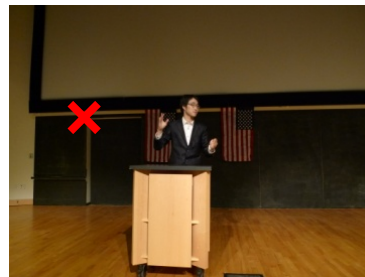
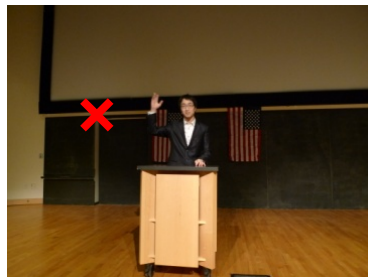
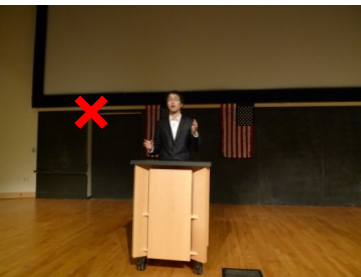
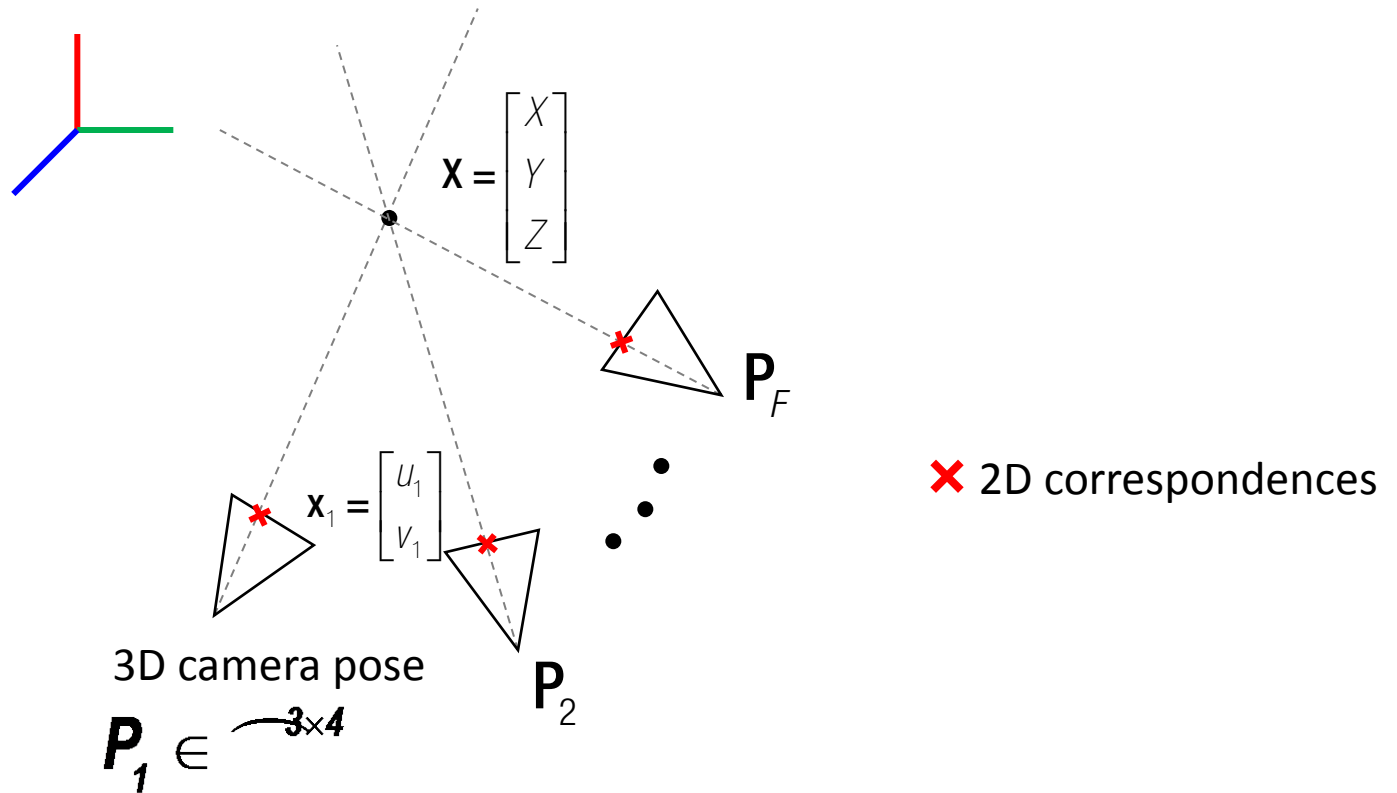
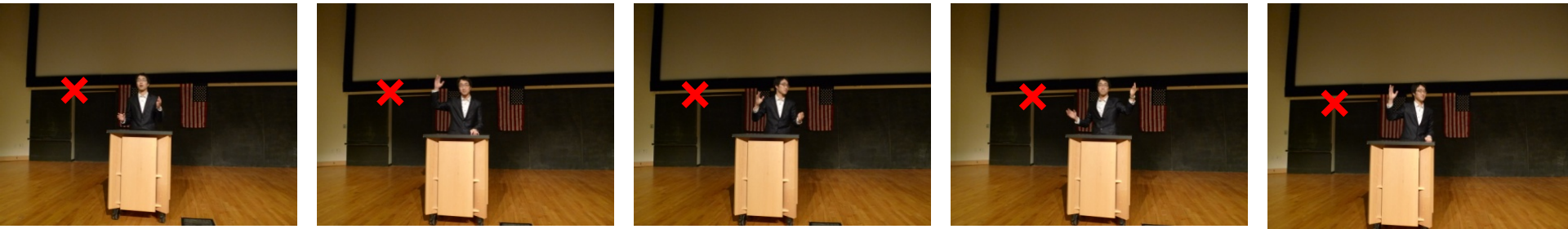


