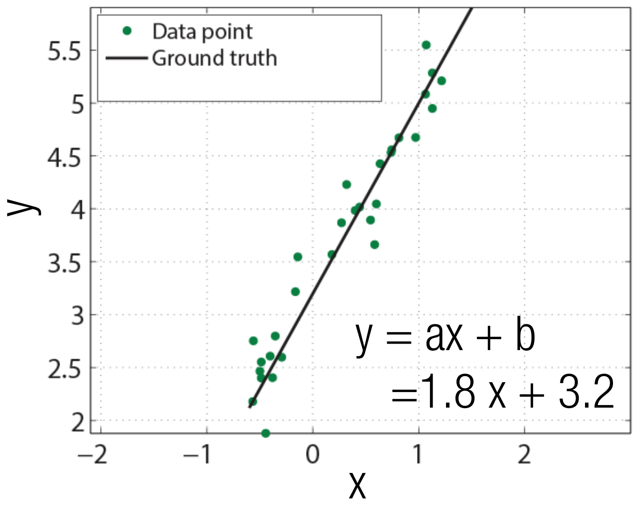
How to Solve? (Linear)

Linear least squares:

$$\min_{x} \|Ax - b\|^2$$



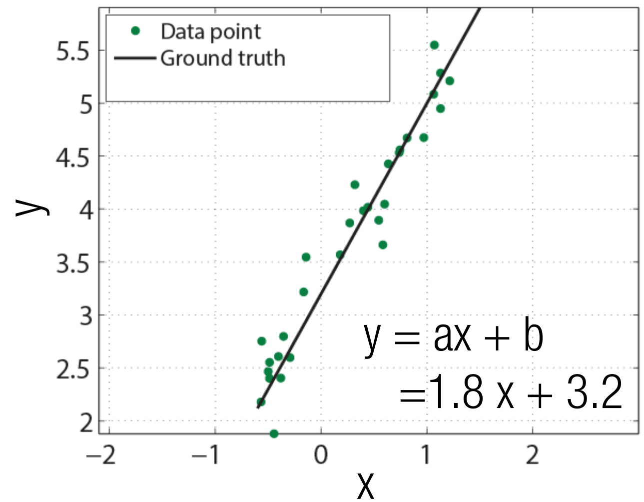
How to Solve? (Linear)

Linear least squares:

$$\min_{x} \|Ax - b\|^2 = \min_{x} (Ax - b)^\top (Ax - b)$$

$$= \min_{x} x^\top A^\top Ax - 2b^\top Ax - b^\top b$$

$$= \min_{x} x^\top A^\top Ax - 2b^\top Ax$$



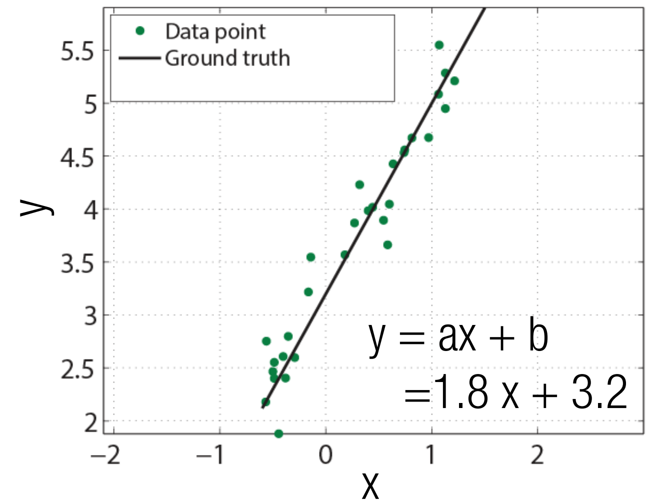
How to Solve? (Linear)

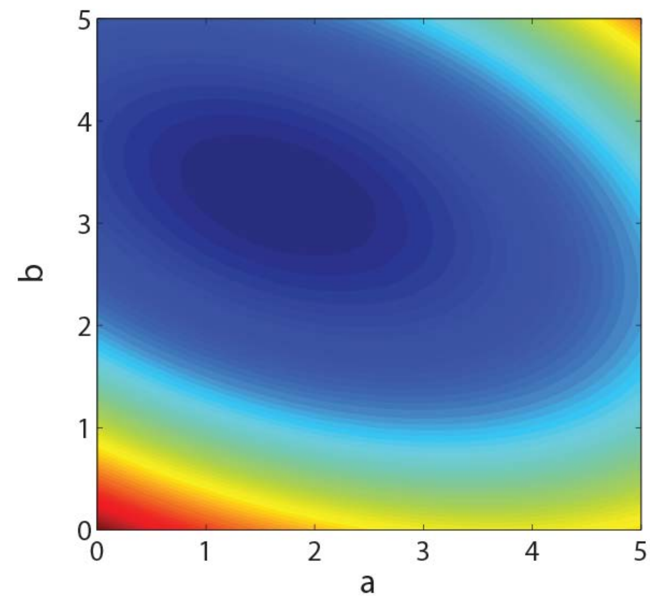
Linear least squares:

$$\min_{x} \|Ax - b\|^2 = \min_{x} (Ax - b)^{\top} (Ax - b)$$
$$= \min_{x} x^{\top}A^{\top}Ax - 2b^{\top}Ax - b^{\top}b$$
$$= \min_{x} x^{\top}A^{\top}Ax - 2b^{\top}Ax$$

Error:

$$E = x^{\top}A^{\top}Ax - 2b^{\top}Ax$$





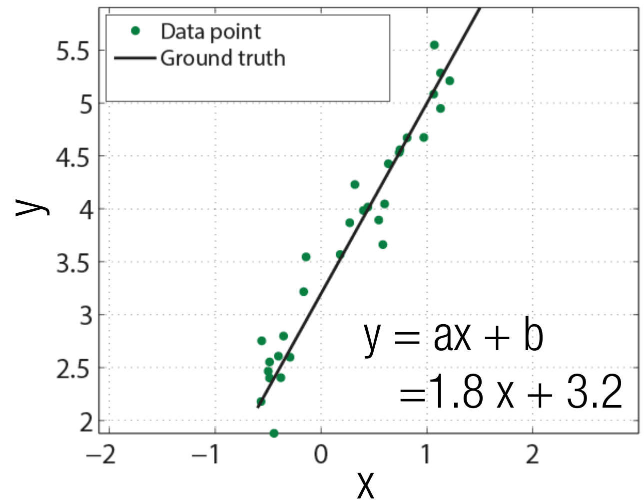
How to Solve? (Linear)

Linear least squares:

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2 = \min_{\mathbf{x}} (\mathbf{A}\mathbf{x}-\mathbf{b})^{\mathsf{T}}(\mathbf{A}\mathbf{x}-\mathbf{b})$$

$$= \min_{\mathbf{x}} \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} - 2\mathbf{b}^{\mathsf{T}}\mathbf{A}\mathbf{x} - \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$= \min_{\mathbf{x}} \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} - 2\mathbf{b}^{\mathsf{T}}\mathbf{A}\mathbf{x}$$

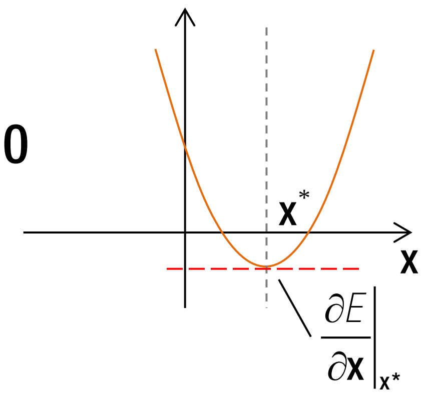


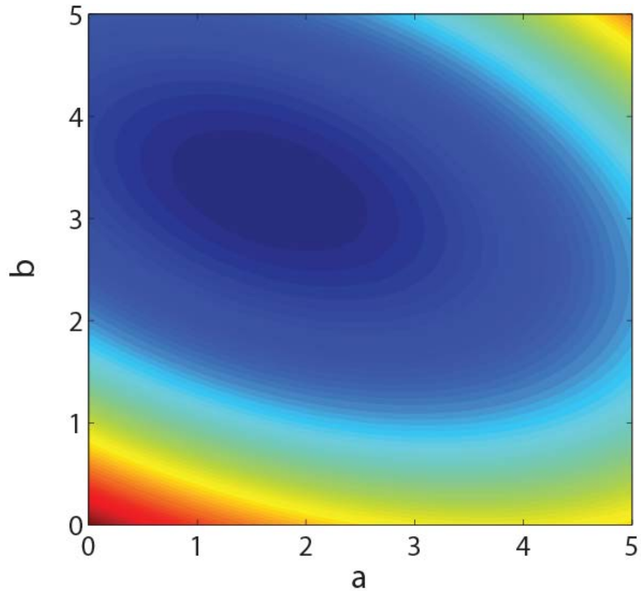
Error:

$$E = \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} - 2\mathbf{b}^{\mathsf{T}}\mathbf{A}\mathbf{x}$$

