THE UNIVERSITY OF ARIZONA

Lang-Nishimura theorem

Gaurish Korpal

Department of Mathematics

September 13, 2021

Outline

Statement

Theorem

Lang-Nishimura theorem

Let $X \dashrightarrow Y$ be rational map between k-varieties, where Y is proper. If X has a smooth k-point, then Y has a k-point.

Corollary

Corollary

Let X and Y be smooth, proper, integral k-varieties that are birational to each other. Then X has a k-point if and only if Y has a k-point.

Proof

Overview

$$
\begin{tikzcd}
 & & \operatorname{Spec} k((t_1))\cdots((t_d)) \arrow[r, hook, "\text{Cohen}", "\text{structure}"'] \arrow[d, hook, "\text{valuative}"', "\text{criterion}"] & \operatorname{Spec} \mathbf{k}(X) \arrow[r, hook, "\text{locally finite}", "\text{presentation}"'] & U \subset X \arrow[r, "\phi"] \arrow[r, dashed, bend left, "f"] & Y \\
 & \operatorname{Spec} k((t_1))\cdots((t_{d-1})) \arrow[r, hook, "\text{model}", "\text{DVR}"'] \arrow[d, no head, squiggly] & \operatorname{Spec} k((t_1))\cdots((t_{d-1}))[[t_d]] \arrow[urrr, bend right=10, "\tilde{\sigma}"] & & & \\
\operatorname{Spec} k \arrow[r, hook, "\text{model over}", "\text{DVR}"'] & \operatorname{Spec} k[[t_1]] \arrow[uurrrr, bend right=20, "\tilde{\sigma}"] & & & &
\end{tikzcd}
$$

Step 1

We first replace X with an open neighborhood of the given smooth k-point x, i.e. we can assume X to be an integral scheme.

Limits and morphisms of finite presentation

Let X be an integral k-variety, and let Y be an arbitrary k-variety. The natural map
$\{\text{rational maps } X \dashrightarrow Y\} \to Y(\mathbf{k}(X))$

$$[\phi : U \to Y] \mapsto (\text{the composition } \operatorname{Spec} \mathbf{k}(X) \hookrightarrow U \xrightarrow{\phi} Y)$$

is a bijection.

Step 1

The given rational map $X \dashrightarrow Y$ corresponds to a $\mathbf{k}(X)$-point of Y.

Step 1.a

$$\{\text{rational maps } X \dashrightarrow Y\} := \varinjlim_{U} \mathrm{Hom}_{\mathrm{Spec}\, k}(U, Y) = \varinjlim_{U} Y(U)$$

where the direct limit is indexed over dense open subschemes U of X, with order relation induced by reverse inclusion.

Step 1.b

- X is an irreducible scheme of finite type.
- Every dense open subscheme U of X contains a dense affine open subscheme.
- The filtered inverse system of dense affine open subschemes $\operatorname{Spec} A_i$ of X is cofinal in the system of all dense open subschemes.

$$\varinjlim_U Y(U) \cong \varinjlim_i Y(\operatorname{Spec} A_i) := \varinjlim_i Y(A_i)$$

Step 1.c

- Y is a scheme of finite type over a locally Noetherian base.
- Y is locally of finite presentation.
- The functor of points of Y commutes with taking direct limits of rings

$$\varinjlim_{i} Y(A_i) = Y(\varinjlim_{i} A_i)$$

Step 1.d

- X is an integral scheme.
- $\mathbf{k}(X) = \mathrm{Frac}(A)$, for some affine open subset $U = \operatorname{Spec} A$.
- The localization by a multiplicative set equals the direct limit of localization at each element of that multiplicative set

$$\mathrm{Frac}(A) = \varinjlim_{t} A_t$$

where $t \in A \setminus \{0\}$ with the order relation induced by divisibility.
- $A_t = \mathcal{O}_U(D(t))$ where $D(t) = U \setminus V(tA)$ is a principal open subset of U, and $A_i = \mathcal{O}_X(\operatorname{Spec} A_i)$ where $\operatorname{Spec} A_i$ is dense affine open subscheme of X

$$\varinjlim_{i} A_i = \mathbf{k}(X)$$

Step 2

Cohen structure theorem

Let X be a k-scheme of finite type, with $x \in X(k)$ a regular point of X. Let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_{X,x}$ and $\widehat{\mathcal{O}}_{X,x}$ the $\mathfrak{m}_x$-adic completion of $\mathcal{O}_{X,x}$. Then we have an isomorphism of k-algebras

$$\widehat{\mathcal{O}}_{X,x} \cong k[[t_1, \ldots, t_d]]$$

where $d = \dim \mathcal{O}_{X,x}$.

