Lebesgue Differentiation Theorem

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1 Motivation

The fundamental theorem of calculus, which equates a Riemann integrable function and the derivative of its (indefinite) integral, states that:

Theorem (Fundamental Theorem of Calculus)**.** *Suppose f is integrable on $[a, b]$ and F be defined as*

$$F(x) = \int_a^x f(y)dy$$

for all $x \in [a, b]$. Then for $I = (x, x + h)$ we get

$$F'(x) = \lim_{\substack{|I|\to 0\\ x\in I}} \frac{1}{|I|}\int_I f(y)dy = f(x)$$

whenever f is continuous at $x \in (a, b)$, where $|I|$ is the length of the interval.

Lebesgue differentiation theorem is an analogue, and a generalization, of the fundamental theorem of calculus in higher dimensions. It is also possible to show a converse – that every differentiable function is equal to the integral of its derivative, but this requires a Henstock–Kurzweil integral[1] in order to be able to integrate an arbitrary derivative [3, pp. 114].

2 Tools

2.1 Approximations of the Identity

Let ϕ be an integrable function on $\mathbb{R}^n$ such that $\int \phi = 1$. Then for $t > 1$ define

$$\phi_t(x) = t^{-n}\phi(t^{-1}x)$$

As $t \to 0$, ϕ_t converges in $\mathcal{S}(\mathbb{R}^n)^*$ to δ, the Dirac measure at origin[2], since if $g \in \mathcal{S}(\mathbb{R}^n)$ (by dominated convergence theorem):

$$\lim_{t\to 0}\phi_t(g) = \lim_{t\to 0}\int_{\mathbb{R}^n} t^{-n}\phi(t^{-1}x)g(x)dx = \lim_{t\to 0}\int_{\mathbb{R}^n} \phi(x)g(tx)dx = g(o) = \delta(g)$$

Since $\delta * g = g$ for $g \in \mathcal{S}(\mathbb{R}^n)$, we have the pointwise limit

$$\lim_{t\to 0}\phi_t * g(x) = g(x)$$

Because of this we say that $\{\phi_t : t > 0\}$ is an *approximation of the identity.*

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[1] It is a generalization of the Riemann integral, and in some situations is more general than the Lebesgue integral.
[2] For $f \in \mathcal{S}(\mathbb{R}^n)$, $\delta(f) = f(0)$

Theorem. *Let $\{\phi_t : t > 0\}$ be an approximation of the identity. Then*

$$\lim_{t\to 0} \|\phi_t * f - f\|_p = 0$$

if $f \in L^p$, $1 \le p < \infty$, and uniformly (i.e. when $p = \infty$) if $f \in C_0(\mathbb{R}^n)$ (the space of continuous functions vanishing at infinity).

As a consequence of this theorem, we know that there exists a sequence $\{t_k\}$, depending on f, such that $t_k \to 0$ and

$$\lim_{t\to 0} \phi_t * f = f \quad \text{a.e.}$$

Hence, if $\lim_{t\to 0} \phi_t * f$ exists then it must be equal $f(x)$ almost everywhere [2, pp. 25].

2.2 Weak-Type Inequalities and Almost Everywhere Convergence

If f is a measurable function on $(X, \mathcal{M}, \mu)$, we define its *distribution function* $\lambda_f : (0, \infty) \to [0, \infty]$ by

$$\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\})$$

Further, for $1 \le q < \infty$ we define

$$[f]_q = \left(\sup_{\alpha>0} \alpha^q \lambda_f(\alpha)\right)^{1/q}$$

Then, the set of all f such that $[f]_q < \infty$ is called the *weak L^q space*, denoted by L^q_w. It's easy to observe that $L^q \subset L^q_w$ and $[f]_q \le \|f\|_q$ (a restatement of Chebyshev's inequality). Note that the containment is strict, since the function $f(x) = 1/x$ on $(0, \infty)$ is in L^1_w but not in L^1.

Then, a sublinear map[3] T is *weak-type* (p, q), $1 \le p \le \infty$ and $1 \le q < \infty$, if $T : L^p(X, \mu) \to L^q_w(Y, \nu)$ and there exist $c > 0$ such that $[Tf]_q \le c \|f\|_p$ for all $f \in L^p$ [4, pp. 198]. If $q = \infty$, then an operator is of *weak-type* (p, ∞) if there exists a constant c such that $\|Tf\|_\infty \le c \|f\|_p$ [1, pp. 34].

Theorem. *Let $\{T_t\}$ be a family of linear operators on $L^p(X, \mu)$ and define*

$$T^* f(x) = \sup_t |T_t f(x)|$$

If T^* is weak (p, q) then the set

$$\{f \in L^p(X, \mu) : \lim_{t\to t_0} T_t f(x) = f(x) \quad a.e.\}$$

is closed in $L^p(X, \mu)$.

T^* is called the *maximal operator* associated with the family $\{T_t\}$.

Since for $f \in \mathcal{S}(\mathbb{R}^n)$ approximations of the identity converge pointwise to f (subsection 2.1), we can apply this theorem to show pointwise convergence almost everywhere for $f \in L^p$, $1 \le p < \infty$, or for $f \in C_0$, if we can show that the maximal operator $\sup_{t>0} |\phi_t * f(x)|$ is weakly bounded [2, pp. 28].

