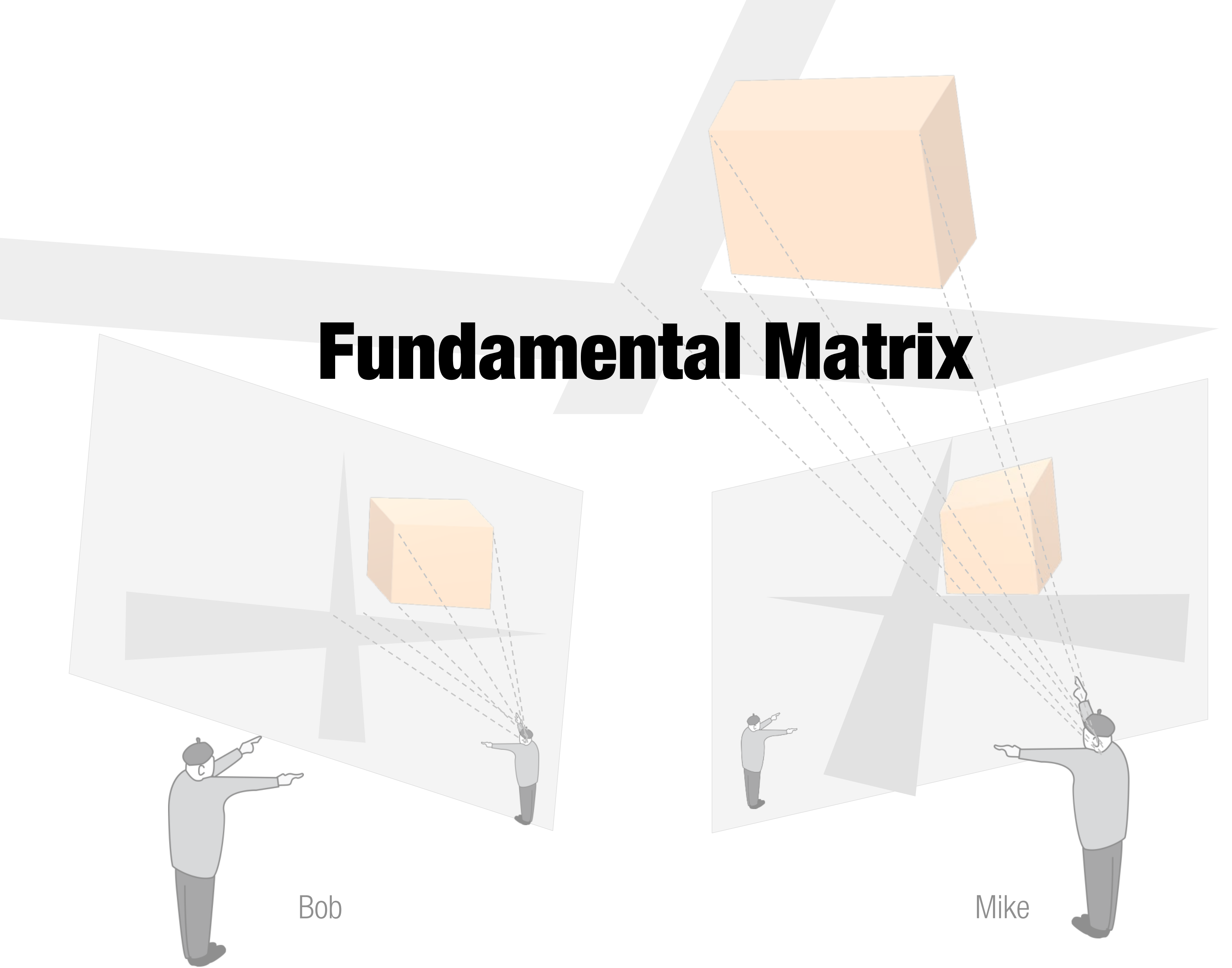
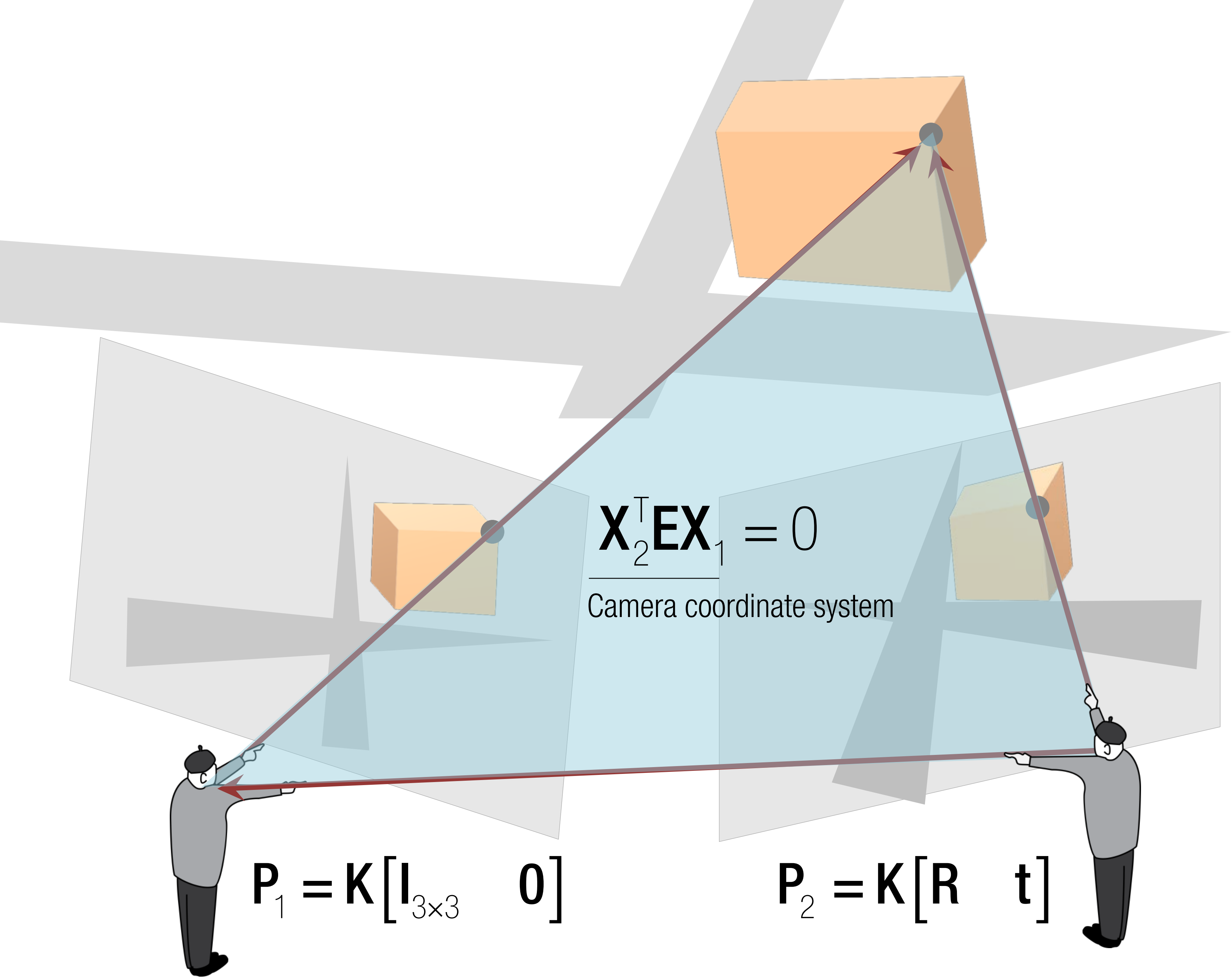


