Experimental determination of gravitational acceleration with a pendulum

Griffin Kowash

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1 Introduction

Using a pendulum to measure g, the acceleration due to gravity at Earth's surface, is a staple laboratory of introductory physics courses. In addition to providing a concrete example of simple harmonic motion, the lab itself can be performed with little more than a string, a weight, and a timer.

However, this exercise rarely yields a satisfying result. Errors may be large and are often unquantified, so getting within 30% or so of the accepted value is generally considered good enough. This report attempts to give a more careful treatment to the classic pendulum lab, both to obtain a more accurate estimate of g and to illustrate the principles of uncertainty analysis in a real-world context.

2 Theoretical background

The equation of motion for a simple pendulum is

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{1}$$

where θ is the angle of the pendulum from vertical, L is the length of the pendulum, and g is the acceleration due to gravity. To simplify the problem, the small-angle approximation $\sin\theta \approx \theta$ is typically invoked under the condition that the amplitude of oscillation remains below 10-20°.

With this approximation, the solution to the differential equation is that of a simple harmonic oscillator,

$$\theta(t) = \theta_0 \cos\omega t \tag{2}$$

where θ_0 is the amplitude and $\omega = \sqrt{\frac{g}{L}}$ is the angular frequency. Solving for g gives the equation

$$g = 4\pi^2 \frac{L}{T^2} \tag{3}$$

which allows the gravitational acceleration to be calculated by measuring its length and its period.

This approach yields reasonable estimates of g in introductory physics labs, but for greater precision, the small-angle approximation is not adequate. Equation 1 can be solved numerically, but another approach is to use a more precise approximation to account for the $\sin\theta$ term, such as the approximation proposed by Lima and Arun [1]. Their result gives

$$g = 4\pi^2 \frac{L}{T^2} (\frac{\ln a}{1-a})^2 \tag{4}$$

where $a = \cos\frac{\theta_0}{2}$. Since the first several terms of the expression are identical to the result derived using the small-angle approximation, we can rewrite this as

$$g = g_0\beta \tag{5}$$

where β, the term in parentheses at the end of Equation 4, acts on g_0, the result from the small-angle approximation, to correct for the amplitude. The table below lists values of β for a selection of angles.

θ_0	β
5°	1.001
10°	1.004
15°	1.009
20°	1.015
30°	1.035
45°	1.082
60°	1.153

Table 1: Values of the correction factor β for a selection of amplitudes.

3 Experimental apparatus

The experimental apparatus (Figure 1) consists of a length of wood, a string, and a medicine bottle filled with coins used as the weight. The length of the pendulum can be precisely adjusted by rotating the spool fixed to the end of the plank (Figure 2b). The string passes through a small hole in the plank, which is tapered to reduce contact between the string and the wood as the pendulum oscillates. The length is measured by aligning black marks on the string, spaced every 10cm, with the lip of the hole. A medicine bottle is used as the weight to allow for convenient adjustment of the mass by adding or removing coins, while maintaining a constant geometry to control for drag.

To measure amplitude, black marks corresponding to different angles are drawn on the cardboard box just behind the pendulum's plane of oscillation (Figure 3). The positions of the marks are calculated using the vertical distance

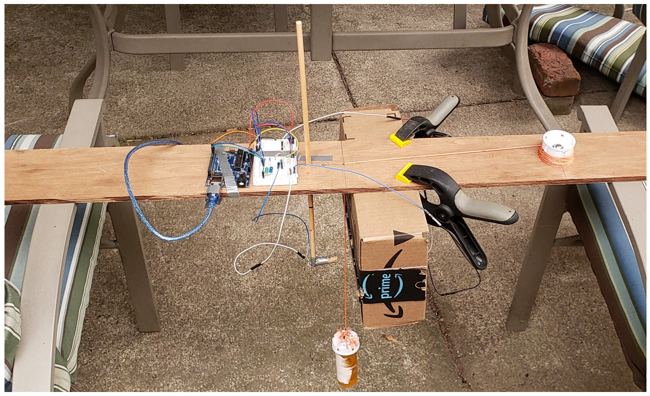
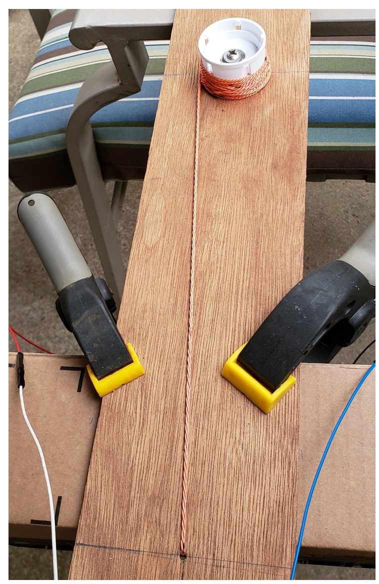


Figure 1: Overview of experimental apparatus.

