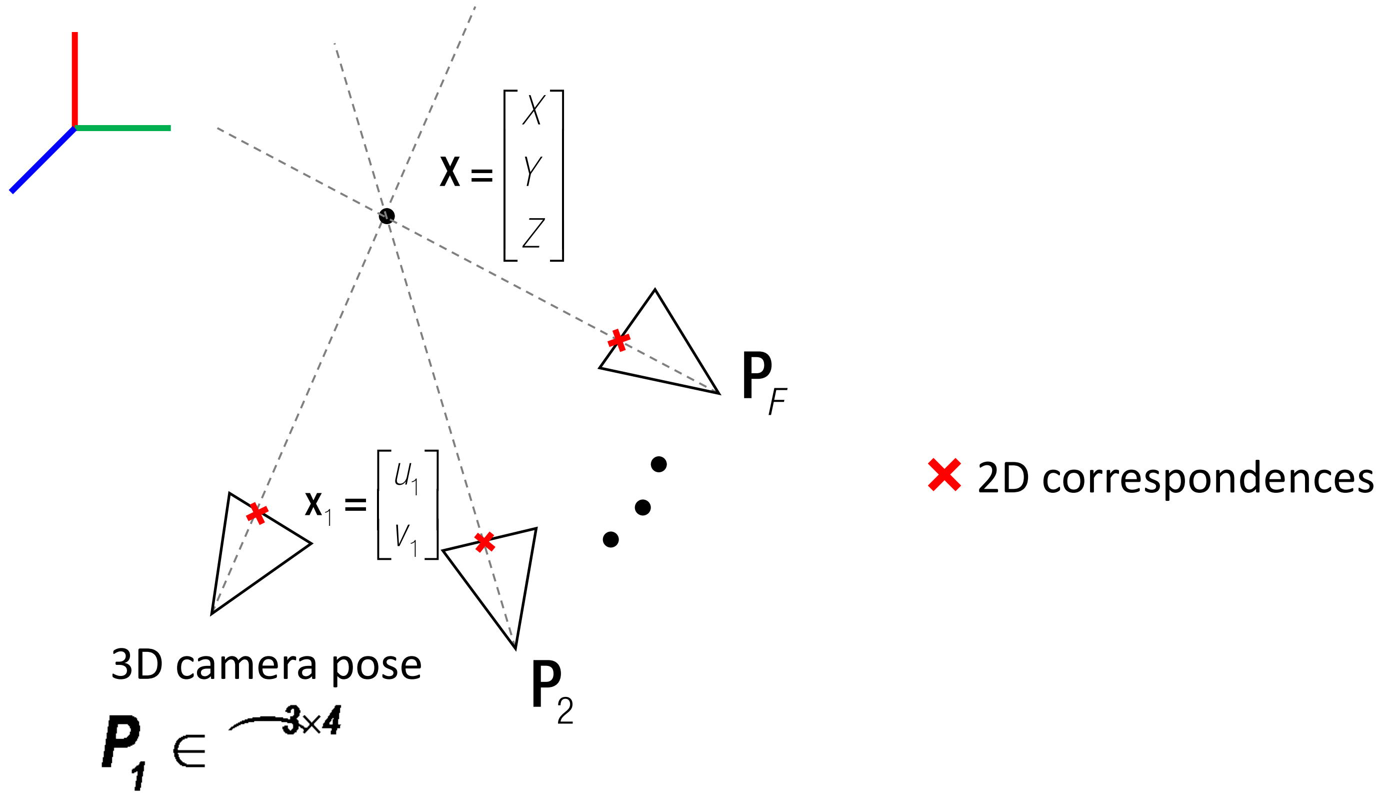
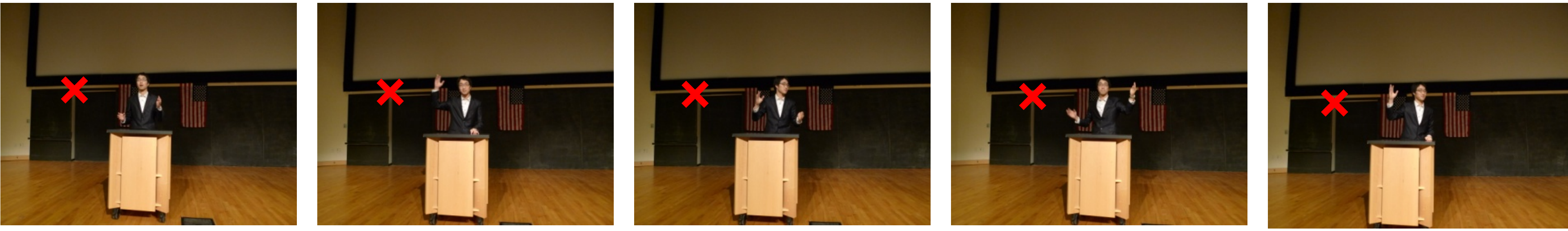
Optical Flow: 2D point correspondences





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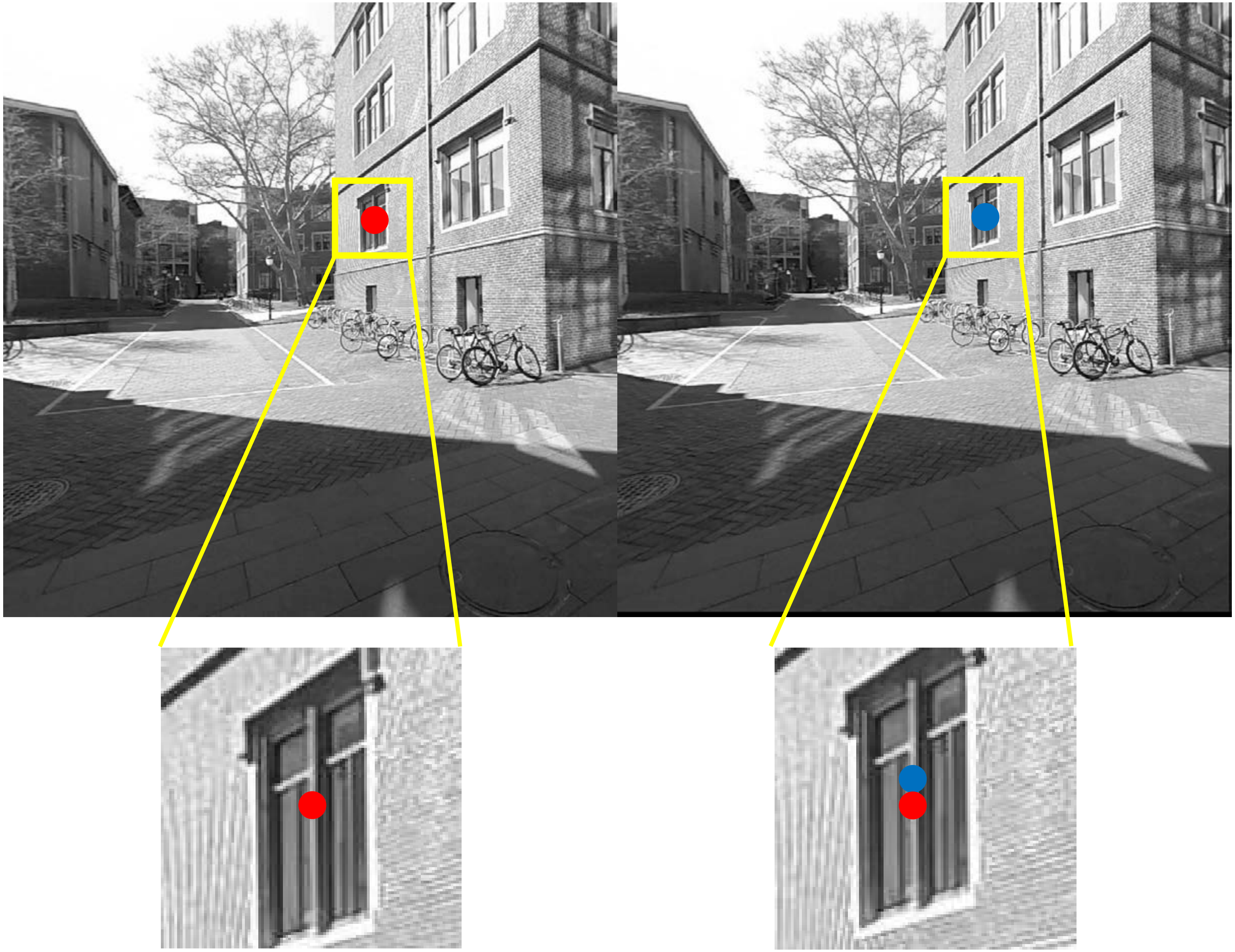
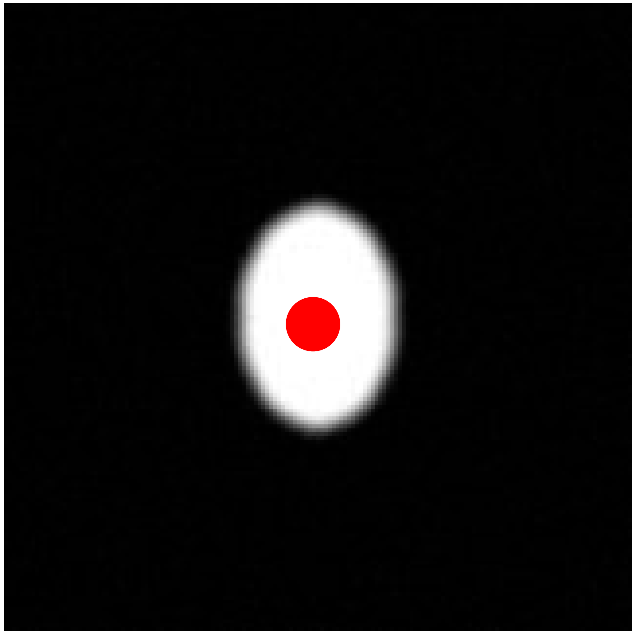
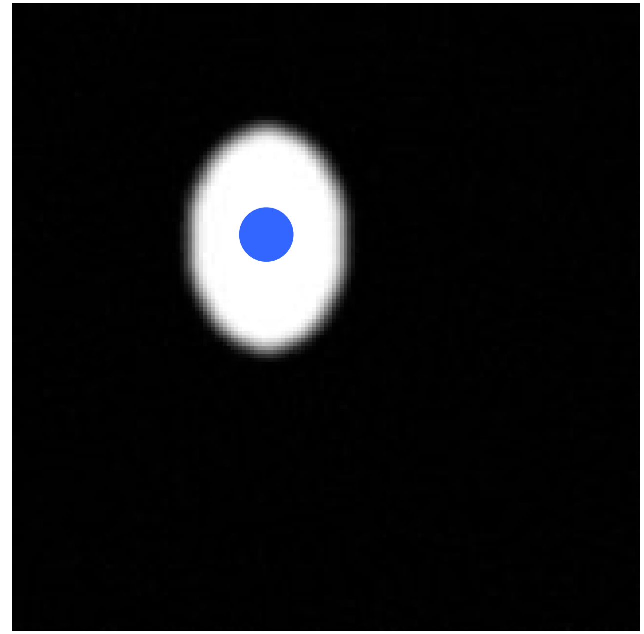
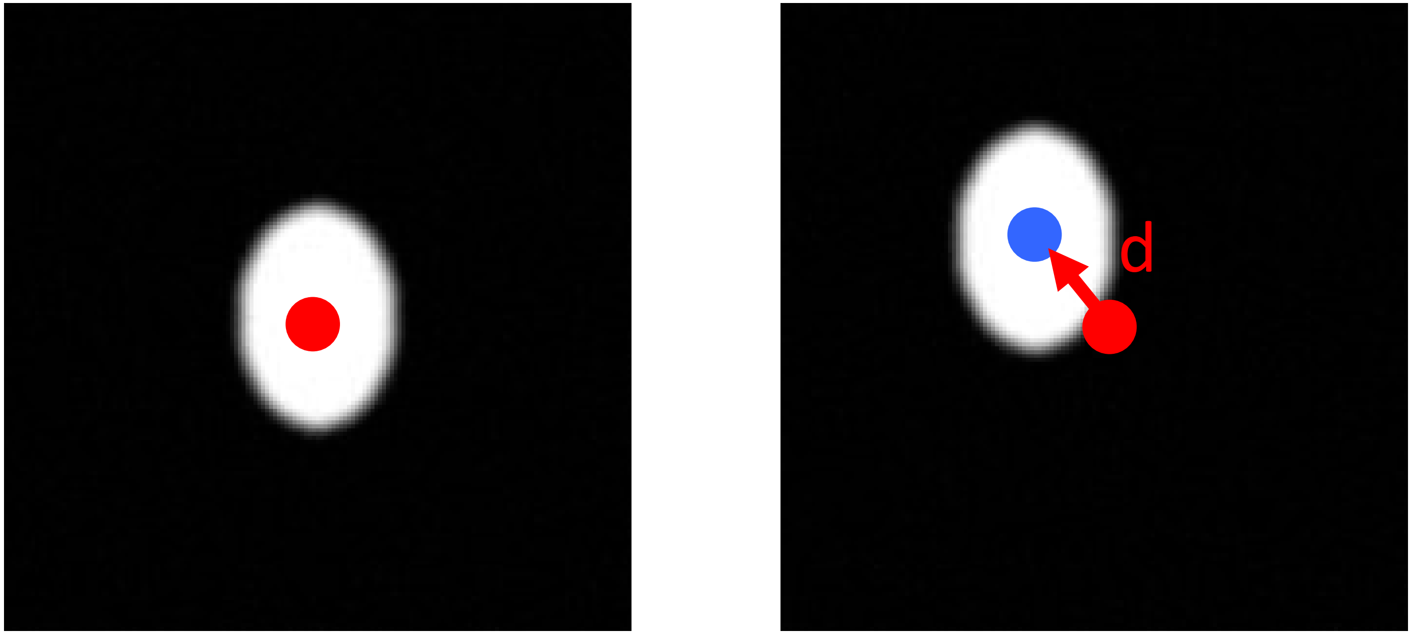






