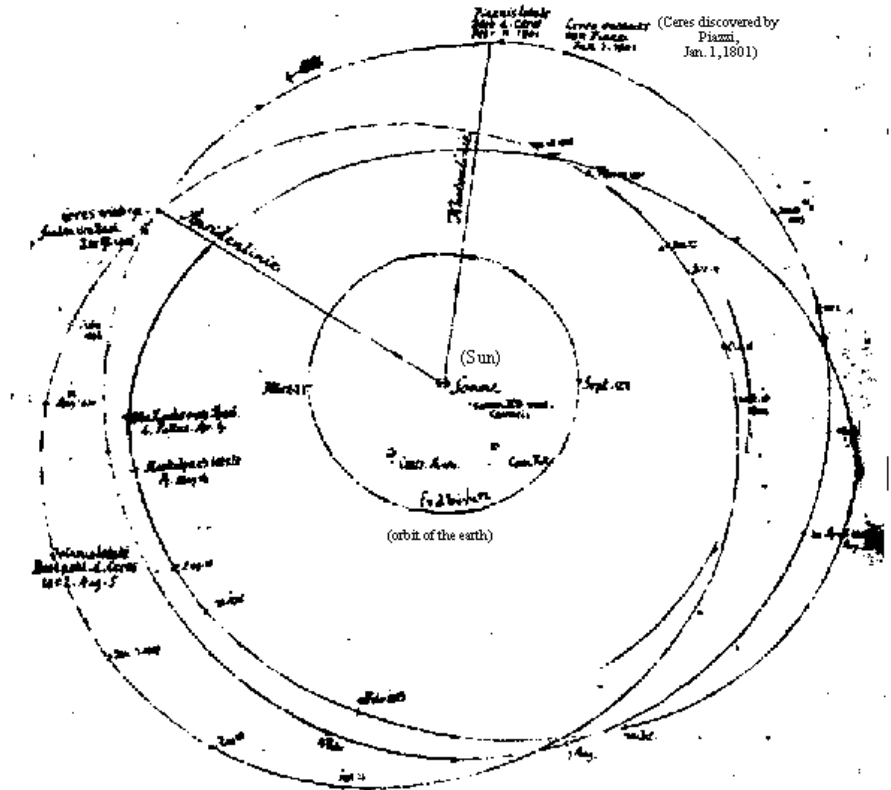
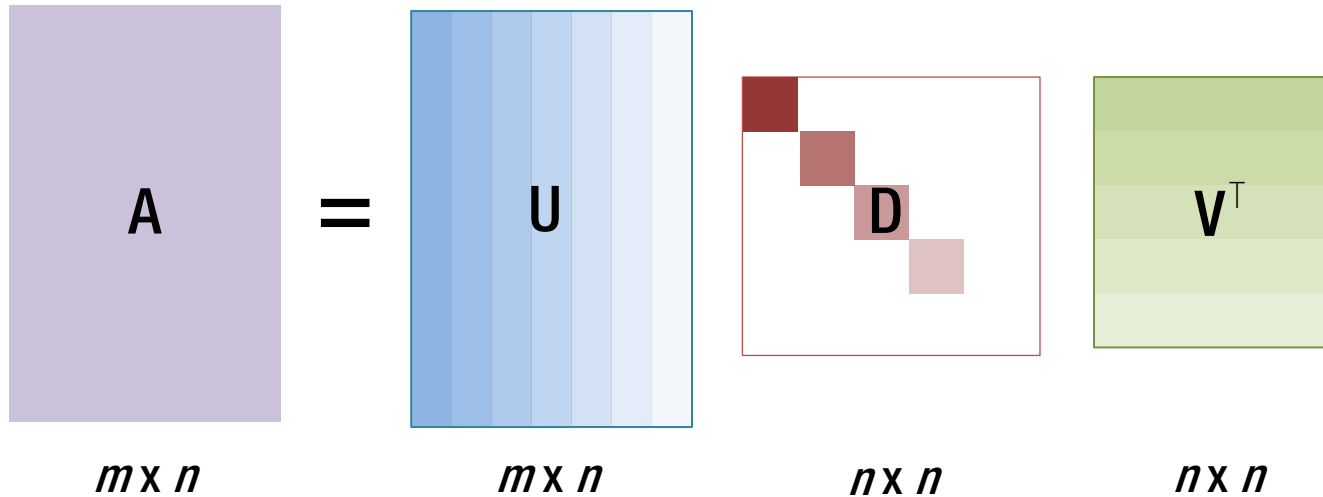


# 1809, Carl Friedrich Gauss

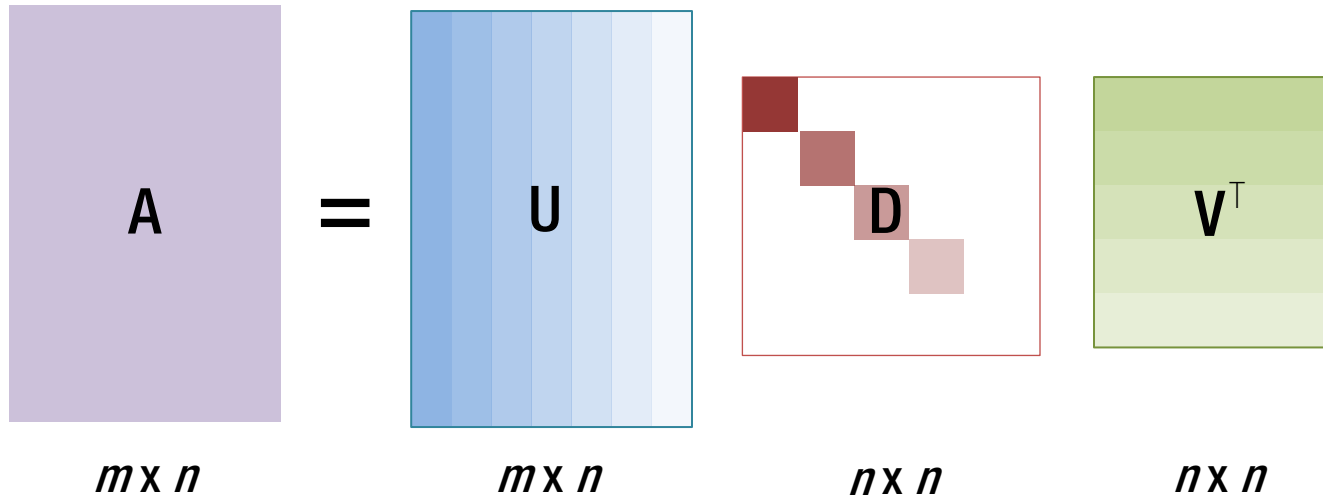


Sketch of the orbits of Ceres and Pallas (nachlaß Gauß, Handb. 4). Courtesy of Universitätsbibliothek Göttingen.

# Singular Value Decomposition



# Singular Value Decomposition

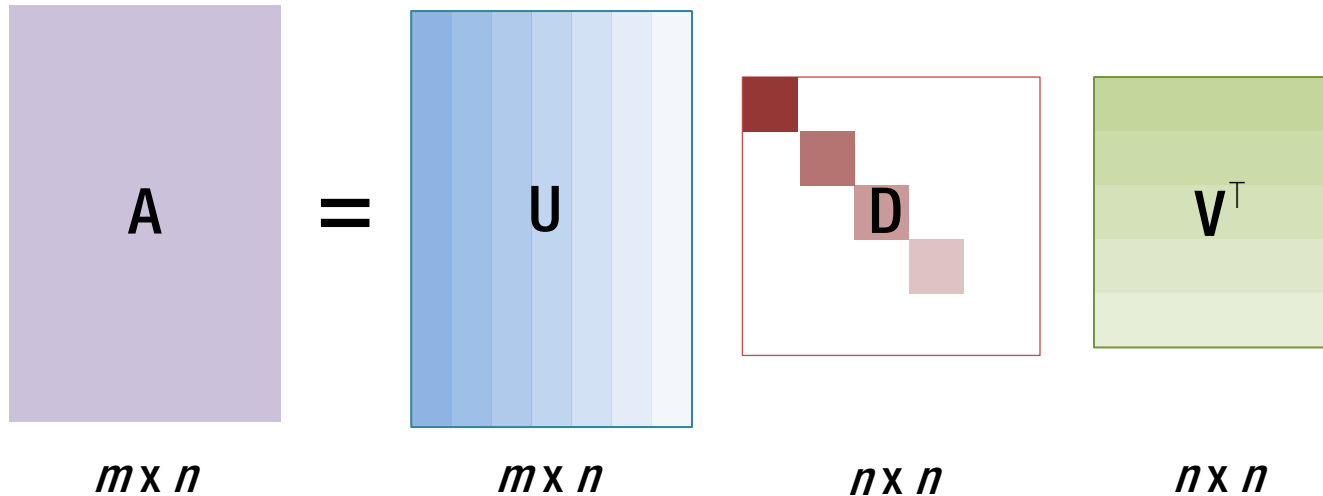


Column orthogonal matrix

$$\mathbf{u}_i^T \mathbf{u}_i = \|\mathbf{u}_i\| = 1$$

$$\mathbf{u}_i^T \mathbf{u}_j = \mathbf{u}_j^T \mathbf{u}_i = 0 \quad \text{for } i \neq j$$

# Singular Value Decomposition



Column orthogonal matrix

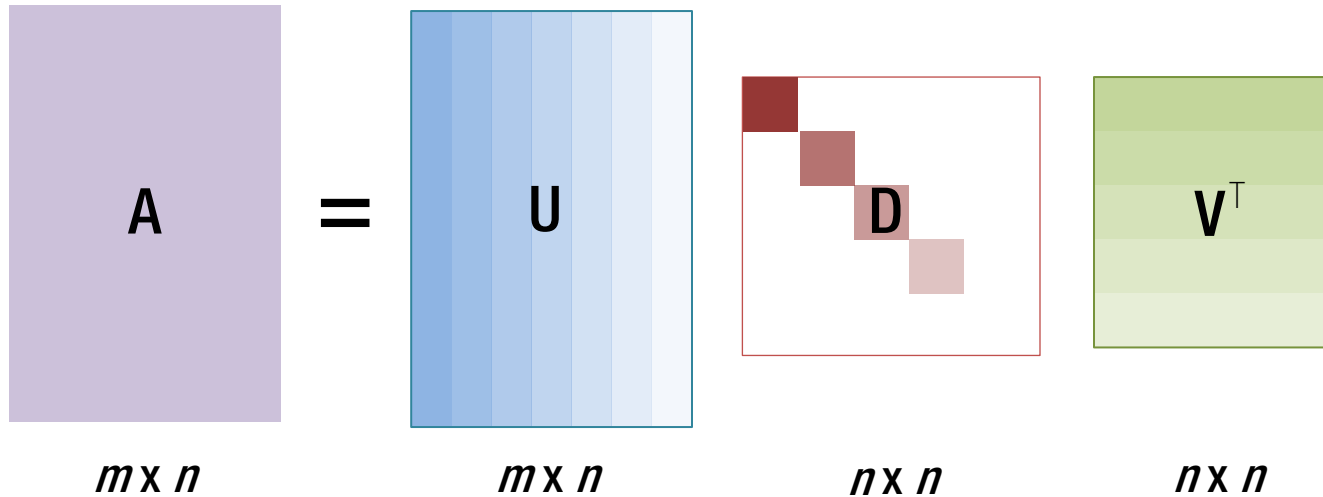
$$\mathbf{u}_i^T \mathbf{u}_i = \|\mathbf{u}_i\| = 1$$

$$\mathbf{U}^T \mathbf{U} = \mathbf{I}_{m \times m}$$

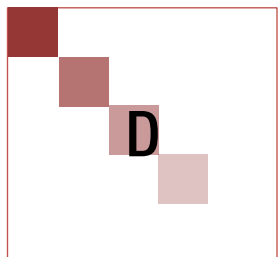
$$\mathbf{u}_i = \mathbf{u}_j^T \mathbf{u}_i = 0 \quad \text{for } i \neq j$$

