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Lang–Nishimura theorem

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Abstract

The Lang–Nishimura theorem implies that the property of having a k-point is a birational invariant of smooth, proper, integral k-varieties. In this report we will discuss Poonen's proof of this result [Poo17, §3.6.4].

Contents

Notation

A	a commutative ring with unit
$\operatorname{Spec} A$	the set of prime ideals of A; affine scheme
A-scheme	a scheme over the scheme $S = \operatorname{Spec} A$
$\mathbb{A}^n_A$	$\operatorname{Spec} A[t_1, \ldots, t_n]$; affine space (of relative dimension n) over A
$\operatorname{Proj} B$	the set of homogeneous prime ideals of a graded A-algebra B; projective scheme over A
$\mathbb{P}^n_A$	$\operatorname{Proj} A[t_0, \ldots, t_n]$; projective space (of relative dimension n) over A
$\mathcal{O}_X$	the sheaf of rings on scheme X

$\mathbf{k}(x)$	$\mathcal{O}_{X,x}/\mathfrak{m}_x$ where $\mathfrak{m}_x$ is the maximal ideal; residue field of a scheme X at $x \in X$
$\mathbf{k}(X)$	Function field of an integral scheme X
$X(T)$	$\mathrm{Hom}_S(T, X)$ where T and X are S-schemes; T-points of X
$X(A)$	$\mathrm{Hom}_S(\mathrm{Spec}\, A, X)$; A-points of X when $T = \mathrm{Spec}\, A$
k	the ground field
$X(k)$	$\mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, k, X)$ where X is a k-scheme; k-points of X
$X(K)$	$\mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, K, X)$ where K/k is a field extension and X is a k-scheme; K-points of X
$X_{S'}$	$X \times_S S'$; the base change of a S-scheme X by $S' \to S$
X_A	$X \times_S \mathrm{Spec}\, A$; base extension when $S' = \mathrm{Spec}\, A$
$\mathbb{P}^n_S$	$\mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} S$; we get $\mathbb{P}^n_A$ for $S = \mathrm{Spec}\, A$

1 Introduction

If $f : X \to Y$ is a morphism of k-varieties and X has a k-point x, then Y has a k-point, namely $f(x)$. Surprisingly, under mild hypothesis, the same conclusion can be drawn if f is only a rational map that can be undefined at x.

Theorem 1.1 (Lang–Nishimura theorem)**.** *Let $X \dashrightarrow Y$ be rational map between k-varieties, where Y is proper. If X has a smooth k-point, then Y has a k-point.*

Sketch of proof. We first replace X with an open neighborhood of the given smooth k-point x, i.e. we can assume X to be an integral scheme. Then, since Y is locally of finite presentation, we get that the given rational map $X \dashrightarrow Y$ corresponds to a $\mathbf{k}(X)$-point of Y.

Next, the Cohen structure theorem implies that $\mathbf{k}(X)$ is embedded in the iterated Laurent series field $F := k((t_1))((t_2))\cdots((t_d))$ where d is the dimension of X. That is, the given rational map $X \dashrightarrow Y$ corresponds to a F-point of Y.

Finally, we observe that $F = \mathrm{Frac}(A)$ where $A = k((t_1))\cdots((t_{d-1}))[[t_d]]$ is a (discrete) valuation ring. Therefore, since Y is proper, the valuative criterion gives us an A-point of Y corresponding to the F-point above, that reduces modulo t_d to a $k((t_1))\cdots((t_{d-1}))$-point of Y. By repeating this argument another $d-1$ times we will end up with a k-point of Y, as desired. $\square$

Corollary 1.1. *Let X and Y be smooth, proper, integral k-varieties that are birational to each other. Then X has a k-point if and only if Y has a k-point.*

2 Foundation

Let's review some foundational algebraic geometry before preparing for the proof.

Definition 2.1 (Sections of a scheme)**.** Let $\pi : X \to S$ be a S-scheme. A *section* of X is a morphism of S-schemes $\sigma : S \to X$. That is, $\pi \circ \sigma = \mathrm{Id}_S$. The set of sections of X is denoted by $X(S) = \mathrm{Hom}_S(S, X)$.

Remark 2.1. Let X be a scheme over a field k. Then we can identify $X(\mathrm{Spec}\, k)$ with the set of points $x \in X$ such that $\mathbf{k}(x) = k$, where the equality means that the canonical map

$$k \to \mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x$$

is an isomorphism [Liu10, Example 2.3.18 and 2.3.29].

Definition 2.2 (Rational points)**.** Let X be a k-scheme. Then the *k-rational points* (or k-points) of X, denoted by $X(k)$, are

$$X(k) = \{x \in X \mid \mathbf{k}(x) = k\} \cong \{\sigma : \operatorname{Spec} k \to X \mid \pi \circ \sigma = \operatorname{Id}_{\operatorname{Spec} k}\} = X(\operatorname{Spec} k)$$

Remark 2.2. If $X = \operatorname{Spec} k[t_1, \dots, t_n]/I$ is an affine k-scheme and $Z = \{\alpha = (\alpha_1, \dots, \alpha_n) \in k^n \mid p(\alpha) = 0 \ \forall p \in I\}$. Then there is a canonical bijection

$$\begin{aligned} \lambda : Z &\to X(k) = \{x \in X \mid \mathbf{k}(x) = k\} \\ \alpha &\mapsto x \leftrightarrow \mathfrak{m}_\alpha = \langle t_1 - \alpha_1, \dots, t_n - \alpha_n \rangle \end{aligned}$$

where $\mathbf{k}(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x = k[t_1, \dots, t_n]/\mathfrak{m}_\alpha = k$. Thus, solving a system of polynomial equations in k amounts to determining the set of rational points of a scheme of the type $\operatorname{Spec} k[t_1, \dots, t_n]/I$.

Remark 2.3. If $X = \operatorname{Proj} k[t_0, t_1, \dots, t_n]/\langle p_1, \dots, p_m \rangle$ such that $p_1, \dots, p_m \in k[t_0, t_1, \dots, t_n]$ are homogeneous polynomials, and $Z_+(p_1, \dots, p_m) = \{\alpha = [\alpha_0 : \dots : \alpha_n] \in \left(k^{n+1} \setminus \{0\}\right)/k^* \mid p_j(\alpha) = 0 \ \forall j\}$. Then there exists a bijection between $Z_+(p_1, \dots, p_m)$ and $X(k)$ [Liu10, Corollary 2.3.44].

Remark 2.4. The k-points are closed in X [Liu10, Exercise 2.5.9].

Definition 2.3 (Scheme-valued points)**.** We can generalize the above definition to the set of *scheme-valued points*. Let X be a S-scheme. If T is a S-scheme, then the set of T-points on X is $X(T) = \operatorname{Hom}_S(T, X)$.

Remark 2.5. Let X be a S-scheme and $X_{S'}$ the base change by $S' \to S$. Then $X(S') = X_{S'}(S')$, where $X(S') = \operatorname{Hom}_S(S', X)$ and $X_{S'}(S') = \operatorname{Hom}_{S'}(S', X_{S'})$ [Liu10, Remark 3.1.6] [Poo17, Proposition 2.3.15].

