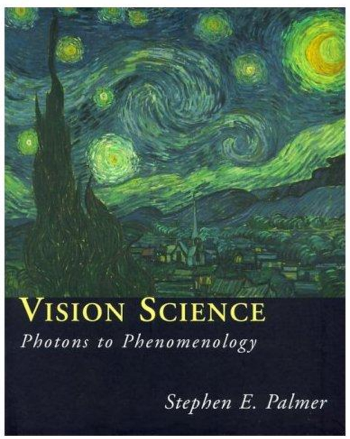
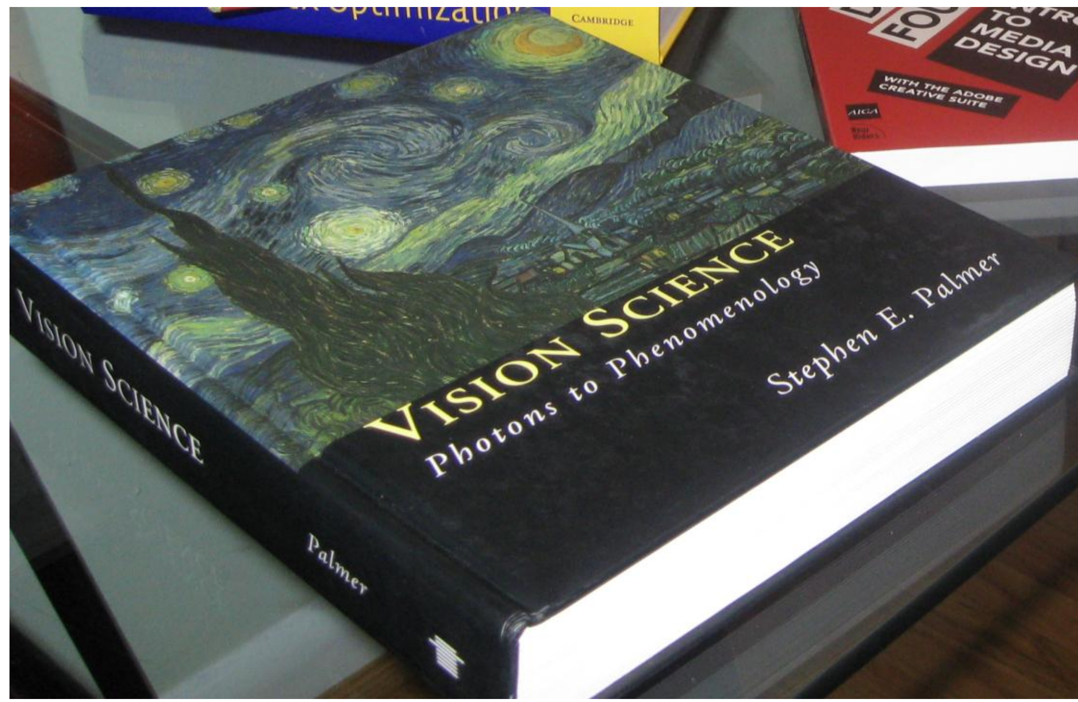
Robot Perception: Compute Projective Transformations

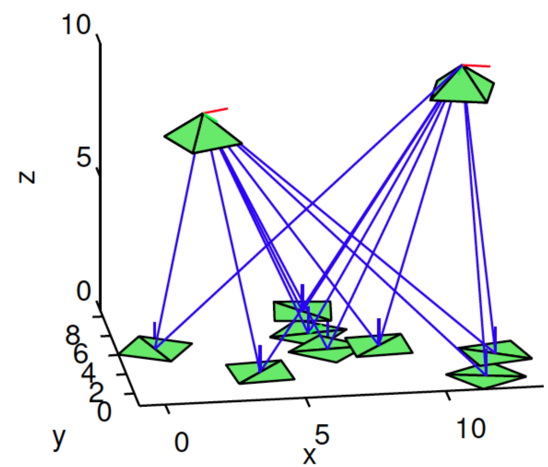
Kostas Daniilidis

A perspective projection of a plane (like a camera image) is always a projective transformation





Using the projective transformation the pose of a robot with respect to a planar pattern:



Projective Transformation

Definition

A **projective transformation** is any invertible matrix transformation $\mathbb{P}^2 \to \mathbb{P}^2$.

A projective transformation A maps p to $p' \sim Ap$.

