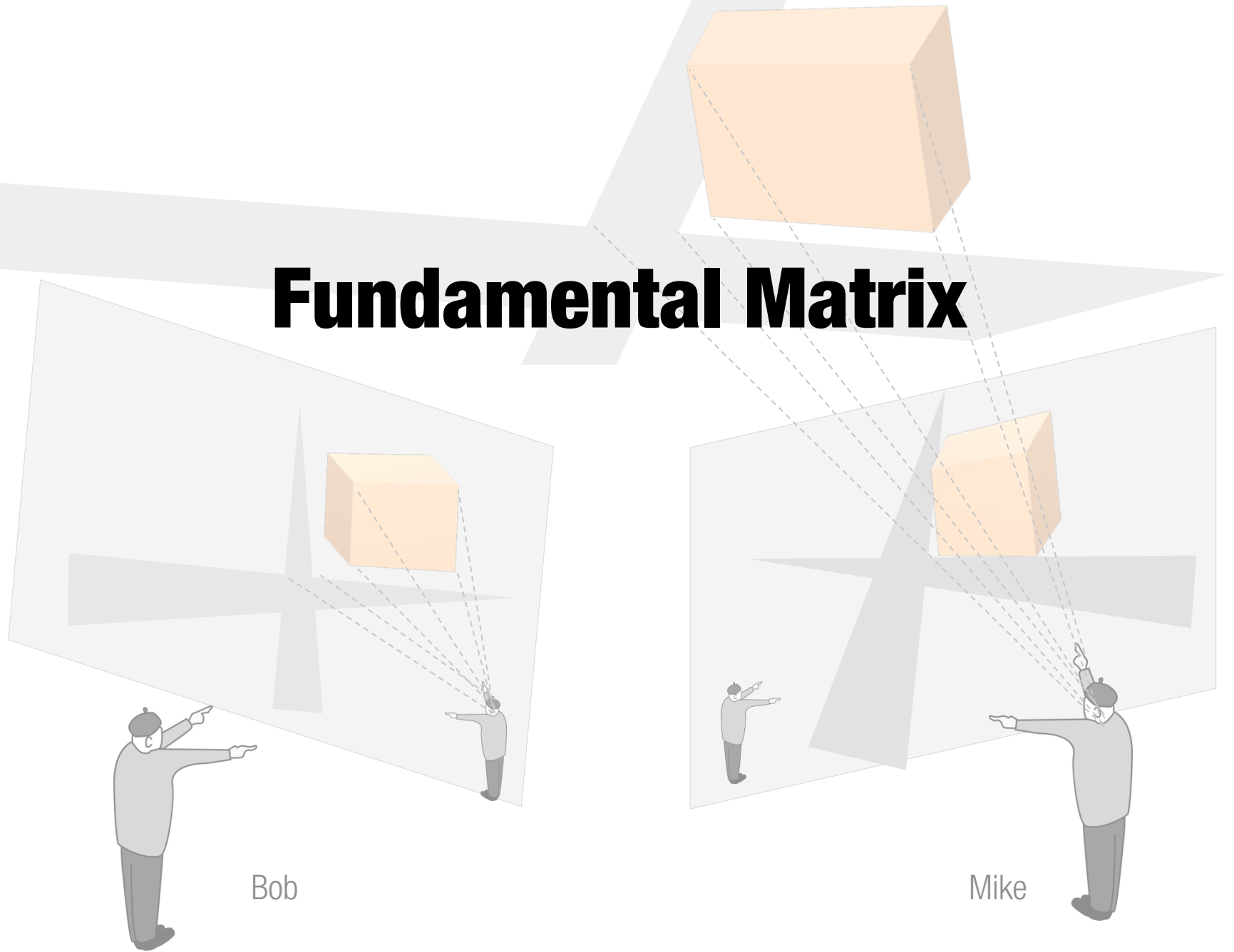
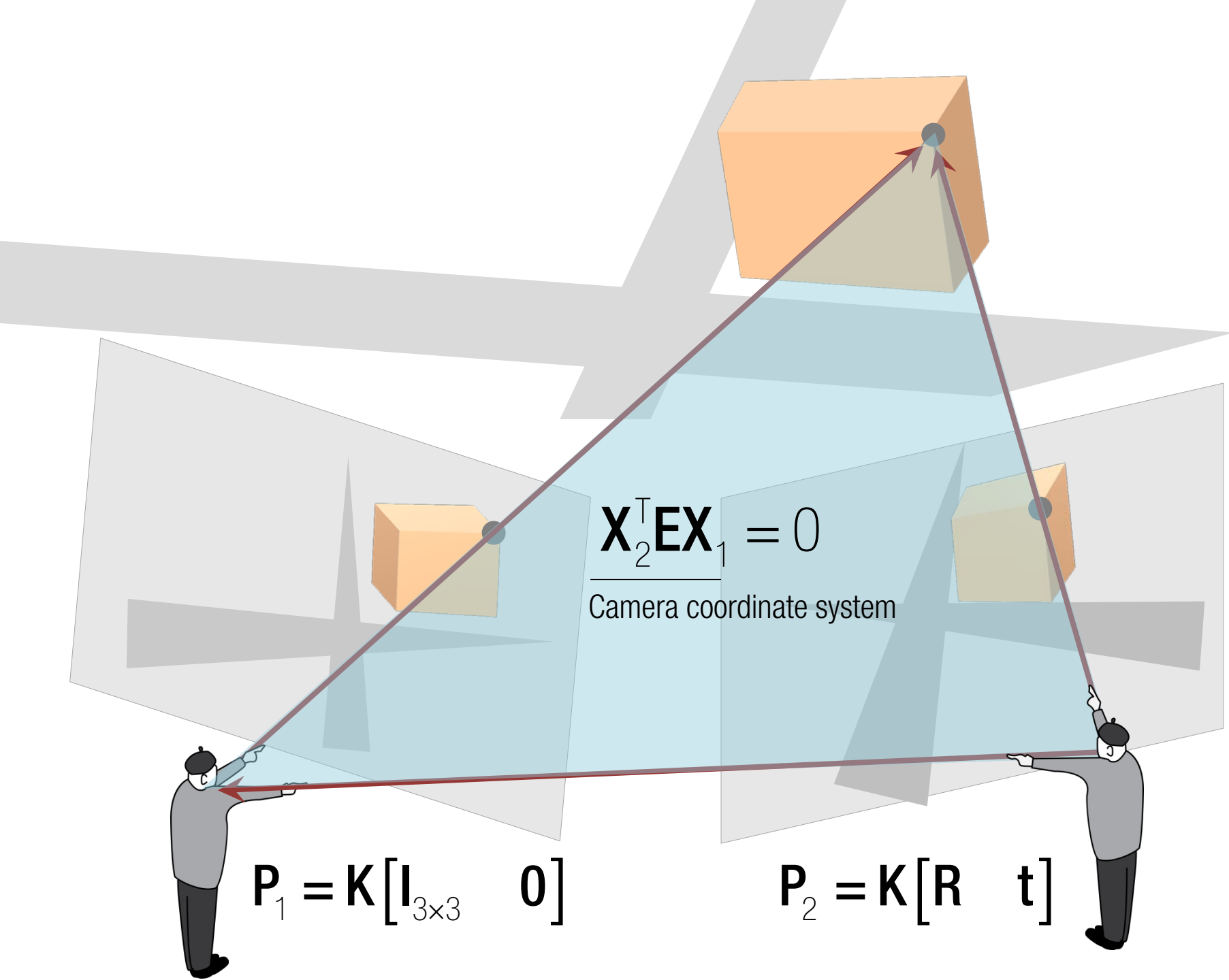