Definition 2.4 (Connected scheme)**.** A scheme X is called *connected* if the underlying topological spaces of X is connected[1].

Definition 2.5 (Irreducible scheme)**.** A scheme X is called *irreducible* if the underlying topological space of X is irreducible[2].

Remark 2.6. An irreducible scheme is connected.

Definition 2.6 (Dimension of scheme)**.** Let X be a scheme. The *dimension* of X is the Krull dimension of the underlying topological space of X. That is, $\dim X$ is the supremum of the lengths of the chains of irreducible closed subsets of X. By convention, the empty set is of dimension $-\infty$.

Remark 2.7. We have $\dim \operatorname{Spec} A = \dim A = \dim A/\operatorname{nil}(A)$ [Liu10, Proposition 2.5.8].

Definition 2.7 (Reduced scheme)**.** A scheme X is called *reduced* if $\mathcal{O}_X(U)$ is reduced[3] for every open subset U of X.

Remark 2.8. Given a scheme X, there exists a unique reduced closed subscheme $i : X_{\text{red}} \to X$ having the same underlying topology as X [Liu10, Proposition 2.4.2(c)].

[1] A toplogical space X is said to be *connected* if the condition $X = X_1 \sqcup X_2$ with disjoint closed (or open) subsets X_i implies that $X_1 = X$ or $X_2 = X$.

[2] A topolgical space X is called *irreducible* if the condition $X = X_1 \cup X_2$ with closed subsets X_i implies that $X_1 = X$ or $X_2 = X$.

[3] A ring A is called *reduced* if 0 is the only nilpotent element of A, i.e. $\operatorname{nil}(A) = 0$.

Definition 2.8 (Integral scheme)**.** A scheme X is called *integral* if $\mathcal{O}_X(U)$ is an integral domain for every open subset U of X.

Remark 2.9. Equivalently, a scheme X is integral if and only if:

1. X is connected, and $\mathcal{O}_{X,x}$ is an integral domain for every $x \in X$ [Liu10, Exercise 2.4.4].
2. X is both irreducible and reduced.

Definition 2.9 (Function field)**.** Let X be an integral scheme. Then its *function field* $\mathbf{k}(X)$ is the field of fractions $\operatorname{Frac}(A)$ for any affine open subset $U = \operatorname{Spec} A$ of X.

Remark 2.10. If X is irreducible but not necessarily reduced, the above definiton of function field should be modified to $\operatorname{Frac}(A/\operatorname{nil}(A))$ [Poo17, Remark 2.2.12].

Remark 2.11. We can also define the function field as the residue field $\mathcal{O}_{X,\xi}$ at the generic point[4] ξ, for details see [Liu10, Definition 2.4.10, Proposition 2.4.18 and Definition 2.4.19]. Then, an element of $\mathbf{k}(X)$ is called a *rational function* on X. Therefore, $\mathbf{k}(X)$ is also called the *field of rational functions* of X.

Definition 2.10 (Rational map)**.** Let X and Y be S-schemes. Consider the pairs (U, ϕ) in which U is a dense open subscheme of X and $\phi : U \to Y$ is an S-morphism. We call pairs (U, ϕ) and (V, ψ) equivalent if ϕ and ψ agree on a dense open subscheme of $U \cap V$. A *S-rational map* $X \dashrightarrow Y$ is an equivalence class of such pairs. That is,

$$\{S\text{-rational maps } X \dashrightarrow Y\} := \varinjlim_{U} \operatorname{Hom}_S(U, Y) = \varinjlim_{U} Y(U)$$

where the direct limit is indexed over dense open subschemes U of X, with order relation induced by reverse inclusion, i.e. $U < V$ if $V \subset U$.

Remark 2.12. A rational function on a scheme X is a rational map from X to $\mathbb{A}^1_{\mathbb{Z}}$ [Sta21, Tag 01RR].

Remark 2.13. The *domain of definition* of a rational map is the union of the U as (U, ϕ) ranges over the equivalence class [Poo17, Proposition 3.6.3].

Definition 2.11 (Dominant rational map)**.** A S-rational map $X \dashrightarrow Y$ is said to be *dominant* if any representative (U, ϕ) is a dominant morphism of schemes (i.e. the image $\phi(U)$ is dense in Y).

Remark 2.14. In general it does not make sense to compose rational maps, since the image of a representative of the first rational map may have empty intersection with the domain of definition of the second. However, if we assume that our schemes are irreducible then we can compose dominant rational maps [Sta21, Tag 01RR].

Definition 2.12 (Birational schemes)**.** Let X and Y be irreducible S-schemes. Then X and Y are said to be *S-birational* if X and Y are isomorphic in the category of irreducible S-schemes and dominant S-rational maps.

Remark 2.15. If X and Y are birational irreducible S-schemes, then the set of S-rational maps from X to Z is bijective with the set of S-rational maps from Y to Z for all S-schemes Z [Sta21, Tag 01RR].

[4] For a comparison with closed points see this discussion: `https://math.stackexchange.com/a/984`

Definition 2.13 (Separated scheme)**.** A scheme X is called *separated* if the diagonal morphism Δ defined as $(\mathrm{Id}_X, \mathrm{Id}_X) : X \to X \times_{\operatorname{Spec}\mathbb{Z}} X$ is a closed immersion of schemes.

Remark 2.16. In the above definition we used that fact that every scheme is in a unique way a $\mathbb{Z}$-scheme, because every commutative ring has one and only one $\mathbb{Z}$-algebra structure [Liu10, Example 2.3.27]. Therefore, $\mathbb{A}^1_k$ is separated as a scheme, even though it is not separated[5] as a topological space due to the existence of non-closed (generic) point [Liu10, Example 2.1.4 and Proposition 3.3.4].

Definition 2.14 (Separated morphism)**.** Let $f : X \to Y$ be a morphism of schemes. Then f is said to be separated (or, X is *separated over* Y) if the diagonal morphism $\Delta_{X/Y}$ defined as $(\mathrm{Id}_X, \mathrm{Id}_X) : X \to X \times_Y X$ is a closed immersion of schemes.

Definition 2.15 (Morphism of finite type)**.** Let $f : X \to Y$ be a morphism of schemes. Then f is said to be of *finite type* if for every affine open subset V of Y, (1) $f^{-1}(V)$ is quasi-compact, and (2) for every affine open subset U of $f^{-1}(V)$ the canonical homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ into a finitely generated $\mathcal{O}_Y(V)$-algebra (i.e. $\mathcal{O}_X(U)$ is an $\mathcal{O}_Y(V)$-algebra of finite type).

Definition 2.16 (Scheme of finite type)**.** A S-scheme X is said to be of *finite type* if the structural morphism $\pi : X \to S$ is of finite type.

Remark 2.17. If X is a k-scheme of finite type, then $x \in X$ is closed if and only if $\mathbf{k}(x)$ is algebraic over k [Poo17, Proposition 2.4.3]. Moreover, if X is irreducible, and x is a closed point of X, then $\dim \mathcal{O}_{X,x} = \dim X$ [Liu10, Corollary 2.5.24].

Definition 2.17 (Variety)**.** A *k-variety* is a separated scheme X of finite type over $\operatorname{Spec} k$.