Fundamental Matrix

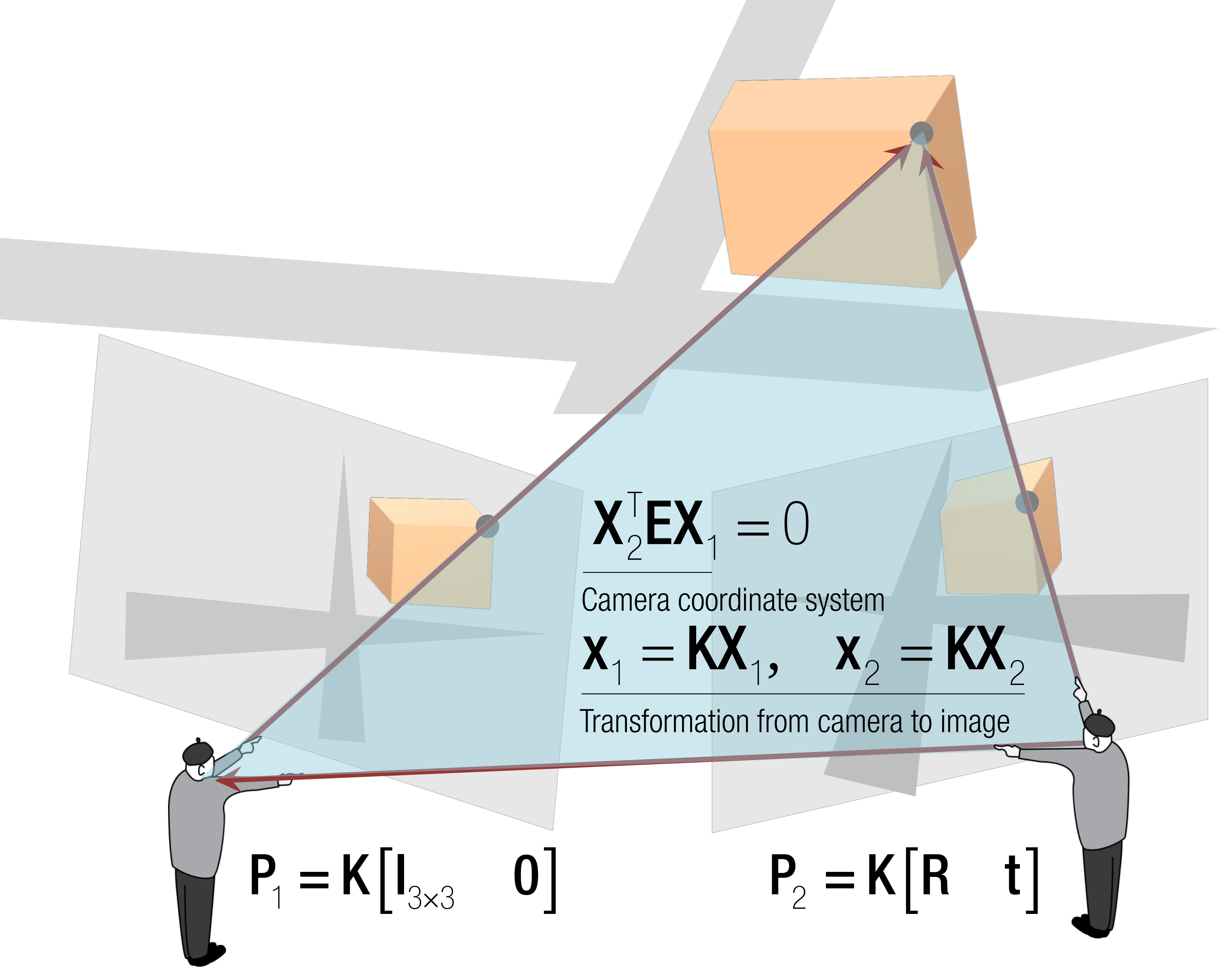


$$X_2^{\top} \mathbf{E} X_1 = 0$$

Camera coordinate system

$$\mathbf{P}_1 = \mathbf{K}\begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{P}_2 = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$



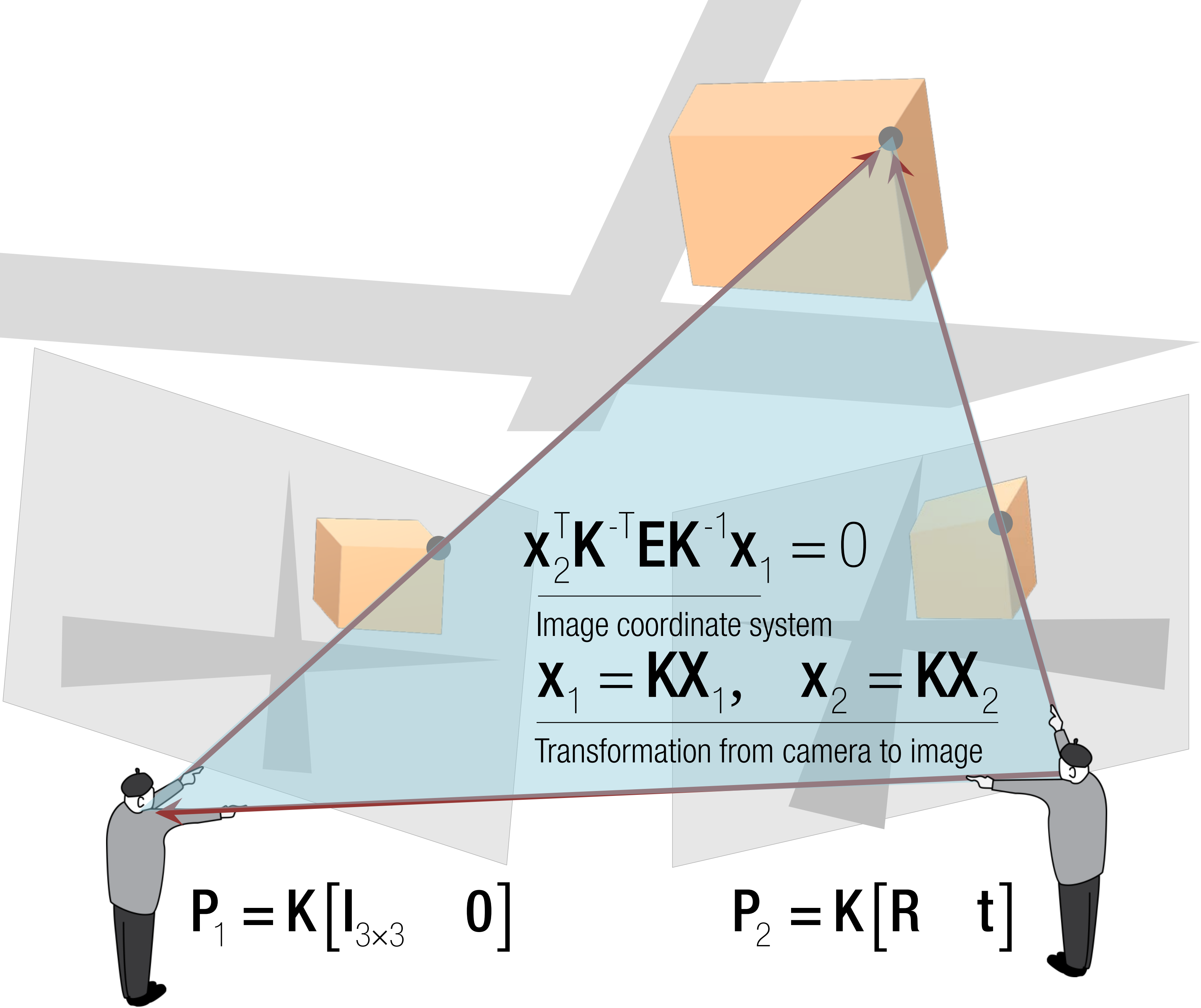
$$\mathbf{X}_2^{\mathsf{T}}\mathbf{E}\mathbf{X}_1 = 0$$