(a)



(b)

Figure 2: Photos of experimental apparatus showing (a) light sensor and pendulum mass, and (b) string adjustment mechanism.

to the pendulum's pivot point. Due to a lack of space, marks for angles above 30°are located closer to the pivot point, which adds significantly more error for those measurements.

To measure the period, an LED and photoresistor are positioned on either side of the pendulum at the bottom of its arc and connected to an Arduino board (Figure 2a). When the pendulum string passes between the LED and photoresistor, the Arduino detects a drop in the light level and records the timestamp

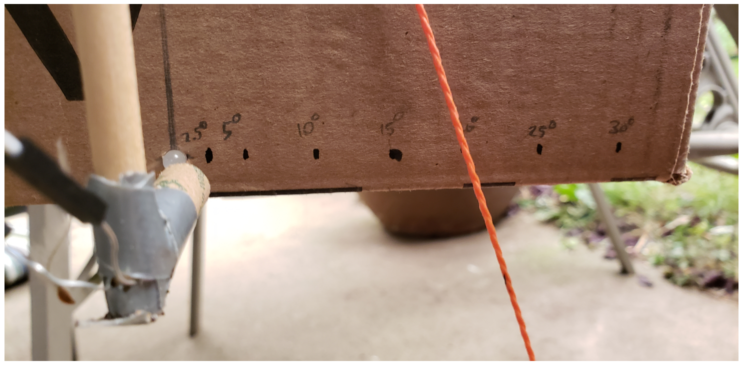
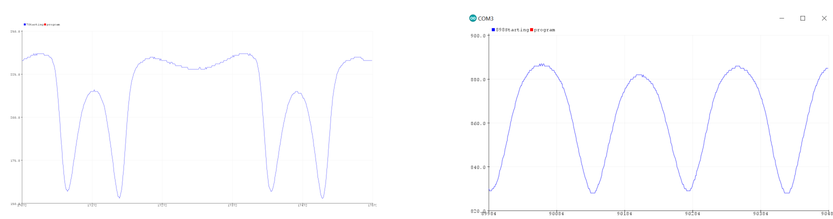


Figure 3: Angle markings used to measure pendulum amplitude.

of the event. Since each full oscillation triggers two of these events, finding the difference between pairs separated by one event gives the period. Alternatively, if several events are recorded, the entire duration can be divided by the number of full oscillations to obtain the average period. Seven full oscillations are averaged for most measurements, although results are very consistent with as few as three.

Before recording measurements, the sensor is aligned at the bottom of the pendulum's arc by adjusting its position until consecutive half-periods are equal in length. This can be achieved fairly accurately by creating a very low amplitude oscillation and observing the output in the Arduino's serial plotter, as shown in Figure 4.



(a)

(b)

Figure 4: Plots used for sensor alignment, showing the output from (a) a misaligned sensor and (b) a properly aligned sensor.

4 Uncertainty analysis

It is crucial to analyze the sources of error to establish a sense of the experiment's accuracy. For Equation 3, derived using the small-angle approximation, the relevant measurements are the period and length of the pendulum. In addition, while the equation does not explicitly depend on mass, it is used in the center of mass calculation described below, so its error must also be considered. The more advanced approximation in Equation 4 also depends on period, length, and mass, but the a term introduces an amplitude dependence as well.

The Arduino board is programmed to check the sensor output every 1ms, so a reasonable error estimate for the period is 0.5ms. For a typical period of 1.5s, this corresponds to a fractional error of around 0.03%. Given the much larger errors present in other measurements, the error in the period is neglected.

