \beta$. We have

$$\begin{aligned}(ca_1 + b_1) + 3(ca_2 + b_2) &= c(a_1 + 3a_2) + (b_1 + 3b_2) \\ &= ca_3 + b_3.\end{aligned}$$

Therefore $c\alpha + \beta$ is in the subset, so it is a subspace by Theorem 1. □

(c) all α such that $a_2 = a_1^2$

Solution. This is not a subspace since it is not closed under vector addition. For example, $(1, 1, 1, \ldots)$ is in the set, but the sum of this vector with itself is not. □

(d) all α such that $a_1 a_2 = 0$

Solution. This is not a subspace since it is not closed under vector addition. For example $(1, 0, \ldots)$ and $(0, 1, \ldots)$ are each in the set, but their sum is not. □

(e) all α such that a_2 is rational

Solution. This is not a subspace because it is not closed under scalar multiplication: Multiplication of any vector in the set having $a_2 \neq 0$ by an irrational scalar produces a vector that is not in the set. □

2.2.2 Exercise 2

Let V be the (real) vector space of all functions f from R into R. Which of the following sets of functions are subspaces of V?

(a) all f such that $f(x^2) = f(x)^2$

Solution. The functions f and g given by

$$f(x) = x \quad \text{and} \quad g(x) = 1$$

each belong to this set, but their sum $f + g$ does not. Therefore this is not a subspace. □

(b) all f such that $f(0) = f(1)$

Solution. Suppose f and g both belong to this set. Then

$$\begin{aligned}(cf + g)(0) &= cf(0) + g(0)\\ &= cf(1) + g(1)\\ &= (cf + g)(1),\end{aligned}$$

so the set satisfies the subspace criterion of Theorem 1 and is thus a subspace of V. □

(c) all f such that $f(3) = 1 + f(-5)$

Solution. Take any f and g in this set. Then

$$(f + g)(3) = f(3) + g(3) = 2 + (f + g)(-5),$$

which does not belong to the set. Therefore this set is not a subspace. □

(d) all f such that $f(-1) = 0$

Solution. Let f and g be such functions. Then

$$(cf + g)(-1) = cf(-1) + g(-1) = 0 + 0 = 0,$$

so this set is a subspace of V by Theorem 1. □

(e) all f which are continuous

Solution. If f and g are continuous, then $cf + g$ is also continuous, so this is a subspace. □

2.2.3 Exercise 3

Is the vector $(3, -1, 0, -1)$ in the subspace of R^5 spanned by the vectors

$$(2, -1, 3, 2), \quad (-1, 1, 1, -3), \quad \text{and} \quad (1, 1, 9, -5)?$$

Solution. The subspace spanned by these three vectors consists of all linear combinations

$$x_1(2, -1, 3, 2) + x_2(-1, 1, 1, -3) + x_3(1, 1, 9, -5).$$

Therefore $(3, -1, 0, -1)$ is in this subspace if and only if the system of equations

$$\begin{aligned}2x_1 - x_2 + x_3 &= 3\\ -x_1 + x_2 + x_3 &= -1\\ 3x_1 + x_2 + 9x_3 &= 0\\ 2x_1 - 3x_2 - 5x_3 &= -1\end{aligned}$$

has a solution. However, the augmented matrix can be row-reduced to

$$\begin{bmatrix} 2 & -1 & 1 & 3\\ -1 & 1 & 1 & -1\\ 3 & 1 & 9 & 0\\ 2 & -3 & -5 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0\\ 0 & 1 & 3 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, this system of equations has no solution and the vector $(3, -1, 0, -1)$ is not in the subspace spanned by the other three given vectors. □

2.2.4 Exercise 4

Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in R^5 which satisfy

$$\begin{aligned} 2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \phantom{{}-3x_5} &= 0 \\ x_1 \phantom{{}-3x_2} + \tfrac{2}{3}x_3 \phantom{{}-3x_4} - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0. \end{aligned}$$

Find a finite set of vectors which spans W.

Solution. After performing the necessary elementary row operations, the coefficient matrix becomes

$$\begin{bmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, letting $x_3 = 3t$, $x_4 = u$, and $x_5 = v$, we see that the elements of W have the form

$$(v - 2t, 2v - u, 3t, u, v).$$

Therefore, a spanning set for W is given by the vectors

$$(-2, 0, 3, 0, 0), \quad (0, -1, 0, 1, 0), \quad \text{and} \quad (1, 2, 0, 0, 1).$$ □

2.2.5 Exercise 5

Let F be a field and let n be a positive integer $(n \geq 2)$. Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A in V are subspaces of V?

(a) all invertible A

Solution. This cannot be a subspace since the zero matrix is not invertible. □

(b) all non-invertible A

Solution. This is also not a subspace since it is possible for the sum of two non-invertible matrices to be invertible. For example, in the 2×2 case, the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not invertible, but their sum is the identity matrix, which is invertible. □

(c) all A such that $AB = BA$, where B is some fixed matrix in V

Solution. Let A_1 and A_2 be matrices in V such that $A_1B = BA_1$ and $A_2B = BA_2$. Then, since matrix multiplication is distributive,

$$\begin{aligned}(cA_1 + A_2)B &= cA_1B + A_2B \\ &= cBA_1 + BA_2 \\ &= B(cA_1) + BA_2 \\ &= B(cA_1 + A_2).\end{aligned}$$

Therefore, by Theorem 1, this set is a subspace of V. □

(d) all A such that $A^2 = A$

Solution. We will assume that the field F has more than two elements. In that case, this set cannot be a subspace since the identity matrix has the property that $I^2 = I$, but the sum of the identity with itself does not have this property. □

2.2.6 Exercise 6

(a) Prove that the only subspaces of R^1 are R^1 and the zero subspace.