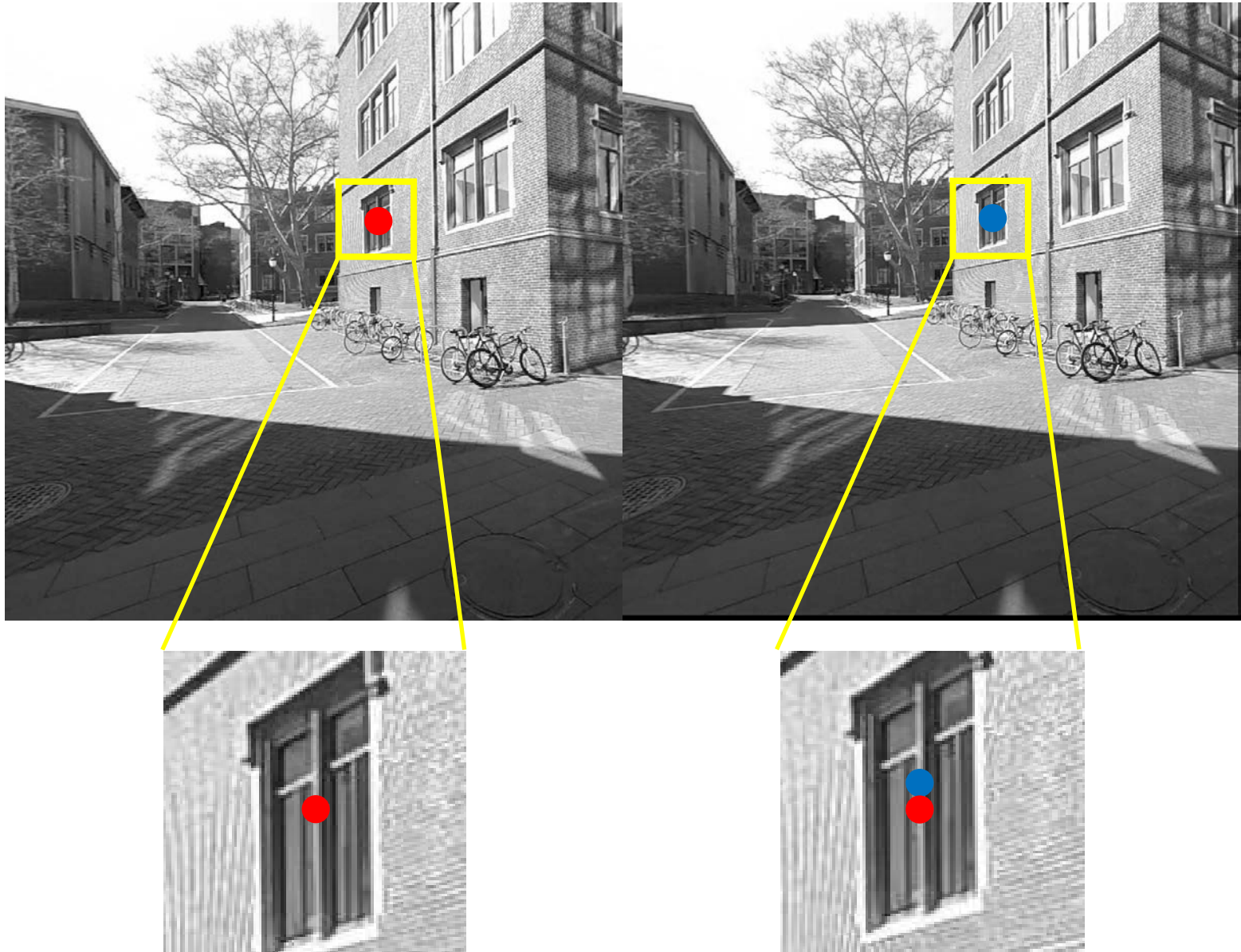
Optical Flow: 2D point correspondences

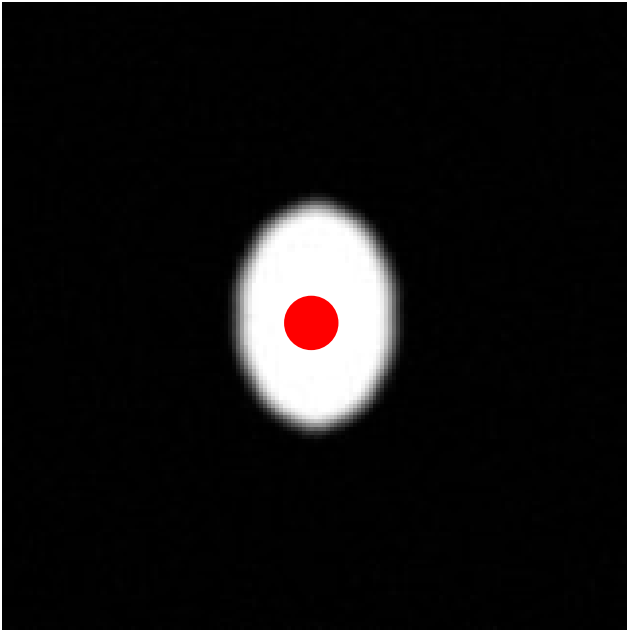


Optical Flow: 2D point correspondences



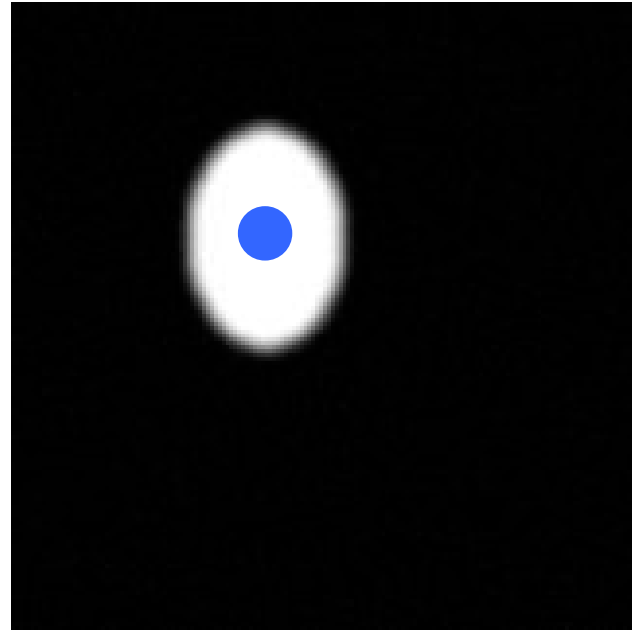
Optical Flow: 2D point correspondences





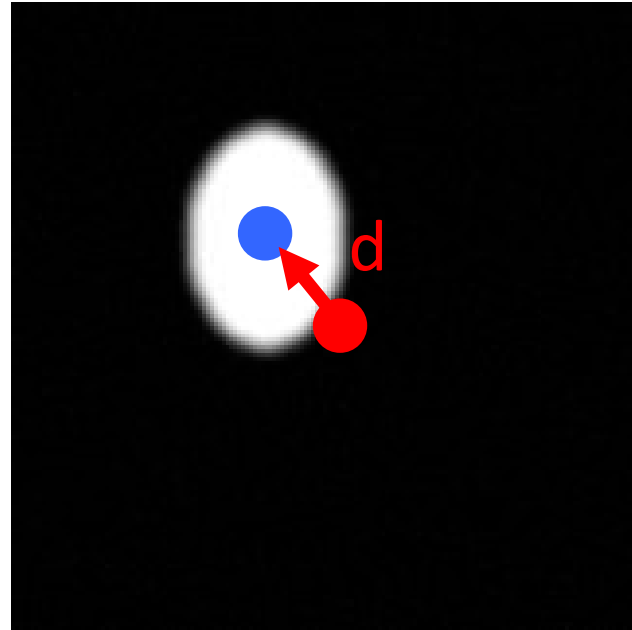
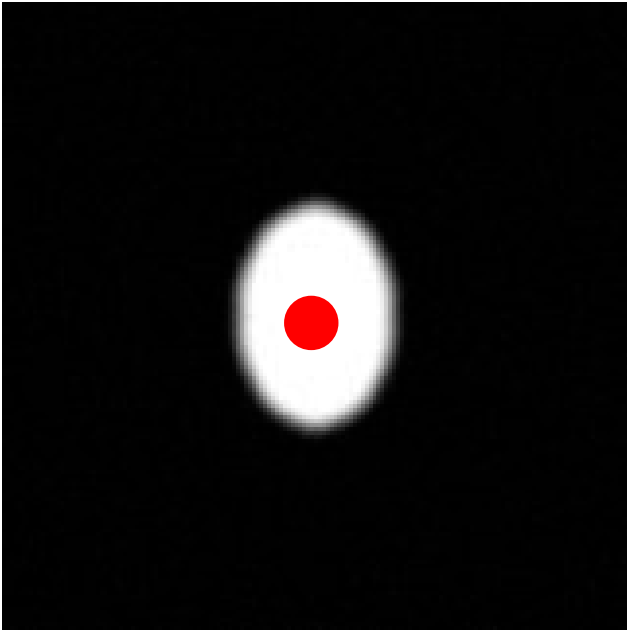
$I(\mathbf{x})$

$t = 0$



$J(\mathbf{x})$

$t = 1$

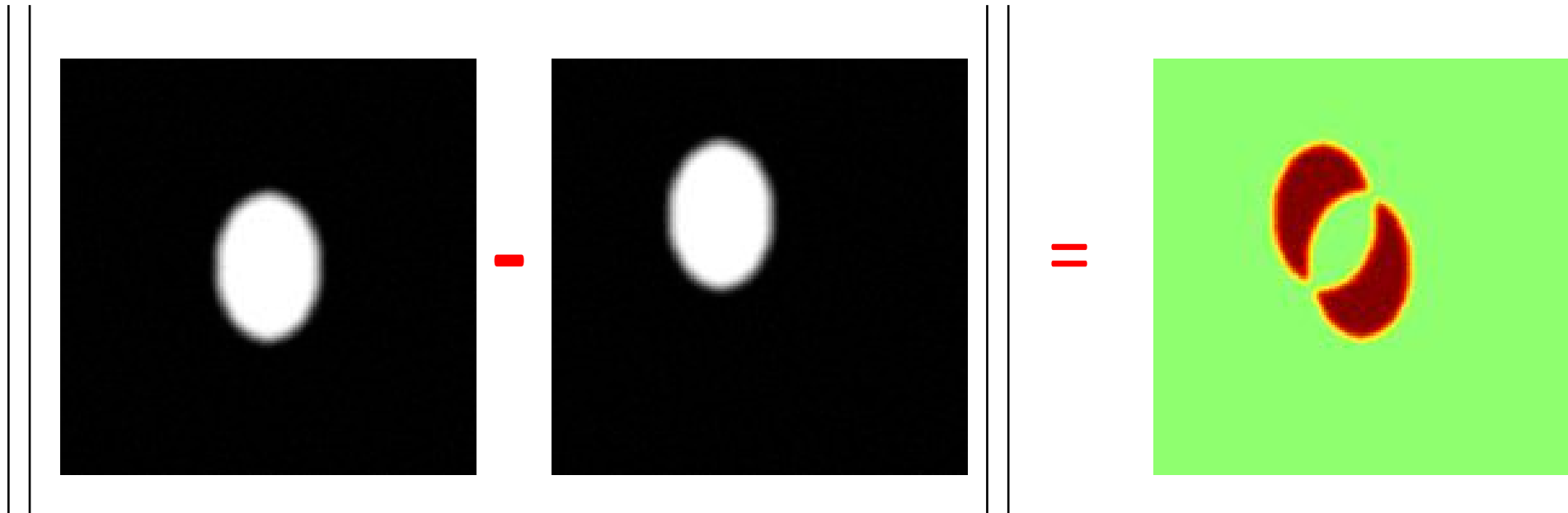


$I(\mathbf{x})$

$J(\mathbf{x})$

$$I(\mathbf{x}) = J(\mathbf{x} + \mathbf{d})$$

$$\min_{\mathbf{d}} E(\mathbf{d}) = ||\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})||^2$$



