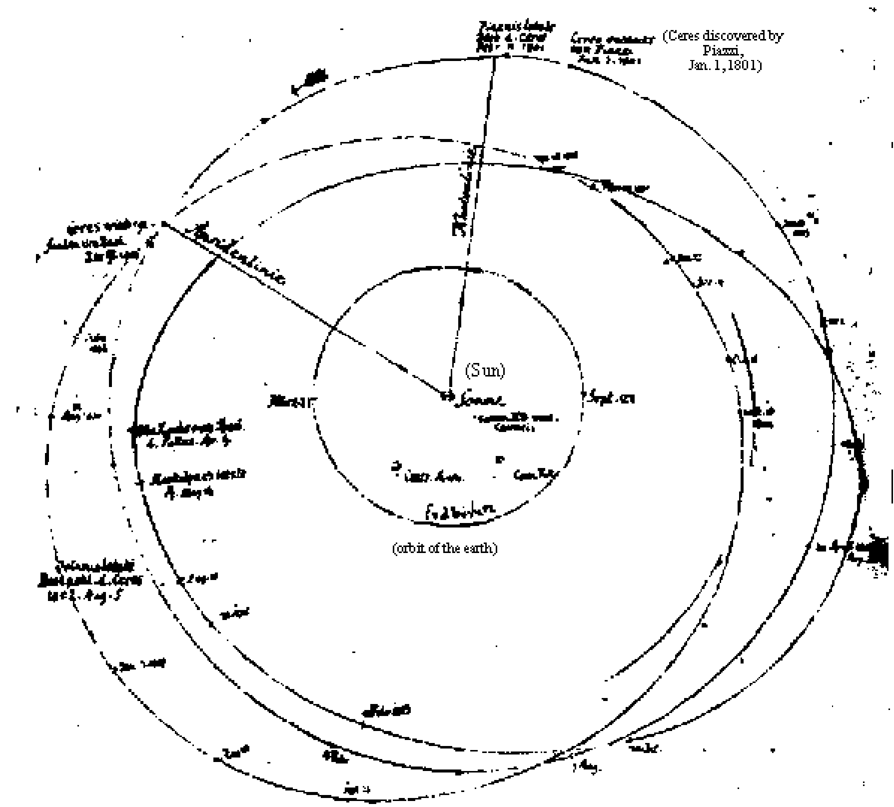
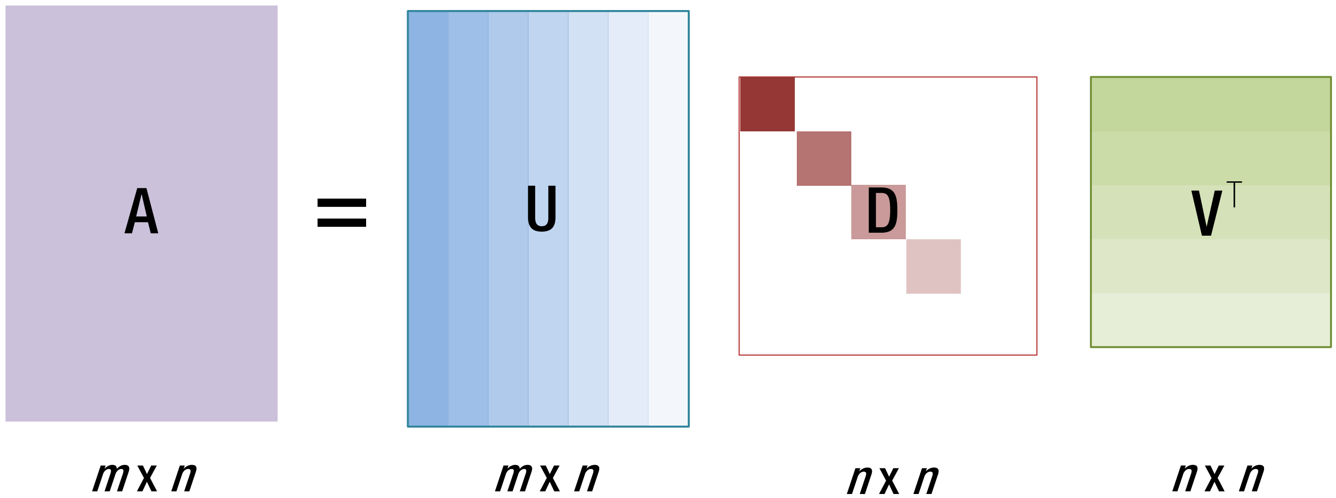
1809, Carl Friedrich Gauss

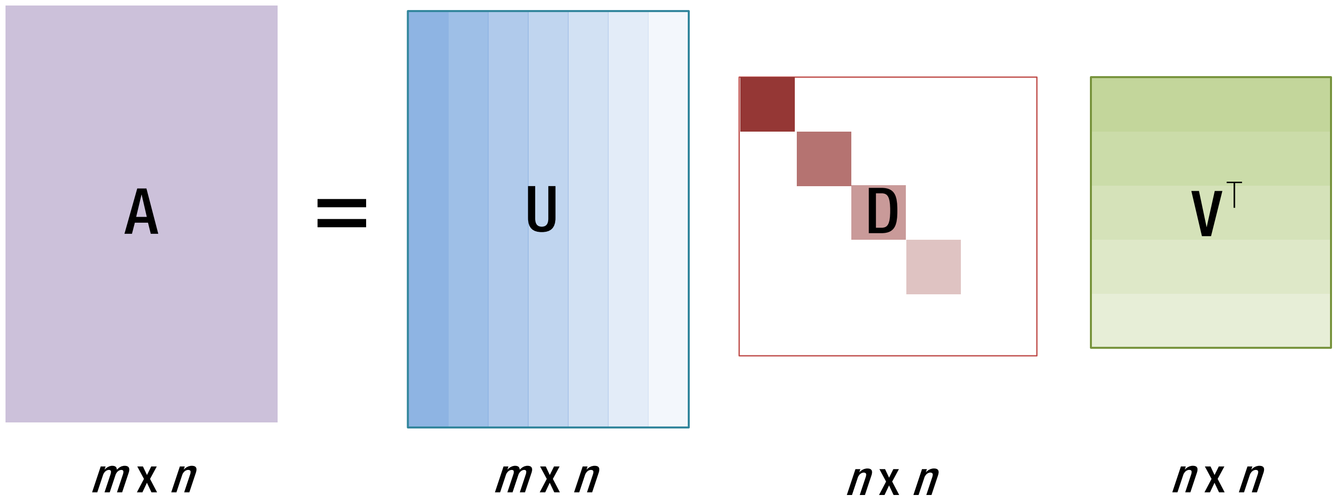


Sketch of the orbits of Ceres and Pallas (nachlaß Gauß, Handb. 4). Courtesy of Universitätsbibliothek Göttingen.

Singular Value Decomposition



Singular Value Decomposition

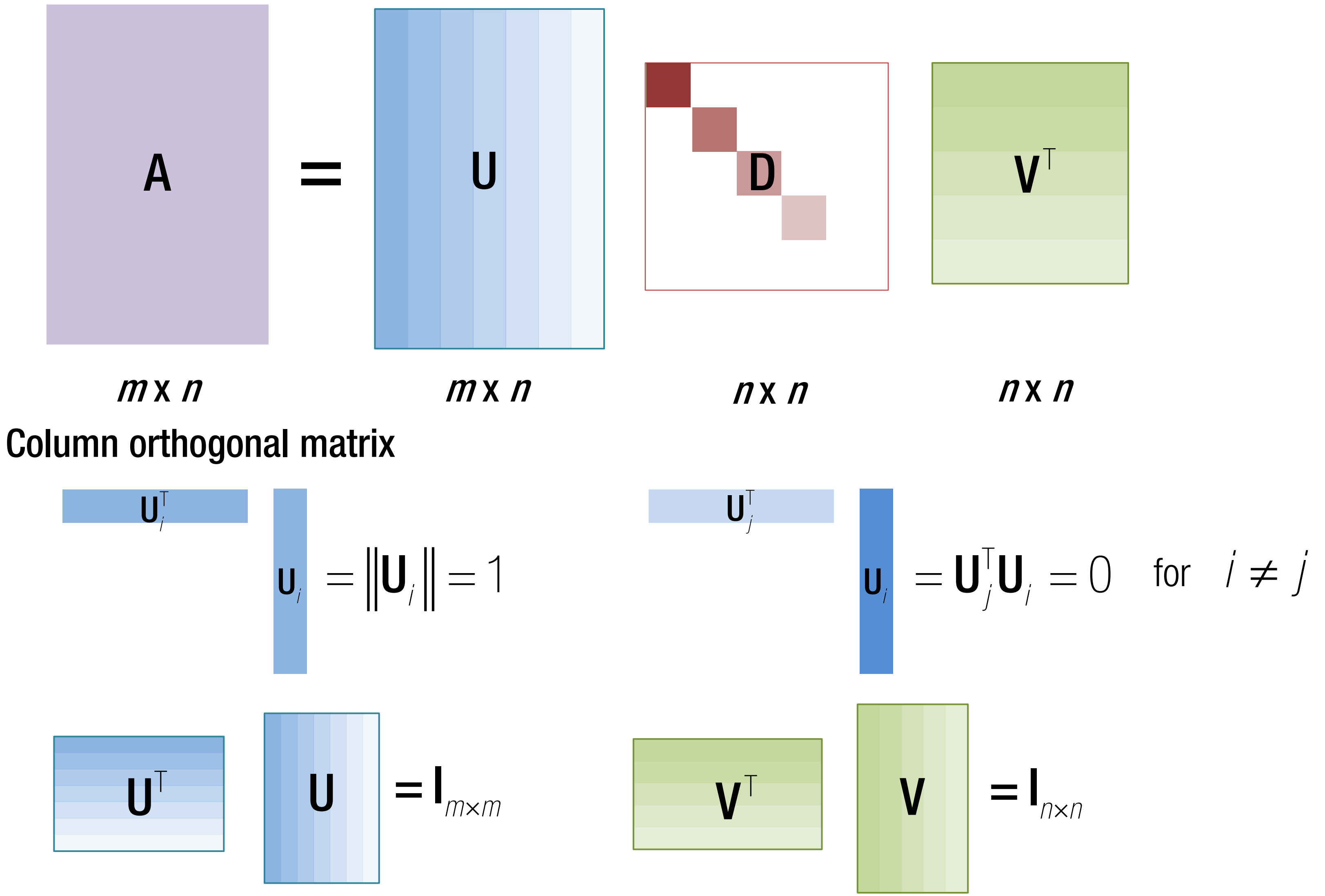


Column orthogonal matrix

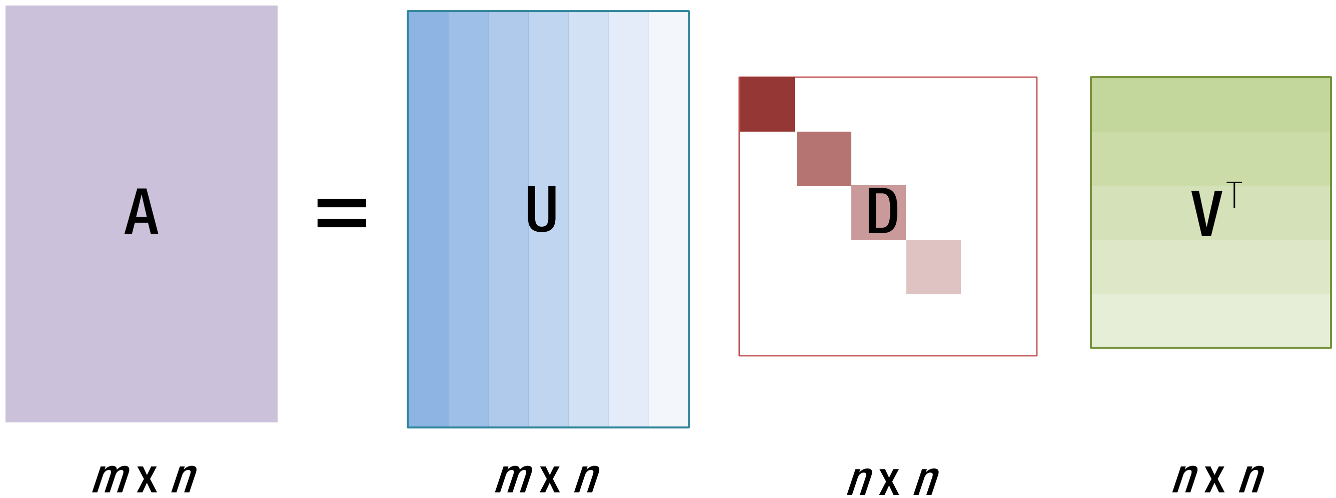
$$\mathbf{U}_i^\top \mathbf{U}_i = \left\| \mathbf{U}_i \right\| = 1$$

$$\mathbf{U}_j^\top \mathbf{U}_i = \mathbf{U}_j^\top \mathbf{U}_i = 0 \quad \text{for} \quad i \neq j$$