Proof. Suppose W is a subspace of R^1. If $W = \{0\}$ we are done, so suppose W contains a nonzero element x. Then W must contain cx for any real number c. In particular, if r is any real number, then W must contain r since $r = (rx^{-1})x$. This shows that $W = R^1$. □

(b) Prove that a subspace of R^2 is R^2, or the zero subspace, or consists of all scalar multiples of some fixed vector in R^2. (The last type of subspace is, intuitively, a straight line through the origin.)

Proof. Let W be a subspace of R^2. If $W = \{(0, 0)\}$ we are done, so assume W contains a nonzero vector α. Then W must contain all scalar multiples of α. If these are the only elements in W, then we are again finished. If, however, W contains two nonzero elements α and β such that β is *not* a scalar multiple of α, then we must show that $W = R^2$.

Let $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$. Also let $\gamma = (c_1, c_2)$ be any element in R^2. Then γ is a linear combination of α and β if and only if the system of equations

$$\begin{aligned}a_1x_1 + b_1x_2 &= c_1 \\ a_2x_1 + b_2x_2 &= c_2\end{aligned}$$

has a solution. Suppose the coefficient matrix

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is not invertible. By Exercise 1.6.8, we then know that $a_1b_2 - a_2b_1 = 0$. Now, one of a_1 and a_2 is nonzero. If $a_1 \neq 0$, then

$$b_2 = \frac{b_1}{a_1} \cdot a_2.$$

Also

$$b_1 = \frac{b_1}{a_1} \cdot a_1,$$

and we have a contradiction since β was assumed to not be a scalar multiple of α. Similarly $a_2 \neq 0$ also leads to a contradiction. This shows that the system of equations above has a solution, so that $W = R^2$. □

(c) Can you describe the subspaces of R^3?

Solution. The subspaces of R^3 are the zero subspace, the set of all scalar multiples of a fixed nonzero vector (i.e., a line through the origin), the set of all linear combinations of two linearly independent vectors (i.e., a plane through the origin), and R^3 itself. □

2.2.7 Exercise 7

Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

Proof. Let W_1 and W_2 be as stated, but assume that neither is contained in the other. Then there is a vector $u \in W_1$ such that $u \notin W_2$, and there is a vector $v \in W_2$ such that $v \notin W_1$. Since $W_1 \cup W_2$ is a subspace, $u + v \in W_1 \cup W_2$. Now either $u + v \in W_1$ or $u + v \in W_2$. In the first case, since $-u \in W_1$ we must have

$$(u + v) + (-u) = v \in W_1,$$

which is a contradiction. But then $u + v \in W_2$ leads to a similar contradiction. Therefore one of the subspaces W_i must be contained in the other. □

2.2.8 Exercise 8

Let V be the vector space of all functions from R into R; let V_e be the subset of even functions,

$$f(-x) = f(x);$$

let V_o be the subset of odd functions,

$$f(-x) = -f(x).$$

(a) Prove that V_e and V_o are subspaces of V.

Proof. Suppose f and g are even functions. Then for any scalar c,

$$\begin{aligned}(cf + g)(-x) &= cf(-x) + g(-x)\\ &= cf(x) + g(x)\\ &= (cf + g)(x),\end{aligned}$$

so $cf + g$ is also even and therefore V_e is a subspace of V. Similarly, if f and g are both odd functions, then

$$\begin{aligned}(cf + g)(-x) &= cf(-x) + g(-x)\\ &= -cf(x) - g(x)\\ &= -(cf + g)(x),\end{aligned}$$

so V_o is also a subspace. □

(b) Prove that $V_e + V_o = V$.

Proof. Let $f \in V$ be arbitrary. Let g be the function in V defined by

$$g(x) = \frac{f(x) + f(-x)}{2}$$

and let h be the function given by

$$h(x) = \frac{f(x) - f(-x)}{2}.$$

It is clear that $g \in V_e$ and $h \in V_o$. Since $f(x) = g(x) + h(x)$ for all x, we see that $V = V_e + V_o$. □

(c) Prove that $V_e \cap V_o = \{0\}$.

Proof. Suppose $f \in V_e \cap V_o$ and fix a particular $x \in R$. Since f is even, $f(-x) = f(x)$. And since f is odd, $f(-x) = -f(x)$. Therefore we have $f(x) = -f(x)$, which is only possible if $f(x) = 0$. Since x was arbitrary, f must be the zero function. This shows that $V_e \cap V_o$ is the zero subspace. □

2.2.9 Exercise 9

Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are *unique* vectors α_1 in W_1 and α_2 is W_2 such that $\alpha = \alpha_1 + \alpha_2$.

