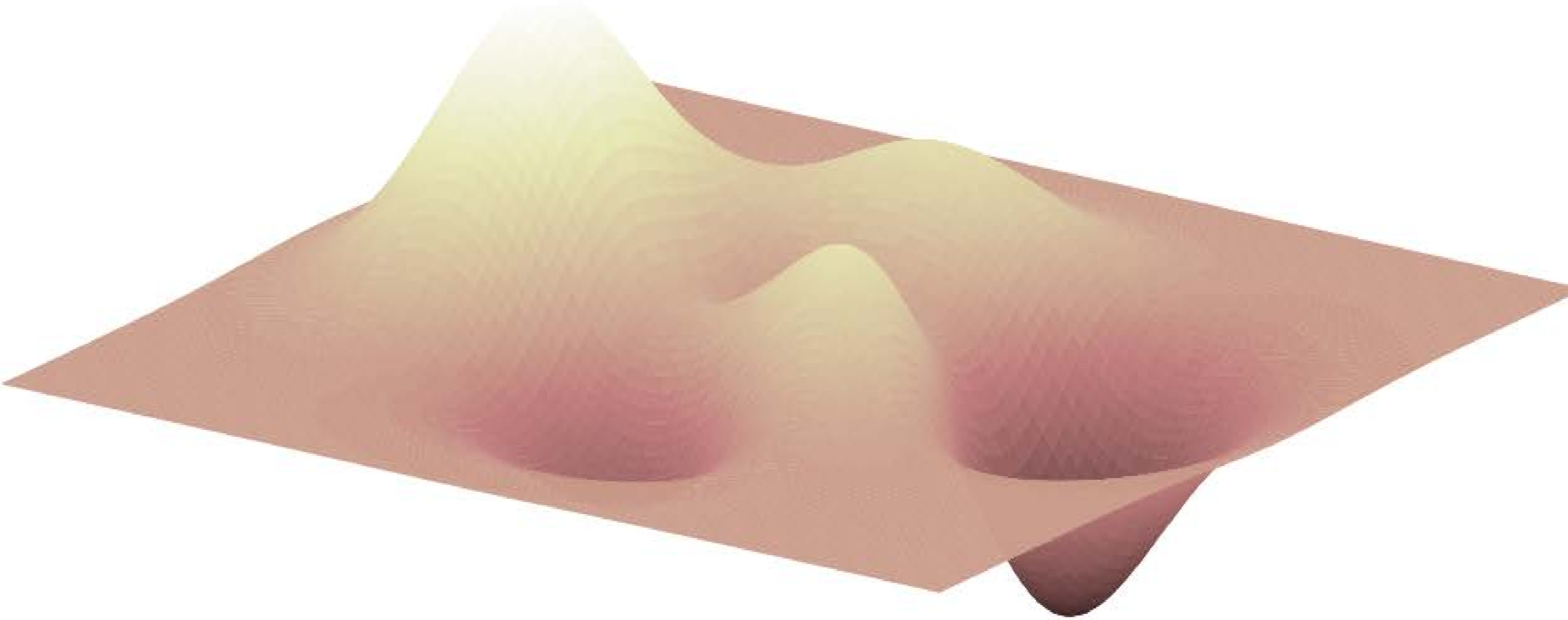
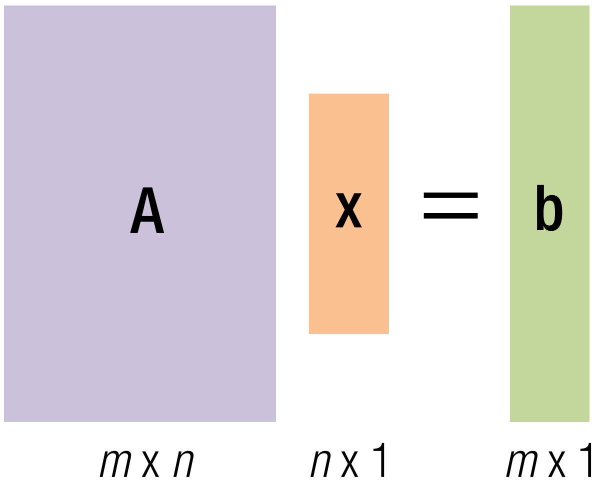
Nonlinear Least Squares