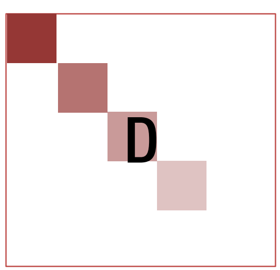
Singular Value Decomposition



Singular Value Decomposition

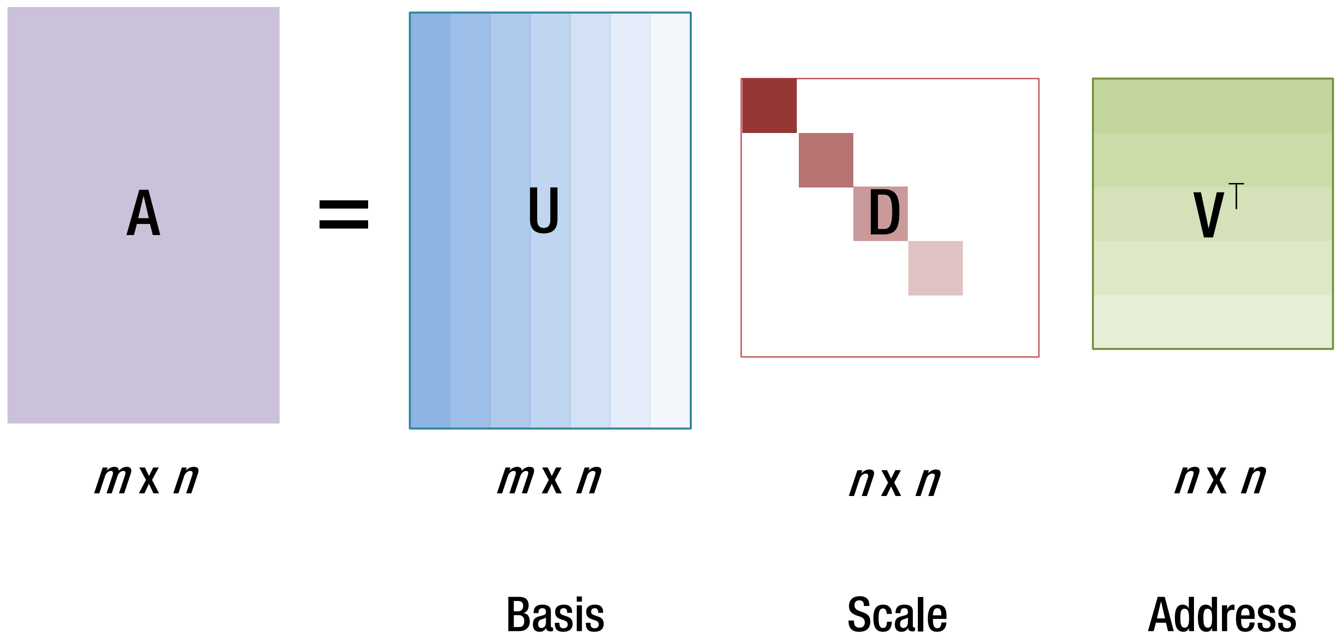


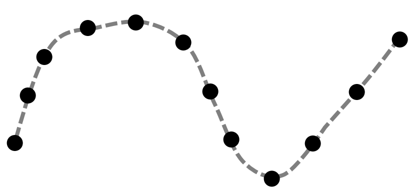
Singular value matrix

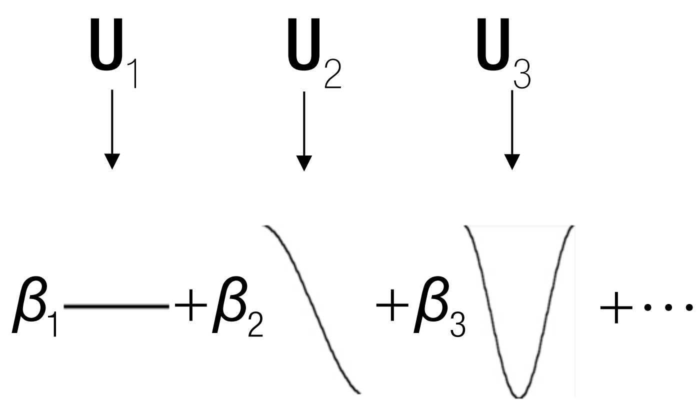


$= \mathrm{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$

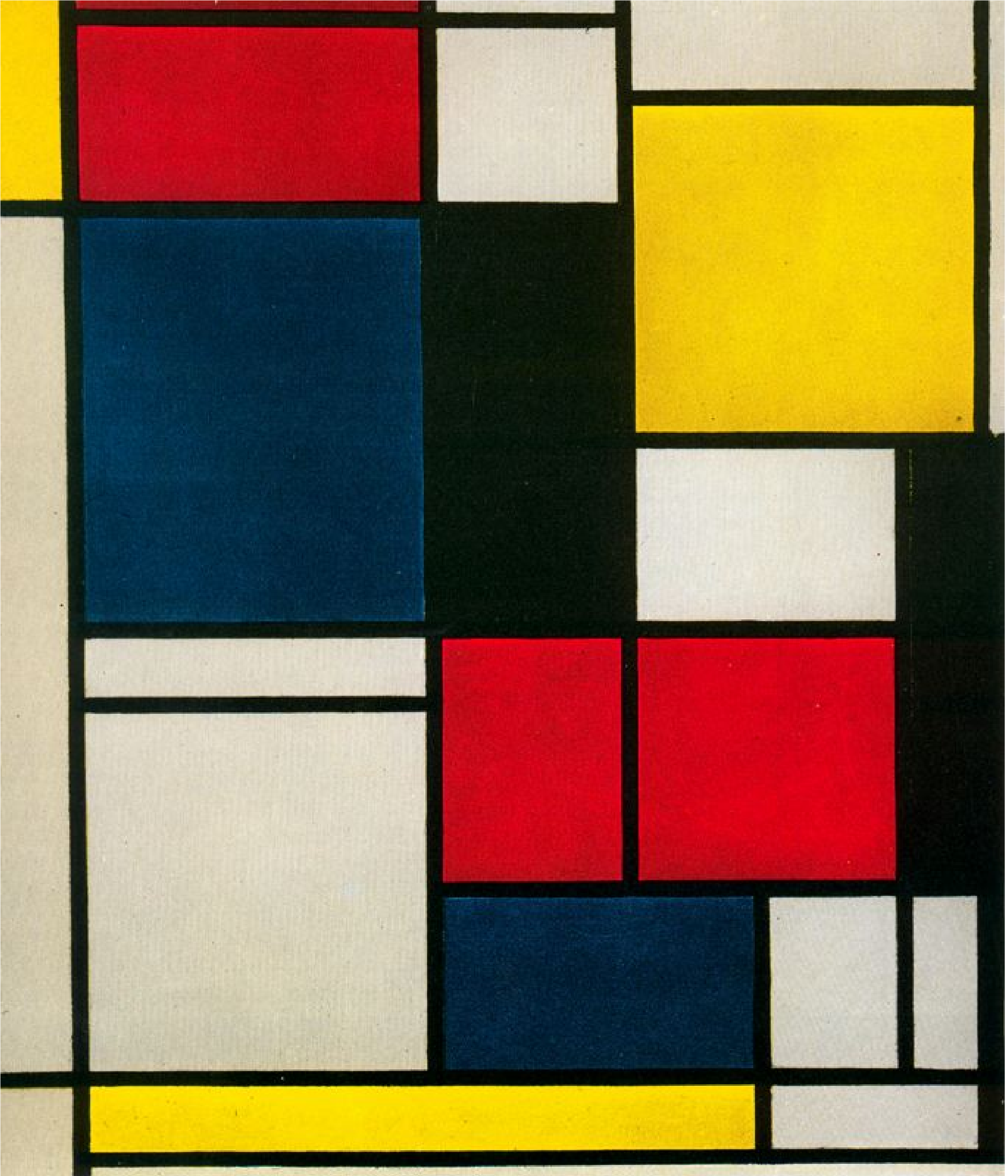
Singular Value Decomposition



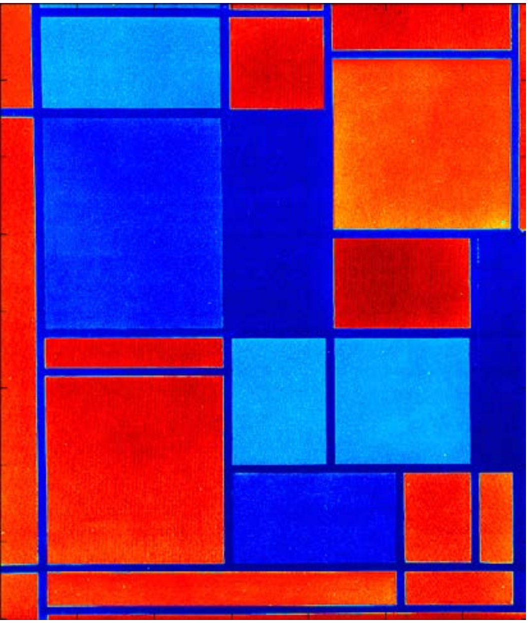




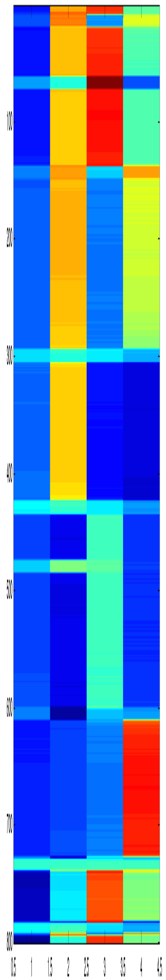
SVD as basis + transformed Address

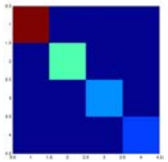


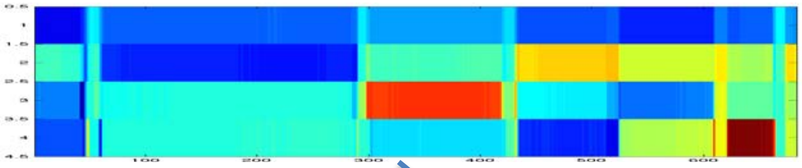
SVD of this?

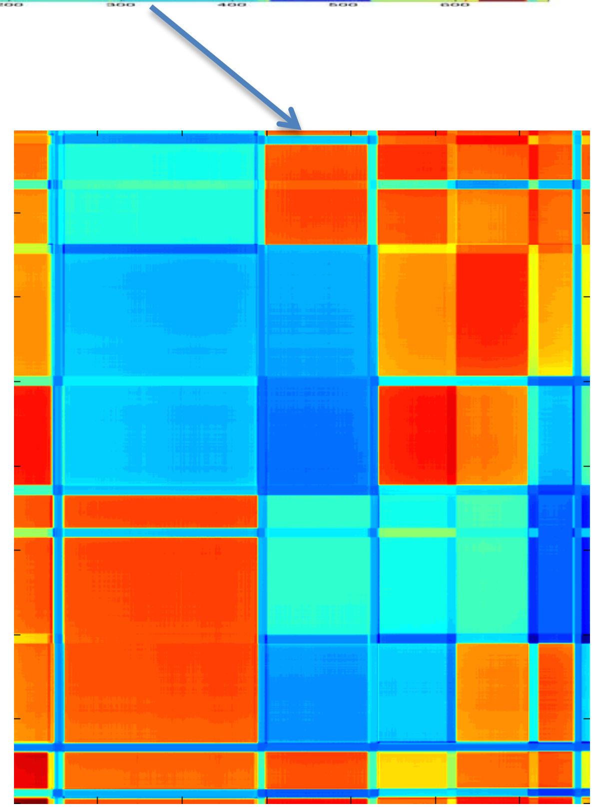


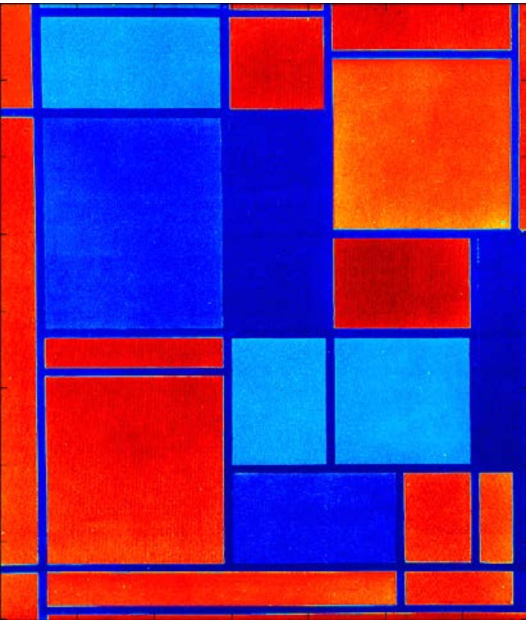
U(:,1:4) D(1:4,1:4) V(:,1:4)’