Proof. Since $W_1 + W_2 = V$, we may find α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$. Now suppose there is also α_3 in W_1 and α_4 in W_2 with $\alpha = \alpha_3 + \alpha_4$. Then

$$\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4.$$

Rearranging, we get

$$\alpha_1 - \alpha_3 = \alpha_4 - \alpha_2.$$

But the vector on the left-hand side must belong to W_1, and the vector on the right-hand side must belong to W_2. Therefore $\alpha_1 - \alpha_3$ belongs to the intersection of W_1 and W_2, which implies that $\alpha_1 - \alpha_3 = 0$ or $\alpha_1 = \alpha_3$. And $\alpha_4 = \alpha_2$ also. This shows that the vectors α_1 and α_2 are unique. □

2.3 Bases and Dimension

2.3.1 Exercise 1

Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

Proof. Let α_1 and α_2 be linearly dependent vectors in the space V. Then, by definition, there are scalars c_1, c_2 not both zero such that

$$c_1\alpha_1 + c_2\alpha_2 = 0.$$

If c_1 is nonzero, then we may write

$$\alpha_1 = -\frac{c_2}{c_1}\alpha_2$$

so that α_1 is a scalar multiple of α_2. If $c_1 = 0$, then c_2 is nonzero and a similar argument will do. □

2.3.2 Exercise 2

Are the vectors

$$\begin{aligned} \alpha_1 &= (1, 1, 2, 4), \quad \alpha_2 = (2, -1, -5, 2) \\ \alpha_3 &= (1, -1, -4, 0), \quad \alpha_4 = (2, 1, 1, 6) \end{aligned}$$

linearly independent in R^4?

Solution. Suppose $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0$. This leads to the system of equations

$$\begin{aligned} c_1 + 2c_2 + c_3 + 2c_4 &= 0 \\ c_1 - c_2 - c_3 + c_4 &= 0 \\ 2c_1 - 5c_2 - 4c_3 + c_4 &= 0 \\ 4c_1 + 2c_2 \quad\quad + 6c_4 &= 0. \end{aligned}$$

Using the method of elimination developed in Chapter 1, we find that this system has the general solution

$$(c_1, c_2, c_3, c_4) = \left(\frac{s - 4t}{3}, \frac{-2s - t}{3}, s, t\right),$$

where $s, t \in R^4$ are arbitrary. For example, we may take $s = 3$ and $t = 0$ to get $c_1 = 1$, $c_2 = -2$, $c_3 = 3$, and $c_4 = 0$. This shows that the vectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are linearly dependent. □

2.3.3 Exercise 3

Find a basis for the subspace of R^4 spanned by the four vectors of Exercise 2.3.2.

Solution. Since α_2 is not a scalar multiple of α_1, the set $\{\alpha_1, \alpha_2\}$ is linearly independent (by Exercise 2.3.1). We also see that it spans the subspace since we can write

$$\alpha_3 = \frac{2}{3}\alpha_2 - \frac{1}{3}\alpha_1$$

and

$$\alpha_4 = \frac{4}{3}\alpha_1 + \frac{1}{3}\alpha_2.$$

So $\{\alpha_1, \alpha_2\}$ is a basis for the subspace. □

2.3.4 Exercise 4

Show that the vectors

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 2, 1), \quad \alpha_3 = (0, -3, 2)$$

form a basis for R^3. Express each of the standard basis vectors as linear combinations of α_1, α_2, and α_3.

Solution. Since $\dim R^3 = 3$, we need only show that the three vectors are independent. Let c_1, c_2, c_3 be scalars such that

$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0.$$

Then we arrive at the homogeneous system of equations $Ax = 0$, where A is the 3×3 matrix whose jth column is α_j. By row-reducing this matrix, we see that it is row-equivalent to the identity matrix. Hence the system $Ax = 0$ has only the trivial solution, i.e. $c_1 = c_2 = c_3 = 0$. Therefore $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis for R^3.

To write the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$, we may solve the systems $Ax = \epsilon_i$. This gives

$$\begin{aligned}
(1, 0, 0) &= \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \\
(0, 1, 0) &= -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 \\
(0, 0, 1) &= -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3.
\end{aligned}$$

□

2.3.5 Exercise 5

Find three vectors in R^3 which are linearly dependent, and are such that any two of them are linearly independent.

Solution. Consider the vectors

$$\alpha_1 = (1, 0, 0), \quad \alpha_2 = (0, 1, 0), \quad \text{and} \quad \alpha_3 = (1, 1, 0).$$

These vectors are pairwise-independent since neither is a scalar multiple of another. But they are clearly linearly dependent since $\alpha_1 + \alpha_2 - \alpha_3 = 0$. □

2.3.6 Exercise 6

Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

Proof. We may simply take the standard basis:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Later, in Exercise 2.3.12, we will prove that this set is a basis for V in the more general case where V is the space of $m \times n$ matrices.

Since V has a basis with four elements, it has dimension 4. □

2.3.7 Exercise 7

Let V be the vector space of Exercise 6. Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

(a) Prove that W_1 and W_2 are subspaces of V.

Proof. Both sets are nonempty. Consider the arbitrary matrices

$$A_1 = \begin{bmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{bmatrix}$$

in W_1 and let c be an arbitrary scalar. Then

$$cA_1 + A_2 = \begin{bmatrix} cx_1 + x_2 & -cx_1 - x_2 \\ y_1 + y_2 & z_1 + z_2 \end{bmatrix} = \begin{bmatrix} cx_1 + x_2 & -(cx_1 + x_2) \\ y_1 + y_2 & z_1 + z_2 \end{bmatrix}$$

which is again in W_1. This shows that W_1 is a subspace of V.