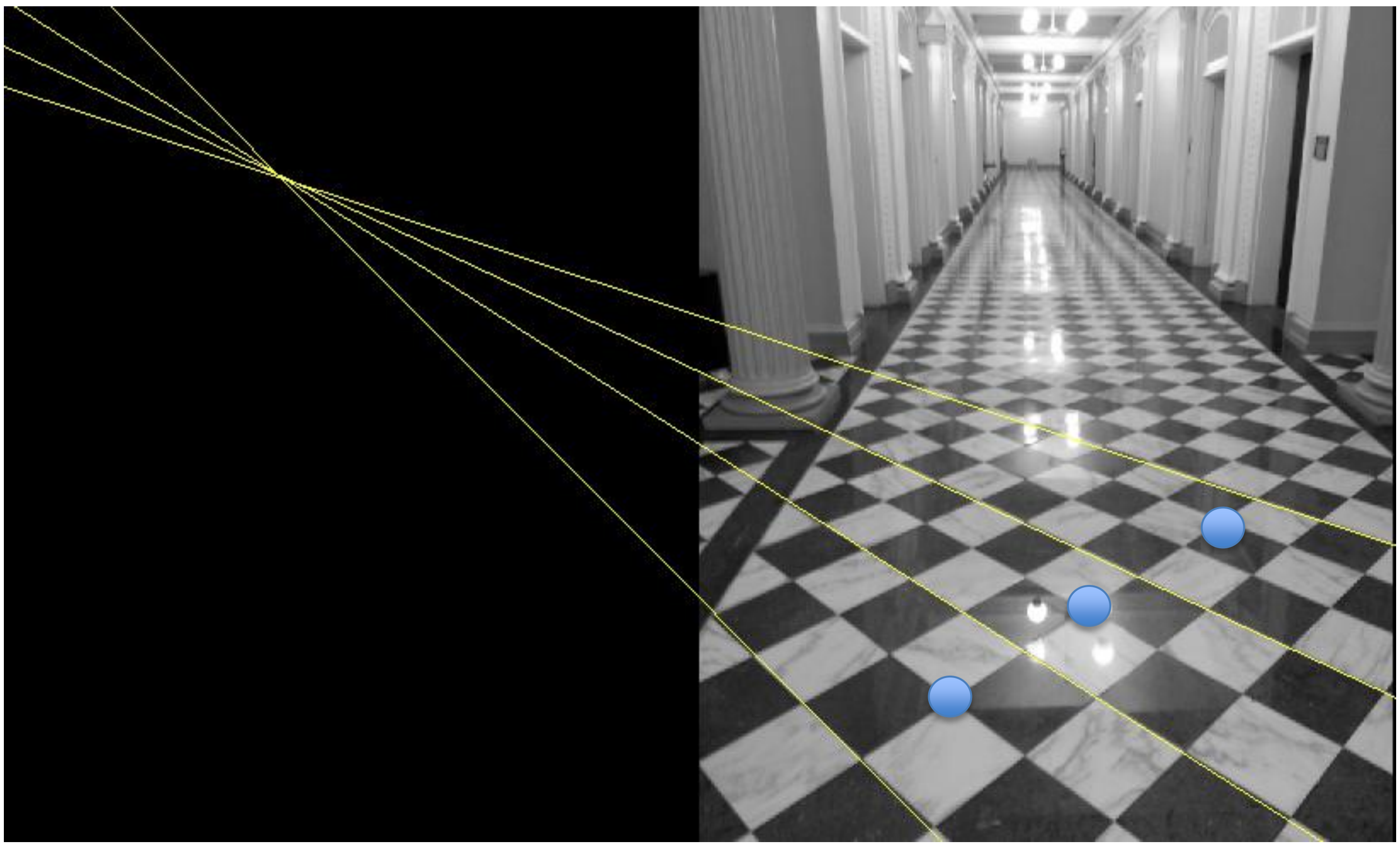
Invertibility means that $\det(A) \neq 0$ and that there exists $\lambda \neq 0$ such that $\lambda p' = Ap$.

Observe that we will write either $p' \sim Ap$ or $\lambda p' = Ap$.

A projective transformation is also known as **collineation or homography.**

A projective transformation preserves incidence:

- Three collinear points are mapped to three collinear points.
- and three concurrent lines are mapped to three concurrent lines.



Projective transformation of lines

If A maps a point to Ap, then where does a line l map to?

Line equation in original plane

$$l^T p = 0$$

Line equation in image plane $p' \sim Ap$

$$l^T A^{-1} p' = 0$$

implies that $l' = A^{-T} l$.

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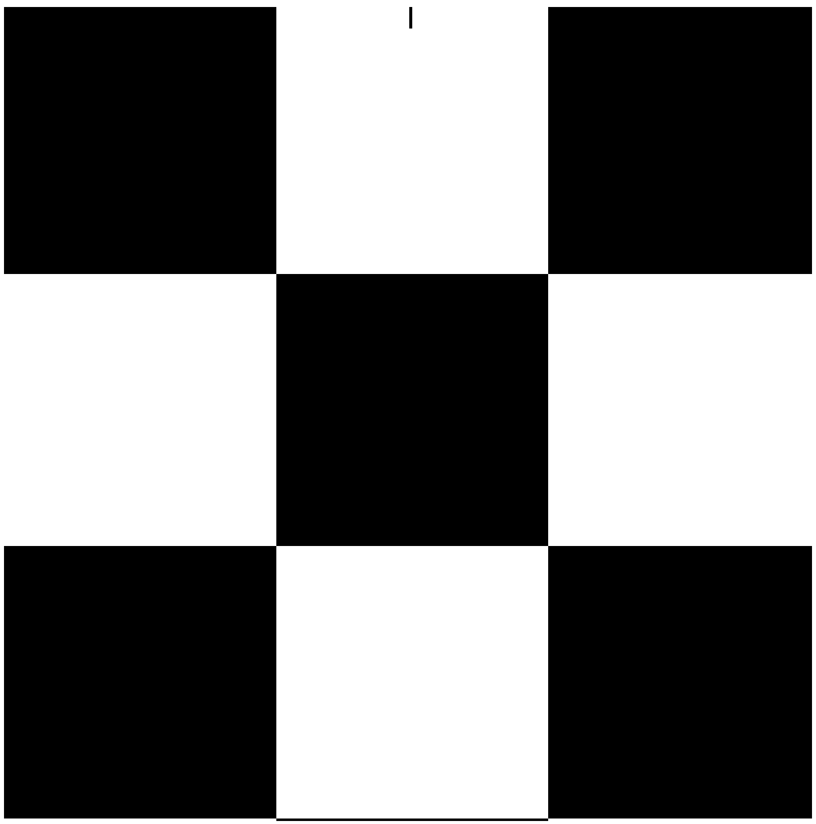
How can we compute the projective transformation between