$$\mathbf{V}^T \mathbf{V} = \mathbf{I}_{n \times n}$$

# Singular Value Decomposition



Singular value matrix

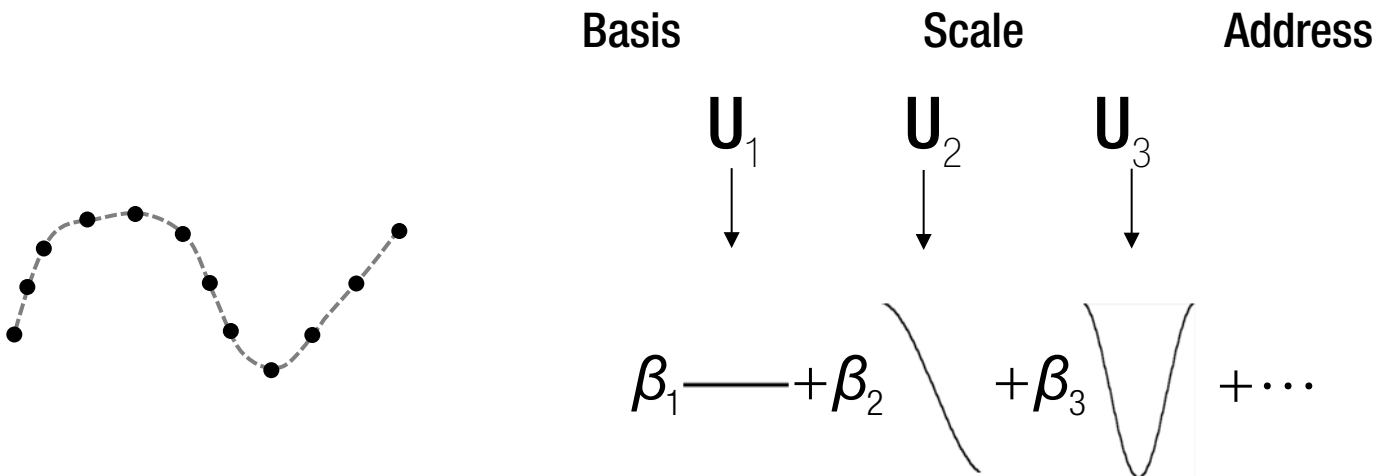
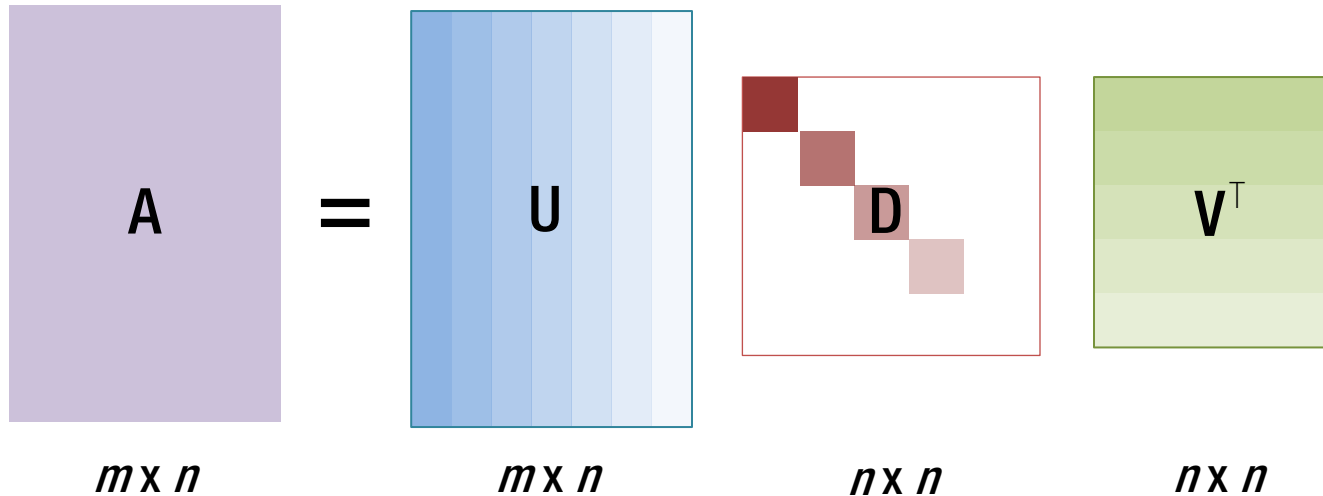


The singular value matrix  $D$  is shown as a square with a red border and a diagonal of red squares. It is defined as:

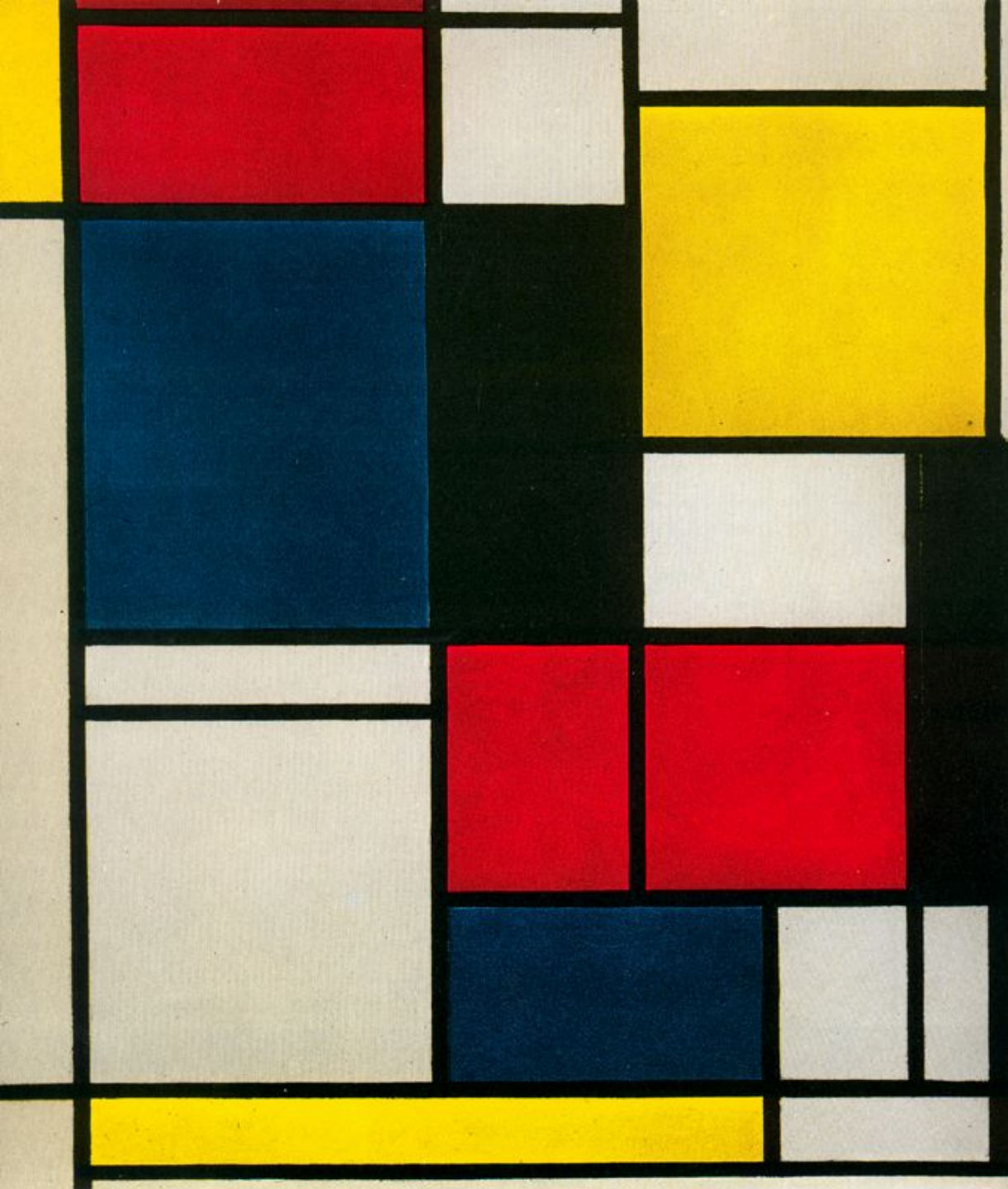
$$D = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\}$$

where  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$

# Singular Value Decomposition

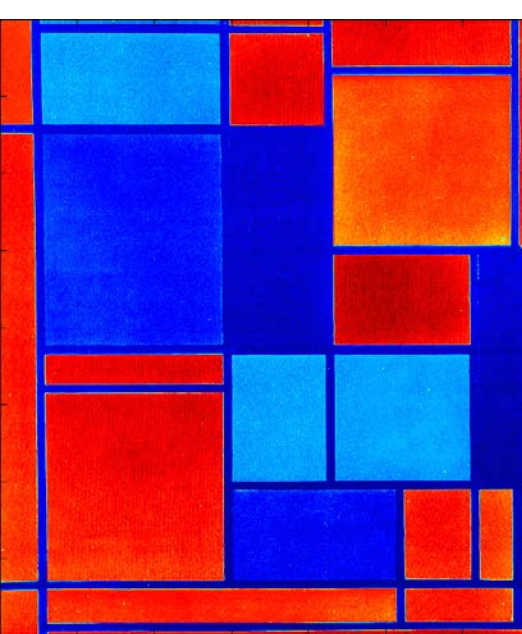


**SVD as basis + transformed Address**



**SVD of  
this?**

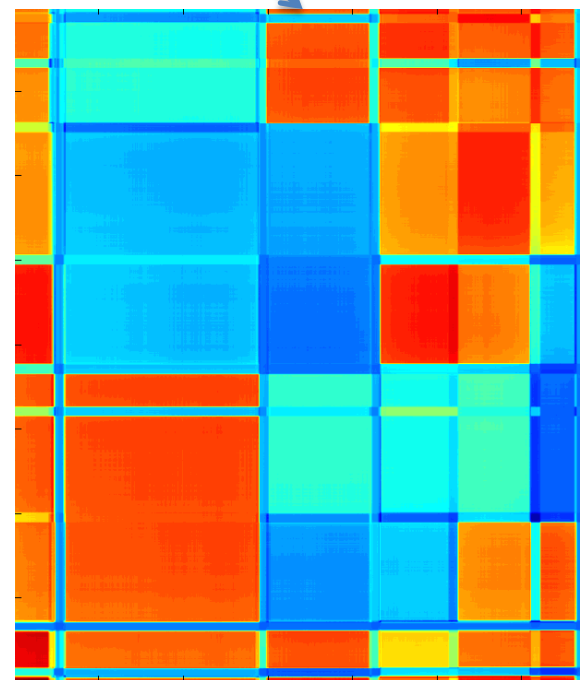
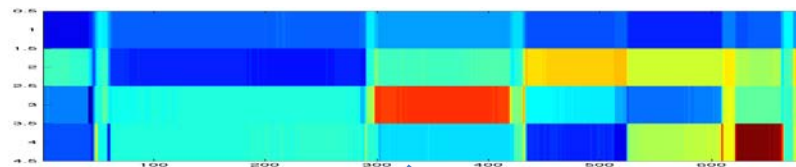
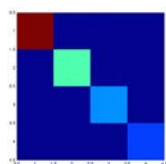
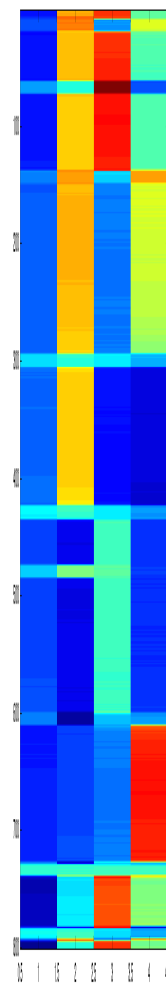