Camera coordinate system

$$\mathbf{x}_1 = \mathbf{K}\mathbf{X}_1, \quad \mathbf{x}_2 = \mathbf{K}\mathbf{X}_2$$

Transformation from camera to image

$$\mathbf{P}_1 = \mathbf{K}\begin{bmatrix}\mathbf{I}_{3\times3} & \mathbf{0}\end{bmatrix} \qquad \mathbf{P}_2 = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$



$$\mathbf{x}_2^{T}\mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1}\mathbf{x}_1 = 0$$

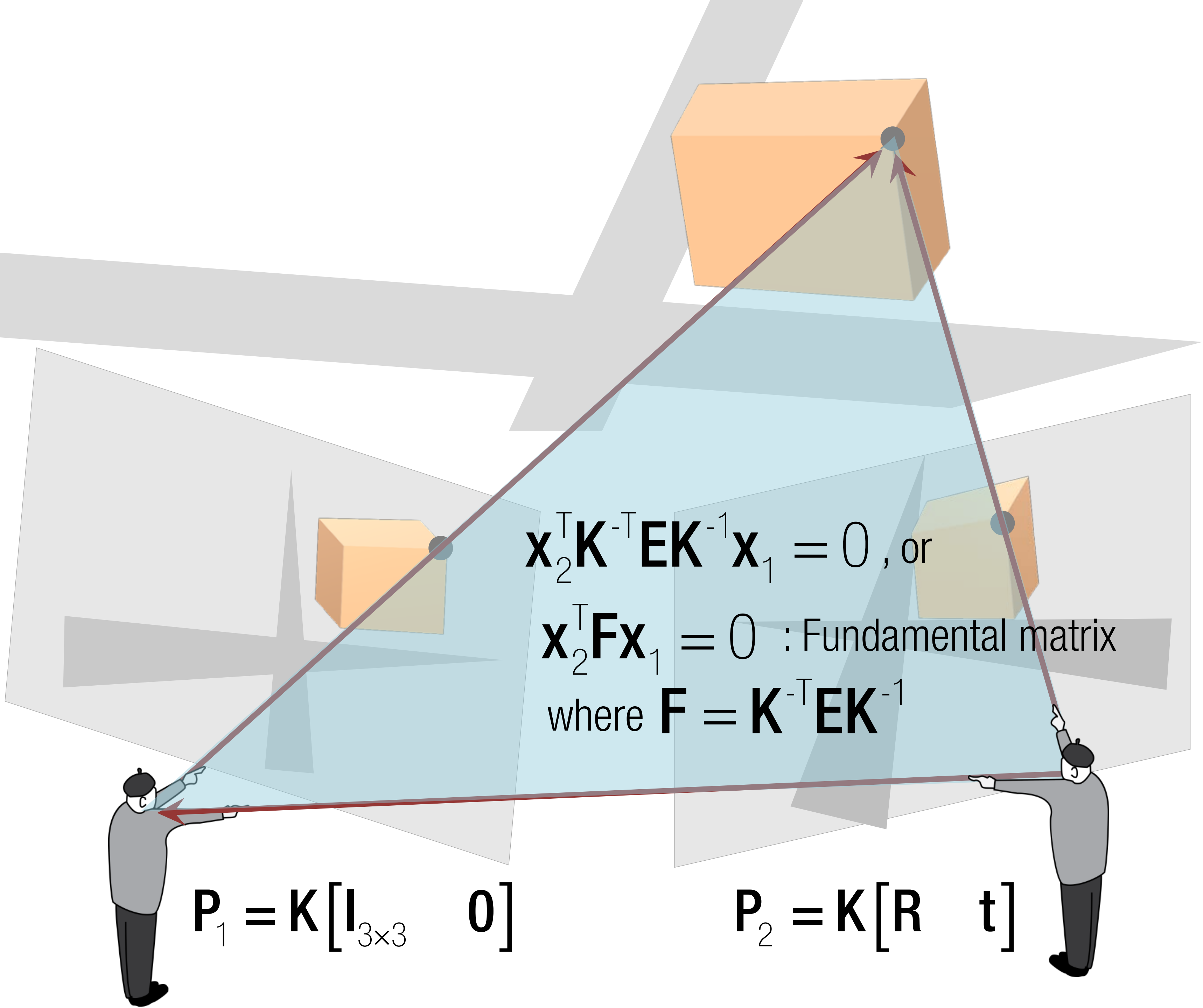
Image coordinate system

$$\mathbf{x}_1 = \mathbf{K}\mathbf{X}_1, \quad \mathbf{x}_2 = \mathbf{K}\mathbf{X}_2$$

