Econ Problem Set #2-1: Dynamic Programming I

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Note: All Python codes for this problem set are in the file "dp1.py."

Exercise 1.

Solution. The spectral radius condition that all of the eigenvalues are less than 1 in absolute value is satisfied. This means that the equation has a unique solution. We first solve this with matrix algebra, rearranging the equation to solve $A_1 X = b$ where:

$$A_1 = \begin{pmatrix} 0.4 & -0.1 & 0.3 \\ -0.5 & 1.4 & -0.2 \\ -1.0 & 0.2 & -0.1 \end{pmatrix}$$

Matrix b remains the same as before. By multiplying both sides by the inverse of A_1, we obtain:

$$x = \begin{pmatrix} -0.89552239 \\ 13.34328358 \\ 45.64179104 \end{pmatrix}$$

This is the same solution that we obtain by the method of successive approximations in the attached Python code.

Exercise 2.

Solution. We show that there always exists a unique solution to the equation. Define:

$$T = c(1-\beta) + \beta \sum_{k=1}^{K} max\{w_k, x\} p_k.$$

Then because $\mathbb{R}$ is a complete metric space and T maps from $\mathbb{R}$ to $\mathbb{R}$, there is always a unique solution if T is a contraction map by Banach's Fixed Point Theorem. Fix any x_1 and x_2 in $\mathbb{R}$. Then:

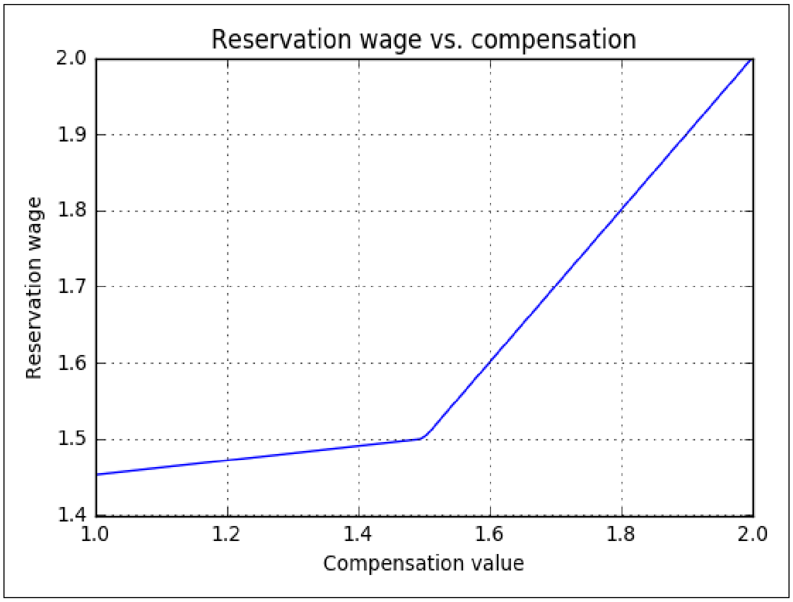
$$\begin{aligned} |T(x_1) - T(x_2)| &= |\beta \sum_{k=1}^{K} max\{w_k, x_1\} p_k - \beta \sum_{k=1}^{K} max\{w_k, x_2\} p_k| \\ &= \beta \sum_{k=1}^{K} p_k (max\{w_k, x_1\} - max\{w_k, x_2\})| \\ &\leq \beta \sum_{k=1}^{K} p_k |(max\{w_k, x_1\} - max\{w_k, x_2\})| \\ &\leq \beta \sum_{k=1}^{K} p_k |x_1 - x_2| = \beta |x_1 - x_2| \end{aligned}$$

Therefore T satisfies the definition of a contraction mapping, and there will always be a unique solution to the equation. To solve the equation, we can use a process of successive approximations, selecting an initial value for x and iterating until a solution converges (a process that we follow exactly in Exercise 3).

Exercise 3.

Solution.

The plot of the reservation wage as a function of compensation values is shown below:



The reservation wage is increasing in the compensation value. This coincides with our intuition because if an individual is being offered more money for not working, the opportunity cost of working at a lower salary is greater. Notable in the plot is the corner at $c = 1.5$. This occurs because 1.5 is the highest possibility for a wage, so at that point the reservation wage is the same as the compensation value.