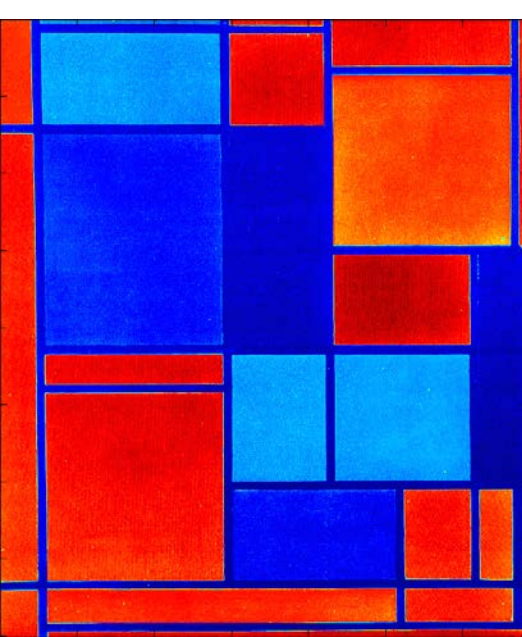


$U(:,1:4)$

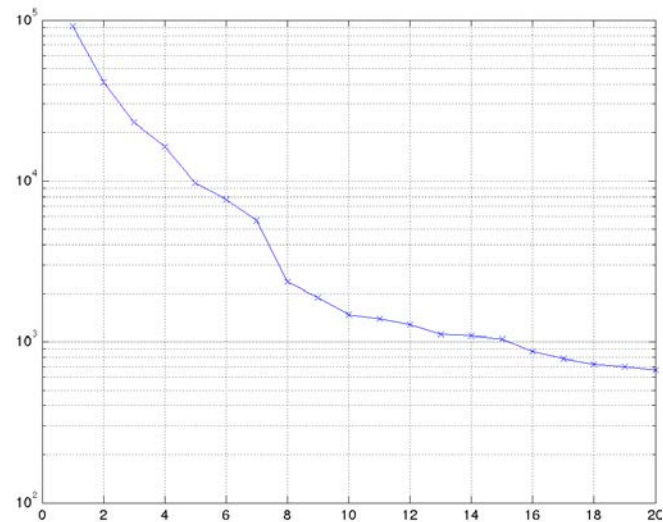
$D(1:4,1:4)$

$V(:,1:4)'$



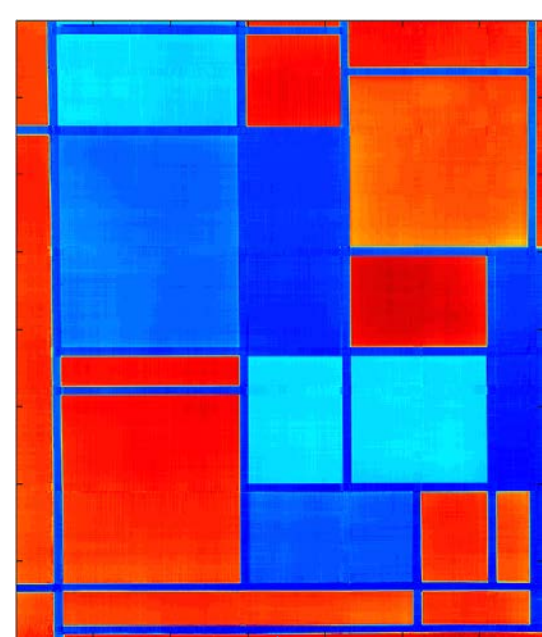


`[u,d,v] = svd(L);`

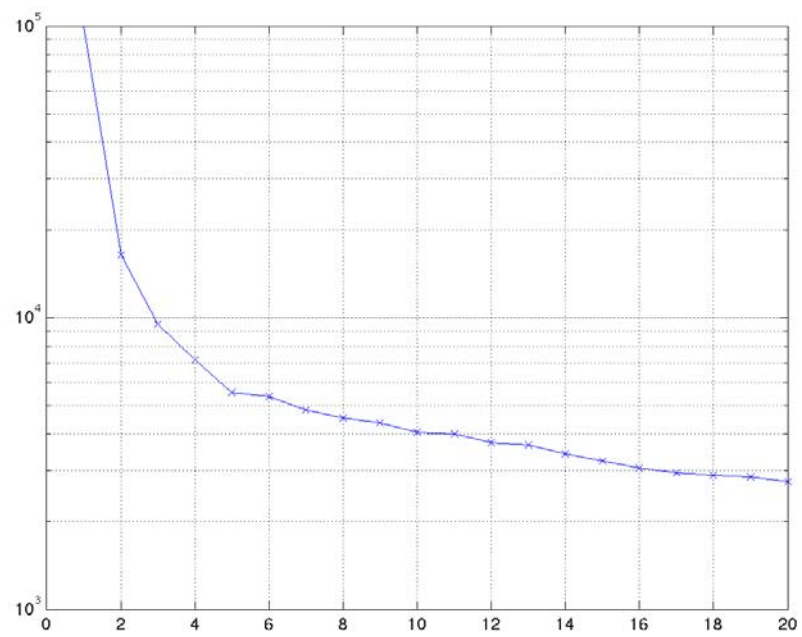
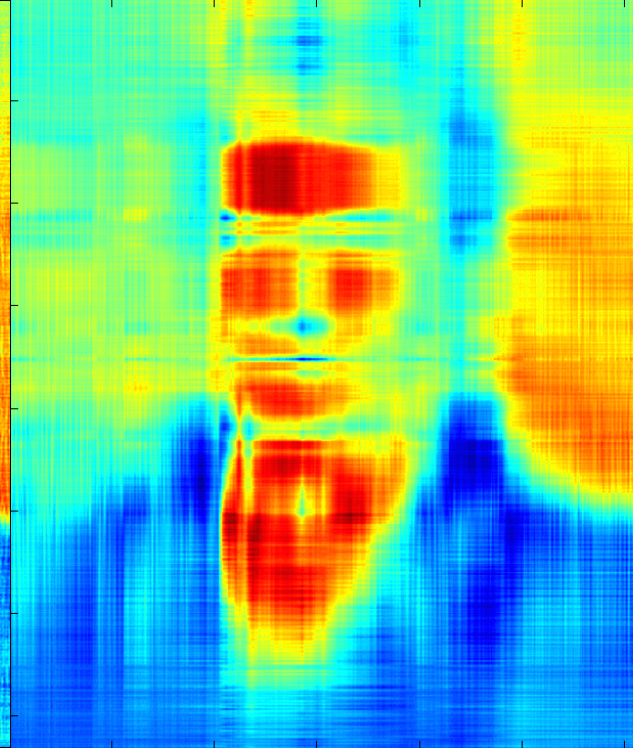
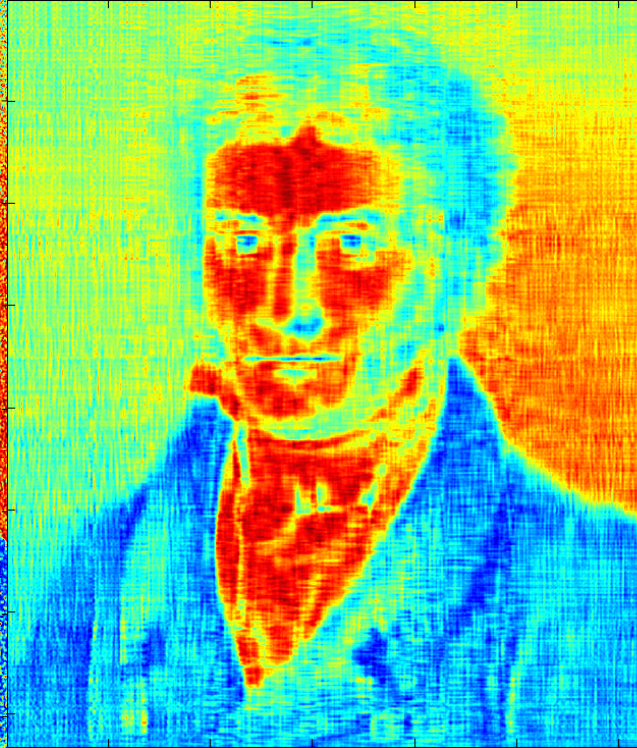


`semilogy(diag(d(1:20,1:20)), 'x-')`

`lm2 = u(:,1:20)*d(1:20,1:20)*v(:,1:20)';`

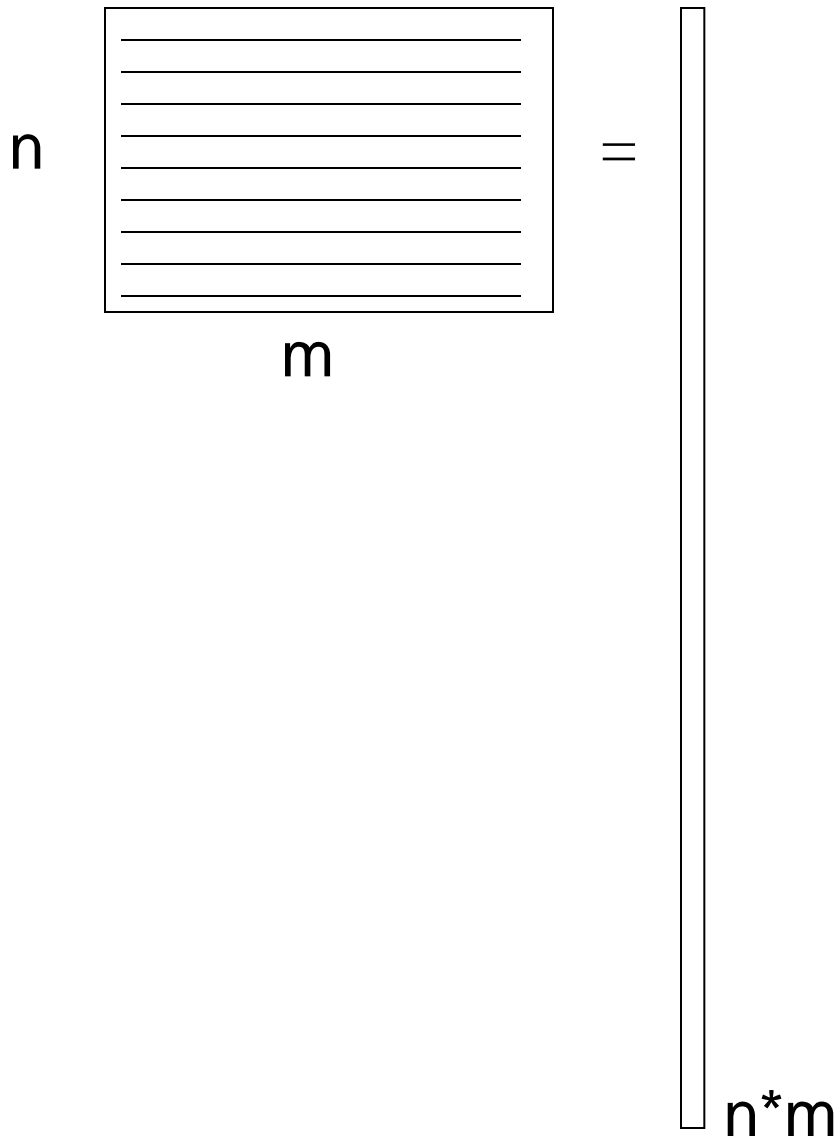




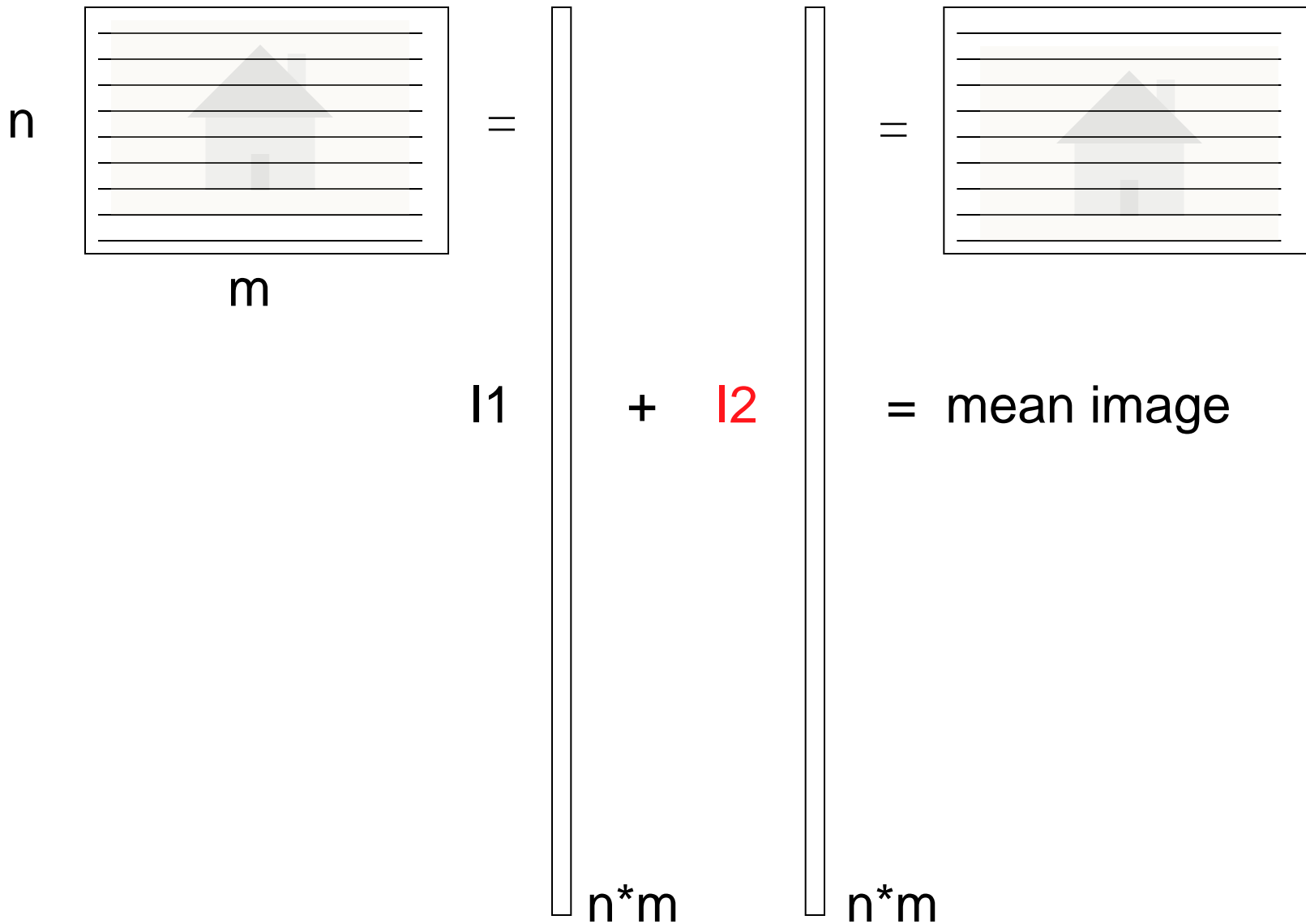




# Images as Vectors



# Vector Mean





# Eigenfaces

---



Eigenfaces look somewhat like generic faces.

