Economics Problem Set #5-1: DSGE Models

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Exercise 1.

Solution. We seek a policy function solution to the Brock-Mirman model of the form: $K_{t+1} = Ae^{z_t}K_t^\alpha$. The associated Euler equation for the Brock-Mirman model is given by:

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \beta E_t\{\frac{\alpha e^{z_{t+1}}(K_{t+1})^{\alpha-1}}{e^{z_t+1}(K_{t+1})^\alpha - K_{t+2}}\}$$

With our form of the policy function solution, the left hand side of the Euler equation reduces to:

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \frac{1}{e^{z_t}K_t^\alpha(1-A)}.$$

To reduce the right hand side, we note that are form of the policy function gives $K_{t+2} = Ae^{z_t+1}(k_{t+1})^\alpha$. Then the right hand side of the Euler equation reduces to:

$$\begin{aligned}
\beta E_t\{\frac{\alpha e^{z_{t+1}}(K_{t+1})^{\alpha-1}}{e^{z_t+1}(K_{t+1})^\alpha - K_{t+2}}\} &= \beta E_t\{\frac{\alpha e^{z_{t+1}}(K_{t+1})^{\alpha-1}}{e^{z_t+1}(K_{t+1})^\alpha - Ae^{z_{t+1}}K_{t+1}^\alpha}\} \\
&= \beta E_t\{\frac{\alpha K_{t+1}^{-1}}{1-A}\} = \beta E_t\{\frac{\alpha}{K_{t+1}(1-A)}\} \\
&= \beta E_t\{\frac{\alpha}{Ae^{z_t}K_t^\alpha(1-A)}\} \\
&= \frac{\alpha\beta}{e^{z_t}K_t^\alpha A(1-A)}
\end{aligned}$$

We note that in order for this to equal the left hand side of the Euler equation, we need to have $\frac{\alpha\beta}{A} = 1$. Therefore the solution of the policy function is given when $A = \alpha\beta$ so the policy function has an algebraic solution $K_{t+1} = \alpha\beta e^{z_t}K_t^\alpha$.

Exercise 2.

Solution. The first four characterizing equations will always be the market clearing conditions: $\ell_t = L_t, k_t = K_t, w_t = W_t, r_t = R_t$. We note that the given utility function and production function have the following partial derivatives:

$$u_c(c_t, \ell_t) = \frac{1}{c_t}$$

$$u_\ell(c_t, \ell_t) = \frac{-a}{1-\ell_t}$$

$$f_K(k_t, \ell_t, z_t) = \alpha e^{z_t}K_t^{\alpha-1}L_T^{1-\alpha}$$

$$f_L(k_t, \ell_t, z_t) = (1-\alpha)e^{z_t}K_t^\alpha L_t^{-\alpha}$$

We are then left with the following equations 1-7:

$$c_t = (1-\tau)(w_t\ell_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1} \tag{1}$$

$$\frac{1}{c_t} = \beta E_t\{\frac{1}{c_{t+1}}((r_{t+1} - \delta)(1-\tau) + 1)\} \tag{2}$$

$$\frac{a}{1-\ell_t} = \frac{w_t}{c_t}(1-\tau) \tag{3}$$

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L_T^{1-\alpha} \tag{4}$$

$$w_t = (1-\alpha)e^{z_t} K_t^{\alpha} L_t^{-\alpha} \tag{5}$$

$$\tau(w_t\ell_t + (r_t - \delta)k_t) = T_t \tag{6}$$

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \epsilon_t \sim \text{i.i.d.}(0, \sigma_z^2) \tag{7}$$

In this case, we cannot use the same guess and check method to obtain the policy function as in Exercise 1 because we have additional non-state variables that were not included in Exercise 1, so it would be difficult to obtain a closed form solution.

Exercise 3.

Solution. The first four characterizing equations will always be the market clearing conditions: $\ell_t = L_t, k_t = K_t, w_t = W_t, r_t = R_t$. We note that the given utility function and production function have the following partial derivatives:

$$u_c(c_t, \ell_t) = c_t^{-\gamma}$$

$$u_\ell(c_t, \ell_t) = \frac{-a}{1-\ell_t}$$

$$f_K(k_t, \ell_t, z_t) = \alpha e^{z_t} K_t^{\alpha-1} L_T^{1-\alpha}$$

$$f_L(k_t, \ell_t, z_t) = (1-\alpha)e^{z_t} K_t^{\alpha} L_t^{-\alpha}$$

We are then left with the following equations 8-14:

$$c_t = (1-\tau)(w_t\ell_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1} \tag{8}$$

$$c_t^{-\gamma} = \beta E_t\{{c_{t+1}}^{-\gamma}((r_{t+1} - \delta)(1-\tau) + 1)\} \tag{9}$$

$$\frac{a}{1-\ell_t} = w_t {c_t}^{-\gamma}(1-\tau) \tag{10}$$

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L_T^{1-\alpha} \tag{11}$$

$$w_t = (1-\alpha)e^{z_t} K_t^{\alpha} L_t^{-\alpha} \tag{12}$$

$$\tau(w_t\ell_t + (r_t - \delta)k_t) = T_t \tag{13}$$

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \epsilon_t \sim \text{i.i.d.}(0, \sigma_z^2) \tag{14}$$

Exercise 4.

Solution. The first four characterizing equations will always be the market clearing conditions: $\ell_t = L_t, k_t = K_t, w_t = W_t, r_t = R_t$. We note that the given utility function and production function have the following partial derivatives:

$$u_c(c_t, \ell_t) = c_t^{-\gamma}$$

$$u_\ell(c_t, \ell_t) = -a(1-\ell_t)^{-\xi}$$

$$f_K(k_t, \ell_t, z_t) = \alpha e^{z_t}(\alpha K_t^n + (1-\alpha)L_t^n)^{1/n-1}({K_t}^{n-1})$$

$$f_L(k_t, \ell_t, z_t) = (1-\alpha)e^{z_t}(\alpha K_t^n + (1-\alpha)L_t^n)^{1/n-1}({L_t}^{n-1})$$

We are then left with the following equations 15-21:

$$c_t = (1-\tau)(w_t\ell_t + (r_t-\delta)k_t) + k_t + T_t - k_{t+1} \tag{15}$$

$$c_t^{-\gamma} = \beta E_t\{{c_{t+1}}^{-\gamma}((r_{t+1}-\delta)(1-\tau)+1)\} \tag{16}$$

$$a(1-\ell_t)^{-\xi} = w_t {c_t}^{-\gamma}(1-\tau) \tag{17}$$

$$r_t = \alpha e^{z_t}(\alpha K_t^n + (1-\alpha)L_t^n)^{1/n-1}({K_t}^{n-1}) \tag{18}$$

$$w_t = (1-\alpha)e^{z_t}(\alpha K_t^n + (1-\alpha)L_t^n)^{1/n-1}({L_t}^{n-1}) \tag{19}$$

$$\tau(w_t\ell_t + (r_t - \delta)k_t) = T_t \tag{20}$$

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \epsilon_t \sim \text{i.i.d.}(0, \sigma_z^2) \tag{21}$$

Exercise 5.

Solution. The first four characterizing equations will always be the market clearing conditions: $\ell_t = L_t, k_t = K_t, w_t = W_t, r_t = R_t$. We note that the given utility function and production function have the following partial derivatives:

$$u_c(c_t, \ell_t) = c_t^{-\gamma}$$

$$u_\ell(c_t, \ell_t) = 0$$

$$f_K(k_t, \ell_t, z_t) = \alpha K_t^{\alpha-1}(L_t e^{z_t})^{1-\alpha} = \alpha K_t^{\alpha-1}(e^{z_t})^{1-\alpha}$$

$$f_L(k_t,\ell_t,z_t) = (1-\alpha)K_t^{\alpha}L_t^{-\alpha}(e^{z_t})^{1-\alpha} = (1-\alpha)K_t^{\alpha}(e^{z_t})^{1-\alpha}$$

We are then left with the following equations 22-28:

$$c_t = (1-\tau)(w_t + (r_t-\delta)k_t) + k_t + T_t - k_{t+1} \tag{22}$$

$$c_t^{-\gamma} = \beta E_t\{{c_{t+1}}^{-\gamma}((r_{t+1}-\delta)(1-\tau)+1)\} \tag{23}$$

$$w_t {c_t}^{-\gamma}(1-\tau) = 0 \tag{24}$$

$$r_t = \alpha K_t^{\alpha-1}(e^{z_t})^{1-\alpha} \tag{25}$$

$$w_t = (1-\alpha)K_t^{\alpha}(e^{z_t})^{1-\alpha} \tag{26}$$

$$\tau(w_t + (r_t-\delta)k_t) = T_t \tag{27}$$

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \epsilon_t \sim \text{i.i.d.}(0,\sigma_z^2) \tag{28}$$