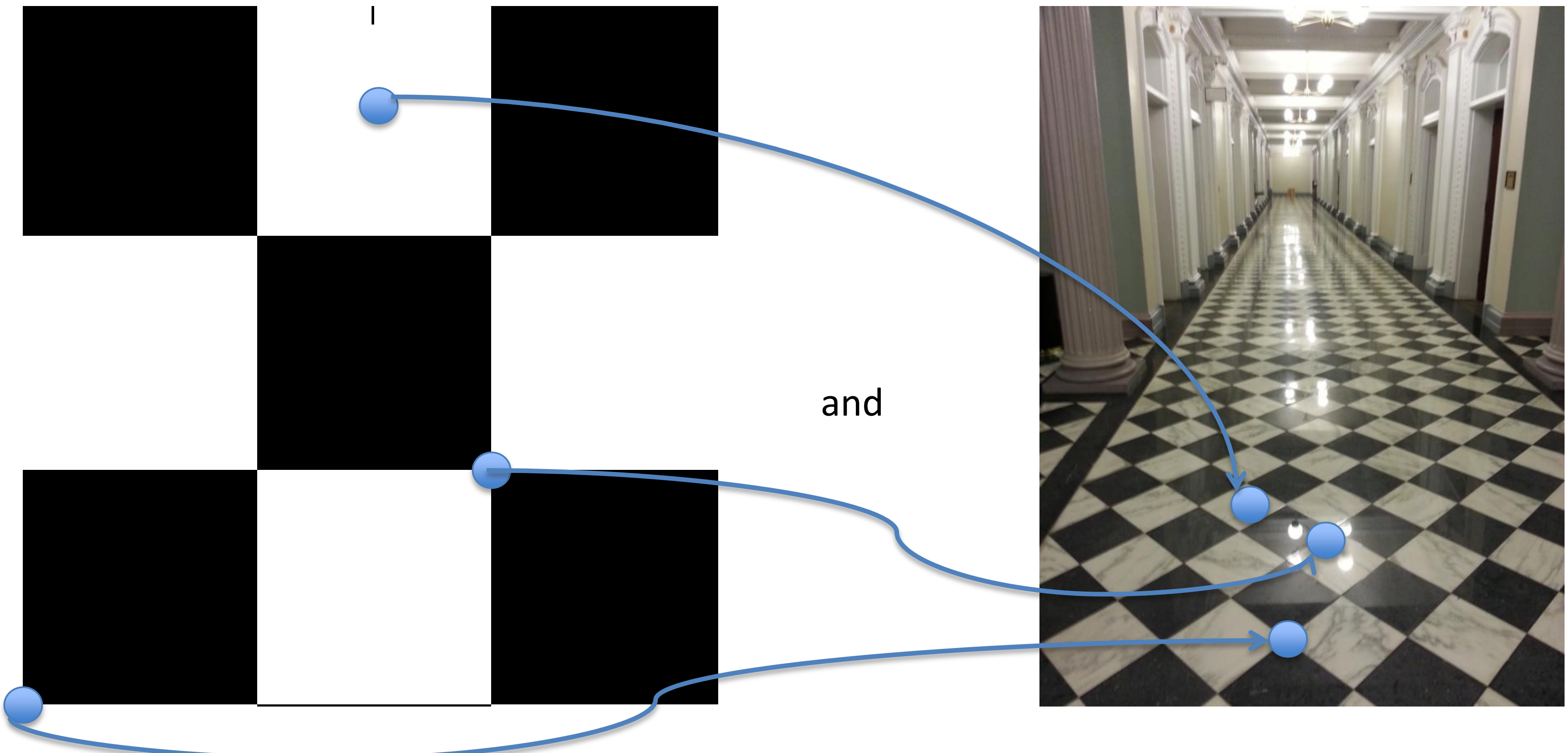


and

Floor tiles measured in [m]

Points in pixel coordinates

The result of such a transformation would map any point in one plane to the corresponding point in the other