Remark 2.18. Here we are following the definition given in [Poo17, Definition 2.1.1]. There is no standard definition, for example, in [Liu10, Definition 2.3.47 and Example 3.2.3] a variety is not even assumed to be separated, whereas in [Sta21, Tag 020C] a variety is also required to be integral.

Definition 2.18 (Noetherian scheme)**.** A scheme X is said to be *Noetherian* if it is a finite union of affine open X_i such that $\mathcal{O}(X_i)$ is a Noetherian ring for every i. Moreover, a scheme is called *locally Noetherian* if every point has a Noetherian open neighborhood.

Remark 2.19. A k-scheme of finite type is Noetherian [Liu10, Remark 2.3.48].

Definition 2.19 (Regular scheme)**.** A scheme X is called *regular* if X is locally Noetherian and $\mathcal{O}_{X,x}$ is a regular local ring[6] for every $x \in X$.

Remark 2.20. Every regular scheme is reduced, since regular local rings are reduced [Poo17, Example 2.2.10]. Also, a connected regular scheme is integral [Poo17, Corollary 3.5.6].

Definition 2.20 (Geometry of scheme)**.** Let X be a scheme over a field k. Then X is said to be *geometrically* connected (resp. irreducible, reduced, integral, regular) if $X_{\overline{k}}$ is connected (resp. irreducible, reduced, integral, regular).

[5] Let X be a topological space. Then X is called separated (Hausdorff) if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed as a subset of the product space $X \times X$.

[6] A Noetherian local ring $(A, \mathfrak{m})$ is called *regular* if the minimal number of generators of its maximal ideal is equal to its Krull dimension, i.e. $\dim_K \mathfrak{m}/\mathfrak{m}^2 = \dim A$ where $K = A/\mathfrak{m}$.

Remark 2.21. Following examples illustrate the difference between the arithmetic and geometry of schemes [Poo17, Remark 2.2.5]:

1. The $\mathbb{Q}$-scheme $X = \operatorname{Spec} \mathbb{Q}[x,y]/\langle y^2 - 2\rangle$ is connected but not geometrically connected.
2. The $\mathbb{Q}$-scheme $X = \operatorname{Spec} \mathbb{Q}[x,y]/\langle x^2 - 2y^2\rangle$ is irreducible but not geometrically irreducible.
3. The $\mathbb{F}_p(t)$-scheme $X = \operatorname{Spec} \mathbb{F}_p(t)[x,y]/\langle y^p - tx^p\rangle$ is reduced but not geometrically reduced.
4. The k-scheme $\mathbb{P}^1_K$, for any non-trivial finite extension K/k, is integral but not geometrically integral.
5. The $\mathbb{F}_p(t)$-scheme $X = \operatorname{Spec} \mathbb{F}_p(t)[x,y]/\langle y^2 - (x^p - t)\rangle$ is regular but not geometrically regular (though geometrically reduced).

Definition 2.21 (Smooth scheme)**.** Let X be a scheme of finite type over k. Then X is said to be *smooth* at $x \in X$ if the points of $X_{\overline{k}}$ lying above x are regular points of $X_{\overline{k}}$. Furthermore, X is said to be *smooth* over k if it is geometrically regular (i.e. smooth at all points).

Remark 2.22. A smooth scheme is also regular. Moreover, by Remark 2.4 and [Liu10, Proposition 4.3.30], $x \in X(k)$ is smooth iff it is regular.

Definition 2.22 (Proper morphism)**.** Let $f : X \to Y$ be a morphism of schemes. Then f is said to be *proper* if it is of finite type, separated and closed under any base change $Y' \to Y$, i.e. $X \times_Y Y' \to Y'$ is a closed morphism (universally closed).

Definition 2.23 (Proper scheme)**.** A S-scheme X is called *proper* if the structural morphism $\pi : X \to S$ is proper.

Remark 2.23. The property of properness is a topological property like separatedness [Liu10, §3.3.2, p. 103]. For example, $\mathbb{A}^1_k$ is not a proper k-scheme [Liu10, Proposition 3.3.18].

3 Preparation

In this section we will discuss the three key ingredients needed for the proof.

3.1 Schemes locally of finite presentation

If a S-scheme satisfies the restrictive notion "locally of finite presentation" then "its functor of points commutes with taking direct limits of rings" [Poo17, Remark 3.1.11]. This can be viewed as a generalization of the fact that a ring homomorphism $\phi : A \to B$ is of finite presentation if and only if for every directed system A_λ of A-algebras we have

$$\varinjlim_{\lambda} \operatorname{Hom}_A(B, A_\lambda) = \operatorname{Hom}_A(B, \varinjlim_{\lambda} A_\lambda)$$

Definition 3.1 (Morphism locally of finite presentation)**.** Let $f : X \to Y$ be a morphism of schemes, with $x \in X$ and $y = f(x)$. Then f is said to be *locally of finite presentation* at x if there exist affine open neighborhoods $V = \operatorname{Spec} A$ of y and $U = \operatorname{Spec} B$ of x such that $\mathcal{O}_X(U) = B$ is of finite presentation over $\mathcal{O}_Y(V) = A$.

Definition 3.2 (Scheme locally of finite presentation)**.** A S-scheme X is said to be *locally of finite presentation* if $\pi : X \to S$ is locally of finite presentation at every point $x \in X$.

Remark 3.1. Over a locally Noetherian base, (locally of) finite type is same as (locally of) finite presentation [Sta21, Tag 01TX] [Poo17, Remark 3.1.13].

Remark 3.2. Over non-Noetherian base, like $S = \operatorname{Spec} A$ where A is an adèle ring of a global field, finitely presented has better properties than finite type [Poo17, Proposition 3.1.8].

Definition 3.3 (Affine S-scheme)**.** A S-scheme X is called *affine S-scheme* if $\pi : X \to S$ is an affine morphism, i.e. $\pi^{-1}S_0$ is an affine scheme for each affine open subscheme S_0 of S.

Remark 3.3. An affine S-scheme is not necessarily affine as a scheme [Poo17, Warning 4.3.3].

Proposition 3.1. *A S-scheme X is locally of finite presentation if and only if for any directed, or filtered set I, and any inverse system (T_i, f_{ij}) of affine S-schemes, we have*

$$\varinjlim_i \operatorname{Hom}_S(T_i, X) = \operatorname{Hom}_S(\varprojlim_i T_i, X)$$

Proof. [Sta21, Tag 01ZB] $\square$

Remark 3.4. The terms "direct limit" and "inverse limit" are also known as "filtered colimit" and "filtered limit", respectively [Sta21, Tag 04AX].

Theorem 3.1. *Let X be an integral k-variety, and let Y be an arbitrary k-variety. The natural map*

$$\{\textit{rational maps from } X \textit{ to } Y\} \to Y(\mathbf{k}(X))$$
$$[\phi : U \to Y] \mapsto (\textit{the composition } \operatorname{Spec} \mathbf{k}(X) \hookrightarrow U \xrightarrow{\phi} Y)$$

is a bijection.