# Eigen-images of Berlin





# Eigen-images

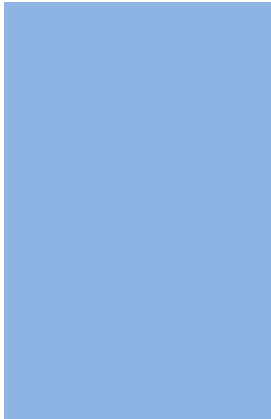
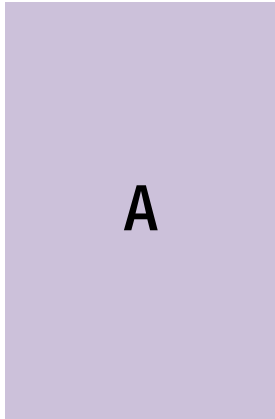


Average of 16 individuals transformed via  
biometrical data of different ethnics

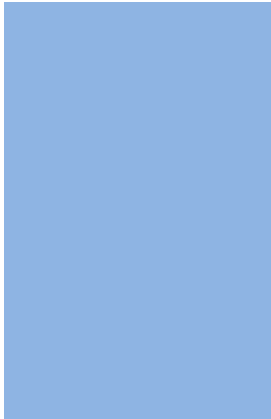
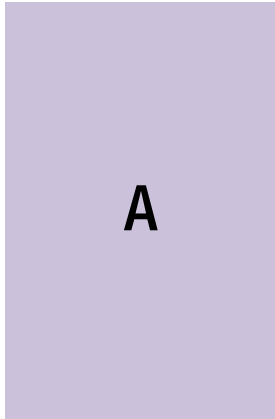


**Average of 16 individuals transformed via  
biometrical data of different ages**

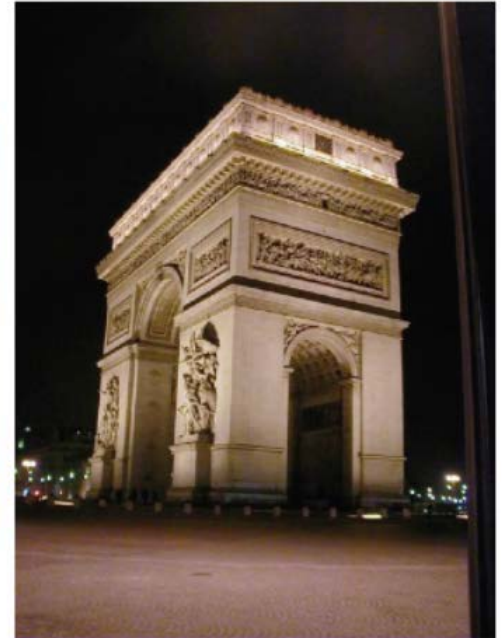
# Rank



# Nullspace



# Principal Component Analysis



=

```

^ ^ ^ ^ 3 -0.9660  0.2586
-0.0029 -0.2586 -0.9660
-1.0000 -0.0005  0.0032
    
```

d(1,1)

ans = 1.0000e+004

d(2,2)

ans = 0.0021

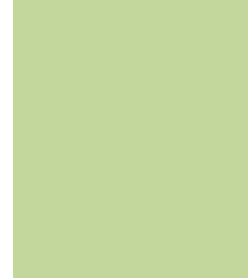
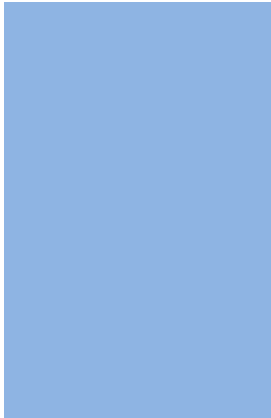
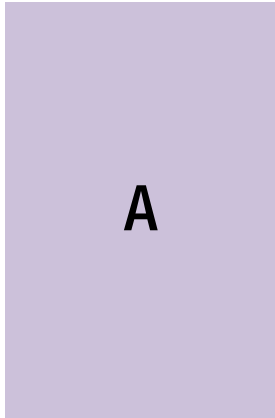
d(3,3)

ans = 2.7838e-016

Rank(F) = 2



# Matrix Inversion with SVD

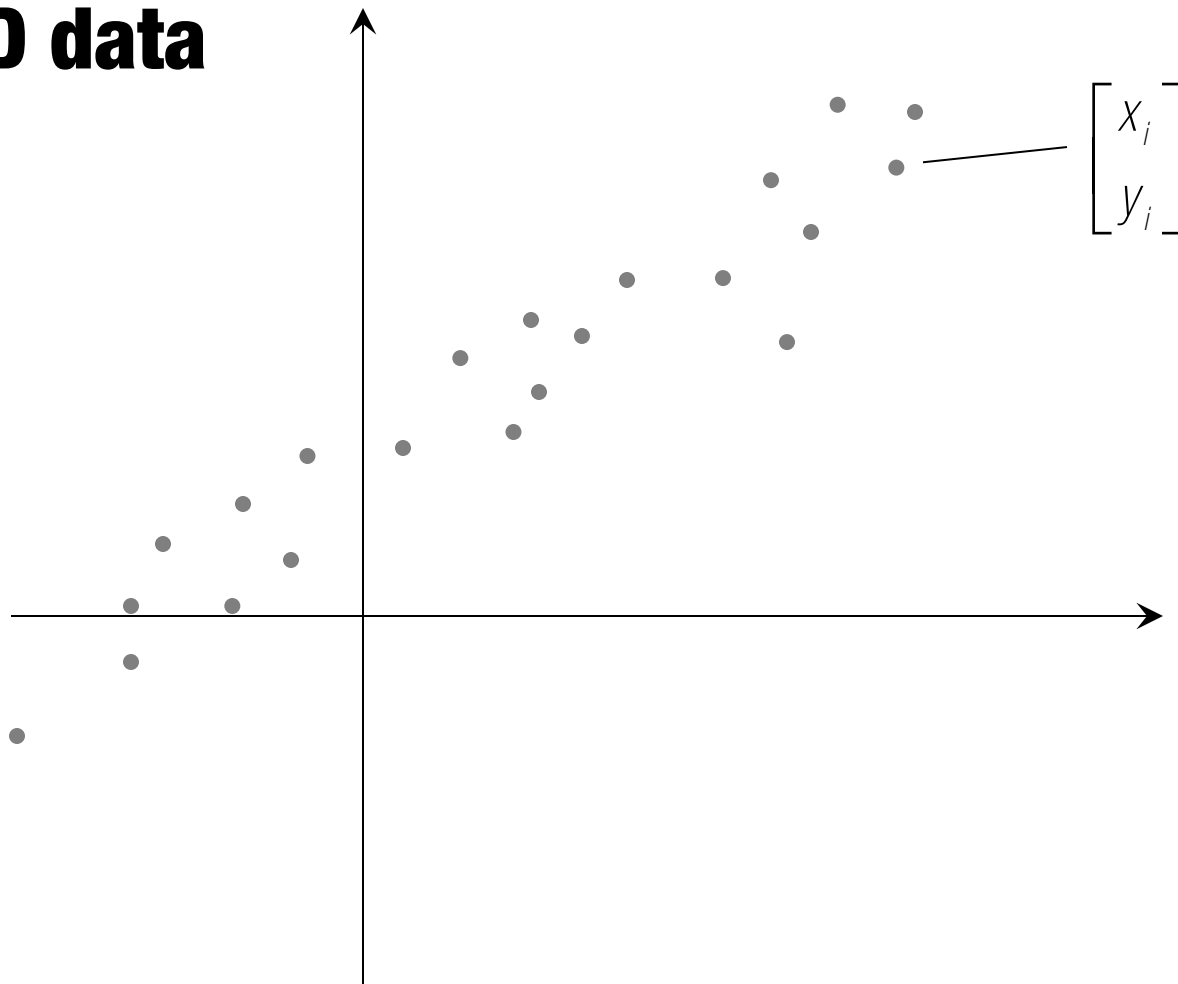


# Two types of Least Square Problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax} - \mathbf{b}\|^2 \quad \text{with} \quad \|\mathbf{b}\| \neq 0$$

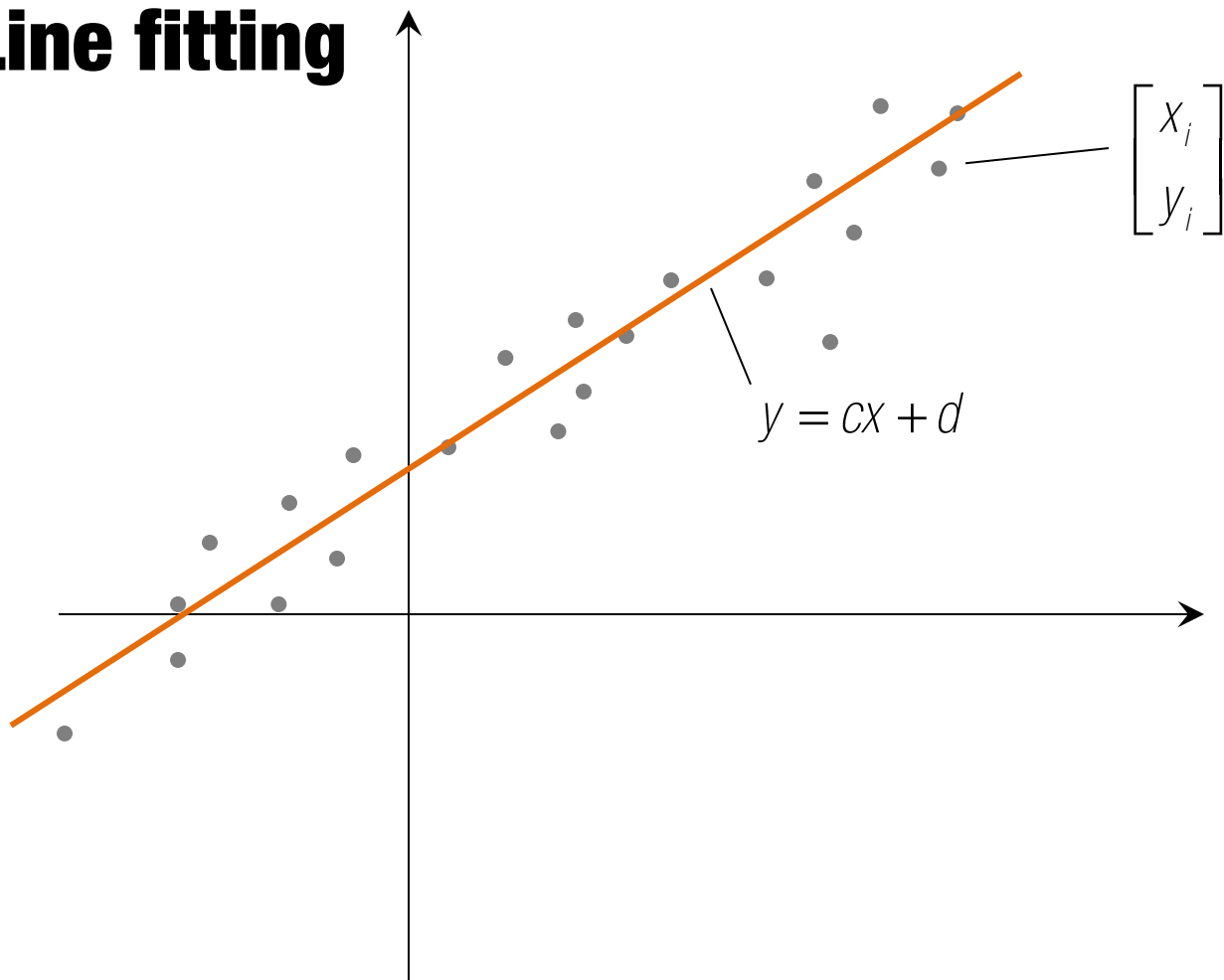
$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax}\|^2 \quad \text{s.t.} \quad \|\mathbf{x}\| = 1$$