Condition for the solution:

$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} - 2\mathbf{A}^{\mathsf{T}}\mathbf{b} = 0$$

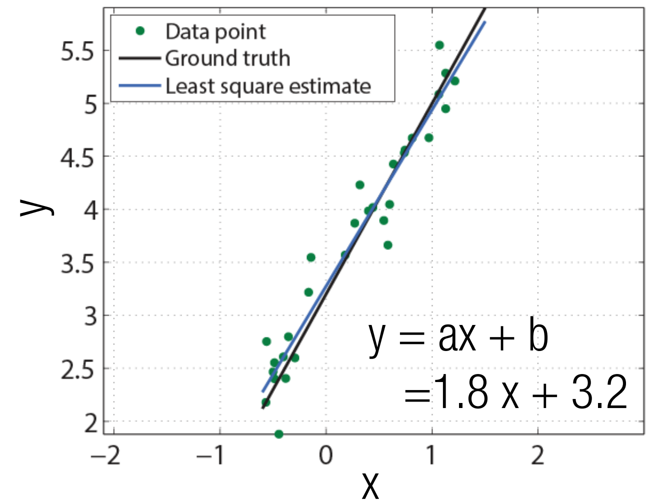




How to Solve? (Linear)

Linear least squares:

$$\min_{\mathbf{x}} \|\mathbf{Ax}-\mathbf{b}\|^2 = \min_{\mathbf{x}} (\mathbf{Ax}-\mathbf{b})^{\top}(\mathbf{Ax}-\mathbf{b})$$
$$= \min_{\mathbf{x}} \mathbf{x}^{\top}\mathbf{A}^{\top}\mathbf{Ax} - 2\mathbf{b}^{\top}\mathbf{Ax} - \mathbf{b}^{\top}\mathbf{b}$$
$$= \min_{\mathbf{x}} \mathbf{x}^{\top}\mathbf{A}^{\top}\mathbf{Ax} - 2\mathbf{b}^{\top}\mathbf{Ax}$$

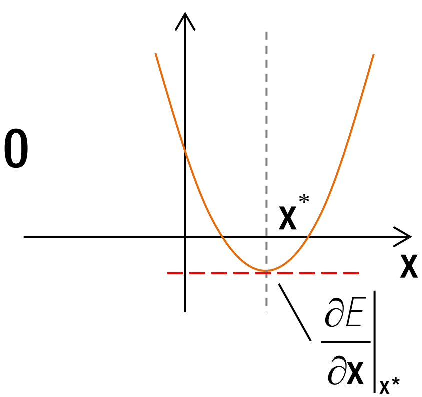


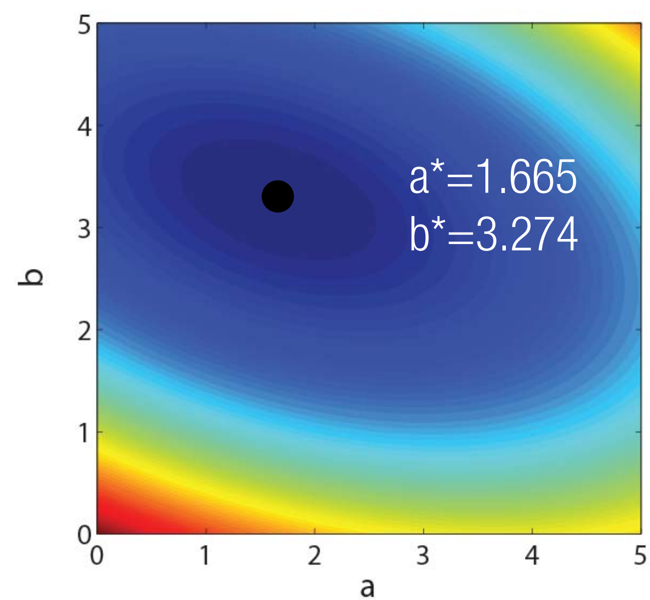
Error:

$$E = \mathbf{x}^{\top}\mathbf{A}^{\top}\mathbf{Ax} - 2\mathbf{b}^{\top}\mathbf{Ax}$$

Condition for the solution:

$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\mathbf{A}^{\top}\mathbf{Ax} - 2\mathbf{A}^{\top}\mathbf{b} = 0$$





Solution:

$$\mathbf{x}^* = \left(\mathbf{A}^{\top}\mathbf{A}\right)^{-1}\mathbf{A}^{\top}\mathbf{b}$$

How to Solve? (Nonlinear)

Nonlinear least squares:

$$\min_{x} \left\| f(x) - b \right\|^2 = \min_{x} \left(f(x) - b \right)^{\top} \left(f(x) - b \right)$$

$$= \min_{x} f(x)^{\top} f(x) - 2b^{\top} f(x) - b^{\top} b$$

$$= \min_{x} f(x)^{\top} f(x) - 2b^{\top} f(x)$$

How to Solve? (Nonlinear)

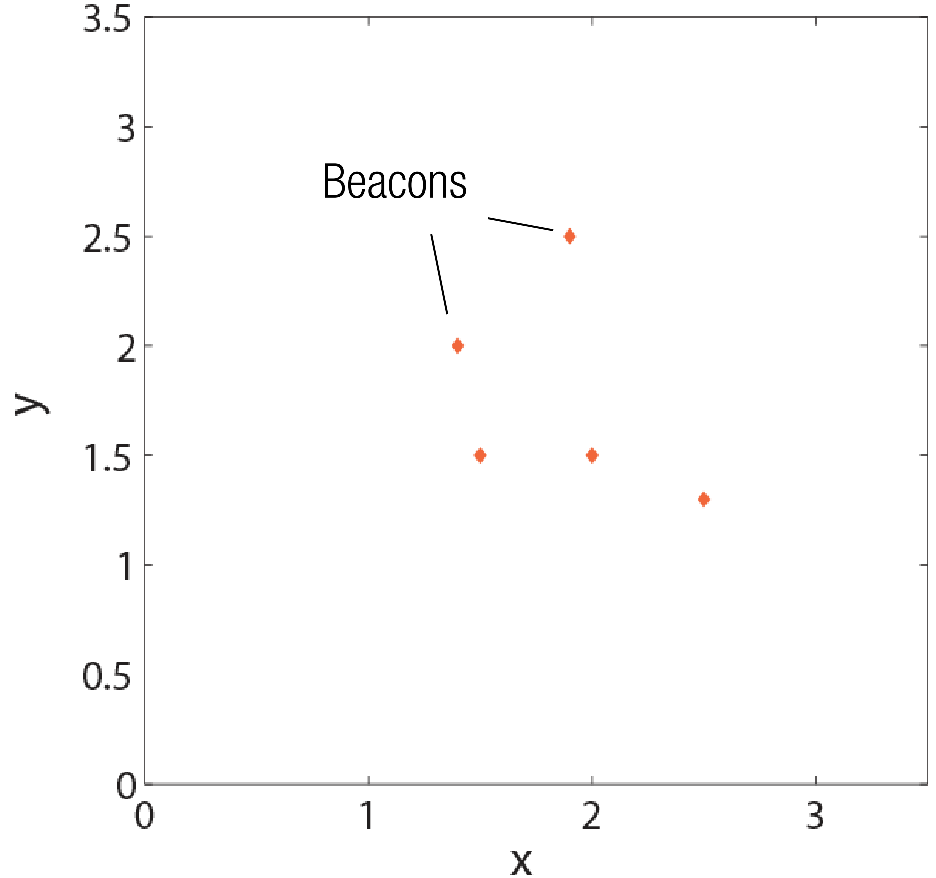
Nonlinear least squares:

$$\min_x \|f(x) - b\|^2 = \min_x \left(f(x) - b\right)^T \left(f(x) - b\right)$$

$$= \min_x f(x)^T f(x) - 2b^T f(x) - b^T b$$

$$= \min_x f(x)^T f(x) - 2b^T f(x)$$

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

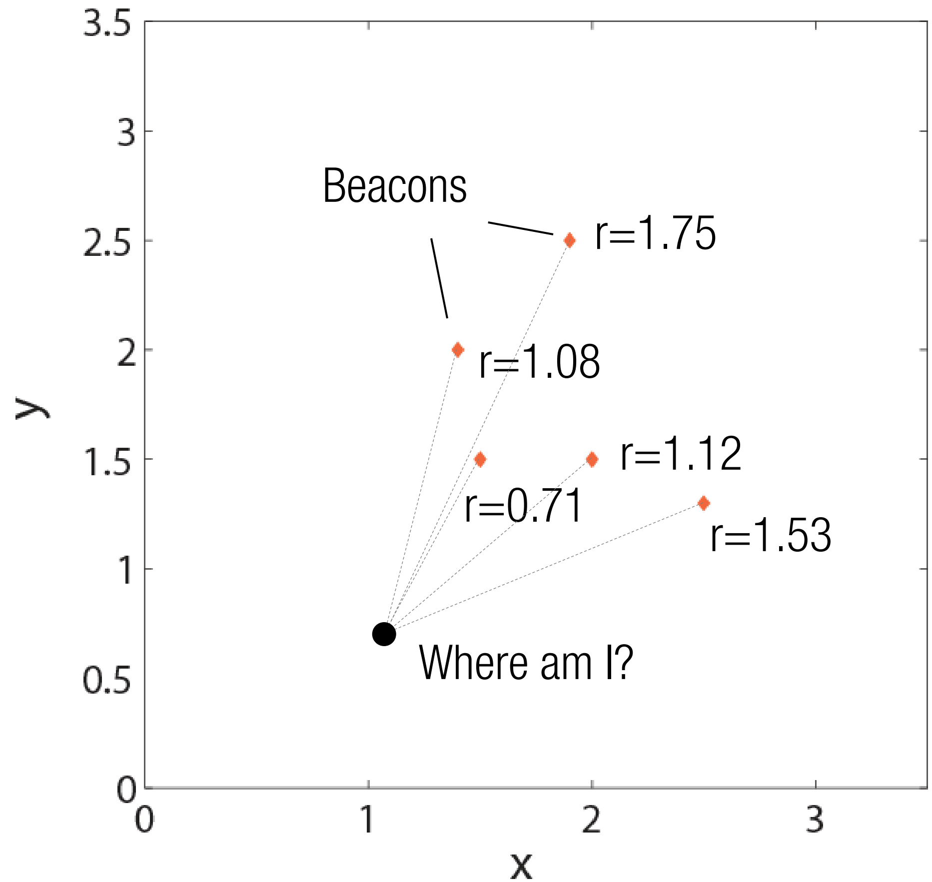
Nonlinear least squares:

$$\min_x \|f(x) - b\|^2 = \min_x (f(x) - b)^T (f(x) - b)$$

$$= \min_x f(x)^T f(x) - 2b^T f(x) - b^T b$$

$$= \min_x f(x)^T f(x) - 2b^T f(x)$$

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

Nonlinear least squares:

$$\min_{\mathbf{x}} \|\mathbf{f(x)} - \mathbf{b}\|^2 = \min_{\mathbf{x}} \left(\mathbf{f(x)} - \mathbf{b}\right)^\top \left(\mathbf{f(x)} - \mathbf{b}\right)$$

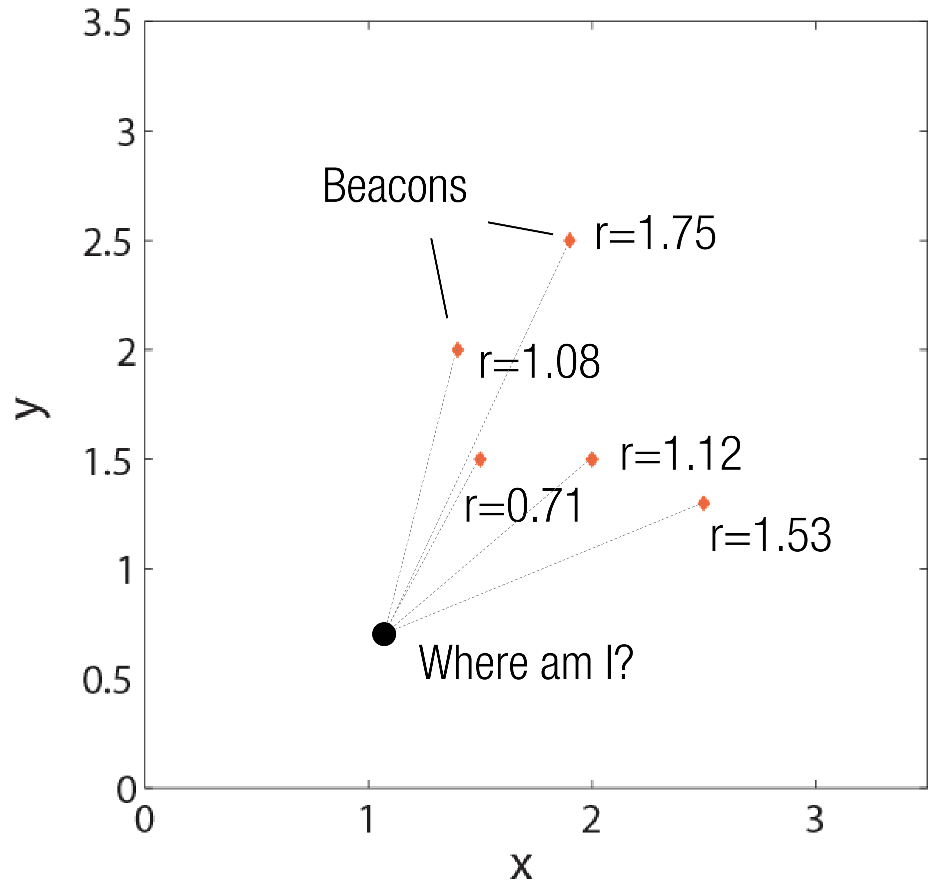
$$= \min_{\mathbf{x}} \mathbf{f(x)}^\top \mathbf{f(x)} - 2\mathbf{b}^\top \mathbf{f(x)} - \mathbf{b}^\top \mathbf{b}$$

$$= \min_{\mathbf{x}} \mathbf{f(x)}^\top \mathbf{f(x)} - 2\mathbf{b}^\top \mathbf{f(x)}$$

Error:

$$E = \mathbf{f(x)}^\top \mathbf{f(x)} - 2\mathbf{b}^\top \mathbf{f(x)}$$

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

Nonlinear least squares:

$$\min_{\mathbf{x}} \left\| \mathbf{f(x) - b} \right\|^2 = \min_{\mathbf{x}} \left(\mathbf{f(x) - b}\right)^\top \left(\mathbf{f(x) - b}\right)$$

$$= \min_{\mathbf{x}} \mathbf{f(x)}^\top \mathbf{f(x)} - 2\mathbf{b}^\top \mathbf{f(x)} - \mathbf{b}^\top \mathbf{b}$$

$$= \min_{\mathbf{x}} \mathbf{f(x)}^\top \mathbf{f(x)} - 2\mathbf{b}^\top \mathbf{f(x)}$$

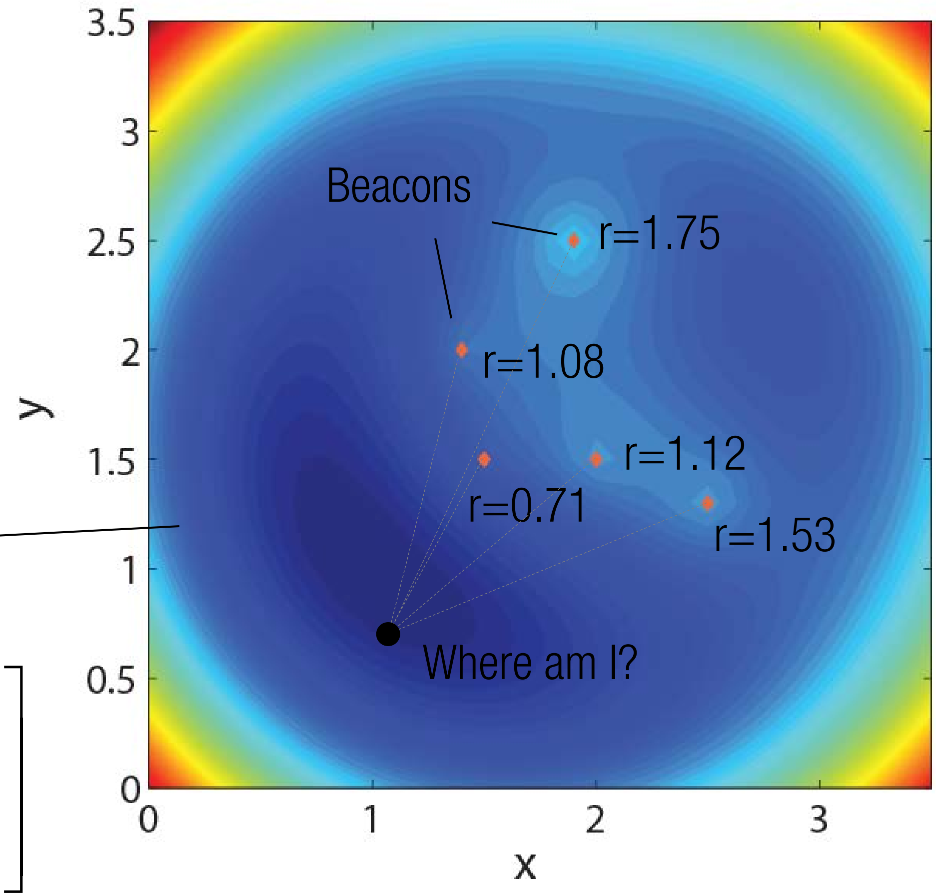
Error:

$$E = \mathbf{f(x)}^\top \mathbf{f(x)} - 2\mathbf{b}^\top \mathbf{f(x)}$$

$$E = \mathbf{S}^\top \mathbf{S}$$

where $\mathbf{S} = \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}$

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

Nonlinear least squares:

$$\min_{\mathbf{x}} \left\| \mathbf{f}(\mathbf{x}) - \mathbf{b} \right\|^2 = \min_{\mathbf{x}} \left(\mathbf{f}(\mathbf{x}) - \mathbf{b} \right)^\top \left(\mathbf{f}(\mathbf{x}) - \mathbf{b} \right)$$

$$= \min_{\mathbf{x}} \mathbf{f}(\mathbf{x})^\top \mathbf{f}(\mathbf{x}) - 2\mathbf{b}^\top \mathbf{f}(\mathbf{x}) - \mathbf{b}^\top \mathbf{b}$$

$$= \min_{\mathbf{x}} \mathbf{f}(\mathbf{x})^\top \mathbf{f}(\mathbf{x}) - 2\mathbf{b}^\top \mathbf{f}(\mathbf{x})$$

Error:

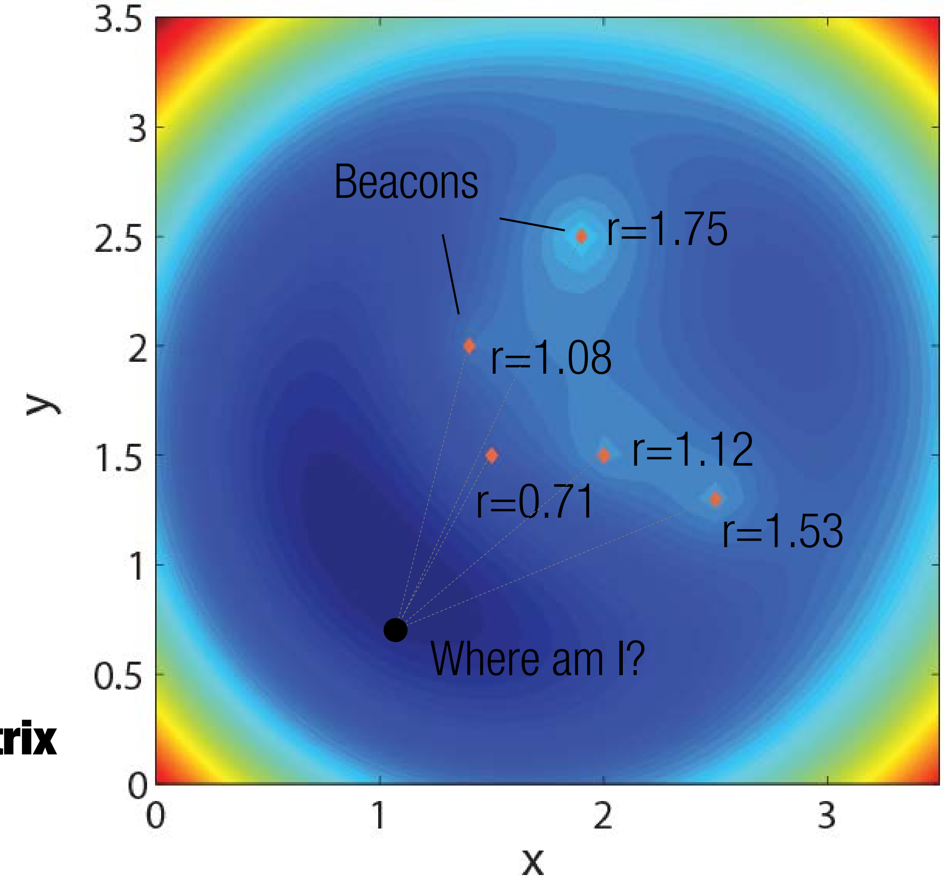
$$E = \mathbf{f}(\mathbf{x})^\top \mathbf{f}(\mathbf{x}) - 2\mathbf{b}^\top \mathbf{f}(\mathbf{x})$$

Condition for the solution:

$$\left. \frac{\partial E}{\partial \mathbf{x}} \right|_{\mathbf{x}^*} = 2 \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^\top \mathbf{f}(\mathbf{x}) - 2 \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^\top \mathbf{b} = 0$$

where $\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$ **: Jacobian matrix**

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

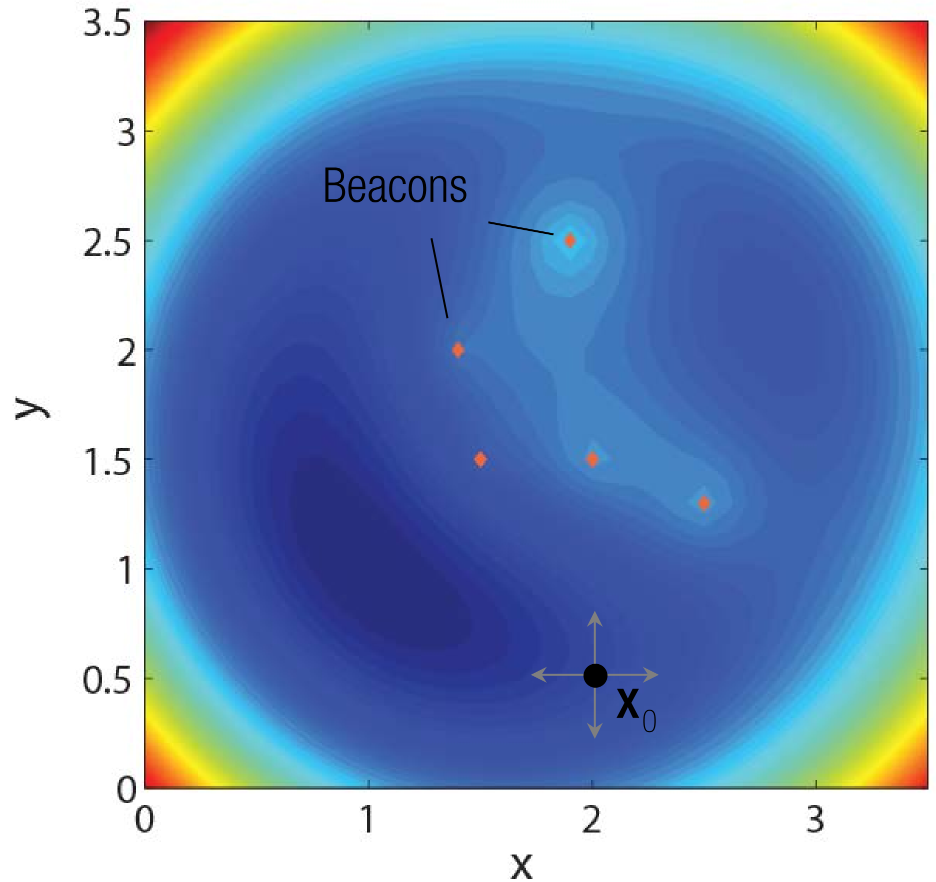
$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\frac{\partial \mathbf{f(x)}}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{f(x)} - 2\frac{\partial \mathbf{f(x)}}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b} = \mathbf{0} \qquad (1)$$

Objective:

Given $\mathbf{x}_0$, move $\mathbf{x}$ such that $\mathbf{f(x)} < \mathbf{f(x_0)}$.