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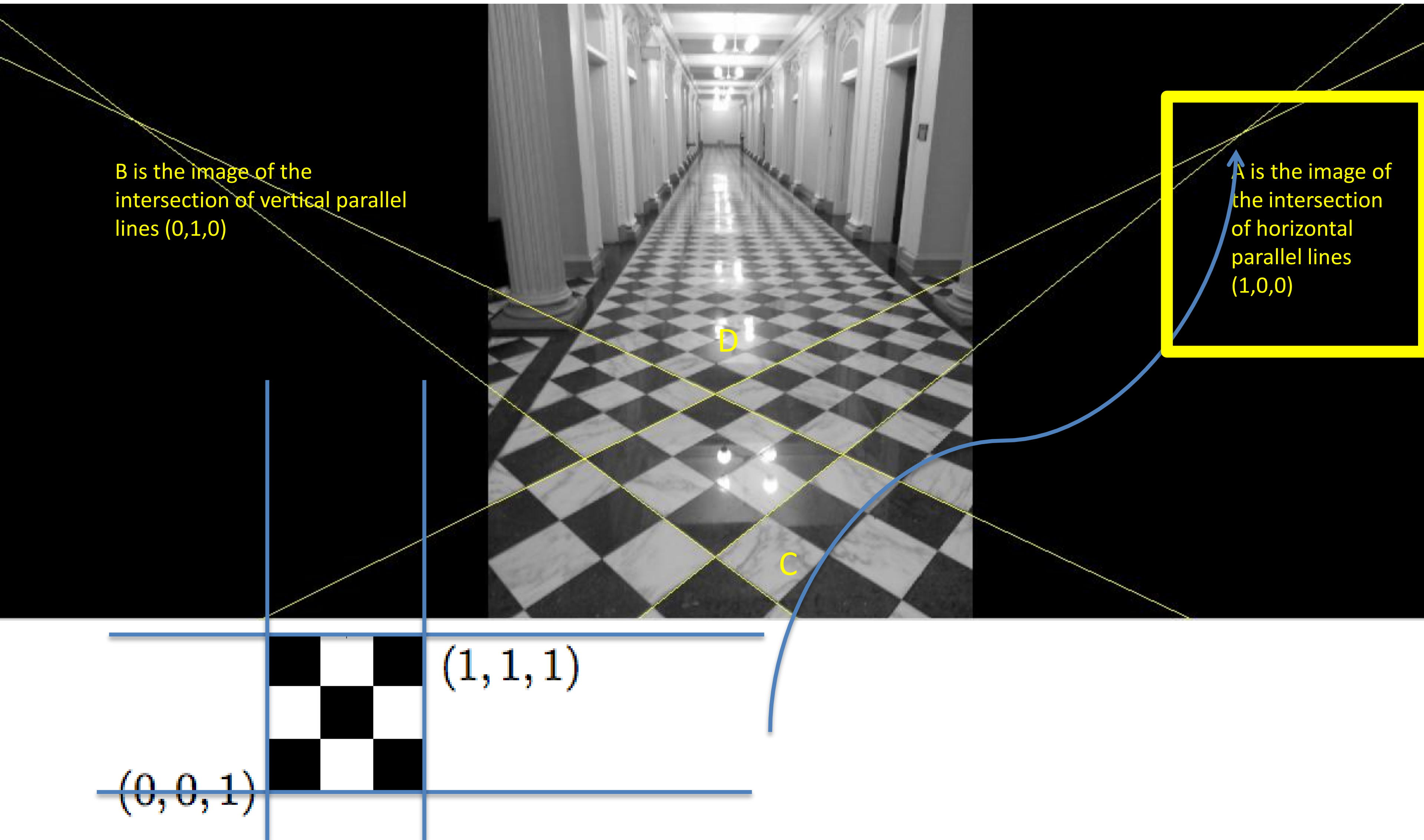