A similar argument will show that W_2 is a subspace of V. □

(b) Find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.

Solution. First we find bases for W_1 and W_2. We may take

$$\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

as a basis for W_1 and

$$\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

as a basis for W_2. Consequently, we see that $\dim W_1 = \dim W_2 = 3$.

Next, observe that matrices in $W_1 \cap W_2$ must have the form

$$\begin{bmatrix} x & -x \\ -x & y \end{bmatrix}.$$

A basis for this space is then

$$\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

so that $\dim(W_1 \cap W_2) = 2$. Finally, we may apply Theorem 6 to determine that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = 3 + 3 - 2 = 4.$$

It follows that $W_1 + W_2 = V$. $\square$

2.3.8 Exercise 8

Again let V be the space of 2×2 matrices over F. Find a basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A_j^2 = A_j$ for each j.

Solution. Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

A simple check will show that $A_j^2 = A_j$ for each j. To show that $\{A_1, A_2, A_3, A_4\}$ is a basis for V, we need only show that it spans V (since any spanning set with four vectors must be linearly independent).

Let

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

be an arbitrary 2×2 matrix over F. Then we can write B as a linear combination of A_1, A_2, A_3, A_4 as follows:

$$B = (x - y)A_1 + (w - z)A_2 + yA_3 + zA_4.$$

Therefore the set $\{A_1, A_2, A_3, A_4\}$ is indeed a basis for V. $\square$

2.3.9 Exercise 9

Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, and γ are linearly independent vectors in V. Prove that $(\alpha + \beta)$, $(\beta + \gamma)$, and $(\gamma + \alpha)$ are linearly independent.

Proof. Let c_1, c_2, and c_3 be scalars in F such that

$$c_1(\alpha + \beta) + c_2(\beta + \gamma) + c_3(\gamma + \alpha) = 0.$$

By rearranging, this becomes

$$(c_1 + c_3)\alpha + (c_1 + c_2)\beta + (c_2 + c_3)\gamma = 0.$$

Since α, β, and γ are linearly independent, we must have

$$c_1 + c_3 = 0, \quad c_1 + c_2 = 0, \quad \text{and} \quad c_2 + c_3 = 0.$$

But this system of equations has the unique solution $(c_1, c_2, c_3) = (0, 0, 0)$. Therefore $(\alpha + \beta)$, $(\beta + \gamma)$, and $(\gamma + \alpha)$ are linearly independent. $\square$

2.3.10 Exercise 10

Let V be a vector space over the field F. Suppose there are a finite number of vectors $\alpha_1, \dots, \alpha_r$ in V which span V. Prove that V is finite-dimensional.

Proof. We know by Theorem 4 that any independent set of vectors in V can have at most a finite number r of elements. Thus, if a basis exists, it must be finite.

We can explicitly construct such a basis: if $\alpha_1, \dots, \alpha_r$ are linearly independent, then we are done. If not, one of the vectors α_i can be written in terms of the other α_j. So remove α_i from the set. This will not affect the span. If the set is now linearly independent, then we have a basis. If not, continue removing elements that are linear combinations of the remaining vectors. This process must eventually terminate since we started with a finite number of vectors in the set. Consequently, a finite basis exists. $\square$

2.3.11 Exercise 11

Let V be the set of all 2×2 matrices A with *complex* entries which satisfy $A_{11} + A_{22} = 0$.

(a) Show that V is a vector space over the field of *real* numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.

Proof. Let A and B be members of V. Then

$$(A+B)_{11} + (A+B)_{22} = (A_{11} + A_{22}) + (B_{11} + B_{22}) = 0 + 0 = 0.$$

And for any scalar c in R,

$$(cA)_{11} + (cA)_{22} = c(A_{11} + A_{22}) = c0 = 0.$$

This shows that V is closed under matrix addition and scalar multiplication.

Next, we already know that matrix addition is commutative and associative. The zero matrix belongs to V, and for any A in V, the matrix $-A$ is also in V.

The remaining vector space axioms follow from the properties of matrix addition and scalar multiplication. Therefore V is a vector space. $\square$

(b) Find a basis for this vector space.

Solution. One basis is given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} \right\}.$$

It is fairly straightforward to check that $\mathcal{B}$ both spans V and is linearly independent. $\square$

(c) Let W be the set of all matrices A in V such that $A_{21} = -\overline{A}_{12}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W.

Solution. First, the zero matrix belongs to W so W is nonempty. Now for any A and B in V and c in R, consider the matrix $cA+B$. We must have

$$(cA+B)_{21} = cA_{21} + B_{21} = -c\overline{A}_{12} - \overline{B}_{12} = -\overline{(cA+B)}_{12}.$$

This shows that W is a subspace of V. A basis for W is given by

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}.$$ □

2.3.12 Exercise 12

Prove that the space of all $m \times n$ matrices over the field F has dimension mn, by exhibiting a basis for this space.

Proof. Let $F^{m \times n}$ denote the space of $m \times n$ matrices over F.

