Economics Problem Set #5-4

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Exercise 1

Solution. By the properties of the linearity of limits and of matrix transposes, we apply the definition of the spectral density to obtain:

$$\begin{aligned} S_{x+y}(\omega) &= \lim_{T\to\infty} E(|\hat{X}^T(\omega) + \hat{Y}^T(\omega)|^2) \\ &= \lim_{T\to\infty} (E(|\hat{X}^T(\omega)|^2) + E(|\hat{Y}^T(\omega)|^2) + 2E(|\hat{X}^T(\omega)\hat{Y}^T(\omega)|)) \\ &= S_X(\omega) + S_Y(\omega) + 2Re(S_{XY}(\omega)) \end{aligned}$$

Exercise 2

Solution. The minimization function for an HP filter is given by:

$$\min_{g_t}\{\sum_{t=1}^{T}(y_t - g_t)^2 + \lambda\sum_{t=1}^{T}((g_{t+1} - g_t) - (g_t - g_{t-1}))^2\}$$

We note that as λ approaches infinity, the second term in the sum will become infinitely large so the expression will be minimized only if the second term equals 0. We verify that this is given by the solution $g_t = g_0 + \beta t$. Given this expression, the second term becomes:

$$\begin{aligned} &\lambda\sum_{t=1}^{T}((g_0 + \beta(t+1) - g_0 - \beta t) - (g_0 - \beta t - g_0 - \beta(t-1)))^2 \\ &= \lambda\sum_{t=1}^{T}(\beta t + \beta - \beta t - \beta t + \beta t - \beta)^2 = 0. \end{aligned}$$

Therefore the linear filtered series will satisfy the minimization problem.

Exercises 3-6

Solution. See the attached Jupyter notebook for the code in this section.