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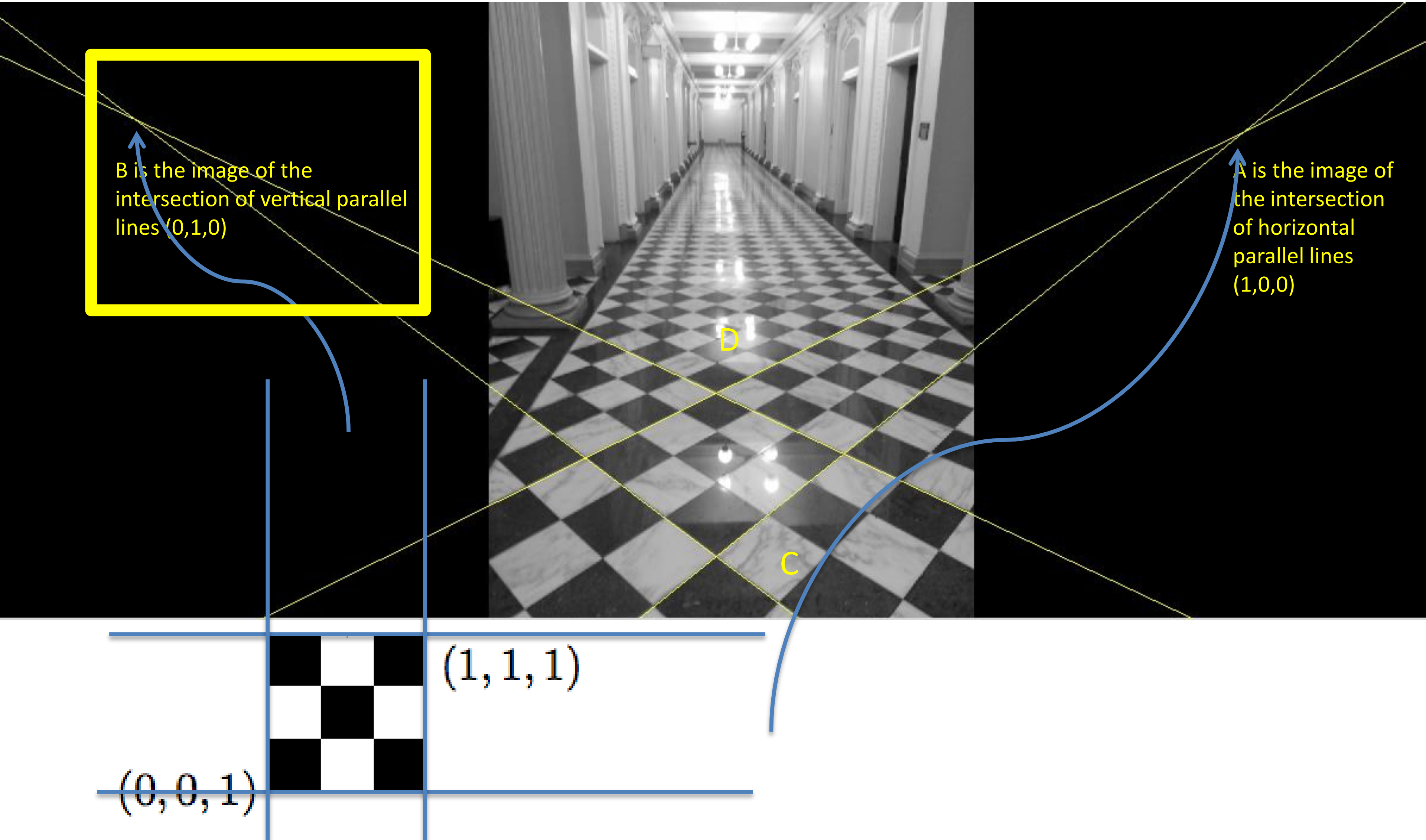

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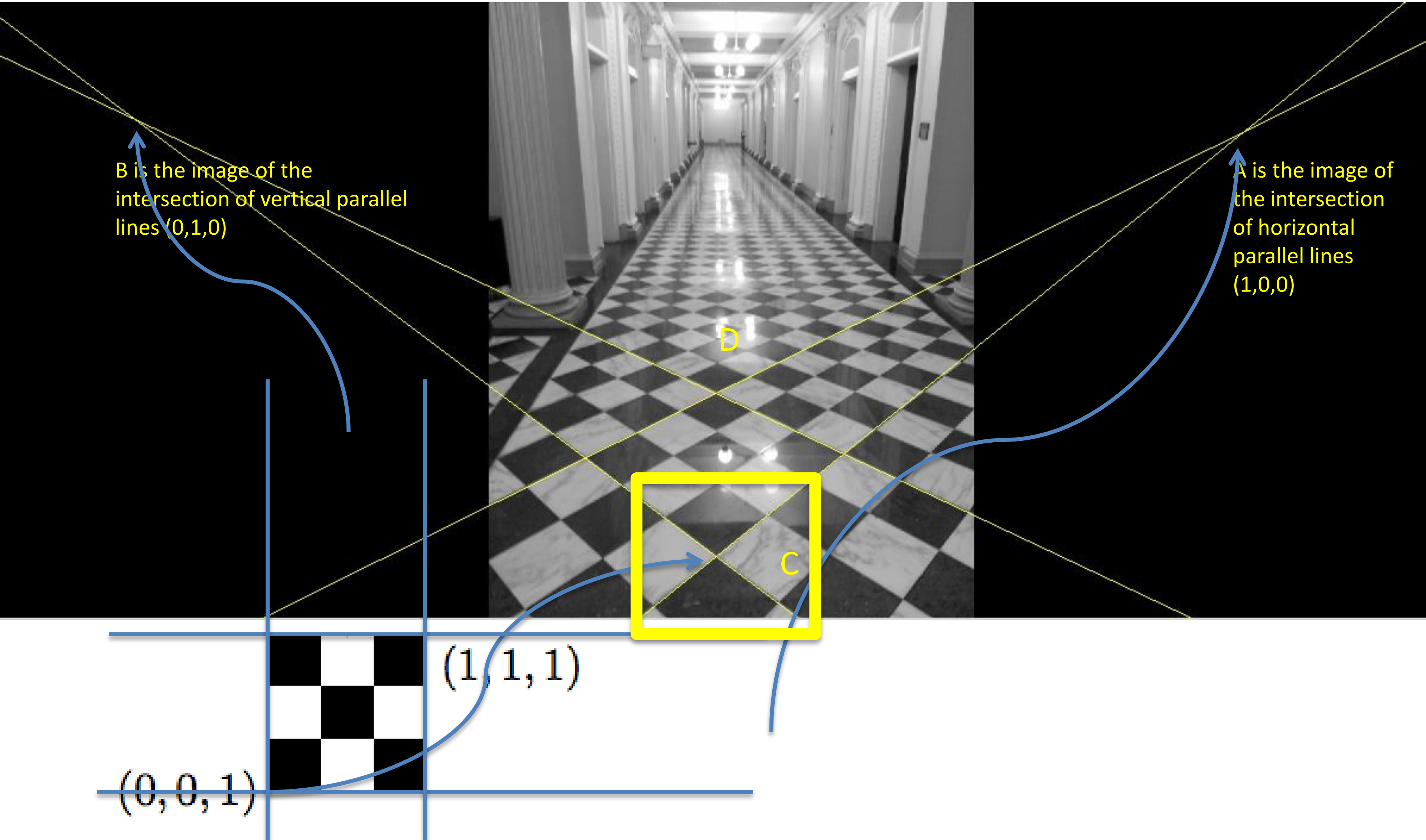