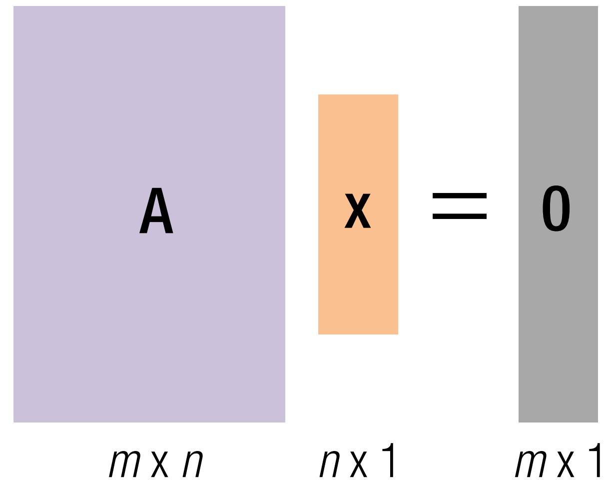
So far,

Type I



$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|^2$$

Type II



$$\min_{\mathbf{x}} \|\mathbf{Ax}\|^2 \quad \text{subject to} \quad \|\mathbf{x}\| = 1$$

Why Linear Least Squares?

$$\min_{x} \left\| Ax - b \right\|^2 = \min_{x} \left(Ax - b \right)^{T} \left(Ax - b \right)$$

Why Linear Least Squares?

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2 = \min_{\mathbf{x}} (\mathbf{A}\mathbf{x}-\mathbf{b})^\top (\mathbf{A}\mathbf{x}-\mathbf{b})$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x} - 2\mathbf{b}^\top \mathbf{A}\mathbf{x} - \mathbf{b}^\top \mathbf{b}$$

Why Linear Least Squares?

$$\min_{x} \|Ax - b\|^2 = \min_{x} (Ax - b)^\top (Ax - b)$$

$$= \min_{x} x^\top A^\top Ax - 2b^\top Ax - b^\top b$$

$$= \min_{x} x^\top A^\top Ax - 2b^\top Ax$$

Why Linear Least Squares?

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|^2 = \min_{\mathbf{x}} (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b})$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax} - \mathbf{b}^\top \mathbf{b}$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax}$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{Qx} + \mathbf{cx} \qquad \text{where } \mathbf{Q} = \mathbf{A}^\top \mathbf{A}, \quad \mathbf{c} = -2\mathbf{b}^\top \mathbf{A}$$

Why Linear Least Squares?

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2 = \min_{\mathbf{x}} (\mathbf{A}\mathbf{x}-\mathbf{b})^\top(\mathbf{A}\mathbf{x}-\mathbf{b})$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top\mathbf{A}^\top\mathbf{A}\mathbf{x} - 2\mathbf{b}^\top\mathbf{A}\mathbf{x} - \mathbf{b}^\top\mathbf{b}$$

$$= \min_{\mathbf{x}} \mathbf{x}^\top\mathbf{A}^\top\mathbf{A}\mathbf{x} - 2\mathbf{b}^\top\mathbf{A}\mathbf{x}$$

$$= \min_{\mathbf{x}} \underbrace{\mathbf{x}^\top\mathbf{Q}\mathbf{x} + \mathbf{c}\mathbf{x}}_{\text{Quadratic equation in } \mathbf{x}.} \qquad \text{where } \mathbf{Q} = \mathbf{A}^\top\mathbf{A}, \quad \mathbf{c} = -2\mathbf{b}^\top\mathbf{A}$$