Fundamental Matrix





$$\mathbf{X}_2^T \mathbf{E} \mathbf{X}_1 = 0$$

Camera coordinate system

$$\mathbf{P}_1 = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{P}_2 = \mathbf{K} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$

A diagram illustrating the geometry of camera projection. Two cameras, represented by stylized figures, are positioned at different locations and orientations. They project a 3D scene, which includes an orange cube and a yellow cube, onto a common image plane. The projection rays from the camera centers through the image points converge at the corresponding points in the 3D scene. A light blue trapezoidal region represents the image plane. The diagram is used to explain the mathematical relationships between camera coordinates, image coordinates, and the projection matrices.
$$\mathbf{x}_2^T \mathbf{E} \mathbf{x}_1 = 0$$

Camera coordinate system

$$\mathbf{x}_1 = \mathbf{K} \mathbf{X}_1, \quad \mathbf{x}_2 = \mathbf{K} \mathbf{X}_2$$

Transformation from camera to image

$$\mathbf{P}_1 = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{P}_2 = \mathbf{K} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$


$$\mathbf{x}_2^T \mathbf{K}^{-T} \mathbf{E} \mathbf{K}^{-1} \mathbf{x}_1 = 0$$

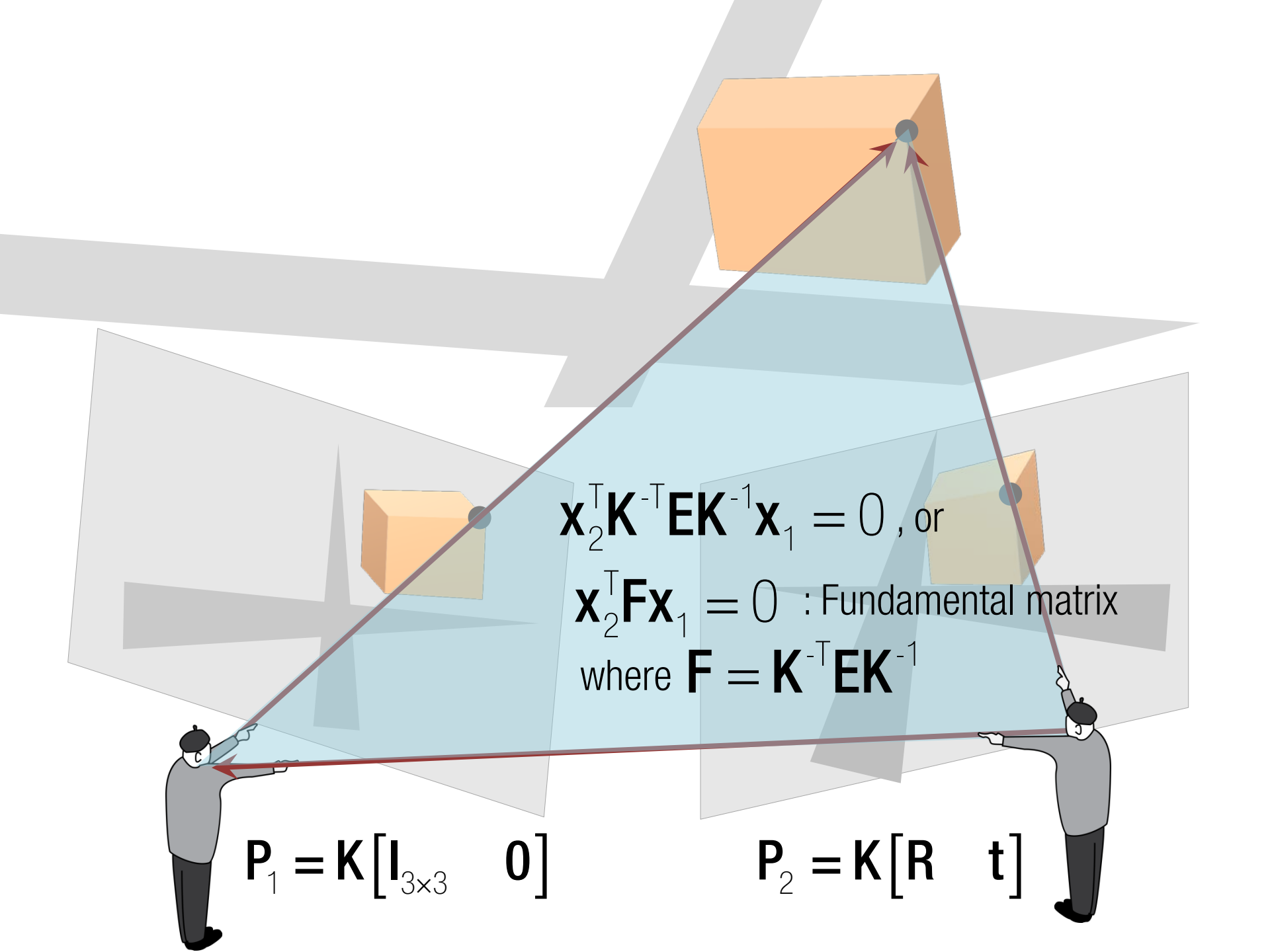
Image coordinate system

$$\mathbf{x}_1 = \mathbf{K} \mathbf{X}_1, \quad \mathbf{x}_2 = \mathbf{K} \mathbf{X}_2$$

