Math Problem Set #4: Introduction to Optimization

OSM Lab - University of Chicago
Geoffrey Kocks

Problem 1: HJ 6.1.

Solution. We can rewrite the problem in standard form as:

$$\min_{\mathbf{w}} -e^{-\mathbf{w}^T\mathbf{x}}$$

$$\text{subject to:}$$

$$\mathbf{w}^T A\mathbf{w} - \mathbf{w}^T A\mathbf{y} - \mathbf{w}^T\mathbf{x} \leq -a$$

$$\mathbf{y}^T - \mathbf{w}^T\mathbf{x} = b$$

Problem 2: HJ 6.5.

Solution. Let K denote the number of knobs produced and let M denote the number of milk cartons produced. Let L denote the minutes of labor and let P denote the grams of plastic. Then in standard form, the optimization problem can be written as:

$$\min_{K,M} -0.07M - 0.05K$$

$$\text{subject to:}$$

$$4M + 3K \leq 240000$$

$$2M + K \leq 6000$$

Problem 3: HJ 6.6.

Solution. The first order conditions for critical points are given by:

$$f_x = 6xy + 4y^2 + y = 0$$

$$f_y = 3x^2 + 8xy + x = 0$$

This system can easily be solved by substitution: $8xy = -3x^2 - x \implies x = 0$ or $y = \frac{-3x-1}{8} \implies 6x(\frac{-3x-1}{8}) + 4(\frac{-3x-1}{8})^2 + \frac{-3x-1}{8} = 0$. Solving this quadratic equation for x gives $x = -1/3$ and $x = -1/9$. Plugging these in and solving for y we obtain the following critical points:

$$(x, y) = (-1/3, 0)$$

$$(x, y) = (0, -1/4)$$

$$(x, y) = (0, 0)$$

$$(x, y) = (-1/9, -1/12)$$

To determine whether they are a maximum, minimum, or saddle point, we take the second derivatives of f:

$$f_{xx} = 6y$$

$$f_{yy} = 8x$$

$$f_{xy} = f_{yx} = 6x + 8y + 1$$

Define:

$$D^2 = \begin{pmatrix} 6y & 6x + 8y + 1 \\ 6x + 8y + 1 & 8x \end{pmatrix}$$

Then $\det D = 48xy - (6x + 8y + 1)(6x + 8y + 1) = 48xy - 36x^2 - 64y^2 - 16y - 12x - 1 - 96xy = -36x^2 - 64y^2 - 16y - 12x - 1 - 48xy$.

When $(x, y) = (-1/3, 0) : \det D < 0$ so (x, y) is a saddle point.

When $(x, y) = (0, -1/4) : \det D < 0$ so (x, y) is a saddle point.

When $(x, y) = (0, 0) : \det D < 0$ so (x, y) is a saddle point.

When $(x, y) = (-1/9, -1/12) : \det D > 0$ and $f_{xx} < 0$ so (x, y) is a local maximum of f.

Problem 4: HJ 6.11.

Solution. The first iteration of Newton's Method gives $x_1 = x_0 - \frac{f'(x_0)}{f"(x_0)}$. For the function $f(x) = ax^2 + bx + c$ the first and second derivatives are $f'(x) = 2ax + b$ and $f"(x) = 2a$. Therefore in the first iteration, regardless of the initial x_0:

$$x_1 = x_0 - \frac{f'(x_0)}{f"(x_0)} = x_0 - \frac{2ax_0 + b}{2a} = \frac{-b}{2a}$$

We can easily verify that $-b/2a$ is the unique minimizer of $f(x)$. The first order condition is $f'(x) = 2ax+b = 0 \implies x = \frac{-b}{2a}$. The second derivative is $f"(x) = 2a > 0$ when $a > 0$ so we know that $\frac{-b}{2a}$ is indeed a local minimum. The local minimum must be unique because the first derivative is linear and will therefore have only one solution to the first order condition. Therefore for any initial guess, one iteration of Newton's method will land at the unique minimizer of f.

Problem 5: HJ 6.14.

Solution. See the Jupyter notebook `newton.ipynb` in the homework solution folder for the implementation of Newton's Method.