Logarithmic Functions

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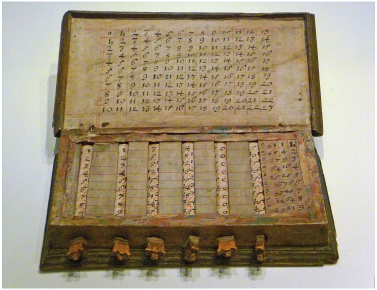
Logarithm

The word *logarithm* was coined by **John Napier** in 1614. It has been derived from the the Greek words **logos** and **arithmes**, which roughly translates to: ratio-number.

(a) Bust of John Napier (1550 - 1617) at Edinburgh Napier University, Scotland. (Credit: Kim Traynor [CC BY-SA 3.0](https://upload.wikimedia.org/wikipedia/commons/8/82/Bust_of_the_mathematician_John_Napier_of_Merchiston.jpg))



(b) A box of Napier's mathematical calculating tables from around 1680. An exhibit in the National Museum of Scotland. (Credit: Kim Traynor [CC BY-SA 3.0](https://upload.wikimedia.org/wikipedia/commons/a/af/Napier%27s_calculating_tables.JPG))

Notation

If b is any positive number not equal to 1, and $x > 0$ then

$$y = \log_b(x)$$

is called the "log base b of x." Use your calculator to find the value of

- $\log_2(2)$
- $\log_{0.5}(2)$
- $\log_{100}(1)$
- $\log_{21}(1)$
- $\log_{18}(0.5)$
- $\log_9\left(\frac{1}{9}\right)$

Steps: Press **math** and then select the eleventh option from the menu, **logBASE(**. If using the CLASSIC mode, to evaluate $\log_b(x)$ enter **logBASE(x,b)**.

If your calculator doesn't have **logBASE(** option, then use the base change formula: $\log_b(x) = \dfrac{\log(x)}{\log(b)} = \dfrac{\ln(x)}{\ln(b)}$ (ratio-number?).

Definition

The logarithm is the **inverse function** to exponentiation. That means the logarithm of a given number x is the exponent to which another fixed number, the base b, must be raised, to produce that number x:

$$y = \log_b(x) \quad \text{is equivalent to} \quad b^y = x$$

Therefore, we have:

- $y = \log_2(2)$ is equivalent to $2^y = 2$, hence $y = 1$.
- $y = \log_{0.5}(2)$ is equivalent to $0.5^y = 2$, hence $y = -1$.
- $y = \log_{100}(1)$ is equivalent to $100^y = 1$, hence $y = 0$.
- $y = \log_{21}(1)$ is equivalent to $21^y = 1$, hence $y = 0$.
- $y = \log_{18}(0.5)$ is equivalent to $18^y = 0.5$, hence $y \approx -0.24$.
- $y = \log_9\left(\frac{1}{9}\right)$ is equivalent to $9^y = \frac{1}{9}$, hence $y = -1$.

Practice problem 1

Solve for x exactly: $4(3)^x = 20$. (page 129 of the workbook)

Hints:

- Simplify and re-write is as $b^x = a$
- Use calculator to solve for $x \approx 1.465$

Fun Fact: *The exponential function was discovered about 70 years after the logarithm function. So, actually, exponential should be defined as the inverse of logarithm function. However, exponential is easier to understand than logarithm.*

Practice problem 2

Suppose \$100 is invested in an account earning 5% interest compounded annually. How long will it take the amount in the account to grow to \$200? (page 125 of the workbook)

Hints:

- Recall that discrete compound interest formula is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

- Among P, r, n and t, which values are known and which are unknown? That is, if amount is the output, then what is the input?
- The answer is 14.21 (approx.). What are the measurement units? What does it mean?