When $\mathbf{d}=0$

$E(\mathbf{d}=0)$

Three steps for solving this problem

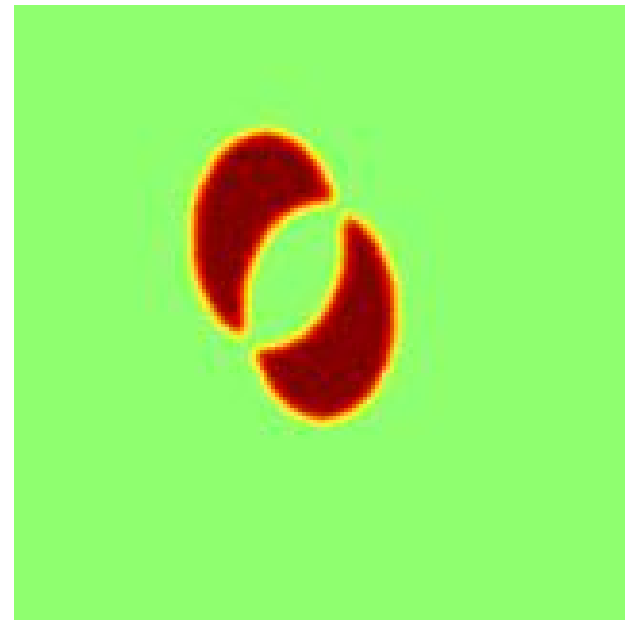
1: Solve for $\left. \frac{\delta E}{\delta \mathbf{d}} \right|_{\mathbf{d}^*} = 0$

2: Taylor expansion on $\mathbf{J}(\mathbf{x} + \mathbf{d})$

3: Solve for \mathbf{d} , warp image, iterate

Step 1: Solve for $\left. \frac{\delta E}{\delta \mathbf{d}} \right|_{d^*} = 0$

$$E(\mathbf{d}) = \|\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})\|^2$$

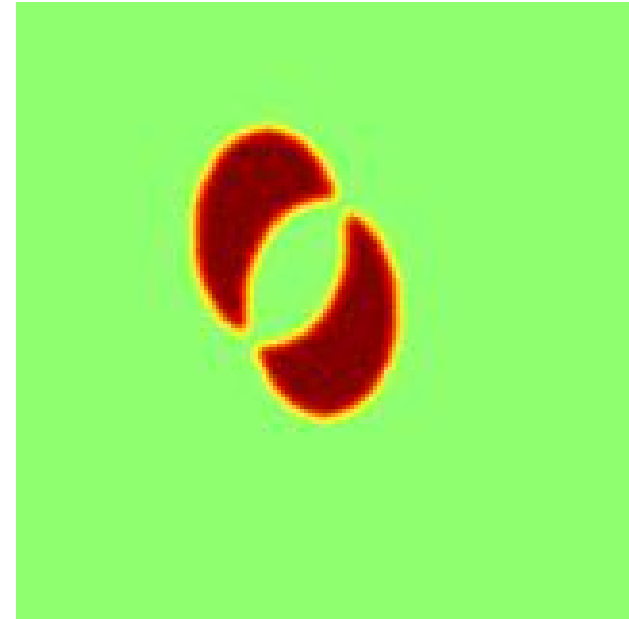


$$\mathbf{E}(\mathbf{d}^* = \mathbf{0})$$

Step 1: Solve for $\left. \frac{\delta E}{\delta \mathbf{d}} \right|_{d^*} = 0$

$$E(\mathbf{d}) = ||\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})||^2$$

$$\left. \frac{\delta E}{\delta \mathbf{d}} \right|_{d^*} = 2 \frac{\delta \mathbf{J}(\mathbf{x} + \mathbf{d})^T}{\delta \mathbf{d}} (\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})) = 0$$



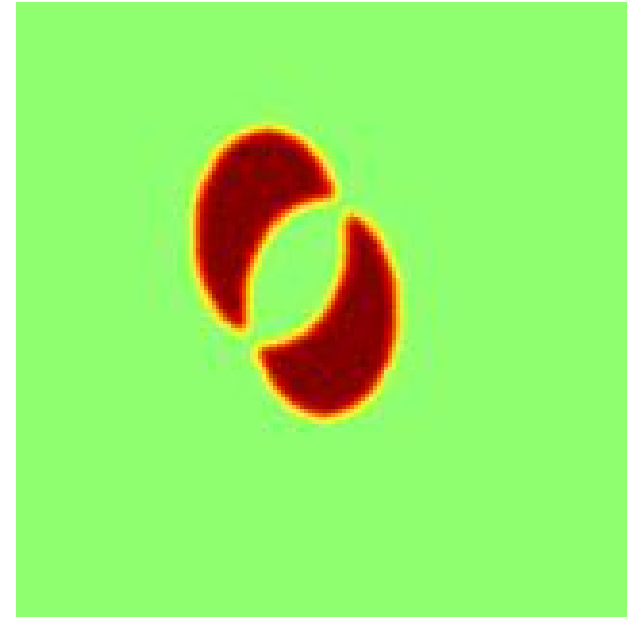
$$\mathbf{E}(\mathbf{d}^* = \mathbf{0})$$

Step 1: Solve for $\left. \frac{\delta E}{\delta \mathbf{d}} \right|_{d^*} = 0$

$$E(\mathbf{d}) = ||\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})||^2$$

$$\left. \frac{\delta E}{\delta \mathbf{d}} \right|_{d^*} = 2 \frac{\delta \mathbf{J}(\mathbf{x} + \mathbf{d})}{\delta \mathbf{d}}^T (\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})) = 0$$

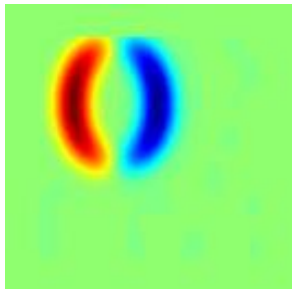
$$\left. \frac{\delta E}{\delta \mathbf{d}} \right|_{d^*} = 2 \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})) = 0$$



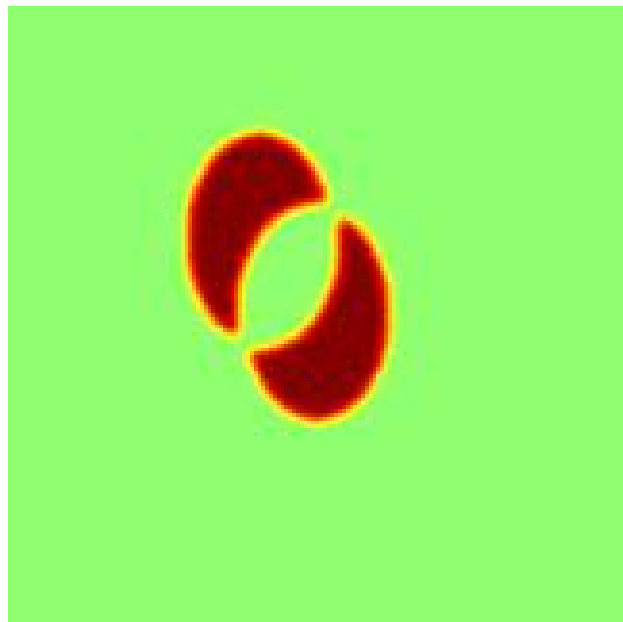
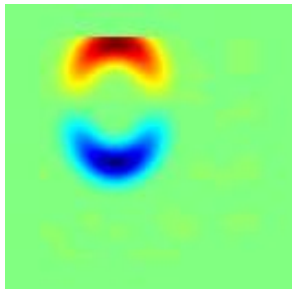
$$\mathbf{E}(\mathbf{d}^* = \mathbf{0})$$

$$\left. \frac{\delta E}{\delta \mathbf{d}} \right|_{d^*} = 2 \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})) = 0$$

$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta x} =$$



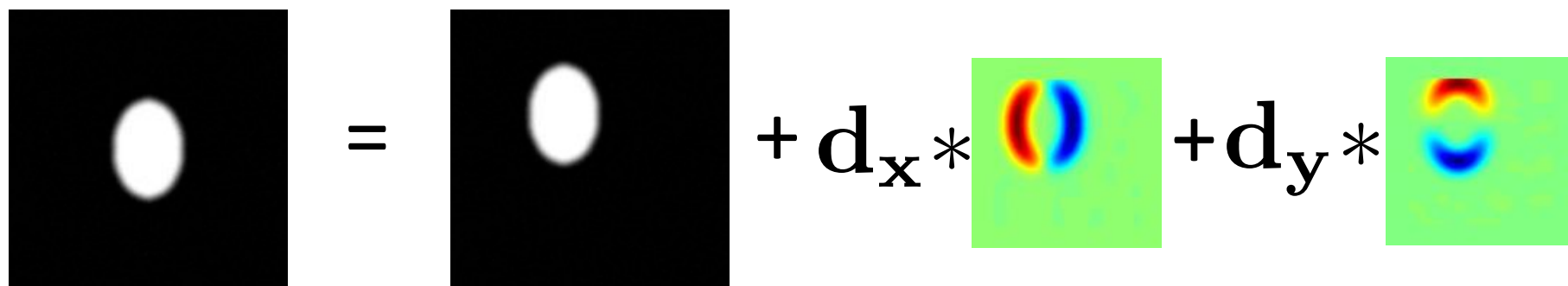
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta y} =$$



$$\mathbf{E}(\mathbf{d}^* = \mathbf{0})$$

Step 2: Taylor expansion $\mathbf{J}(\mathbf{x} + \mathbf{d})$

$$\mathbf{J}(\mathbf{x} + \mathbf{d}) = \mathbf{J}(\mathbf{x}) + \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d}$$