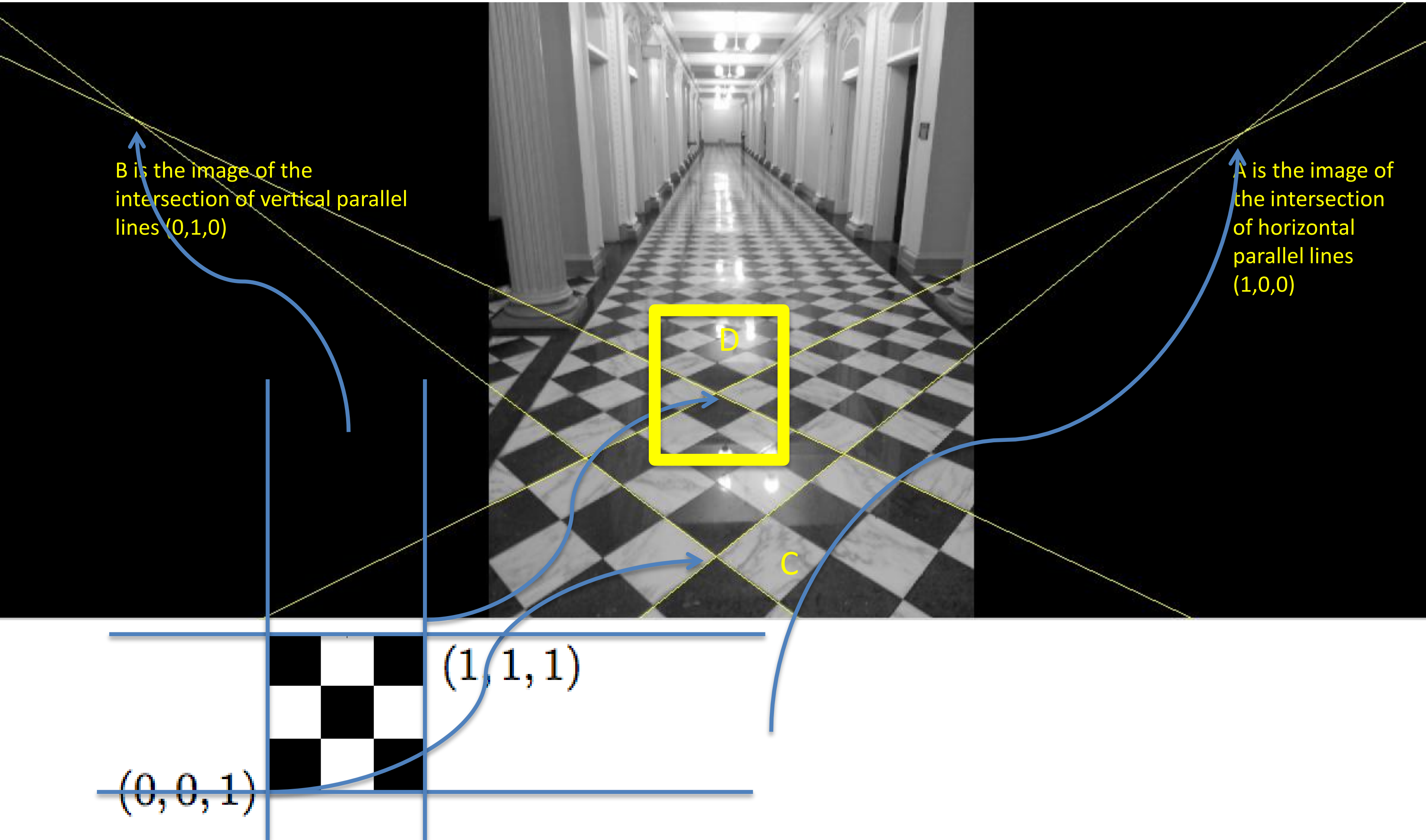
Assume that a mapping A maps the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ to the non-collinear points A,B,C

with coordinate vectors a, b and $c \in \mathbb{P}^2$. Then the following is a possible projective transformation:

$$\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with 3 degrees of freedoms α, β and γ. This means 3 points do not suffice to compute a projective transformation.