$\mathbf{I}(\mathbf{x})$ $\quad$ $\mathbf{J}(\mathbf{x})$

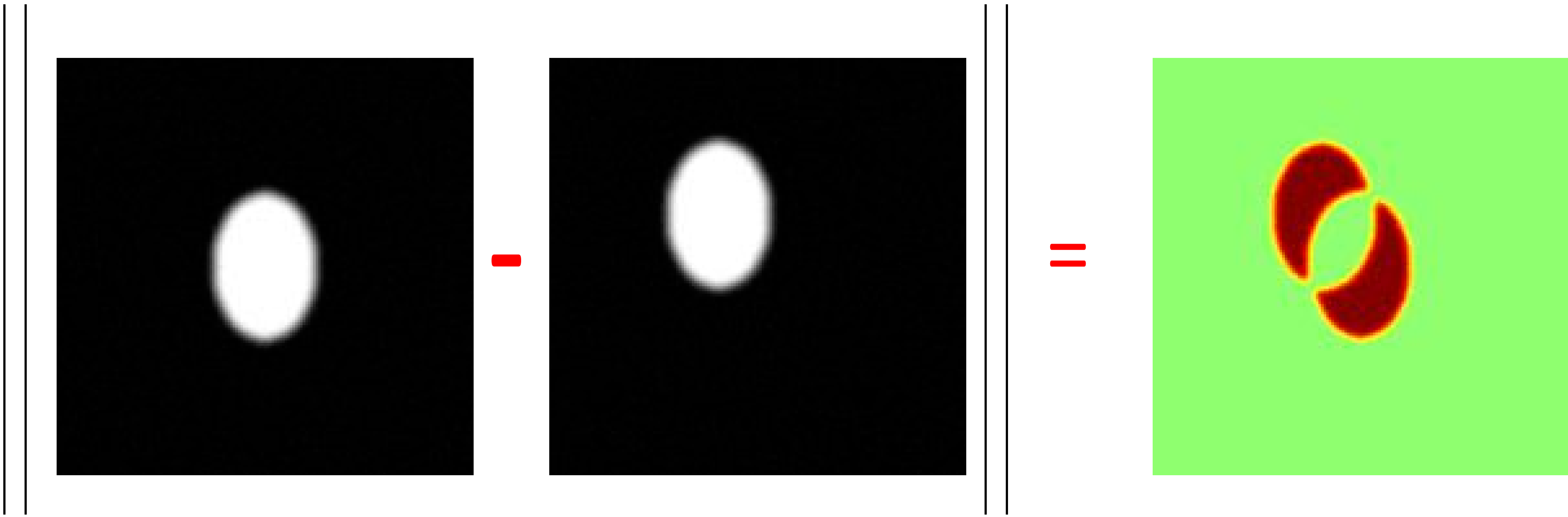
$t = 0$ $\quad$ $t = 1$



$\mathbf{I}(\mathbf{x})$ $\mathbf{J}(\mathbf{x})$

$$\mathbf{I}(\mathbf{x}) = \mathbf{J}(\mathbf{x} + \mathbf{d})$$

$$\min_{\mathbf{d}} E(\mathbf{d}) = \|\mathbf{J}(\mathbf{x}+\mathbf{d}) - \mathbf{I}(\mathbf{x})\|^2$$



When d=0

E(d=0)

Three steps for solving this problem

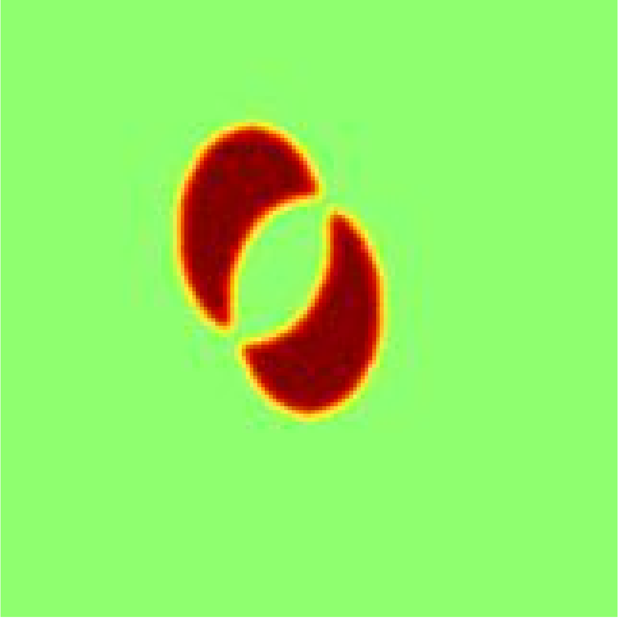
$$1: \text{ Solve for } \left.\frac{\delta E}{\delta \mathbf{d}}\right|_{\mathbf{d}^*} = 0$$

$$2: \text{ Taylor expansion on } \mathbf{J}(\mathbf{x} + \mathbf{d})$$

$$3: \text{ Solve for d, warp image, iterate}$$

Step 1: Solve for $\left.\frac{\delta E}{\delta \mathbf{d}}\right|_{d^*} = 0$

$$E(\mathbf{d}) = ||\mathbf{J}(\mathbf{x}+\mathbf{d}) - \mathbf{I}(\mathbf{x})||^{\mathbf{2}}$$