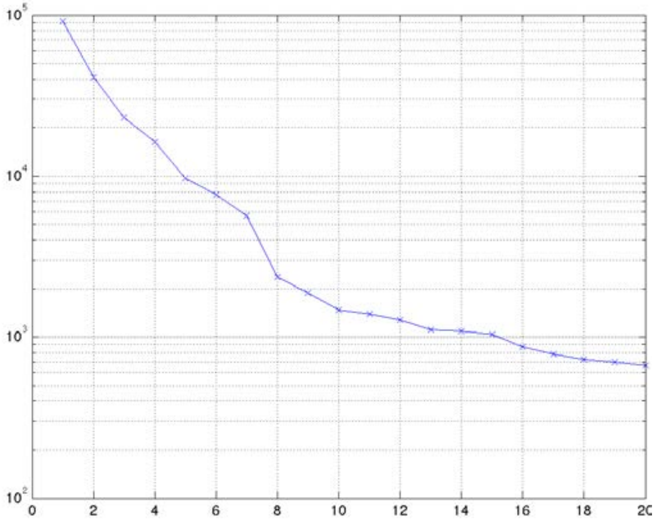


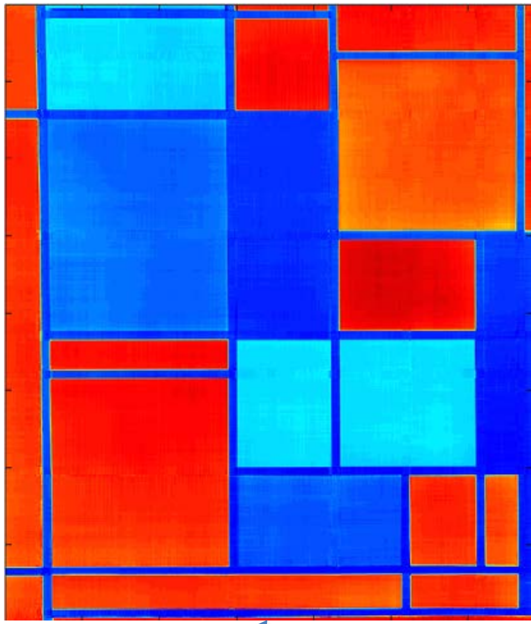








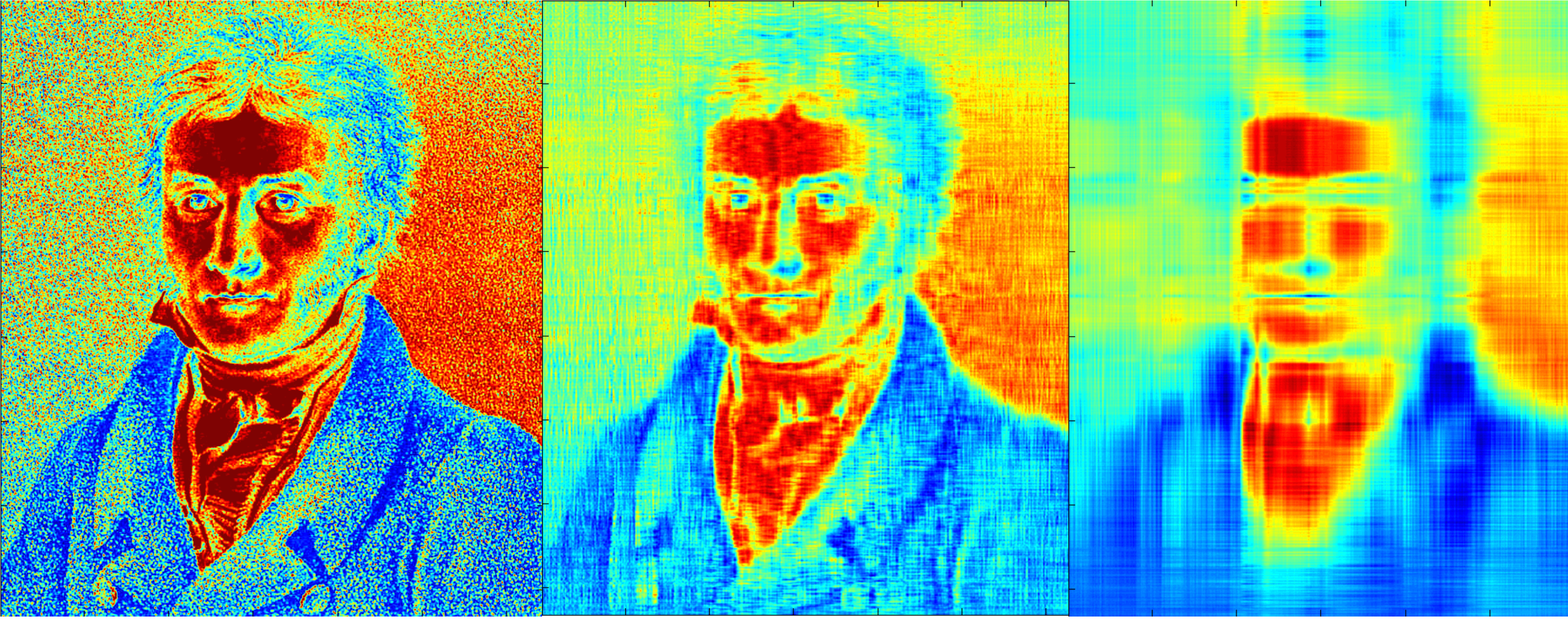


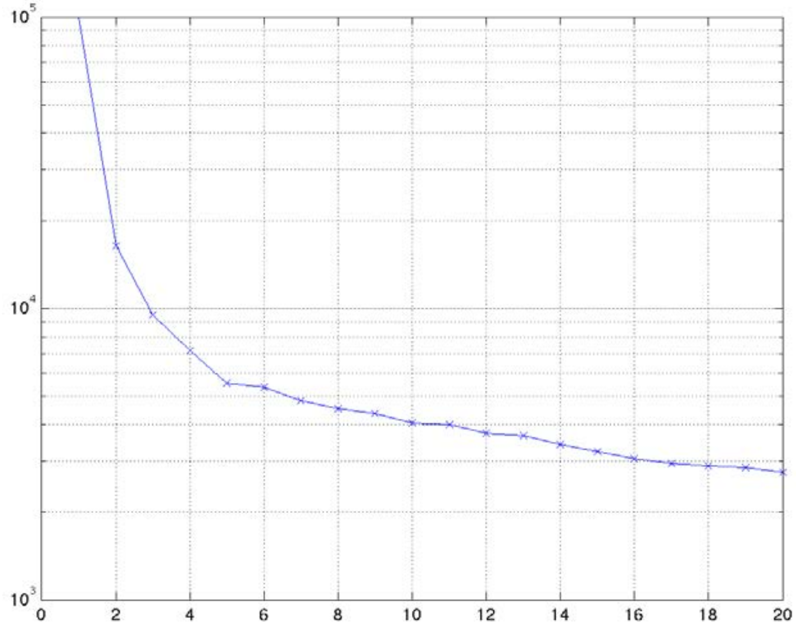


[u,d,v] = svd(I);

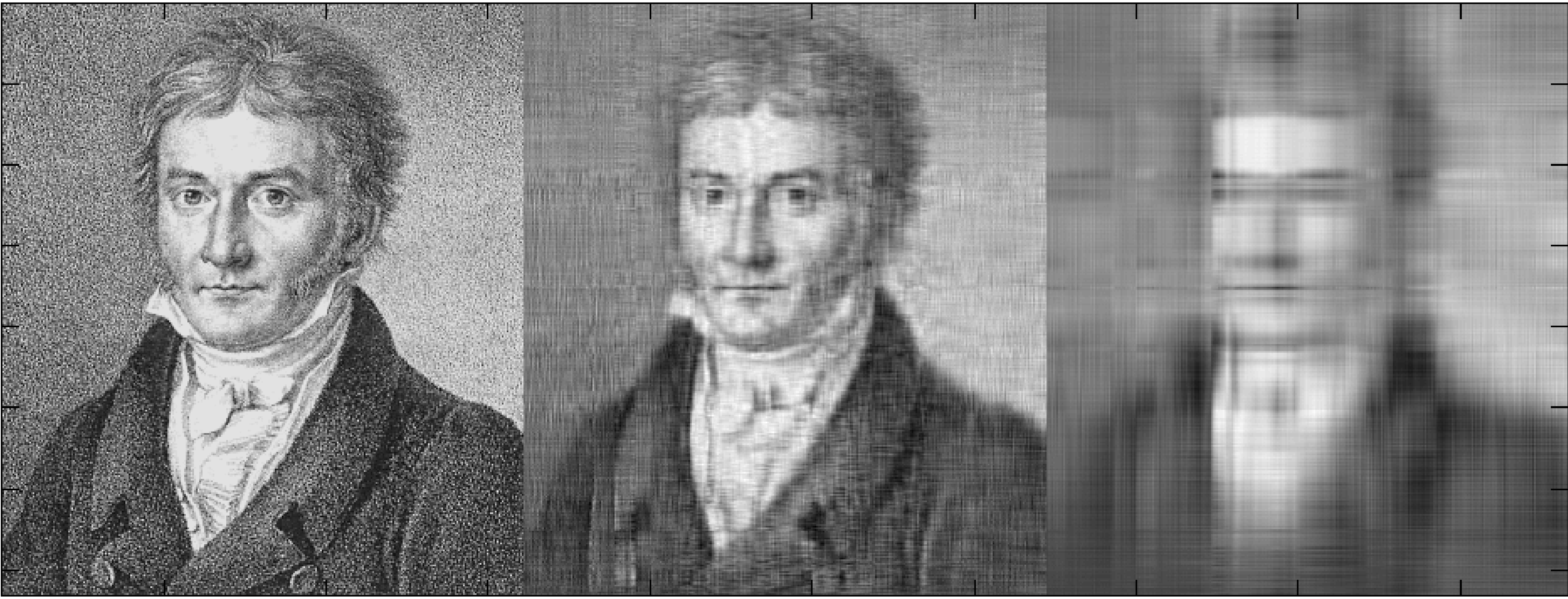
semilogy(diag(d(1:20,1:20)),'x-')

Im2 = u(:,1:20)*d(1:20,1:20)*v(:,1:20)';