For each i and j with $1 \le i \le m$ and $1 \le j \le n$, let ϵ_{ij} denote the $m \times n$ matrix over F whose ijth entry is 1, with all other entries 0. Let $\mathcal{B}$ denote the set of all ϵ_{ij}. We will show that $\mathcal{B}$ is a basis for $F^{m \times n}$, so that the dimension of this space is mn.

First, let

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \epsilon_{ij},$$

where each c_{ij} is an arbitrary scalar in F. Then A is the matrix whose ijth entry is c_{ij}. By choosing these scalars appropriately, we see that any $m \times n$ matrix over F can be written as a linear combination of the matrices in $\mathcal{B}$. Therefore $\mathcal{B}$ spans $F^{m \times n}$.

Moreover, $A = 0$ if and only if each $c_{ij} = 0$, so $\mathcal{B}$ is linearly independent. This shows that $\mathcal{B}$ is a basis for $F^{m \times n}$. □

2.3.13 Exercise 13

Discuss Exercise 2.3.9, when V is a vector space over the field with two elements described in Exercise 1.2.5.

Solution. In Exercise 2.3.9, it was stated that the field F should be a subfield of the complex numbers (in particular, a field with characteristic 0). When this restriction is taken away, the result does not necessarily hold, as we will now demonstrate.

Let V be the vector space F^3, where F is the field with 2 elements. Let $\alpha = (1,0,0)$, $\beta = (0,1,0)$, and $\gamma = (0,0,1)$. We see that α, β, and γ are linearly independent (in fact they form the standard basis of F^3).

Now consider the vectors

$$\alpha + \beta = (1,1,0), \quad \beta + \gamma = (0,1,1), \quad \text{and} \quad \gamma + \alpha = (1,0,1).$$

These are *not* linearly independent, since

$$(1,1,0) + (0,1,1) + (1,0,1) = (0,0,0).$$

So the result from Exercise 2.3.9 does not hold in this more general setting. □

2.3.14 Exercise 14

Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.

Proof. Assume the contrary, and let $\{x_1, x_2, \ldots, x_n\}$ be a finite basis for V. Then every real number can be expressed as a linear combination

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

where $c_1, \ldots, c_n$ are rational numbers. Thus we can establish a one-to-one correspondence between the n-tuples of rational numbers and the set of real numbers. Since the rational numbers are countable, this implies that the reals are also countable. But this is clearly a contradiction. Therefore V is *not* finite-dimensional. $\square$

2.4 Coordinates

2.4.1 Exercise 1

Show that the vectors

$$\alpha_1 = (1,1,0,0), \quad \alpha_2 = (0,0,1,1)$$
$$\alpha_3 = (1,0,0,4), \quad \alpha_4 = (0,0,0,2)$$

form a basis for R^4. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Solution. Let

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix}.$$

P is invertible and has inverse

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

By Theorem 8, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis for R^4. Moreover, the jth column of P^{-1} gives the coordinates of the standard basis vector ϵ_j in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. □

2.4.2 Exercise 2

Find the coordinate matrix of the vector $(1,0,1)$ in the basis of C^3 consisting of the vectors $(2i,1,0)$, $(2,-1,1)$, $(0,1+i,1-i)$, in that order.

Solution. Let

$$P = \begin{bmatrix} 2i & 2 & 0 \\ 1 & -1 & 1+i \\ 0 & 1 & 1-i \end{bmatrix}.$$

Then

$$P^{-1} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i & -i & -1 \\ -\frac{1}{2}i & -1 & i \\ -\frac{1}{4} + \frac{1}{4}i & \frac{1}{2} + \frac{1}{2}i & 1 \end{bmatrix}.$$

Since

$$P^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i & -i & -1 \\ -\frac{1}{2}i & -1 & i \\ -\frac{1}{4} + \frac{1}{4}i & \frac{1}{2} + \frac{1}{2}i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2}i \\ \frac{3}{4} + \frac{1}{4}i \end{bmatrix},$$

the vector $(1,0,1)$ has coordinates $\left(-\frac{1}{2} - \frac{1}{2}i, \frac{1}{2}i, \frac{3}{4} + \frac{1}{4}i\right)$ in the given basis. □

2.4.3 Exercise 3

Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for R^3 consisting of

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1) \quad \alpha_3 = (1, 0, 0).$$

What are the coordinates of the vector (a, b, c) in the ordered basis $\mathcal{B}$?

Solution. Let

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Then

$$P^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

and

$$P^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b - c \\ b \\ a - 2b + c \end{bmatrix}.$$

So (a, b, c) has coordinates $(b - c, b, a - 2b + c)$ in the ordered basis $\mathcal{B}$. □

2.4.4 Exercise 4

Let W be the subspace of C^3 spanned by $\alpha_1 = (1, 0, i)$ and $\alpha_2 = (1 + i, 1, -1)$.

(a) Show that α_1 and α_2 form a basis for W.

Solution. Since neither α_1 nor α_2 is a scalar multiple of the other, the set $\{\alpha_1, \alpha_2\}$ is linearly independent. Hence this set is a basis for W. □

(b) Show that the vectors $\beta_1 = (1, 1, 0)$ and $\beta_2 = (1, i, 1 + i)$ are in W and form another basis for W.