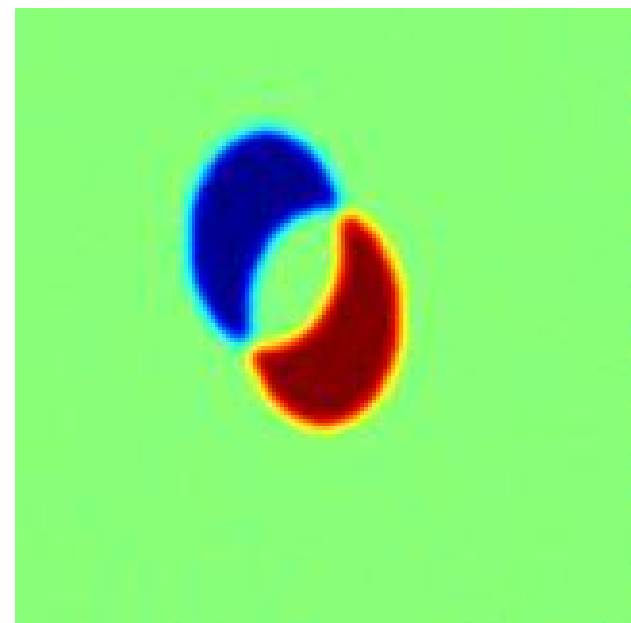
Putting all together

$$\left. \frac{\delta E}{\delta \mathbf{d}} \right|_{d^*} = 2 \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})) = 0$$

$$\mathbf{J}(\mathbf{x} + \mathbf{d}) = \mathbf{J}(\mathbf{x}) + \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d}$$

$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

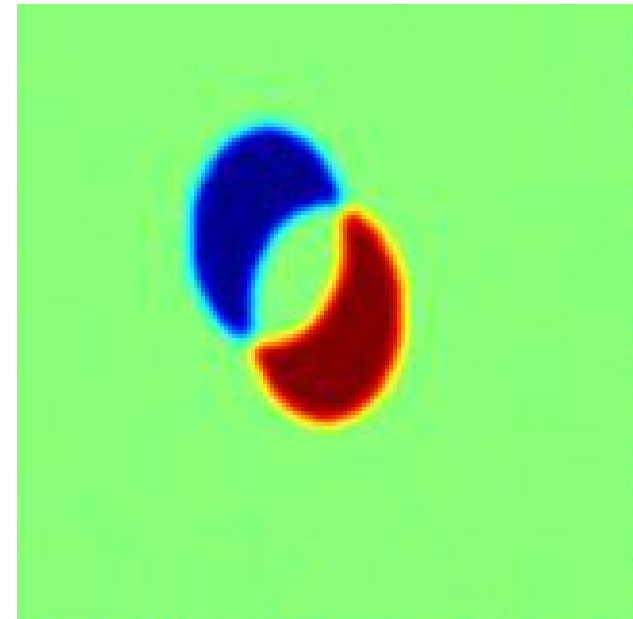
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$



$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$



2D unknowns flow vector per pixel
2 equations



$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

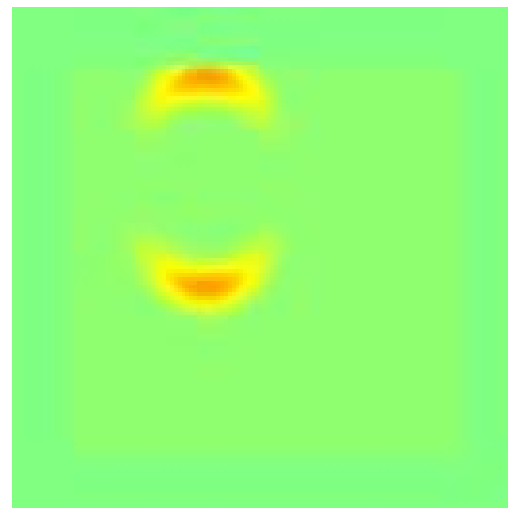
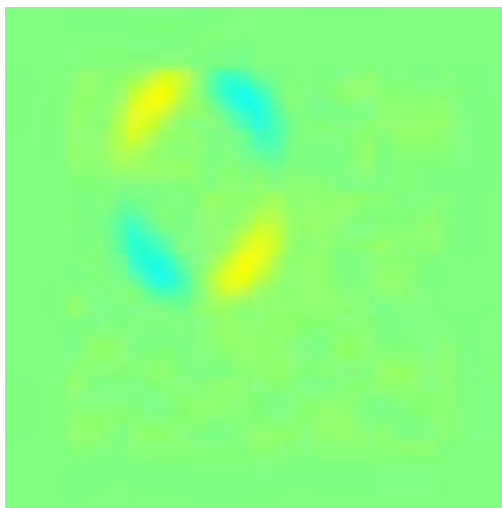
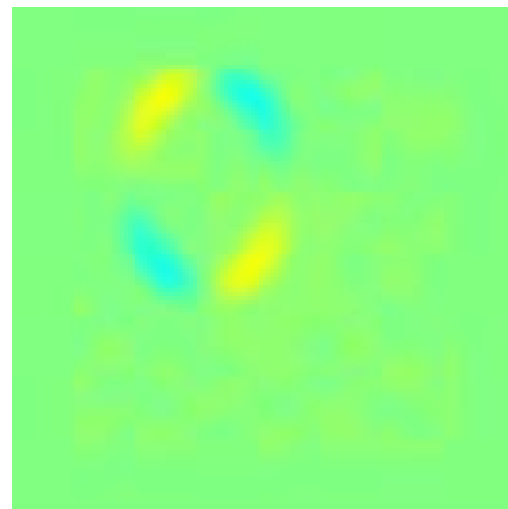
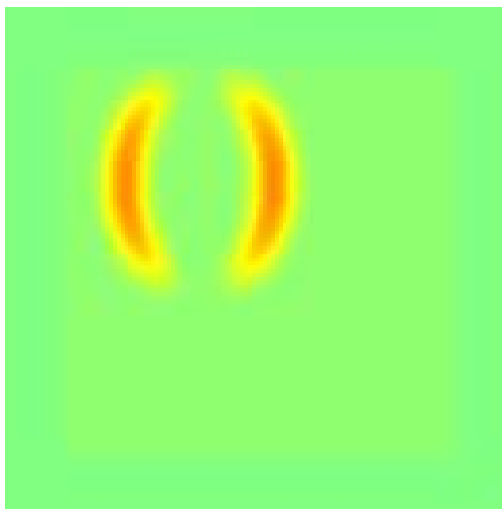


Also known as second moment matrix

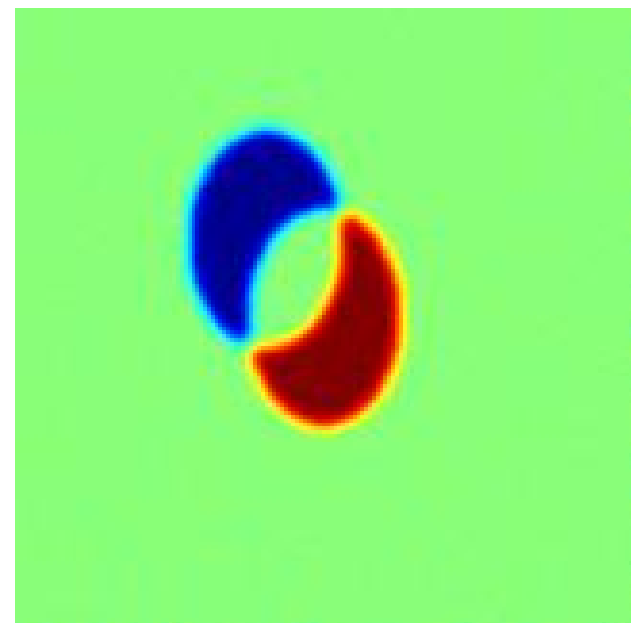
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}$$