The length of the pendulum is a more difficult value to measure. To obtain an accurate estimate, the shape and mass distribution of the system must be taken into account to locate the center of mass. As discussed in the Experimental Apparatus section, the pendulum mass is a medicine bottle holding up to 39 US quarters. The center of mass of the bottle is estimated by balancing it on the edge of a ruler, while the center of mass of the quarter stack is assumed to be halfway along its length. The measured positions and masses of the system are listed below.

	Bottle	Quarters (39)
COM position	3.34±0.2 cm	5.16±0.1 cm
Mass	11.2±0.05 g	220.1±0.05 g

Table 2: Measurements of mass and center of mass locations for the components of the pendulum mass. Center of mass positions are measured relative to the lid of the bottle.

The string of the pendulum also contributes slightly to the center of mass. To investigate this effect, the string's linear mass density was measured to be about 0.0002g/cm. For a length of 50cm, this corresponds to a mass of 0.01g. Treating it as a point mass located at -25cm, it would raise the total center of mass by about 10μm, an amount that is dwarfed by the other sources of error. As a result, the contribution from the string was omitted from the final calculation.

Using these values and their associated errors, the center of mass of the pendulum system is estimated to be at 5.1±0.1 cm relative to the lid of the bottle. The gap between the lid and the arbitrary "0 cm" mark on the string is 2.0±0.1 cm. The distance from the "0 cm" mark to the pendulum's pivot point carries an additional error of approximately 0.2cm (although the error likely increases slightly with length due to the way it was measured). Therefore, when the pendulum is set to a length L_0 in centimeters as measured by the markings on the string, the physical length is

$$L = (L_0 + 7.1) \pm 0.4cm \tag{6}$$

Finally, the uncertainty in the angle must be evaluated. The calculation begins with the measurements x and y, which respectively are the horizontal and vertical distances from each angle marking to the pivot point. These values are used to calculate the angle θ, which in turn goes into the calculation of β through the intermediate variable a. The uncertainties at each step (denoted by

δ) are calculated by multiplying the corresponding uncertainty by the derivative of the function with respect to the relevant variable. The relationships used in the error propagation are shown below in Equations 7-10, while the numerical results are listed in Table 3.

$$r = \frac{x}{y} \qquad \delta r = \frac{x}{y}(\frac{\delta x}{x} + \frac{\delta y}{y}) \tag{7}$$

$$\theta = \arctan r \qquad \delta\theta = \frac{1}{1+r^2}\delta r \tag{8}$$

$$a = \cos\frac{\theta}{2} \qquad \delta a = \frac{1}{2}\sin\frac{\theta}{2}\delta\theta \tag{9}$$

$$\beta = (\frac{\ln a}{1-a})^2 \qquad \delta\beta = 2\frac{\ln a}{(1-a)^3}(\ln(a) + \frac{1}{a} - 1)\delta a \tag{10}$$

θ_0	$\delta\theta$	$\frac{\delta\beta}{\beta}$
5°	1.7°	0.06%
10°	1.8°	0.1%
15°	1.8°	0.2%
20°	1.9°	0.2%
25°	1.9°	0.3%
30°	1.8°	0.4%
35°	6.7°	1.8%
40°	6.3°	2.0%
45°	5.8°	2.1%
50°	5.3°	2.1%
55°	4.7°	2.1%
60°	4.0°	2.0%

Table 3: Uncertainty $\delta\theta$ in measured amplitude θ_0 and resulting fractional uncertainty in correcting factor β.

The significant increase in error between 30° and 35° is due to a quirk of the experimental setup that required the marks for larger angles to be located closer to the pivot point.

The last step in evaluating the uncertainties is to add the fractional uncertainties in β and L to find the uncertainty in the calculated value of g. This is the procedure used to create shaded error regions around the g versus θ_0 curves in the Results sections below.

An important factor neglected in the error analysis is the impact of air resistance. While it may introduce a systematic error in the results by increasing the calculated value of g, steps have been taken to reduce its impact. For instance, the design of the pendulum weight ensures that the drag coefficient remains constant as mass is added or removed. In addition, the majority of tests were performed with a relatively large mass, which helps reduce the impact of drag forces. The brief exploration of lower mass regimes discussed in the Results section below confirms that lower masses result in a much greater effect from drag on the calculated value of g.

