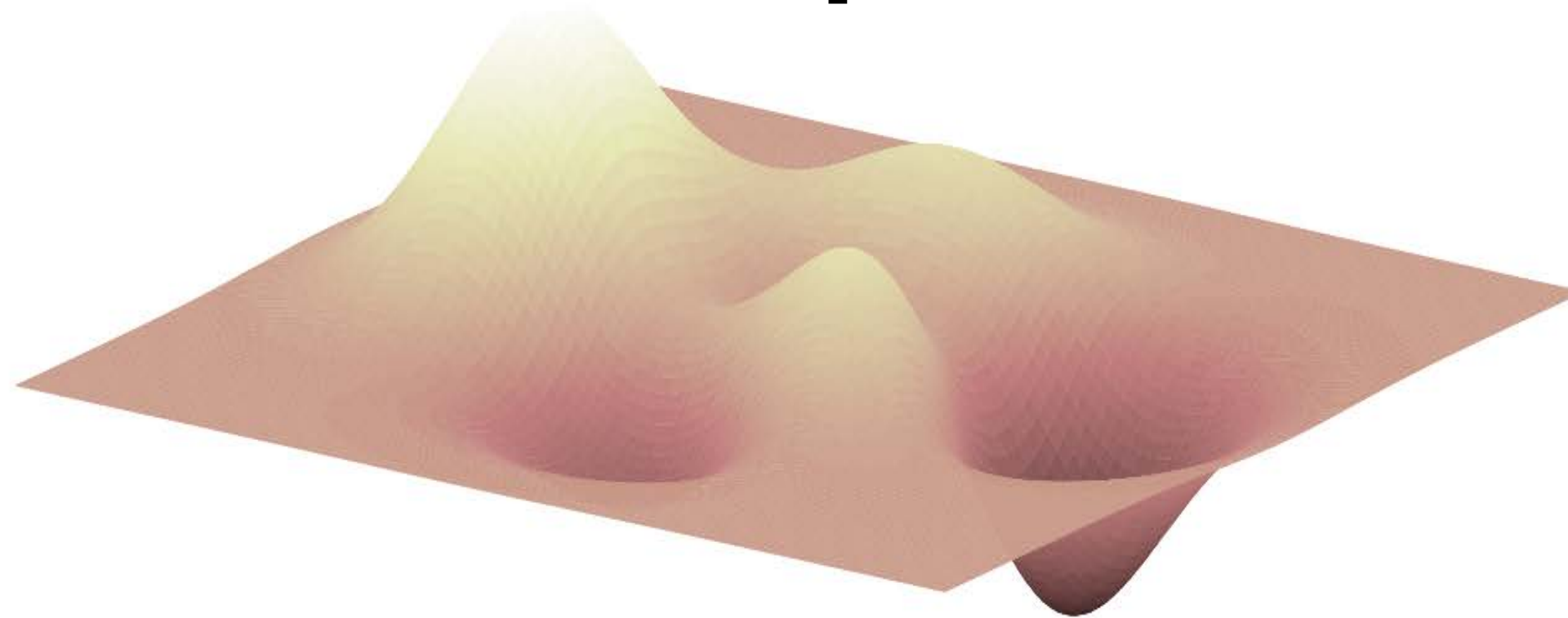


Nonlinear Least Squares



So far,

Type I

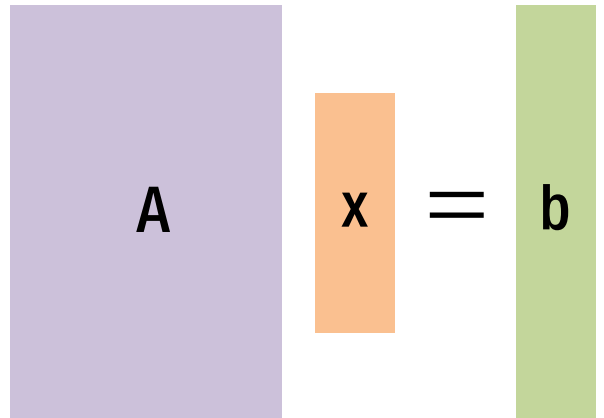


Diagram illustrating the Type I problem. A purple square labeled A is shown with dimensions $m \times n$ below it. To its right is an orange rectangle labeled x with dimensions $n \times 1$ below it. To the right of x is an equals sign, followed by a green rectangle labeled b with dimensions $m \times 1$ below it.