Example:

Localization using range data from beacons



How to Solve? (Nonlinear)

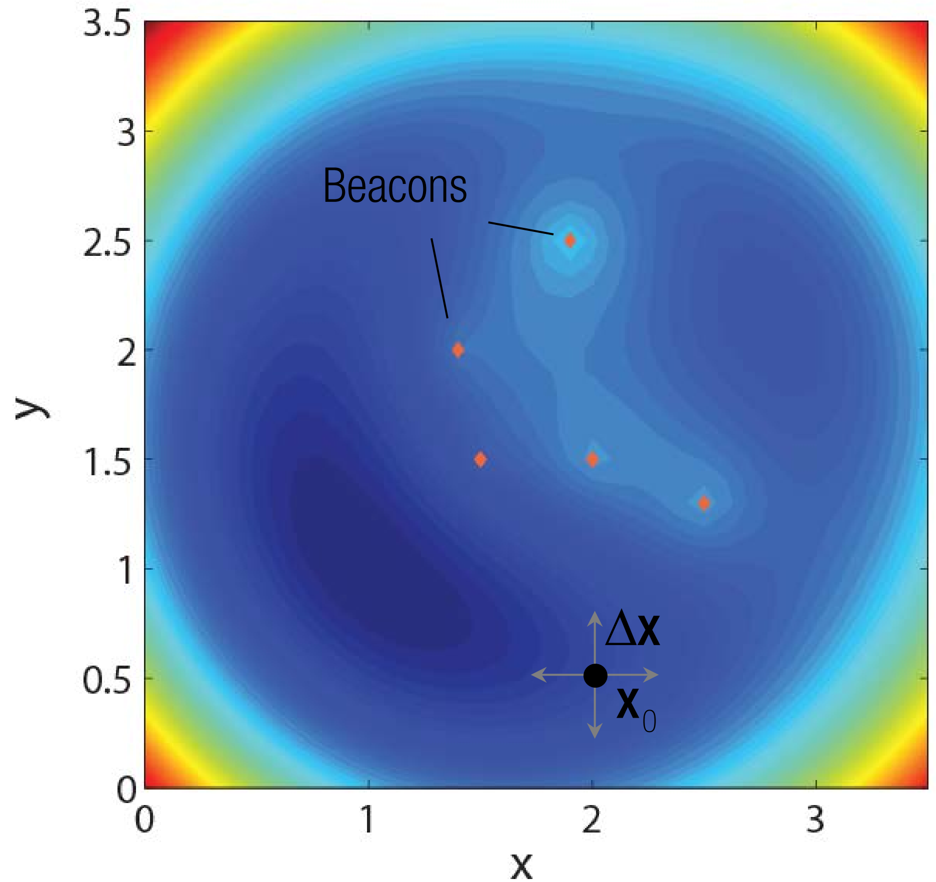
$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \mathbf{f}(\mathbf{x}) - 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \mathbf{b} = \mathbf{0} \qquad (1)$$

Example:
Localization using range data from beacons

Objective:
Given $\mathbf{x}_0$, move $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x}) < \mathbf{f}(\mathbf{x}_0)$.

Taylor expansion around $\mathbf{x}$:

$$\mathbf{f}(\mathbf{x}+\Delta\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x}$$



How to Solve? (Nonlinear)

$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \mathbf{f}(\mathbf{x}) - 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \mathbf{b} = \mathbf{0} \qquad (1)$$

Objective:

Given $\mathbf{x}_0$, move $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x}) < \mathbf{f}(\mathbf{x}_0)$.

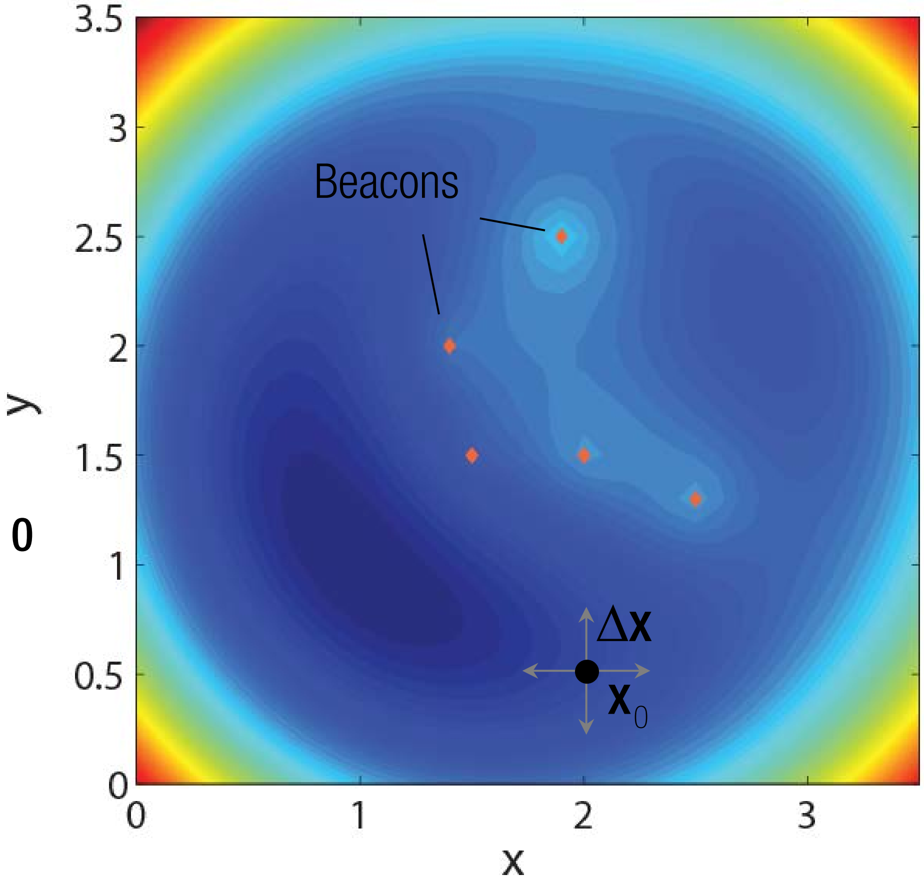
Taylor expansion around $\mathbf{x}$:

$$\mathbf{f}(\mathbf{x}+\Delta\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x}$$

Plugging into Equation (1):

$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x}\right) - 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \mathbf{b} = \mathbf{0}$$

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{f}(\mathbf{x}) - 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b} = \mathbf{0} \qquad (1)$$

Objective:
Given $\mathbf{x}_0$, move $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x}) < \mathbf{f}(\mathbf{x}_0)$.

Taylor expansion around $\mathbf{x}$:

$$\mathbf{f}(\mathbf{x} + \Delta\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x}$$

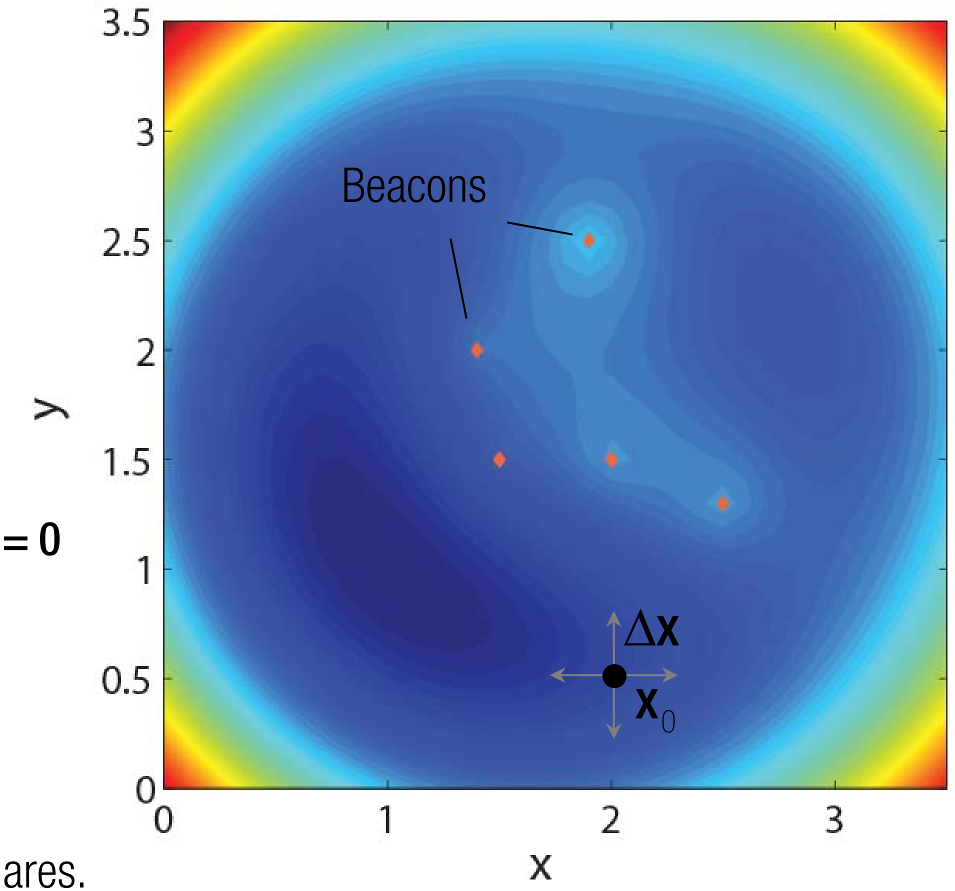
Plugging into Equation (1):

$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} \right) - 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b} = \mathbf{0}$$

$$\longrightarrow \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - \mathbf{f}(\mathbf{x}) \right)$$

Note that $\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$ for linear least squares.

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{f}(\mathbf{x}) - 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b} = \mathbf{0} \qquad (1)$$

Objective:

Given $\mathbf{x}_0$, move $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x}) < \mathbf{f}(\mathbf{x}_0)$.