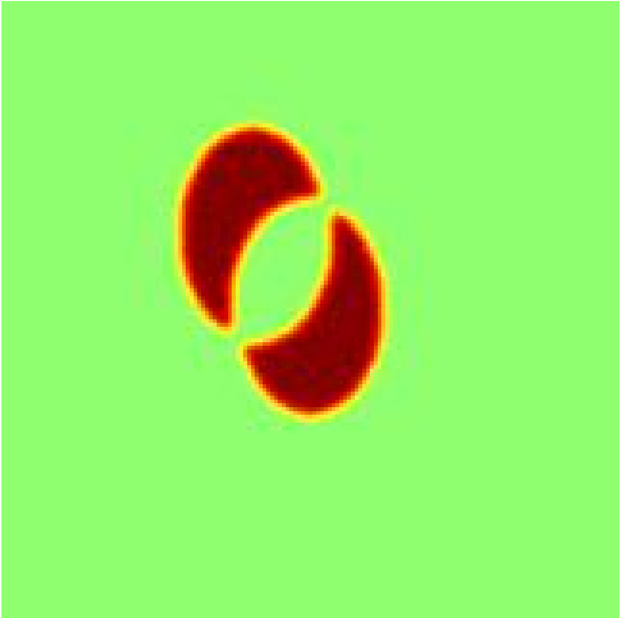


$$\mathbf{E}(\mathbf{d}^* = \mathbf{0})$$

Step 1: Solve for $\left.\frac{\delta E}{\delta \mathbf{d}}\right|_{d^*} = 0$

$$E(\mathbf{d}) = ||\mathbf{J}(\mathbf{x}+\mathbf{d}) - \mathbf{I}(\mathbf{x})||^{\mathbf{2}}$$

$$\left.\frac{\delta E}{\delta \mathbf{d}}\right|_{d^*} = 2\frac{\delta \mathbf{J}(\mathbf{x}+\mathbf{d})}{\delta \mathbf{d}}^T (\mathbf{J}(\mathbf{x}+\mathbf{d}) - \mathbf{I}(\mathbf{x})) = 0$$



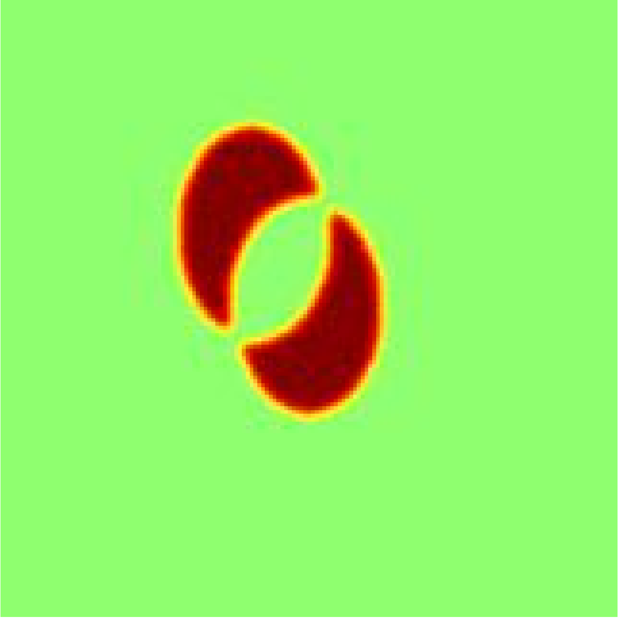
$$\mathbf{E}(\mathbf{d}^* = \mathbf{0})$$

Step 1: Solve for $\left.\frac{\delta E}{\delta \mathbf{d}}\right|_{d*} = 0$

$$E(\mathbf{d}) = ||\mathbf{J}(\mathbf{x}+\mathbf{d}) - \mathbf{I}(\mathbf{x})||^{\mathbf{2}}$$

$$\left.\frac{\delta E}{\delta \mathbf{d}}\right|_{d*} = 2\frac{\delta \mathbf{J}(\mathbf{x}+\mathbf{d})}{\delta \mathbf{d}}^T (\mathbf{J}(\mathbf{x}+\mathbf{d}) - \mathbf{I}(\mathbf{x})) = 0$$

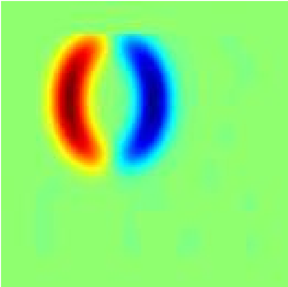
$$\left.\frac{\delta E}{\delta \mathbf{d}}\right|_{d*} = 2\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{J}(\mathbf{x}+\mathbf{d}) - \mathbf{I}(\mathbf{x})) = 0$$



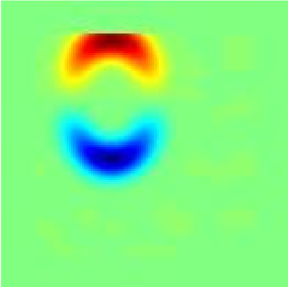
$$\mathbf{E}(\mathbf{d}^* = \mathbf{0})$$

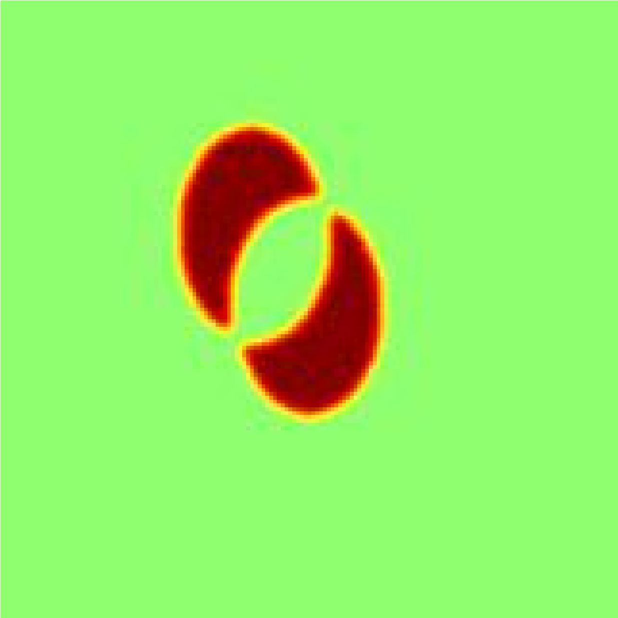
$$\left.\frac{\delta E}{\delta \mathbf{d}}\right|_{d^*} = 2\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{J}(\mathbf{x}+\mathbf{d}) - \mathbf{I}(\mathbf{x})) = 0$$

$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta x} =$$



$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta y} =$$

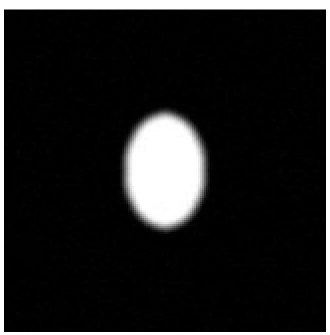
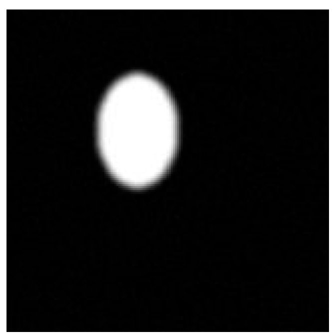
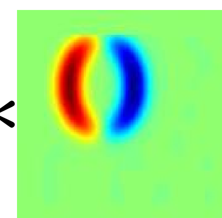
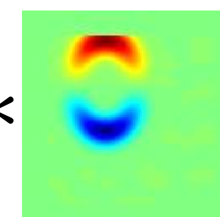




$$\mathbf{E}(\mathbf{d}^* = \mathbf{0})$$

Step 2: Taylor expansion $\mathbf{J}(\mathbf{x}+\mathbf{d})$

$$\mathbf{J}(\mathbf{x}+\mathbf{d}) = \mathbf{J}(\mathbf{x}) + \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}\mathbf{d}$$

 =  $+\mathbf{d}_{\mathbf{x}} *$  $+\mathbf{d}_{\mathbf{y}} *$ 

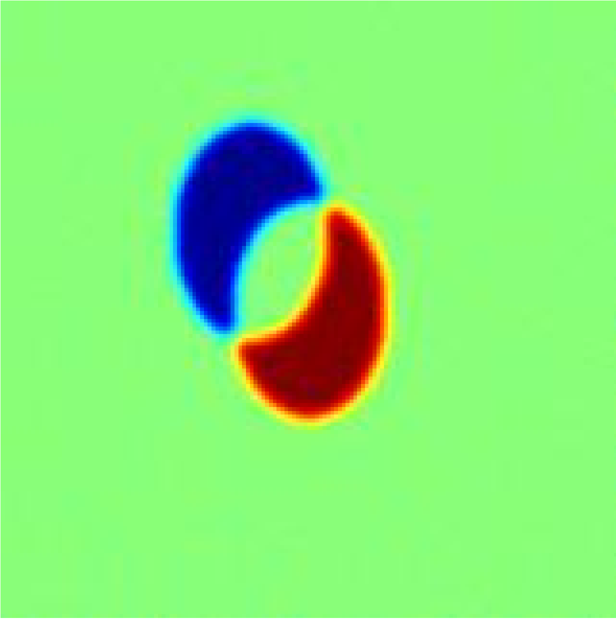
Putting all together

$$\left.\frac{\delta E}{\delta \mathbf{d}}\right|_{d*} = 2\boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}}^{T}\left(\boxed{\mathbf{J}(\mathbf{x}+\mathbf{d})} - \mathbf{I}(\mathbf{x})\right) = 0$$

$$\boxed{\mathbf{J}(\mathbf{x}+\mathbf{d})} = \mathbf{J}(\mathbf{x}) + \boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}}\mathbf{d}$$