Transformation from camera to image

$$\mathbf{P}_1 = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{P}_2 = \mathbf{K} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$



The diagram illustrates the geometry of two cameras, P_1 and P_2 , observing a 3D scene. A semi-transparent blue plane, representing the fundamental matrix, is positioned between the cameras and the scene. This plane contains the epipoles of both cameras, marked with blue dots. Red lines represent the projection rays from each camera through the epipoles to corresponding points in the 3D scene. The scene includes a large orange cube and two smaller yellow cubes. The background shows a stylized city street with buildings and a road.

$$\mathbf{x}_2^T \mathbf{K}^{-T} \mathbf{E} \mathbf{K}^{-1} \mathbf{x}_1 = 0, \text{ or}$$

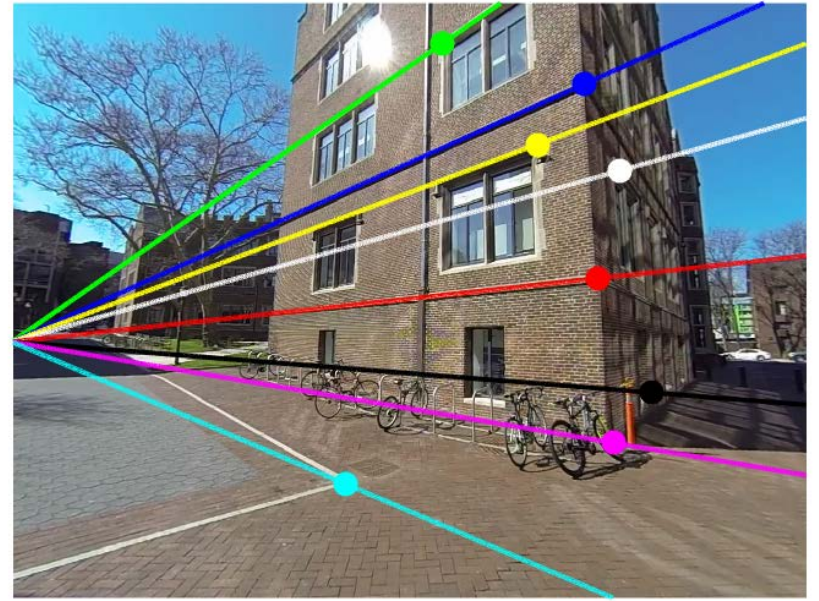
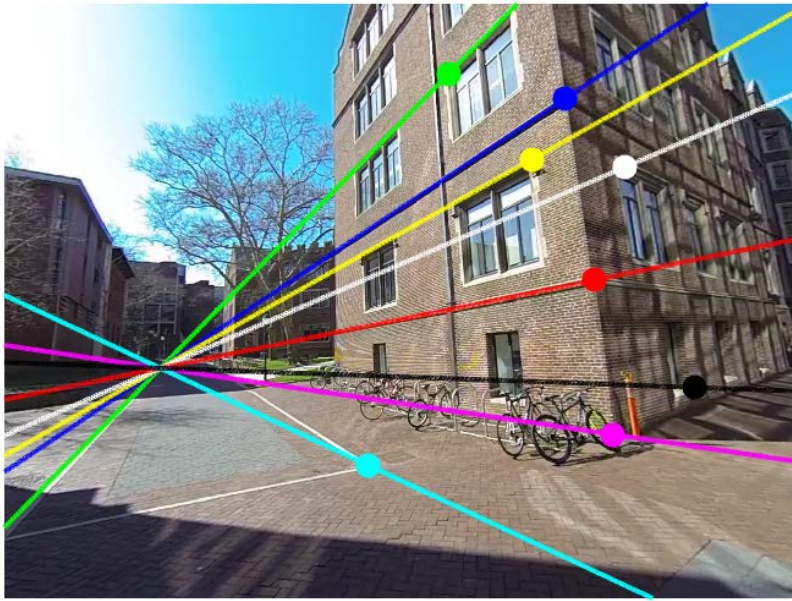
$$\mathbf{x}_2^T \mathbf{F} \mathbf{x}_1 = 0 : \text{Fundamental matrix}$$