Transformation from camera to image

$$\mathbf{P}_1 = \mathbf{K}\left[\mathbf{I}_{3\times3} \quad \mathbf{0}\right]$$

$$\mathbf{P}_2 = \mathbf{K}\left[\mathbf{R} \quad \mathbf{t}\right]$$



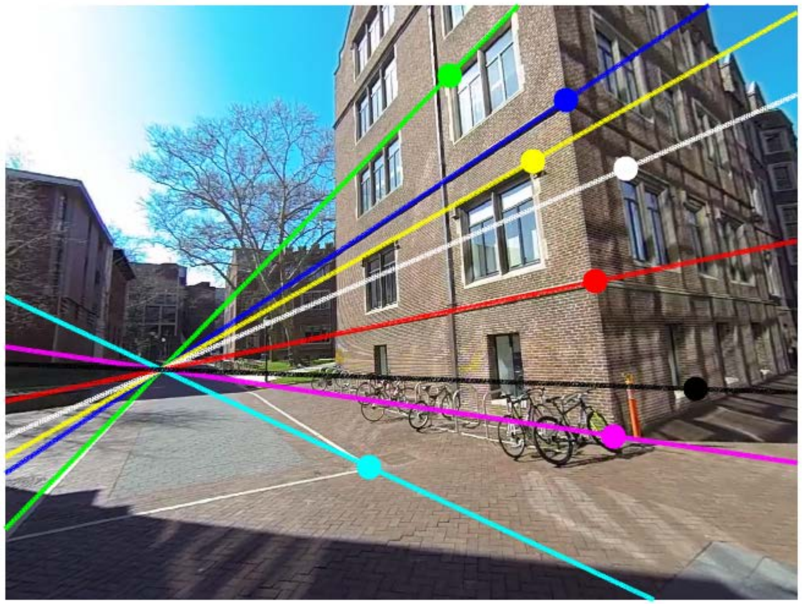
$\mathbf{x}_2^{T}\mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1}\mathbf{x}_1 = 0$, or

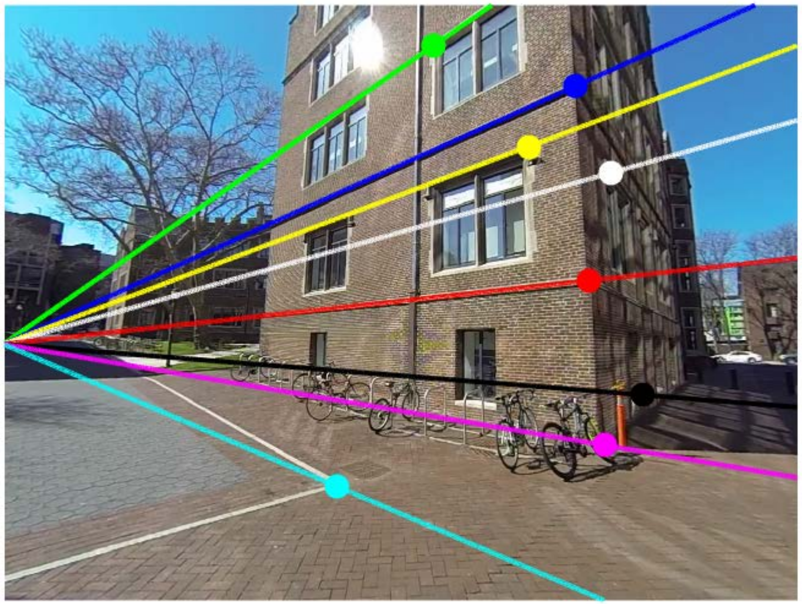
$\mathbf{x}_2^{T}\mathbf{F}\mathbf{x}_1 = 0$: Fundamental matrix

where $\mathbf{F} = \mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1}$

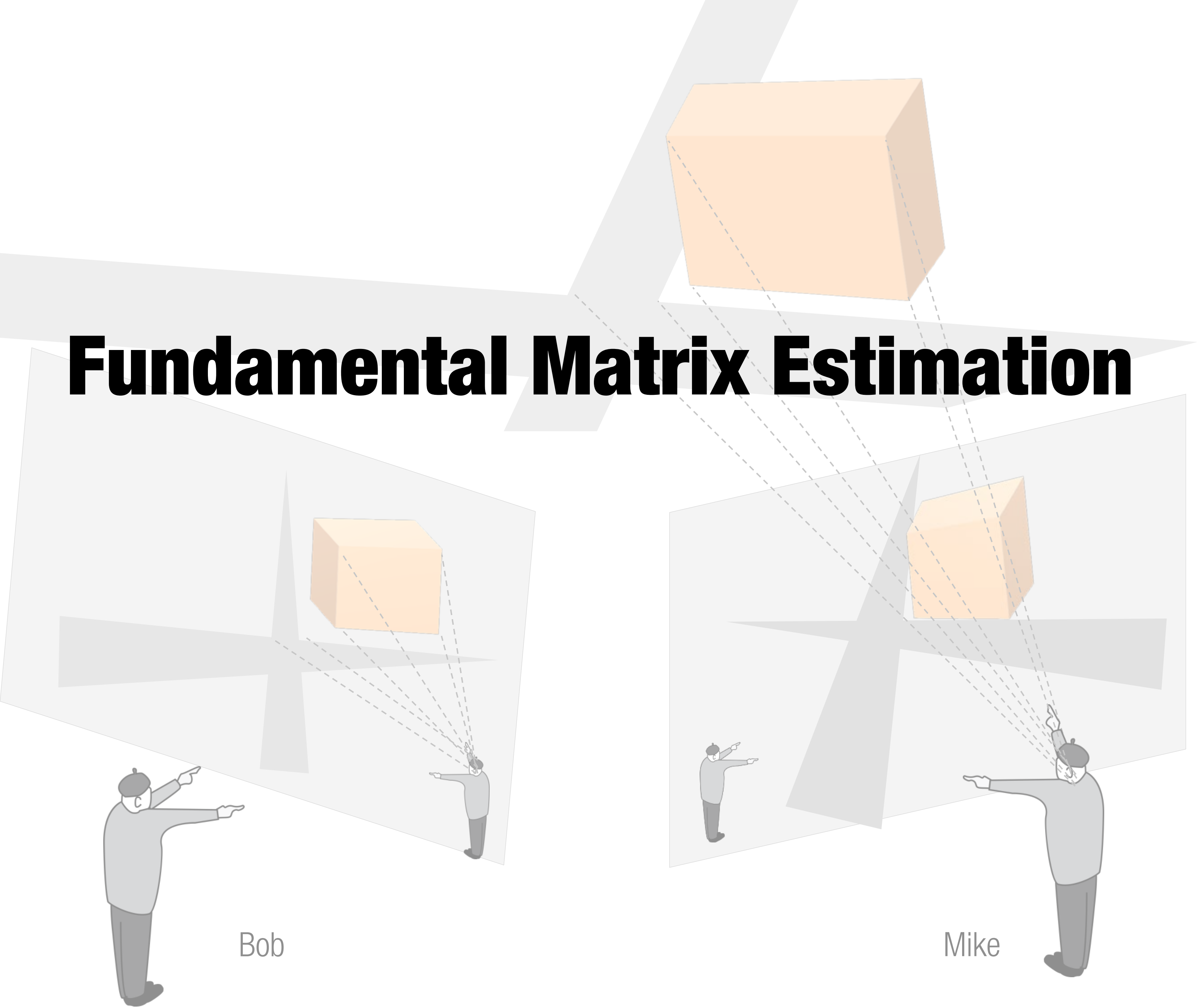
$\mathbf{P}_1 = \mathbf{K}\left[\mathbf{I}_{3\times3} \quad \mathbf{0}\right]$

