Math Problem Set #6: Linear Optimization

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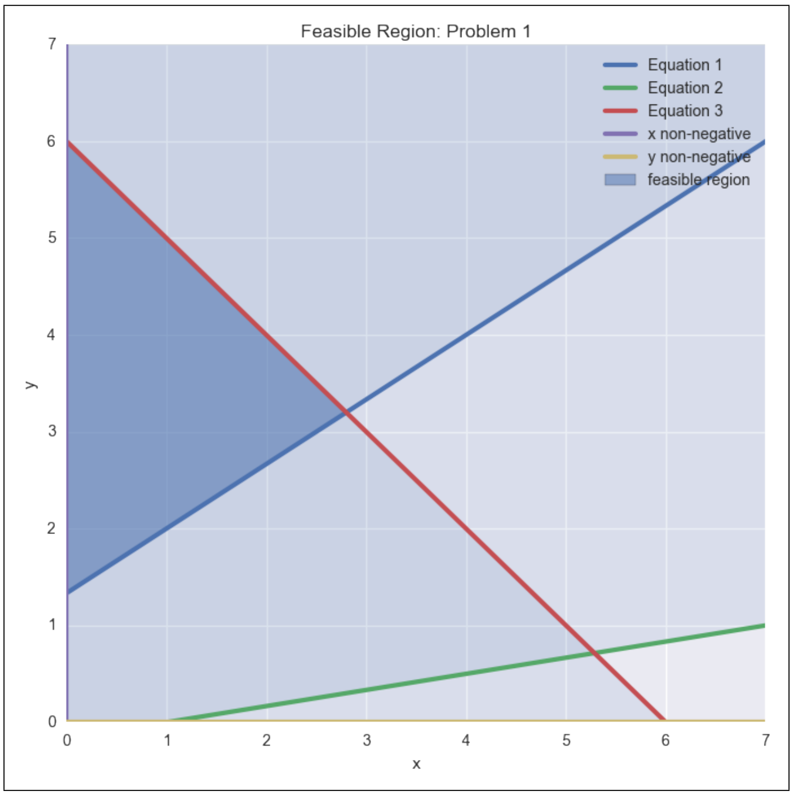
Problem 1: HJ 8.1

Solution. The plot of the feasible set (Figure 1) is shown below with the corresponding objective function (Figure 2):

Figure 1: Feasible Set



The optimizer occurs for x = 2.8, y = 3.2.