Let us introduce a 4th point D



Let us assume that the same A maps $(1, 1, 1)$ to the point d. Then, the following should hold:

$$\lambda d = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

hence

$$\lambda d = \alpha a + \beta b + \gamma c.$$

There always exist such $\lambda, \alpha, \beta, \gamma$ because four elements of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ are always linearly dependent.

Because a, b, c are not collinear, there exist unique $\alpha/\lambda, \beta/\lambda, \gamma/\lambda$ for writing this linear combination.

Since A is the same as A/λ we solve for α, β, γ such that $d = \alpha a + \beta b + \gamma c$, which can be written as a linear system

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = d.$$

Since a, b, c are not collinear we can always find a unique triple α, β, γ. The resulting projective transformation is $A = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix}$.

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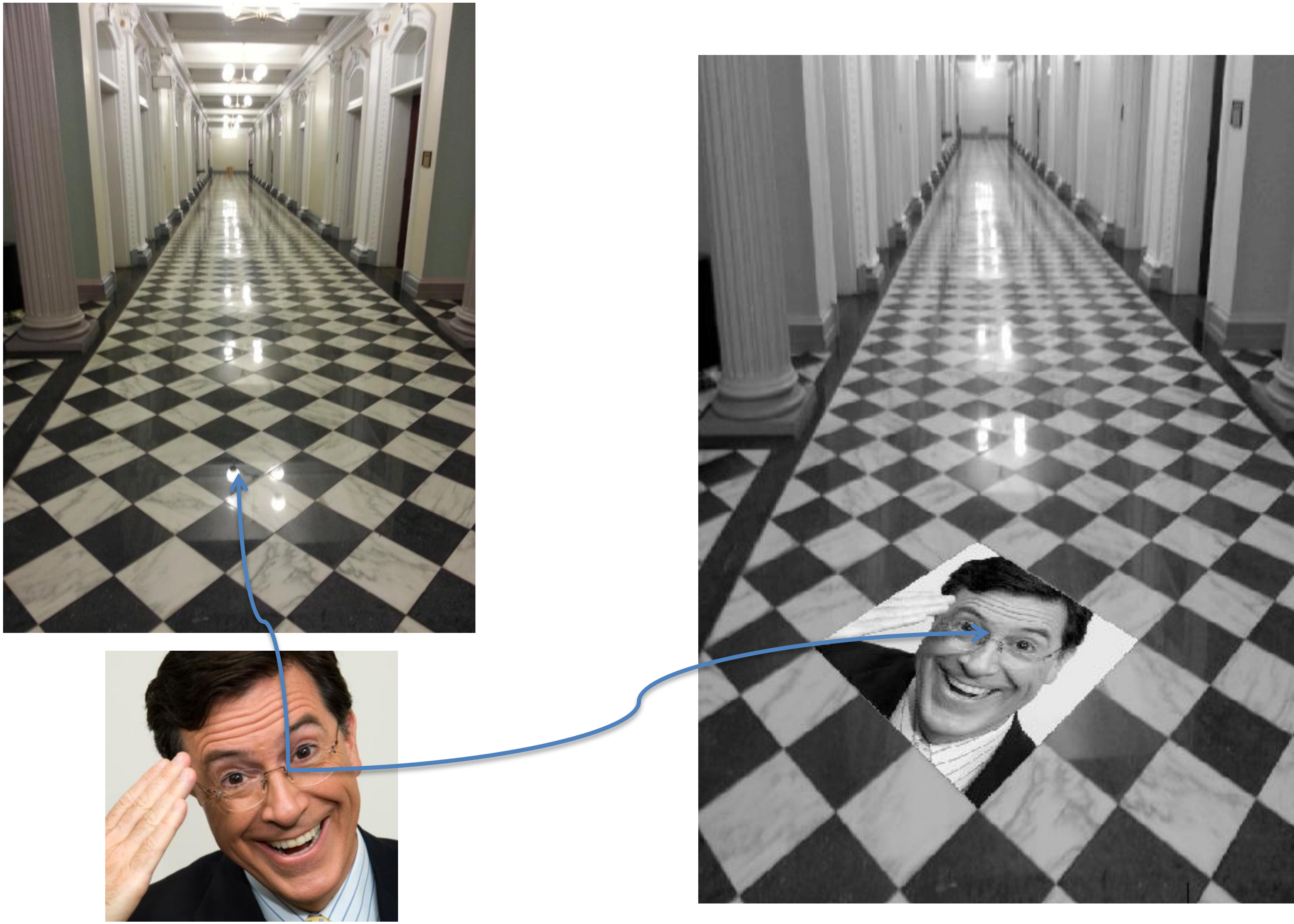
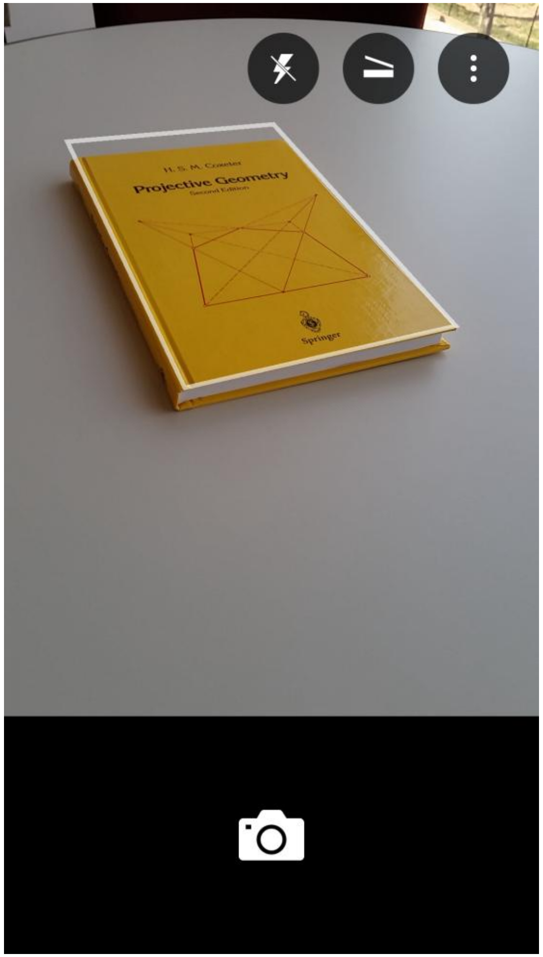
Four points not three of them collinear suffice to recover unambiguously a projective transformation.