Solution. If $c_1(1, 0, i) + c_2(1 + i, 1, -1) = (1, 1, 0)$, then equating coordinates and solving the resulting system gives $c_1 = -i$ and $c_2 = 1$. Therefore β_1 is in W and its coordinates in $\{\alpha_1, \alpha_2\}$ are $(-i, 1)$.

In a similar way, we can determine that β_2 is in W and has coordinates $(2 - i, i)$ in the given basis.

Neither β_1 nor β_2 is a scalar multiple of the other, so the set $\{\beta_1, \beta_2\}$ is linearly independent. Since W has dimension 2, the set $\{\beta_1, \beta_2\}$ is also a basis for W. □

(c) What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for W?

Solution. From the coordinates for β_1 and β_2 that we found previously, we get the transition matrix

$$P = \begin{bmatrix} -i & 2-i \\ 1 & i \end{bmatrix}.$$

This matrix has inverse

$$P^{-1} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i & \frac{3}{2} + \frac{1}{2}i \\ \frac{1}{2} + \frac{1}{2}i & -\frac{1}{2} + \frac{1}{2}i \end{bmatrix},$$

so

$$\alpha_1 = \left(\frac{1}{2} - \frac{1}{2}i\right)\beta_1 + \left(\frac{1}{2} + \frac{1}{2}i\right)\beta_2$$

and

$$\alpha_2 = \left(\frac{3}{2} + \frac{1}{2}i\right)\beta_1 + \left(-\frac{1}{2} + \frac{1}{2}i\right)\beta_2. \qquad \square$$

2.4.6 Exercise 6

Let V be the vector space over the complex numbers of all functions from R into C, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

(a) Prove that f_1, f_2, and f_3 are linearly independent.

Proof. Let c_1, c_2, and c_3 be complex numbers such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

for all x in R. Then

$$c_1 + c_2 e^{ix} + c_3 e^{-ix} = 0.$$

Using Euler's formula, we can write

$$c_1 + c_2(\cos x + i \sin x) + c_3(\cos x - i \sin x) = 0$$

or, rearranging,

$$c_1 + (c_2 + c_3)\cos x + (c_2 - c_3) i \sin x = 0. \tag{2.1}$$

If $x = 0$, then

$$c_1 + c_2 + c_3 = 0, \tag{2.2}$$

while if $x = \pi$ we get

$$c_1 - c_2 - c_3 = 0. \tag{2.3}$$

Equations (2.2) and (2.3) together imply that $c_1 = 0$.

Next, letting $x = \pi/2$ in (2.1), we get

$$(c_2 - c_3)i = 0, \tag{2.4}$$

which implies that $c_2 = c_3$. Equation (2.2) then implies that $c_2 = c_3 = 0$.

Since it is necessary that $c_1 = c_2 = c_3 = 0$, it follows that $\{f_1, f_2, f_3\}$ is a linearly independent set. $\square$

(b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible 3×3 matrix P such that

$$g_j = \sum_{i=1}^{3} P_{ij} f_i.$$

Solution. First, we have $g_1 = f_1$. Next, since

$$f_2(x) + f_3(x) = (\cos x + i\sin x) + (\cos x - i\sin x) = 2\cos x,$$

we have $g_2 = \frac{1}{2}f_2 + \frac{1}{2}f_3$. And since

$$f_2(x) - f_3(x) = 2i\sin x,$$

we see that $g_3 = -\frac{1}{2}if_2 + \frac{1}{2}if_3$. Therefore the desired matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}i \\ 0 & \frac{1}{2} & \frac{1}{2}i \end{bmatrix},$$

and this matrix is invertible. □

2.4.7 Exercise 7

Let V be the (real) vector space of all polynomial functions from R into R of degree 2 or less, i.e., the space of all functions f of the form

$$f(x) = c_0 + c_1x + c_2x^2.$$

Let t be a fixed real number and define

$$g_1(x) = 1, \quad g_2(x) = x + t, \quad g_3(x) = (x+t)^2.$$

Prove that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V. If

$$f(x) = c_0 + c_1x + c_2x^2$$

what are the coordinates of f in this ordered basis $\mathcal{B}$?

Solution. Let

$$A = a + b(x+t) + c(x+t)^2 = (a + bt + ct^2) + (b + 2ct)x + cx^2,$$

where a, b, c are real numbers. First, if $A = 0$ then, by equating coefficients, we have

$$\begin{aligned} a + bt + ct^2 &= 0, \\ b + 2ct &= 0, \\ c &= 0. \end{aligned}$$

Working backward through the equations, we see that a, b, and c must all be 0. This shows that $\mathcal{B}$ is linearly independent.

If we now set $A = c_0 + c_1x + c_2x^2$, where c_1, c_2, c_3 are arbitrary, and equate coefficients, we get

$$\begin{aligned} a + bt + ct^2 &= c_0, \\ b + 2ct &= c_1, \\ c &= c_2. \end{aligned}$$

Through back-substitution, we find that

$$b = c_1 - 2tc_2 \quad \text{and} \quad a = c_0 - tc_1 + t^2c_2.$$

This shows that any polynomial of degree 2 or less can be written as a linear combination of g_1, g_2, and g_3. $\mathcal{B}$ is therefore a basis for V.