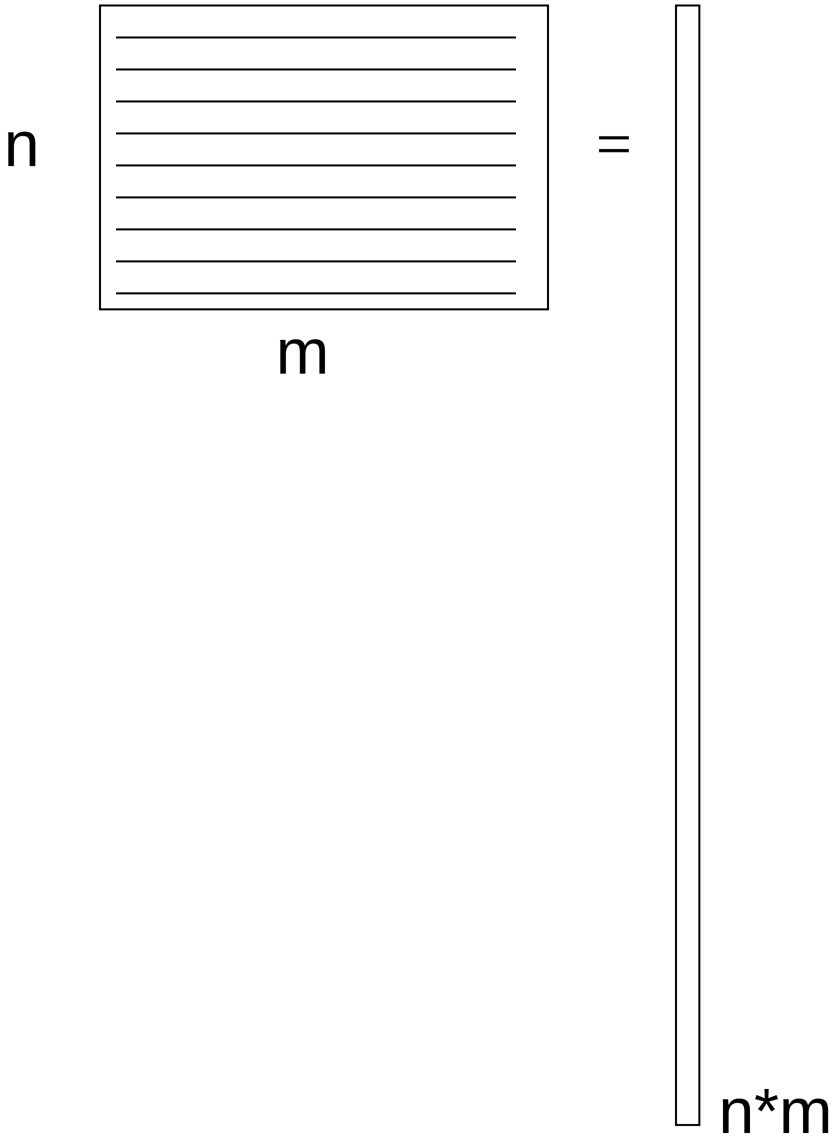




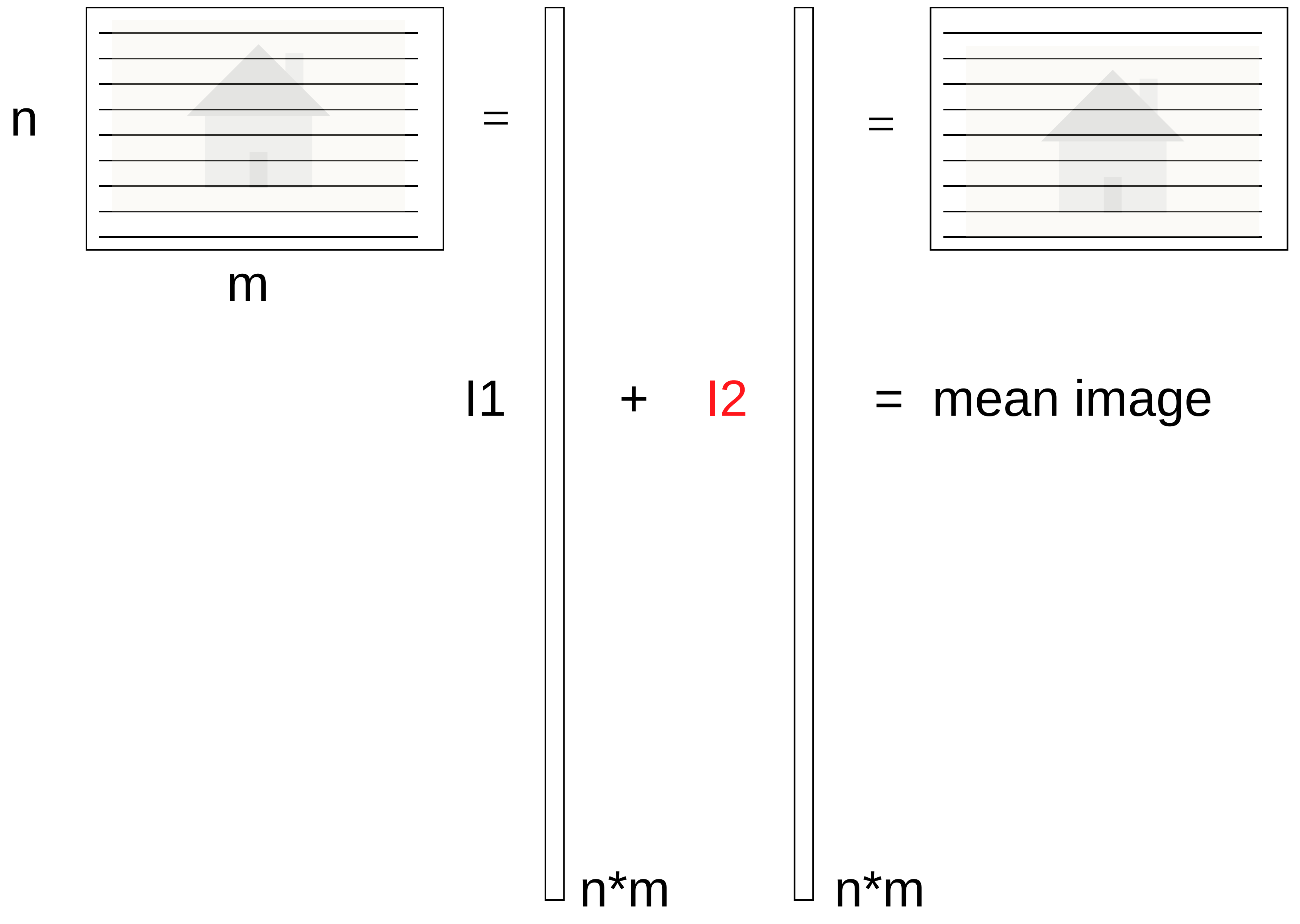
10^5
10^4
10^3
0
2
4
6
8
10
12
14
16
18
20



Images as Vectors



Vector Mean



Eigenfaces

Eigenfaces look somewhat like generic faces.

Eigen-images of Berlin

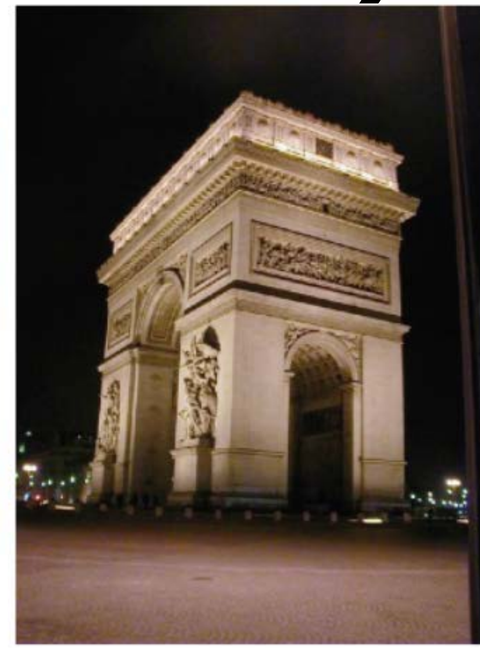
Eigen-images

Average of 16 individuals transformed via
biometrical data of different ethnics

Average of 16 individuals transformed via
biometrical data of different ages

Rank

Nullspace



=

[illegible]3	-0.9660	0.2586
-0.0029	-0.2586	-0.9660
-1.0000	-0.0005	0.0032

d(1,1)
ans = 1.0000e+004

d(2,2)
ans =   0.0021

d(3,3)
ans = 2.7838e-016

Rank(F) = 2

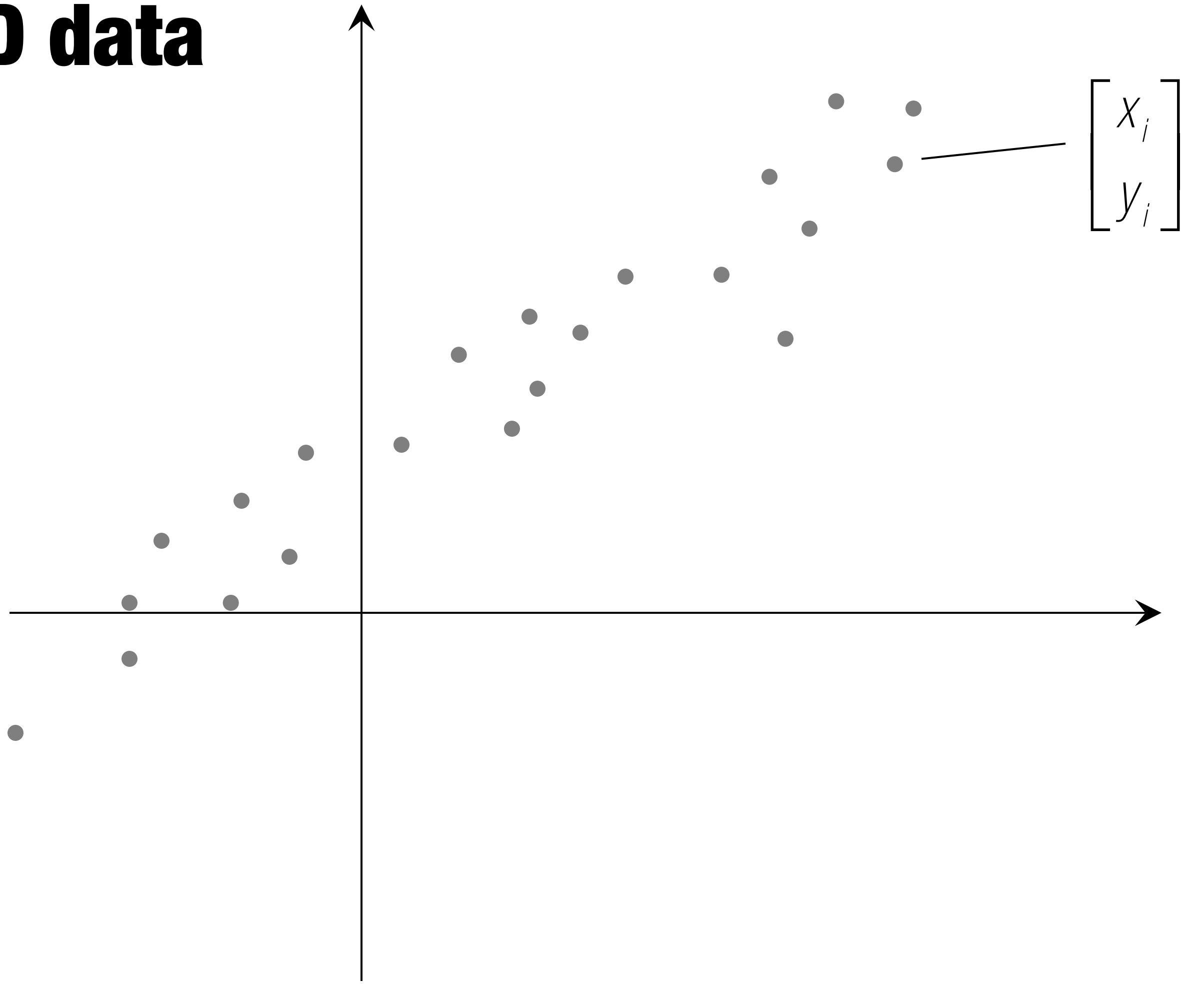
Matrix Inversion with SVD

Two types of Least Square Problem:

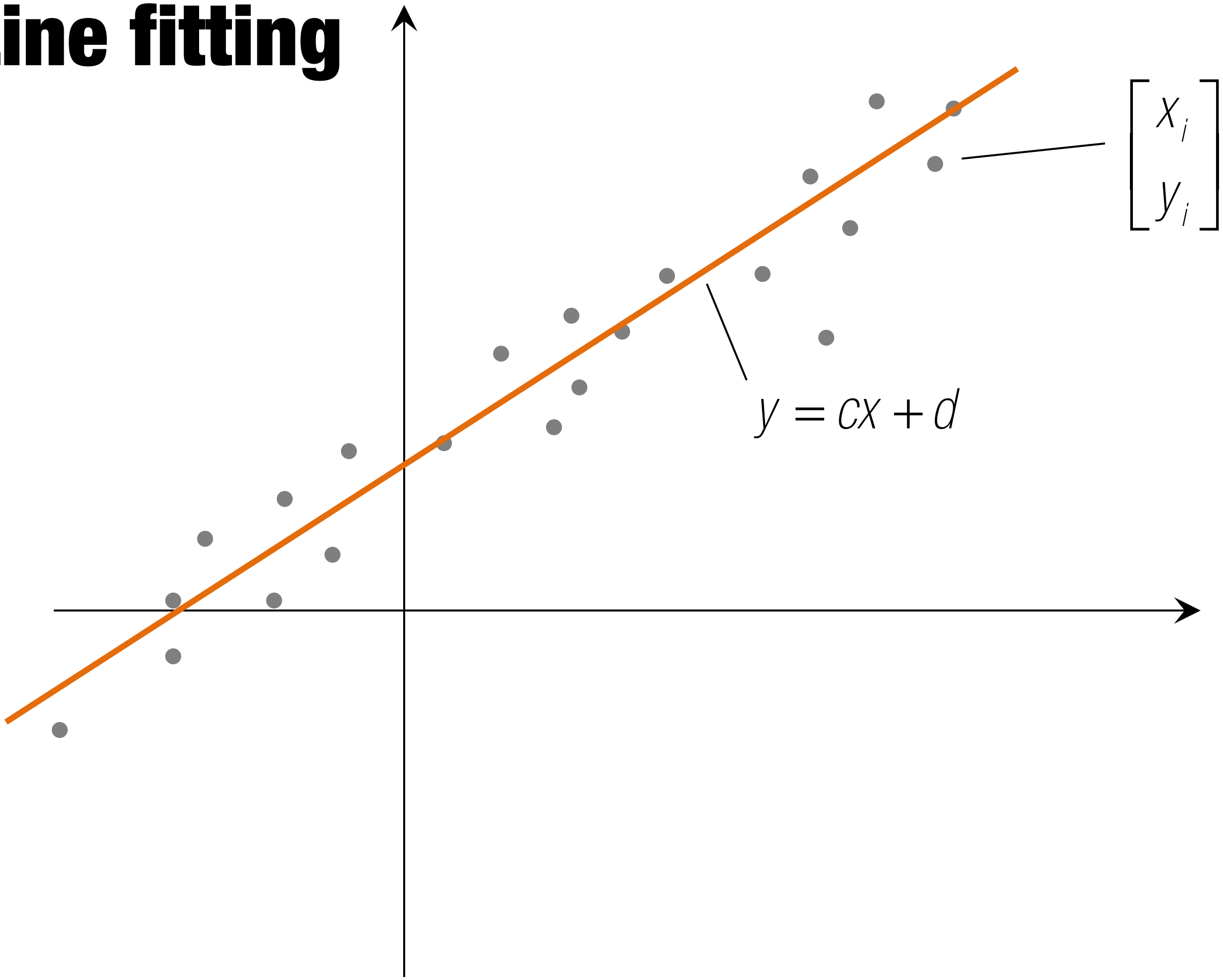
$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathsf{A}\mathbf{x} - \mathbf{b}\|^2 \qquad \text{with} \qquad \|\mathbf{b}\| \neq 0$$

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathsf{A}\mathbf{x}\|^2 \qquad \text{s.t.} \qquad \|\mathbf{x}\| = 1$$

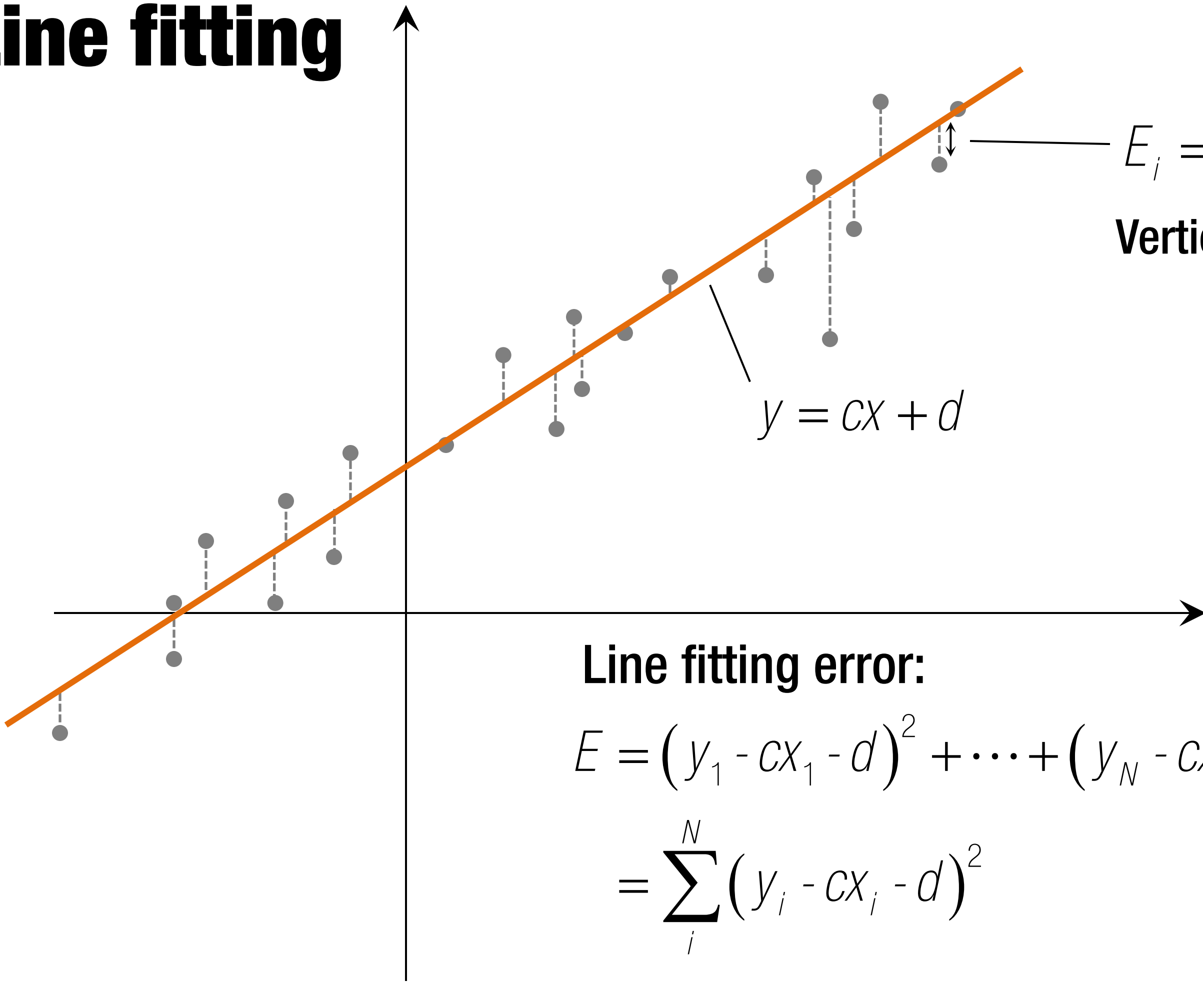
2D data



Line fitting



Line fitting



$$E_i = \left| y_i - cx_i - d \right|$$

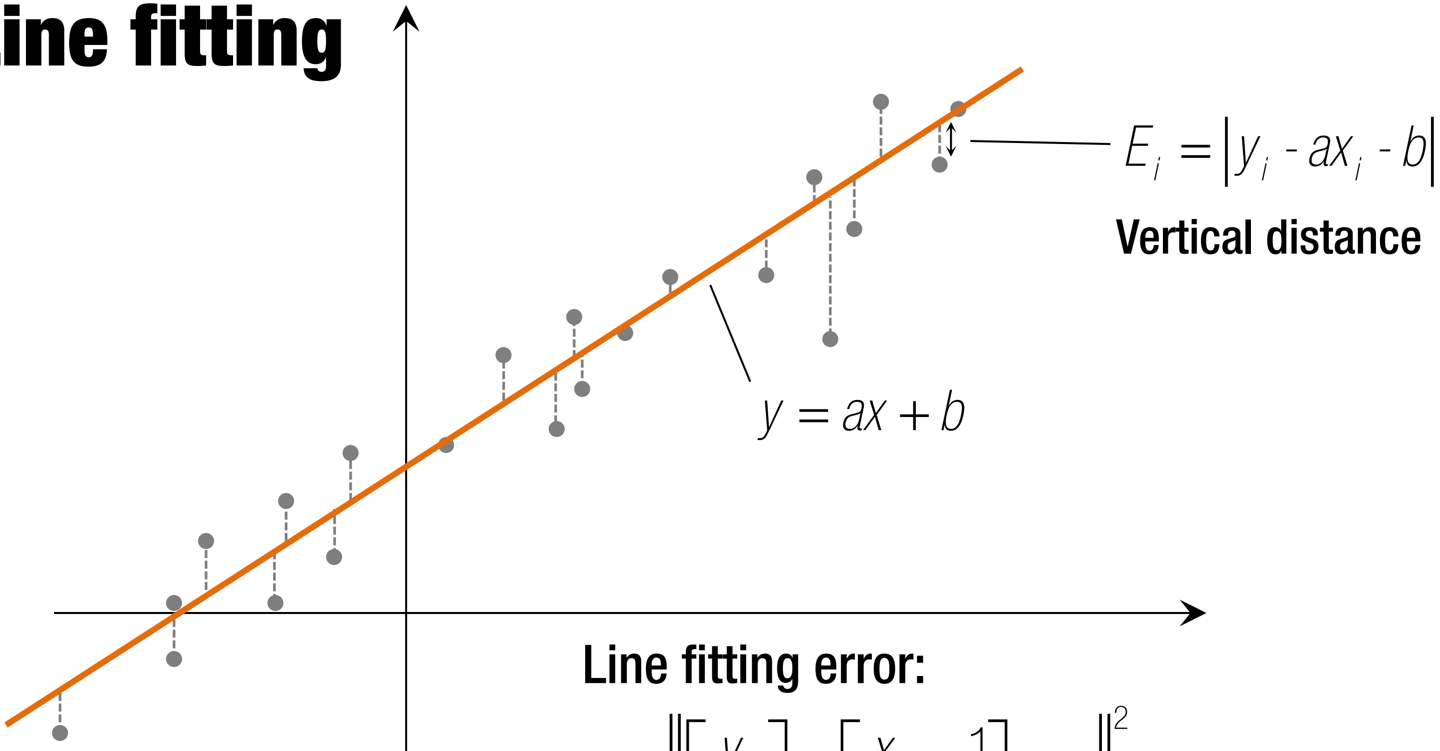
Vertical distance

Line fitting error:

$$E = \left(y_1 - cx_1 - d\right)^2 + \cdots + \left(y_N - cx_N - d\right)^2$$

$$= \sum_i^N \left(y_i - cx_i - d\right)^2$$

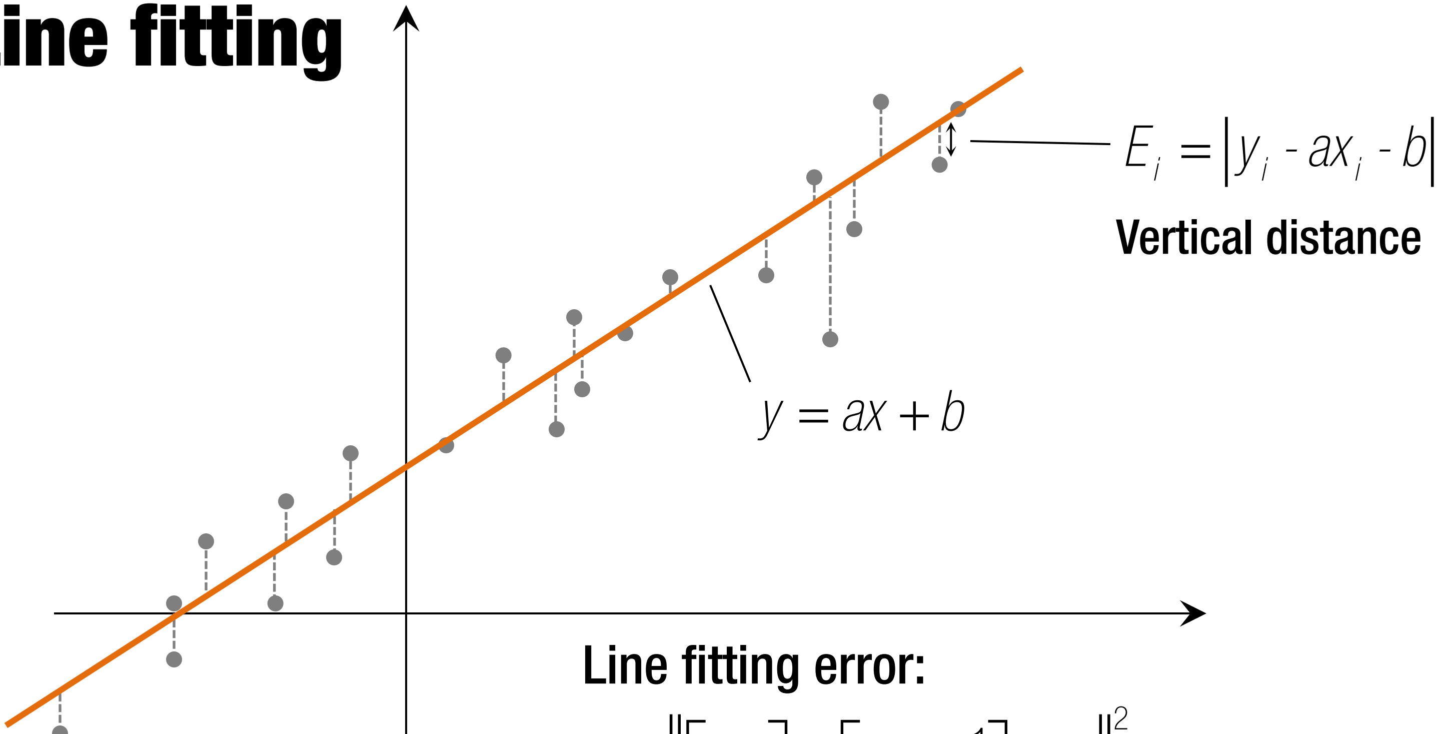
Line fitting



Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \left\| \mathbf{b} - \mathbf{Ax} \right\|^2$$