$$\text{where } \mathbf{F} = \mathbf{K}^{-T} \mathbf{E} \mathbf{K}^{-1}$$

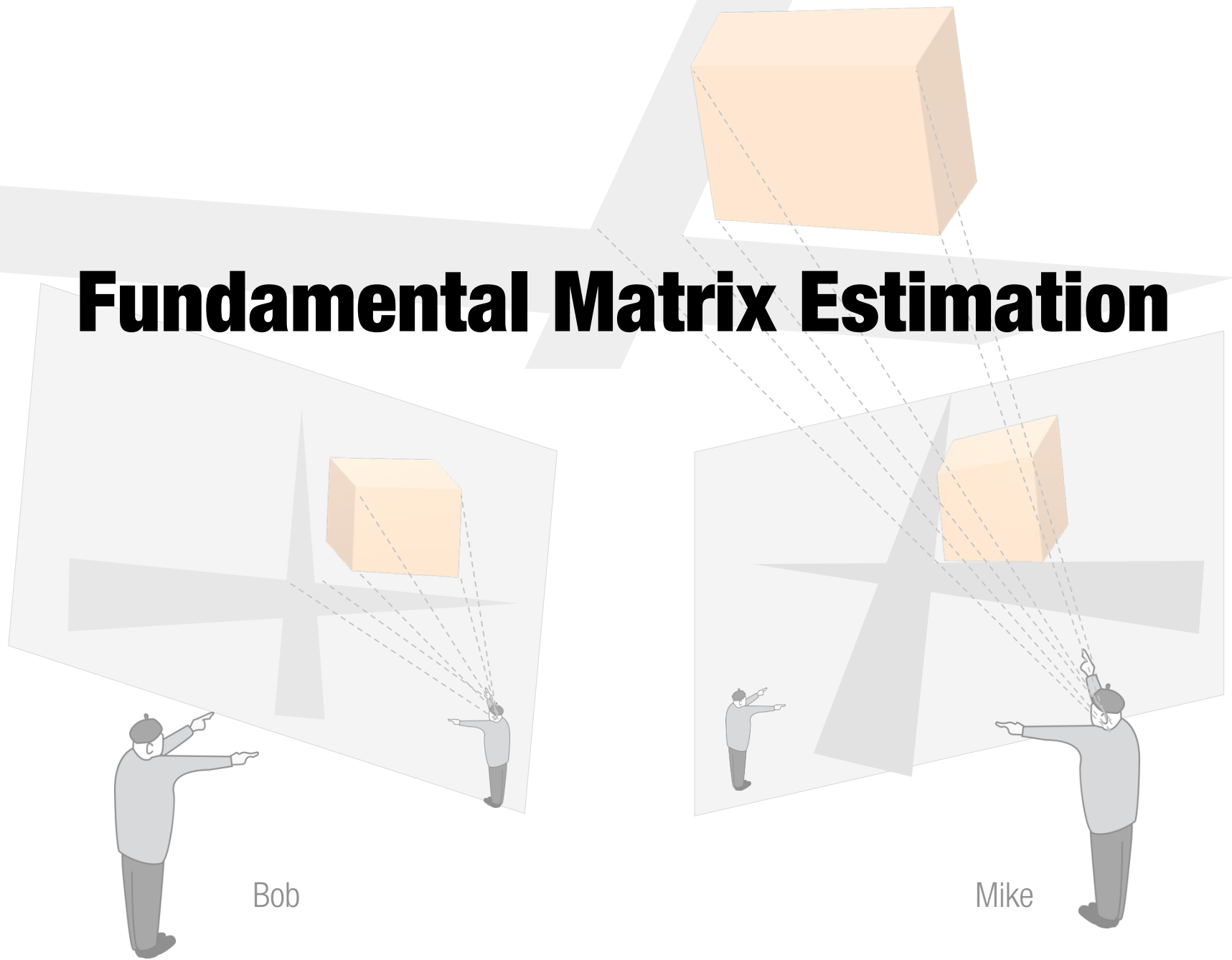
$$\mathbf{P}_1 = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{P}_2 = \mathbf{K} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$





Fundamental Matrix Estimation



Fundamental Matrix

$$\mathbf{x}_2^T \mathbf{F} \mathbf{x}_1 = 0$$

$$\mathbf{F} \in \mathbb{R}^{3 \times 3}$$

$$\text{rank}(\mathbf{F}) = 2$$

$\underbrace{\hspace{1.5cm}}$ Matrix dimensions
 Degree of freedoms: $\underbrace{3 \times 3 - 1}_{\text{scale factor}} = 8$

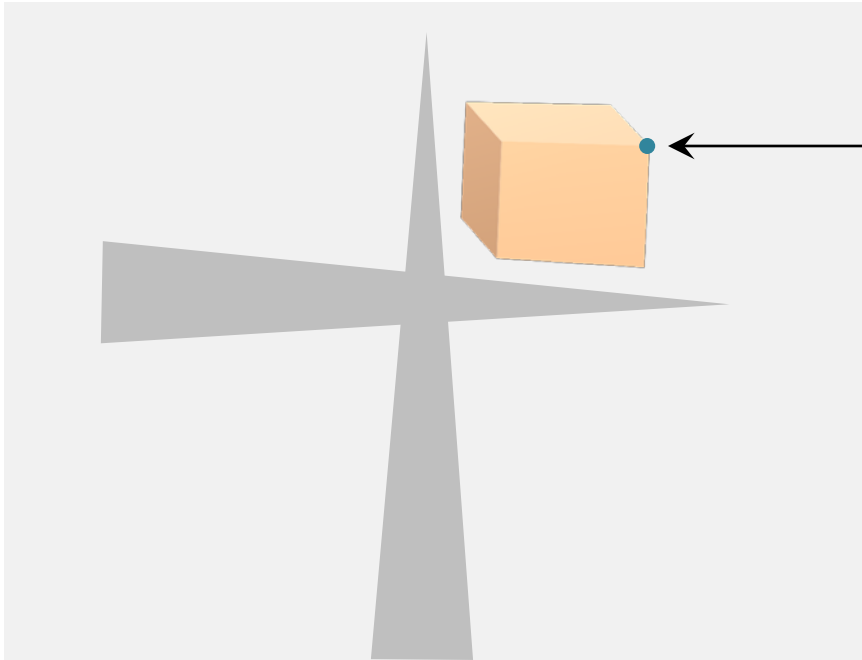
of unknowns: 8

of required equations: 8

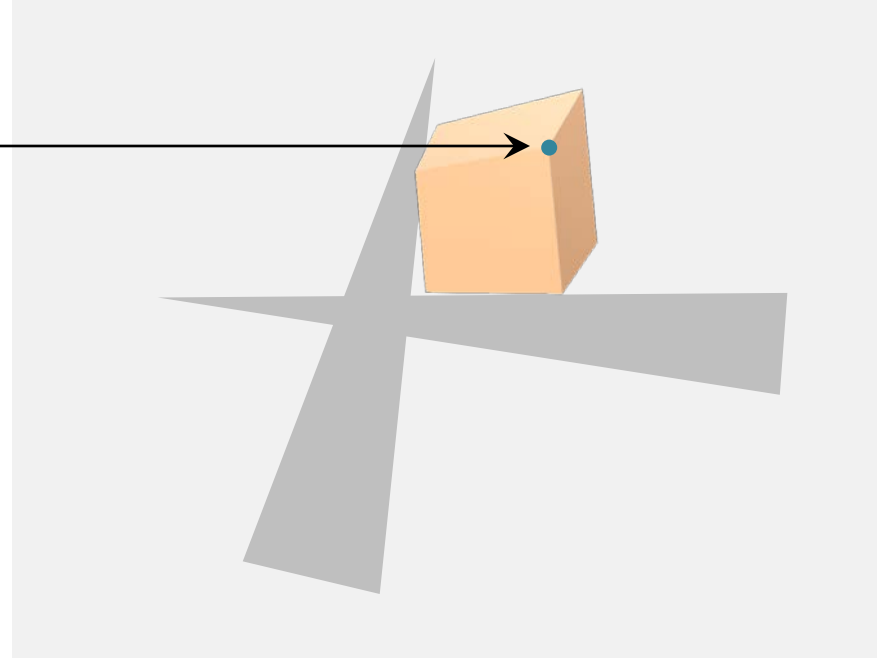
$$\begin{array}{c} \mathbf{x}_{2,1}^T \mathbf{F} \mathbf{x}_{1,1} = 0 \\ \vdots \\ \mathbf{x}_{2,8}^T \mathbf{F} \mathbf{x}_{1,8} = 0 \end{array}$$