Friction, particularly at the pivot point, is another factor neglected in the analysis. While the apparatus was designed to fix the string to a single contact point and reduce sliding against surfaces, the design can certainly be improved. One possibility is to suspend the pendulum using a strong clamp holding the end of the string.

A final consideration is the behavior of the detection mechanism below about 10°. At small amplitudes, there is increased noise present in the data, as well as double-counting errors by the sensor program. Double-counted points are corrected in the final analysis, but the higher noise in this regime may make the signal less reliable.

5 Results

5.1 Effect of pendulum amplitude on period

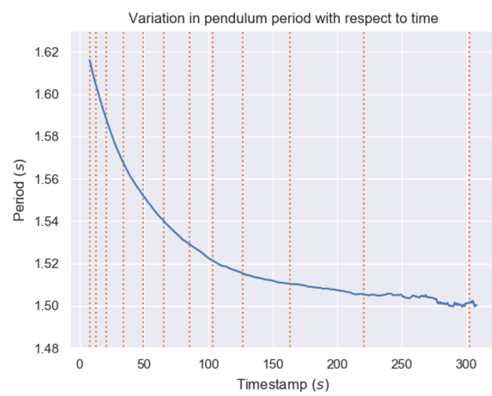


Figure 5: Decay of pendulum period over time. Vertical dotted lines denote the measured amplitude, starting at 60°on the left and decreasing in intervals of 5°.

The amplitude of oscillation is the major variable examined in this lab. As discussed in the Background section, the calculation of g is performed twice, once using the small-angle approximation and again using the approximation proposed by Lima and Arun. The generally accepted upper limit for the small-angle approximation is 10-20°, so to quantify its effect in this experiment, the pendulum period was recorded as its amplitude decayed from 60°to 5°over several minutes. The period data (with outliers removed and a boxcar average applied) is shown above in Figure 5.

To convert this time series into a period-amplitude relationship, the 5°intervals were linearly interpolated to assign each period measurement an estimated amplitude. For example, the midpoint of the 40-45°interval, with a measured period slightly under 1.56s, was assigned an angle of 42.5°. While not perfect, the curve within each interval is linear enough to make this method a reliable approximation. The result of the interpolation is shown in Figure 6 below.

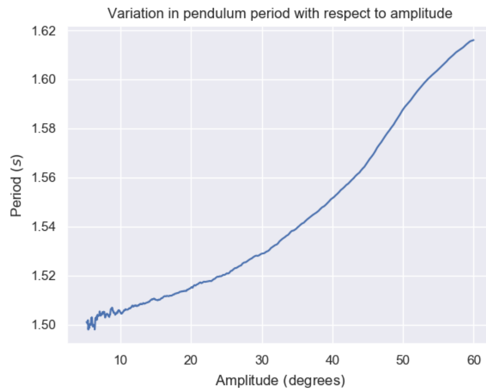
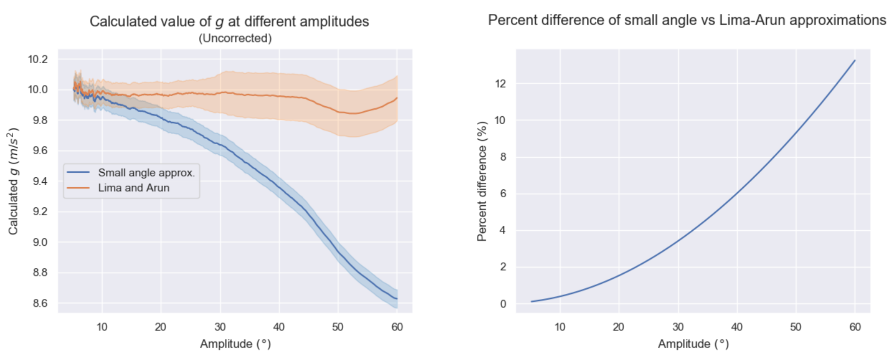


Figure 6: Approximate period as a function of pendulum amplitude.

Next, the period-amplitude data was used to calculate the gravitational acceleration g using the small-angle and Lima-Arun approximations. Uncertainties, depicted by the shaded regions around each curve, were calculated using the uncertainty in L and, for the Lima-Arun curve, the uncertainty in θ. The results of these calculations, as well as the percent difference between the two methods, are shown below in Figure 7.