Line fitting

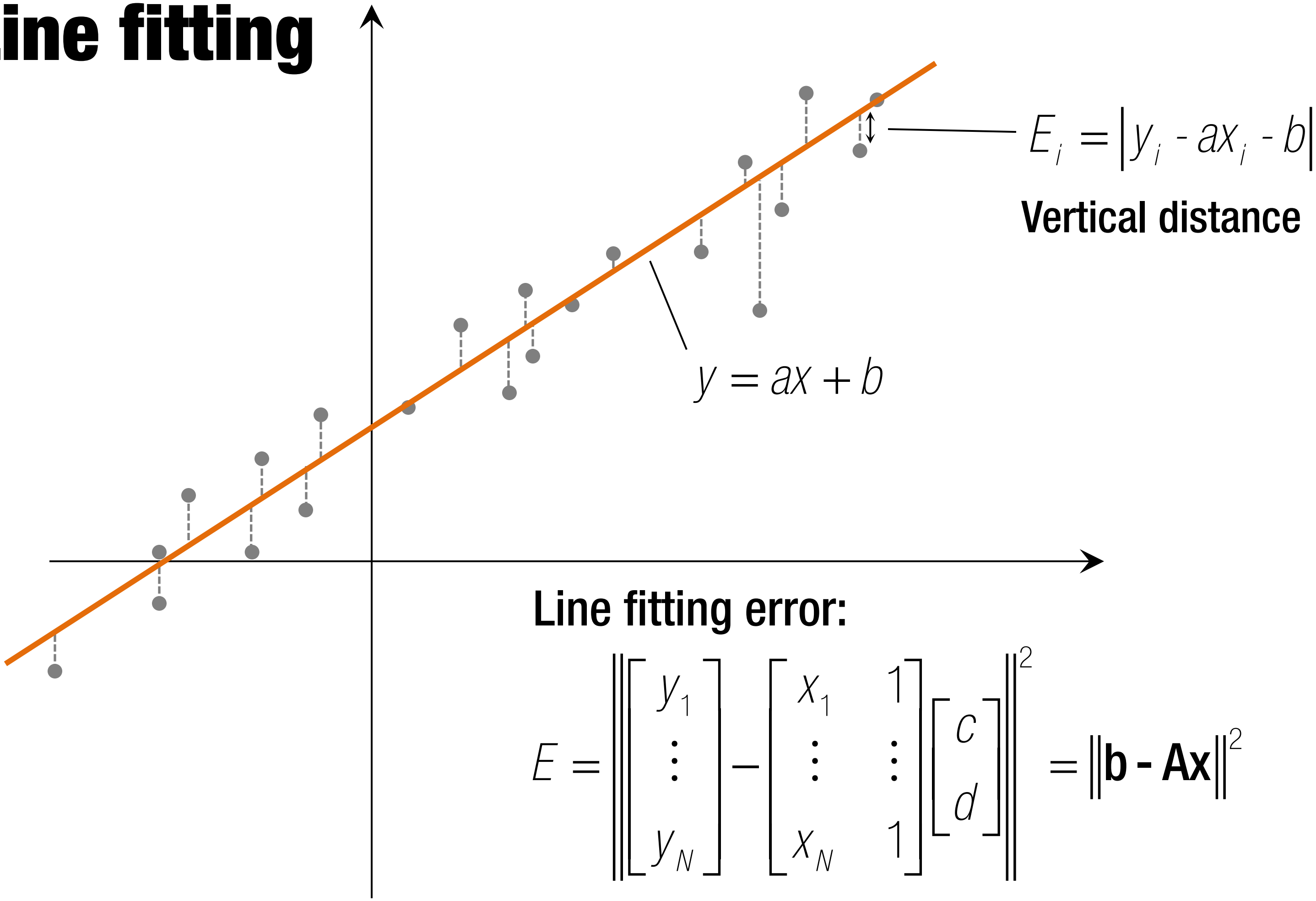


Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \left\| \mathbf{b} - \mathbf{Ax} \right\|^2$$

$$= (\mathbf{b} - \mathbf{Ax})^\top (\mathbf{b} - \mathbf{Ax}) = \mathbf{b}^\top \mathbf{b} - 2\mathbf{b}^\top \mathbf{Ax} + \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax}$$

Line fitting



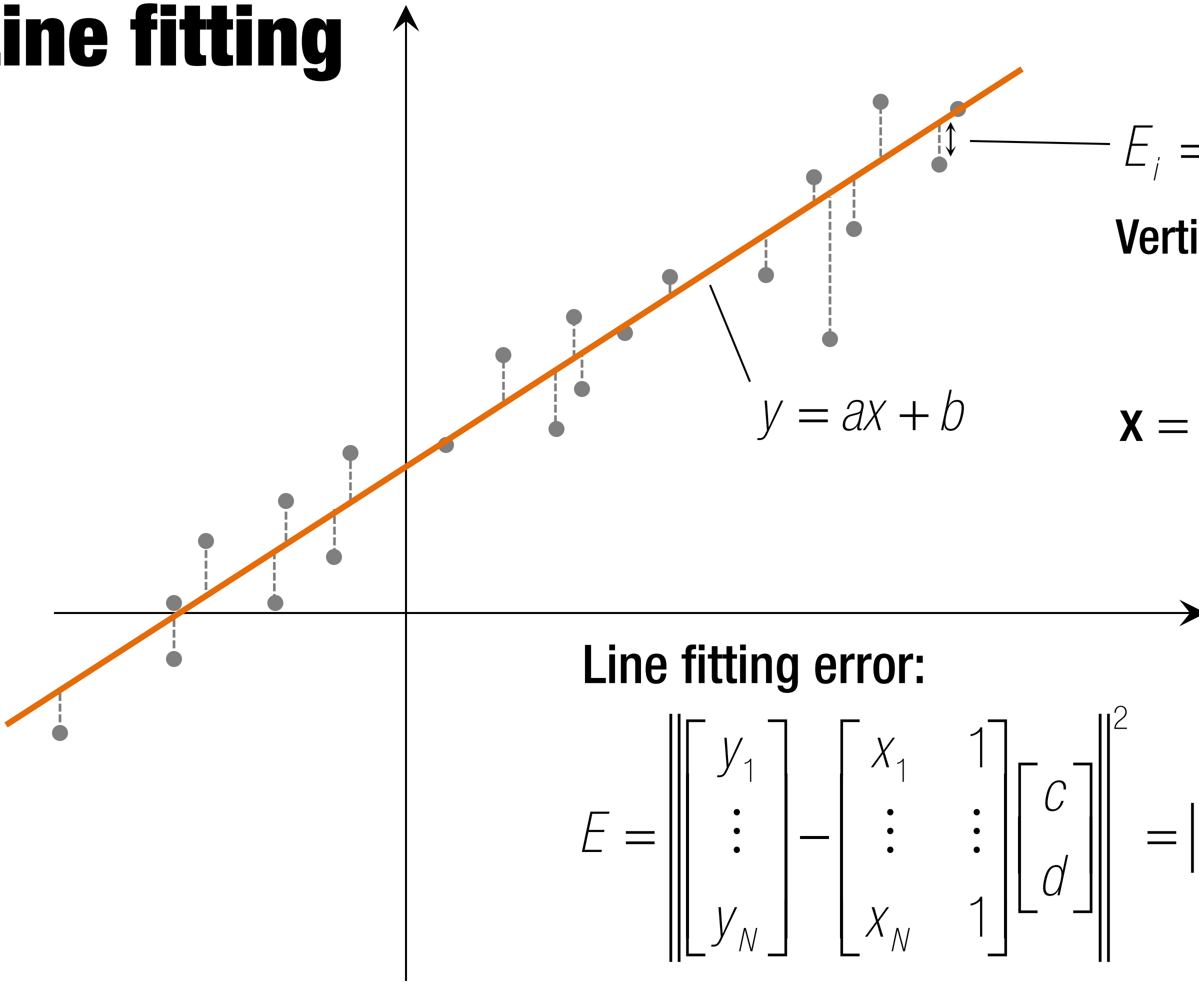
Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \left\| \mathbf{b} - \mathbf{A}\mathbf{x} \right\|^2$$

$$= (\mathbf{b} - \mathbf{A}\mathbf{x})^\top (\mathbf{b} - \mathbf{A}\mathbf{x}) = \mathbf{b}^\top \mathbf{b} - 2\mathbf{b}^\top \mathbf{A}\mathbf{x} + \mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x}$$

$$\frac{\partial E}{\partial \mathbf{x}} = -2\mathbf{b}^\top \mathbf{A} + 2\mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{0} \;\rightarrow\; \mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{b}^\top \mathbf{A}$$

Line fitting



$$E_i = \left| y_i - ax_i - b \right|$$

Vertical distance

$$\mathbf{x} = \begin{bmatrix} c \\ d \end{bmatrix} = \left(\mathbf{A}^{\top}\mathbf{A}\right)^{-1}\mathbf{b}^{\top}\mathbf{A}$$

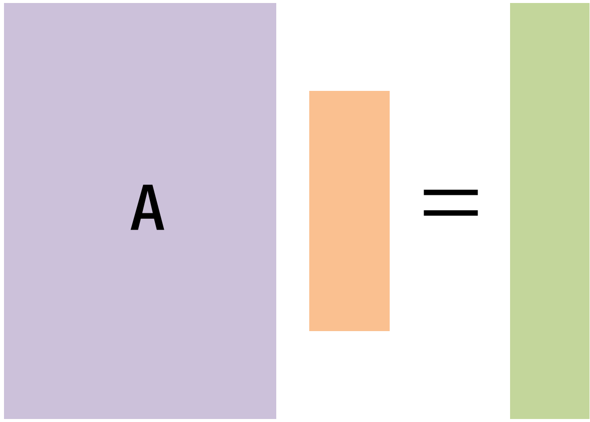
Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \left\| \mathbf{b} - \mathbf{Ax} \right\|^2$$

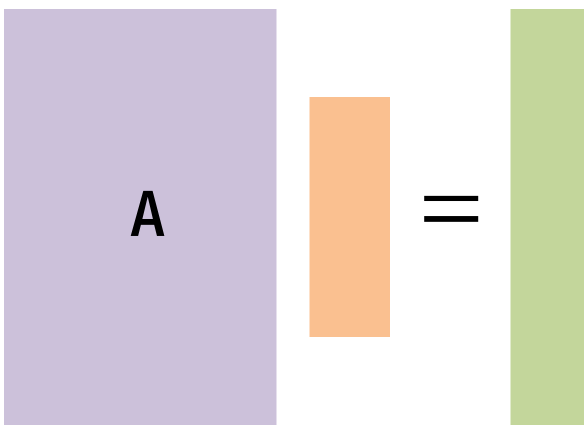
$$= \left(\mathbf{b} - \mathbf{Ax}\right)^{\top}\left(\mathbf{b} - \mathbf{Ax}\right) = \mathbf{b}^{\top}\mathbf{b} - 2\mathbf{b}^{\top}\mathbf{Ax} + \mathbf{x}^{\top}\mathbf{A}^{\top}\mathbf{Ax}$$

$$\frac{\partial E}{\partial \mathbf{x}} = -2\mathbf{b}^{\top}\mathbf{A} + 2\mathbf{A}^{\top}\mathbf{Ax} = \mathbf{0} \;\longrightarrow\; \mathbf{A}^{\top}\mathbf{Ax} = \mathbf{b}^{\top}\mathbf{A}$$