Taylor expansion around $\mathbf{x}$:

$$\mathbf{f}(\mathbf{x}+\Delta\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x}$$

Plugging into Equation (1):

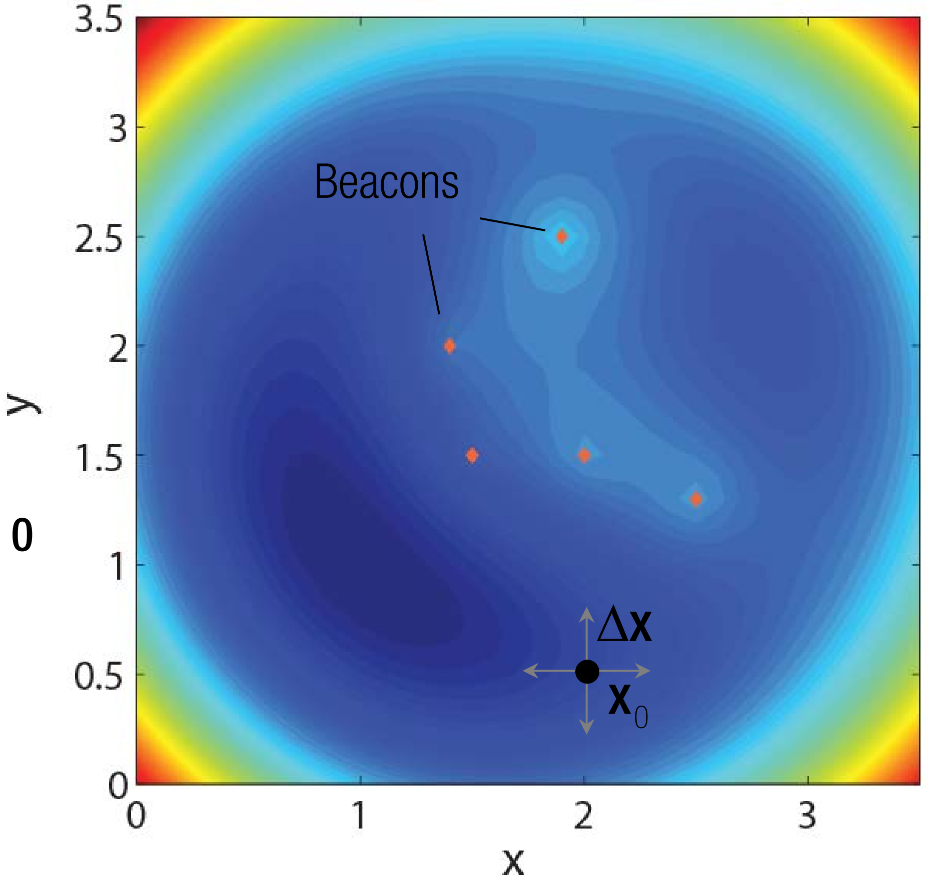
$$\left.\frac{\partial E}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} = 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}}\left(\mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x}\right) - 2\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b} = \mathbf{0}$$

$$\longrightarrow \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}}\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$

$$\boxed{\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta\mathbf{x}}$$

where $\Delta\mathbf{x} = \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right)^{-1} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}}\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$

Example:
Localization using range data from beacons

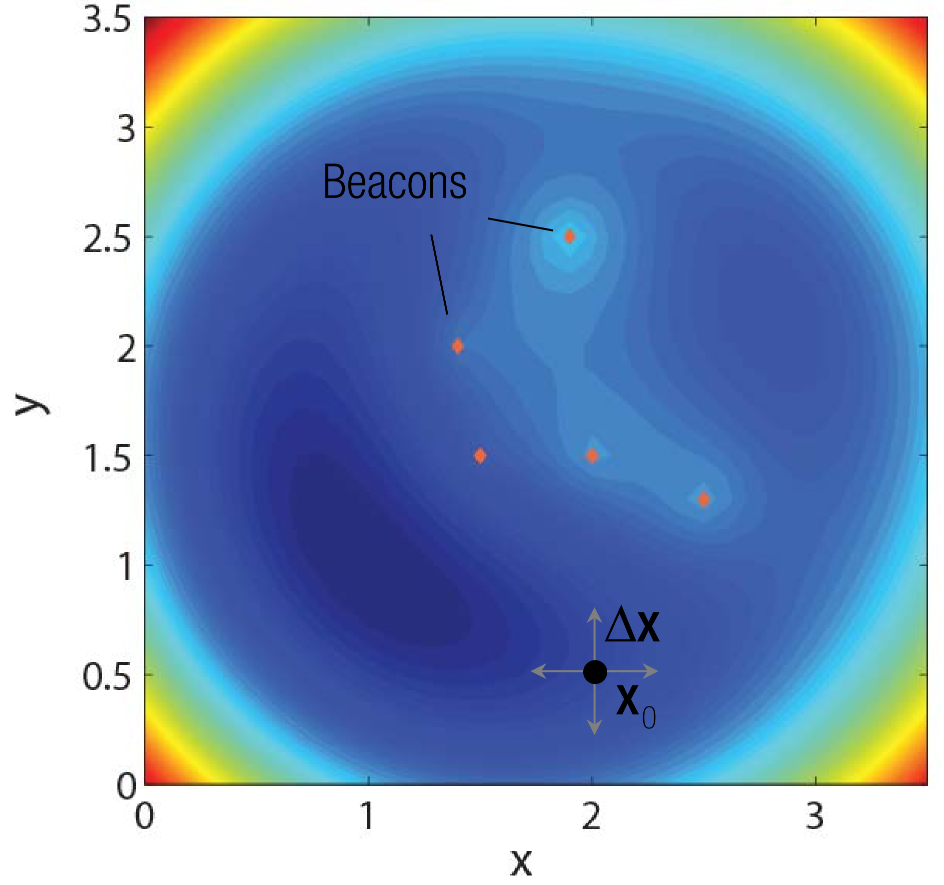


How to Solve? (Nonlinear)

$$\boxed{\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta\mathbf{x}}$$

$$\text{where } \Delta\mathbf{x} = \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right)^{-1} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$

Example:
Localization using range data from beacons



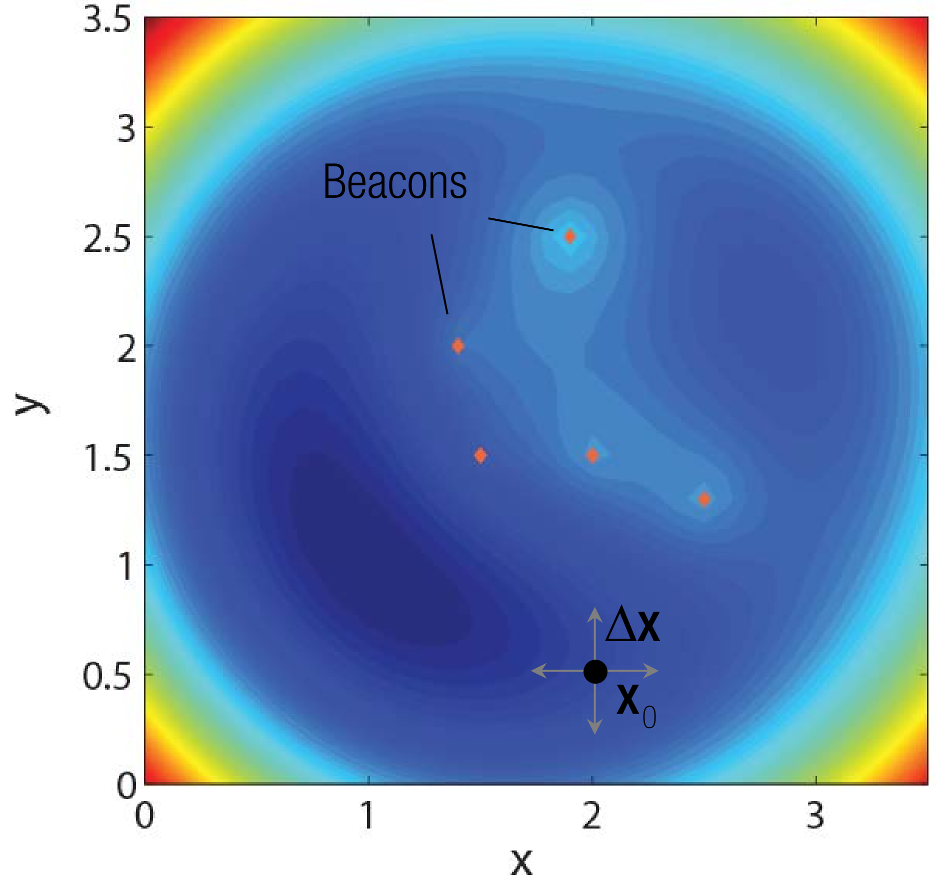
How to Solve? (Nonlinear)