Proof. The set of k-rational maps is $\varinjlim_U Y(U)$, where U ranges over dense open subschemes of X ordered by reverse inclusion. Moreover, since X is an irreducible scheme of finite type, every dense open subscheme U of X contains a dense affine open subscheme [Liu10, §2.3.4 & Proposition 2.4.5(a)]. That is, the filtered inverse system[7] of dense affine open subschemes $\operatorname{Spec} A_i$ of X is cofinal in the system of all dense open subschemes. Hence, we have

$$\varinjlim_U Y(U) \cong \varinjlim_i Y(A_i) \tag{3.1}$$

where the direct system is indexed over $\mathbb{N}$, with the order relation induced by reverse inclusion, i.e. $i < j$ if $\operatorname{Spec} A_j \subset \operatorname{Spec} A_i$.

Now, since Y is a scheme of finite type over a locally Noetherian base, Y is locally of finite presentation by Remark 3.1. Therefore, we can apply Proposition 3.1 to get

$$\varinjlim_i Y(A_i) = Y(\varinjlim_i A_i) \tag{3.2}$$

where $Y(\varinjlim_i A_i) = \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \varinjlim_i A_i, Y) = \operatorname{Hom}_{\operatorname{Spec} k}(\varprojlim_i \operatorname{Spec} A_i, Y)$ since we have filtered colimit[8].

[7] Note that by definition of scheme morphism we implicitly have a direct system of commutative rings A_i's.

[8] Related discussion on StackExchange: `https://math.stackexchange.com/q/4199661/`

Finally, since X is an integral scheme, we have $\mathbf{k}(X) = \mathrm{Frac}(A)$, for some affine open subset $\operatorname{Spec} A$. Moreover, since the localization by a multiplicative set equals to the direct limit of localization at each element of that multiplicative set[9], we have $\mathrm{Frac}(A) = \varinjlim_t A_t$, where the direct limit is indexed over $t \in A \setminus \{0\} \leftrightarrow \mathbb{N}$ with the order relation induced by divisibility, i.e. $t < r$ if $t \mid r$. Note that $A_t = \mathcal{O}_{\operatorname{Spec} A}(D(t))$ where $D(t) = \operatorname{Spec} A \setminus V(tA)$ is a principal open subset of U. Therefore, since $\operatorname{Spec} A_i$ are dense affine open subschemes of X [Liu10, Proposition 2.3.12], we have

$$Y(\varinjlim_i A_i) = Y(\mathbf{k}(X)) \tag{3.3}$$

Hence, combining (3.1), (3.2) and (3.3) we get the desired bijection. □

3.2 Cohen structure theorem

Here we will look at a special case of the Cohen structure theorem [Sta21, Tag 0323] which states that if A is a complete regular local ring of dimension d containing some field, then

$$A \cong K[[t_1, \dots, t_d]]$$

where K is the residue field of A.

Definition 3.4 (I-adic topology)**.** Let A be a commutative ring, and I an ideal. Then the descending filteration $(I^n)_n$ defines a unique structure of topological ring[10] for which I^n form a fundamental system of neighborhoods of 0. This topology on A defined by ideals I^n is called the *I-adic topology.*

Remark 3.5. Similarly, if M is an A-module, then the filteration $(I^n M)_n$ defines a structure of topological A-module, called the *I-adic topology on M* [Liu10, p. 16].

Definition 3.5 (I-adic completion)**.** Let A be a ring endowed with the I-adic topology. Then the ring $\widehat{A} := \varprojlim_n (A/I^n)$ is called the *formal completion of A for the I-adic topology.*

Remark 3.6. The ring of formal power series in d variables $A[[t_1, \dots, t_d]]$ is isomorphic to the $\mathfrak{m}$-adic completion of the polynomial ring $A[t_1, \dots, t_d]$, for $\mathfrak{m} = \langle t_1, \dots, t_d \rangle$ [Liu10, Example 1.3.6]. Moreover, it has the following useful properties:

1. If A is an integral domain, then $A[[t_1, \dots, t_d]]$ is also an integral domain.

2. If A is a local ring, then $A[[t_1, \dots, t_d]]$ is also local[11] with the maximal ideal $\mathfrak{m} + \langle t_1, \dots, t_d \rangle$, where $\mathfrak{m}$ is the maximal ideal of A.

3. If A is a Noetherian ring, then $A[[t_1, \dots, t_d]]$ is also Noetherian. Furthermore, the I-adic completion of a Noetherian ring is Noetherian [Liu10, Corollary 1.3.8].

4. If A is a Noetherian ring, then $\dim A[[t_1, \dots, t_d]] = \dim A + d$ [Liu10, Corollary 2.5.17].

Remark 3.7. Similarly, if M is an A-module with the I-adic topology, then the $\widehat{A}$-module $\widehat{M} := \varprojlim_n (M/I^n M)$ is called the *completion of M for the I-adic topology* [Liu10, p. 18].

[9] Relevant StackExchange discussions: (1) `https://math.stackexchange.com/q/664471/` and (2) `https://math.stackexchange.com/a/1811677/`

[10] A *topological ring* is a ring A endowed with a topology for which the maps $A \times A \to A$ defined by $(a_1, a_2) \mapsto a_1 - a_2$ and $(a_1, a_2) \mapsto a_1 a_2$ are continuous.

[11] The proof follows from the fact that $f \in A[[t_1, \dots, t_d]]$ is a unit iff $f(0)$ is a unit in A: `https://math.stackexchange.com/a/989896`

Lemma 3.1. *Let M and N be A-modules endowed with I-adic and J-adic topology, respectively. Let $\phi : M \to N$ be a homomorphism such that for some $n_0 \geq 0$, we have $N = \phi(M) + J^n N$ and $J^n N = \phi(I^n M) + J^{n+1} N$ for every $n \geq n_0$. Then the canonical homomorphism $\widehat{M} \to \widehat{N}$ is surjective.*

Proof. Since the inverse limit of a system does not depend on its first terms, we may assume that $n_0 = 0$. Then the hypothesis implies that the canonical homomorphisms $\phi_n : M/I^n M \to N/J^n N$ and $\ker \phi_{n+1} \to \ker \phi_n$, where $\ker \phi_\ell = \phi^{-1}(J^\ell N)/I^\ell M$, are surjective for all $n \geq 0$. Hence we have the exact sequence of inverse systems

$$0 \longrightarrow (\ker \phi_n)_n \longrightarrow (M/I^n M)_n \longrightarrow (N/J^n N)_n \longrightarrow 0$$

on which inverse limit is an exact functor [Liu10, Lemma 1.3.1 & Exercise 1.3.15], proving the desired surjectivity. □

Definition 3.6 (Stable I-filtration of module)**.** Let M be an A-module. An *I-filteration of M* is a filteration $(M_n)_n$ of M by submodules M_n such that $IM_n \subseteq M_{n+1}$. This filtration is called *stable* if there exists n_0 such that $M_{n+1} = IM_n$ for every $n \geq n_0$.