# 2D data

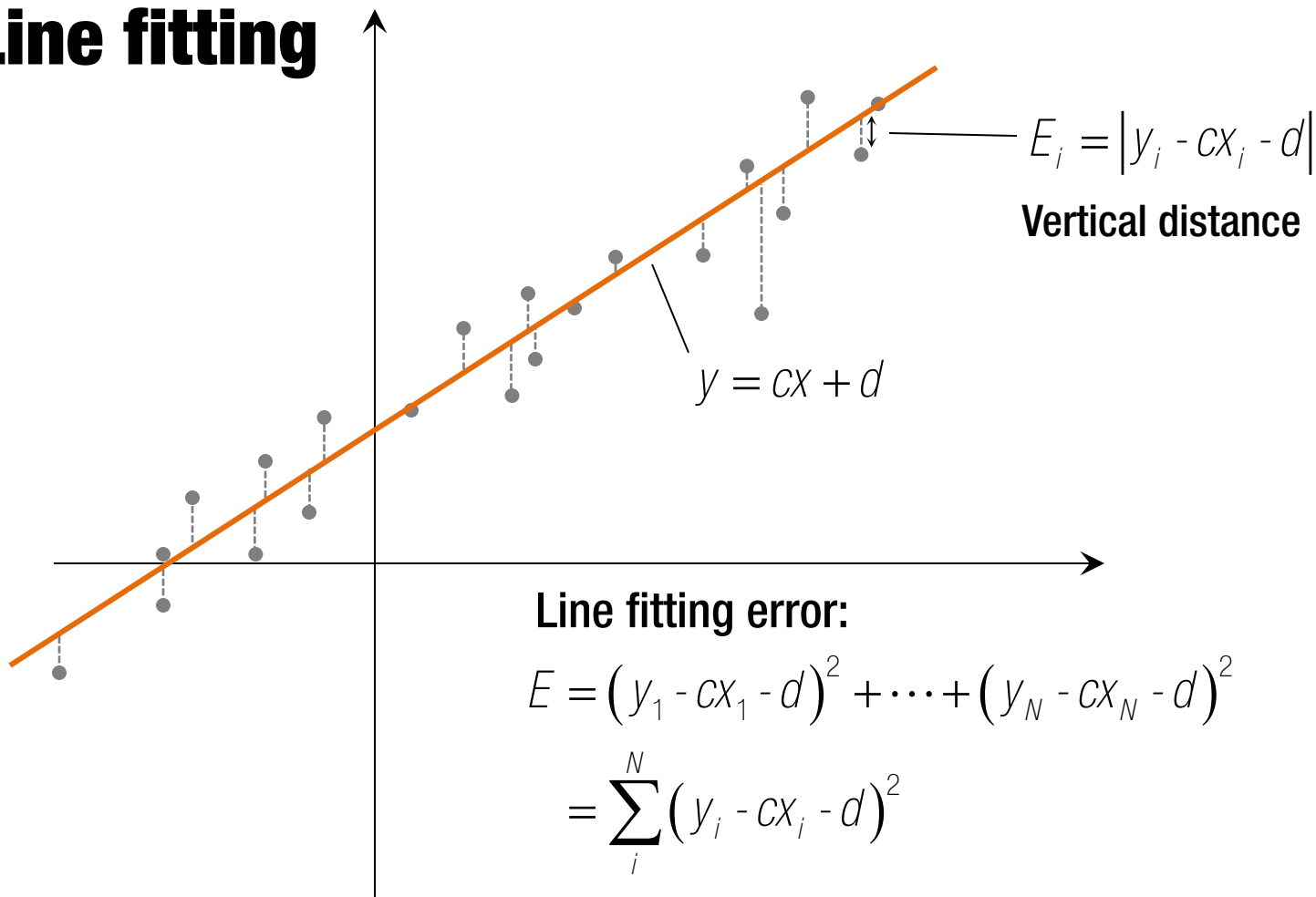




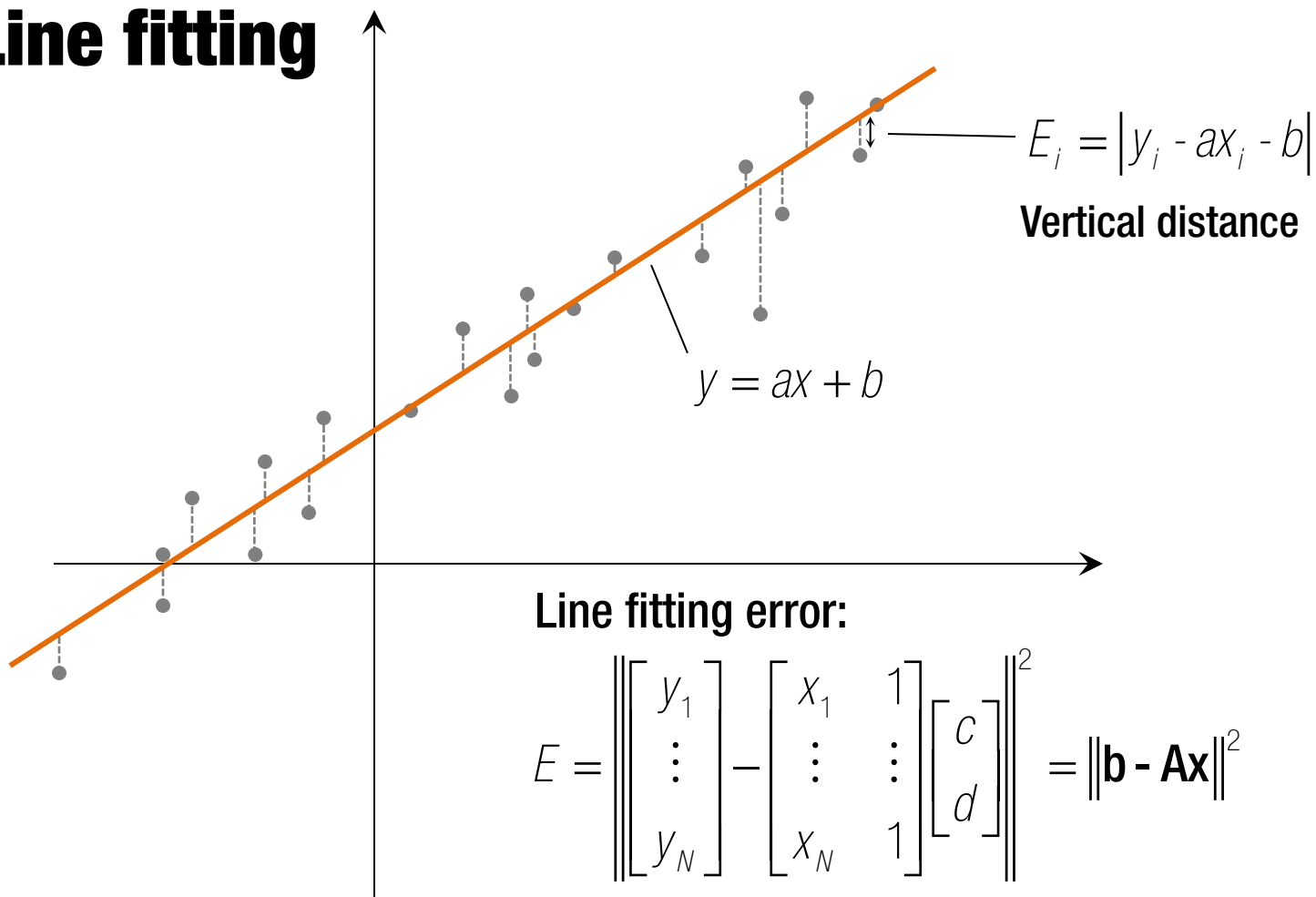
# Line fitting



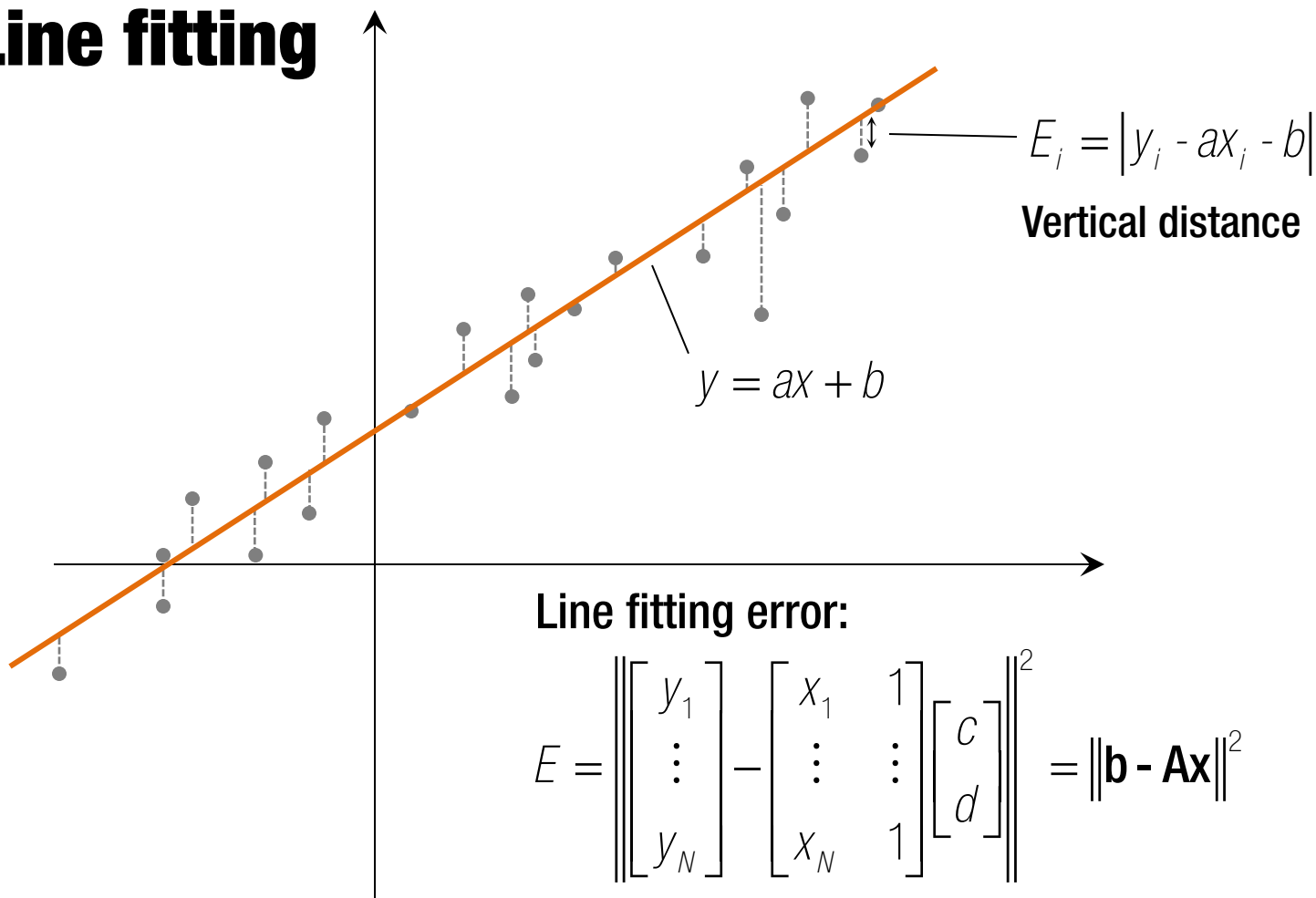
# Line fitting



# Line fitting



# Line fitting

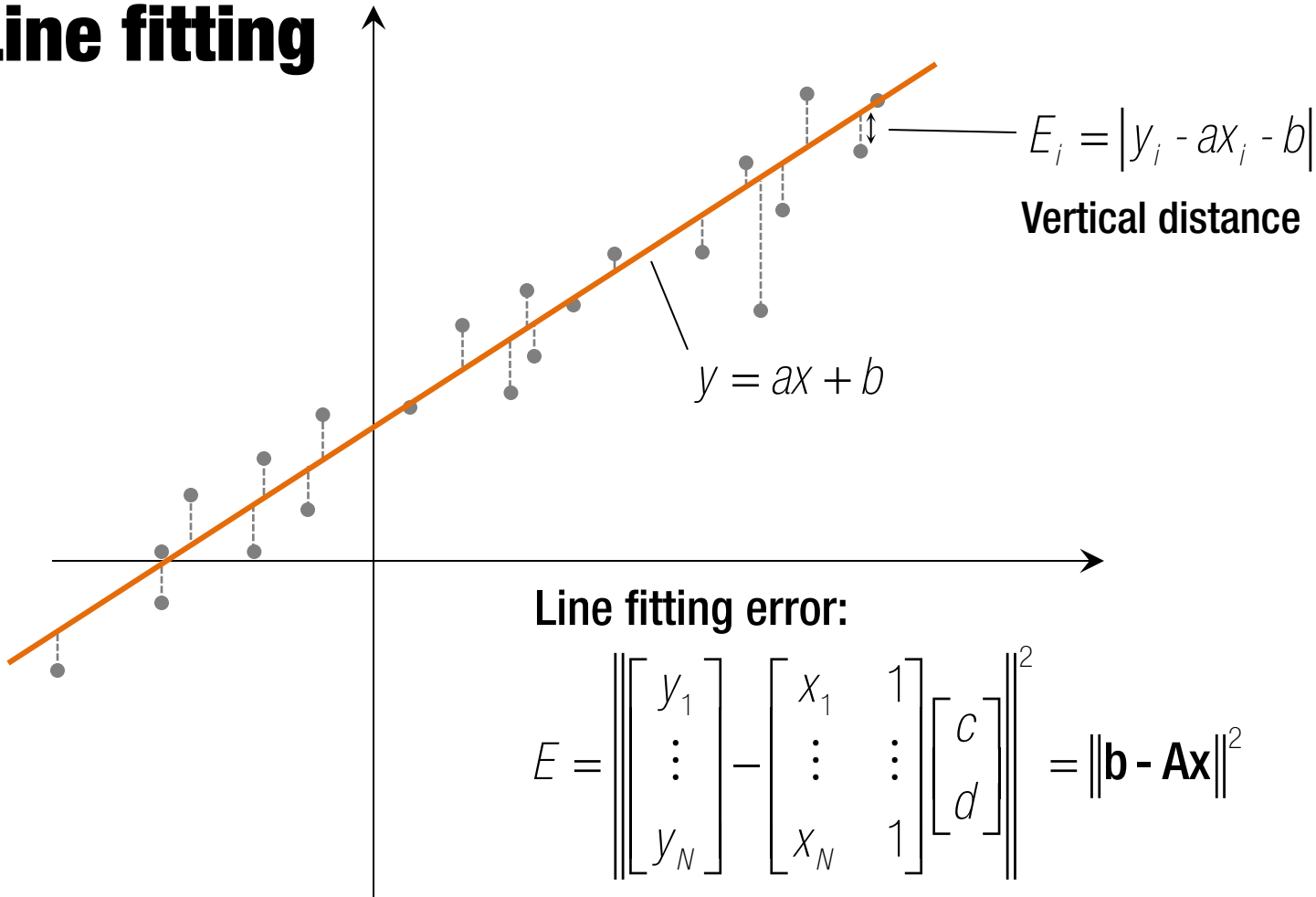


Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \|\mathbf{b} - \mathbf{Ax}\|^2$$

$$= (\mathbf{b} - \mathbf{Ax})^\top (\mathbf{b} - \mathbf{Ax}) = \mathbf{b}^\top \mathbf{b} - 2\mathbf{b}^\top \mathbf{Ax} + \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax}$$

# Line fitting



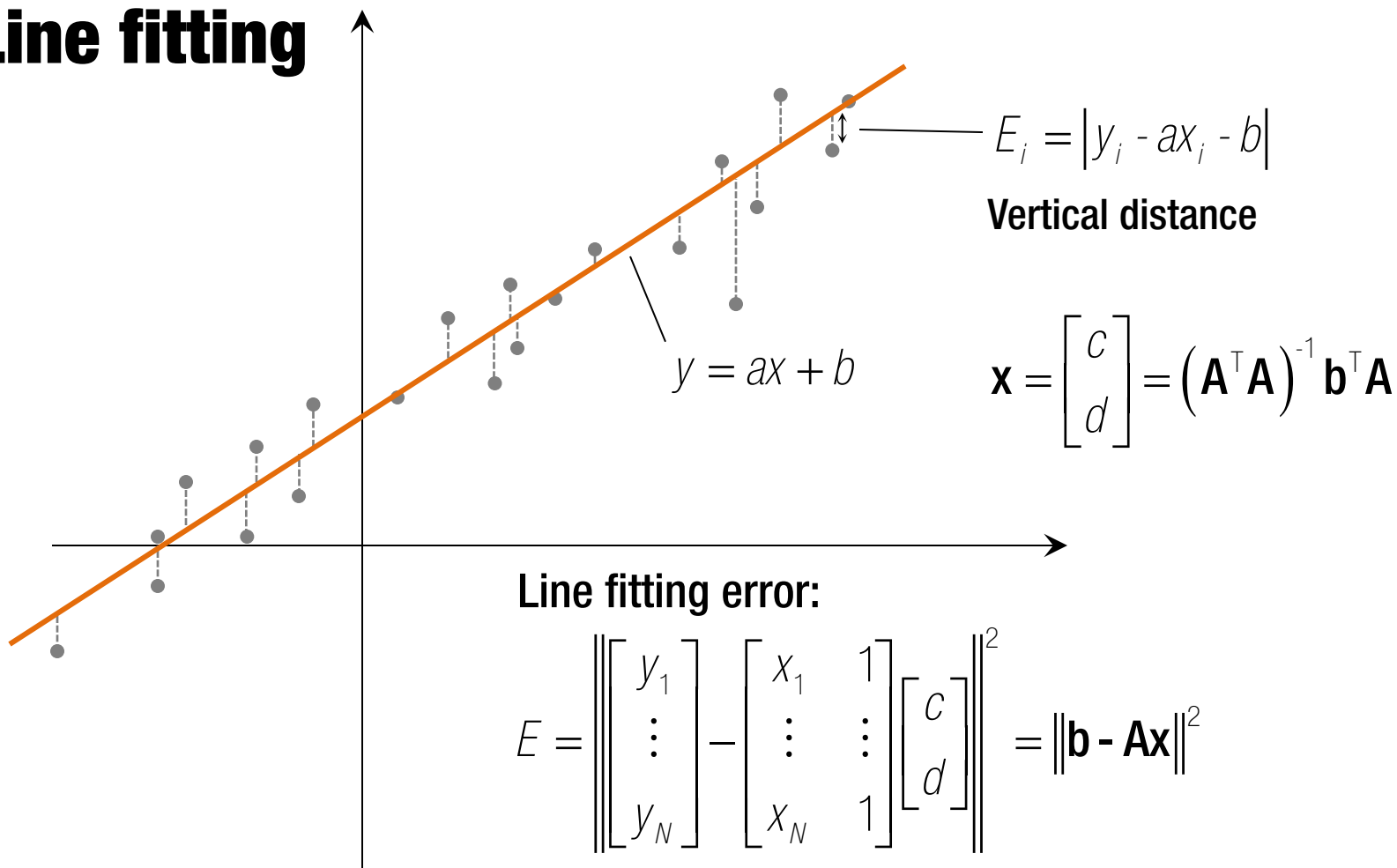
Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \|\mathbf{b} - \mathbf{Ax}\|^2$$

$$= (\mathbf{b} - \mathbf{Ax})^\top (\mathbf{b} - \mathbf{Ax}) = \mathbf{b}^\top \mathbf{b} - 2\mathbf{b}^\top \mathbf{Ax} + \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax}$$

$$\frac{\partial E}{\partial \mathbf{x}} = -2\mathbf{b}^\top \mathbf{A} + 2\mathbf{A}^\top \mathbf{Ax} = 0 \rightarrow \mathbf{A}^\top \mathbf{Ax} = \mathbf{b}^\top \mathbf{A}$$

# Line fitting



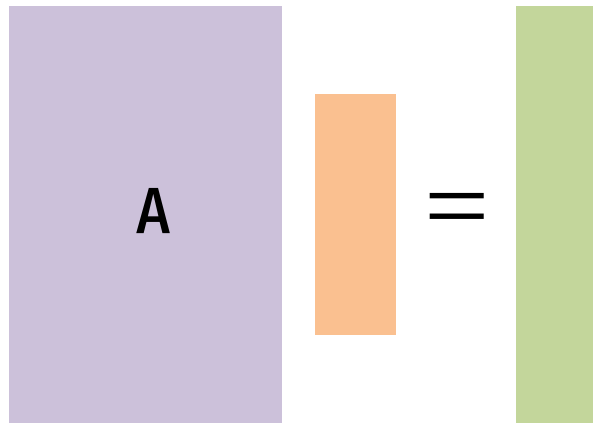
Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \|\mathbf{b} - \mathbf{Ax}\|^2$$