$$\min_x \|Ax - b\|^2$$

Type II

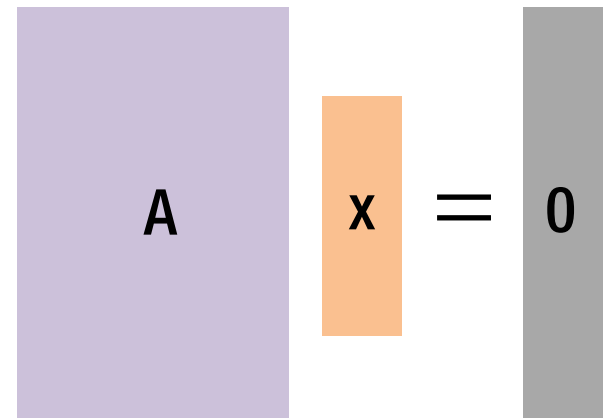


Diagram illustrating the Type II problem. A purple square labeled A is shown with dimensions $m \times n$ below it. To its right is an orange rectangle labeled x with dimensions $n \times 1$ below it. To the right of x is an equals sign, followed by a gray rectangle labeled 0 with dimensions $m \times 1$ below it.

$$\min_x \|Ax\|^2 \text{ subject to } \|x\| = 1$$

Why Linear Least Squares?

$$\min_x \|Ax - b\|^2 = \min_x (Ax - b)^\top (Ax - b)$$

Why Linear Least Squares?

$$\begin{aligned}\min_x \|Ax - b\|^2 &= \min_x (Ax - b)^\top (Ax - b) \\ &= \min_x x^\top A^\top Ax - 2b^\top Ax - b^\top b\end{aligned}$$

Why Linear Least Squares?

$$\begin{aligned}\min_x \|Ax - b\|^2 &= \min_x (Ax - b)^\top (Ax - b) \\ &= \min_x x^\top A^\top Ax - 2b^\top Ax - b^\top b \\ &= \min_x x^\top A^\top Ax - 2b^\top Ax\end{aligned}$$

Why Linear Least Squares?

$$\min_x \|Ax - b\|^2 = \min_x (Ax - b)^\top (Ax - b)$$

$$= \min_x x^\top A^\top Ax - 2b^\top Ax - b^\top b$$

$$= \min_x x^\top A^\top Ax - 2b^\top Ax$$

$$= \min_x x^\top Qx + cx$$

$$\text{where } Q = A^\top A, \quad c = -2b^\top A$$

Why Linear Least Squares?

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|^2 = \min_{\mathbf{x}} (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b})$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax} - \mathbf{b}^\top \mathbf{b}$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax}$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{Qx} + \mathbf{cx} \quad \text{where } \mathbf{Q} = \mathbf{A}^\top \mathbf{A}, \quad \mathbf{c} = -2\mathbf{b}^\top \mathbf{A}$$

Quadratic equation in \mathbf{x} .

\mathbf{Q} is positive definite (all eigen values of \mathbf{Q} are positive). \longrightarrow There exists the global solution \mathbf{x} .

Why Linear Least Squares?

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|^2 = \min_{\mathbf{x}} (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b})$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{A} \mathbf{x} - 2\mathbf{b}^\top \mathbf{A} \mathbf{x} - \mathbf{b}^\top \mathbf{b}$$

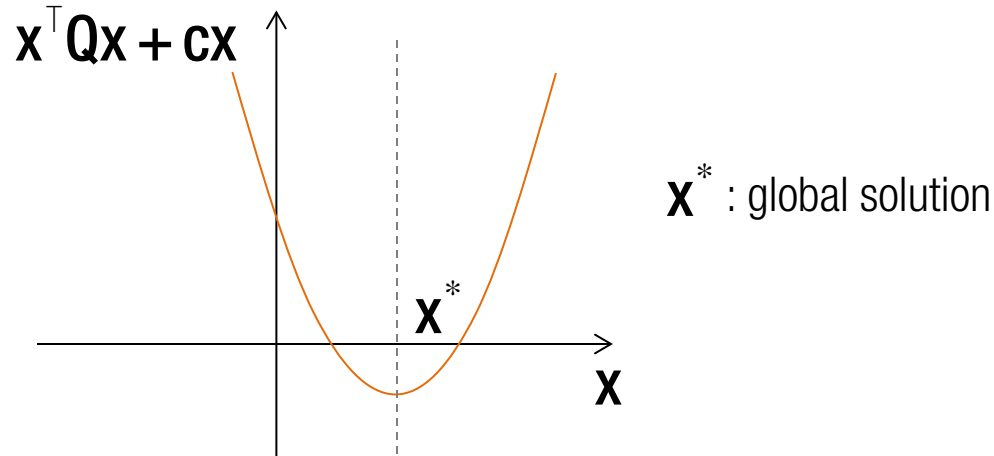
$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{A} \mathbf{x} - 2\mathbf{b}^\top \mathbf{A} \mathbf{x}$$