In the steady state form, these can be represented as follows (equations 29-35):

$$\bar{c} = (1-\tau)(\bar{w} + (\bar{r}-\delta)\bar{k}) + \bar{T} \tag{29}$$

$$\bar{c}^{-\gamma} = \beta E_t\{\bar{c}^{-\gamma}((\bar{r}-\delta)(1-\tau)+1)\} \tag{30}$$

$$\bar{w}\bar{c}^{-\gamma}(1-\tau) = 0 \tag{31}$$

$$\bar{r} = \alpha\bar{k}^{\alpha-1}(e^{\bar{z}})^{1-\alpha} \tag{32}$$

$$\bar{w} = (1-\alpha)\bar{k}^{\alpha}(e^{\bar{z}})^{1-\alpha} \tag{33}$$

$$\tau(\bar{w} + (\bar{r}-\delta)\bar{k}) = \bar{T} \tag{34}$$

We know immediately from Equation 31 that $\bar{w} = 0$. We also know from the initial conditions that $\bar{z} = 0$ so we can solve for each of the remaining variables by recursive substitution. We note that this will yield the same answers as solving the equation numerically. The code is attached in the file `Part1_Exercises5_6`.

Exercise 6.

Solution. The first four characterizing equations will always be the market clearing conditions: $\ell_t = L_t, k_t = K_t, w_t = W_t, r_t = R_t$. We note that the given utility function and production function have the following partial derivatives:

$$u_c(c_t,\ell_t) = c_t^{-\gamma}$$

$$u_\ell(c_t,\ell_t) = -a(1-\ell_t)^{-\xi}$$
$$f_K(k_t,\ell_t,z_t) = \alpha K_t^{\alpha-1}(L_t e^{z_t})^{1-\alpha} = \alpha K_t^{\alpha-1}(e^{z_t})^{1-\alpha}$$
$$f_L(k_t,\ell_t,z_t) = (1-\alpha)K_t^{\alpha}L_t^{-\alpha}(e^{z_t})^{1-\alpha} = (1-\alpha)K_t^{\alpha}(e^{z_t})^{1-\alpha}$$

We are then left with the following equations 35-41:

$$c_t = (1-\tau)(w_t + (r_t-\delta)k_t) + k_t + T_t - k_{t+1} \tag{35}$$

$$c_t^{-\gamma} = \beta E_t\{{c_{t+1}}^{-\gamma}((r_{t+1}-\delta)(1-\tau)+1)\} \tag{36}$$

$$a(1-\ell_t)^{-\xi} = w_t {c_t}^{-\gamma}(1-\tau) \tag{37}$$

$$r_t = \alpha K_t^{\alpha-1}(e^{z_t})^{1-\alpha} \tag{38}$$

$$w_t = (1-\alpha)K_t^{\alpha}(e^{z_t})^{1-\alpha} \tag{39}$$

$$\tau(w_t + (r_t-\delta)k_t) = T_t \tag{40}$$

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \epsilon_t \sim \text{i.i.d.}(0,\sigma_z^2) \tag{41}$$

In the steady state form, these can be represented as follows (equations 42-47):

$$\bar{c} = (1-\tau)(\bar{w} + (\bar{r}-\delta)\bar{k}) + \bar{T} \tag{42}$$

$$\bar{c}^{-\gamma} = \beta E_t\{\bar{c}^{-\gamma}((\bar{r}-\delta)(1-\tau)+1)\} \tag{43}$$

$$\bar{w}\bar{c}^{-\gamma}(1-\tau) = a(1-\bar{\ell})^{-\xi} \tag{44}$$

$$\bar{r} = \alpha\bar{k}^{\alpha-1}(e^{\bar{z}})^{1-\alpha} \tag{45}$$

$$\bar{w} = (1-\alpha)\bar{k}^{\alpha}(e^{\bar{z}})^{1-\alpha} \tag{46}$$

$$\tau(\bar{w} + (\bar{r}-\delta)\bar{k}) = \bar{T} \tag{47}$$

We know from the initial conditions that $\bar{z} = 0$ so we can simplify some of the equations, however we would not be able to solve for the solutions algebraically in this case because we are not given the value for $\bar{\ell}$. The code is attached in the file `Part1_Exercises5_6`.

Exercise 8.

Solution. See the Jupyter notebook `Part1_Exercise8.ipynb` in this folder for the implementation of value function iteration of the Brock-Mirman model. When we compare our solution with the closed form solution obtained in Exercise 1, we note that both solutions are increasing in z and K. However the algebraic solution is continuous whereas our value function iteration solution only applies on a discrete grid.