$$= (\mathbf{b} - \mathbf{Ax})^\top (\mathbf{b} - \mathbf{Ax}) = \mathbf{b}^\top \mathbf{b} - 2\mathbf{b}^\top \mathbf{Ax} + \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax}$$

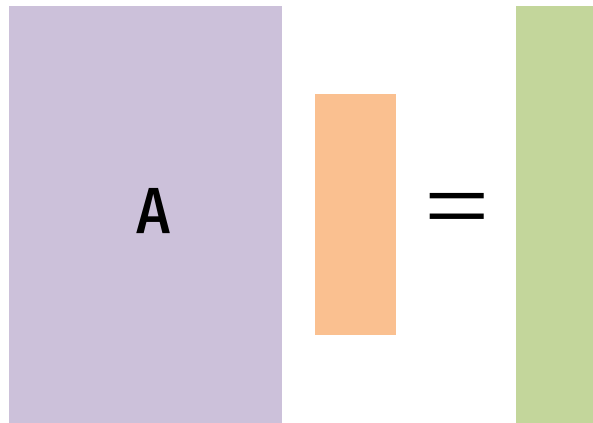
$$\frac{\partial E}{\partial \mathbf{x}} = -2\mathbf{b}^\top \mathbf{A} + 2\mathbf{A}^\top \mathbf{Ax} = 0 \rightarrow \mathbf{A}^\top \mathbf{Ax} = \mathbf{b}^\top \mathbf{A}$$

# Linear Inhomogeneous Equations



A diagram illustrating a linear inhomogeneous equation. It consists of three vertical rectangles and an equals sign. The first rectangle is purple and contains the letter 'A'. The second rectangle is orange. The third rectangle is green. The rectangles are arranged horizontally, separated by the equals sign, representing the equation  $Ax = b$ .

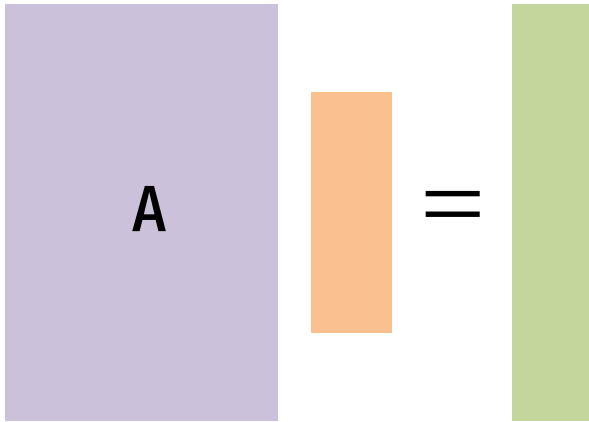
# Linear Inhomogeneous Equations



A diagram illustrating a linear inhomogeneous equation. It consists of three vertical rectangles and an equals sign. The first rectangle on the left is purple and contains the letter 'A' in its center. To its right is a smaller orange rectangle. To the right of the orange rectangle is an equals sign (=). To the right of the equals sign is a green rectangle. The purple rectangle is the tallest, followed by the green rectangle, and the orange rectangle is the shortest.