Characteristics

Observe that, the logarithm function $f(x) = \log_b(x)$ has the following properties (why?):

- **Domain:** all positive real numbers, i.e. the interval $(0, \infty)$
- **Range:** all real numbers, i.e. the interval $(-\infty, \infty)$
- **x-intercept:** only one point $(1, 0)$
- **y-intercept:** does not exist
- **Horizontal asymptote:** does not exist
- **Vertical asymptote:** the line $x = 0$

Why can't the base of logarithm be equal to 1? Why can't the base of logarithm be a negative number?

Euler's number: e

The mathematical constant $e \approx 2.71$ "naturally" pops up when one tries to understand the "smoothness" of $\log_b(x)$ (i.e. calculus). This constant was discovered by **Jacob Bernoulli** in 1683, when he was trying to find the formula for the "continuous compound interest", i.e. $A = Pe^{rt}$. In 1728, **Leonhard Euler** introduced the letter e for this constant.

(c) Portrait of Jacob Bernoulli (1654-1705) by Niklaus Bernoulli (public domain (https://upload.wikimedia.org/wikipedia/commons/1/19/Jakob_Bernoulli.jpg))

(d) Portrait of Leonhard Euler (1707-1783) by Jakob Emanuel Handmann (public domain (https://upload.wikimedia.org/wikipedia/commons/d/d7/Leonhard_Euler.jpg))

Special logarithms

The following two special logarithms occur on a regular basis:

Base b	**Name of** $\log_b(x)$	**Notation and Calculator key**	**Examples**
e	natural logarithm	$\ln(x)$	used in Physics (melting or evaporation): $$W_{\mathrm{iso}} = -nRT\ln\left(\frac{V_2}{V_1}\right)$$
10	common logarithm	$\log(x)$	used in Chemistry (fruit juice v. household cleaners): $$\mathrm{pH} = -\log[H_3O^+]$$

Graphs

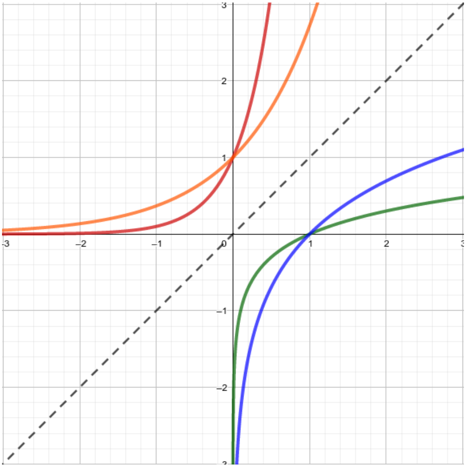


Figure: The graphs of $y = 10^x$, $y = e^x$, $y = ln(x)$ and $y = log(x)$.

Practice problem 3

For the function $f(x) = \log(3x + 2)$, determine the domain, range, intercept(s) and vertical asymptote. Graph the given function. (page 128 of the workbook)

Hints:

- The domain of $y = \log_b(\square)$ is $\square > 0$.
- The range of $y = \log_b(\square)$ is all real numbers.
- The x-intercept of a function $f(x)$ is the point $(m, 0)$ such that $f(m) = 0$. And $\log_b(\square) = 0$ when $\square = 1$.
- The y-intercept of a function $f(x)$ is the point $(0, t)$ such that $f(0) = t$.
- The vertical asymptote of $y = \log_b(\square)$ is at $\square = 0$.
- Use this information to sketch the graph.

Practice problem 4

A population of 1000 bacteria is on the kitchen counter. The amount of time (T), in hours, that it takes the population to reach x bacteria is given by

$$T = f(x) = 16 \ln\left(\frac{x}{1000}\right)$$

How long will it take for the population to double? How many bacteria will there be in 3 hours? (page 126 of the workbook)

Hints:

- What is the input and output of the given function?
- What is the first question asking us to find? What do you mean by $f(1000)$ and $f(2000)$?
- The answer to the first question is 11 (approx.). What are the measurement units? What does it mean?
- What is the second question asking us to find? What do you mean by $3 = f(x)$?
- The answer to the second question is 1206 (approx.). What are the measurement units? What does it mean?