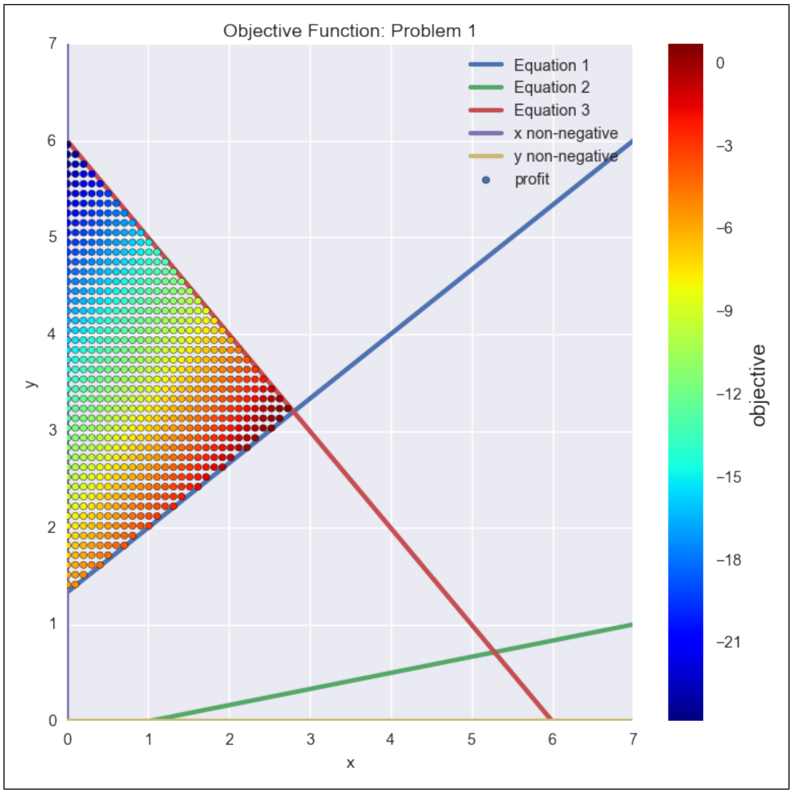
Problem 2: HJ 8.2

Solution. The objective functions of the feasible polygons for parts (i) and (ii) are shown in Figure 3 and Figure 4 respectively. By the Fundamental Theorem, the maximum will occur at one of the vertices. Checking each of the vertices this gives us:

For part (i):

$$f(0,5) = 5$$
$$f(3,4) = 13$$
$$f(6,2) = 20$$
$$f(4,0) = 12$$

Figure 2: Objective Function



$$f(0,0) = 0$$

Therefore the optimal value of 20 occurs at the optimal point $(x_1, x_2) = (6, 2)$.

For part (ii):

$$f(0,0) = 0$$

$$f(27,0) = 108$$

$$f(0,11) = 66$$

$$f(5,16) = 116$$

$$f(15,12) = 132$$

Therefore the optimal value of 132 occurs at the optimal point $(x, y) = (15, 12)$.

Problem 3: HJ 8.5

Solution. *(i.)* By the Simplex Algorithm:
Initial with slack variables:

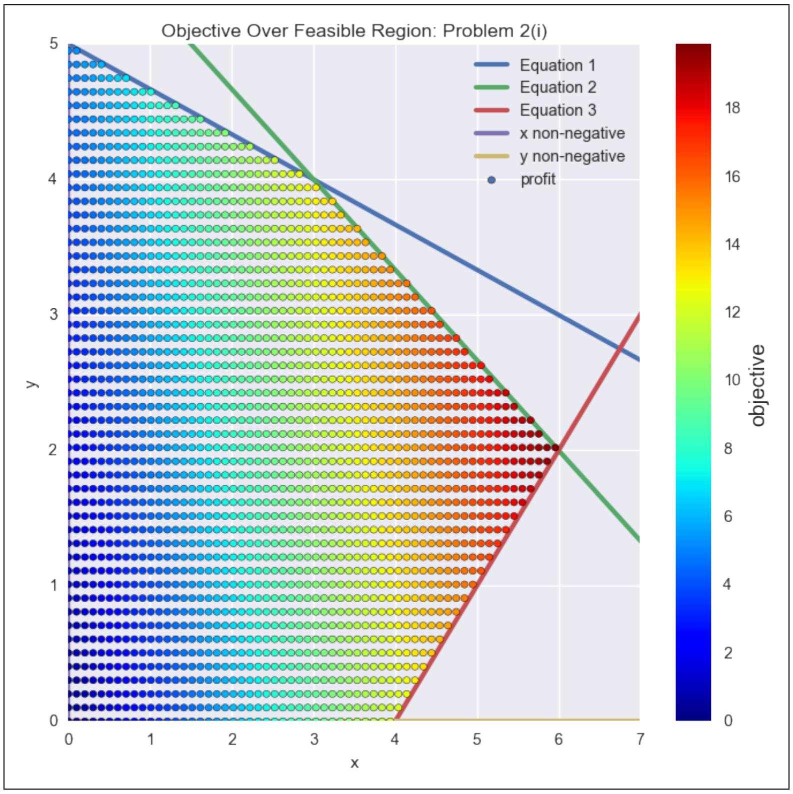
$$f = 3x_1 + x_2$$

$$w_1 = 15 - x_1 - 3x_2$$

$$w_2 = 18 - 2x_1 - 3x_2$$

$$w_3 = 4 - x_1 + x_2$$

Figure 3: Objective Function



Iteration 1:

$$f = 12 - 3w_3 + 4x_2$$

$$w_1 = 11 + w_3 - 4x_2$$

$$w_2 = 10 + 2w_3 - 5x_2$$

$$x_1 = 4 - w_3 + x_2$$

Iteration 2:

$$f = 20 - 0.8w_2 - 1.4w_3$$

$$w_1 = 3 + 0.8w_2 - 0.6w_3$$

$$x_2 = 2 - 0.2w_2 + 0.4w_3$$

$$x_1 = 6 - 0.2w_2 - 0.6w_3$$

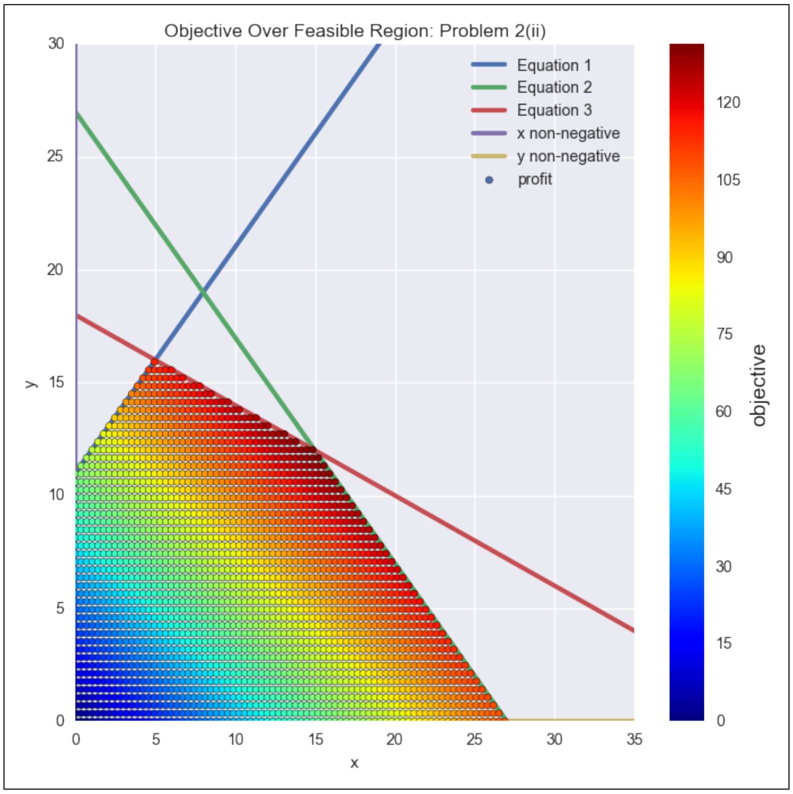
After the second iteration, all of the coefficients in the objective function are negative so we finish the process. A maximum of 20 is achieved when $w_2 = w_3 = 0 \implies x_1 = 6, x_2 = 2$.

(ii.) By the Simplex Algorithm:
Initial with slack variables:

$$f = 4x + 6y$$

$$w_1 = 11 + x - y$$

Figure 4: Objective Function



$$w_2 = 27 - x - y$$
$$w_3 = 90 - 2x - 5y$$

Iteration 1:

$$f = 108 - 4w_2 + 2y$$
$$w_1 = 38 - w_2 - 2y$$
$$x = 27 - w_2 - y$$
$$w_3 = 36 + w2_2 - 3y$$

Iteration 2:

$$f = 132 - \frac{8}{3}w_2 - \frac{2}{3}w_3$$
$$w_1 = 14 - \frac{7}{3}w_2 + \frac{2}{3}w_3$$
$$x = 15 - \frac{5}{3}w_2 + \frac{1}{3}w_3$$
$$y = 12 + \frac{2}{3}w_2 - \frac{1}{3}w_3$$

Consistent with our answer in Problem 2, this gives a maximum of 132 at $(x, y) = (15, 12)$.