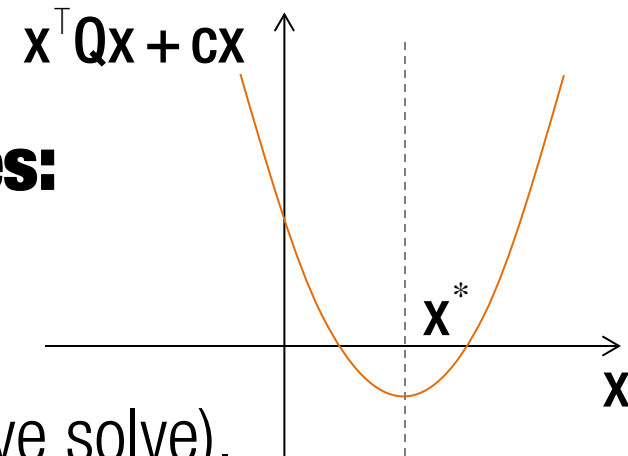
$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{c} \mathbf{x}$$

$$\text{where } \mathbf{Q} = \mathbf{A}^\top \mathbf{A}, \quad \mathbf{c} = -2\mathbf{b}^\top \mathbf{A}$$

Quadratic equation in \mathbf{x} .

\mathbf{Q} is positive definite (all eigen values of \mathbf{Q} are positive). \longrightarrow There exists the global solution \mathbf{x} .

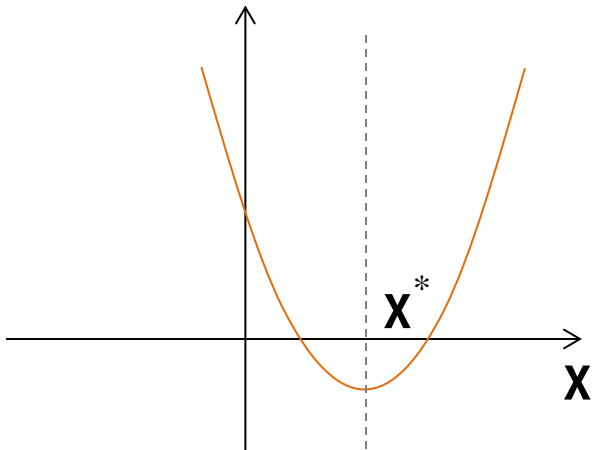
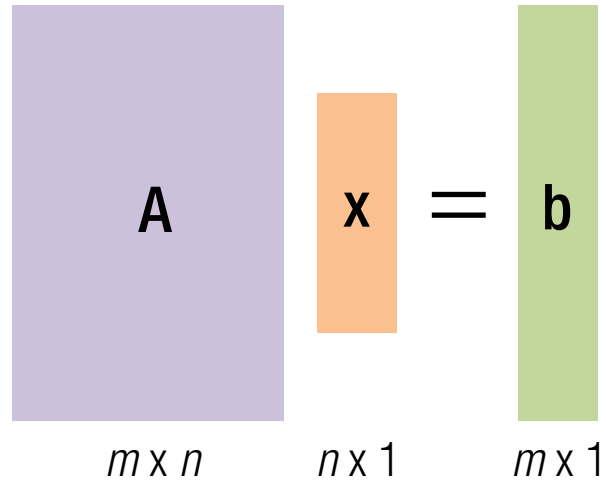




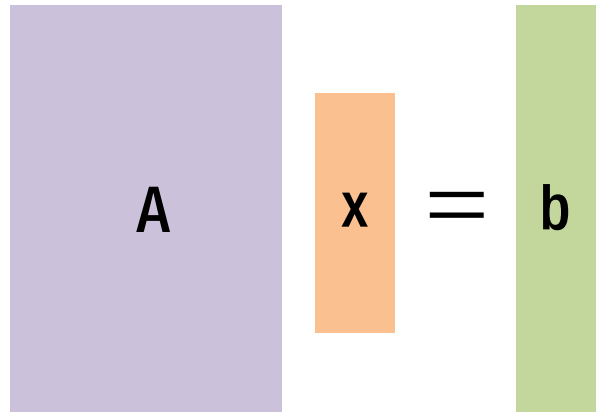
The properties of *linear* least squares:

- Has the global/unique solution.
- Has the closed form solution (non-iterative solve).
- Is solved efficiently (SVD).
- Requires no extra parameters such as initialization.