8 correspondences

Point correspondence

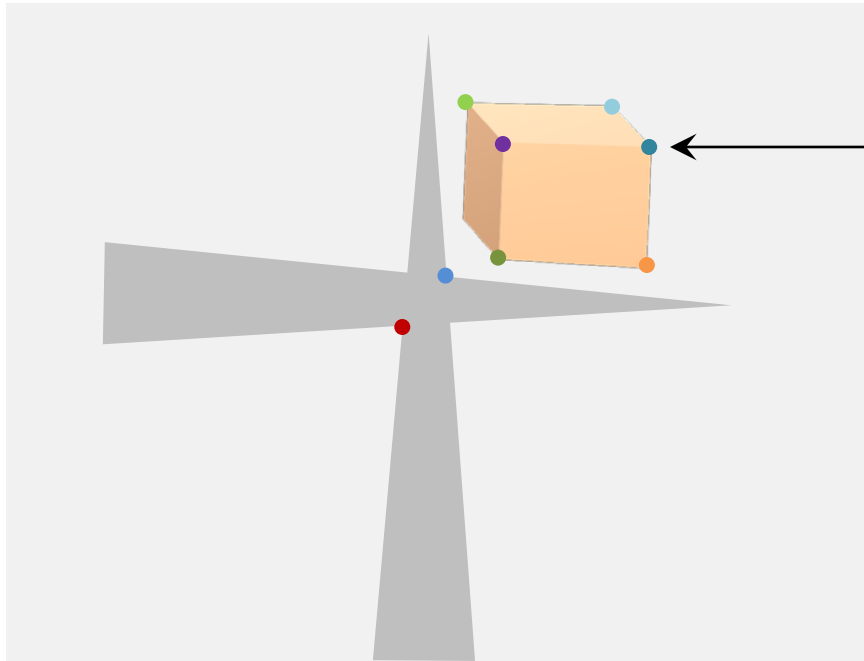


Bob's view

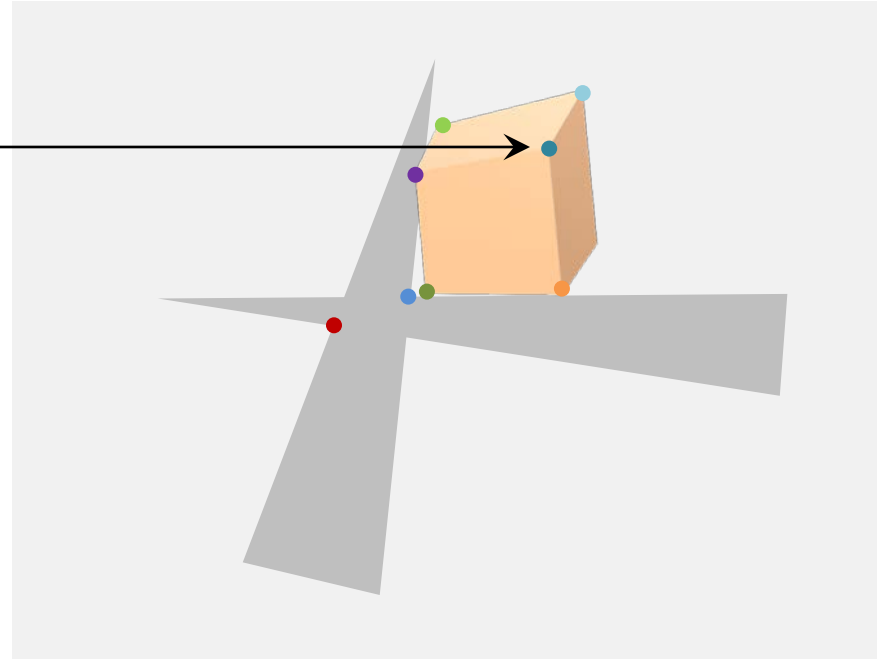


Mike's view

Point correspondence



Bob's view



Mike's view

8 correspondences

$$\mathbf{x}_2^T \mathbf{F} \mathbf{x}_1 = 0$$

$$\begin{bmatrix} u_i^2 & v_i^2 & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u_i^1 \\ v_i^1 \\ 1 \end{bmatrix} = 0$$

Linear equation in F:

$$u_i^1 u_i^2 f_{11} + u_i^1 v_i^2 f_{21} + u_i^1 f_{31} + v_i^1 u_i^2 f_{12} + v_i^1 v_i^2 f_{22} + v_i^1 f_{32} + u_i^2 f_{13} + v_i^2 f_{23} + f_{33} = 0$$

$$\mathbf{x}_2^\top \mathbf{F} \mathbf{x}_1 = 0$$

$$\begin{bmatrix} u_i^2 & v_i^2 & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u_i^1 \\ v_i^1 \\ 1 \end{bmatrix} = 0$$

Linear equation in F:

$$\begin{bmatrix} u_1^1 u_1^2 & u_1^1 v_1^2 & u_1^1 & v_1^1 u_1^2 & v_1^1 v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_8^1 u_8^2 & u_8^1 v_8^2 & u_8^1 & v_8^1 u_8^2 & v_8^1 v_8^2 & v_8^1 & u_8^2 & v_8^2 & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

$$\mathbf{x}_2^\top \mathbf{F} \mathbf{x}_1 = 0$$

$$\begin{bmatrix} u_i^2 & v_i^2 & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u_i^1 \\ v_i^1 \\ 1 \end{bmatrix} = 0$$