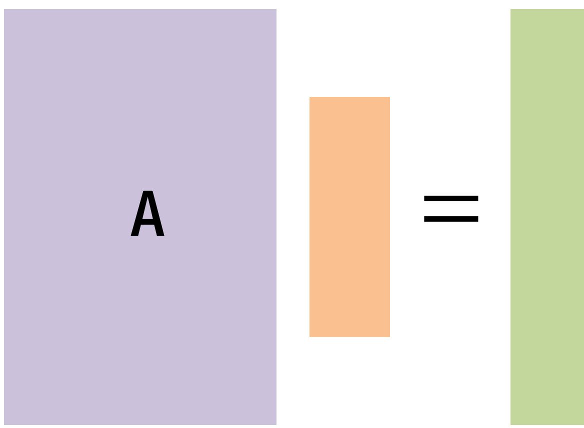
Linear Inhomogeneous Equations



Linear Inhomogeneous Equations



Linear Inhomogeneous Equations



1 $\cdots$

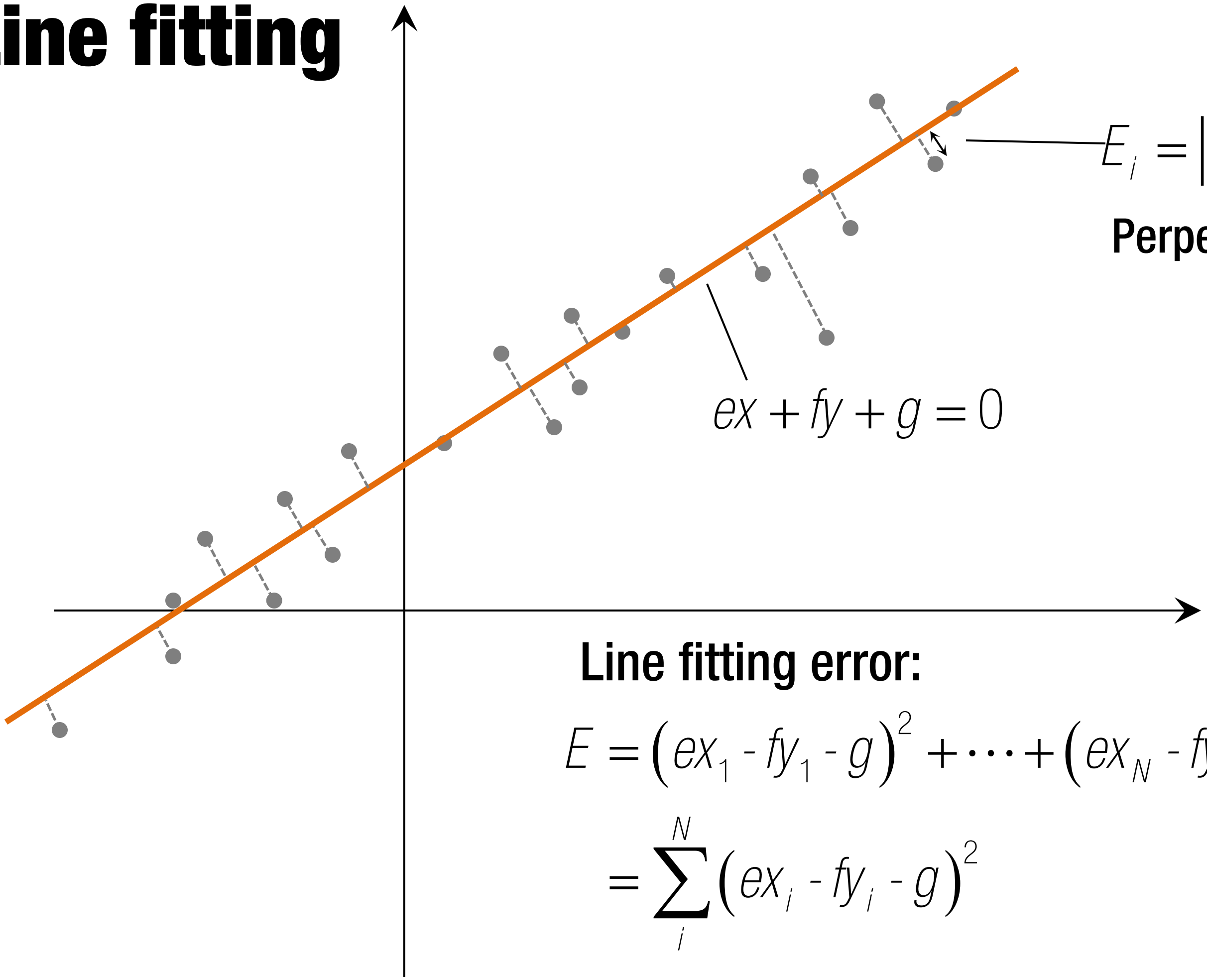
$$x = A^{-1}b$$

Two types of Least Square Problem:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathsf{A}\mathbf{x} - \mathbf{b}\|^2 \qquad \text{with} \qquad \|\mathbf{b}\| \neq 0$$

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathsf{A}\mathbf{x}\|^2 \qquad \text{s.t.} \qquad \|\mathbf{x}\| = 1}$$

Line fitting

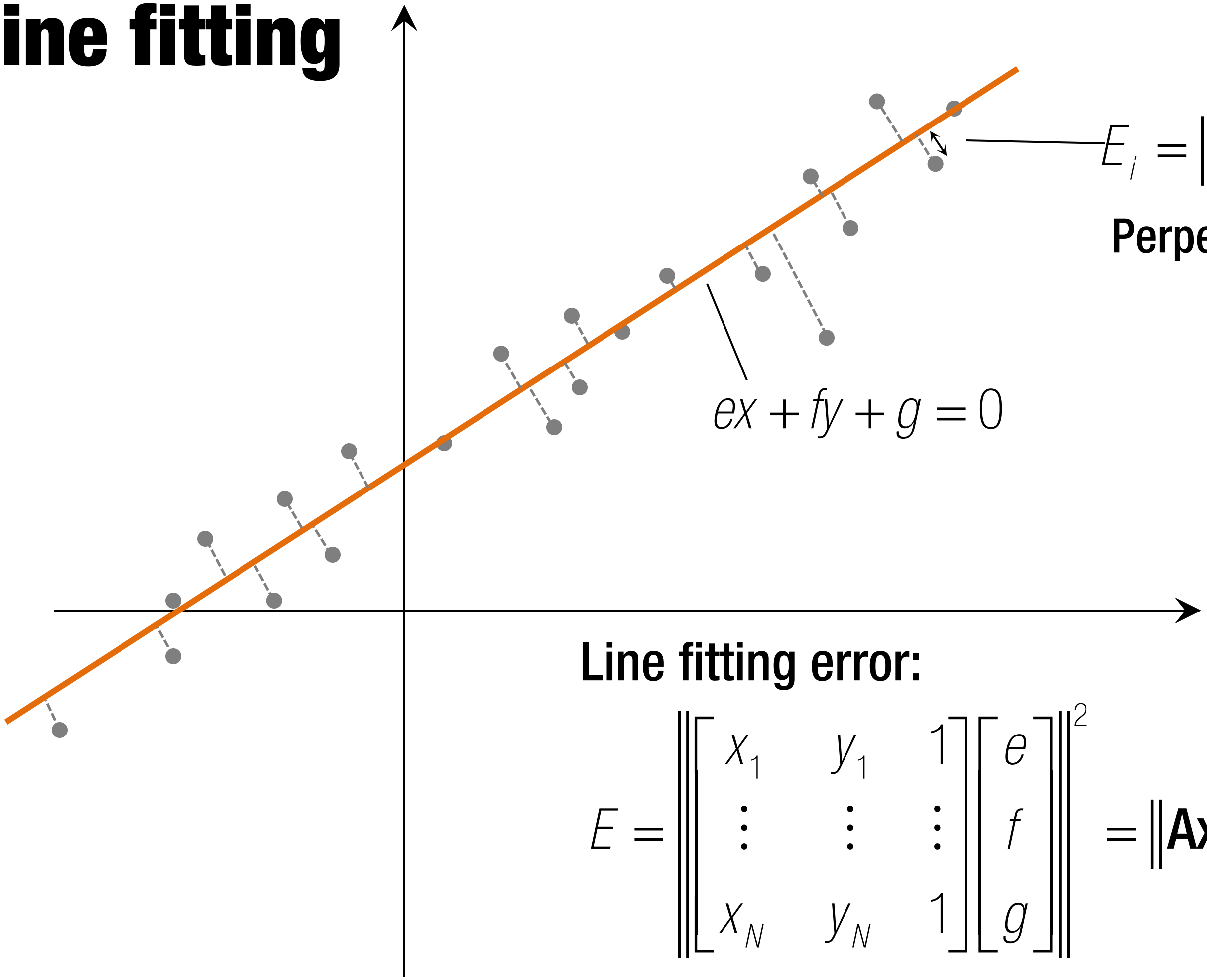


Line fitting error:

$$E = (ex_1 - fy_1 - g)^2 + \cdots + (ex_N - fy_N - g)^2$$

$$= \sum_{i}^{N} (ex_i - fy_i - g)^2$$

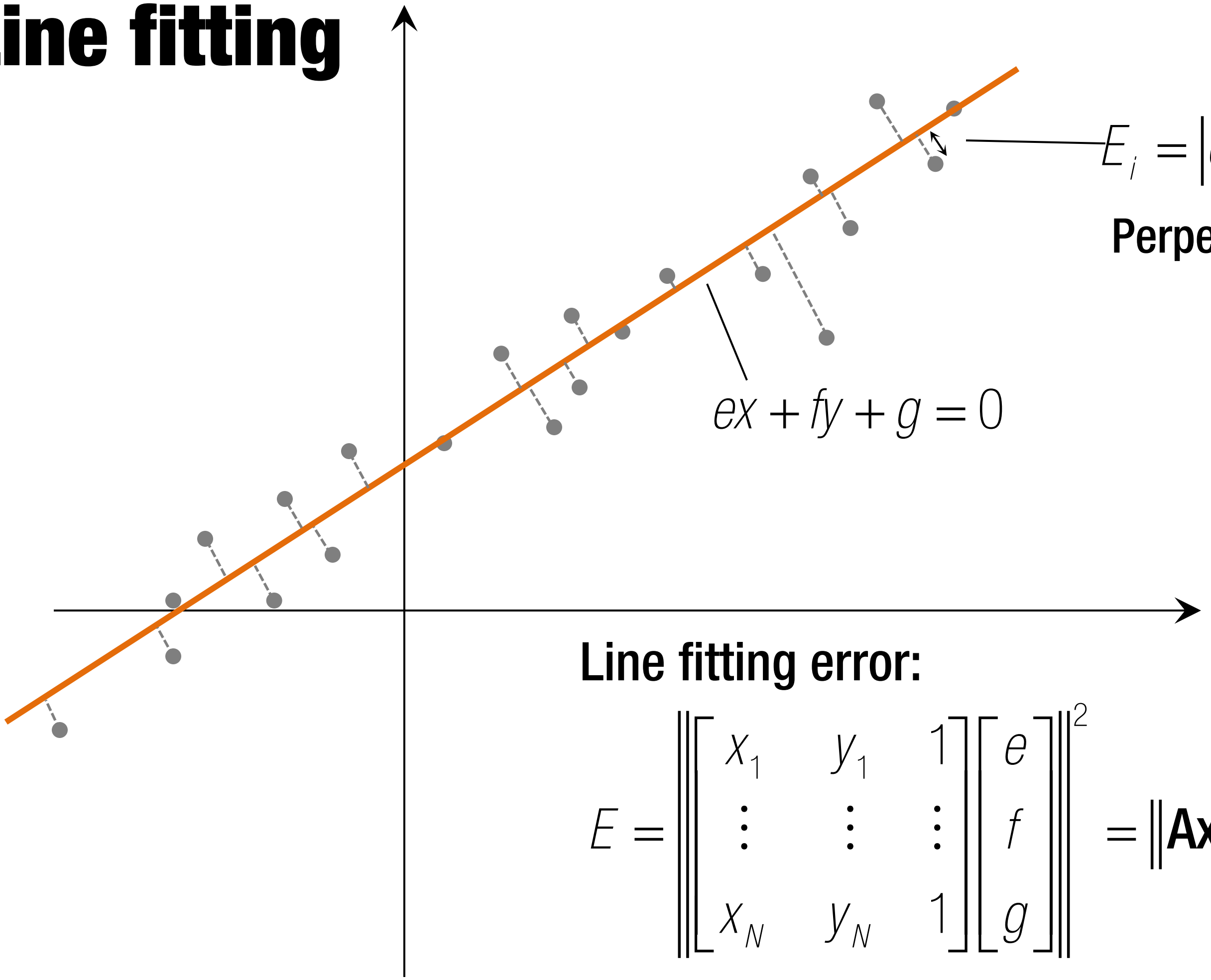
Line fitting



Line fitting error:

$$E = \left\| \begin{bmatrix} x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & 1 \end{bmatrix} \begin{bmatrix} e \\ f \\ g \end{bmatrix} \right\|^2 = \left\| \mathbf{Ax} \right\|^2$$