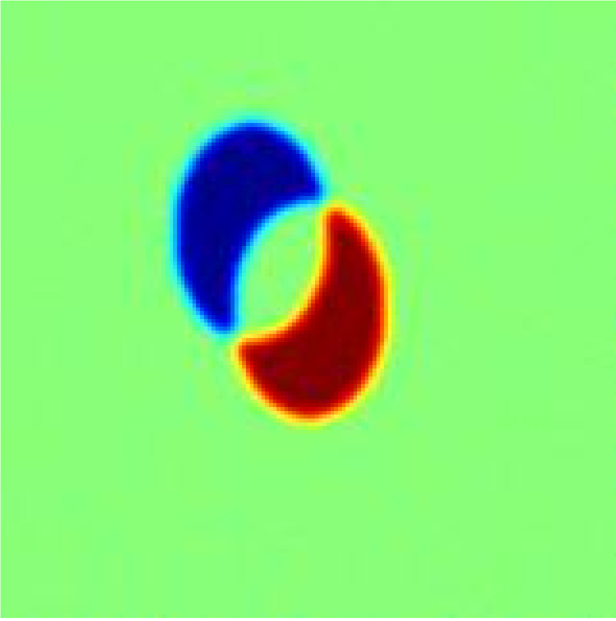
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$



$$\boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}}^T \boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}} \mathbf{d} = \boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

2D unknowns flow vector per pixel
2 equations



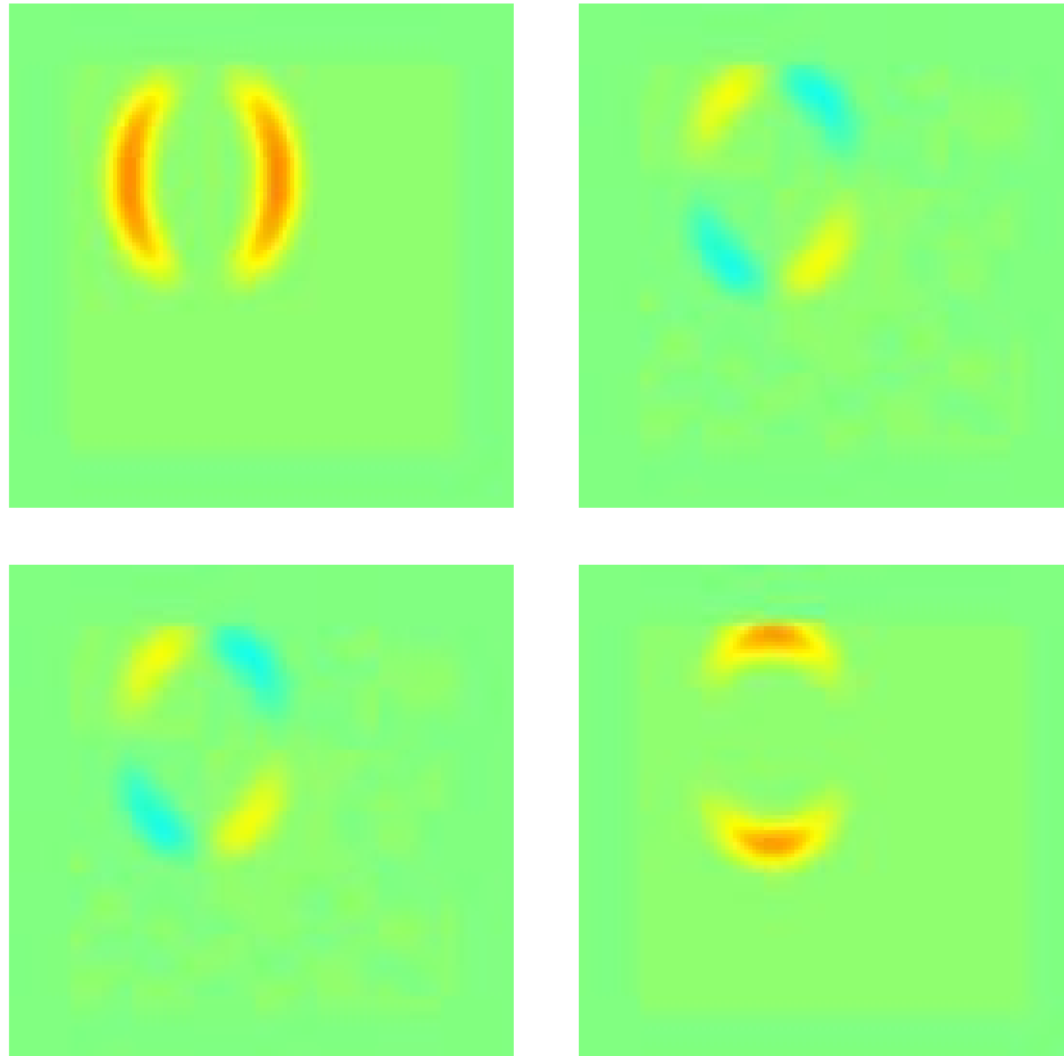
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

Also known as second moment matrix

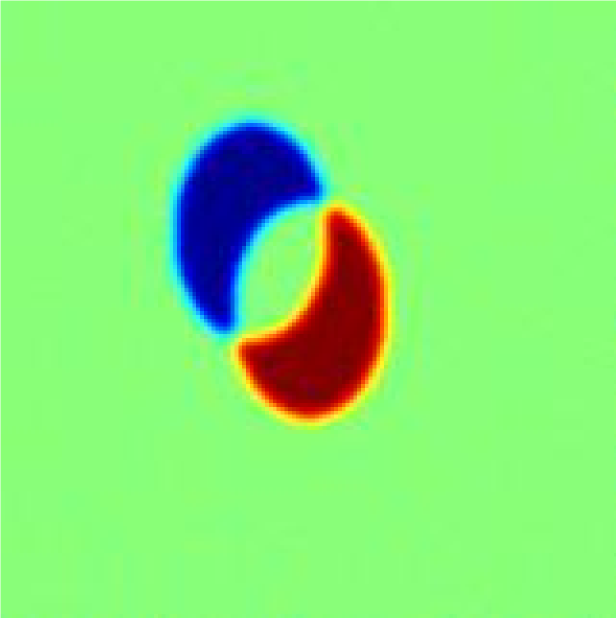
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}$$

$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta x}^{2} \qquad \frac{\delta \mathbf{J}(\mathbf{x})}{\delta x}\frac{\delta \mathbf{J}(\mathbf{x})}{\delta y}$$

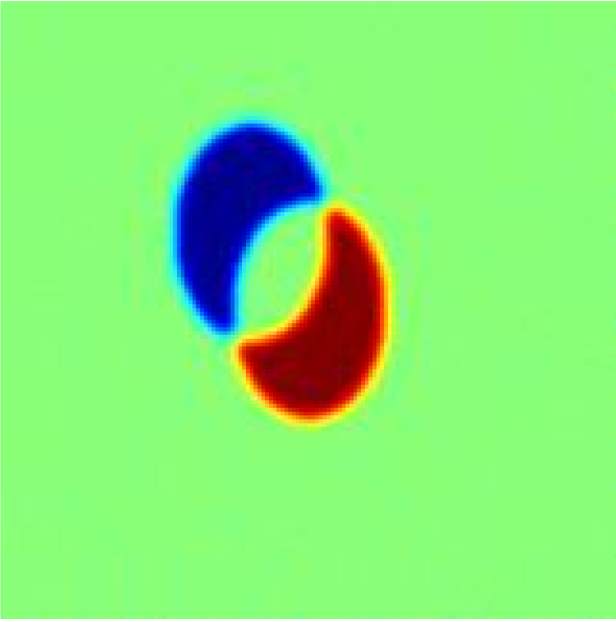
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta x}\frac{\delta \mathbf{J}(\mathbf{x})}{\delta y} \qquad \frac{\delta \mathbf{J}(\mathbf{x})}{\delta y}^{2}$$

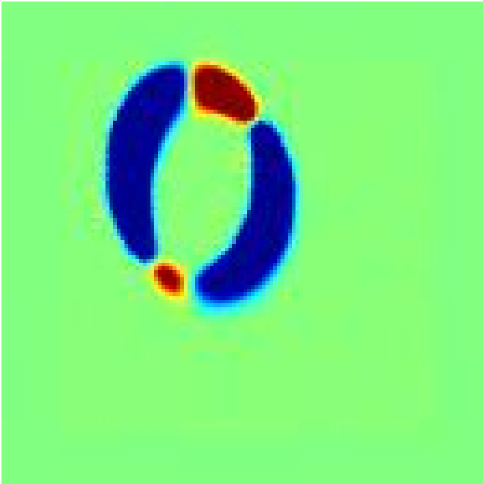


$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

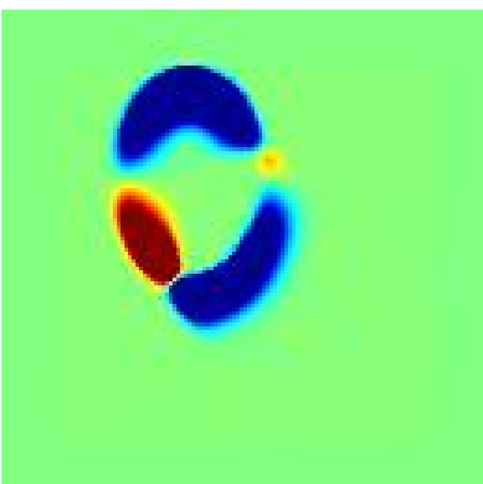


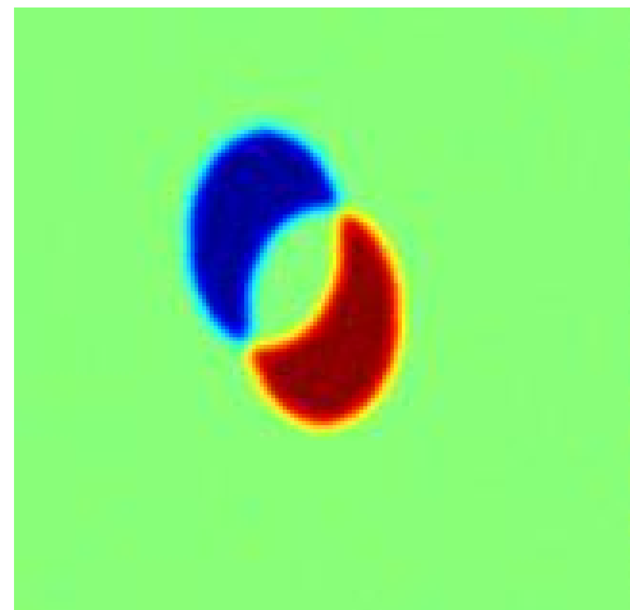
$$\boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}} \mathbf{d} = \boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))}$$