$\mathbf{P}_2 = \mathbf{K}\left[\mathbf{R} \quad \mathbf{t}\right]$





Fundamental Matrix Estimation



Fundamental Matrix

$$\mathbf{x}_2^{\top}\mathbf{F}\mathbf{x}_1 = 0$$

$$\mathbf{F} \in \mathbb{R}^{3\times3}$$

$$\text{rank}(\mathbf{F}) = 2$$

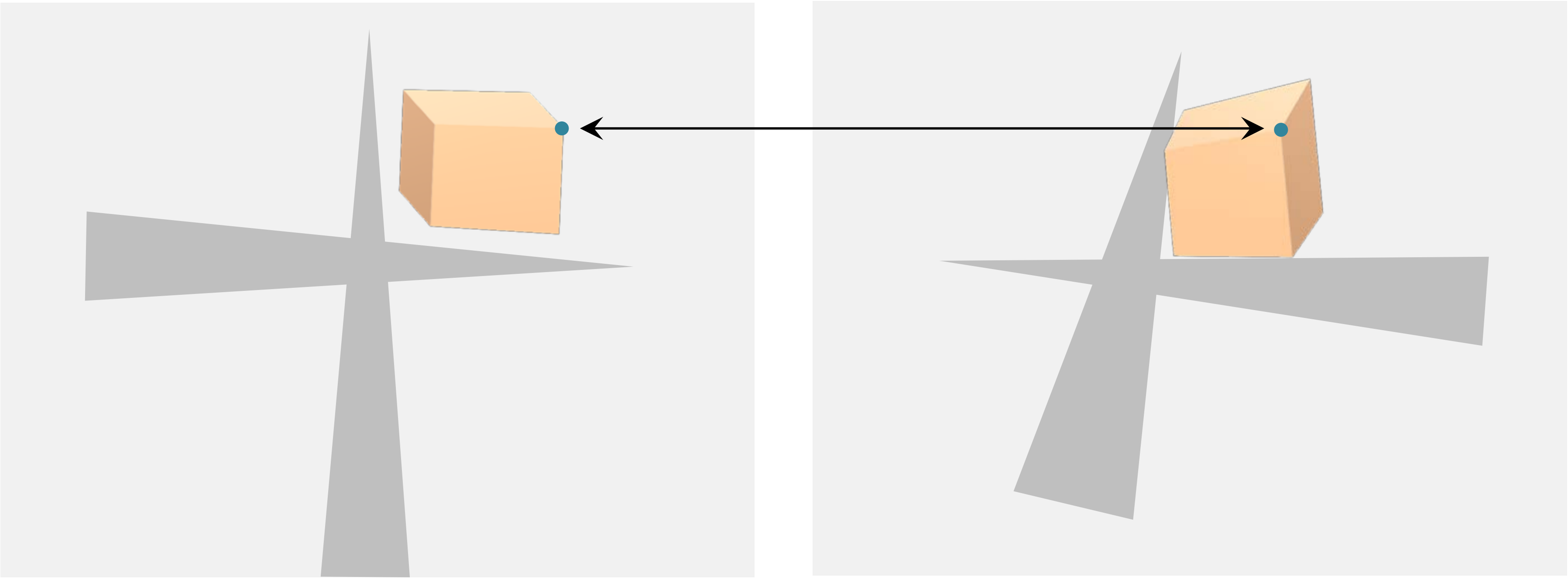
Degree of freedoms: $\overbrace{3\text{x}3}^{\text{Matrix dimensions}}\underbrace{-1}_{\text{scale factor}} = 8$

\# of unknowns: 8

\# of required equations: 8

$$\left.\begin{array}{c} \mathbf{x}_{2,1}^{\top}\mathbf{F}\mathbf{x}_{1,1} = 0 \\ \vdots \\ \mathbf{x}_{2,8}^{\top}\mathbf{F}\mathbf{x}_{1,8} = 0 \end{array}\right| \text{8 correspondences}$$

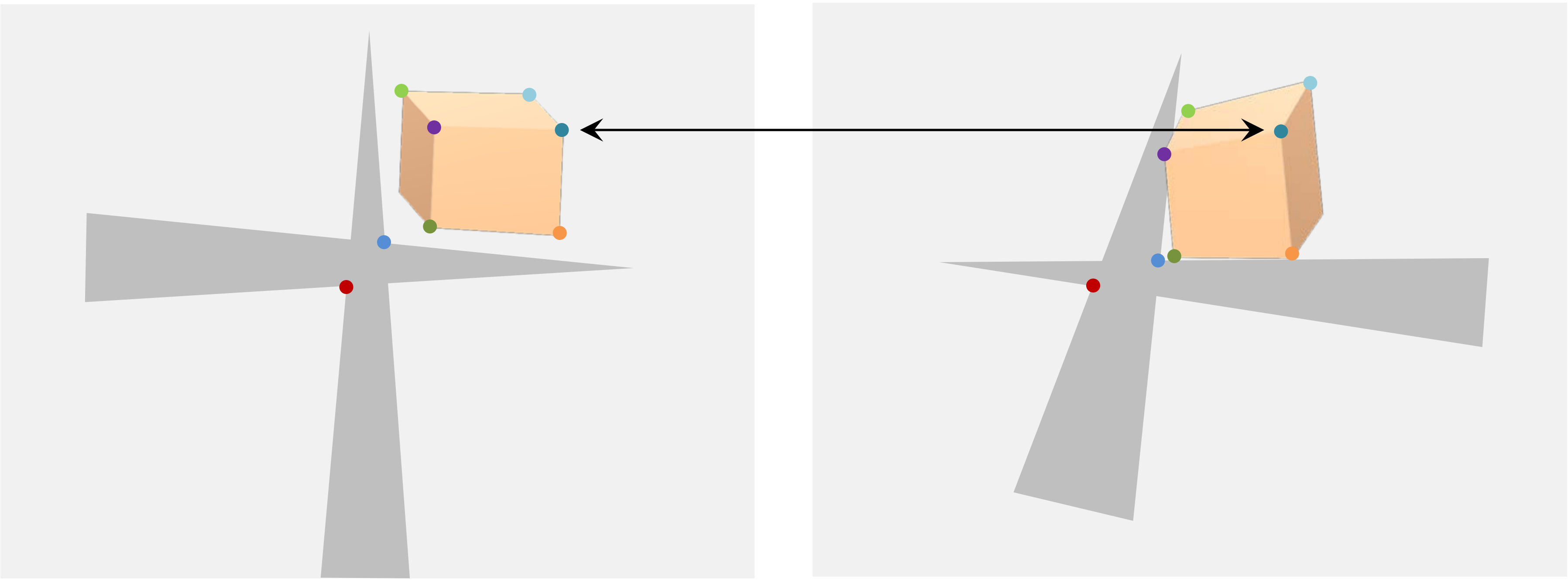
Point correspondence



Bob's view

Mike's view

Point correspondence



Bob's view

Mike's view

8 correspondences

$$\mathbf{x}_2^{\mathsf{T}}\mathbf{F}\mathbf{x}_1 = 0$$

$$\begin{bmatrix} u_i^2 & v_i^2 & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u_i^1 \\ v_i^1 \\ 1 \end{bmatrix} = 0$$