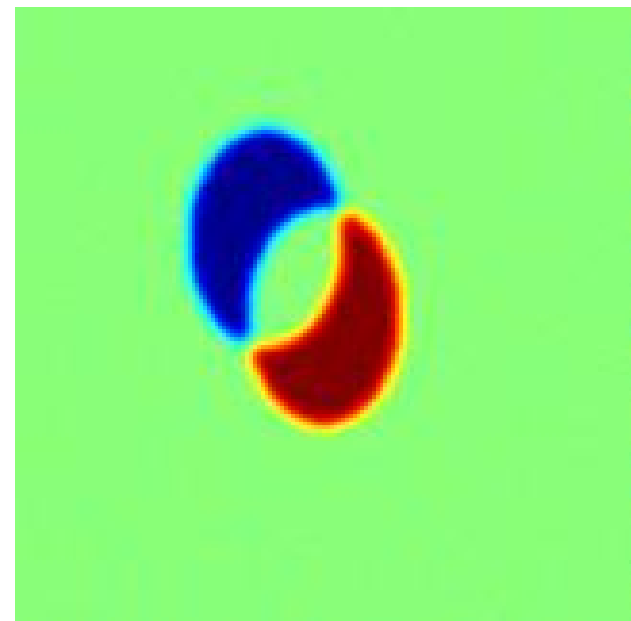
$$\begin{array}{cc} \frac{\delta \mathbf{J}(\mathbf{x})^2}{\delta x} & \frac{\delta \mathbf{J}(\mathbf{x})}{\delta x} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta y} \\ \frac{\delta \mathbf{J}(\mathbf{x})}{\delta x} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta y} & \frac{\delta \mathbf{J}(\mathbf{x})^2}{\delta y} \end{array}$$

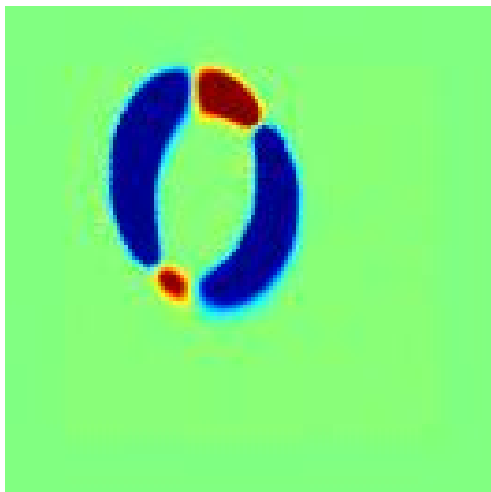


$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

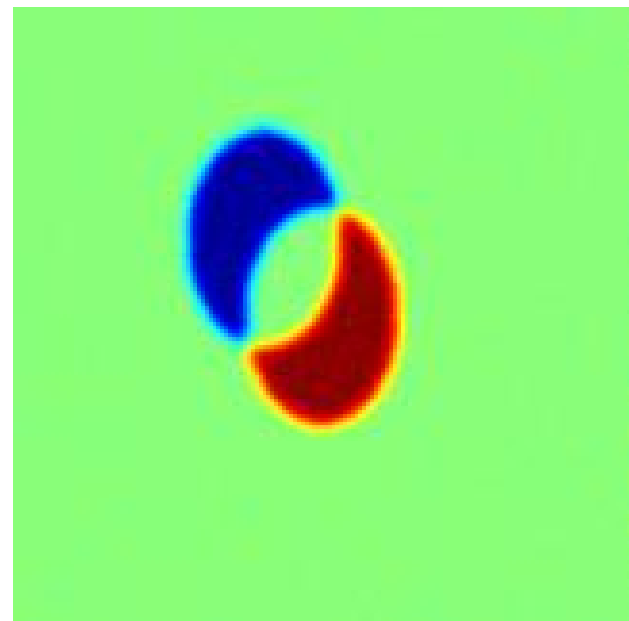
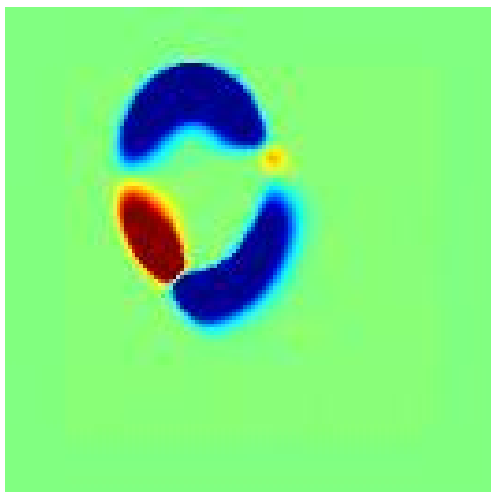