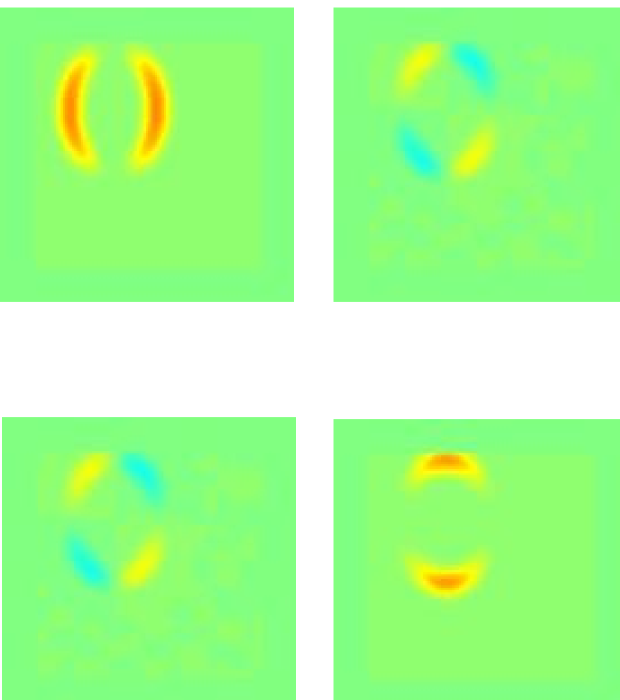


$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} \left(\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x})\right)$$

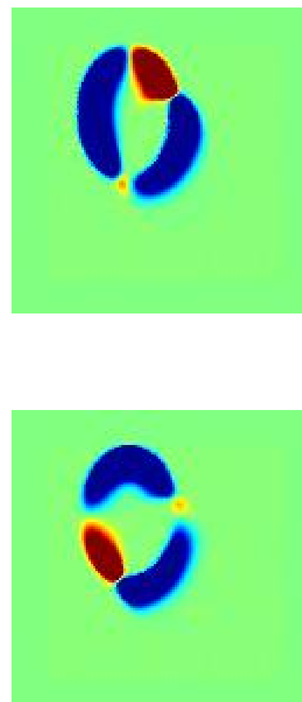




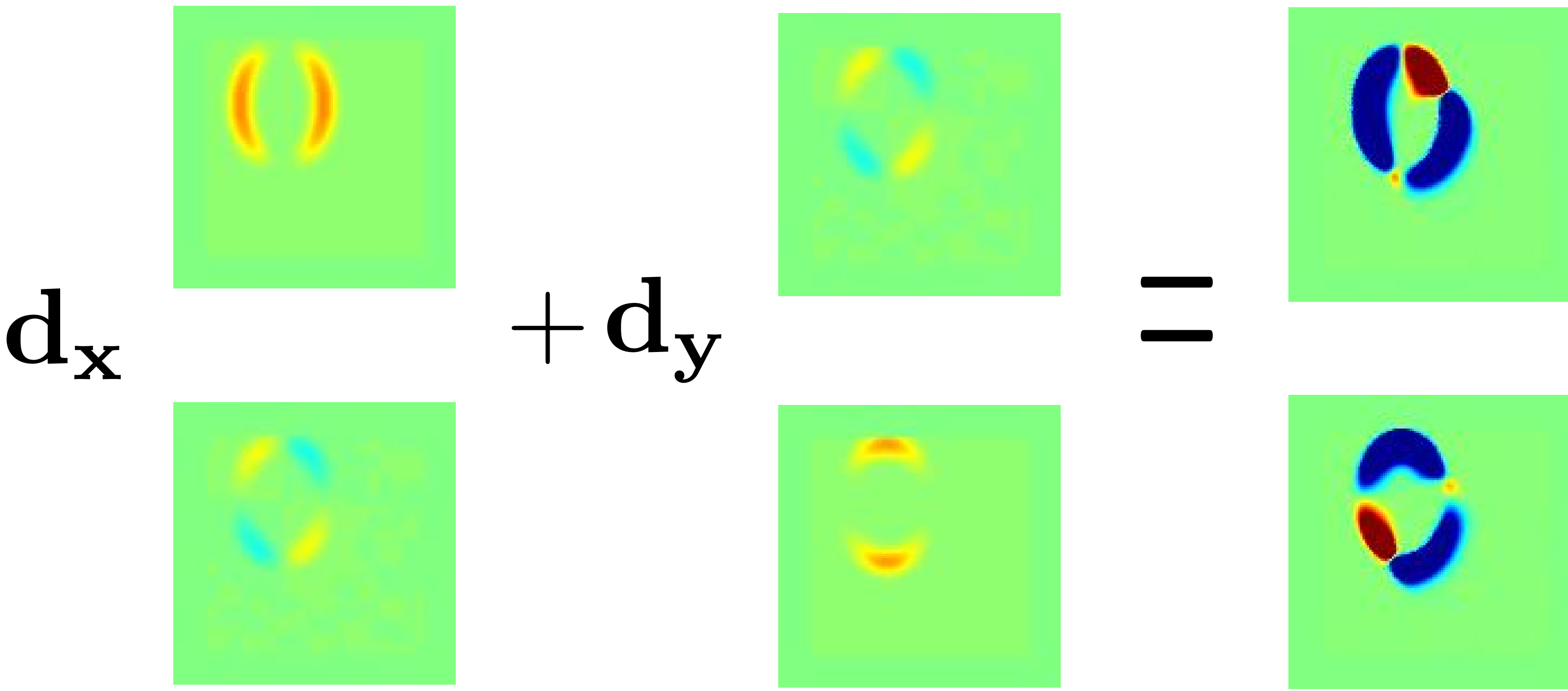
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$



$$\begin{matrix} \mathbf{d}_\mathbf{x} \\ \mathbf{d}_\mathbf{y} \end{matrix} =$$



$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

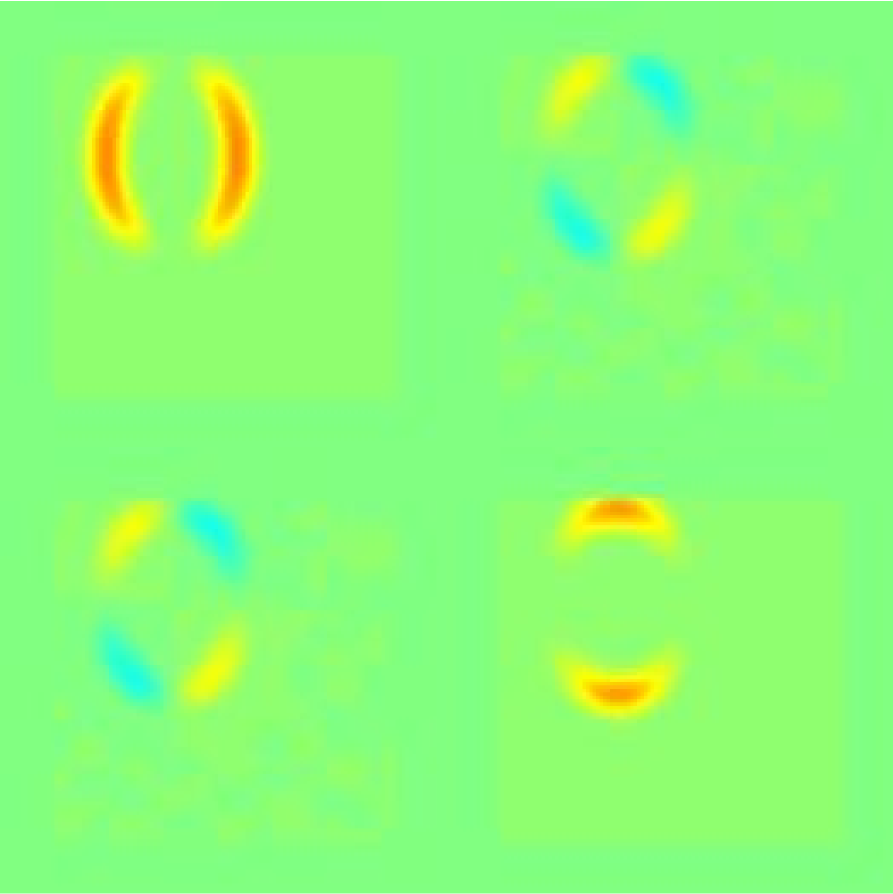


$$\mathbf{min}_{\mathrm{d}} E(\mathbf{d}) = \sum_{\mathbf{x}} ||\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})||^{\mathbf{2}}$$

$$\mathbf{min}_{\mathbf{d}} E(\mathbf{d}) = \sum_{\mathbf{x}} \|\mathbf{J}(\mathbf{x}+\mathbf{d}) - \mathbf{I}(\mathbf{x})\|^{\mathbf{2}}$$

$$\sum_{\mathbf{x}} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \sum_{\mathbf{x}} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