Linear equation in F:

$$u_i^1 u_i^2 f_{11} + u_i^1 v_i^2 f_{21} + u_i^1 f_{31} + v_i^1 u_i^2 f_{12} + v_i^1 v_i^2 f_{22} + v_i^1 f_{32} + u_i^2 f_{13} + v_i^2 f_{23} + f_{33} = 0$$

$$\mathbf{x}_2^{\mathsf{T}}\mathbf{F}\mathbf{x}_1 = 0$$

$$\begin{bmatrix} u_i^2 & v_i^2 & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u_i^1 \\ v_i^1 \\ 1 \end{bmatrix} = 0$$

Linear equation in F:

$$\begin{bmatrix} u_1^1 u_1^2 & u_1^1 v_1^2 & u_1^1 & v_1^1 u_1^2 & v_1^1 v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_8^1 u_8^2 & u_8^1 v_8^2 & u_8^1 & v_8^1 u_8^2 & v_8^1 v_8^2 & v_8^1 & u_8^2 & v_8^2 & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

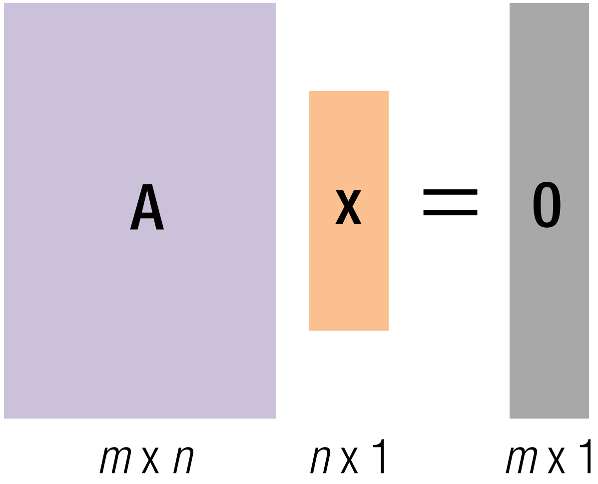
$$\mathbf{x}_2^{\mathsf{T}}\mathbf{F}\mathbf{x}_1 = 0$$

$$\begin{bmatrix} u_i^2 & v_i^2 & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u_i^1 \\ v_i^1 \\ 1 \end{bmatrix} = 0$$

Linear equation in F:

$$\underbrace{\begin{bmatrix} u_1^1 u_1^2 & u_1^1 v_1^2 & u_1^1 & v_1^1 u_1^2 & v_1^1 v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_8^1 u_8^2 & u_8^1 v_8^2 & u_8^1 & v_8^1 u_8^2 & v_8^1 v_8^2 & v_8^1 & u_8^2 & v_8^2 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ \text{[illegible]} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix}}_{\mathbf{X}} = \mathbf{0}$$

Linear Homogeneous Equations



Linear least square solve produces a trivial solution:

$$\mathbf{x} = \left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{b} \rightarrow \quad \mathbf{x} = \mathbf{0}$$

An additional constraint on $\mathbf{X}$ to avoid the trivial solution: $\left\|\mathbf{x}\right\| = 1$

1) $\mathbf{rank(A)} = r < n - 1$: infinite number of solutions

$$\mathbf{x} = \lambda_{r+1}\mathbf{V}_{r+1} + \cdots + \lambda_n \mathbf{V}_n \qquad \text{where} \quad \sum_{i=r+1}^{n} \lambda_i^2 = 1$$

2) $\mathbf{rank(A)} = n - 1$: one exact solution

$$\mathbf{x} = \mathbf{V}_n$$

3) $n < m$: no exact solution in general (needs least squares)

$$\boxed{\min_{\mathbf{x}} \left\|\mathbf{Ax}\right\|^2 \text{ subject to } \left\|\mathbf{x}\right\| = 1 \rightarrow \quad \mathbf{x} = \mathbf{V}_n}$$

8 Point Algorithm

- Construct 8x9 matrix **A**.

$$\begin{bmatrix} u_1^1 u_1^2 & u_1^1 v_1^2 & u_1^1 & v_1^1 u_1^2 & v_1^1 v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_8^1 u_8^2 & u_8^1 v_8^2 & u_8^1 & v_8^1 u_8^2 & v_8^1 v_8^2 & v_8^1 & u_8^2 & v_8^2 & 1 \end{bmatrix}$$

A

8 Point Algorithm

- Construct 8x9 matrix **A**.
- Solving linear homogeneous equations via SVD:

  $\mathbf{x} = \mathbf{V}_{:,9}$ where $\mathbf{A} = \mathbf{UDV}^\top$

  $\mathbf{F} = \mathrm{reshape}(\mathbf{x}, 3, 3)$: constructing matrix from vector.

8 Point Algorithm

- Construct 8x9 matrix **A**.
- Solving linear homogeneous equations via SVD:

  $\mathbf{x} = \mathbf{V}_{:,8}$ where $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}$

  $\mathbf{F} = \mathrm{reshape}(\mathbf{x}, 3, 3)$: constructing matrix from vector.
- Applying rank constraint, i.e., $\mathrm{rank}(\mathbf{F}) = 2$.

$\mathbf{F}_{\mathrm{rank2}} = \mathbf{U}\tilde{\mathbf{D}}\mathbf{V}^{\mathrm{T}}$ where $\tilde{\mathbf{D}}$:$\mathbf{D}$ with the last element zero.