$$\boxed{\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta\mathbf{x}}$$

$$\text{where } \Delta\mathbf{x} = \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - \mathbf{f}(\mathbf{x}) \right)$$

$$\mathbf{f}(x, y) = \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}$$

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

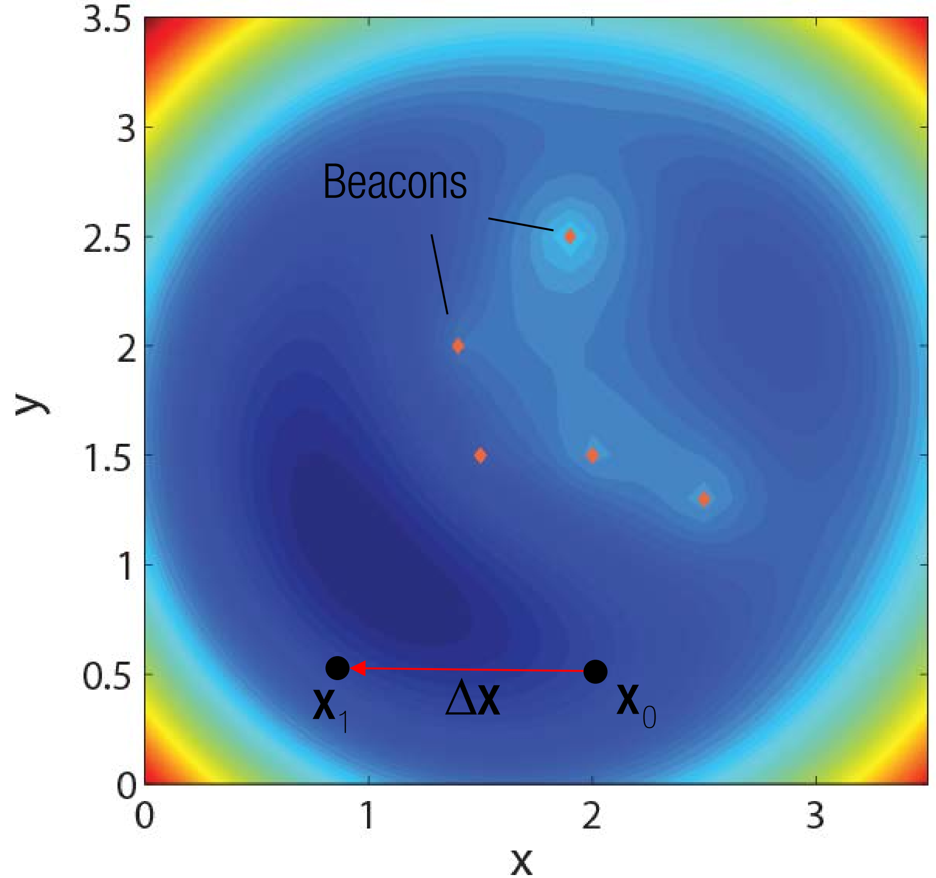
$$\boxed{\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta\mathbf{x}}$$

$$\text{where } \Delta\mathbf{x} = \left(\frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}}^{T}\frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}}\right)^{-1}\frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}}^{T}\left(\mathbf{b}-\mathbf{f}(\mathbf{x})\right)$$

$$\mathbf{f}(x,y) = \begin{bmatrix} \sqrt{(u_1-x)^2+(v_1-y)^2} \\ \vdots \\ \sqrt{(u_5-x)^2+(v_5-y)^2} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}$$

$$\underbrace{\frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}} = \begin{bmatrix} \frac{u_1-x}{\sqrt{(u_1-x)^2+(v_1-y)^2}} & \frac{v_1-y}{\sqrt{(u_1-x)^2+(v_1-y)^2}} \\ \\ \frac{u_5-x}{\sqrt{(u_5-x)^2+(v_5-y)^2}} & \frac{v_5-y}{\sqrt{(u_5-x)^2+(v_5-y)^2}} \end{bmatrix}}_{\text{Jacobian matrix}}$$

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

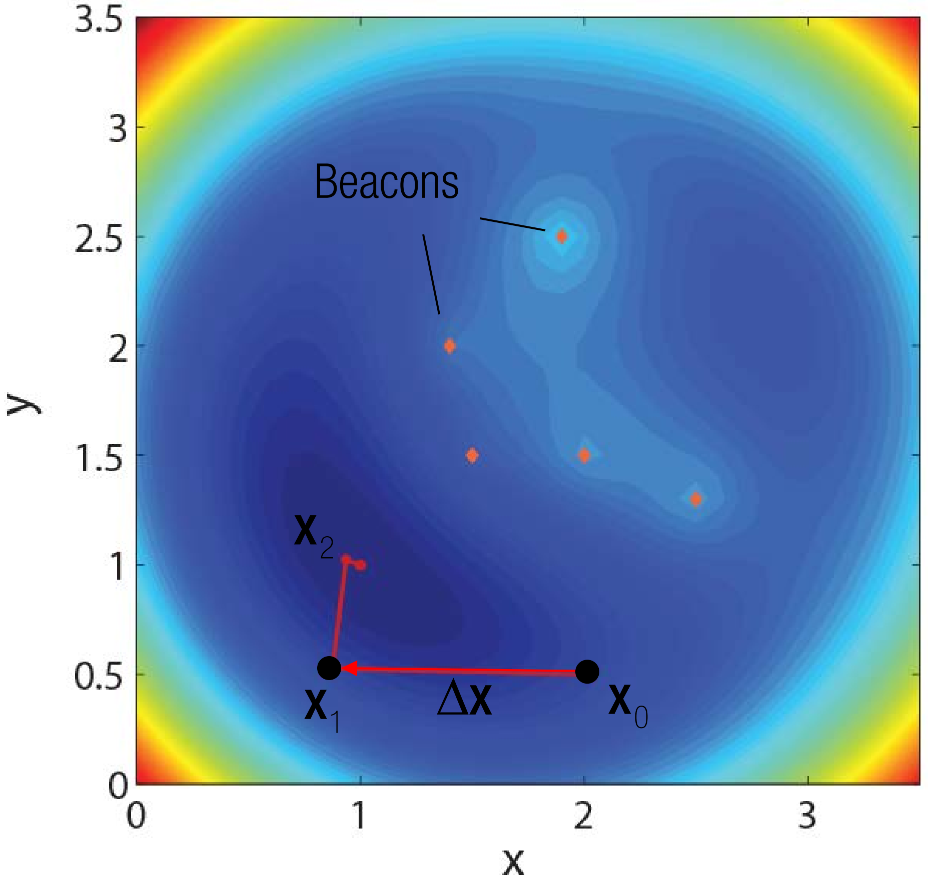
$$\boxed{\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta\mathbf{x}}$$

$$\text{where } \Delta\mathbf{x} = \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - \mathbf{f}(\mathbf{x}) \right)$$

$$\mathbf{f}(x, y) = \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}$$

$$\underbrace{\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{u_1 - x}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} & \frac{v_1 - y}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} \\ \frac{u_5 - x}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} & \frac{v_5 - y}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} \end{bmatrix}}_{\text{Jacobian matrix}}$$

Example:
Localization using range data from beacons



How to Solve? (Nonlinear)

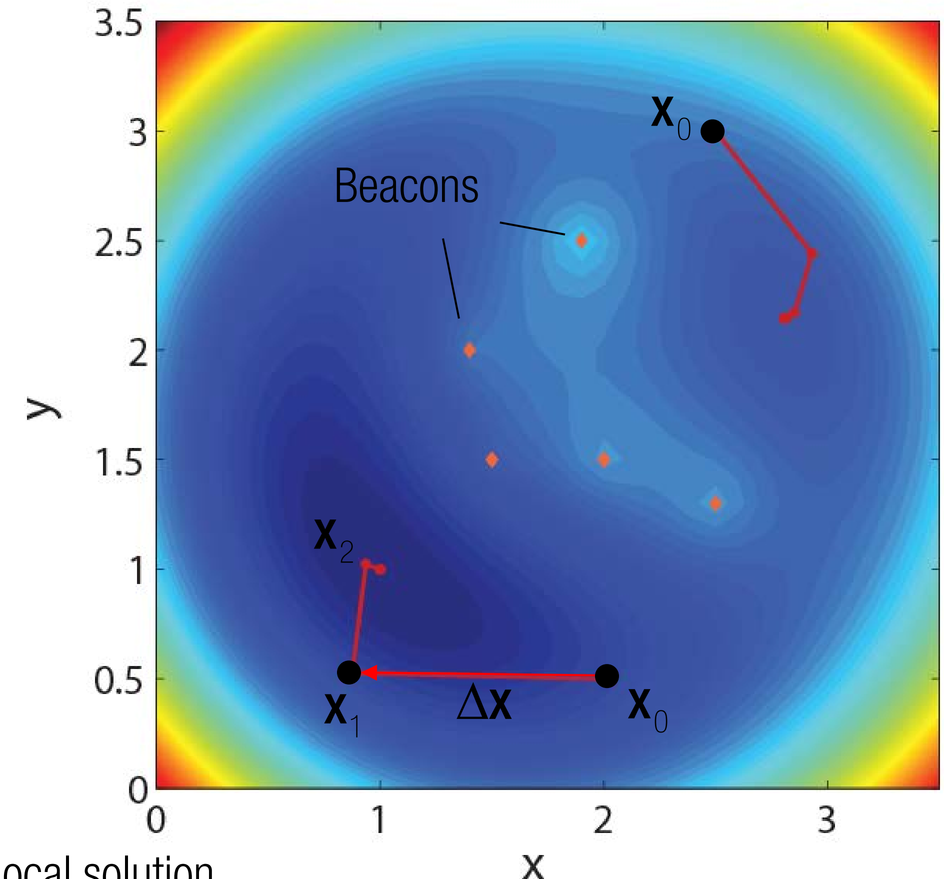
$$\boxed{\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta\mathbf{x}}$$