Linear equation in F:

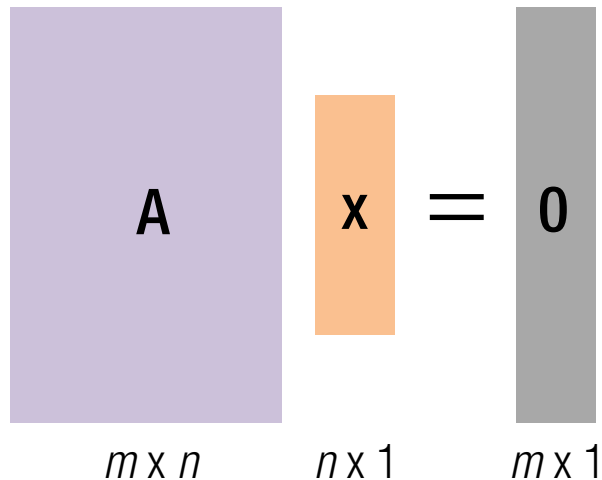
$$\begin{bmatrix} u_1^1 u_1^2 & u_1^1 v_1^2 & u_1^1 & v_1^1 u_1^2 & v_1^1 v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_8^1 u_8^2 & u_8^1 v_8^2 & u_8^1 & v_8^1 u_8^2 & v_8^1 v_8^2 & v_8^1 & u_8^2 & v_8^2 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

Linear Homogeneous Equations

Linear least square solve produces a trivial solution:

$$\mathbf{x} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b} \rightarrow \mathbf{x} = \mathbf{0}$$

An additional constraint on \mathbf{x} to avoid the trivial solution: $\|\mathbf{x}\| = 1$



A diagram illustrating the matrix equation $\mathbf{A}\mathbf{x} = \mathbf{0}$. It consists of three vertical rectangles: a purple one labeled \mathbf{A} with dimensions $m \times n$ below it, an orange one labeled \mathbf{x} with dimensions $n \times 1$ below it, and a gray one labeled $\mathbf{0}$ with dimensions $m \times 1$ below it. An equals sign is placed between the orange and gray rectangles.

1) $\text{rank}(\mathbf{A}) = r < n - 1$: infinite number of solutions

$$\mathbf{x} = \lambda_{r+1} \mathbf{V}_{r+1} + \dots + \lambda_n \mathbf{V}_n \quad \text{where} \quad \sum_{i=r+1}^n \lambda_i^2 = 1$$

2) $\text{rank}(\mathbf{A}) = n - 1$: one exact solution

$$\mathbf{x} = \mathbf{V}_n$$

3) $n < m$: no exact solution in general (needs least squares)

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \quad \text{subject to} \quad \|\mathbf{x}\| = 1 \rightarrow \mathbf{x} = \mathbf{V}_n$$

8 Point Algorithm

- Construct 8x9 matrix **A**.

$$\mathbf{A} = \begin{bmatrix} u_1^1 u_1^2 & u_1^1 v_1^2 & u_1^1 & v_1^1 u_1^2 & v_1^1 v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_8^1 u_8^2 & u_8^1 v_8^2 & u_8^1 & v_8^1 u_8^2 & v_8^1 v_8^2 & v_8^1 & u_8^2 & v_8^2 & 1 \end{bmatrix}$$

8 Point Algorithm

- Construct 8x9 matrix **A**.
- Solving linear homogeneous equations via SVD:
 $\mathbf{x} = \mathbf{V}_{:,9}$ where $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$
 $\mathbf{F} = \text{reshape}(\mathbf{x}, 3, 3)$: constructing matrix from vector.

8 Point Algorithm

- Construct 8x9 matrix **A**.
- Solving linear homogeneous equations via SVD:
 $\mathbf{x} = \mathbf{V}_{:,8}$ where $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$
 $\mathbf{F} = \text{reshape}(\mathbf{x}, 3, 3)$: constructing matrix from vector.
- Applying rank constraint, i.e., $\text{rank}(\mathbf{F}) = 2$.

$$\mathbf{F}_{\text{rank2}} = \mathbf{U}\tilde{\mathbf{D}}\mathbf{V}^\top \text{ where } \tilde{\mathbf{D}} : \mathbf{D} \text{ with the last element zero.}$$

$$\mathbf{F}_{\text{rank2}} = \mathbf{U} \tilde{\mathbf{D}} \mathbf{V}^\top \quad \mathbf{F} = \mathbf{U} \mathbf{D} \mathbf{V}^\top$$

SVD cleanup

1.2 Match Outlier Rejection via RANSAC

Goal Given N correspondences between two images ($N \geq 8$), $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$, implement the following function that estimates inlier correspondences using fundamental matrix based RANSAC:

`[y1 y2 idx] = GetInliersRANSAC(x1, x2)`

(INPUT) \mathbf{x}_1 and \mathbf{x}_2 : $N \times 2$ matrices whose row represents a correspondence.