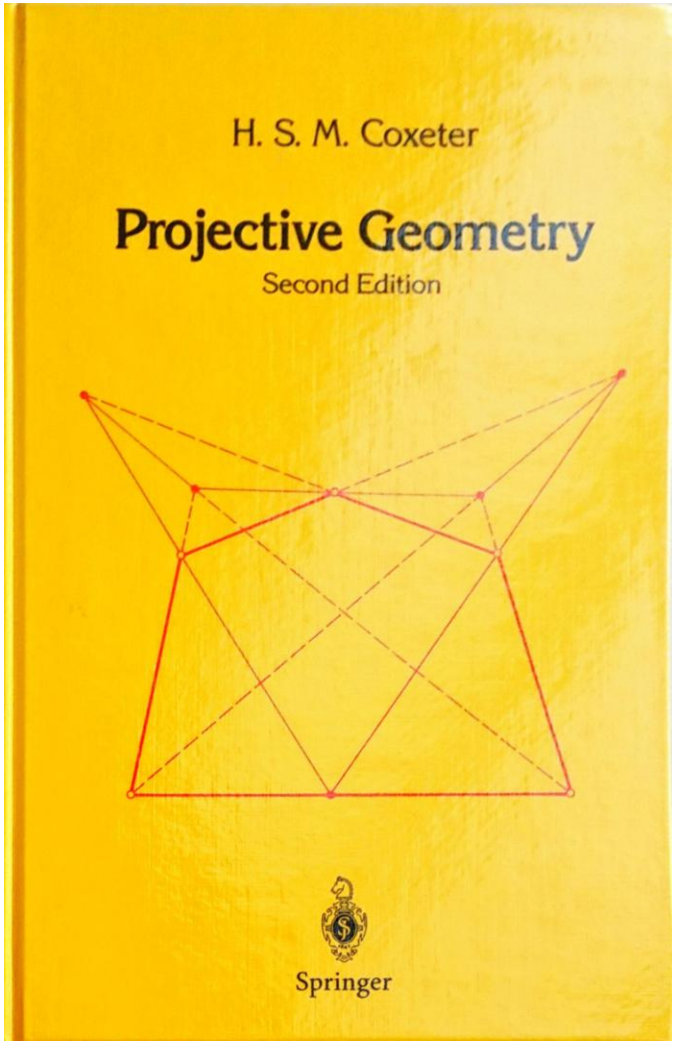
Knowledge of this projective transformation makes Virtual Billboards possible!



Microsoft Office Lens App

Office Lens


H. S. M. Coxeter
Projective Geometry
Second Edition
Springer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What happens when the original set of points is not a square?

Find projective transformation mapping $(a, b, c, d) \rightarrow (a', b', c', d')$:

To determine this mapping we go through the four canonical points.

We find the mapping from $(1, 0, 0), etc$ to (a, b, c, d) and we call it T:

$$a \sim T(1, 0, 0)^T, etc$$

We find the mapping from $(1, 0, 0), etc$ to (a', b', c', d') and we call it T':

$$a' \sim T'(1, 0, 0)^T, etc$$

Then, back-substituing $(1, 0, 0)^T \sim T^{-1}a, etc$ we obtain that

$$a' = T'T^{-1}a, etc$$

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