Quadratic Functions and Applications

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Parabola

The word *parabola* has been derived from the the Greek word **parabolē**, which roughly translates to: the path followed by something thrown under the influence of gravity.

(a) GuidoB [CC BY-SA 3.0 (https://commons.wikimedia.org/wiki/File:ParabolicWaterTrajectory.jpg)]

(b) MichaelMaggs Edit by Richard Bartz [CC BY-SA 3.0 (https://commons.wikimedia.org/wiki/File:Bouncing_ball_strobe_edit.jpg)]

Parabola

This shape is essential for research and survival.

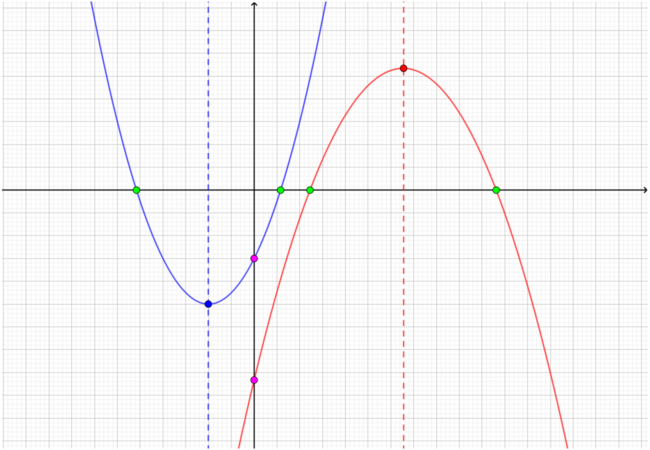
(c) The new 12m Telescope located 50 miles southwest of Tucson on Kitt Peak. Thomas Folkers [The Arizona Radio Observatory (ARO) (http://aro.as.arizona.edu/images/12_meter/Completed_12Meter-1.jpg)]

(d) Solana Generating Station located near Gila Bend, Arizona, about 70 miles southwest of Phoenix. ENERGY.GOV [Public domain]

Therefore, we should try to understand it.

Parabola

We will be dealing with only two types of parabolas: **open upwards** and **open downwards**. We can observe the key features like and "y-intercept", "x-intercepts" (at most 2), "axis of symmetry", and "vertex" (minimum/maximum).



General Form: $f(x) = ax^2 + bx + c$, $a \neq 0$

A quadratic function in general form is written as $f(x) = ax^2 + bx + c$ with $a \neq 0$.

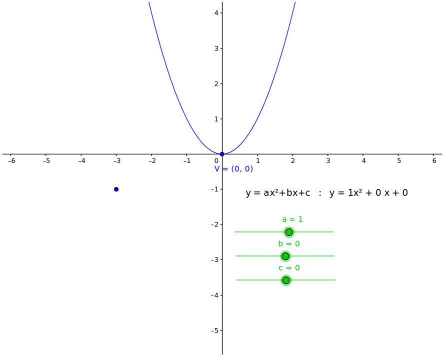


Figure: The graph contains three green sliders which we can drag to the left or right, and observe the effect it has on the graph. We can access the applet here: https://www.geogebra.org/m/FhP89QWZ#material/vkWFfTSr

General Form: $f(x) = ax^2 + bx + c$, $a \neq 0$

Therefore, we observe that:

- "a" determines how much the graph is stretched away from, or compressed towards, the x-axis. Note what happens to the graph when you set "a" to a negative value.
- "b" alters the the graph in a complex way. The graph moves along a parabolic trajectory as "b" is changed. In particular, the change in "b" causes the x-coordinate of the **vertex** to change by the same percentage, and the y-coordinate of the **vertex** to move along a parabola.
- "c" shifts (translates) the graph vertically.

General Form: $f(x) = ax^2 + bx + c$, $a \neq 0$

In particular, we have:

- Y-intercept: $(0, f(0)) \equiv (0, c)$.
- X-intercepts: if $b^2 - 4ac \geq 0$, then use the `Quadratic formula`

$$A \equiv \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, 0 \right), \qquad B \equiv \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}, 0 \right)$$

- Axis of symmetry: $x = -\dfrac{b}{2a}$
- Vertex: $V \equiv \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a} \right) \right)$

General Form: $f(x) = ax^2 + bx + c,\ a \neq 0$

Let's consider the following example:

Example 1

A stone is thrown upward. Its height in meters t seconds after release is given by

$$h(t) = -4.9t^2 + 49t + 277.4$$

1. How many seconds does it take the stone to reach its maximum height? What is the maximum height the stone reaches?
2. When does the stone strike the ground?

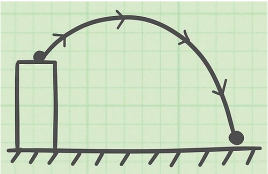


Figure: Wikivisual [CC BY-NC-SA 3.0 (https://www.wikihow.com/images/thumb/8/86/Solve-a-Projectile-Motion-Problem-Step-1.jpg/aid7279687-v4-900px-Solve-a-Projectile-Motion-Problem-Step-1.jpg)]

Factored Form: $f(x) = a(x - M)(x - N)$

A quadratic function in factored form is written as
$f(x) = a(x - M)(x - N)$ with $a \neq 0$.