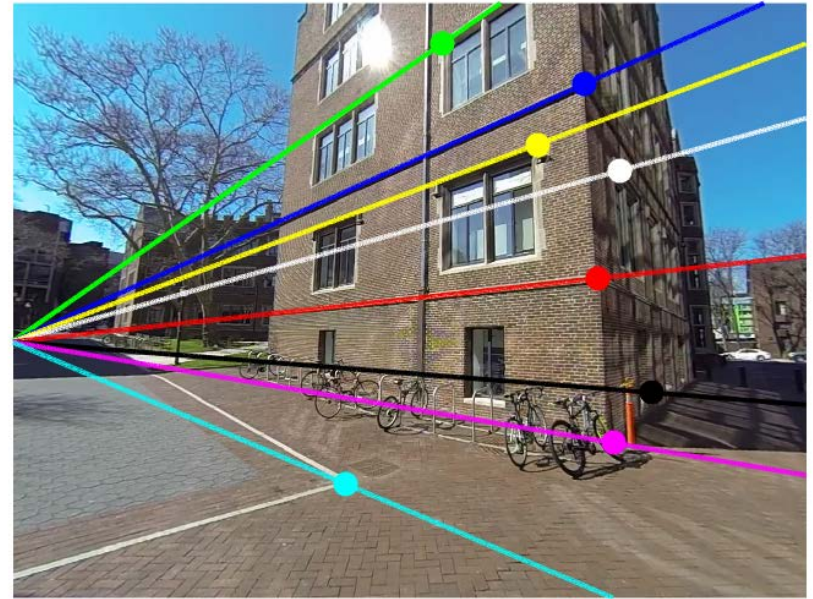
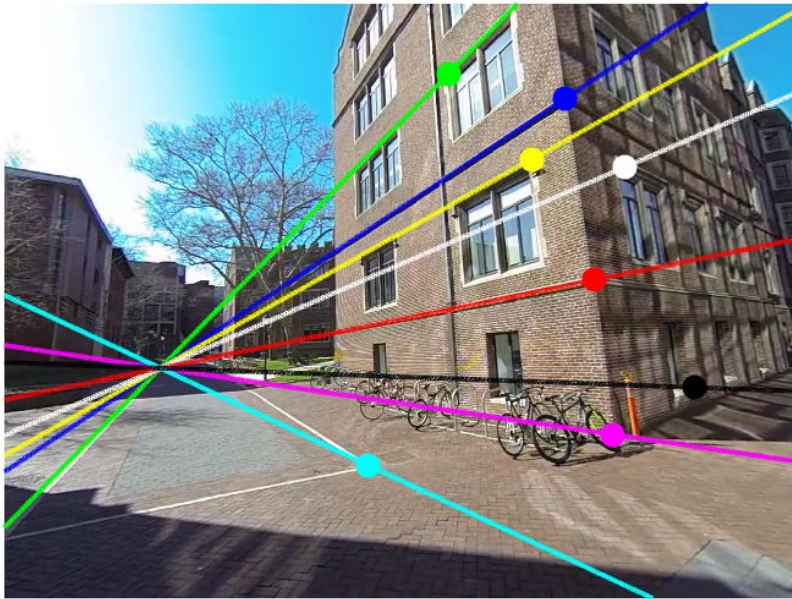
(OUTPUT) \mathbf{y}_1 and \mathbf{y}_2 : $N_i \times 2$ matrices whose row represents an inlier correspondence where N_i is the number of inliers.

(OUTPUT) \mathbf{idx} : $N \times 1$ vector that indicates ID of inlier \mathbf{y}_1 .

A pseudo code the RANSAC is shown in Algorithm 2.

Algorithm 2 GetInliersRANSAC

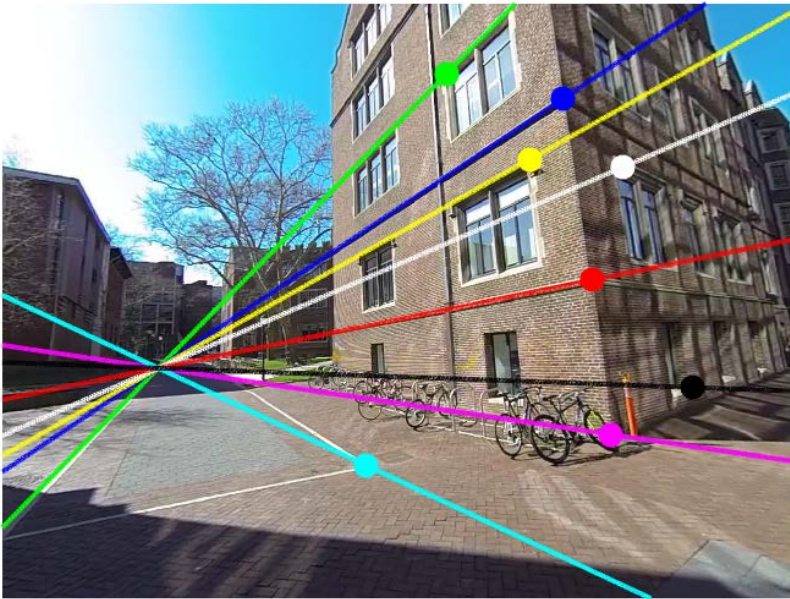
```
1:  $n \leftarrow 0$ 
2: for  $i = 1 : M$  do
3:   Choose 8 correspondences,  $\hat{\mathbf{x}}_1$  and  $\hat{\mathbf{x}}_2$ , randomly
4:    $\mathbf{F} = \text{EstimateFundamentalMatrix}(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ 
5:    $\mathcal{S} \leftarrow \emptyset$ 
6:   for  $j = 1 : N$  do
7:     if  $|\mathbf{x}_{2j}^\top \mathbf{F} \mathbf{x}_{1j}| < \epsilon$  then
8:        $\mathcal{S} \leftarrow \mathcal{S} \cup \{j\}$ 
9:     end if
10:  end for
11:  if  $n < |\mathcal{S}|$  then
12:     $n \leftarrow |\mathcal{S}|$ 
13:     $\mathcal{S}_{in} \leftarrow \mathcal{S}$ 
14:  end if
15: end for
```