(a) (b)

Figure 7: Plots showing (a) calculated value of g as a function of amplitude, and (b) percent difference between the small-angle and Lima-Arun approximations.

It is apparent from the plots that the small-angle approximation diverges dramatically from the Lima-Arun approximation at larger amplitudes, and even at small angles the difference is significant. While the small-angle approximation consistently decreases as the amplitude increases, the Lima-Arun approximation remains nearly constant up to at least 30°. Based on these results, the small angle approximation does not seem sufficient for more precise measurements.

The best estimate for g was obtained by averaging the Lima-Arun values and uncertainties from 10°to 30°. This region was selected because it is approximately constant and it has minimal noise and uncertainty. The calculation gives a value of

$$g = 9.97 \pm 0.08 m/s^2 \tag{11}$$

However, there is a crucial problem with this result. Unfortunately, the period data was collected outside on a day with strong winds, and the periods measured were consistently different from those collected previously without wind. Due to the consistency of the discrepancy, we make the major assumption that the wind introduced a systematic error to the period measurements. Using the average period of 1.5190s measured on a windless day at 15°with a 50cm string, there is an 8.8ms difference from the average 1.5102s period measured on the windy day under otherwise identical conditions. In an attempt to correct this systematic error, 8.8ms was added to each period measurement and used to recalculate g. The resulting plot and final estimate are shown below in Figure 8 and Equation 12.

$$g = 9.85 \pm 0.08 m/s^2 \tag{12}$$

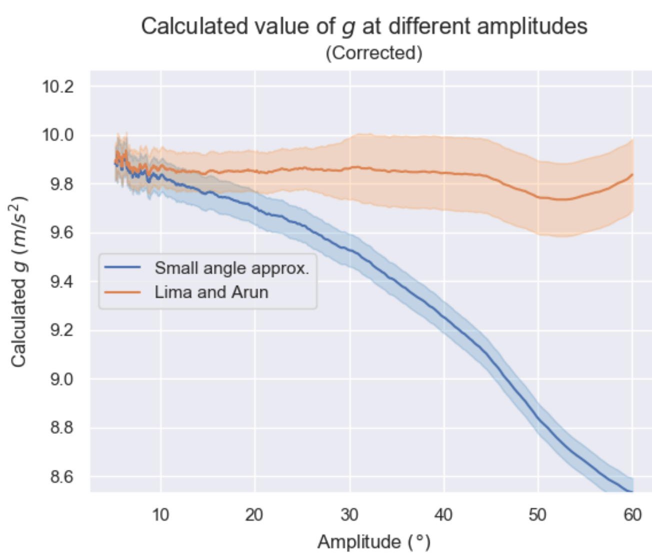


Figure 8: Calculated value of g as a function of amplitude, with an 8.8ms correction added to each period measurement.

While there is reasonable justification for this correction, it should be regarded with some skepticism. To ensure that it is not a result of a flawed assumption or experimenter bias, the measurements should be repeated either on a windless day or indoors.

If we accept the correction as accurate, the corrected estimate and uncertainty is consistent with the accepted value of $g = 9.81m/s^2$. Otherwise, the uncorrected estimate in Equation 11 is inconsistent with the accepted value. This would indicate either that there is a systematic error in the measurements other than the effect of wind, or that the uncertainties were underestimated.

5.2 Effect of pendulum mass

From the derivation described in the Background section, the mass would ideally have no effect on the measured period. In reality, factors such as air resistance and friction may affect the measured period over different mass regimes for two reasons. First, a lower mass may lead to a higher acceleration due to drag, leading to an underestimate of g. Second, the greater effect of drag may make it more susceptible to fluctuations in wind or air density.