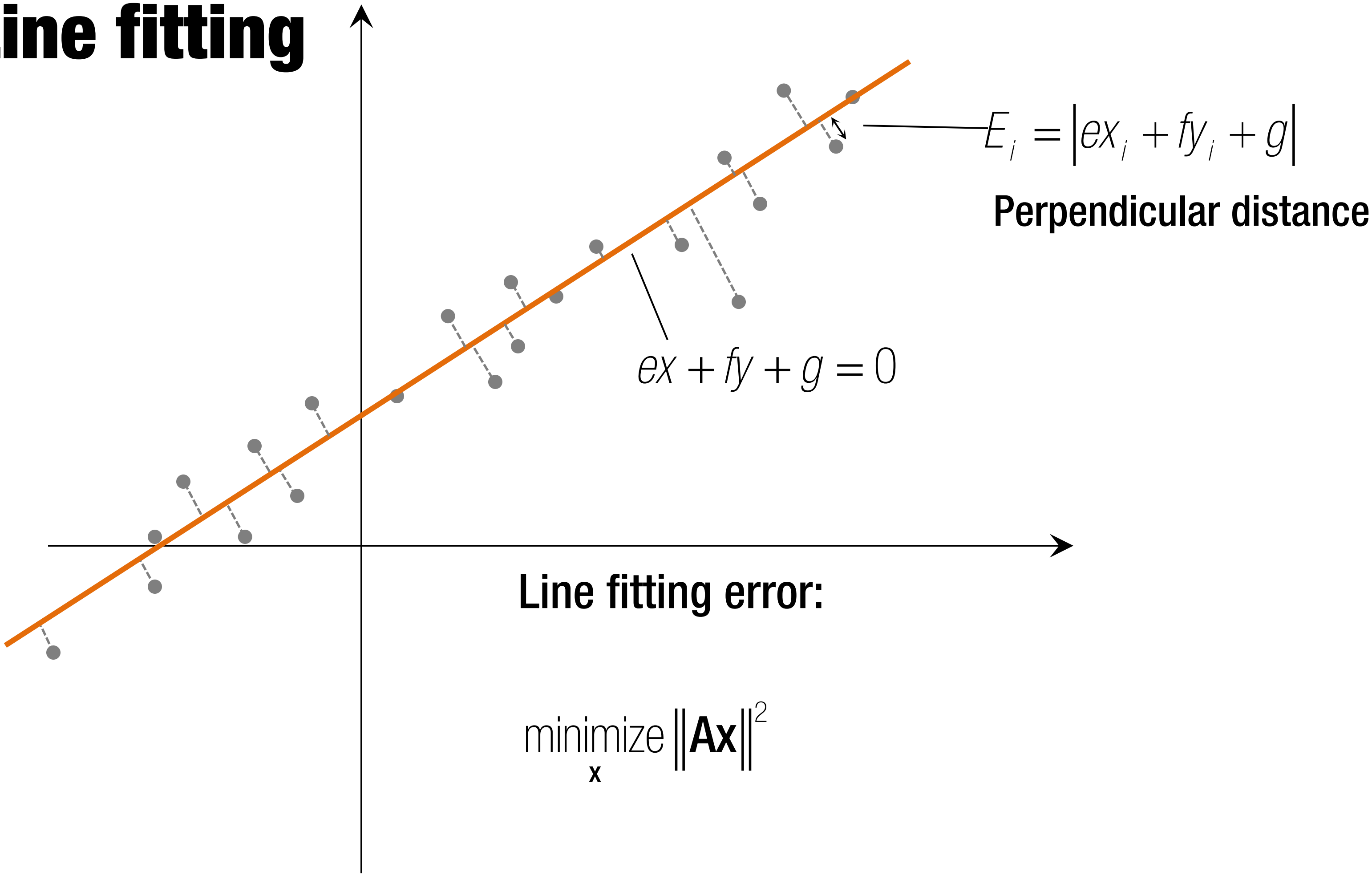
Line fitting



Line fitting error:

$$E = \left\| \begin{bmatrix} x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & 1 \end{bmatrix} \begin{bmatrix} e \\ f \\ g \end{bmatrix} \right\|^2 = \|\mathbf{Ax}\|^2$$

Line fitting

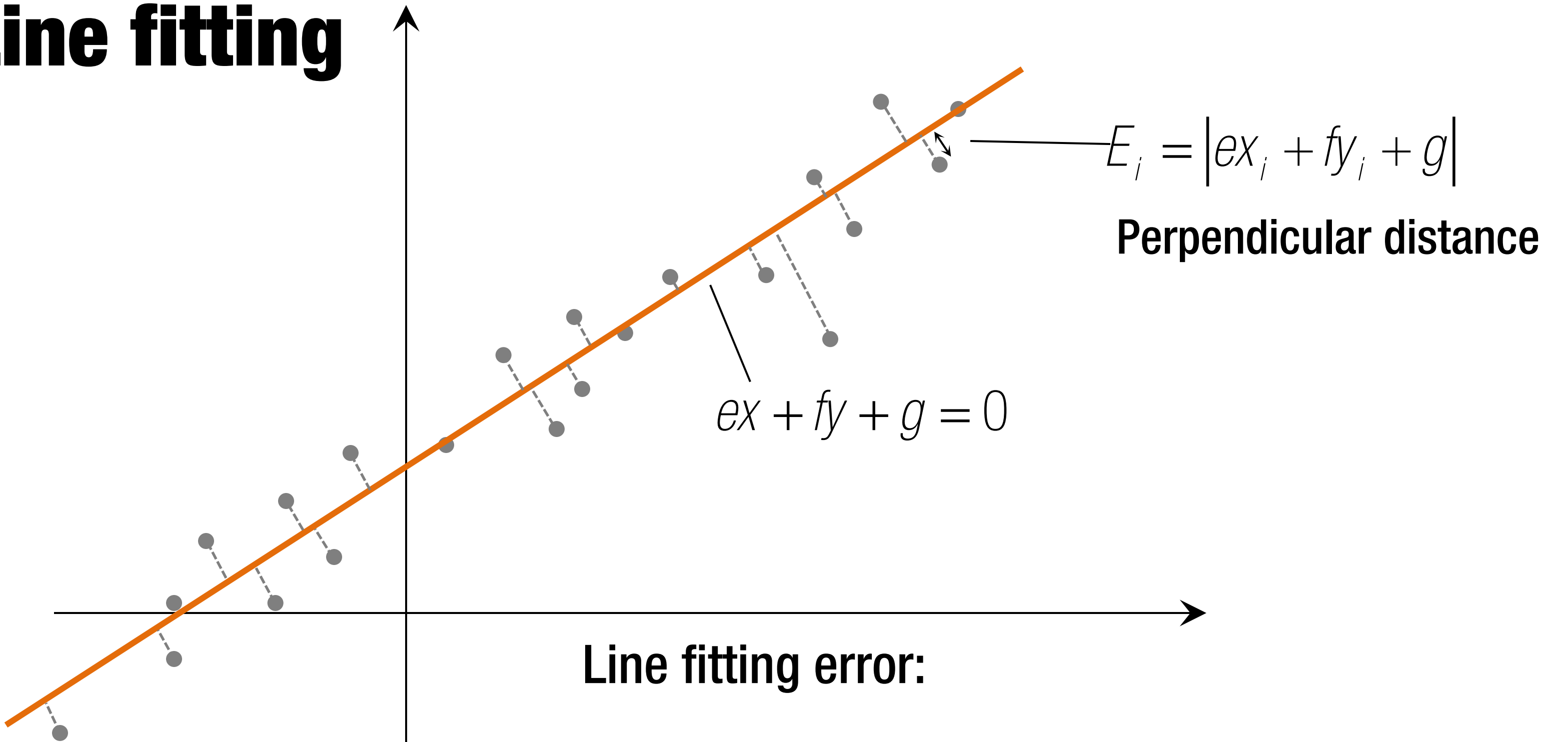


Line fitting error:

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

Trivial solution: $\mathbf{x} = \mathbf{0}$

Line fitting



Line fitting error:

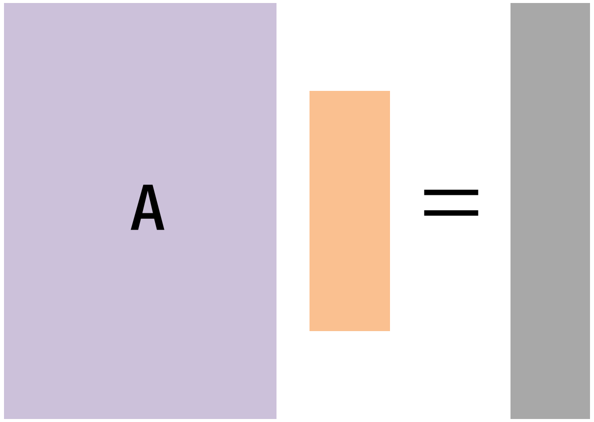
$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

$$\text{subject to } \|\mathbf{x}\| = 1$$

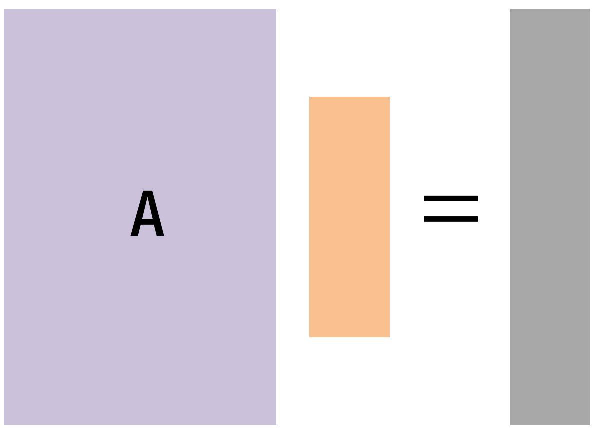
$$\mathbf{x} = \mathbf{V}_3 \quad \text{where } \mathbf{A} = \mathbf{UDV}^{\mathsf{T}}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 \end{bmatrix}$$

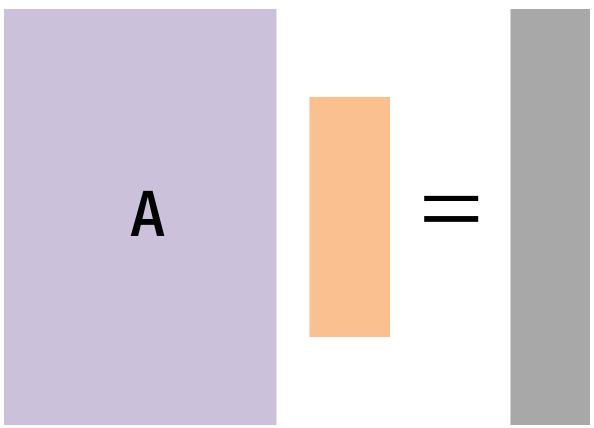
Linear Homogeneous Equations



Linear Homogeneous Equations



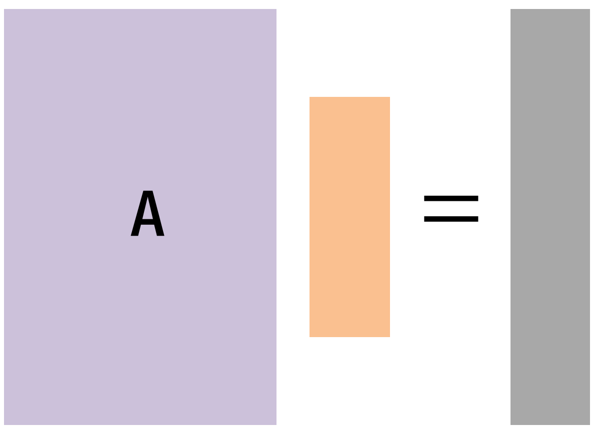
Linear Homogeneous Equations



lution

$$\mathbf{x} = \mathbf{V}_n$$

Linear Homogeneous Equations



lution

$$\mathbf{x} = \mathbf{V}_n$$