$$\text{where } \Delta\mathbf{x} = \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right)^{-1} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$

$$\mathbf{f}(x, y) = \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}$$

$$\underbrace{\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{u_1 - x}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} & \frac{v_1 - y}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} \\ \frac{u_5 - x}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} & \frac{v_5 - y}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} \end{bmatrix}}_{\text{Jacobian matrix}}$$

Example:
Localization using range data from beacons



A different initialization converges to a different local solution.

How to Solve? (Nonlinear)

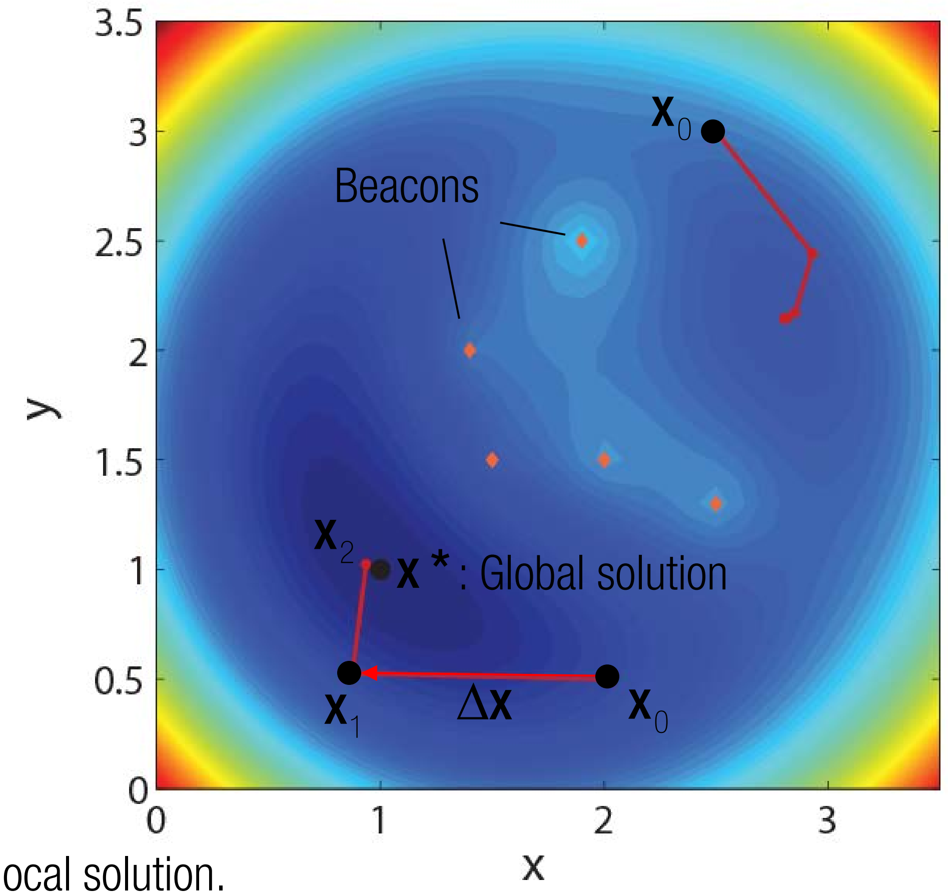
$$\boxed{\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta\mathbf{x}}$$

$$\text{where } \Delta\mathbf{x} = \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - \mathbf{f}(\mathbf{x}) \right)$$

$$\mathbf{f}(x, y) = \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}$$

$$\underbrace{\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{u_1 - x}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} & \frac{v_1 - y}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} \\ \frac{u_5 - x}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} & \frac{v_5 - y}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} \end{bmatrix}}_{\text{Jacobian matrix}}$$

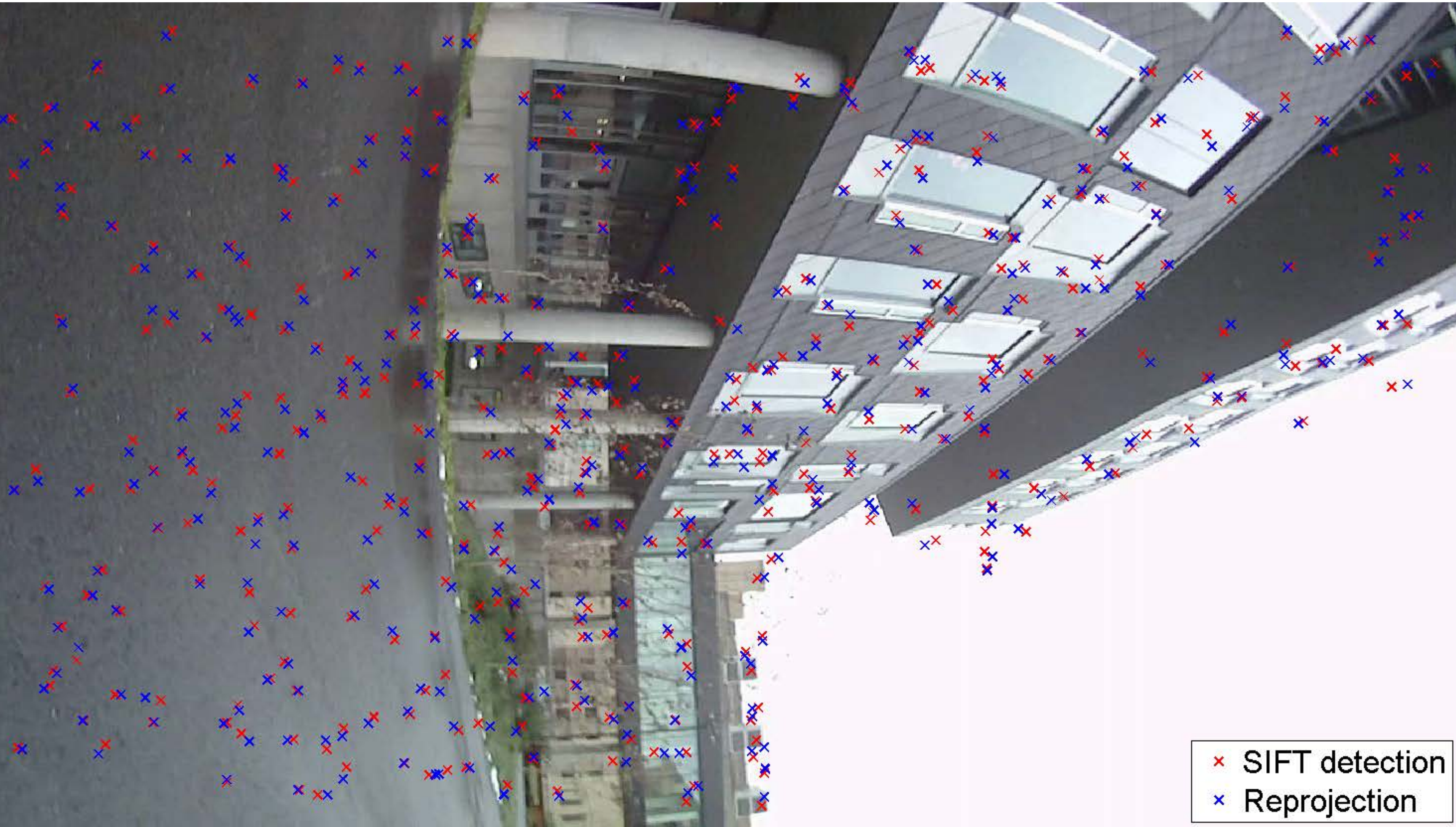
Example:
Localization using range data from beacons



A different initialization converges to a different local solution.

Geometric Refinement

Before Bundle Adjustment



Geometric Refinement

After Bundle Adjustment