$\mathbf{F}_{\mathrm{rank2}} = \mathbf{U}\,\tilde{\mathbf{D}}\,\mathbf{V}^{\mathrm{T}}$ $\quad$ $\mathbf{F} = \mathbf{U}\,\mathbf{D}\,\mathbf{V}^{\mathrm{T}}$

SVD cleanup

1.2 Match Outlier Rejection via RANSAC

Goal Given N correspondences between two images ($N \geq 8$), $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$, implement the following function that estimates inlier correspondences using fundamental matrix based RANSAC:

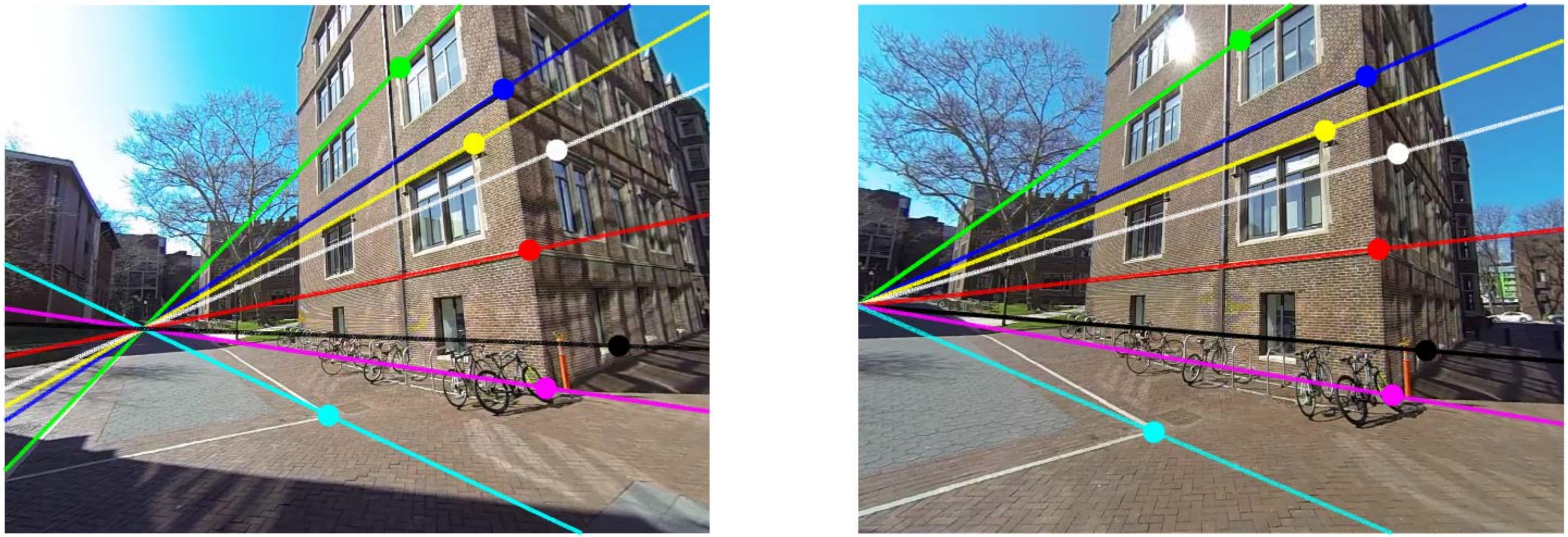
`[y1 y2 idx] = GetInliersRANSAC(x1, x2)`

> (INPUT) `x1` and `x2`: $N \times 2$ matrices whose row represents a correspondence.
>
> (OUTPUT) `y1` and `y2`: $N_i \times 2$ matrices whose row represents an inlier correspondence where N_i is the number of inliers.
>
> (OUTPUT) `idx`: $N \times 1$ vector that indicates ID of inlier `y1`.

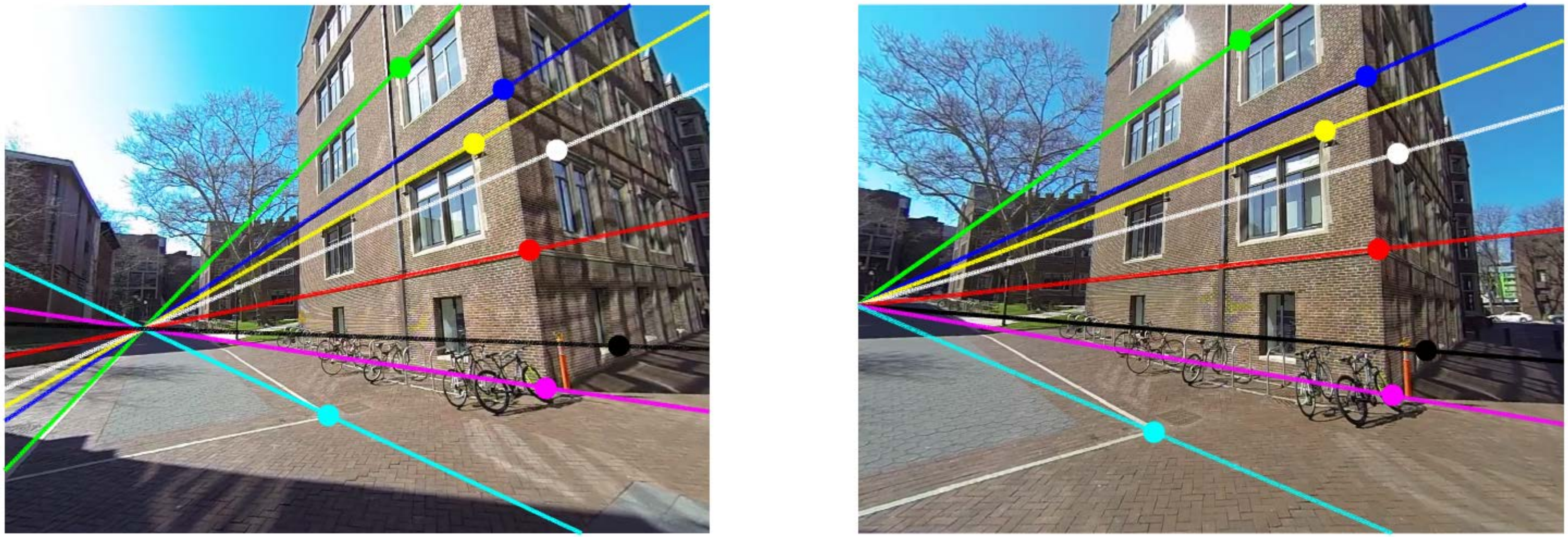
A pseudo code the RANSAC is shown in Algorithm 2.

Algorithm 2 GetInliersRANSAC

```
$n \leftarrow 0$
for $i = 1 : M$ do
    Choose 8 correspondences, $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$, randomly
    F = EstimateFundamentalMatrix($\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$)
    $\mathcal{S} \leftarrow \emptyset$
    for $j = 1 : N$ do
        if $|\mathbf{x}_{2j}^{\mathsf{T}} \mathbf{F} \mathbf{x}_{1j}| < \epsilon$ then
            $\mathcal{S} \leftarrow \mathcal{S} \cup \{j\}$
        end if
    end for
    if $n < |\mathcal{S}|$ then
        $n \leftarrow |\mathcal{S}|$
        $\mathcal{S}_{in} \leftarrow \mathcal{S}$
    end if
end for
```



```
F =
  1.0e+003 *
   0.0000    0.0001   -0.0463
  -0.0001    0.0000    0.0181
   0.0519   -0.0043   -9.9997
```



```
>> rank(F)
ans =
     3
>> [u,d,v] = svd(F);
>> d(3,3) = 0;
```

```
>> F = u * d * v'          : SVD cleanup
F =
  1.0e+003 *
    0.0000    0.0001   -0.0463
   -0.0001    0.0000    0.0181
    0.0519   -0.0043   -9.9997
>> rank(F)
ans =
     2
```

```
x1 =
   950   450
```