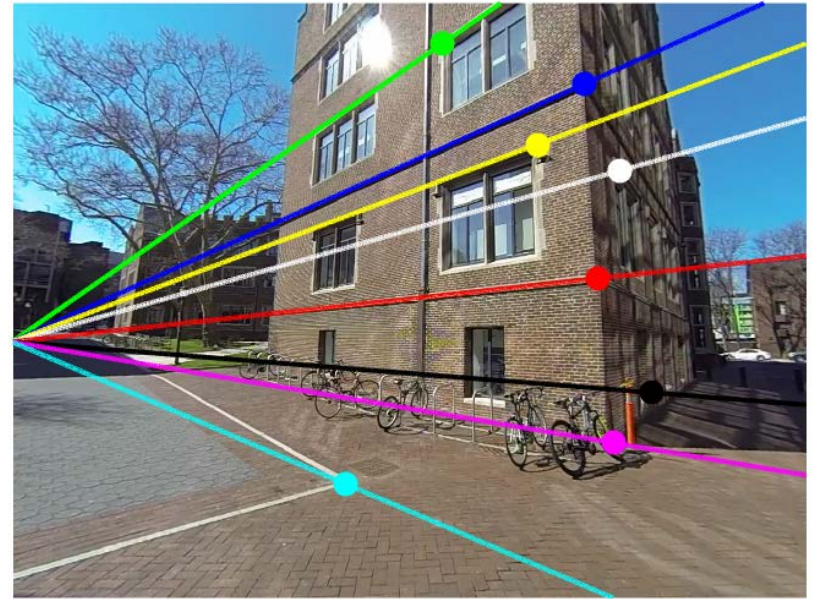
$F =$

$1.0e+003 *$

| | | |
|---------|---------|---------|
| 0.0000 | 0.0001 | -0.0463 |
| -0.0001 | 0.0000 | 0.0181 |
| 0.0519 | -0.0043 | -9.9997 |



```
>> rank(F)
ans =
     3
>> [u,d,v] = svd(F);
>> d(3,3) = 0;
```



```
>> F = u * d * v'      : SVD cleanup
F =
  1.0e+003 *
    0.0000    0.0001   -0.0463
   -0.0001    0.0000    0.0181
    0.0519   -0.0043   -9.9997
>> rank(F)
ans =
     2
```



$$\mathbf{x}_1 = \begin{bmatrix} 950 \\ 450 \end{bmatrix}$$



$$\mathbf{L}_2 = \begin{bmatrix} -0.1024 & -0.9947 & 547.0942 \end{bmatrix}$$

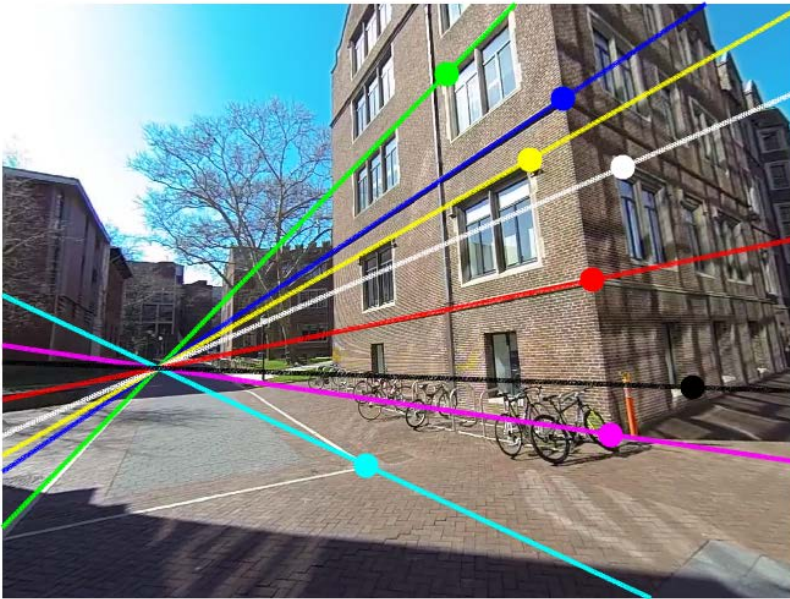
$$\mathbf{L}_2 = \mathbf{F} \mathbf{x}_1$$



$$\mathbf{L}_1 = \begin{bmatrix} 0.5489 & 0.8359 & -627.0515 \end{bmatrix}$$

$$\mathbf{L}_1 = \mathbf{F}^T \mathbf{x}_2$$

$$\mathbf{x}_2 = \begin{bmatrix} 920 & 130 \end{bmatrix}$$



$[u, d] = \text{eigs}(F^T F);$

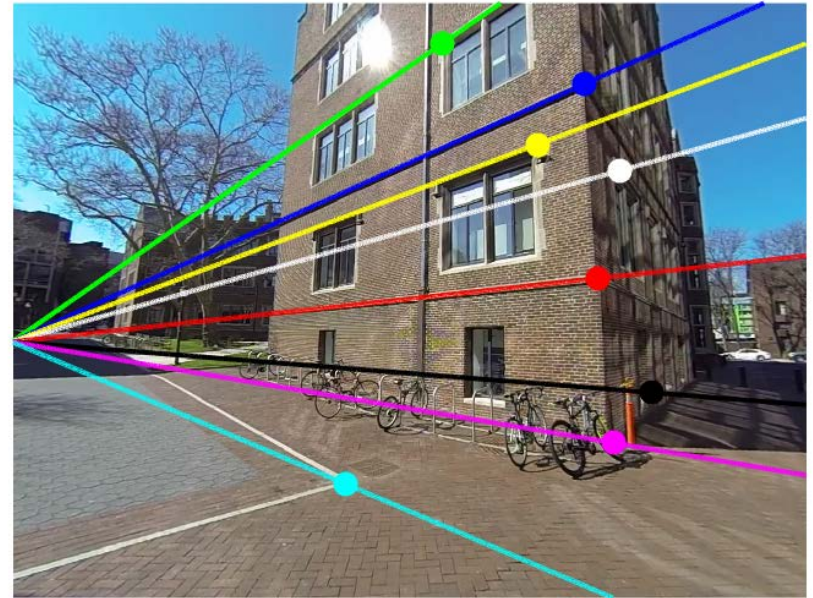
$u =$

| | | |
|---------|---------|---------|
| -0.0052 | 0.9258 | -0.3780 |
| 0.0004 | -0.3780 | -0.9258 |
| 1.0000 | 0.0050 | -0.0016 |

$d =$

| | | |
|--------|------------|-------------|
| 1.0000 | 0 | 0 |
| 0 | 6.4719e-10 | 0 |
| 0 | 0 | -7.6511e-22 |

$uu = u(:, 3) = [-0.3780, -0.9258, -0.0016]$



$[u,d] = \text{eigs}(F^*F');$

$u =$

| | | |
|---------|---------|---------|
| 0.0046 | 1.0000 | 0.0029 |
| -0.0018 | 0.0029 | -1.0000 |
| 1.0000 | -0.0046 | -0.0018 |

$d =$

| | | |
|--------|------------|-------------|
| 1.0000 | 0 | 0 |
| 0 | 6.4719e-10 | 0 |
| 0 | 0 | -5.6583e-21 |

$uu = u(:, 3) = [0.0029, -1.0000, -0.0018]$