$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

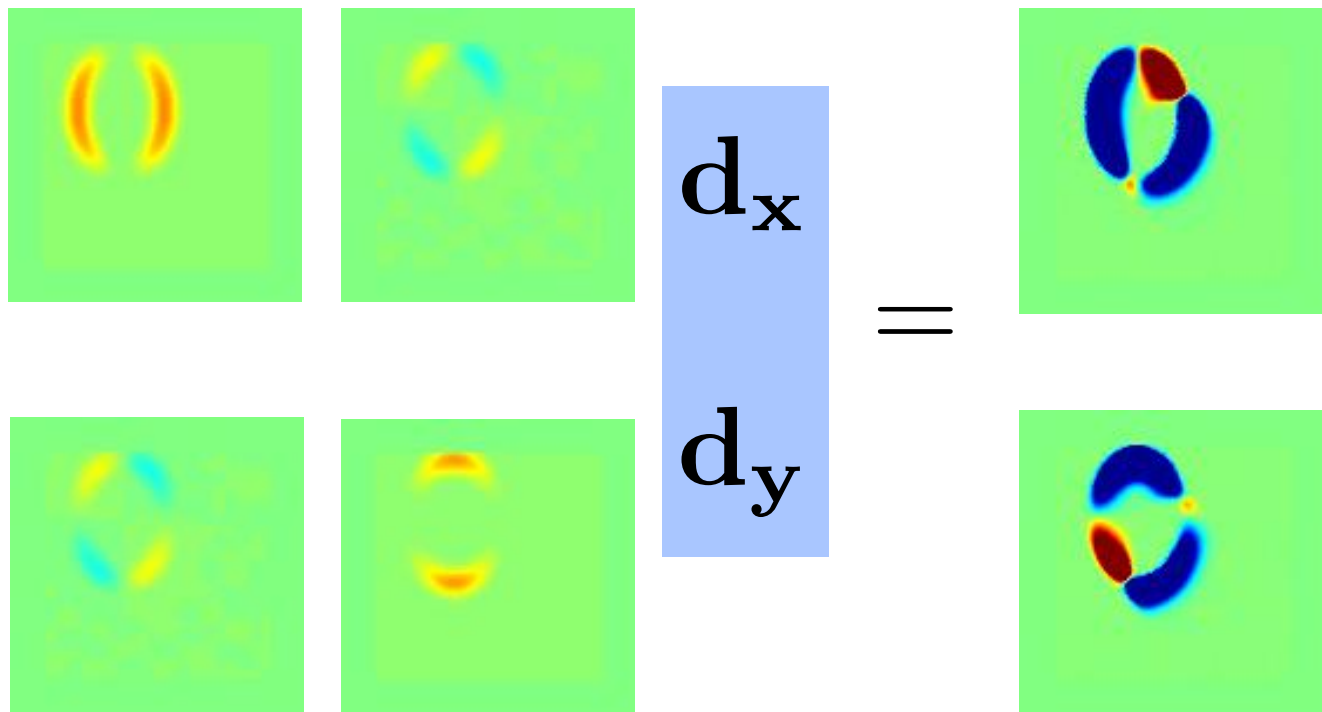




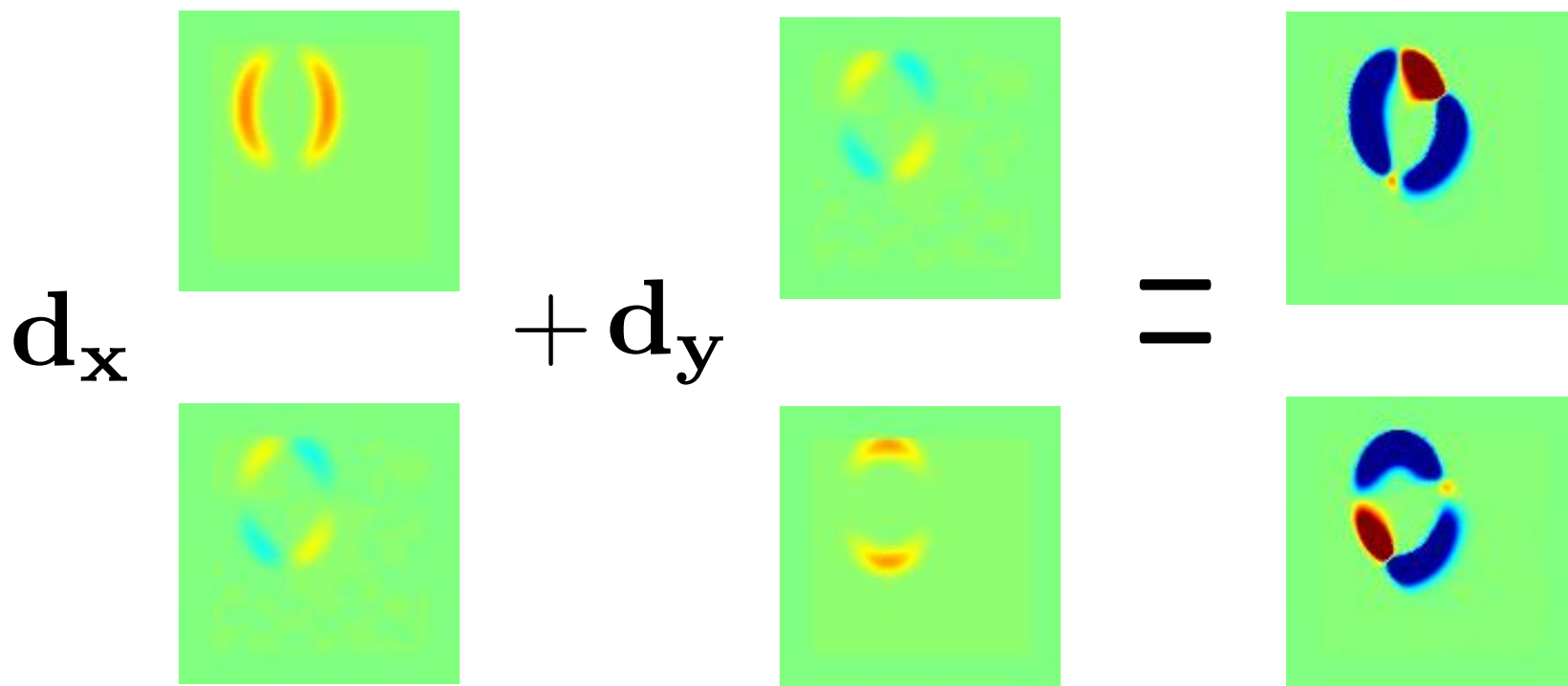
$$\frac{\delta \mathbf{J}(\mathbf{x})^T}{\delta \mathbf{x}} (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$
Two blue arrows originate from the left side of the equation. One arrow points horizontally to the left, towards the top-left heatmap. The other arrow points diagonally down and to the left, towards the bottom-left heatmap.



$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$



$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

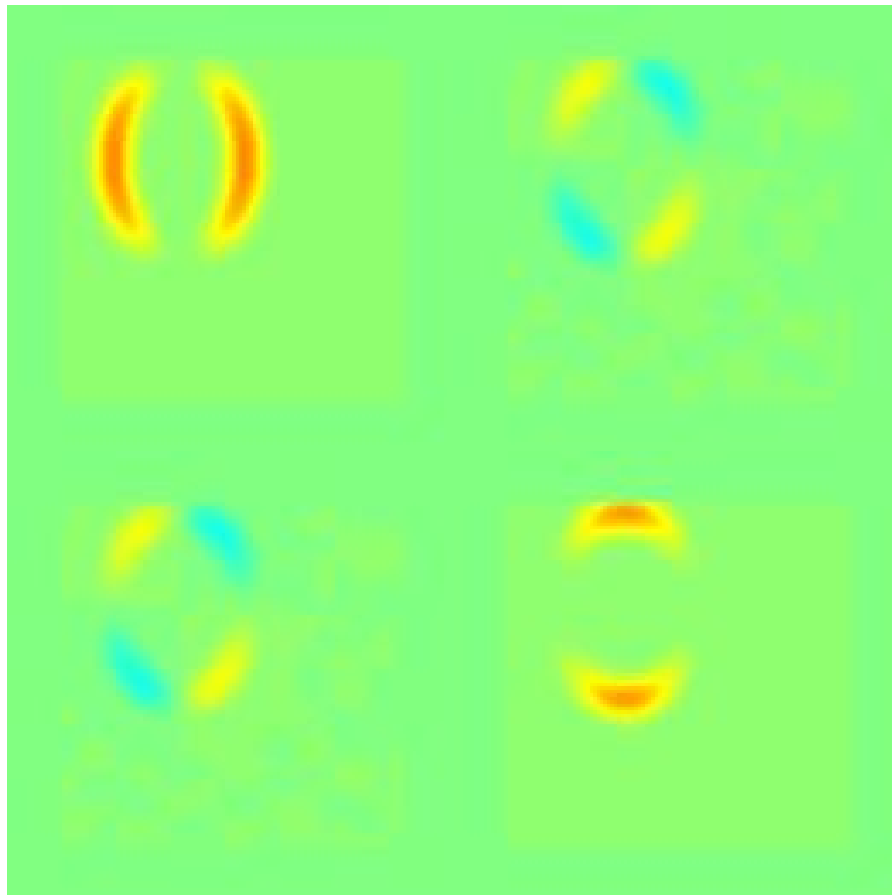


$$\min_{\mathbf{d}} E(\mathbf{d}) = \sum_{\mathbf{x}} ||\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})||^2$$

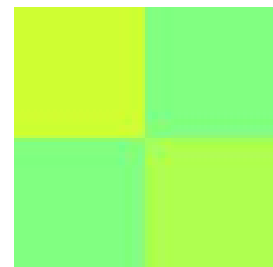
$$\min_{\mathbf{d}} E(\mathbf{d}) = \sum_{\mathbf{x}} ||\mathbf{J}(\mathbf{x} + \mathbf{d}) - \mathbf{I}(\mathbf{x})||^2$$

$$\sum_{\mathbf{x}} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \sum_{\mathbf{x}} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

Summing over pixels

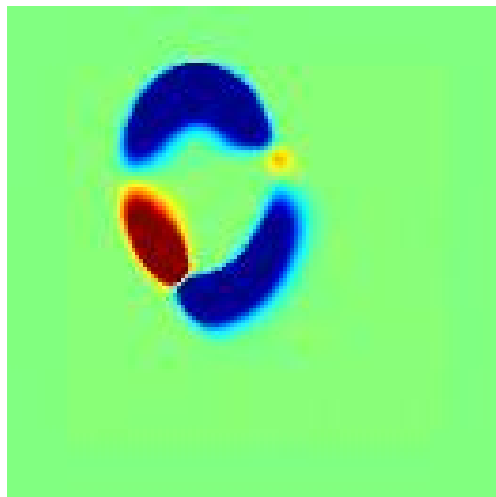
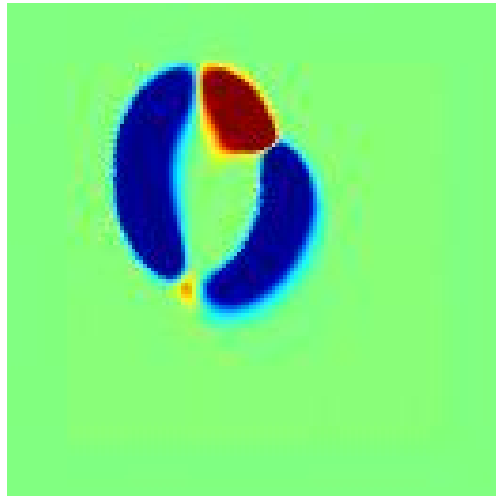


=

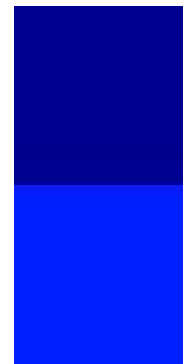


2×2 matrix

Summing over pixels



=



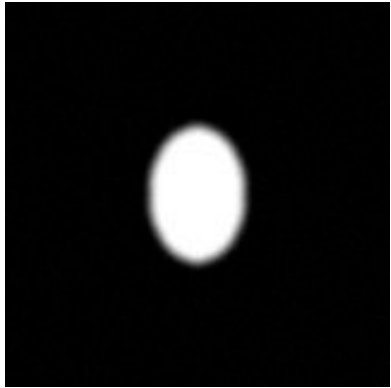
2×1 matrix

$$\sum_{\mathbf{x}} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \sum_{\mathbf{x}} \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

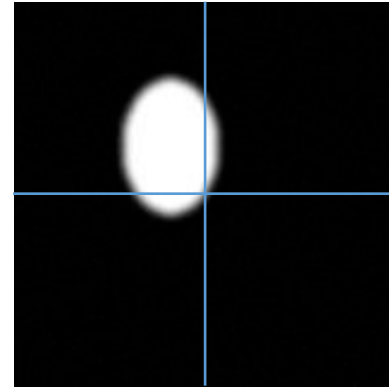
$$\mathbf{d}_{\mathbf{x}} \begin{array}{|c|} \hline \text{red} \\ \hline \text{blue} \\ \hline \end{array} + \mathbf{d}_{\mathbf{y}} \begin{array}{|c|} \hline \text{blue} \\ \hline \text{red} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{dark blue} \\ \hline \text{dark red} \\ \hline \end{array}$$