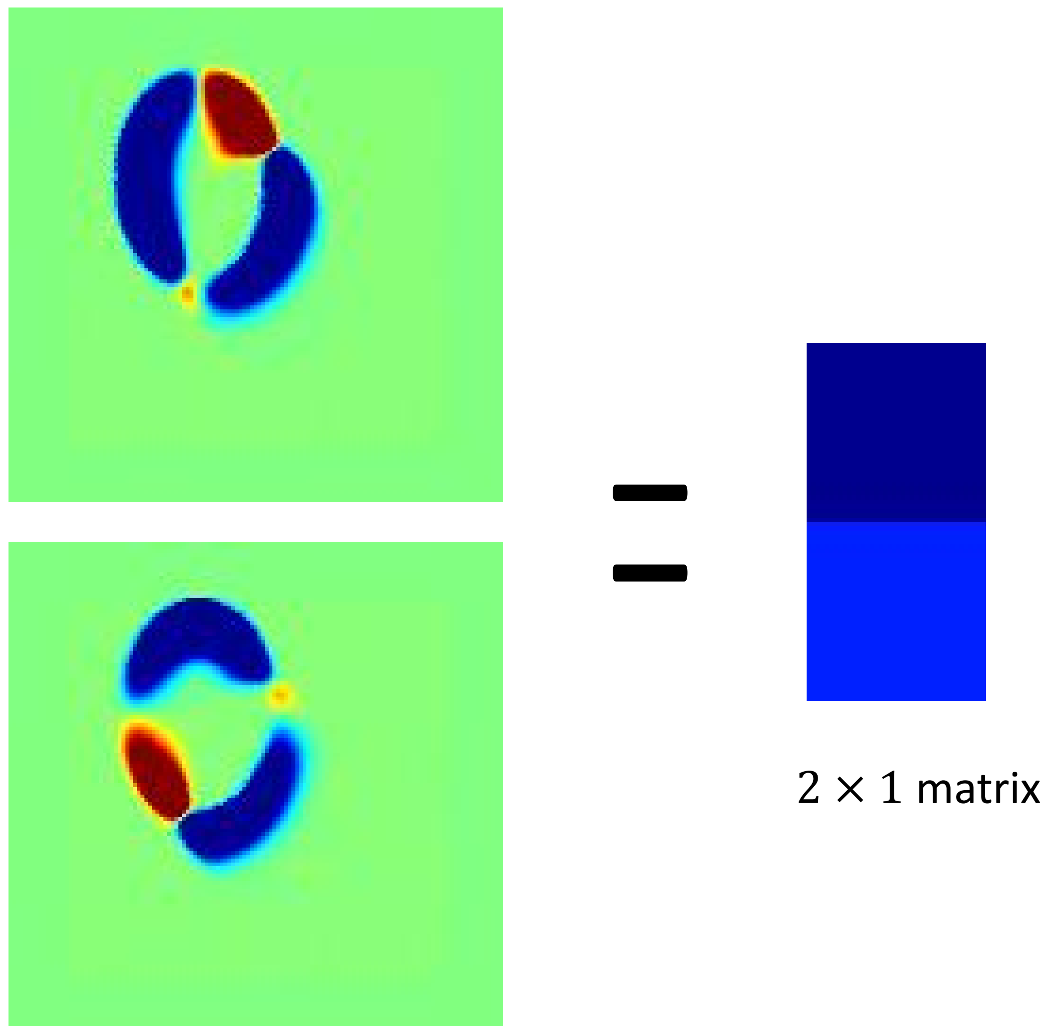
Summing over pixels



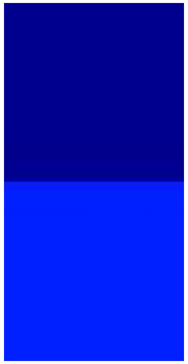
=

2×2 matrix

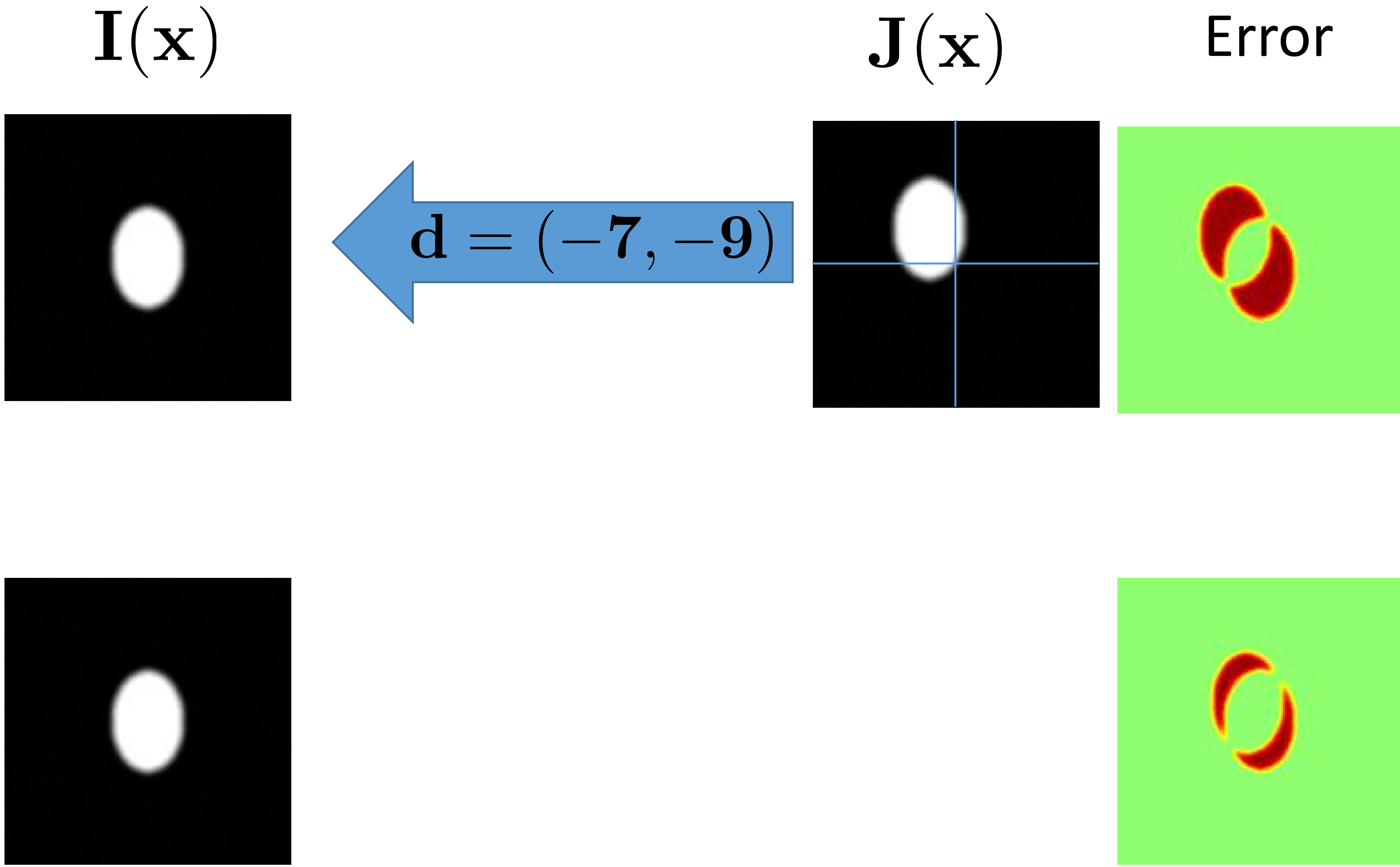
Summing over pixels



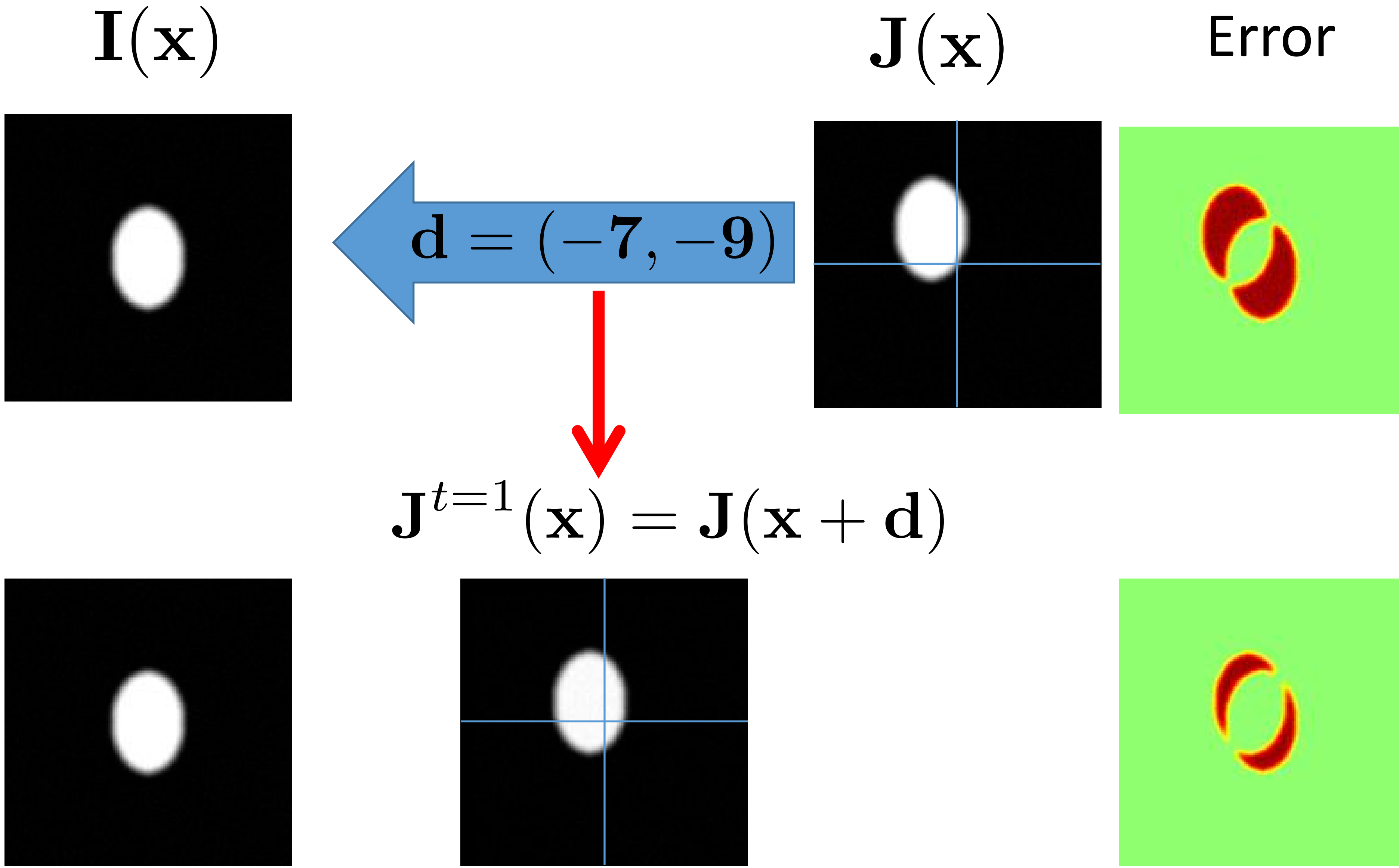
$$\sum_{\mathbf{x}} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \sum_{\mathbf{x}} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} \left(\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x})\right)$$