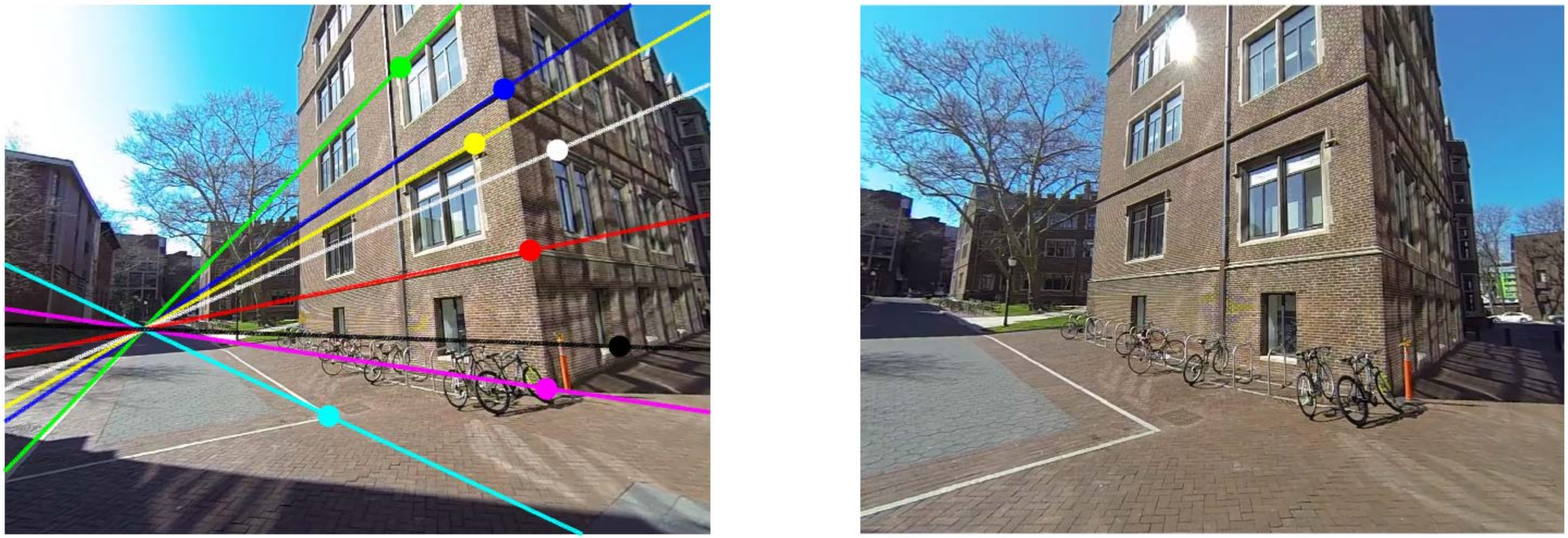
```
L2 =
   -0.1024   -0.9947  547.0942
```

$$\mathbf{L}_2 = \mathbf{F}\mathbf{x}_1$$

```
L1 =
   0.5489    0.8359 -627.0515
```

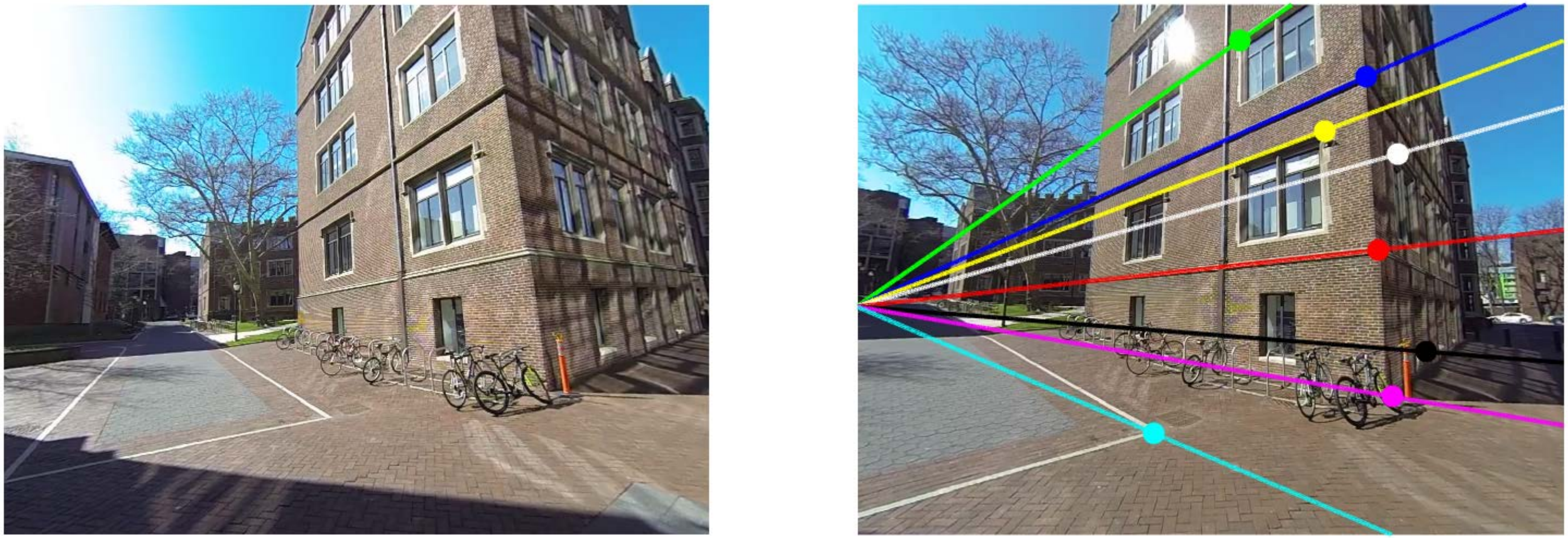
$$\mathbf{L}_1 = \mathbf{F}^\top \mathbf{x}_2$$

```
x2 =
   920   130
```



```
[u,d] = eigs(F'*F);
u =
   -0.0052    0.9258   -0.3780
    0.0004   -0.3780   -0.9258
    1.0000    0.0050   -0.0016
d =
   1.0000         0               0
        0    6.4719e-10           0
        0         0    -7.6511e-22
```

```
uu = u(:, 3) = [-0.3780, -0.9258, -0.0016]
```



```
[u,d] = eigs(F*F');
u =
    0.0046    1.0000    0.0029
   -0.0018    0.0029   -1.0000
    1.0000   -0.0046   -0.0018
d =
   1.0000         0                0
        0    6.4719e-10         0
        0         0    -5.6583e-21
```

```
uu = u(:, 3) = [0.0029, -1.0000, -0.0018]
```