Proposition 3.2 (Artin-Rees lemma)**.** *Let A be a Noetherian ring, I an ideal of A, M a finitely generated A-module, and N a submodule of M. Then the I-filteration $(I^n M \cap N)_n$ of N is stable.*

Proof. [Liu10, Corollary 1.3.12] □

Remark 3.8. Following are some useful properties one can be obtain using the Artin-Rees lemma:

1. Let A be a Noetherian ring, I an ideal of A and $\widehat{}$ denote the I-adic completions:
   (a) If M is a finitely generated A-module with the canonical map $j_M : M \to \widehat{M}$, then the A-bilinear mapping $\widehat{A} \times M \to \widehat{M}$ defined by $(a, \ell) \mapsto a j_M(\ell)$ gives the canonical A-linear isomorphism $\widehat{A} \otimes_A M \to \widehat{M}$ [Liu10, Corollary 1.3.14].
   (b) $\widehat{A}$ is a flat ring[12] over A [Liu10, Theorem 1.3.15].

   Proof of (b). Let J be an ideal of A. Then the first part implies that $\widehat{A} \otimes_A J \cong \widehat{J}$. Next, consider the canonical homomorphism $\varphi : \widehat{J} \to \widehat{A}$. If this map is injective then $\widehat{A}$ is flat over A [Liu10, Theorem 1.2.4]. Let $\alpha = (\alpha_n)_n \in \ker(\varphi) \subseteq \varprojlim_n (J/I^n J)$ and fix some $n \geq 1$. Then by Artin-Rees lemma there exists an $m \geq n$ such that $I^m \cap J \subseteq I^n J$. Now, let $\beta_m \in J$ be an element whose image in $J/I^m J$ equals α_m. Then $\beta_m \in I^n J$ and $\alpha_n = 0$ is the image of β_m in $J/I^n J$. Therefore, $\alpha = 0$. □

2. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $\widehat{A}$ its $\mathfrak{m}$-adic completion [Liu10, Theorem 1.3.16]:
   (a) For every $n \geq 1$, we have a canonical isomorphism $A/\mathfrak{m}^n \cong \widehat{A}/\mathfrak{m}^n \widehat{A}$.
   (b) $\widehat{A}$ is a local ring[13] with maximal ideal $\mathfrak{m}\widehat{A}$.

   Proof of (b). The first part implies that $\widehat{A}/\mathfrak{m}\widehat{A} \cong A/\mathfrak{m}$ is a field, hence $\mathfrak{m}\widehat{A}$ is a maximal ideal. It remains to prove that this is the only maximal ideal of $\widehat{A}$. Let $\alpha = (\alpha_n)_n \in \widehat{A} = \varprojlim_n (A/\mathfrak{m}^n)$. Now if $\alpha \notin \mathfrak{m}\widehat{A}$ then $\alpha_1 \notin \mathfrak{m}$ and hence α_1 is a unit. Therefore, since $\alpha_n \equiv \alpha_{n+1} \pmod{\mathfrak{m}^n}$ for every n, α_n is a unit for each n. Then $\alpha^{-1} = (\alpha_n^{-1})_n$ where $\alpha_n^{-1} \in A/\mathfrak{m}^n$ is the inverse of α_n. Hence, $\widehat{A}$ is local. □

[12] It is same as saying that the canonical homomorphism $A \to \widehat{A}$ of the A-algebra $\widehat{A}$ is flat [Liu10, Corollary 1.2.15]. That is, $\widehat{A}$ is a flat A-module.

[13] source: `https://math.stackexchange.com/q/169056`

Definition 3.7 (Flat morphism)**.** Let $f : X \to Y$ be a morphism of schemes. The f is said to be *flat at the point* $x \in X$ if the ring homomorphism $f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. Furthermore, f is said to be *flat* if it is flat at every point of X.

Remark 3.9. The structural morphism $\pi : X \to \operatorname{Spec} k$ of a k-variety if flat. In fact, any algebra over a field is flat over the field [Liu10, Example 4.3.4].

Proposition 3.3. *Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes. Let $x \in X$ and $y = f(x)$. Then*

$$\dim \mathcal{O}_{X_y,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

where $X_y = X \times_Y \operatorname{Spec} \mathbf{k}(y)$ is the fiber[14] *of f over y.*

Proof. [Liu10, Theorem 4.3.12] □

Corollary 3.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, and $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A. Then* $\dim \widehat{A} = \dim A$. *Moreover, A is regular if and only if $\widehat{A}$ is regular.*

Proof. Applying Proposition 3.3 to the flat morphism $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} A$ (Remark 2.4(1b)) and to the closed fiber [Liu10, Proposition 3.1.2]

$$\operatorname{Spec} \widehat{A} \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbf{k}(y) = \operatorname{Spec}(\widehat{A} \otimes_A \mathbf{k}(y)) = \operatorname{Spec}(\widehat{A} \otimes_A A/\mathfrak{m})$$

of dimension 0 (Remark 2.4(1a)), we get that $\dim \widehat{A} = \dim A$.

Since $\mathfrak{m}\widehat{A}$ is the maximal ideal of $\widehat{A}$ (Remark 2.4(2b)), and $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}\widehat{A}/\mathfrak{m}^2\widehat{A}$, the second assertion follows. □

Lemma 3.2. *Let $\phi : A \to B$ be a surjective ring homomorphism such that A and B are commutative rings of the same finite Krull dimension. If A is an integral domain then ϕ is an isomorphism*[15].

Proof. The surjective ring homomorphism $\phi : A \to B$ gives rise to a closed immersion $f : \operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$ [Liu10, Lemma 2.3.17], i.e. $f(\operatorname{Spec} B)$ is homeomorphic to a closed subset of $\operatorname{Spec} A$. Moreover, if A is an integral domain then $\operatorname{Spec} A$ is reduced [Liu10, Proposition 2.4.2(a)] and irreducible [Liu10, Proposition 2.4.7(c)]. Thus, A having finite Krull dimension implies that a strict closed subset $\Pi \subsetneq \operatorname{Spec} A$ must have dimension smaller than that of $\operatorname{Spec} A$ [Liu10, Proposition 2.5.5]. However, since $\dim \operatorname{Spec} B = \dim \operatorname{Spec} A$, we get that $f(\operatorname{Spec} B) = \operatorname{Spec} A$, i.e. f is surjective. Moreover, since $\operatorname{Spec} A = V(0)$ and $f(\operatorname{Spec} B) = V(\ker(\phi))$ [Liu10, Exercise 2.3.3(b)], we have $V(\ker(\phi)) = V(0)$, i.e. $\ker(\phi) \subset \sqrt{0} = \operatorname{nil}(A)$ [Liu10, Lemma 2.1.6(b)]. However, $\operatorname{nil}(A) = 0$ since $\operatorname{Spec} A$ is reduced. Therefore, $\ker(\phi) = 0$ and ϕ is injective. □

Theorem 3.2. *Let X be a k-scheme of finite type, with $x \in X(k)$ a regular point of X. Let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_{X,x}$ and $\widehat{\mathcal{O}}_{X,x}$ the $\mathfrak{m}_x$-adic completion of $\mathcal{O}_{X,x}$. Then we have an isomorphism of k-algebras*

$$\widehat{\mathcal{O}}_{X,x} \cong k[[t_1, \dots, t_d]]$$

where $d = \dim \mathcal{O}_{X,x}$.

[14] This is homeomorphic to $f^{-1}(y)$ [Liu10, Proposition 3.1.16] and is a scheme over $\mathbf{k}(y)$ [Liu10, Definition 3.1.13 and Remark 3.1.17].