Step 2

The $\mathbf{k}(X)$-point of Y corresponds to a F-point of Y, where F is iterated Laurent series field. Since

$$\mathcal{O}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{X,x} \cong k[[t_1, \ldots, t_d]] \hookrightarrow F := k((t_1))((t_2)) \cdots ((t_d))$$

implies that $\mathbf{k}(X) = \mathrm{Frac}(\mathcal{O}_{X,x})$ is embedded in F.

Step 2.a

- $\mathcal{O}_{X,x}$ is a regular local ring of dimension d containing its residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x = \mathbf{k}(x) = k$.
- Fix a coordinate system $\{f_1, \ldots, f_d\}$ for $\mathcal{O}_{X,x}$, i.e. $\mathfrak{m}_x = \langle f_1, \ldots, f_d \rangle$
- We have the k-algebra homomorphism:

$$\begin{aligned} \varphi : A = k[t_1, \ldots, t_d] &\to \mathcal{O}_{X,x} \\ t_i &\mapsto f_i \end{aligned}$$

Step 2.b

- φ induces k-algebra isomorphism

$$k[t_1, \ldots, t_d] \xrightarrow{\sim} \bigoplus_{n \geq 0} \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}$$

- In particular

$$\mathcal{O}_{X,x} = \varphi(A) + \mathfrak{m}_x^n \quad \text{and} \quad \mathfrak{m}_x^n = \varphi(\mathfrak{n}^n) + \mathfrak{m}_x^{n+1}$$

for every $n \geq 1$, where $\mathfrak{n} = \langle t_1, \ldots, t_d \rangle$.

Step 2.c

- The canonical map $\varphi_n : A/\mathfrak{n}^n \to \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is surjective because of the above equality.
- $\ker \varphi_n = \varphi^{-1}(\mathfrak{m}_x^n)/\mathfrak{n}^n$.
- $\ker \varphi_{n+1} \to \ker \varphi_n$ is also surjective.
- Inverse limit is an exact functor on

$$0 \longrightarrow (\ker \varphi_n)_n \longrightarrow (A/\mathfrak{n}^n)_n \longrightarrow (\mathcal{O}_{X,x}/\mathfrak{m}_x^n)_n \longrightarrow 0$$

- The canonical homomorphism

$$\widehat{\varphi} : \widehat{A} = k[[t_1, \dots, t_d]] \to \widehat{\mathcal{O}}_{X,x}$$

  is surjective.

Step 2.d

- $\widehat{A}$ is an integral domain.
- $\dim \widehat{A} = \dim \widehat{\mathcal{O}}_{X,x} = d < \infty$.
- $\widehat{\phi}$ is an isomorphism.

Step 3

Valuative criterion for properness

Let Y be a proper S-scheme. Then for any S-scheme $\operatorname{Spec} A$, where A is a valuation ring with $F = \operatorname{Frac}(A)$, the canonical map $Y(A) \to Y(F)$ is bijective.

Step 3 (applied d times)

The F-point of Y extends to a A-point of Y, where $F = k((t_1))\cdots((t_d))$ and $A = k((t_1))\cdots((t_{d-1}))[[t_d]]$.
The A-point of Y reduces modulo t_d to a K-point of Y, where t_d is a uniformizer of A and $K = k((t_1))\cdots((t_{d-1})) = A/\langle t_d\rangle$ is the residue field such that A-model Y of $Y_F = Y \times_{\operatorname{Spec} k} \operatorname{Spec} F$ reduces modulo t_d to the K-scheme $Y_K = Y \times_{\operatorname{Spec} k} \operatorname{Spec} K$.

Step 3.a

- Consider the A-scheme $Z = Y_A = Y \times_S \operatorname{Spec} A$.
- Base change independence of the functor of points gives

$$Y(A) = Z(A)$$

- Similarly, for the F-scheme $Z_F = Z \times_{\operatorname{Spec} A} \operatorname{Spec} F$

$$Y(F) = Z_F(F)$$

Thus it is sufficient to prove that the canonical map $Z(A) \to Z_F(F)$ is bijective, where Z is a proper A-scheme.

Step 3.b

- Let $\alpha, \beta \in Z(A) = \mathrm{Hom}_{\mathrm{Spec}\, A}(\mathrm{Spec}\, A, Z)$ be two distinct sections such that $\alpha|_{\mathrm{Spec}\, F} = \beta|_{\mathrm{Spec}\, F} \in Z_F(F)$.
- Note that $\mathrm{Spec}\, A$ is reduced and Z is separated A-scheme.
- $\mathrm{Spec}\, F$ is dense in $\mathrm{Spec}\, A$ (generic point).
- Hence $\alpha = \beta$ and the map $Z(A) \to Z_F(F)$ is injective.