Life isn't that easy.



Life isn't that easy.

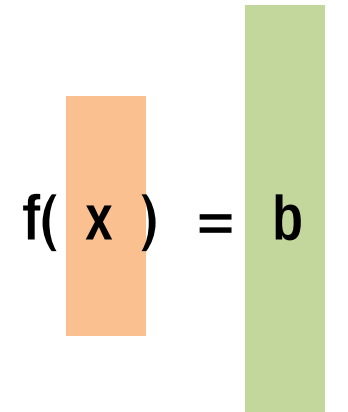


A diagram illustrating a linear system $Ax = b$. On the left is a large purple rectangle labeled A with dimensions $m \times n$ below it. To its right is an orange rectangle labeled x with dimensions $n \times 1$ below it. Further right is an equals sign, followed by a green rectangle labeled b with dimensions $m \times 1$ below it.

$$A \quad x = b$$

$m \times n \quad n \times 1 \quad m \times 1$

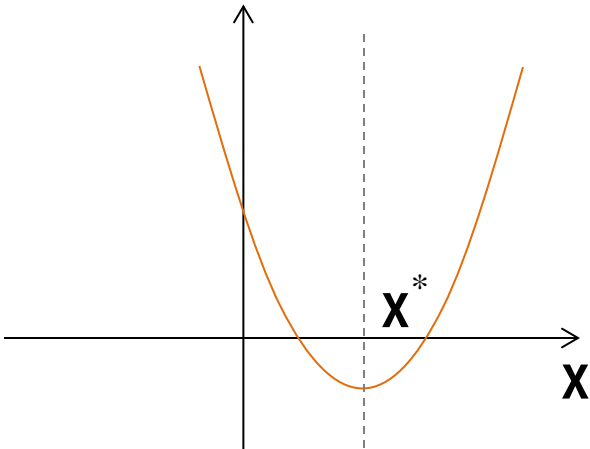
$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$



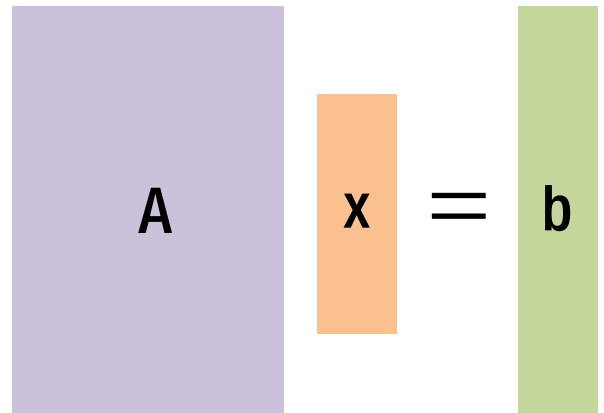
A diagram illustrating a function equation $f(x) = b$. On the left is an orange rectangle labeled x with dimensions $n \times 1$ below it. To its right is an equals sign, followed by a green rectangle labeled b with dimensions $m \times 1$ below it. The function f is indicated above the x rectangle.

$$f(x) = b$$

$n \times 1 \quad m \times 1$



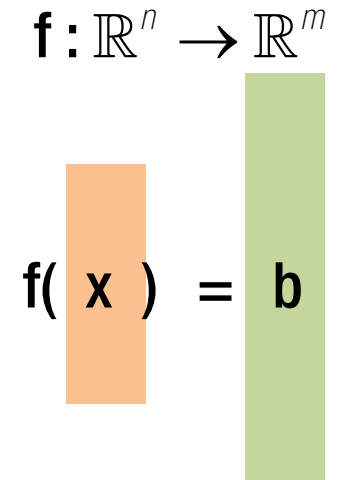
Life isn't that easy.



A diagram illustrating a linear system. A large purple rectangle labeled A is on the left, with dimensions $m \times n$ written below it. To its right is an orange rectangle labeled x with dimensions $n \times 1$ below it. Further right is a green rectangle labeled b with dimensions $m \times 1$ below it. An equals sign is placed between the orange and green rectangles.

$$A \quad x = b$$

$m \times n$ $n \times 1$ $m \times 1$



A diagram illustrating a nonlinear system. Above the diagram is the function notation $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$. Below it, an orange rectangle labeled x with dimensions $n \times 1$ below it is followed by a green rectangle labeled b with dimensions $m \times 1$ below it. An equals sign is placed between the orange and green rectangles, with the function f written to the left of the orange rectangle.

$$f(x) = b$$

$n \times 1$ $m \times 1$