[15] source: `https://math.stackexchange.com/a/604091/`

Proof. Note that $\mathcal{O}_{X,x}$ is a regular local ring of dimension d containing its residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x = \mathbf{k}(x) = k$ [Liu10, Corollary 2.3.26]. Let us fix a coordinate system $\{f_1, \dots, f_d\}$ for $\mathcal{O}_{X,x}$, i.e. $\mathfrak{m}_x = \langle f_1, \dots, f_d \rangle$, and consider the following k-algebra homomorphism:

$$\begin{aligned} \varphi : k[t_1, \dots, t_d] &\to \mathcal{O}_{X,x} \\ t_i &\mapsto f_i \end{aligned}$$

Next, since φ also induces the isomorphism $k[t_1, \dots, t_d] \cong \bigoplus_{n \geq 0} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ [Liu10, Exercise 4.2.13], we have $\mathcal{O}_{X,x} = \varphi(k[t_1, \dots, t_d]) + \mathfrak{m}_x^n$ and $\mathfrak{m}_x^n = \varphi(\langle t_1, \dots, t_d \rangle^n) + \mathfrak{m}_x^{n+1}$ for every $n \geq 1$. Then, from Lemma 3.1 and Remark 3.6, it follows that

$$\widehat{\varphi} : k[[t_1, \dots, t_d]] \to \widehat{\mathcal{O}}_{X,x}$$

is surjective. From Remark 3.6 we also get that $k[[t_1, \dots, t_d]]$ is a local Noetherian[16] integral domain of dimension d. Moreover, $d = \dim \widehat{\mathcal{O}}_{X,x}$ by Corollary 3.1. Then, from Lemma 3.2 it follows that $\widehat{\varphi}$ is an isomorphism[17]. □

3.3 Valuative criteria

The final ingredient is the valuative criterion for properness of schemes [Sta21, Tag 03IW].

Definition 3.8 (Valuation)**.** Let Γ be a totally ordered abelian group (written additively) and k be a field. A *valuation* of k with values in Γ is a mapping $v : k^* \to \Gamma$ such that

1. $v(\alpha\beta) = v(\alpha) + v(\beta)$, i.e. v is a group homomorphism
2. $v(\alpha + \beta) \geq \min(v(\alpha), v(\beta))$

for all $\alpha, \beta \in k^*$. Moreover, we set $v(0) = +\infty$ by convention.

Remark 3.10. The subgroup $\Gamma_v = v(k^*)$ of Γ is called the *value group*. Usually, v is surjective and $\Gamma = \Gamma_v$.

Definition 3.9 (Valuation ring of a field)**.** Let k be a field with valuation v. Then the set

$$O_k = \{\alpha \in k \mid v(\alpha) \geq 0\}$$

is a subring of k called the *valuation ring of k*.

Remark 3.11. O_k is a local integral domain with the maximal ideal $\mathfrak{m}_k = \{\alpha \in k \mid v(\alpha) > 0\}$ called the *valuation ideal*.

Definition 3.10 (Valuation ring)**.** An integral domain A is called *valuation ring* if there is a valuation v of its field of fractions $F = \mathrm{Frac}(A)$, such that $A = O_F$.

Lemma 3.3. *Let A be a valuation ring with $F = \mathrm{Frac}(A)$, and B a local ring contained in F which dominates*[18] *A. Then $B = A$.*

[16] Being local Noetherian ensures that the Krull dimension will be finite [Liu10, Corollary 2.5.14(b)].

[17] Here since $\widehat{\varphi}$ is a k-algebra homomorphism, and not just a ring homomorphism, it is enough to show that it is injective just like in the above lemma: `https://math.stackexchange.com/a/4208827/`

[18] Let K be a field with local rings $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ contained in K. Then we say B *dominates* A if $A \subset B$ and $\mathfrak{m} = A \cap \mathfrak{n}$.

Proof. Let v be the valuation of F and $\mathfrak{n}$ be the maximal ideal of B. If there exist $b \in B \setminus A$, then $v(1/b) > 0$. Hence $1/b$ is contained in the maximal ideal $\mathfrak{m}$ of A. That is, since B dominates A, $1/b \in \mathfrak{n}$. However, then $1 = b \cdot 1/b \in \mathfrak{n}$, which is impossible. $\square$

Remark 3.12. There exists a valuation ring A having 'very many' prime ideals [Liu10, Exercise 3.3.26] such that the scheme $\operatorname{Spec} A \setminus \{\mathfrak{m}\}$, where $\mathfrak{m}$ is the maximal ideal of A, has no closed points [Liu10, Exercise 3.3.27].

Definition 3.11 (Discrete valuation ring)**.** A valuation ring A with the value group Γ_v isomorphic to $\mathbb{Z}$ (under addition) is called *discrete valuation ring*.

Remark 3.13. A valuation ring is Noetherian if and only if it is a discrete valuation ring or a field [Sta21, Tag 00II].

Remark 3.14. If k is a field, then the ring of formal power series $k[[t]]$ is a discrete valuation ring since it is a local principal ideal domain which is not a field [Liu10, Example 3.3.23]. Furthermore, the localization of $k[[t]]$ with respect to the set of positive powers of t is also a valuation ring called the field of formal Laurent power series[19] [Sta21, Tag 0BI0]. In fact, we have $k((t)) = \operatorname{Frac}(k[[t]])$ since it is the smallest field in which the domain $k[[t]]$ can be embedded[20].

Remark 3.15. Let A be a discrete valuation ring, with fraction field $F = \operatorname{Frac}(A)$, residue field K and maximal ideal $\mathfrak{m}$. Then $\operatorname{Spec} A$ has exactly two points: the generic point $\xi = \operatorname{Spec} F$ corresponding to the ideal $\langle 0 \rangle$, and the special point or closed point $s = \operatorname{Spec} K$ corresponding to the maximal ideal $\mathfrak{m}$ of A [Poo17, §3.2.3]. The subset $\{\xi\}$ is open in $\operatorname{Spec} A$ [Liu10, Example 2.4.13].

Definition 3.12 (Generic and special fibers)**.** Let X be an scheme over a (discrete) valuation ring A with fraction field $F = \operatorname{Frac}(A)$ and residue field K. The *generic fiber* of X is the F-scheme $X_F = X \times_{\operatorname{Spec} A} \operatorname{Spec} F$, and the *special fiber* of X is the K-scheme $X_K = X \times_{\operatorname{Spec} A} \operatorname{Spec} K$.

Remark 3.16. We can also define the *generic fiber* as in Proposition 3.3. We have $X_F = X_\xi = X \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbf{k}(\xi)$ where ξ is the generic point of $\operatorname{Spec} A$ [Liu10, Definition 3.1.15], since $\mathbf{k}(\xi) = \mathbf{k}(\operatorname{Spec} A) = \operatorname{Frac} A = F$ (Remark 2.11). Similarly, $X_K = X_s = X \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbf{k}(s)$ since $\mathbf{k}(s) = \mathcal{O}_{\operatorname{Spec} A, s}/\mathfrak{m}_s = A/\mathfrak{m} = K$ [Liu10, Example 3.1.18].