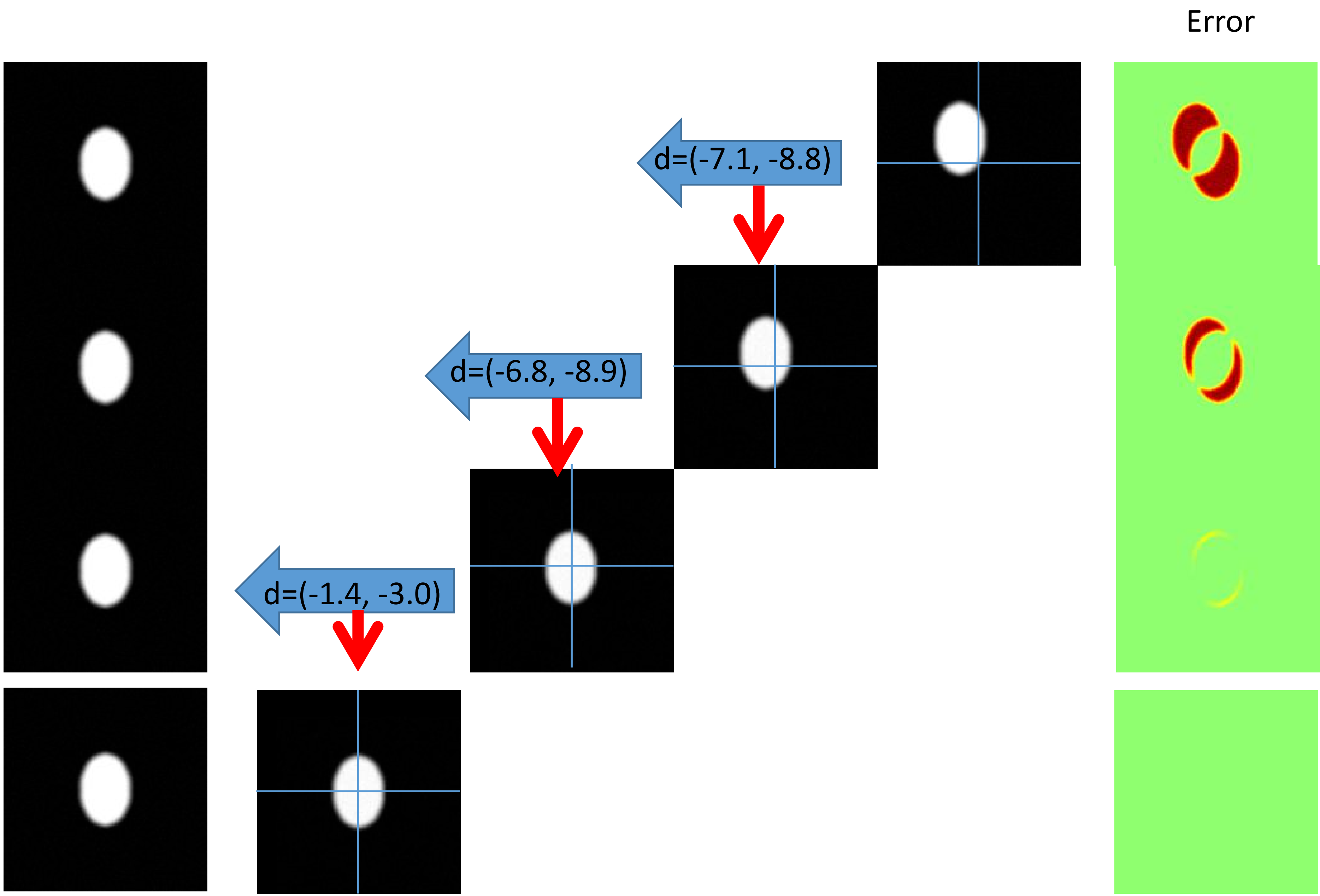
$\mathbf{d}_{\mathbf{x}}$ $+\,\mathbf{d}_{\mathbf{y}}$ $=$ 

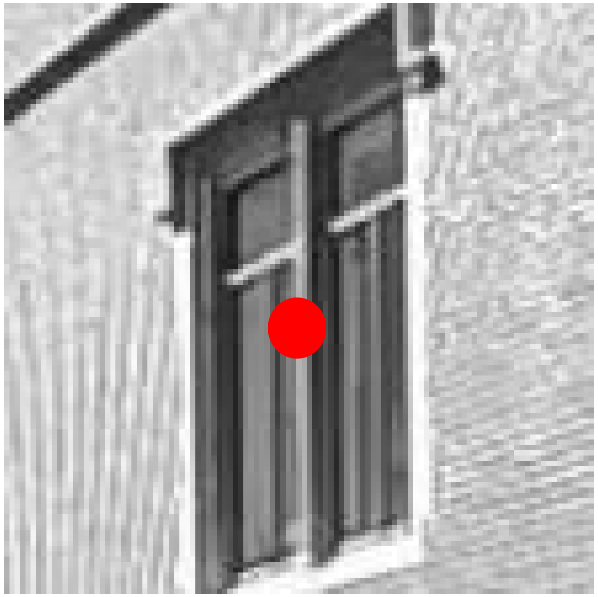
3: Solve for d, warp image, iterate

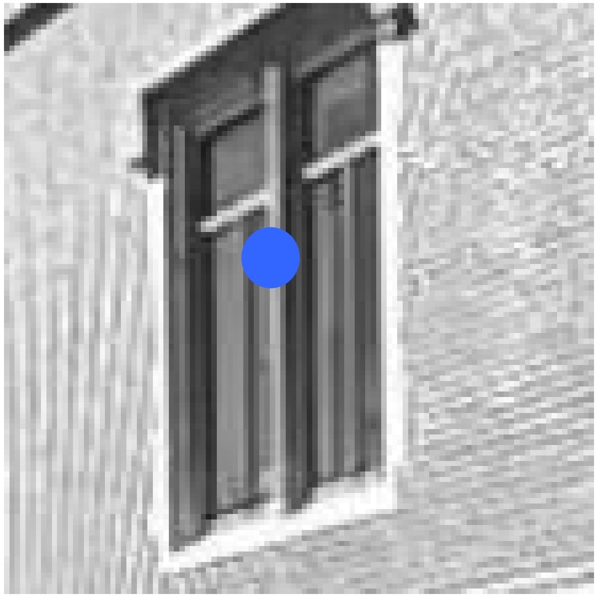


3: Solve for d, warp image, iterate




Error
d=(-7.1, -8.8)
d=(-6.8, -8.9)
d=(-1.4, -3.0)





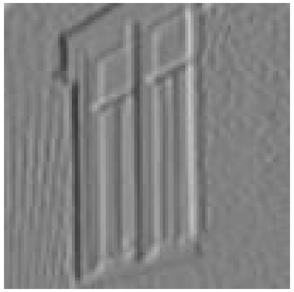
$\mathbf{I}(\mathbf{x})$ $\mathbf{J}(\mathbf{x})$

$t = 0$ $t = 1$

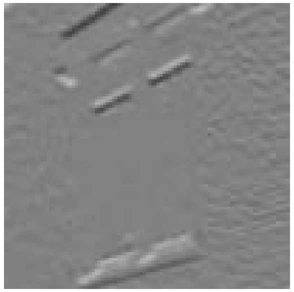
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

$$\boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}}^T \boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}} \mathbf{d} = \boxed{\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