2.3 Hardy-Littlewood Maximal Function

The maximal function that we will discuss, arose first in the one-dimensional situation treated by Hardy and Littlewood. It seems that they were led to the study of this function by toying with the question of how a batsman's score in cricket may best be distributed to maximize his satisfaction [3, pp. 100].

Let $B_r(x) = B(x, r)$ be the Euclidean ball of radius r, centered at $x \in \mathbb{R}^n$. Let $\phi = \frac{1}{|B_1(0)|}\chi_{B_1(0)}$, be the characteristic function of the unit ball, normalized so that $\int \phi = 1$, and then we set $\phi_r(x) = r^{-n}\phi(r^{-1}x)$. If f is measurable function, we define the *Hardy-Littlewood maximal function* by

$$Mf(x) = \sup_{r>0} |f| * \phi_r(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy$$

[3] T be a map from some some vector space $\mathcal{D}$ of measurable functions on $(X, \mathcal{M}, \mu)$ to space of all measurable functions on $(Y, \mathcal{N}, \nu)$. Then T is called sublinear if $|T(f + g)| \le |Tf| + |Tg|$ and $|T(cf)| = c|Tf|$ for all $f, g \in \mathcal{D}$ and $c > 0$.

Theorem. *If f is measurable, then Mf is measurable*[4].

Indeed, the set $E_\alpha = \{x \in \mathbb{R}^n : Mf(x) > \alpha\}$ is open, because if $\bar{x} \in E_\alpha$, there exists a ball B such that $\bar{x} \in B$ and

$$\frac{1}{|B|}\int_B |f(y)|dy > \alpha$$

The Hardy-Littlewood maximal function is a tool which can be used to study the identity operator. The identity operator is interesting since by using the Hardy-Littlewood maximal function we will prove the Lebesgue differentiation theorem – the identity operator is a pointwise limit of averages on balls [1, pp. 41].

2.4 Vitali Covering Argument

The Vitali covering argument is commonly used in measure theory of Euclidean spaces. This argument is frequently used when we are, for instance, considering the n-dimensional Lebesgue measure, μ, of a set $E \subset \mathbb{R}^n$, which we know is contained in the union of a certain collection of balls $\{B_j : j \in J\}$, each of which has a measure we can more easily compute, or has a special property one would like to exploit. Hence, if we compute the measure of this union, we will have an upper bound on the measure of E.

Theorem. *Suppose $\mathcal{B} = \{B_1, B_2, \ldots B_N\}$ is a finite collection of open balls in $\mathbb{R}^n$. Then there exists a disjoint sub-collection $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ of $\mathcal{B}$ that satisfies*

$$\mu\left(\bigcup_{\ell=1}^{N} B_\ell\right) \leq 3^n \sum_{j=1}^{k} \mu(B_{i_j})$$

Loosely speaking, we may always find a disjoint sub-collection of balls that covers a fraction of the region covered by the original collection of balls [3, pp. 102].

3 Differentiation Theorem

As we shall observe, one has that $Mf(x) \geq |f(x)|$ for a.e. x; using the Vitalli covering argument (subsection 2.4) we can prove that Mf is not much larger than $|f|$ [3, pp. 103].

Theorem. *If f is measurable and $\alpha > 0$, then there exists a constant C, which depends on n but is independent of α and f, such that*

$$\mu\left(\{x \in \mathbb{R}^n : Mf(x) > \alpha\}\right) \leq \frac{C}{\alpha}\int_{\mathbb{R}^n} |f(x)|dx$$

This theorem asserts that the Hardy-Littlewood maximal operator is of weak-type $(1,1)$. It is easy to see that it is sub-linear and of weak type (∞,∞) since from the definition of Mf we get that $\|Mf\|_\infty \leq \|f\|_\infty$. Thus by the Marcinkiewicz interpolation theorem, we can conclude it is of strong-type[5] (p,p), where $p = (1-t)^{-1}$ for some $0 < t < 1$.

Now $Mf(x) = \sup_{r>0} |f| * \phi_r(x)$ being weak $(1,1)$ implies that $\sup_{r>0} |f * \phi_r(x)|$ is weak $(1,1)$. If we replace $T_t f(x)$ by $f * \phi_r(x)$ in subsection 2.2 and let $f \in L^1_{loc}$ (space of locally integrable functions[6]) we get the desired differentiation theorem [2, pp. 36]:

Theorem (Lebesgue Differentiation Theorem)**.** *If $f \in L^1_{loc}(\mathbb{R}^n)$ then*

$$\lim_{r\to 0^+} \frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)dy = f(x) \quad a.e.$$

[4] A function between two measurable spaces such that the preimage of any measurable set is measurable.

[5] We say that a sublinear map T is *strong-type* (p,q), $1 \leq p,q \leq \infty$, if $T : L^p \to L^q$ and there exists a constant c such that $\|Tf\|_q \leq c\|f\|_p$ for all $f \in L^p$.

[6] $f \in L^1_{loc}$ if $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable function with respect to the lebesgue measure, and $\int_K |f(x)|dx < \infty$ for any bounded measurable set $K \subset \mathbb{R}^n$.

References

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