# Linear Inhomogeneous Equations



$$x = A^{-1}b$$

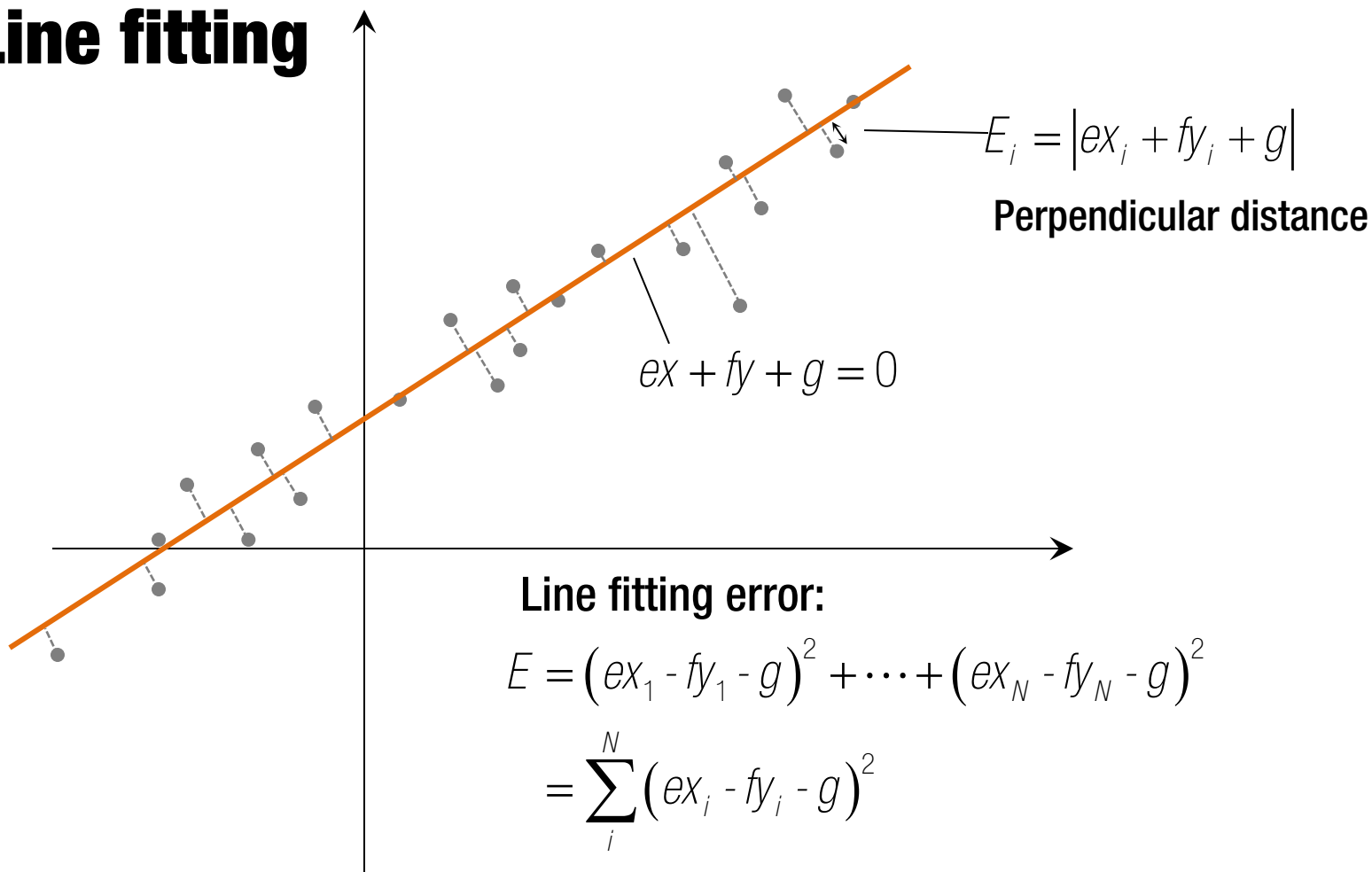
1 ...

# Two types of Least Square Problem:

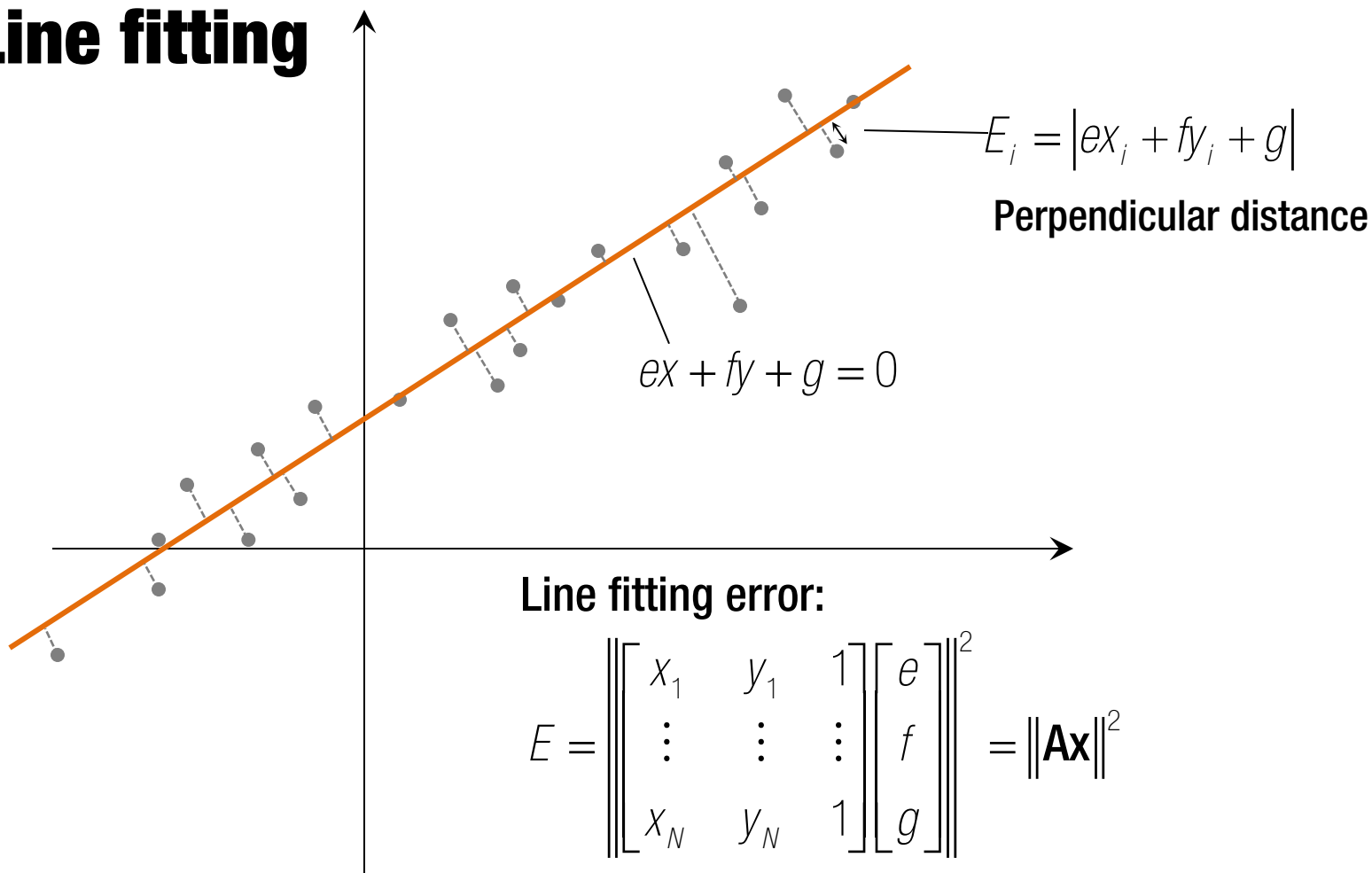
$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax} - \mathbf{b}\|^2 \quad \text{with} \quad \|\mathbf{b}\| \neq 0$$

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax}\|^2 \quad \text{s.t.} \quad \|\mathbf{x}\| = 1$$