3: Solve for \mathbf{d} , warp image, iterate

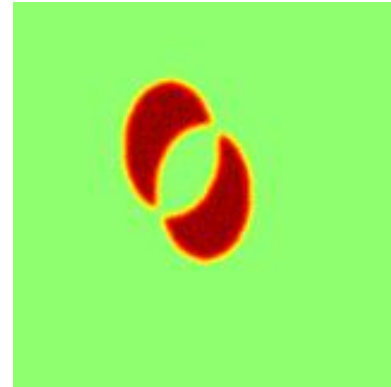
$\mathbf{I}(\mathbf{x})$



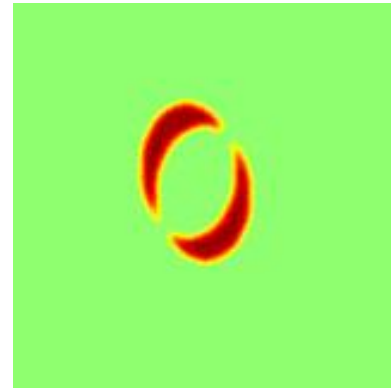
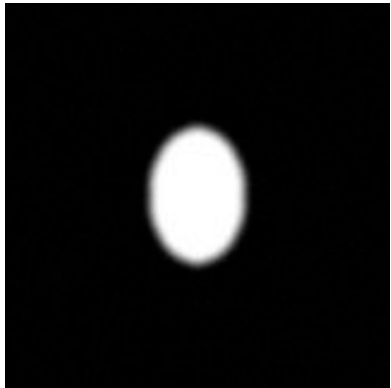
$\mathbf{J}(\mathbf{x})$



Error

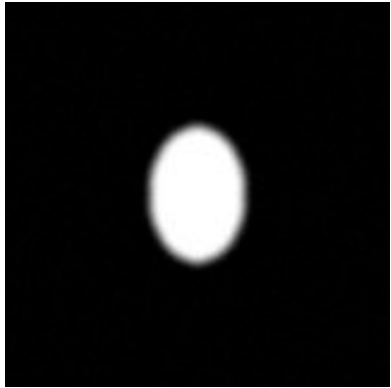


$\mathbf{d} = (-7, -9)$

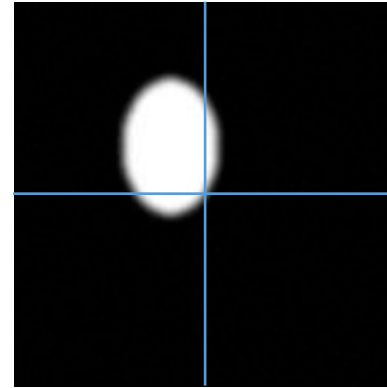


3: Solve for \mathbf{d} , warp image, iterate

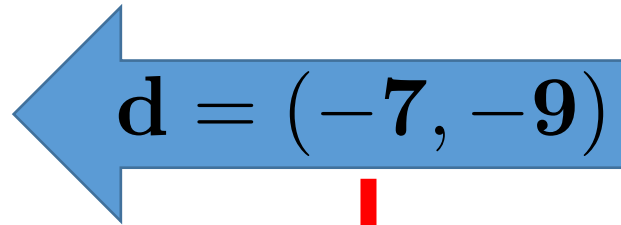
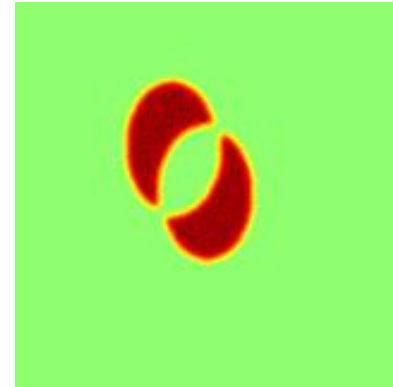
$\mathbf{I}(\mathbf{x})$



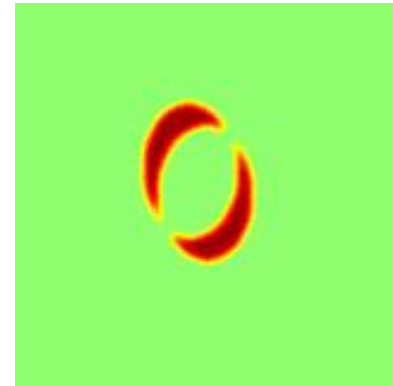
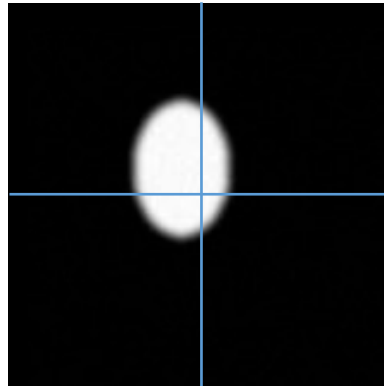
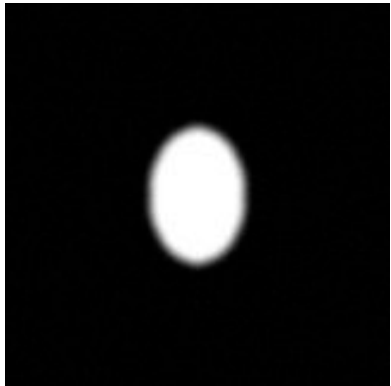
$\mathbf{J}(\mathbf{x})$



