Measurement of wave harmonics with a lock-in amplifier

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28 September 2020

1 Abstract

This report describes the use of a lock-in amplifier to measure harmonics of periodic waveforms, with example measurements presented for square and triangle waves. The simulated results agree almost perfectly with the theoretical Fourier coefficients, the only exception being a 3% discrepancy in the $n = 11$ harmonic of the square wave.

2 Introduction

Lock-in amplifiers are extremely useful when trying to isolate a weak signal of known frequency in the presence of noise several orders of magnitude larger. However, its ability to pick out a narrow frequency range from a waveform also gives it the ability to identify harmonics. In this lab, a simulated lock-in amplifier is used to identify the first several harmonics of a square wave and a triangle wave, and the resulting signals are compared with the theoretical coefficients of their respective Fourier series.

3 Background

Square waves and triangle waves can both be represented by an infinite sum of the form

$$f(t) = \sum_{n=1}^{\infty} a_n \sin(nt) \tag{1}$$

where f represents a wave with zero phase shift and period 2π, and the index n determines the angular frequency of each component. The coefficients a_n, known as the Fourier coefficients, are found by taking the inner product of the wave function with the corresponding sine term:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt)\ dt \tag{2}$$

The first several coefficients for a sine wave and triangle wave of amplitude one are shown in Table 1.

n	a_n (square)	a_n (triangle)
1	$\frac{4}{\pi}$	$\frac{8}{\pi^2}$
2	0	0
3	$\frac{4}{3\pi}$	$-\frac{8}{9\pi^2}$
4	0	0
5	$\frac{4}{5\pi}$	$\frac{8}{25\pi^2}$
6	0	0
7	$\frac{4}{7\pi}$	$-\frac{8}{49\pi^2}$
8	0	0
9	$\frac{4}{9\pi}$	$\frac{8}{81\pi^2}$

Table 1: First nine Fourier coefficients for square and triangle waves. The general expressions for the odd terms are $a_n = \frac{4}{n\pi}$ and $a_n = \frac{8}{n^2\pi^2}(-1)^{(n-1)/2}$ for the square and triangle wave, respectively, while the even terms are zero.

4 Measurements

When the signal and reference frequencies are perfectly aligned, the amplifier output should be a constant voltage (after a settling period of 50∼150ms). This steady output voltage is used to produce the values below in Table 3. However, when the frequencies are slightly misaligned, the output oscillates, with the amplitude depending on the frequency difference and the 3dB point of the amplifier. To characterize this response, the output amplitude was measured for a 1.0kHz reference and 1.1kHz signal over three 3dB settings. The results, shown in Table 2 below, indicate that the oscillation amplitude is large when the signal is roughly within $\pm f_{3dB}$ of the reference.

f_{3db} (Hz)	Amplitude (V)
10	0.019
100	0.176
500	0.263

Table 2: Output oscillation amplitude for f_{ref}=1.0kHz and f_{sig}=1.1kHz over a selection of f_{3dB} values.

To identify the harmonics, the signal (1kHz square or triangle wave) was passed into the input of a simulated lock-in amplifier. The reference frequency f_{ref} was varied in steps of 1kHz from the fundamental mode of 1kHz up to 11kHz. The results, as well as the theoretical coefficients scaled to match the measured value for $n = 1$, are shown below in Tables 3 and 4.

The signs and relative magnitudes of the measured responses for both waveforms match almost perfectly with the scaled theoretical predictions. The only

f_{ref} (kHz)	V_{square} (V)	$V_{triangle}$ (V)
1.00	0.637	0.406
2.00	0.000	0.000
3.00	0.212	-0.045
4.00	0.000	0.000
5.00	0.127	0.016
6.00	0.000	0.000
7.00	0.091	-0.008
8.00	0.000	0.000
9.00	0.071	0.005
10.0	0.000	0.000
11.0	0.060	-0.003

Table 3: Measured amplifier response for 1.0kHz square and triangle waves over a range of reference frequencies.

n	a_n (square)	a_n (triangle)
1	0.637	0.406
2	0.000	0.000
3	0.212	-0.045
4	0.000	0.000
5	0.127	0.016
6	0.000	0.000
7	0.091	-0.008
8	0.000	0.000
9	0.071	0.005
10	0.000	0.000
11	0.058	-0.003

Table 4: Table of theoretical Fourier coefficients for square and triangle waves, scaled such that the first coefficient matches the measured data.

discrepancy is the $n = 11$ term for the square wave, with a difference of 0.02, corresponding to an error of about 3%. The discrepancy was still present after increasing the time resolution of the simulation, and its cause is unknown.

5 Conclusion

While its intended application is the measurement of weak signals in the presence of noise, this report shows how a lock-in amplifier can instead be used to determine the harmonic frequencies of a periodic waveform. Compared to the theoretical Fourier coefficients, the lock-in amplifier measured the correct values almost perfectly, with the only exception being a 3% difference on the $n = 11$ term of the square wave. Further investigation on this topic could involve determining the cause of this discrepancy, as well as measuring the Fourier coefficients of more complex waveforms.