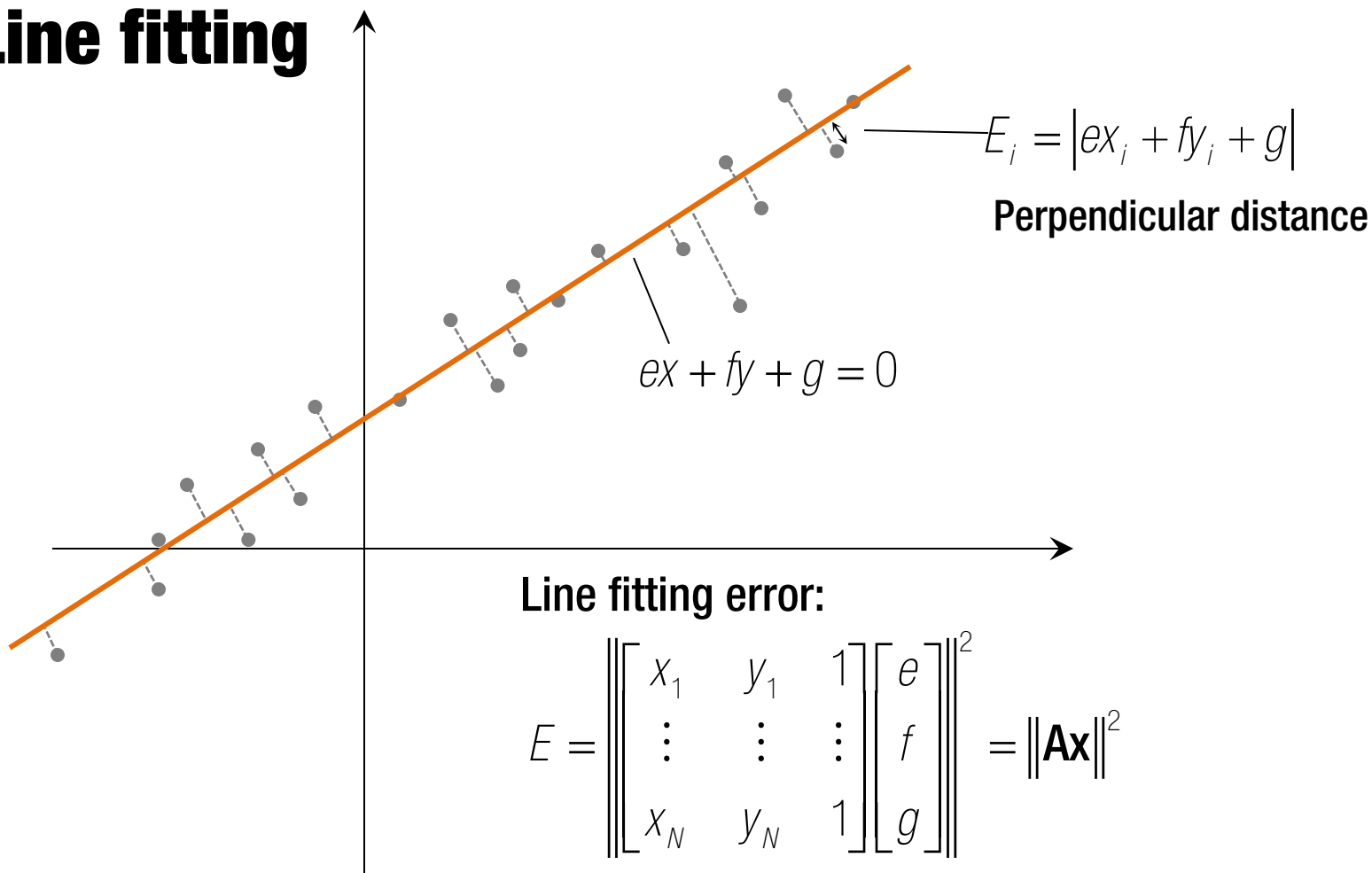
# Line fitting



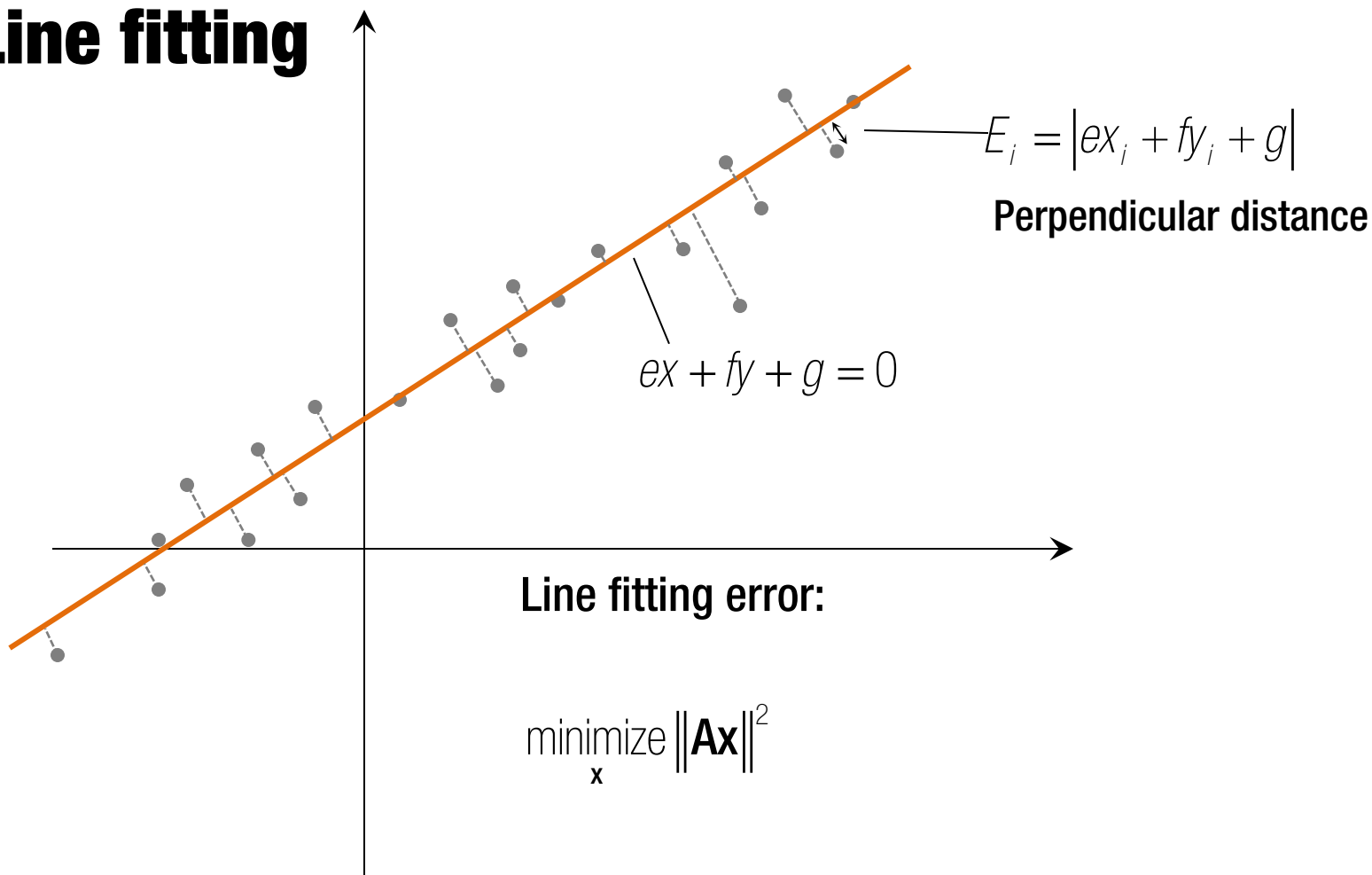
# Line fitting



# Line fitting



# Line fitting

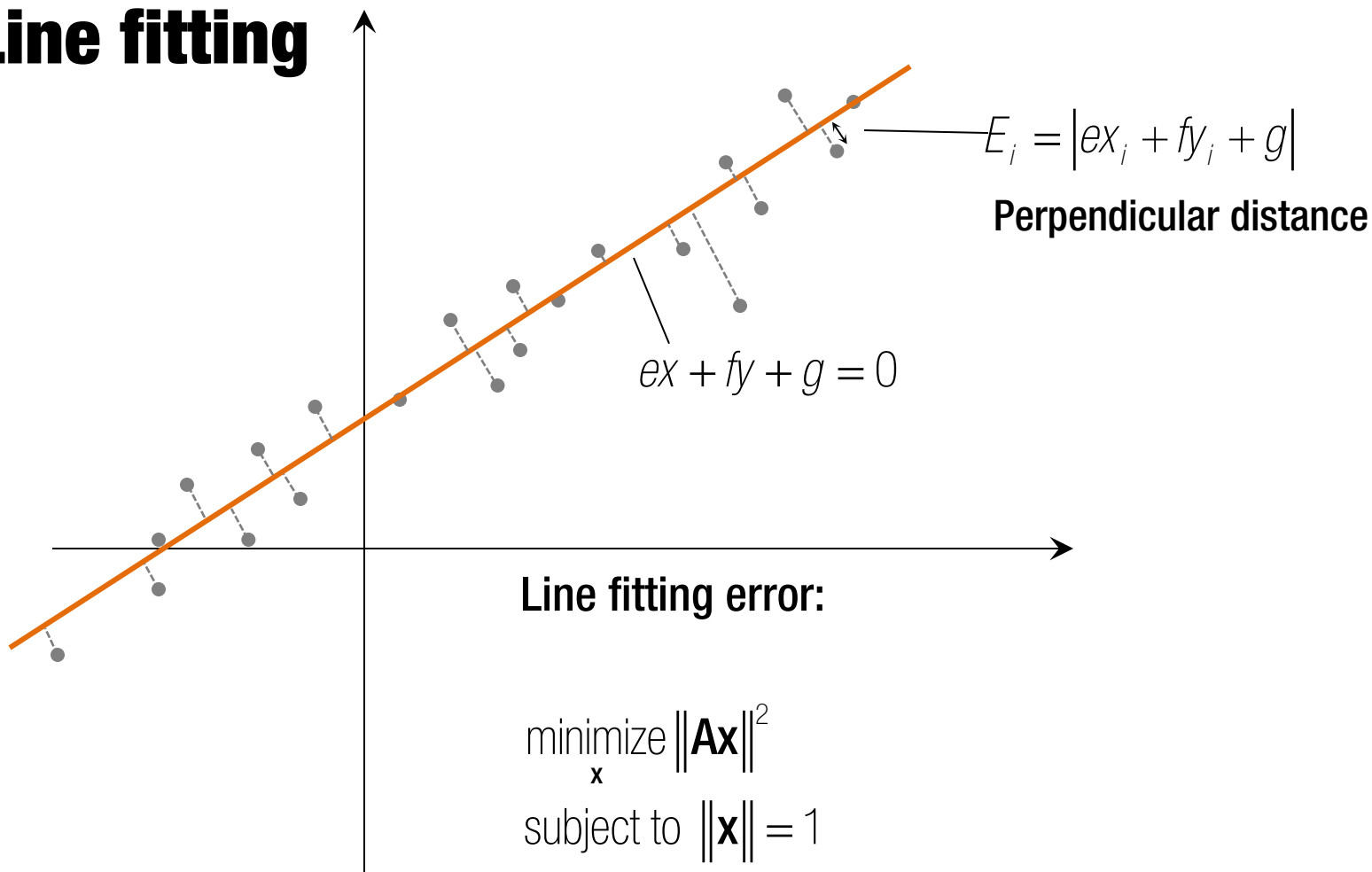


Line fitting error:

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

Trivial solution:  $\mathbf{x} = 0$

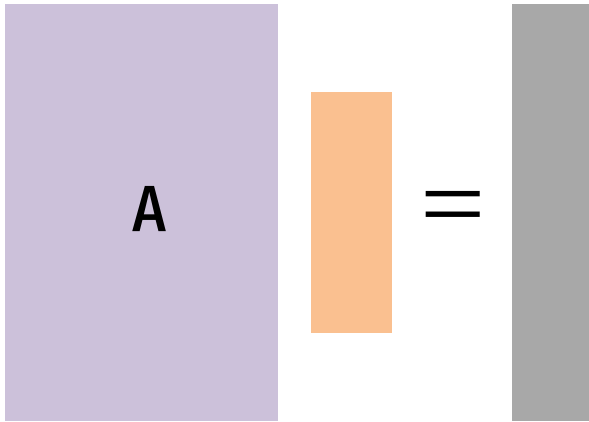
# Line fitting



$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad \|\mathbf{Ax}\|^2 \\ &\text{subject to} \quad \|\mathbf{x}\| = 1 \end{aligned}$$

$$\mathbf{x} = \mathbf{V}_3 \quad \text{where} \quad \mathbf{A} = \mathbf{UDV}^\top$$
$$\mathbf{V} = [\mathbf{V}_1 \quad \mathbf{V}_2 \quad \mathbf{V}_3]$$

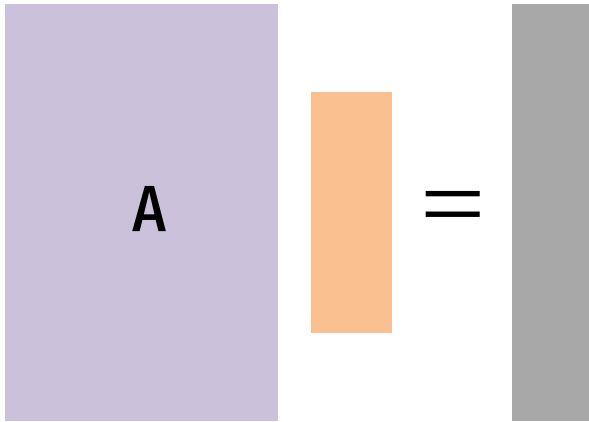
# Linear Homogeneous Equations



A diagram illustrating a linear homogeneous equation. It consists of three main components arranged horizontally: a purple rectangle on the left, an orange rectangle in the middle, and a gray rectangle on the right. The purple rectangle is labeled with the letter 'A' in its center. To the right of the orange rectangle is an equals sign (=). The orange rectangle is significantly shorter than the purple and gray rectangles, which are of equal height. The gray rectangle is positioned to the right of the equals sign, completing the visual representation of the equation  $Ax = 0$ .

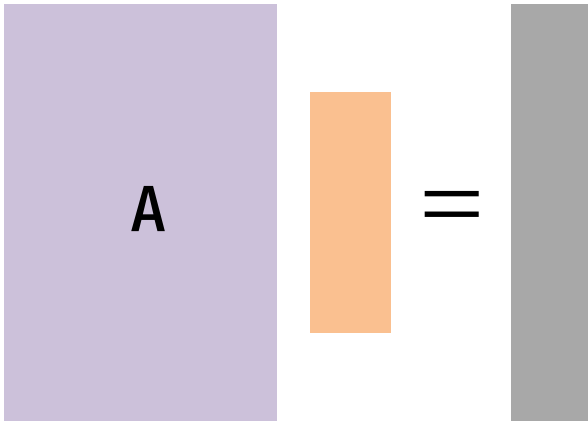


# Linear Homogeneous Equations



A diagram illustrating a linear homogeneous equation. It consists of three main components arranged horizontally: a purple rectangle on the left, an orange rectangle in the middle, and a gray rectangle on the right. The purple rectangle is labeled with the letter 'A' in its center. To the right of the orange rectangle is an equals sign (=). The orange rectangle is significantly shorter than the purple and gray rectangles, which are of equal height. The gray rectangle is positioned to the right of the equals sign, completing the visual representation of the equation  $Ax = 0$ .

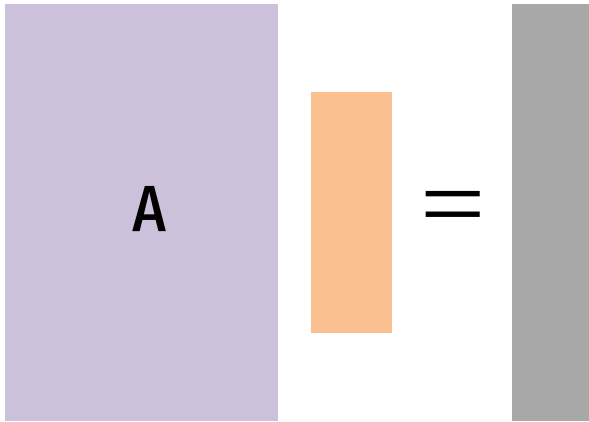
# Linear Homogeneous Equations



$$\mathbf{x} = \mathbf{V}_n$$

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# Linear Homogeneous Equations



$$\mathbf{x} = \mathbf{V}_n$$

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