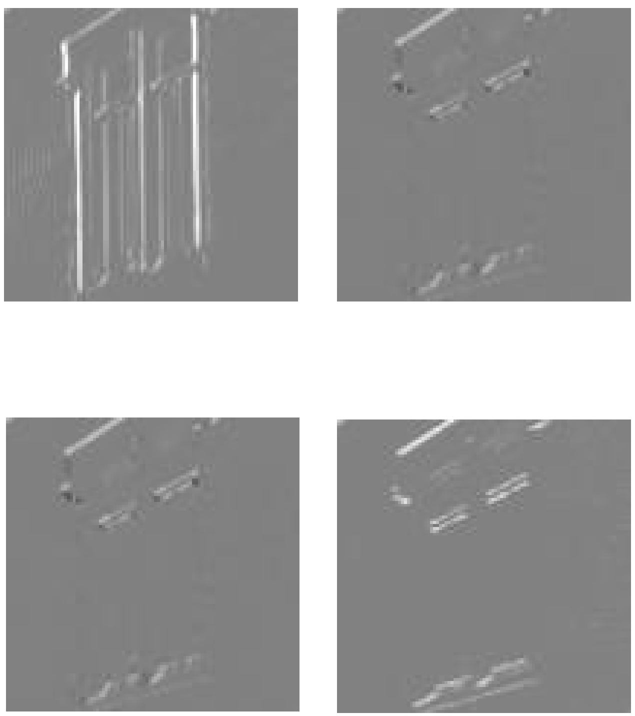
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta x} =$$



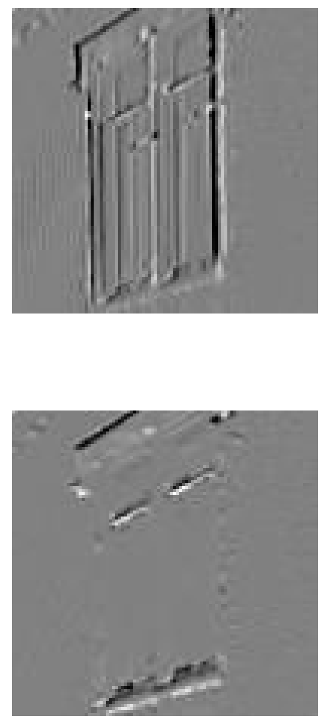
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta y} =$$



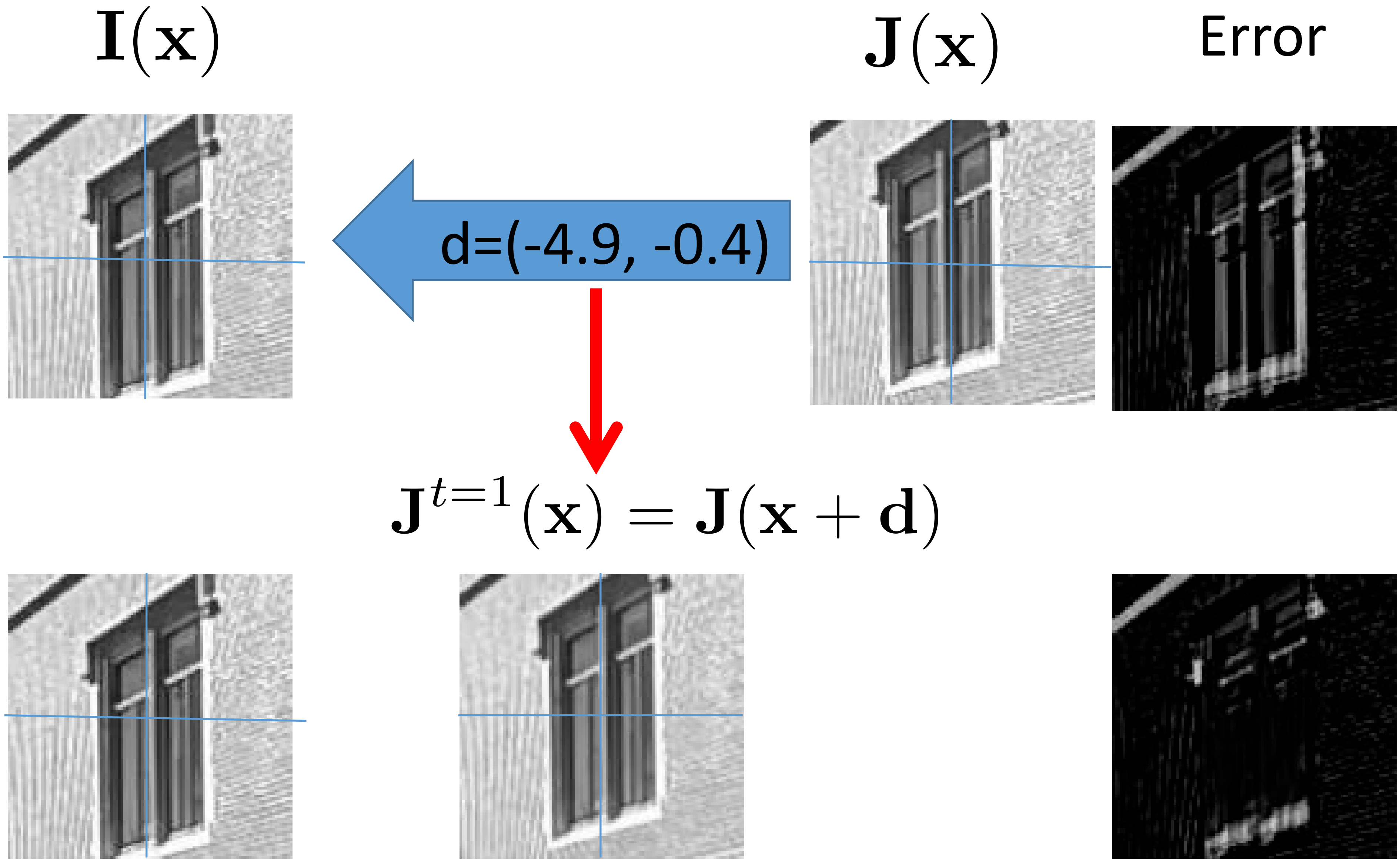
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^{T} \left(\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x})\right)$$

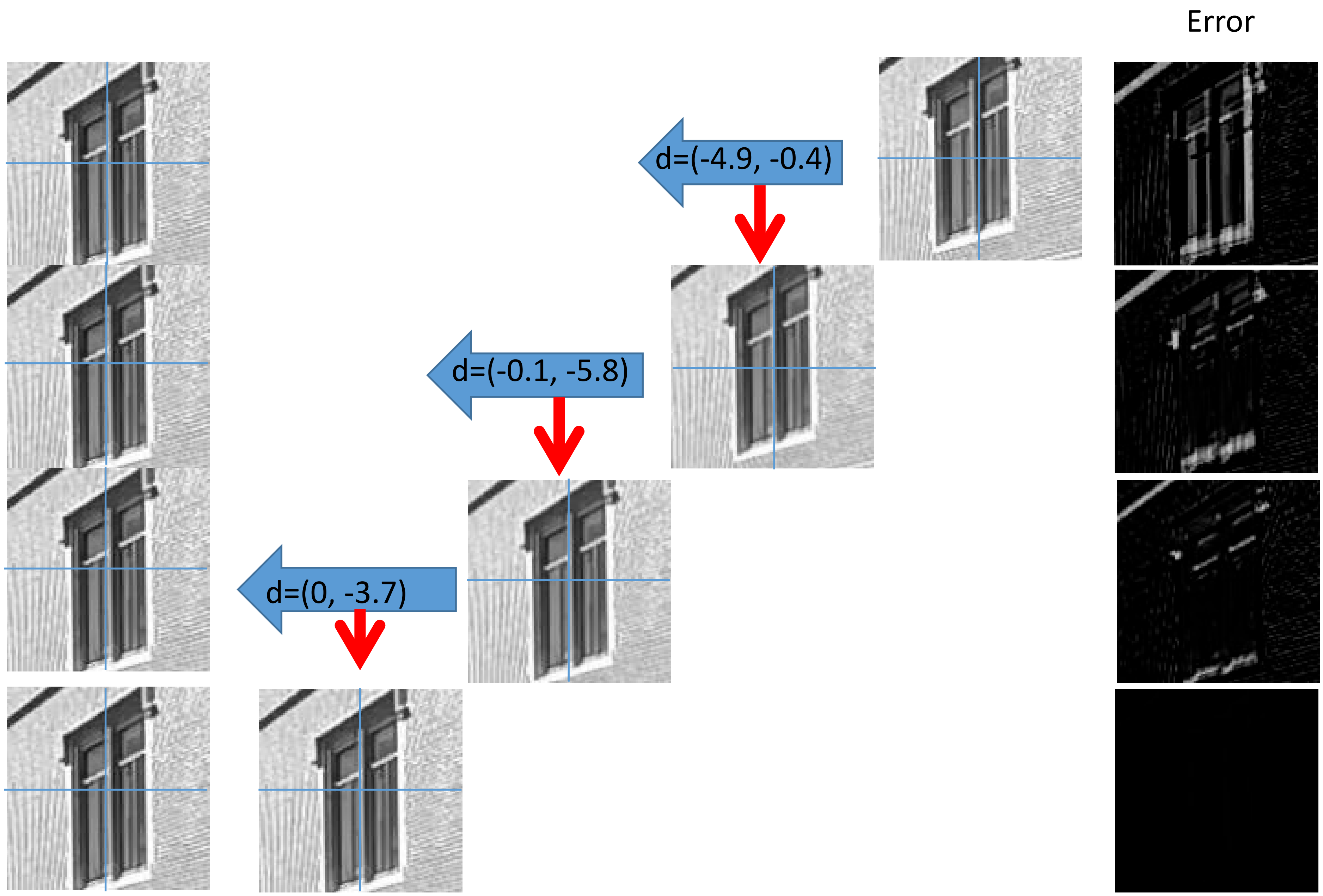


$$\begin{matrix} \mathbf{d}_\mathbf{x} \\ \mathbf{d}_\mathbf{y} \end{matrix} =$$



3: Solve for d, warp image, iterate




Error
d=(-4.9, -0.4)
d=(-0.1, -5.8)
d=(0, -3.7)