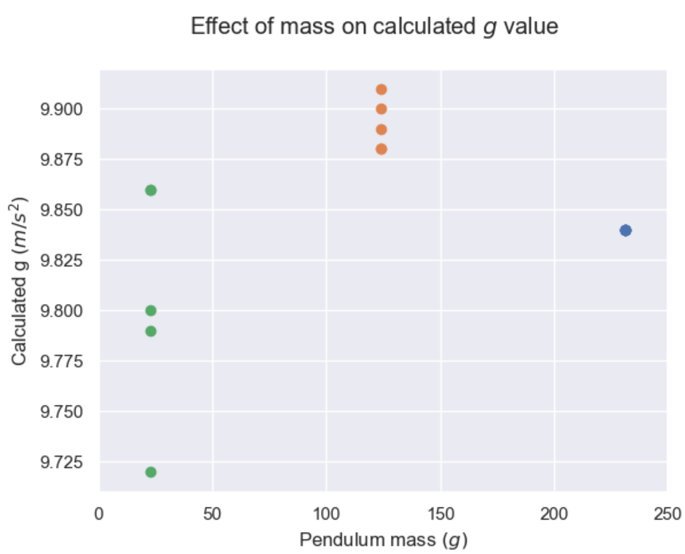


Figure 9: Calculated values of g over three mass regimes. The corresponding number of US quarters for each regime is 2, 20, and 39.

To test this, the pendulum mass was modified by changing the number of coins in the canister, with the drag coefficient held constant by design. The pendulum was set to a length of 50cm and released from an amplitude of 15°. Calculations of g using the small-angle approximation (with the length estimates adjusted to account for shifts in center of mass) over three mass regimes are plotted above in Figure 9.

Based on the plot, a higher mass reduces the spread of measured values, giving a more precise estimate of g. More data is required to determine whether there is a systematic effect due to varying mass, especially because the values were collected on a day with strong winds.

5.3 Effect of pendulum length

To briefly explore the effect of length, g was calculated using the Lima-Arun approximation at five lengths between 0.2m and 0.6m, with an amplitude of 15°and a mass of 39 US quarters. The results are shown below in Figure 10.

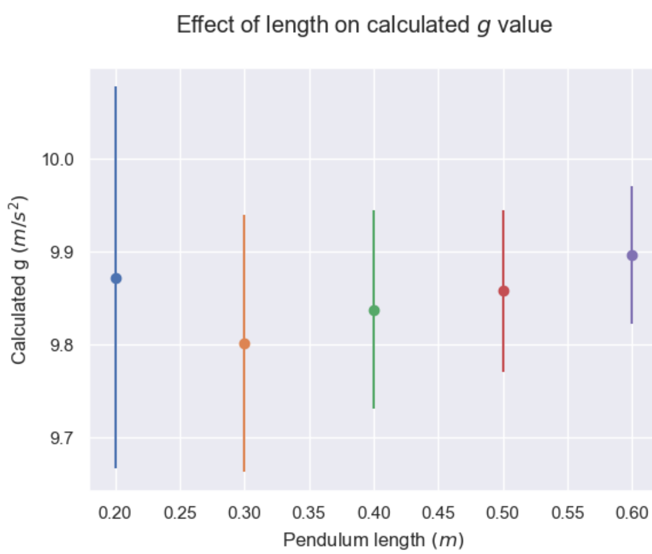


Figure 10: Calculated values of g at five different lengths between 0.2m and 0.6m.

Given the relatively large error margins, it is not clear whether there is any effect due to the pendulum length. Furthermore, the errors on the longer lengths are likely underestimated, as error in length markings on the pendulum string probably increases with length. However, the somewhat linear increase between 0.3m and 0.6m suggests that there may be a trend. This feature is worth investigating further to determine whether it interferes with an accurate estimation of g.

6 Conclusion

The design of this experiment has plenty of room for improvement. With more care, uncertainties can be reduced further, particularly for the angle measurements. In addition, now that mass has been investigated as a variable, the medicine bottle design can be replaced with a simpler geometry whose center of mass can be determined more reliably. Another crucial step is to investigate the effect of length, as the data presented here is inconclusive. Finally, the impact of drag should be examined, since damping forces change the period of an oscillator.

Determining the acceleration due to gravity at Earth's surface using a pendulum is a more complex problem than it seems at first glance. In addition to the non-linearity introduced by the $\sin\theta$ term in the equation of motion, reducing and characterizing error in the experimental design takes careful thought and analysis. However, despite variability due to amplitude, length, and weather conditions, the results of this experiment were consistently within 1-2% of the accepted value. The extra effort for precision and accuracy clearly pays off.

References

[1] Lima, F.M.S & Arun, P. An accurate formula for the period of a simple pendulum oscillating beyond the small angle regime. July 2006. https://arxiv.org/pdf/physics/0510206.pdf