Problem 4: HJ 8.7

Solution. *(i.)* See Figure 5 for the corresponding polygon for this problem. We see from the figure that the problem has a maximum of 11 at $(x_1, x_2) = (3, 4)$. We confirm this with the simplex method:

Initial with slack variables:

$$f = x_1 + 2x_2$$

$$w_1 = -8 + 4x_1 + 2x_2$$

$$w_2 = 6 + 2x_1 - 3x_2$$

$$w_3 = 3 - x_1$$

Iteration 1:

$$f = 3 - w_3 + 2x_2$$

$$w_1 = 4 - 4w_3 + 2x_2$$

$$w_2 = 12 - 2w_3 - 3x_2$$

$$x_1 = 3 - w_3$$

Iteration 2:

$$f = 11 - \frac{7}{3}w_3 - \frac{2}{3}w_2$$

$$w_1 = 12 - \frac{14}{3}w_3 - \frac{2}{3}w_3$$

$$x_2 = 4 - \frac{2}{3}w_3 - \frac{1}{3}w_2$$

$$x_1 = 3 - w_3$$

Therefore the maximum is indeed 11 at $(x_1, x_2) = (3, 4)$.

(ii.) See Figure 6 for the corresponding polygon for this problem. This problem has no solution because the region is infeasible.

(iii.) See Figure 7 for the corresponding graph for this problem. We note that the polygon is unbounded, but it still has a solution because the function is decreasing along the unbounded region. We can see from the figure that the maximum of 2 is obtained at $(x_1, x_2) = (0, 2)$. We confirm this below with the simplex method:

Initial with slack variables:

$$f = -3x_1 + x_2$$

$$w_1 = 4 - x_2$$

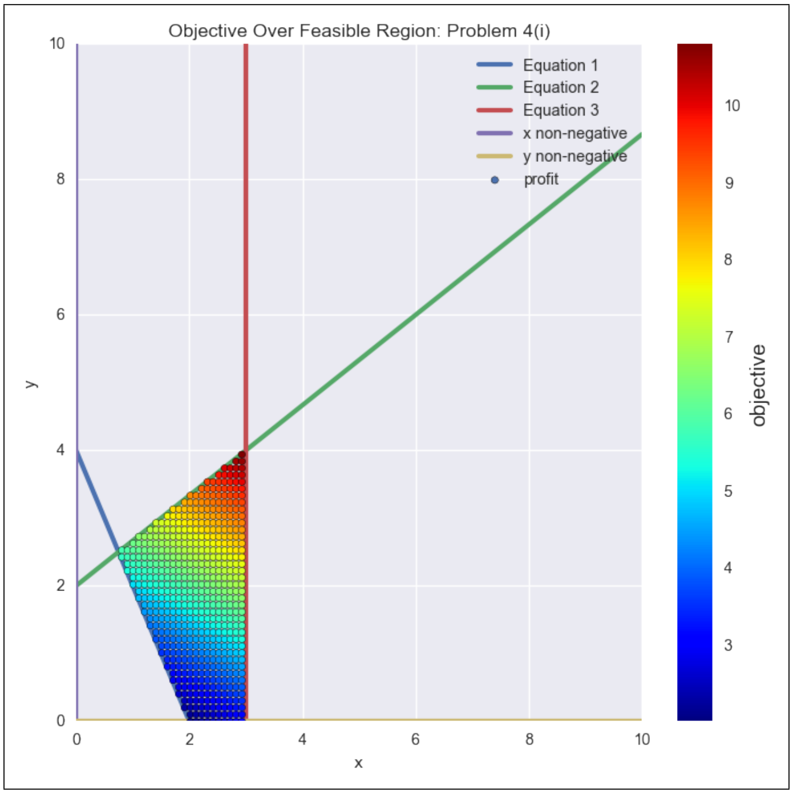
$$w_2 = 6 + 2x_1 - 3x_2$$

Iteration 1:

$$f = 2 - \frac{7}{3}x_1 - \frac{1}{3}w_2$$

$$w_1 = 2 - \frac{2}{3}x_1 + \frac{1}{3}w_2$$

Figure 5: Objective Function



$$x_2 = 2 + \frac{2}{3}x_1 - \frac{1}{3}w_2$$

The algorithm is completed after one iteration and we obtain the solution of 2 at $(x_1, x_2) = (0, 2)$.