Proposition 3.4. *Let X be a reduced S-scheme and Y a separated S-scheme. Let f and g be two morphisms of S-schemes from X to Y such that $f|_U = g|_U$ for some everywhere dense open subset U of X. Then $f = g$.*

Proof. Since Y is a separated S-scheme, $\Delta_{Y/S} = (\mathrm{Id}_Y, \mathrm{Id}_Y) : Y \to Y \times_S Y$ is a closed immersion of schemes (Definition 2.14). Moreover, $\Delta_{Y/S} \circ f = (f, f) : X \to Y \times_S Y$ since $\Delta_{Y/S} \circ f$ verifies the universal property

$$\begin{tikzcd} & Y \arrow[d, phantom, "\circlearrowleft"] & \\ X \arrow[ur, "f"] \arrow[rr, dashed, "\exists!\ (f{,}f)"] & {} & Y \times_S Y \arrow[ul, "p"'] \end{tikzcd}$$

[19]A formal Laurent series is a generalization of a formal power series in which finitely many negative exponents are permitted. The formal Laurent series with coefficients in A form the ring denoted by $A((t))$. This ring is equal to the localization of $A[[t]]$ with respect to the set of positive powers of t. `https://math.stackexchange.com/a/605473`

[20]As in the second part of Remark 3.6, the proof follows from the fact that $f \in k[[t]]$ is a unit iff $f(0)$ is not zero: `https://math.stackexchange.com/a/2930617`

where $p : Y \times_S Y \to Y$ is the projection [Liu10, Definition 3.1.1]. Next consider the map $h = (f, g) : X \to Y \times_S Y$. Then on U, the map $\Delta_{Y/S} \circ f$ coincides with h, i.e. $U \subseteq h^{-1}(\Delta_{Y/S}(Y))$. Moreover, since U is dense in X and $\Delta_{Y/S}(Y)$ is homeomorphic to a closed subset of $Y \times_S Y$, we have $X = h^{-1}(\Delta_{Y/S}(Y))$. Therefore, $f(x) = g(x)$ for every $x \in X$.

Now, it just remains to show that the sheaf maps are equal [Liu10, Definition 2.2.20]. Since the equality of sheaf maps can be checked locally, let's fix some $x \in X$, to get $y = f(x) = g(x) \in Y$. Then we have the local homomorphisms $f_x^{\#}, g_x^{\#} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. We may suppose that y is in an affine piece $\operatorname{Spec} B$ corresponding to a prime ideal $\mathfrak{p}$ and $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B) = g^{-1}(\operatorname{Spec} B)$ with $x \in \operatorname{Spec} A$, for some commutative rings A and B [Liu10, Definition 2.3.8]. Hence, we can restrict ourselves to $\tilde{f}, \tilde{g} : \operatorname{Spec} A \to \operatorname{Spec} B$. Then, by the equivalence of the categories of affine schemes and commutative rings with unit [Liu10, Lemma 2.3.23], it is sufficient to show that the corresponding ring homomrphisms $\varphi, \psi : B \to A$ are equal. Let $b \in B$ and $a = \varphi(b) - \psi(b) \in A$. Then $U \cap \operatorname{Spec} A \subseteq V(\langle a \rangle) \subseteq \operatorname{Spec} A \subseteq X$. Moreover, since $U \cap \operatorname{Spec} A$ is dense in $\operatorname{Spec} A$, we have $V(\langle a \rangle) = \operatorname{Spec} A = V(0)$. Furthermore, since X being reduced implies that $\mathcal{O}_X(\operatorname{Spec} A) = A$ is a reduced ring, we have $\langle a \rangle \subseteq \sqrt{0} = \operatorname{nil}(A) = 0$. That is, $a = 0$ and hence $\varphi = \psi$. Since the choice of x was arbitrary, this completes the proof. □

Remark 3.17. The converse, that Y is a separated S-scheme, holds when $X = \operatorname{Spec} A$ for some discrete valuation ring A, S is Noetherian, and Y is of finite type over S [Liu10, Remark 3.3.12]. This gives us the *valuative criterion of separatedness.*

Theorem 3.3. *Let X be a proper scheme over a valuation ring A with $F = \operatorname{Frac}(A)$. Then the canonical map $X(A) \to X_F(F)$ is bijective.*

Proof. Firstly we will prove that the map is injective. Let $f, g \in X(A) = \operatorname{Hom}_{\operatorname{Spec} A}(\operatorname{Spec} A, X)$ be two distinct sections such that $f|_{\operatorname{Spec} F} = g|_{\operatorname{Spec} F} \in X_F(F) = \operatorname{Hom}_{\operatorname{Spec} F}(\operatorname{Spec} F, X_F)$. Note that $\operatorname{Spec} A$ is reduced and X is a separated A-scheme. Therefore, since $\operatorname{Spec} F$ is a dense in $\operatorname{Spec} A$ [Liu10, Remark 2.4.15], by Proposition 3.4, $f = g$.

Now, to prove surjectivity we need to show that the section $\sigma \in X_F(F)$ can be extended[21] to a section in $X(A)$. Let ξ and s, respectively, be the generic and closed points of $\operatorname{Spec} A$ as described in Remark 3.15. Then $x = \sigma(\xi)$ represents an element of $X_F(F)$ (Remark 2.1). Next, let $Z = \overline{\{x\}} \subseteq X$ be the closed subset endowed with the structure of a reduced (hence integral) subscheme [Liu10, Proposition 2.4.2(e)]. Then Z is a proper A-scheme, since $\pi : Z \to X \to \operatorname{Spec} A$ is a proper morphism [Liu10, Proposition 3.3.16(a,b)]. Therefore, since the image of $\pi : Z \to \operatorname{Spec} A$ is closed and contains ξ, we must have $\pi(Z) = \operatorname{Spec} A$. Moreover, since the point x is closed in X_F (Remark 2.4) and dense in Z_F, we have generic fiber $Z_F = \{x\}$. Next, let $t \in Z_s$, where Z_s is the special fiber of Z. Then $\mathcal{O}_{Z,t}$ is a local ring dominating A, with the field of fractions $\mathcal{O}_{Z,x} = \mathbf{k}(x) = \mathbf{k}(Z) = \operatorname{Frac} A = F$. It follows from Lemma 3.3 that $\mathcal{O}_{Z,t} = A$. Moreover, we have the canonical map $\operatorname{Spec} \mathcal{O}_{Z,t} \to Z$ with z in its image [Liu10, Example 2.3.16; Exercise 2.4.2]. Therefore, we have a section $\widetilde{\sigma} \in X(A)$

$$\widetilde{\sigma} : \operatorname{Spec} A = \operatorname{Spec} \mathcal{O}_{Z,t} \to Z \to X$$

such that $\widetilde{\sigma}|_{\operatorname{Spec} F} = \sigma$. □

Corollary 3.2. *Let X be a proper S-scheme. Then for any S-scheme $\operatorname{Spec} A$, where A is a valuation ring with $F = \operatorname{Frac}(A)$, the canonical map $X(A) \to X(F)$ is bijective.*

[21]Note that $A \hookrightarrow F$ implies that $\operatorname{Spec} F \hookrightarrow \operatorname{Spec} A$, hence we have restrictions and extensions of sections.