Step 3.c

- Now we need to prove surjectivity of $Z(A) \to Z_F(F)$.
- Consider the section $\sigma \in Z_F(F) = \mathrm{Hom}_{\mathrm{Spec}\, F}(\mathrm{Spec}\, F, Z_F)$.
- We wish to show that it can be extended to $\tilde{\sigma} \in Z(A) = \mathrm{Hom}_{\mathrm{Spec}\, A}(\mathrm{Spec}\, A, Z)$.
- Since A is a valuation ring, $\mathrm{Spec}\, A$ has a generic point corresponding to the fraction field and a special point/closed point corresponding to the residue field (maximal ideal).

Step 3.c.i

- Let $\xi = \operatorname{Spec} F$ be the generic point of $\operatorname{Spec} A$.
- Then $z = \sigma(\xi)$ represents an element of $Z_F(F) = \{z \in Z_F \mid \mathbf{k}(z) = F\}$.
- Let $W = \overline{\{z\}} \subseteq Z$ be the closed subset endowed with the structure of a reduced (hence integral) subscheme.
- The canonical map $W \to Z$ is a closed immersion.
- W is a proper A-scheme since $\pi : W \to Z \to \operatorname{Spec} A$ is proper.
- The image of $\pi : W \to \operatorname{Spec} A$ is closed and contains ξ

$$\pi(W) = \operatorname{Spec} A$$

- z is closed in Z_F and dense in W_F

$$W_F = \{z\}$$

Step 3.c.ii

- Let $w \in W_K = W \times_{\operatorname{Spec} A} \operatorname{Spec} K \simeq \pi^{-1}(s)$ where $K = A/\mathfrak{m} = \mathbf{k}(s)$ is the residue field at the closed point s of $\operatorname{Spec} A$ corresponding to the maximal ideal of A.
- The local ring $\mathcal{O}_{W,w}$ dominates the valuation ring A in the field $\mathcal{O}_{W,z} = \mathbf{k}(W) = \operatorname{Frac}(A) = F$

$$\mathcal{O}_{W,w} = A$$

- The canonical map $\operatorname{Spec} \mathcal{O}_{W,w} \to W$ with z in its image.
- $\tilde{\sigma} \in Z(A)$ defined as

$$\tilde{\sigma} : \operatorname{Spec} A = \operatorname{Spec} \mathcal{O}_{W,w} \to W \to Z$$

such that $\tilde{\sigma}|_{\operatorname{Spec} F} = \sigma$.

Epilogue

Verifications

Y must be proper

Let $X = \operatorname{Proj} \mathbb{F}_2[x, y, z]/\langle y^2z - yz^2 + x^2z - x^3\rangle$ be an elliptic curve. Then $\#X(\mathbb{F}_2) = 5$. Next, let $Y = X \setminus X(\mathbb{F}_2)$, which is not a projective curve, hence not proper. Then we have a rational map $X \dashrightarrow Y$ defined by the identity morphism on Y, such that Y doesn't have a $\mathbb{F}_2$-point even though X has smooth $\mathbb{F}_2$-points.

Verifications

The k-point of X must be smooth

Let $X = \operatorname{Spec} \mathbb{R}[x,y]/\langle x^2+y^2\rangle$ and $Y = \operatorname{Proj} \mathbb{R}[u,v]/\langle u^2+v^2\rangle$. Then $X(\mathbb{R})$ has only one $\mathbb{R}$-point (corresponding to the origin) and it is not smooth. Moreover, Y is a proper $\mathbb{R}$-scheme. Then we have a rational map corresponding to $(x,y) \mapsto [x:y]$ (affine to projective coordinates), such that Y doesn't have $\mathbb{R}$-points even though X has a $\mathbb{R}$-point.

Applications

Severi-Brauer varieties (François Châtelet, 1944)

Let X be a Severi-Brauer variety of dimension $n-1$ over the field k. If X is birational to $\mathbb{P}_k^{n-1}$ then $X(k) \neq \emptyset$.

Applications

del Pezzo surfaces

- Let X be a degree 6 del Pezzo surface over a field k. If there exist separable extensions K and L with $[K : k] = 2$ and $[L : k] = 3$ such that X has a K-point and an L-point, then X has a k-point.
- Every degree 5 del Pezzo surface has a k-point.