Moreover, we have also shown that the polynomial $f(x) = c_0 + c_1x + c_2x^2$ has coordinates

$$(c_0 - tc_1 + t^2c_2, c_1 - 2tc_2, c_2)$$

in the ordered basis $\{g_1, g_2, g_3\}$. $\square$

2.6 Computations Concerning Subspaces

2.6.1 Exercise 1

Let $s < n$ and A an $s \times n$ matrix with entries in the field F. Use Theorem 4 (not its proof) to show that there is a non-zero X in $F^{n\times 1}$ such that $AX = 0$.

Proof. Let $\alpha_1, \ldots, \alpha_n$ denote the columns of A. Then each α_i is a member of the vector space F^s, which has dimension strictly less than n. Therefore, by Theorem 4, the α_i are necessarily linearly dependent. Thus we can write

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$$

for $c_1, \ldots, c_n$ in F not all 0. If we let

$$X = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

then $AX = 0$ as required. □

2.6.2 Exercise 2

Let

$$\alpha_1 = (1, 1, -2, 1), \quad \alpha_2 = (3, 0, 4, -1), \quad \alpha_3 = (-1, 2, 5, 2).$$

Let

$$\alpha = (4, -5, 9, -7), \quad \beta = (3, 1, -4, 4), \quad \gamma = (-1, 1, 0, 1).$$

(a) Which of the vectors α, β, γ are in the subspace of R^4 spanned by the α_i?

Solution. Let A be the 3×4 matrix whose ith row is α_i. By performing row reduction on A, we get

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 3 & 0 & 4 & -1 \\ -1 & 2 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -3/13 \\ 0 & 1 & 0 & 14/13 \\ 0 & 0 & 1 & -1/13 \end{bmatrix}.$$

A vector ρ is in the row space of A if and only if

$$\rho = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \left(c_1, c_2, c_3, -\frac{3}{13}c_1 + \frac{14}{13}c_2 - \frac{1}{13}c_3\right).$$

Checking each of α, β, and γ, we see that only α is in the row space. So α is in the subspace spanned by the α_i, while β and γ are not in the subspace. □

(b) Which of the vectors α, β, γ are in the subspace of C^4 spanned by the α_i?

Solution. Our work above is still valid in C^4. α_1, α_2, and α_3 will span a larger subspace due to the scalars being taken from C instead of R, but the members of this subspace will still have the same form as before. Thus, of the three vectors α, β, and γ, only α is in the subspace. □

(c) Does this suggest a theorem?

Solution. This suggests the following theorem: let F be a subfield of the field E. Let α be a vector in F^n, and let $\beta_1, \ldots, \beta_n$ in F^n span some subspace. Then α is in this subspace of F^n if and only if it is in the subspace of E^n spanned by the same vectors β_i. □

2.6.3 Exercise 3

Consider the vectors in R^4 defined by

$$\alpha_1 = (-1, 0, 1, 2), \quad \alpha_2 = (3, 4, -2, 5), \quad \alpha_3 = (1, 4, 0, 9).$$

Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of R^4 spanned by the three given vectors.

Solution. Let A be the 3×4 matrix whose ith row is α_i. We can perform row-reduction on A to get

$$R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a vector ρ in R^4 is in the row space of A if and only if it has the form

$$\rho = \left(r_1, r_2, \frac{1}{4} r_2 - r_1, \frac{11}{4} r_2 - 2r_1 \right),$$

where r_1 and r_2 are real numbers. If we label the components of ρ as

$$\rho = (x_1, x_2, x_3, x_4),$$

then we get the following system of equations:

$$\begin{aligned} x_3 &= \frac{1}{4} x_2 - x_1 \\ x_4 &= \frac{11}{4} x_2 - 2x_1. \end{aligned}$$

Or, we can rearrange these equations to write

$$\begin{aligned} x_1 - \frac{1}{4} x_2 + x_3 \qquad &= 0 \\ 2x_1 - \frac{11}{4} x_2 \qquad + x_4 &= 0. \end{aligned}$$

This system of equations is homogeneous and its solution set is precisely the subspace spanned by α_1, α_2, and α_3. □

2.6.4 Exercise 4

In C^3, let

$$\alpha_1 = (1, 0, -i), \quad \alpha_2 = (1+i, 1-i, 1), \quad \alpha_3 = (i, i, i).$$

Prove that these vectors form a basis for C^3. What are the coordinates of the vector (a, b, c) in this basis?

Solution. Let

$$A = \begin{bmatrix} 1 & 0 & -i \\ 1+i & 1-i & 1 \\ i & i & i \end{bmatrix}.$$

By performing row-reduction, one can verify that A is row-equivalent to the identity matrix. So A has rank 3 and α_1, α_2, and α_3 are linearly independent and span C^3, as required to be a basis.