Proof. Consider the proper A-scheme $Z = X_A = X \times_S \operatorname{Spec} A$ [Liu10, Proposition 3.3.16(c)]. Then from Remark 2.5 it follows that

$$X(A) = \operatorname{Hom}_S(\operatorname{Spec} A, X) = \operatorname{Hom}_{\operatorname{Spec} A}(\operatorname{Spec} A, X_A) = X_A(A) = Z(A).$$

Similarly, for the F-scheme $Z_F = Z \times_{\operatorname{Spec} A} \operatorname{Spec} F$, we have

$$X(F) = \operatorname{Hom}_S(\operatorname{Spec} F, X) = \operatorname{Hom}_{\operatorname{Spec} A}(\operatorname{Spec} F, X_A) = \operatorname{Hom}_{\operatorname{Spec} F}(\operatorname{Spec} F, Z_F) = Z_F(F).$$

The desired result follows by applying the above theorem to Z. $\square$

Remark 3.18. The converse holds when A is a discrete valuation ring, S is Noetherian, and X is of finite type over S [Poo17, Theorem 3.2.12] [Liu10, Remark 3.3.28]. This gives us the *valuative criterion for properness.*

Definition 3.13 (Model over discrete valuation ring)**.** Let A be a discrete valuation ring with fraction field $F = \operatorname{Frac}(A)$. Let X be a F-scheme. Then an A-model of X is an A-scheme $\widetilde{X}$ equipped with an isomorphism $\widetilde{X} \times_{\operatorname{Spec} A} \operatorname{Spec} F \xrightarrow{\sim} X$ of F-schemes.

Definition 3.14 (Reduction modulo uniformizer)**.** Let A be a discrete valuation ring, with fraction field $F = \operatorname{Frac}(A)$, residue field K and uniformizer π. Let X be a F-variety and $\widetilde{X}$ be its A-model. Then the *reduction modulo* π of X is the special fiber of $\widetilde{X}$.

Remark 3.19. Suppose that we have extended a F-scheme to an A-model and we now wish to reduce the F-points on the generic fiber to k-points of the special fiber. Since there is no homomorphism $F \to k$, so to make sense of this, we must first extend the F-point to an A-point of the model. For a proper scheme over a discrete valuation ring, this extension is always possible (and unique), by Theorem 3.3 [Poo17, §3.2.5].

4 Conclusion

Now we are ready to prove and discuss the main result.

Proof of Theorem 1.1. Let x be the given smooth k-point on X. Then, by Remark 2.22, X is regular at x. Hence, we can replace X by an open neighborhood of x, and assume that X is integral (reduced by Remark 2.20). Then, by Theorem 3.1, the rational map $X \dashrightarrow Y$ corresponds to an element of $Y(\mathbf{k}(X))$.

Next, let $d = \dim X$. Then, by Remark 2.4 and Remark 2.17, we get $\dim \mathcal{O}_{X,x} = d$. Therefore, Theorem 3.2 gives the isomorphism in the following chain of embeddings

$$\mathcal{O}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{X,x} \cong k[[t_1, \ldots, t_d]] \hookrightarrow F := k((t_1))((t_2)) \cdots ((t_d))$$

where F is iterated Laurent series field (Remark 3.14). Moreover, since F is a field, the fraction field $\operatorname{Frac}(\mathcal{O}_{X,x}) = \mathbf{k}(X)$ embeds[22] in F, and hence the rational map $X \dashrightarrow Y$ corresponds to an element of $Y(F)$.

Finally, since Y is a proper[23] k-scheme, by Corollary 3.2 the element of $Y(F)$ extends to an element of $Y(k((t_1))((t_2)) \cdots ((t_{d-1}))[[t_d]])$ since $\operatorname{Frac}(k((t_1)) \ldots ((t_{d-1}))[[t_d]]) = k((t_1)) \ldots ((t_d))$

[22] Recall that the field of fractions of an integral domain is the smallest field in which it can be embedded. Therefore, since the field F contains the ring $k[[t_1, \ldots, t_d]] = k[[t_1]] \ldots [[t_d]]$, we must have $\operatorname{Frac}(k[[t_1, \ldots, t_d]]) \subseteq F$ (but not equal unless $d = 1$: `https://math.stackexchange.com/a/2906674`). In general, if A is an integral domain, then $\operatorname{Frac}(A[[t]]) \subseteq \operatorname{Frac}(A)((t))$ (`https://math.stackexchange.com/q/140054`).

[23] We will be using the fact that proper morphisms are stable under base change [Liu10, Proposition 3.3.16(c)].

by Remark 3.14. Furthermore, by Remark 3.19, the element of $Y(k((t_1))((t_2))\cdots((t_{d-1}))[[t_d]])$ reduces modulo t_d to an element of $Y(k((t_1))((t_2))\cdots((t_{d-1})))$. Repeating this argument another $d-1$ times, we get an element in $Y(k)$. □

Remark 4.1. The Lang-Nishimura theorem can fail[24] if

1. *Y is not proper:* Let $X = \operatorname{Proj}\mathbb{F}_2[x,y,z]/\langle y^2z - yz^2 + x^2z - x^3\rangle$ be an elliptic curve [Liu10, Definition 6.1.25]. Then $\#X(\mathbb{F}_2) = 5$. Next, let $Y = X \setminus X(\mathbb{F}_2)$, which is not a projective curve[25], hence not proper [Liu10, Remark 3.3.33]. Then we have a rational map $X \dashrightarrow Y$ defined by the identity morphism on Y, such that Y doesn't have a $\mathbb{F}_2$-point even though X has smooth $\mathbb{F}_2$-points.

2. *k-point of X is not smooth:* Let $X = \operatorname{Spec}\mathbb{R}[x,y]/\langle x^2+y^2\rangle$ and $Y = \operatorname{Proj}\mathbb{R}[u,v]/\langle u^2+v^2\rangle$. Then $X(\mathbb{R})$ has only one $\mathbb{R}$-point (corresponding to the origin) and it is not smooth (not regular by [Liu10, Theorem 4.2.19]). Moreover, Y is a proper $\mathbb{R}$-scheme [Liu10, Theorem 3.3.30]. Then we have a rational map corresponding to $(x,y)\mapsto[x:y]$ (affine to projective coordinates), such that Y doesn't have $\mathbb{R}$-points even though X has a $\mathbb{R}$-point.

Remark 4.2. The Lang-Nishimura theorem is useful for understanding Severi-Brauer varieties (Châtelet's theorem) [Poo17, Proposition 4.5.10] and the arithmetic of del Pezzo surfaces (degree 6 and 5) [Poo17, Lemma 9.4.18 and Corollary 9.4.28].

References

[Liu10] Q. Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002 (corrected paperback printing 2010). Translated from the French by Reinie Erné.

[Poo17] B. Poonen. *Rational points on varieties*, volume 186 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2017.

[Sta21] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2021.

[24] These examples are the result of my discussions of [Poo17, Exercise 3.12] with Tristan Phillips: `https://math.stackexchange.com/a/1872159/`.

[25] Related StackExchange discussion on [Poo17, Exercise 2.1]: `https://math.stackexchange.com/a/1424968/`