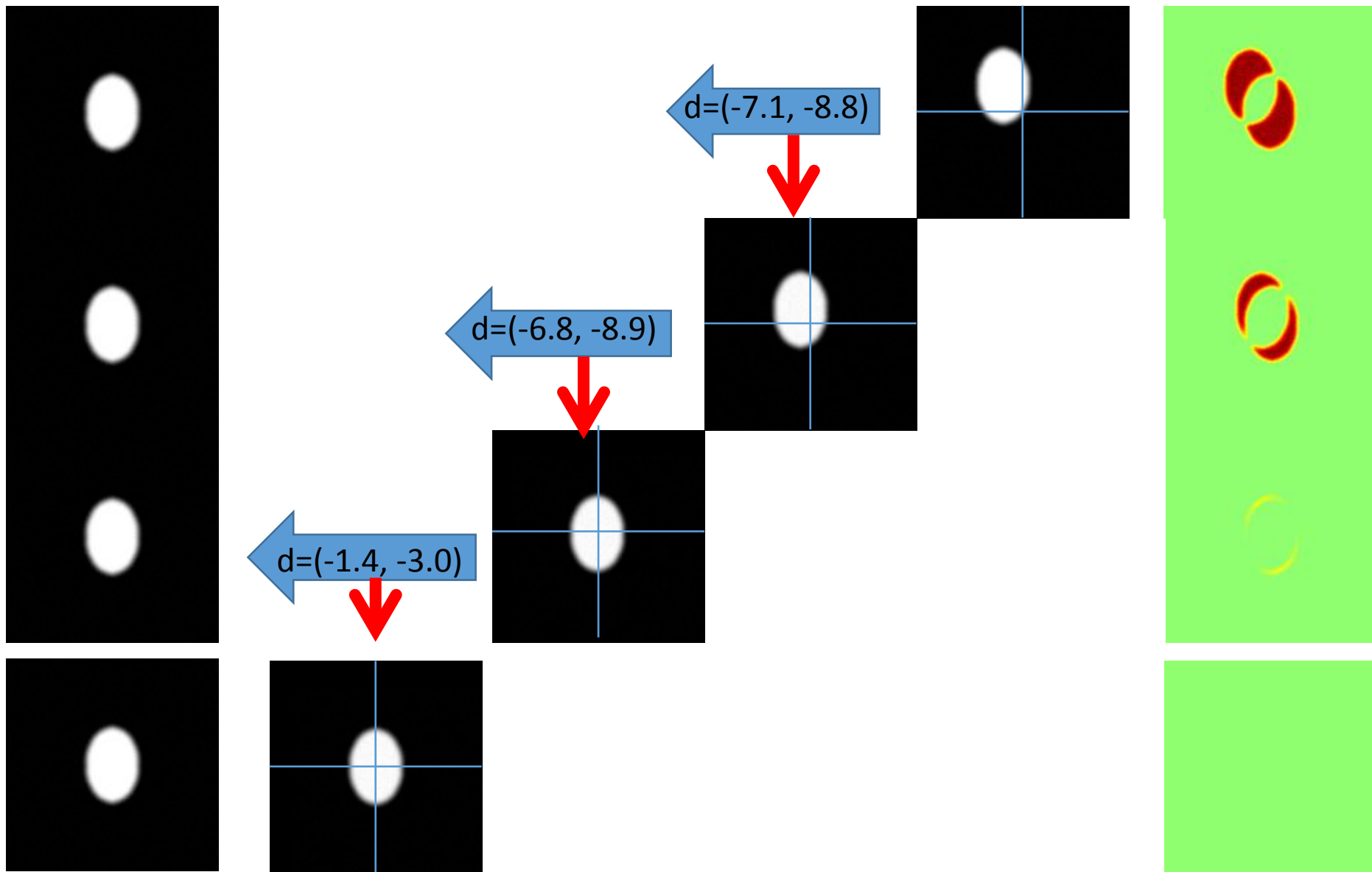
Error

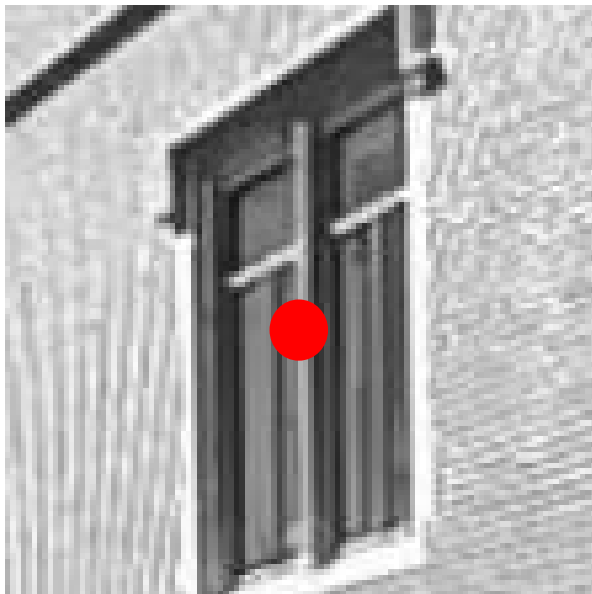


$$\mathbf{J}^{t=1}(\mathbf{x}) = \mathbf{J}(\mathbf{x} + \mathbf{d})$$



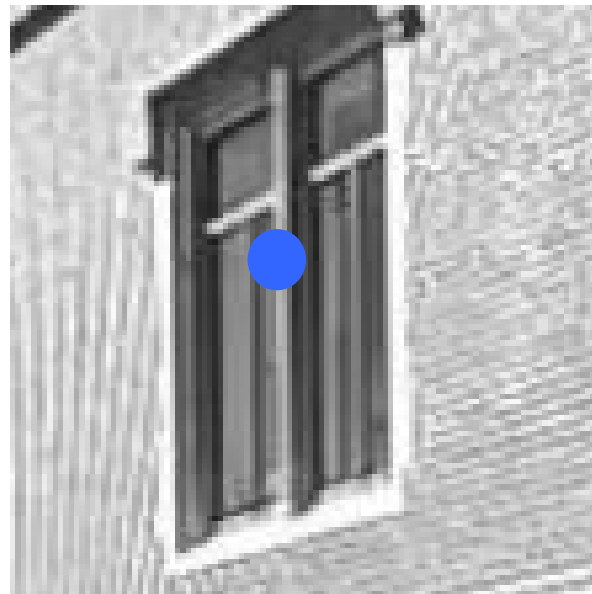
Error





$\mathbf{I}(\mathbf{x})$

$t = 0$



$\mathbf{J}(\mathbf{x})$

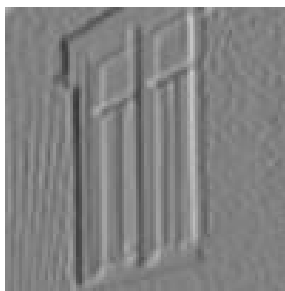
$t = 1$

$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$



$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

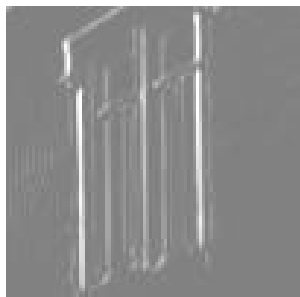
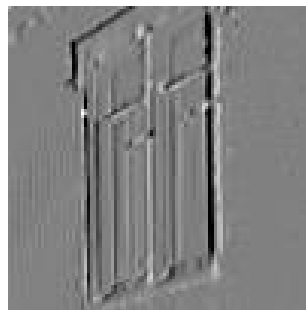
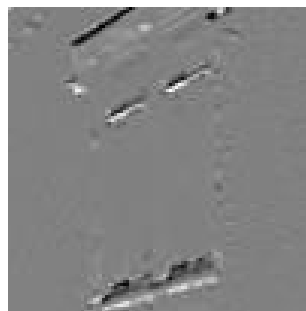
$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta x} =$$



$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta y} =$$

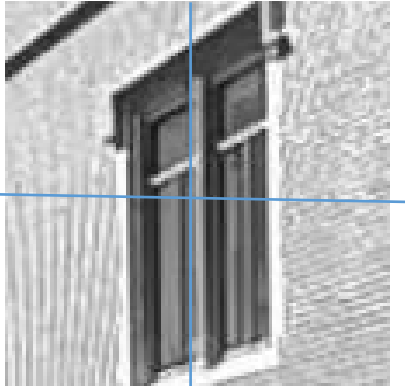


$$\frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}} \mathbf{d} = \frac{\delta \mathbf{J}(\mathbf{x})}{\delta \mathbf{x}}^T (\mathbf{I}(\mathbf{x}) - \mathbf{J}(\mathbf{x}))$$

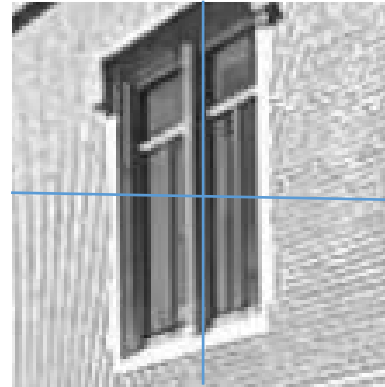

 \mathbf{d}_x
 $=$

 \mathbf{d}_y


3: Solve for \mathbf{d} , warp image, iterate

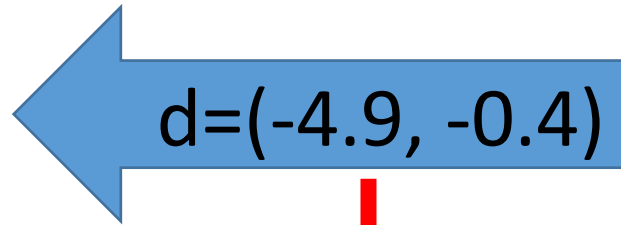
$\mathbf{I}(\mathbf{x})$



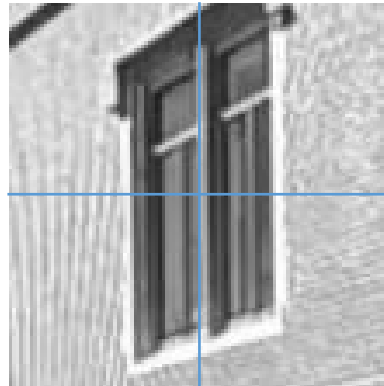
$\mathbf{J}(\mathbf{x})$



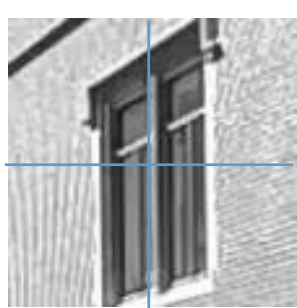
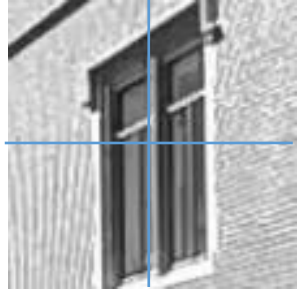
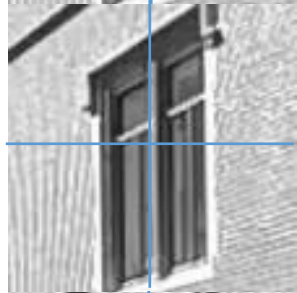
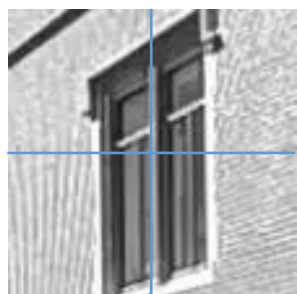
Error



$$\mathbf{J}^{t=1}(\mathbf{x}) = \mathbf{J}(\mathbf{x} + \mathbf{d})$$



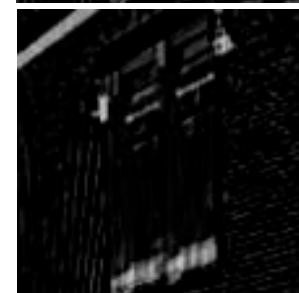
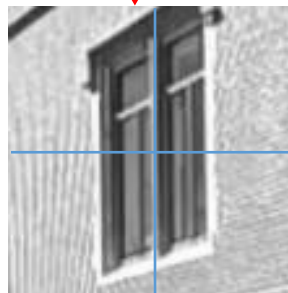
Error



$d=(-4.9, -0.4)$



$d=(-0.1, -5.8)$



$d=(0, -3.7)$