Let the coordinates of (a, b, c) in this basis be (x, y, z). This leads to the following system of equations.

$$\begin{aligned} x + (1+i)y + iz &= a \\ (1-i)y + iz &= b \\ -ix + \quad\quad y + iz &= c. \end{aligned}$$

With a bit of effort, one may determine this system to have the solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{a+b-2c}{5} + \frac{4c-2a-2b}{5}i \\ \frac{a+b-2c}{5} + \frac{3b-2a-c}{5}i \\ \frac{3a-2b-c}{5} - \frac{a+b+3c}{5}i \end{bmatrix}.$$ □

2.6.5 Exercise 5

Give an explicit description for the vectors

$$\beta = (b_1, b_2, b_3, b_4, b_5)$$

in R^5 which are linear combinations of the vectors

$$\alpha_1 = (1, 0, 2, 1, -1), \quad \alpha_2 = (-1, 2, -4, 2, 0)$$
$$\alpha_3 = (2, -1, 5, 2, 1), \quad \alpha_4 = (2, 1, 3, 5, 2).$$

Solution. Performing row-reduction on the augmented matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 2 & b_1 \\ 0 & 2 & -1 & 1 & b_2 \\ 2 & -4 & 5 & 3 & b_3 \\ 1 & 2 & 2 & 5 & b_4 \\ -1 & 0 & 1 & 2 & b_5 \end{bmatrix}$$

produces

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{2}{3}b_1 + \frac{1}{2}b_2 - \frac{1}{6}b_4 - \frac{1}{2}b_5 \\ 0 & 1 & 0 & 0 & \frac{7}{6}b_4 - \frac{5}{3}b_1 - \frac{3}{2}b_2 - \frac{1}{2}b_5 \\ 0 & 0 & 1 & 0 & \frac{3}{2}b_4 - 2b_1 - \frac{5}{2}b_2 - \frac{1}{2}b_5 \\ 0 & 0 & 0 & 1 & \frac{4}{3}b_1 + \frac{3}{2}b_2 - \frac{5}{6}b_4 + \frac{1}{2}b_5 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}.$$

Thus we see that $\beta = (b_1, b_2, b_3, b_4, b_5)$ is in the subspace spanned by α_1, α_2, α_3, α_4 if and only if $b_3 + b_2 - 2b_1 = 0$. For any such β, we have

$$\begin{aligned}\beta = \left(\frac{2}{3}b_1 + \frac{1}{2}b_2 - \frac{1}{6}b_4 - \frac{1}{2}b_5\right)\alpha_1 + \left(\frac{7}{6}b_4 - \frac{5}{3}b_1 - \frac{3}{2}b_2 - \frac{1}{2}b_5\right)\alpha_2 \\ + \left(\frac{3}{2}b_4 - 2b_1 - \frac{5}{2}b_2 - \frac{1}{2}b_5\right)\alpha_3 + \left(\frac{4}{3}b_1 + \frac{3}{2}b_2 - \frac{5}{6}b_4 + \frac{1}{2}b_5\right)\alpha_4.\end{aligned}$$

□

2.6.6 Exercise 6

Let V be the real vector space spanned by the rows of the matrix

$$A = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}.$$

(a) Find a basis for V.

Solution. If we perform row-reduction on the matrix A, we get the row-reduced echelon matrix

$$R = \begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The three nonzero rows ρ_1, ρ_2, and ρ_3 of R form a basis for V. □

(b) Tell which vectors $(x_1, x_2, x_3, x_4, x_5)$ are elements of V.

Solution. If we take linear combinations of ρ_1, ρ_2, and ρ_3, we can see that the vector $(x_1, x_2, x_3, x_4, x_5)$ is in V if and only if

$$x_2 = 7x_1 \quad \text{and} \quad x_4 = 3x_1 + 5x_3.$$ □

(c) If $(x_1, x_2, x_3, x_4, x_5)$ is in V what are its coordinates in the basis chosen in part (a)?

Solution. Let $x = (x_1, x_2, x_3, x_4, x_5)$ be in V. If

$$x = c_1\rho_1 + c_2\rho_2 + c_3\rho_3,$$

then we get the following system of equations:

$$\begin{aligned} c_1 &= x_1 \\ 7c_1 &= x_2 \\ c_2 &= x_3 \\ 3c_1 + 5c_2 &= x_4 \\ c_3 &= x_5. \end{aligned}$$

We see that x has coordinates (x_1, x_3, x_5) in the basis $\{\rho_1, \rho_2, \rho_3\}$. □

2.6.7 Exercise 7

Let A be an $m \times n$ matrix over the field F, and consider the system of equations $AX = Y$. Prove that this system of equations has a solution if and only if the row rank of A is equal to the row rank of the augmented matrix of the system.

Proof. Let R be the row-reduced echelon matrix that is row-equivalent to A. Form the augmented matrix A' and let R' be the row-reduced echelon matrix row-equivalent to A'. Then the nonzero rows of R form a basis for the row space of A, and the nonzero rows of R' form a basis for the row space of A'. We want to show that these bases have the same number of elements.

By the nature of the process of row reduction, it must be that the first n columns of R' will be identical to the n columns of R. Consequently, R' cannot have fewer nonzero rows than R, as any nonzero row of R must correspond to a nonzero row in R'. However, it might be possible for R' to have more nonzero rows than R. Such nonzero rows would need to have zeros in every column except the last. But then such a row would indicate that the system $AX = Y$ has no solutions, which we know to be false. Therefore A and A' have the same row rank.

Now let us consider the converse. Let R and R' be as before, and suppose that the row ranks of A and A' are equal. If $AX = Y$ has no solutions, then R' would necessarily have a row consisting of zeros in every column but the last. But then the corresponding row in R would have only zero entries, resulting in R' having a larger row rank than R. This is impossible, so $AX = Y$ must have a solution. $\square$