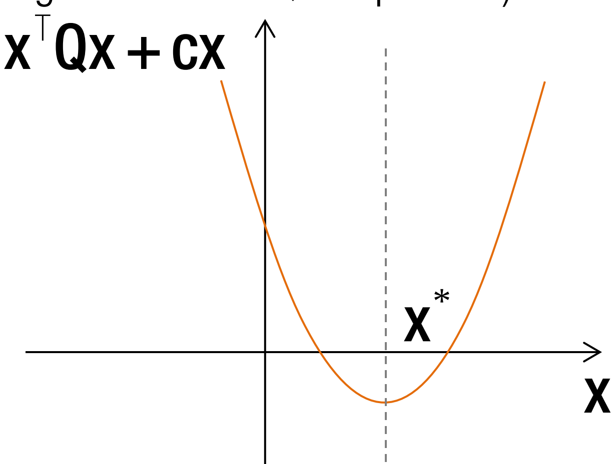
$\mathbf{Q}$ is positive definite (all eigen values of $\mathbf{Q}$ are positive). $\longrightarrow$ There exists the global solution $\mathbf{x}$.

Why Linear Least Squares?

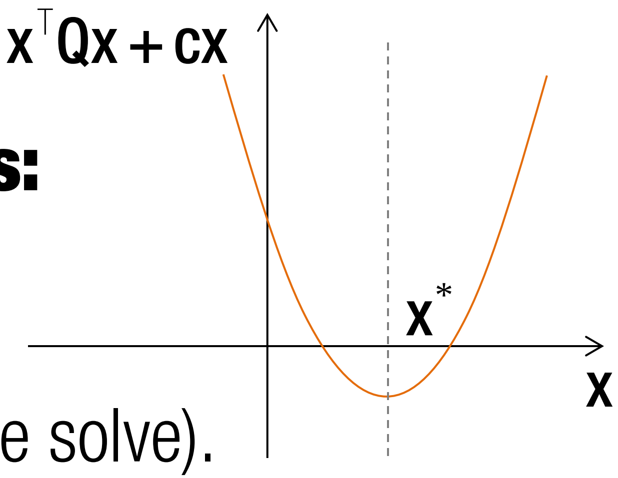
$$
\begin{aligned}
\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|^2 &= \min_{\mathbf{x}} (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) \\
&= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax} - \mathbf{b}^\top \mathbf{b} \\
&= \min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax} \\
&= \min_{\mathbf{x}} \underbrace{\mathbf{x}^\top \mathbf{Qx} + \mathbf{cx}}_{\text{Quadratic equation in } \mathbf{x}.}
\end{aligned}
$$

where $\mathbf{Q} = \mathbf{A}^\top \mathbf{A}, \quad \mathbf{c} = -2\mathbf{b}^\top \mathbf{A}$

$\mathbf{Q}$ is positive definite (all eigen values of $\mathbf{Q}$ are positive). $\longrightarrow$ There exists the global solution $\mathbf{x}$.