Problem 5: HJ 8.13

Solution. Solving the primal problem is equivalent to solving the dual problem. In this case the dual problem is given by:

$$\min 0$$
$$A^T\mathbf{y} \geq 0$$
$$\mathbf{y} \geq 0$$

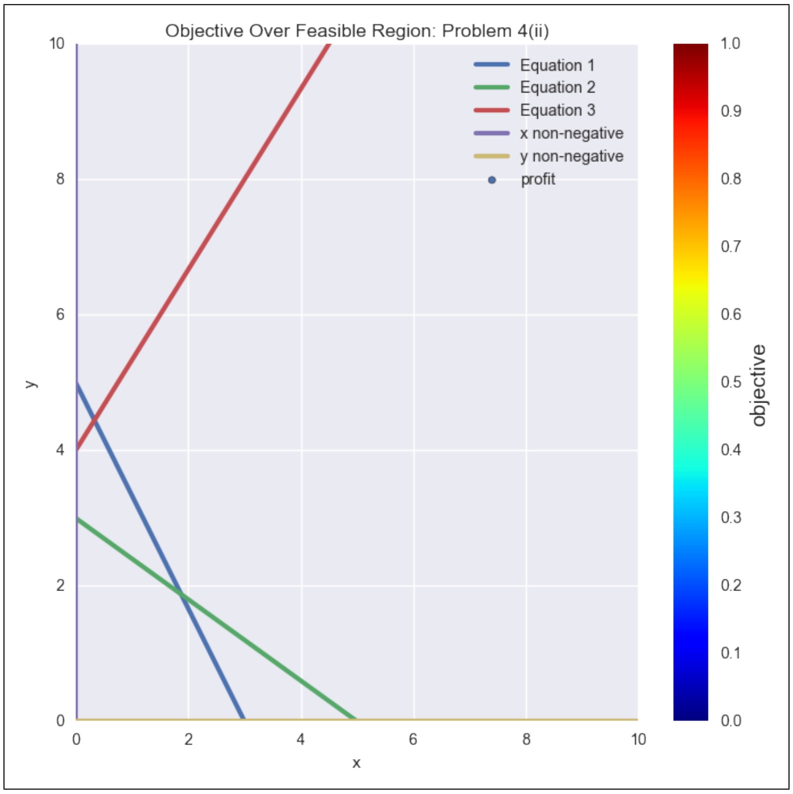
Assume that the problem is bounded. Then it follows from the dual form of the problem that the optimal value will be 0. Based on the primal problem, this will only hold when $\mathbf{x} = 0$ so 0 is an optimum point.

Now suppose that $\mathbf{x} = 0$ is not an optimum point. The dual problem will be the same, so the only way that there can be no solution to the problem is if it is unbounded.

Therefore either $\mathbf{x} = 0$ is an optimum point or the problem is unbounded.

Problem 6: HJ 8.17

Figure 6: Objective Function



Solution.
The standard form linear optimization problem is given by:

$$\max \mathbf{c}^T\mathbf{x}$$

$$A\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

The dual problem is then given by:

$$\min \mathbf{b}^T\mathbf{y}$$

$$A^T\mathbf{y} \geq \mathbf{c}$$

$$\mathbf{y} \geq 0$$

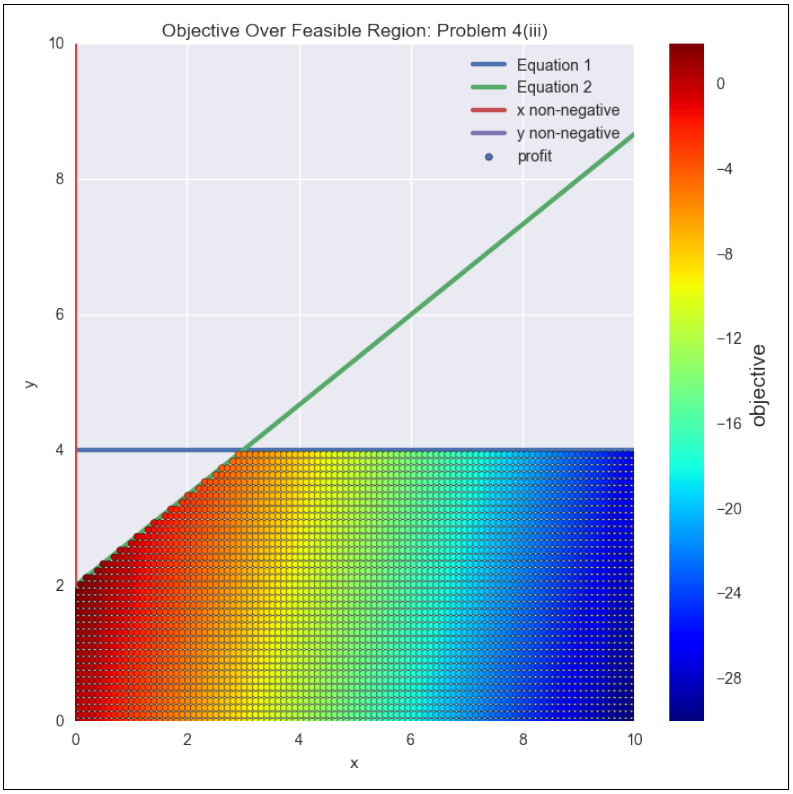
This can be written in standard form as:

$$\max -\mathbf{b}^T\mathbf{y}$$

$$-A^T\mathbf{y} \leq -\mathbf{c}$$

$$\mathbf{y} \geq 0$$

Figure 7: Objective Function



This dual problem has the dual problem:

$$\min -\mathbf{c}^T\mathbf{x}$$

$$-A\mathbf{x} \geq -\mathbf{b}$$

$$\mathbf{y} \geq 0$$

In standard form, this has the form below which is equivalent to our original problem.

$$\max \mathbf{c}^T\mathbf{x}$$

$$A\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

Therefore the dual of the dual of a linear problem is the primal problem.

Problem 7: HJ 8.18

Solution. Solving the original problem by the simplex method, we obtain:
Original with slack variables:

$$f = x_1 + x_2$$

$$w_1 = 3 - 2x_1 - x_2$$

$$w_2 = 5 - x_1 - 3x_2$$
$$w_3 = 4 - 2x_1 - 3x_2$$

Iteration 1:

$$f = 1.5 - 0.5w_1 + -0.5x_2$$
$$x_1 = 1.5 - 0.5w_1 - 0.5x_2$$
$$w_2 = 3.5 + 0.5w_1 - 2.5x_2$$
$$w_3 = 1 + w_1 - 2x_2$$

Iteration 2:

$$f = 1.75 - 0.25w_1 - 0.25w_3$$
$$x_1 = 1.25 - 0.75w_1 + 0.25w_3$$
$$w_2 = 2.25 - 0.75w_1 + 1.25w_3$$
$$x_2 = 0.5 + 0.5w_1 - 0.5w_3$$

After the second iteration, the algorithm is complete and we obtain a maximum of 1.75 at $(x_1, x_2) = (1.25, 0.5)$.

We now convert this to the equivalent dual problem of:

$$\min 3y_1 + 5y_2 + 4y_3$$

$$2y_1 + y_2 + 2y_3 \geq 1$$
$$y_1 + 3y_2 + 3y_3 \geq 1$$

To put this problem in standard form, we would take the negative of each of the equations, leading to the solution $(y_1, y_2, y_3) = (0.25, 0, 0.25)$. This gives the same optimal value of 1.75.