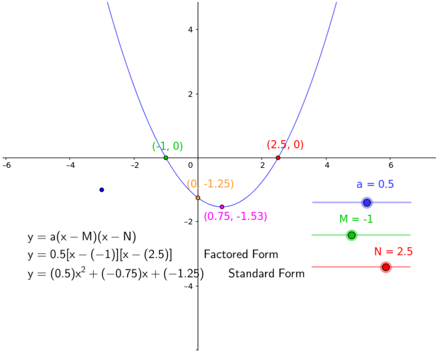


Figure: The graph contains three sliders which we can drag to the left or right, and observe the effect it has on the graph. We can access the applet here: https://www.geogebra.org/m/FhP89QWZ#material/mDmcZjVY

Factored Form: $f(x) = a(x - M)(x - N)$

Therefore, we observe that:

- "a" determines how much the graph is stretched away from, or compressed towards, the x-axis. Just as in the case of general form.
- "M" and "N" determine where the function will cross the x-axis. They also affect the position of **vertex**.

Note that, only those quadratic functions which have x-intercepts can be written in this form. Moreover, from this form we easily get the x-intercepts.

Factored Form: $f(x) = a(x - M)(x - N)$

In particular, we have:

- Y-intercept: $(0, f(0)) \equiv (0, a \times M \times N)$.
- X-intercepts: $(M, 0)$ and $(N, 0)$
- Axis of symmetry: $x = \dfrac{M + N}{2}$
- Vertex: $V \equiv \left(\dfrac{M + N}{2}, f\left(\dfrac{M + N}{2}\right)\right)$

Factored Form: $f(x) = a(x - M)(x - N)$

Let's consider the following example:

Example 2

A concert venue holds a maximum of 1000 people. With ticket prices at \$30, the average attendance is 650 people. It is predicted that for each dollar the ticket price is lowered, approximately 25 more people attend. Create a function to represent the **revenue** generated from ticket sales, and use this to find the maximum possible **revenue**.

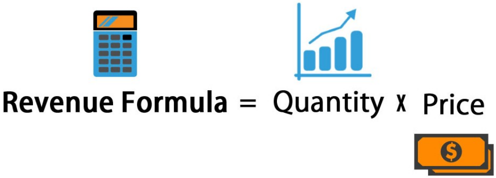


Figure: EDUCBA [all rights reserved (https://cdn.educba.com/academy/wp-content/uploads/2019/01/Revenue-Formula.jpg)]

Vertex Form: $f(x) = a(x-h)^2 + k$, $a \neq 0$

A quadratic function in vertex form is written as $f(x) = a(x-h)^2 + k$ with $a \neq 0$.

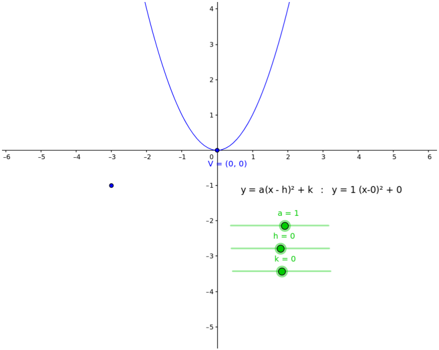


Figure: The graph contains three green sliders which we can drag to the left or right, and observe the effect it has on the graph. We can access the applet here: https://www.geogebra.org/m/FhP89QWZ#material/DKqSmxK9

Vertex Form: $f(x) = a(x-h)^2 + k$, $a \neq 0$

Therefore, we observe that:

- "a" determines how much the graph is stretched away from, or compressed towards, the x-axis. Just as in the case of the previous two forms.
- "h" determines the x-coordinate of the **vertex**.
- "k" determines the y-coordinate of the **vertex**.

Note that, from this form we easily get the vertex. However, unlike the factored form, it doesn't depend on the existence of x-intercepts.

Vertex Form: $f(x) = a(x-h)^2 + k$, $a \neq 0$

In particular, we have:

- Y-intercept: $(0, f(0)) \equiv (0, ah^2 + k)$.
- X-intercepts: if $\frac{k}{a} \leq 0$, then

$$A \equiv \left(h - \sqrt{-\frac{k}{a}}, 0\right), \qquad B \equiv \left(h + \sqrt{-\frac{k}{a}}, 0\right)$$

- Axis of symmetry: $x = h$
- Vertex: $V \equiv (h, k)$

Vertex Form: $f(x) = a(x-h)^2 + k,\ a \neq 0$

Let's consider the following example:

Example 3

Bob made a quilt that is 4 ft. $\times$ 4 ft. He has 10 sq. ft. of fabric to create a border around the quilt. How wide should he make the border to use all the fabric? (The border must be of same width on all the four sides.)

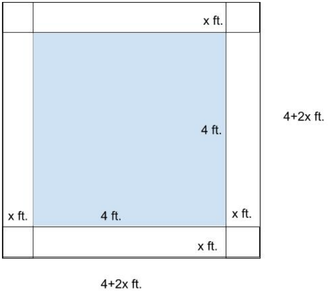


Figure: MITE [CC BY-NC-SA 4.0 (https://www.montereyinstitute.org/courses/Algebra1/COURSE_TEXT_RESOURCE/U10_L2_T1_text_container.html)]