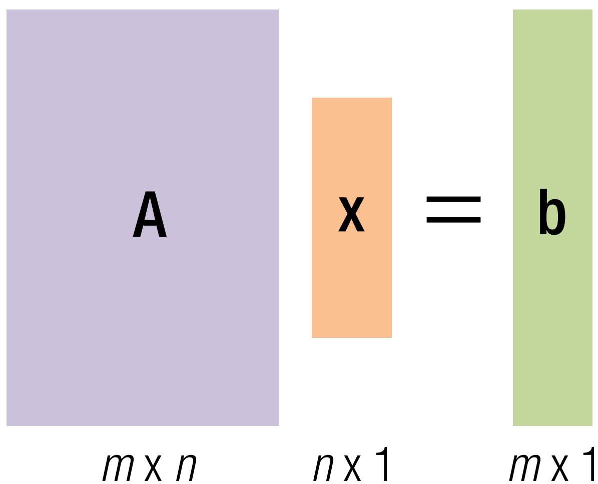
$\mathbf{x}^*$: global solution

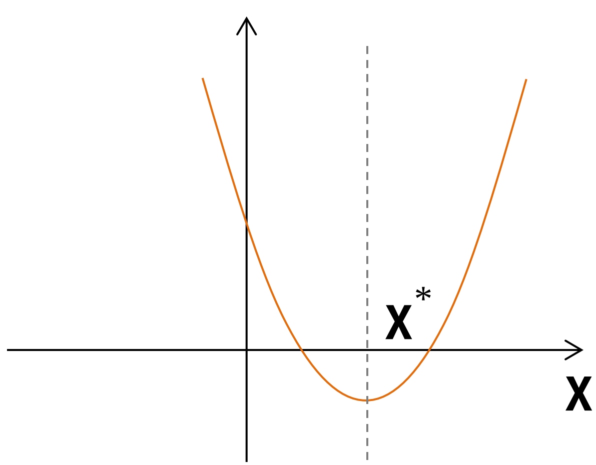


The properties of *linear* least squares:

- Has the global/unique solution.
- Has the closed form solution (non-iterative solve).
- Is solved efficiently (SVD).
- Requires no extra parameters such an initialization.

Life isn't that easy.





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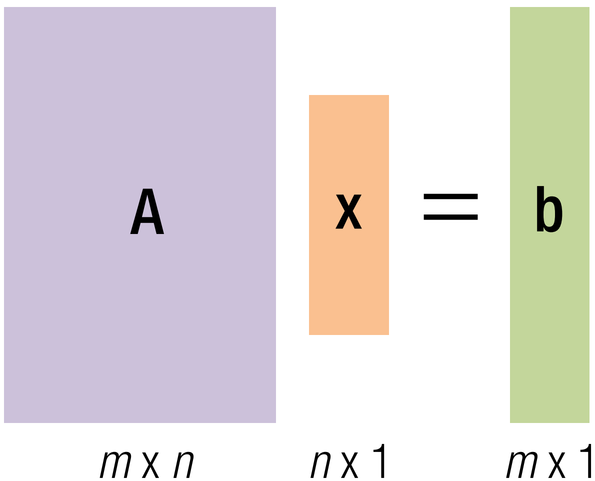

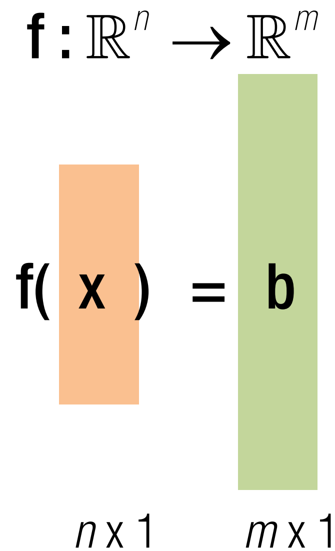

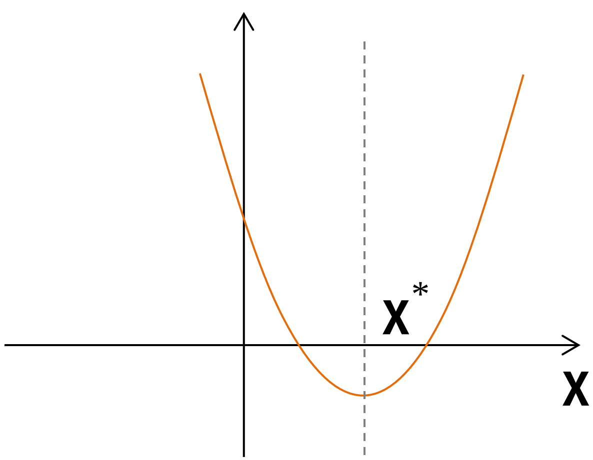

Life isn't that easy.